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Bayesian Semi-parametric Realized-CARE Models for Tail Risk Forecasting Incorporating Range and Realized Measures

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Keywords: Realized-CARE, Realized Variance, Realized Range, Subsampling Realized Range, Target Search, Markov Chain Monte Carlo, Value-at-Risk, Expected Shortfall.

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Keywords: Realized-CARE, Realized Variance, Realized Range, Subsampling Realized Range, Target Search, Markov Chain Monte Carlo, Value-at-Risk, Expected Shortfall.

1 INTRODUCTION

In recent decades, quantitative financial risk measurement has become a fundamental tool for investment decisions, capital allocation and external regulation. The recent Global Financial Crisis (GFC) has once again emphasized the importance of accurate risk measurement and prediction for financial organizations. Value-at-Risk is a quantitative tool to measure and control financial risk. It represents the market risk as one number and has become a standard measurement for capital allocation and risk management since it was proposed in 1993. However, VaR has been criticised, because it cannot measure the expected loss for violating returns. Expected Shortfall, which was proposed by Artzner *et al.* (1997, 1999), gives the expected loss conditional on returns exceeding a VaR threshold, because ES is a coherent measure and it has been used widely for tail risk measurement.

Volatility estimation plays key role in calcualting accurate VaR or ES. Since the introduction of the Auto-Regressive Conditionally Heteroskedastic (ARCH) model of Engle (1982) and the generalised (G)ARCH of Bollerslev (1986) employing squared return as model input, different volatility estimators and GARCH type models were developed in the past decades. Parkinson (1980) and Garman and Klass (1980) considered the daily high-low range as an improved volatility estimator compared to the daily return. The availability of high frequency intra-day data generated several popular and efficient realized measures, including Realized Variance (RV): Andersen and Bollerslev (1998); Andersen *et al.* (2003), Realized Range (RR): Martens and van Dijk (2007); Christensen and Podolskij (2007). In order to deal with the micro-structure noisy accompanied by the high frequency volatility measures, Zhang, Mykland and Aït-Sahalia (2005) and Martens and van Dijk (2007) designed the subsampled and scaled process respectively. Regarding the volatility modelling, Hansen *et al.* (2001) proposed a volatility framework named Realized-GARCH (Re-GARCH), which incorporated a measurement equation that connects the unobserved volatility with RV. Gerlach and Wang (2015) extended Re-GARCH model through employing RR (called RR-GARCH) and proved that the proposed RR-GARCH framework can generate more accurate and efficient VaR and ES forecasts than that of the traditional GARCH and Re-GARCH.

However, the performance of parametric GARCH type models heavily depend on the choice of error distribution. A semi-parametric model named Conditional Autoregressive Expectile (CARE) was proposed by Taylor (2008). The expectile can be estimated with Asymmetric Least Square (ALS), then it is transformed into ES through a connection between expectile and ES (Newey and Powell, 1987). Gerlach, Chen and Lin (2012) developed non-linear family of the CARE model and the Bayesian estimation framework. Further, Gerlach and Chen (2015) extended the CARE type models with employing daily high-low range.

In this paper, we propose a Realized Conditional Autoregressive Expectile (Re-CARE) framework, which is analogous to the Re-GARCH. Re-CARE adds a measurement equation that links the latent conditional expectile with Range & realized measures into the conventional CARE model, and an adaptive Bayesian algorithm is developed to estimate Re-CARE parameters. Comparing to the grid search of calcuating optimal expectile level in Taylor (2008), the quadratic fitting grid search of expectile level we proposed significantly reduces the time of calculation and the maintains the search accuracy. To evaluate the performance of proposed Re-CARE, the forecasting accuracy of VaR and ES with Re-CARE employing Range and various realized measures will be assessed and compared with traditional GARCH and Re-GARCH.

The paper is organized as follows: Section 2 introduces different realized measures and proposes the sub-sampled RR. The expectile, its connection with, existing CARE type models and Re-GARCH type models are reviewed in Section3. Section 4 proposes the Realized-CARE type models, and the adaptive Bayesian for parameters estimation is presented in Section 5. The simulation and empirical studies are discussed in Section 6 and Section 7 respectively. Finally, Section 8 concludes the paper and discusses about the future work.

2 REALIZED MEASURES

This section reviews the different volatility estimators and proposes the sub-sampled Realized Range.

For day t, representing the daily high, low and closing prices as H_t, L_t and C_t, the most commonly used daily log return is:

$$r_t = \log(C_t) - \log(C_{t-1})$$

The high-low range (squared) was proposed by Parkinson (1980) and was proved to be a much more efficient volatility estimator than return, based on the range distribution theory (Feller, 1951):

$$Ra_t^2 = \frac{(\log H_t - \log L_t)^2}{4\log 2}$$

where 4log(2) scales Ra to be a unbiased return variance estimator. Ra was further improved in the following decades (Garman and Klass, 1980; Rogers and Satchell, 1991; Yang and Zhang, 2000), and see Molnár (2012) for a full review regarding the properties of various volatilities estimator.

Range considering the overnight price jump was proposed in Gerlach and Chen (2015):

$$Rao_t = \log\big(\max(H_t, C_{t-1})\big) - \log\big(\min(L_t, C_{t-1})\tag{1}$$

Supposing that the day t is divided into N equally sized intervals of length Δ, we have the subscription of each intra-day set $\Theta = 0, 1, 2, ..., N$ and can calculate the high frequency volatility measures. For day t, denoting the *i-th* interval closing price as $P_{t-1+i\triangle}$, $H_{t,i} = \sup_{(i-1)\triangle<j<i\triangle} P_{t-1+j}$ and $L_{t,i} = \inf_{(i-1)\triangle<j<i\triangle} P_{t-1+j}$ will represent the high and low prices during this time interval. Then RV was proposed as (Andersen and Bollerslev, 1998)):

$$RV_t^{\triangle} = \sum_{i=1}^{N} [log(P_{t-1+i\triangle}) - log(P_{t-1+(i-1)\triangle})]^2 \tag{2}$$

Further, Martens and van Dijk (2007) and Christensen and Podolskij (2007) developed the Realized Range, which sums the squared intra-day range.

$$RR_t^{\triangle} = \frac{\sum_{i=1}^{N}(\log H_{t,i} - \log L_{t,i})^2}{4\log 2} \tag{3}$$

Through theoretical derivation and simulation, Martijns and van Dijk (2007) proved RR is a more efficient volatility estimator than RV. Gerlach and Wang (2015) confirmed that RR can provide extra efficiency in empirical tail risk forecasting.

However, the accuracy of RV and RR is tend to be affected by the microstructure noise. Martens and van Dijk (2007) presented a scaled process to tackle such noise, as in Equations (4) and (5).

$$RV_{S,t}^{\triangle} = \frac{\sum_{l=1}^{q} RV_{t-1}}{\sum_{l=1}^{q} RV_{t-1}^{\triangle}} RV_t^{\triangle}, \tag{4}$$

$$RR_{S,t}^{\triangle} = \frac{\sum_{l=1}^{q} RR_{t-1}}{\sum_{l=1}^{q} RR_{t-1}^{\triangle}} RR_t^{\triangle}, \tag{5}$$

where RV_{t-1} and RR_{t-1} represents the daily return square and range square at day $t-1$. This scaling process is inspired by the fact that the daily return and range are less affected by micro-structure noise than their high frequency counterparts, thus can be used as an unbiased indicator of the actual level of volatility.

In addition, Zhang, Mykland and Aït-Sahalia (2005) proposed a sub-sampled process to deal with the micro-structure noise and it will be studied in this paper as well. For day t, N equally sized samples are grouped into M non-overlapping subsets $\Theta^{(m)}$ with size $N/M = n_k$, which means:

$$\Theta = \bigcup_{m=1}^{M} \Theta^{(m)}, \text{ where } \Theta^{(k)} \cap \Theta^{(l)} = \emptyset, \text{ when } k \neq l.$$

Then sub-sampling will be implemented on the subsets Θ^i with n_k interval:

$$\Theta^i = i, i+n_k, ..., i+n_k(M-2), i+n_k(M-1), \text{ where } i = 0, 1, 2..., n_k - 1.$$

Representing the log closing price at the *i-th* interval of day t as $C_{t,i} = P_{t-1+i\triangle}$, the RV with the subsets Θ^i is:

$$RV_i = \sum_{m=1}^{M} (C_{t,i+n_k m} - C_{t,i+n_k(m-1)})^2; \text{ where } i = 0, 1, 2..., n_k - 1.$$

We have the T/M RV with T/N sub-sampling as (supposing there are T minutes per trading day):

$$RV_{T/M,T/N} = \frac{\sum_{i=0}^{n_k-1} RV_i}{n_k}, \tag{6}$$

Then, denoting the high and low prices during the interval $i+n_k(m-1)$ and $i+n_km$ as $H_{t,i} = \sup_{(i+n_k(m-1))\triangle<j<(i+n_km)\triangle}P_{t-1+j}$ and $L_{t,i} = \inf_{(i+n_k(m-1))\triangle<j<(i+n_km)\triangle}P_{t-1+j}$ respectively, we propose the T/M RR with T/N subsampling as:

$$RR_i = \sum_{m=1}^{M}(H_{t,i} - L_{t,i})^2; \text{ where } i = 0, 1, 2..., n_k - 1. \tag{7}$$

$$RR_{T/M,T/N} = \frac{\sum_{i=0}^{n_k-1} RR_i}{4\text{log}2n_k}, \tag{8}$$

For example, the 5 mins RV and RR with 1 min subsampling can be calculated as below respectively:

$$RV_{5,1,0} = (\text{log}C_{t5} - \text{log}C_{t0})^2 + (\text{log}C_{t10} - \text{log}C_{t5})^2 + ...$$
$$RV_{5,1,1} = (\text{log}C_{t6} - \text{log}C_{t1})^2 + (\text{log}C_{t11} - \text{log}C_{t6})^2 + ...$$
$$RV_{5,1} = \frac{\sum_{i=0}^{4} RV_{5,1,i}}{5}$$

$$RR_{5,1,0} = (\text{log}H_{t0\leq t\leq t5} - \text{log}L_{t0\leq t\leq t5})^2 + (\text{log}H_{t5\leq t\leq t10} - \text{log}L_{t5\leq t\leq t10})^2 + ...$$
$$RR_{5,1,1} = (\text{log}H_{t1\leq t\leq t6} - \text{log}L_{t1\leq t\leq t6})^2 + (\text{log}H_{t6\leq t\leq t11} - \text{log}L_{t6\leq t\leq t11})^2 + ...$$
$$RR_{5,1} = \frac{\sum_{i=0}^{4} RR_{5,1,i}}{4\text{log}(2)5}$$

3 EXPECTILE AND CARE TYPE MODELS

3.1 Expectile

τ level expectile μ_τ, defined by Aigner, Amemiya and Poirier (1976), can be used to estimate the α level quantile Q_α, which means the proportion of observations below μ_τ is α. μ_τ can be estimated through minimising the following expection:

$$\text{E}(|\tau - I(Y < \mu_\tau)|(Y - \mu_\tau)^2)$$

where Y is a continuous r.v., $\tau \in [0, 1]$, $I(Y < \mu_\tau)$ indicates that it equals to 1 when $Y < \mu_\tau$. If $Y = y_1, y_2, ...y_n$, the following Asymmetric Least Square (ALS) is employed to calculate μ_τ in Taylor (2008):

$$\sum_{t=1}^{n}(|\tau - I(y_t < \mu_\tau)|(y_t - \mu_\tau)^2) \tag{9}$$

Thus there is no distribution required to calculate μ_τ. As discussed in Section 1, ES is defined as $\text{ES}_\alpha = E(Y|Y < Q_\alpha)$, which stands for the expected value of Y conditional on the set of Y that is more extreme than Q_α.

Newey and Powell (1987) found relationship between expectile and ES. If $\text{E}(Y) = 0$, Taylor (2008) showed this relationship is formulated as:

$$\text{ES}_\alpha = (1 + \frac{\tau}{(1-2\tau)\alpha_\tau})\mu_\tau \tag{10}$$

here $\mu_\tau = Q_\alpha$.

3.2 CARE type models and Re-GARCH

Taylor (2008) proposed the CARE type models that have the similar form as the CAViaR type models (Engle and Manganelli 2004), i.e. symmetric absolute value (SAV), asymmetric (AS) and indirect GARCH (IG). The CARE type models were extended into fully nonlinear family in Gerlach, Chen and Lin (2012). Here we only present the CARE-SAV model:

CARE-SAV:

$$\mu_t = \beta_1 + \beta_2\mu_{t-1} + \beta_3|r_t|$$

where r_t is the day t return, and μ_t is the τ level expectile for day t, while τ is removed from the notation for the reason of brevity. Further, Gerlach and Chen (2015) employed the Range in the CARE framework, and the Ra-CARE type models demonstrated superiority compared to the CARE using return in the tail risk forecasting.

Range-CARE-SAV

$$\begin{aligned} r_t &= \mu_t + \varepsilon_t \qquad (11) \\ \mu_t &= \beta_1 + \beta_2\mu_{t-1} + \beta_3 Ra_{t-1} \\ \varepsilon_t &\sim AG(\tau, 0, \sigma) \end{aligned}$$

where AG is the Asymmetric Gaussian distribution. However, this AG is only employed to construct a likelihood function for Bayesian algorithm, while it is not used neither in the parameters estimation nor the expectile estimation (Gerlach, Chen, and Lin, 2012; Gerlach and Chen, 2015).

An innovative Realized-GARCH framework was developed in Hansen *et al.* (2011). Comparing to the conventional GARCH model, Re-GARCH employed a measurement equation which captures the connection between unobserved volatility and the realized variance. Re-GARCH demonstrates its superiority in the empirical study comparing to GARCH.

Re-GARCH

$$r_t = \sqrt{h_t} z_t, \tag{12}$$

$$h_t = \omega + \beta h_{t-1} + \gamma x_{t-1},$$

$$x_t = \xi + \varphi h_t + \tau_1 z_t + \tau_2 (z_t^2 - 1) + \sigma_\varepsilon \varepsilon_t \tag{13}$$

$z_t \overset{\text{i.i.d.}}{\sim} D_1(0,1)$ and $\varepsilon_t \overset{\text{i.i.d.}}{\sim} D_2(0,1)$ and $x_t = RV$; $D_1(0,1) = D_2(0,1) \equiv N(0,1)$. Watanabe (2012), Contino and Gerlach (2014) further extended the model through incorporating Student-t or skewed-t (Hansen, 1994).

4 MODEL PROPOSED

Inspired by the CARE type models and Re-GARCH framework, we propose the Realized-CARE-SAV is proposed as following:

Realized-CARE-SAV (Re-CARE-SAV)

$$r_t = \mu_t + \varepsilon_t \tag{14}$$

$$\mu_t = \beta_1 + \beta_2 \mu_{t-1} + \beta_3 x_{t-1}$$

$$x_t = \xi + \phi |\mu_t| + u_t$$

where $r_t = [\log(C_t) - \log(C_{t-1})] \times 100$ is the percentage log-return for day t, $\varepsilon_t \overset{\text{i.i.d.}}{\sim} AG(\tau, 0, \sigma)$, $u_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma_u^2)$. The three equations in the Realized-CARE are named as: the *return equation*, the *expectile equation* and the *measurement equation*, respectively. The measurement equation captures the contemporaneous dependence between the expectile μ_t and realized measure x_t.

Through choosing x_t as R_t, $\sqrt{RV_t}$ and $\sqrt{RR_t}$ respectively, we proposed the Realized-CARE-SAV-Range (Re-CARE-SAV-Ra), Realized-CARE-SAV-Realized Variance (Re-CARE-SAV-RV) and Realized-CARE-SAV-Realized Range (RR-CARE-SAV-RR).

The Re-CARE framework can be easily extended into the asymmetric, indirect GARCH and nonlinear version (Engle and Manganelli, 2004; Taylor, 2008; Gerlach, Chen and Lin, 2012; Gerlach and Chen, 2015), while we focus on the Re-CARE-SAV in this paer. For example, the Realized-CARE with indirect GARCH specification:

Realized-CARE-IG (Re-CARE-IG)

$$
\begin{aligned}
& r_t = \mu_t + \varepsilon_t \\
& \mu_t = -\sqrt{\beta_1 + \beta_2 \mu_{t-1}^2 + \beta_3 x_{t-1}^2} \\
& x_t^2 = \xi + \phi \mu_t^2 + u_t \\
& \varepsilon_t \overset{\text{i.i.d.}}{\sim} AG(\tau, 0, \sigma), u_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma_u^2) \\
& x_t = R_t, \sqrt{RV_t}, \sqrt{RR_t}, \\
& \beta 1 > 0, \beta 2 > 0, \beta 3 > 0
\end{aligned}
$$

In order to guarantee that the μ_t does not diverge, it is logical that a necessary for Re-CARE-SAV type models is $\beta_2 + \beta_3 \phi < 1$. This can be derived through substituting the measurement equation into the expectile equation in Model (14).

In this paper, we also extended the Realized-GARCH (Hansen *et.al* (2011)) with setting the volatility equation in Re-GARCH into an absolute value GARCH specification (Taylor (1986); Schwert (1989)).

Realized-GARCH-Abs (Re-GARCH-Abs)

$$
\begin{aligned}
r_t &= \sigma_t \varepsilon_t^* \\
\sigma_t &= \beta_1^* + \beta_2^* \sigma_{t-1} + \beta_3^* x_{t-1} \\
x_t &= \xi^* + \phi^* \sigma_t + u_t^* \\
\varepsilon_t^* &\overset{\text{i.i.d.}}{\sim} N(0,1), u_t^* \overset{\text{i.i.d.}}{\sim} N(0, \sigma_{u^*}^2) \\
x_t &= R_t, \sqrt{RV_t}, \sqrt{RR_t},
\end{aligned}
$$

5 LIKELIHOOD AND BAYESIAN ESTIMATION

5.1 CARE Likelihood Function with AG

With $\mathbf{r} = (r_1, r_2, ..., r_n)'$, the ALS as specified in Equation 15 is employed by Taylor (2008) to estiamted μ_τ, and expectile level τ is obtained through a grid search. A set of τ values on a grid are used to estimated their corresponding μ_τ with ALS, then τ is chosen to make the violation rate (VRate) of μ_τ closest to the quantile level α. If $\alpha < 0.5$, it means we are working with the left-tail or negative risk, and on the left-tail we have $\tau < \alpha$ or expectile level is more extreme than the corresponding quantile level.

$$
\sum_{t=1}^{n} (|\tau - I(r_t < \mu_\tau)|(r_t - \mu_\tau)^2) \tag{15}
$$

Gerlach, Chen, and Lin (2012), Gerlach and Chen (2015) incorporated $\varepsilon_t \sim AG(\tau, 0, \sigma)$ into CARE framework, as specified in (11), which make the construction of likelihood function (as in 16) and Bayesian estimation feasible. AG distribution assumption is not used in parameters nor expectile estimation, and the scales factor σ is integrated out with a Jeffreys prior. Gerlach, Chen, and Lin (2012) also proved that maximizing this integrated likelihood function will produce identical τ and μ_τ estimation results as ALS approach.

$$
L(\mathbf{r}; \theta) = \big(\sum_{t=1}^{n} |\tau - I(r_t < \mu_t(\beta)|(r_t - \mu_t(\beta))^2 \big)^{-n/2} \tag{16}
$$

5.2 Realized CARE Log Likelihood

Because the Re-CARE framework has a measurement equation with $u_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma_u^2)$, the full log-likelihood function for Re-CARE (as in 14) equals to the sum of log-likelihood $\ell(\mathbf{r};\theta)$ of CARE equation and log-likelihood $\ell(\mathbf{x}|\mathbf{r};\theta)$ of the measurement equation, where $u_t = x_t - \xi - \phi|\mu_t|$. In Re-GARCH framework, the measurement equation contribute to volatility and thus the GRACH equation in-sample and predictive log-likelihood values are improved comparing to traditional GARCH. Therefore, we expect the measurement equation in Re-CARE will also facilitate the grid search of τ and estimation of μ_τ.

$$
\begin{aligned}
&\ell(\mathbf{r},\mathbf{x};\theta) = \ell(\mathbf{r};\theta) + \ell(\mathbf{x}|\mathbf{r};\theta) \\
&= \underbrace{(-n/2)log\Big(\sum_{t=1}^{n} |\tau - I(r_t < \mu_t(\beta)|(r_t - \mu_t(\beta))^2\Big)}_{\ell(\mathbf{r};\theta)} \\
&\underbrace{-\frac{1}{2}\sum_{t=1}^{n}\big(log(2\pi) + log(\sigma_u^2) + u_t^2/\sigma_u^2\big)}_{\ell(\mathbf{x}|\mathbf{r};\theta)}
\end{aligned}
$$

5.3 Quadratic Fitting for the Expectile Level Search

As discussed before, the expectile estimation CARE type model rely on the full grid search of optimal level of τ, e.g 100 equally spaced τ trial values on $[0, \alpha]$. During the empirical study, we discovered that the relationship between τ trial values and their corresponding violation rate of μ_τ is close to monotonic, which makes the relationship between τ and $|\text{VRate} - \alpha|$ close to quadratic (as in the top and bottom plots of Figure 1). Thus we propose the following 2-step quadratic fitting approach to accelerate the grid search process.

Step 1: 4 equally space τ step 1 trials values are generated between 0.0001 and $\alpha/1.5$, e.g. 0.0001, 0.0023, 0.0045, 0.0067. Here $\alpha/1.5$ is used because the empirical study shows final τ is always found to be smaller than $\alpha/1.5$ Employing $|\text{VRate} - \alpha|$ as the objective function, we can fit a quadratic function to these points, as is in Figure ... Then the

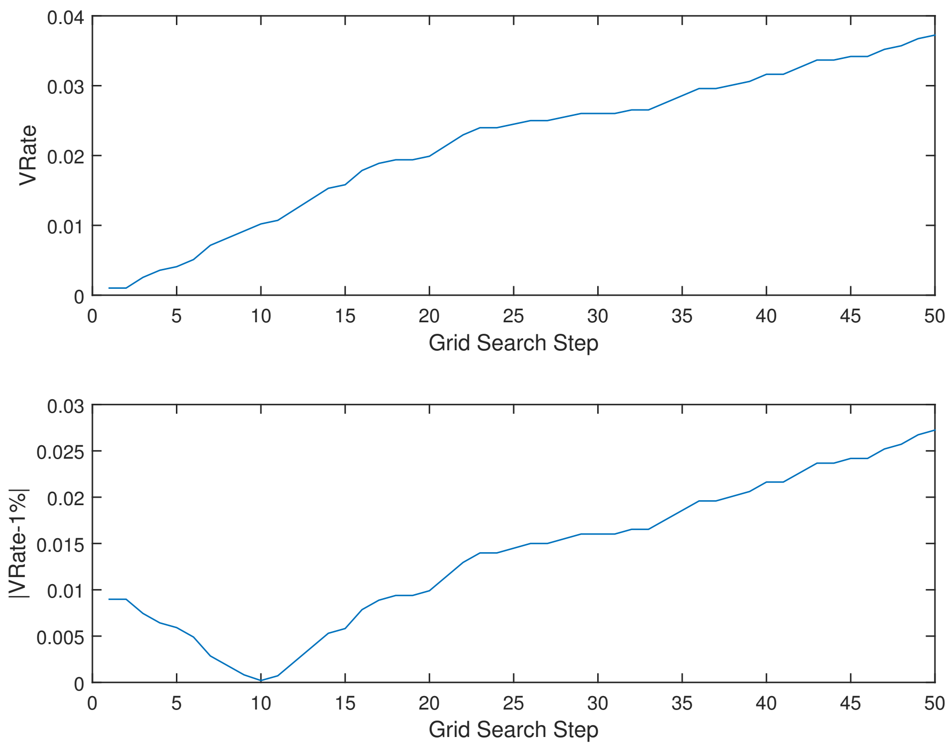

Figure 1: Expectile Grid Search VRate Plot.

stationary point c of these point can be calculated, e.g. $c = 0.0018$.

Step 2: A focused grid search around $c = 0.0018$ was incorporated afterwards. Using grid search step size as 0.0002, 8 points either side of c are used as step 2 τ trial values (as in Table 1 below) and then the final τ (0.0012) is selected. This approach can save the original grid search time approximately by 50% to 60%.

Table 1: Step 2 τ trial values of the quadratic fitting of τ grid search

0.0002	0.0004	0.0006	0.0008	0.0010	0.0012	0.0014	0.0016	0.0018
0.002	0.0022	0.0024	0.0026	0.0028	0.0030	0.0032	0.0034	

5.4 Bayesian Estimation

Through constructing the likelihood function, now we are able to employ the Bayesian algorithm to estimate 6 parameters of Re-CARE. A two-step adaptive Bayesian method that is adapted from Contino and Gerlach (2014) is employed. Firstly, 6 parameters are dived into two blocks: $\boldsymbol{\theta_1} = (\beta_1, \beta_2, \beta_3, \phi)'$ and $\boldsymbol{\theta_2} = (\xi, \sigma)'$. $beta_1, \beta_2$ and β_3 are in group because they are all in the CARE equation and change with high correlation. The stationary constraint $\beta_2 + \beta_3\phi < 1$ may make correlated movement among these 3 parameters, thus we put ϕ in the group $\boldsymbol{\theta_1}$ as well. Further, we choose the priors to be uninformative over the possible stationarity and positivity region, e.g. $\pi(\boldsymbol{\theta}) \propto I(A)$, which is a flat prior for $\boldsymbol{\theta}$ over the region A.

An adaptive MCMC algorithm, adapted from that in Contino and Gerlach (2014), employs random walk Metropolis (RW-M) for burn-in period, and independent kernel Metropolis-Hastings (IK-MH) algorithm (Metropolis et al., 1953; Hastings, 1970) for the sampling period. The burn-in period uses a Gaussian proposal distribution for the random walk process of mean vector, for each block of parameters. The covariance matrix of the proposal distribution in each block is tuned towards a target accept ratio of 23.4% (Roberts, Gelman and Gilks, 1997). Then the IK-MH sampling period incorporates a mixture of three Gaussian proposal distributions. The mean of last 10% of the burn-in period samples are used as the input of the sampling period, and the varaice-covariance matrices of three Gaussian proposal distribution are Σ, 10Σ, 100Σ respectively, where Σ is calculated as the covariance of the last 10% of the burn-in period samples, for each block.

6 SIMULATION STUDY (new simulation with 5000 replicated datasets)

A simulation study is conducted to present the comparative performance of the MCMC and ML estimation approaches, and to testify the grid search accuracy with the quadratic fitting methodology proposed, with respect to the parameter estimation, 1 step ahead VaR and ES forecasting accuracy. Both the mean and Root Mean Square Error (RMSE) values are calculated for MCMC (with quadratic fitting target grid search) and ML to demonstrate the the bias and precision properties of two approaches, and to illustrate the advantages can be realized through employing the MCMC and quadratic fitting grid search.

$N = 5000$ simulated datasets were generated from Realized-GARCH-Abs specified as Model (17). Two simulation studies are performed, with the sample sizes of simulated data sets as n=1500 and n=3000 respectively.

$$
\begin{aligned}
r_t &= \sigma_t \varepsilon_t^* \\
\sigma_t &= 0.02 + 0.75\sigma_{t-1} + 0.25 x_{t-1} \\
x_t &= 0.1 + 0.9\sigma_t + u_t \\
\varepsilon_t^* &\overset{\text{i.i.d.}}{\sim} N(0,1), u_t \overset{\text{i.i.d.}}{\sim} N(0, 0.3^2)
\end{aligned}
\tag{17}
$$

In order to calculate the corresponding Realized-CARE-SAV true values, the parameter mapping between the Realized-GARCH-Abs and Realized-CARE-SAV is required. With $\text{VaR}_t = \mu_t = \sigma_t \Phi^{-1}(\alpha)$, we have $\sigma_t = \frac{\mu_t}{\Phi^{-1}(\alpha)} = \frac{\text{VaR}_t}{\Phi^{-1}(\alpha)}$, where $\Phi^{-1}(\alpha)$ is the standard Normal inverse at α quantile level. Putting it into the GARCH and measurement equations of Model (17), we have

$$
\begin{aligned}
\frac{\mu_t}{\Phi^{-1}(\alpha)} &= 0.02 + 0.75\frac{\mu_{t-1}}{\Phi^{-1}(\alpha)} + 0.25x_{t-1} \\
x_t &= 0.1 + 0.9\frac{\mu_t}{\Phi^{-1}(\alpha)} + u_t
\end{aligned}
\tag{18}
$$

then the corresponding Realized-CARE-SAV specification can be written as:

$$
\begin{aligned}
\mu_t &= 0.02\Phi^{-1}(\alpha) + 0.75\mu_{t-1} + 0.25\Phi^{-1}(\alpha)x_{t-1} \\
x_t &= 0.1 - \frac{0.9}{\Phi^{-1}(\alpha)}|\mu_t| + u_t
\end{aligned}
\tag{19}
$$

In each model the true 1-step-ahead α level VaR forecast is then $\text{VaR}_{n+1} = \sigma_{t+1}\Phi^{-1}(\alpha)$, and the true 1-step-ahead α level ES forcast is $\text{ES}_{n+1} = \sigma_{t+1}\Phi^{-1}(\delta_\alpha)$, where δ_α is the quantile level that ES occurs at with the standard normal distribution (Gerlach and Chen, 2015). Following Basel II and Basel III risk management guidelines, the 1% quantile level is employed (corresponding $\delta_\alpha = 0.0038$ with the standard normal distribution), then the true value of VaR_{n+1} and ES_{n+1} can be calculated as -4.1872 and -4.7970 respectively. Through the one to one relationship between VaR and ES (Equation 10), we can derive the true value of τ as 0.001452. In addition, the quadratic fitting target grid search of τ is incorporated in the MCMC process, while there is no target search for τ before the ML estimation, to testify the accuracy of target search.

The Re-CARE-SAV model is then fit to the 5000 datasets generated, with one using the MCMC method and one using the ML estimator (the 'fmincon' constrained optimisation routine in Matlab software is employed). The MCMC sampler has 10000 iterations for each data set, with a burn-in of 5000 iterations in each case. Then all iterations in the independent MH are used to calculate the posterior mean estimate.

Estimation results are summarised in Tables 2. Boxes indicate the preferred measure comparing MCMC and ML for both bias (Mean) and precision (RMSE). Regarding the simulation results with $n = 1500$, we can see both methods generate close to unbiased and quite reasonably precise parameter estimates and VaR and ES forecasts. The bias results slightly favour the ML method, with 4 out of 7 parameter estimates, and VaR and ES forecasts averaging are quite close their true value. However, the precision clearly favors the MCMC method in 7 out of 9 parameter estimates and VaR&ES forecasts. Extending the sample size to 3000, firstly it can be seen that both MCMC and ML observe the improved bias and precision estimation, while the results are even more in favour of the MCMC method compared to $n = 1500$ results. Again both methods generate close to unbiased and quite reasonably precise parameter estimates and tail risk forecasts. The bias results favor the MCMC approach in 6 out of 9 parameter estimates and VaR& ES forecasts, whilst the precision is clearly lower for the MCMC method for 7 parameters and the tail risk forecasts. Finally, the close to true value estimation results of τ in MCMC results prove the validity of the quadratic target fitting.

7 DATA and EMPIRICAL STUDY

7.1 Data Description

The daily and high frequency data observed at 1-minute and 5-minute frequency, including open, high, low and closing prices, were downloaded from Thomson Reuters Tick History. We collected data for 6 marketing indices S&P500, NASDAQ (both US), Hang Seng (Hong Kong), FTSE 100 (UK), DAX (Germany) and SMI (Swiss) with time range Jan 2000 to Sep 2015, and 3 individual assets IBM, GE (both US) and BHP (AU)) with time range Jan 2000 to Dec 2015. The starting data collection time for BHP is July 2001 since it had a 2: 1 Stock Split in the June 2001, and the starting time for GE is May 2000 due to similar reason.

The daily return, range and range considering overnight price jump can be then calculated with the daily data downloaded. With the 5-minute log-return square and long range square calculated, we can sum them up to get the RV and RR. Then the scaled and subsampled RV and RR are generated with specifications in Section 2. $q = 66$ is employed for the scaling process, which is around 3 month time. Thus the final starting time is 3 months from the starting time of data collection. Figure 2 plots S&P 500

Table 2: Summary statistics for the two estimators of the Realized-CARE-SAV model, with data simulated from Model 17.

$n = 1500$		MCMC-Target Search		ML	
Parameter	True	Mean	RMSE	Mean	RMSE
$\beta 1$	-0.0465	$\boxed{-0.0544}$	$\boxed{0.1923}$	-0.0709	0.3258
$\beta 2$	0.7500	0.7335	0.0417	$\boxed{0.7378}$	$\boxed{0.0393}$
$\beta 3$	-0.5816	-0.6238	$\boxed{0.1485}$	$\boxed{-0.6045}$	0.2048
ξ	0.1000	0.0839	$\boxed{0.2830}$	$\boxed{0.1018}$	0.7508
φ	0.3869	$\boxed{0.3879}$	$\boxed{0.0723}$	0.3852	0.1734
σ_u	0.3000	0.03010	0.0057	$\boxed{0.3005}$	$\boxed{0.0056}$
τ	0.001452	$\boxed{0.001304}$	$\boxed{0.0004}$	0.001303	$\boxed{0.0004}$
1% VaR_{n+1}	-4.1872	$\boxed{-4.2392}$	$\boxed{0.2920}$	-4.2416	0.3241
1% ES_{n+1}	-4.7970	-4.7911	$\boxed{0.3241}$	$\boxed{-4.7935}$	0.3608
$n = 3000$	True	Mean	RMSE	Mean	RMSE
$\beta 1$	-0.0465	$\boxed{-0.0499}$	$\boxed{0.1287}$	-0.0577	0.2059
$\beta 2$	0.7500	$\boxed{0.7422}$	0.0272	0.7411	$\boxed{0.0257}$
$\beta 3$	-0.5816	-0.6014	$\boxed{0.0976}$	$\boxed{-0.5925}$	0.1328
ξ	0.1000	$\boxed{0.0919}$	$\boxed{0.1947}$	0.0880	0.4542
φ	0.3869	$\boxed{0.3876}$	$\boxed{0.0503}$	0.3891	0.1066
σ_u	0.3000	0.3005	0.0040	$\boxed{0.3002}$	$\boxed{0.0039}$
τ	0.001452	$\boxed{0.001378}$	$\boxed{0.0003}$	$\boxed{0.001378}$	$\boxed{0.0003}$
1% VaR_{n+1}	-4.1872	-4.1970	$\boxed{0.1965}$	$\boxed{-4.1969}$	0.2103
1% ES_{n+1}	-4.7970	$\boxed{-4.7759}$	$\boxed{0.2255}$	$\boxed{-4.7759}$	0.2411

Note:A box indicates the favored estimators, based on mean and RMSE.

absulute Return, square root of RV and square root of RR.

7.2 Tail Risk Forecasting

Both Value-at-Risk (VaR) and Expected Shortfall (ES) are tested on the 6 indices and 3 assets series, as recommended in Basel II and III Capital Accord. As discussed in Section

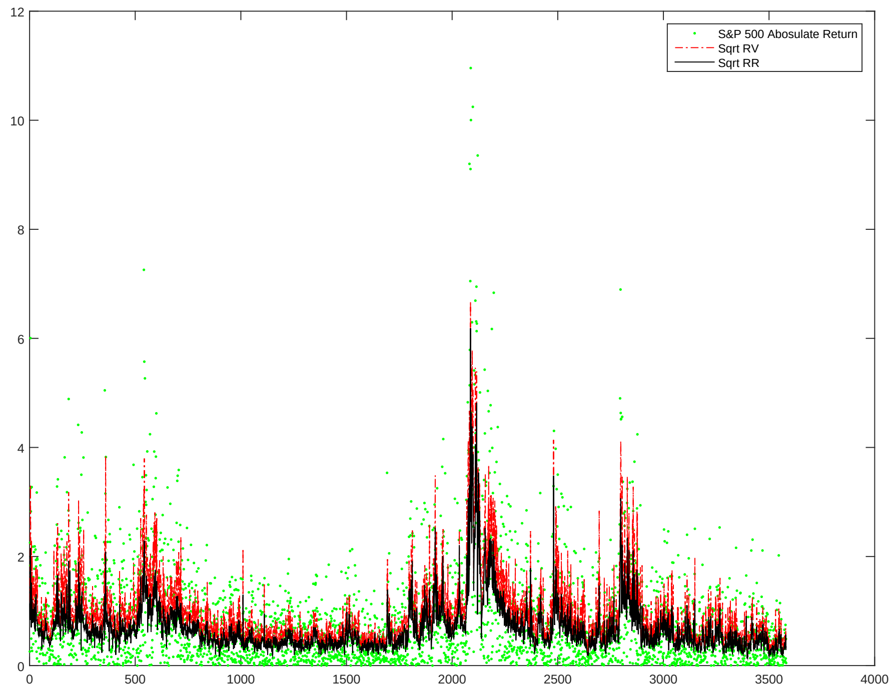

Figure 2: S&P 500 Abs Return, Sqrt RV and Sqrt RR Plots.

3, the VaR, which equals to the α level quantile, can be estimated by the corresponding τ level expectile. Then through employing the one-to-one relationship between expectle and ES (Equation (10)), ES can be calculated.

In all the tail risk forecasting study, the one-step ahead VaR and ES forecasts are calculates through the CARE equation of Re-CARE and Equation (10) with rolling and fixed size in-sample data. In order to see the performance of various models for the GFC period, the start time of each forecasting experiment is chosen as beginning of 2008. On average, across the 9 time series, 1684 VaR and ES forecasts are generated with the proposed Re-CARE type models (estimated with MCMC) with different measures of volatility, including range, range considering overnight jump, RV & RR, scaled RV & RR and subsampled RV & RR. The conventional GARCH with Student-t distribution, CARE-SAV and Re-GARCH with Gaussian or Student-t as the error distributions for its GARCH equation (estimated with ML), are also included in the study for the purpose of comparison.

Then we employ the VaR violation rate (VRate) and ES violation rate (ESRate) to evaluate the VaR and ES forecasting accuracy. VRate and ESRate are simply is the percentage of returns which exceed the forecasted VaR or ES level in the forecasting period (Equation (20) and (21)). Models with VRate the closest to quantile level α are preferred. In addition, Gerlach and Chen (2015) presented the quantile levels where the 1% ES is estimated to fall is between 0.0035% to 0.0038%, for Gaussian and non-Gaussian distributions, which is the expected ESRate for the ES forecasting study.

$$\text{VRate} = \frac{1}{m} \sum_{t=n+1}^{n+m} I(r_t < \text{VaR}_t), \tag{20}$$

$$\text{ESRate} = \frac{1}{m} \sum_{t=n+1}^{n+m} I(r_t < \text{ES}_t), \tag{21}$$

where n is the in-sample size and m is the forecasting sample size.

However, having a VRate or ESRate close to their expected level is not sufficient to ensure an accurate forecasting model. Thus several accuracy and independency tests are also employed, e.g. the unconditional coverage (UC) and conditional coverage (CC) tests of Kupiec (1995) and Christoffersen (1998) respectively, as well as the dynamic quantile (DQ) test of Engle and Manganelli (2004) and the VQR test of Gaglione et al. (2011). With the approach of Gerlach and Chen (2015), the derived expected ES level can be used to treating ES forecasts as quantile forecasts at appropriate quantile levels.

7.2.1 Value at Risk

Table 3 presents the VRate at the 1% quantile for each model for 9 market or assets (also mean and median of the 9 VRates). The estimation period sample size for each forecast is denoted as n, and the forecast sample size is represented with m, in each market. Box indicates the model in each market that has a violation rate (VRate) closest to 1%, while bold indicates the model with VRate furthest away from expected. G-t, CARE-SAVE, Re-GARCH-GG with RV and Re-GARCH-tG with RV are estimated with ML, and the Realized-CARE type models are estimated with MCMC incorporating the quadratic fitting target search.

Chang et al. (2011) and McAleer et al. (2013) proposed using forecast combinations of the VaR series from different models, potentially as a robust combined VaR forecast to the GFC . This approach is also employed in our empirical study since our forecasting period includes the GFC. Specifically, the mean, median, minimum and maximum of each of the VaR forecasts from the 12 models in Table 3 are considered. We consider the lower tail VaR forecasts in this paper, so "Min" is the most extreme of the 12 forecasts (i.e. furthest from 0) and "Max" is the least extreme. The violation rate for "Mean", "Median", "Min" and "Max", series are also presented in Table 3.

Clearly, Re-CARE employing sub-sampled RR has VRate that is the closest to the 1% quantile level based on the mean of VRates on 9 time series studied, and Re-CARE-RR has the closest to the expected VRate with the median. In addition, we can apparently observe the generally improved performance of Re-CARE compared to GARCH, Re-GARCH or CARE-SAV, while Re-GARCH-GG had the VRate that is furthest from that expected, which is not surprising since it is the only parametric model employing the Gaussian error. Further, regarding the combination approach, the "Min" approach is too conservative in each series, while the "Max" series produces anti-conservative VaR forecasts that produce far too many violations. The "Mean" and "Median" of the 12 models produced series that generated competitive violation rates.

Table 3: 1% VaR Forecasting VRate with different models on 6 indices and 3 assets.

Model	S&P 500	NASDAQ	HK	FTSE	DAX	SMI	IBM	GE	BHP	Mean	Median
G-t	1.67%	1.91%	1.59%	1.53%	1.42%	1.62%	1.07%	1.09%	1.19%	1.454%	1.532%
CARE	1.42%	1.61%	0.98%	1.12%	1.24%	1.38%	1.07%	1.60%	0.97%	1.267%	1.242%
RG-RV-GG	2.28%	2.15%	3.00%	1.41%	2.01%	1.86%	1.37%	1.55%	1.36%	**1.888%**	**1.861%**
RG-RV-tG	1.60%	1.56%	2.15%	1.18%	1.66%	1.26%	0.66%	0.97%	1.25%	1.366%	1.261%
RC-Ra	1.23%	1.61%	1.10%	1.00%	1.48%	1.62%	0.90%	1.49%	0.80%	1.248%	1.234%
RC-RaO	1.05%	1.79%	1.23%	1.00%	1.42%	1.44%	0.84%	1.43%	0.97%	1.241%	1.226%
RC-RV	1.42%	1.67%	2.39%	1.06%	1.18%	1.38%	0.84%	1.43%	1.02%	1.377%	1.381%
RC-RR	1.17%	1.56%	1.10%	0.88%	1.06%	1.62%	0.72%	1.20%	0.91%	1.136%	$\boxed{1.104\%}$
RC-ScRV	1.30%	1.61%	1.23%	1.12%	1.42%	1.32%	0.90%	1.43%	1.08%	1.268%	1.295%
RC-ScRR	1.48%	1.85%	0.92%	1.00%	1.36%	1.38%	0.72%	1.37%	0.97%	1.228%	1.360%
RC-SubRV	1.60%	1.73%	1.04%	0.94%	1.54%	1.32%	0.72%	1.60%	0.85%	1.260%	1.321%
RC-SubRR	1.23%	1.50%	1.10%	0.71%	1.12%	1.50%	0.72%	1.49%	0.74%	$\boxed{1.123\%}$	1.124%
Mean	1.36%	1.73%	1.35%	1.12%	1.30%	1.32%	0.78%	0.92%	0.85%	1.192%	1.301%
Median	1.42%	1.67%	1.23%	1.12%	1.36%	1.32%	0.66%	1.20%	0.85%	1.204%	1.226%
Min	0.56%	0.48%	0.31%	0.41%	0.47%	0.48%	0.48%	0.34%	0.45%	0.442%	0.473%
Max	2.84%	3.11%	3.37%	2.06%	2.66%	2.76%	1.73%	3.21%	1.93%	2.631%	2.761%
m	1621	1672	1631	1697	1691	1666	1675	1746	1760	1684.33	1675
n	1960	1892	1890	1944	1936	1930	1916	1839	1569	1875.11	1916

Note:A box indicates the favored model based on average VaR VRate, whilst bold indicates the least favoured model. m is the out-of-sample size, and n is in-sample size. SAV stands for the CARE-SAV model, RG stands for the Realized-GARCH type models, and RC represents the Realized-CARE type models.

However, having a VRate close to 1% on average is not sufficient to guarantee an accurate forecast model. We employed several tests exist in the literature to statistically test the forecast accuracy and independence of violations, a requirement of a proper risk model. The tests include the unconditional coverage (UC) Kupiec (1995), conditional coverage (CC) of Christoffersen (1998), dynamic quantile (DQ) of Engle and Manganelli (2004) and VaR quantile regression (VQR) test of Gaglianone et al. (2011). Table 4 displays the number of markets&assets in which each 1% VaR forecast model is rejected for each test, conducted at 5% significance level. We can clearly see that the Re-CARE type models are generally less likely to be rejected by various back tests, and Re-CARE with Range got the least number of rejections, following by Re-CARE-RR, Re-CARE-ScRV, Re-CARE-ScRR and "Mean" combination approach (rejected by 3 time in total). The "Min" and "Max" combinations are rejected by all 9 series, and G-t and Re-GARCH-GG are rejected by 8 times respectively.

Table 4: Counts of 1% VaR rejections with UC, CC, DQ, VQR tests for different models on 6 indices and 3 assets.

Model	UC	CC	DQ	VQR	Total
G-t	5	3	7	3	**8**
CARE	2	1	6	0	6
RG-RV-GG	6	5	5	6	**8**
RG-RV-tG	4	2	2	4	5
RC-Ra	2	2	4	1	4
RC-RaO	1	1	2	1	$\boxed{2}$
RC-RV	2	2	3	2	5
RC-RR	2	1	3	1	3
RC-ScRV	1	2	2	0	3
RC-ScRR	1	2	3	0	3
RC-SubRV	4	2	4	1	6
RC-SubRR	0	1	3	1	4
Mean	1	2	3	2	3
Median	1	2	2	3	4
Min	9	4	2	5	**9**
Max	9	9	9	8	**9**

Note:A box indicates the model with least number of rejections, whilst bold indicates that with the highest number os rejections. All tests are conducted at 5% significance level.

7.2.2 Expected Shortfall

Employing the VaR forecasts and expectile levels searched in the previous section VaR empirical study, we can use the ES and VaR one to one relationship (Equation 10) to directly generate the 1-step-ahead ES forecast for the same 12 models during the forecast sample in each market&asset. Regarding the expected level of ESRate for different models and distributions, Chen, Gerlach and Lu (2012) discuss how to treat ES forecasts as quantile forecasts in parametric models, where the quantile level that ES falls at can be deduced exactly. Gerlach and Chen (2015) illustrated that the quantile level that the quantile level that the 1% ES was estimated to fall was between 0.0035% and 0.0038%. Specifically, they present the expected violation rate of ES is exactly 0.0038% for models with Gaussian errors, $\approx$ 0.0036% for non-parametric models and is estimated by the quantile level dependent on the degrees of freedom for models with Student-t errors ($\approx$ 0.0036% for the time series considered here). With this approach, we can then treat ES forecasts as quantile forecasts and employ the UC, CC, DQ and VQR tests with corresponding ES nominal level to test the ES forecasting accuracy and independence of violations.

Table 5 presents the ESRate in the forecast period at the 1% quantile for each model in 9 time series. Similar as the VaR study, boxe indicate the model in each market that has an ES violation rate closest to that desired, and bold indicates the model with ESRate furthest from the corresponding nominal level.

Clearly, the Re-CARE-RR and Re-CARE-RaO generate the ESRate closest to that expected for the 1% ES across the six markets and 3 assets, and their ESRate is just lower than their 0.0036% nominal level, which means the ES forecasts from these two models are conservative. Also, we can still see that the Re-CARE type models have obviously better performance than the GARCH, Re-GACCH and CARE-SAV models. The mean, median, min and max of the 12 models' 1% ES forecasts are again calculated and their ES violations are also shown in Table 5. The "Mean" and "Median" approaches are again optimal among the four combination methods, while the ESRates are anti-conservative.

Furthermore, Figure 3 demonstrates the extra efficiency that can be gained by employing the Re-CARE framework with RR. Specifically, the ESRate of the Gt, CARE-SAV and Re-CARE-RR are 0.68%, 0.37% and 0.31% respectively. These violation rates mean Gt generated too anti-conservative ES forecasts that produce far too many violations, and CARE-SAV is clearly more conservative than Gt and have close to nominal level ESRate, and Re-CARE-RR is the most conservative model. Through close inspection of Figure 3, e.g. the close to end of the forecasting period, CARE-SAV have obviously lower level of ES forecasts than Gt, in order to be conservative, but it also means the capital set aside by financial institutions to cover extreme losses are more with CARE-SAV than with Gt. However, we can clearly observe the Re-CARE-RR is less conservative than the other 2 models in the same time period. Thus the efficiency of employing Re-CARE-RR can be deduced in that this model can produce ES forecasts that have far fewer and close to expected violations but are simultaneously less extreme than those of the traditional

GARCH and CARE-SAV model. Since the capital set aside by financial institutions should be directly proportional to the ES forecast, the Re-CARE-RR model is saving the company money, by giving more accurate and often less extreme ES forecasts, and this extra efficiency is frequently observed for Re-CARE type models in other markets/assets.

Further, at times of GFC when there is a persistence of extreme returns, close inspection of Figure 3 reveals that the Re-CARE-RR ES forecasts "recover" the fastest among the 3 model presented, in terms of being marginally the fastest to produce forecasts that again follow the tail of the data. Also, we can see GARCH models are to over-react to extreme events and to be subsequently very slow to recover, due to their oft-estimated very high level of persistence.

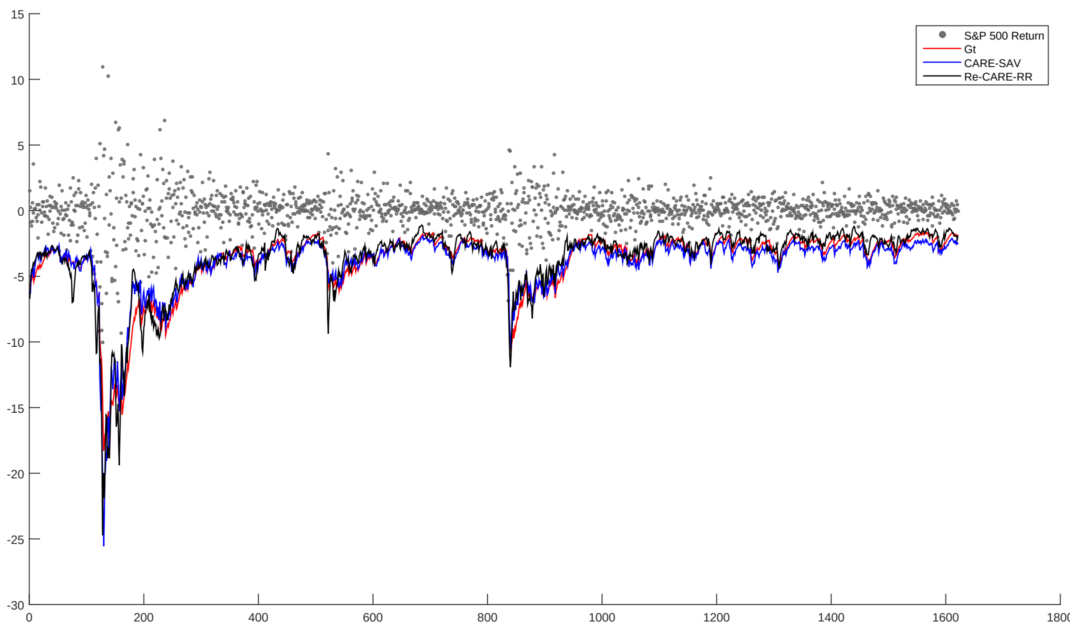

Figure 3: S&P 500 ES Forecasts with Gt, CARE-SAV and Re-CARE-RR.

Similarly, we conducted the back tests on all the ES forecasts and results are shown in Table 6. The UC, CC, DQ and VQR quantile accuracy tests are applied to the ES violations from each model, using that model's nominal 1% ES quantile level. In addition, the averages of the 1% ES forecast residuals, standardised by the 1% VaR forecasts are also calculated. Given an accurate 1% ES forecast model should produce standardised residuals that average approximately 0, a bootstrap test on whether these averages differ from 0 is also performed and presented in Table 6. As can be seen, models with least number of rejections are Re-CARE-RaO and Re-CARE-RV, only rejected 2 out 9 time series. They are followed by Re-GARCH-tG, Re-CARE-Ra, Re-CARE-ScRV, Re-CARE-ScRR, Re-CARE-SubRV, "Mean", "Median" and "Min" approaches (rejected by 3 time in total).

Table 5: 1% ES Forecasting VRate with different models on 6 indices and 3 assets.

Model	S&P 500	NASDAQ	HK	FTSE	DAX	SMI	IBM	GE	BHP	Mean	Median
G-t	0.68%	0.48%	0.49%	0.77%	0.53%	0.96%	0.42%	0.34%	0.80%	0.608%	0.532%
CARE	0.37%	0.78%	0.49%	0.53%	0.30%	0.60%	0.36%	0.40%	0.68%	0.501%	0.490%
RG-RV-GG	1.30%	1.14%	1.96%	0.77%	0.95%	1.02%	0.66%	0.57%	0.57%	**0.993%**	**0.946%**
RG-RV-tG	0.37%	0.54%	0.98%	0.41%	0.47%	0.36%	0.30%	0.11%	0.57%	0.457%	0.412%
RC-Ra	0.37%	0.24%	0.49%	0.29%	0.47%	0.90%	0.24%	0.29%	0.40%	0.410%	0.370%
RC-RaO	0.25%	0.36%	0.37%	0.29%	0.35%	0.66%	0.24%	0.23%	0.57%	0.369%	$\boxed{0.355\%}$
RC-RV	0.43%	0.54%	0.86%	0.35%	0.35%	0.60%	0.24%	0.17%	0.40%	0.438%	0.398%
RC-RR	0.31%	0.24%	0.49%	0.12%	0.35%	0.72%	0.36%	0.11%	0.51%	$\boxed{0.357\%}$	$\boxed{0.355\%}$
RC-ScRV	0.56%	0.48%	0.49%	0.29%	0.41%	0.60%	0.24%	0.23%	0.40%	0.411%	0.414%
RC-ScRR	0.49%	0.42%	0.43%	0.35%	0.41%	0.60%	0.36%	0.11%	0.57%	0.416%	0.419%
RC-SubRV	0.43%	0.36%	0.37%	0.35%	0.35%	0.60%	0.24%	0.23%	0.45%	0.376%	0.359%
RC-SubRR	0.43%	0.18%	0.43%	0.18%	0.24%	0.66%	0.24%	0.11%	0.45%	0.324%	0.239%
Mean	0.37%	0.36%	0.55%	0.41%	0.30%	0.54%	0.24%	0.23%	0.57%	0.396%	0.370%
Median	0.43%	0.36%	0.43%	0.41%	0.35%	0.60%	0.24%	0.17%	0.57%	0.396%	0.412%
Min	0.12%	0.18%	0.12%	0.12%	0.18%	0.12%	0.24%	0.11%	0.06%	0.139%	0.123%
max	1.54%	1.56%	2.15%	1.12%	1.06%	1.50%	0.78%	0.80%	0.97%	1.275%	1.120%

Note:A box indicates the favored model based on average ES VRate, whilst bold indicates the least favoured model.

8 CONCLUSION

In this paper, the Realized-CARE, a new framework to estimate and forecast financial tail risk, is proposed. Through incorporating intra-day and high frequency volatility measures, e.g. Ra, RaO, RV, RR, Scaled RV, Scaled RR, Sub-sampled RV and Sub-sampled RR, we observe significant improvements in the out-of-sample the forecasting of tail risk measures, compared to Re-GARCH models employing realized volatility, and traditional GARCH and CARE-SAV models, as well as forecast combinations of these models. Specifically, Re-CARE models with RR, Sub-sampled RR generate the most accurate VaR forecasts, and Re-CARE models employing RR, RaO, SubRV are most accurate for ES forecasting empirical study. In the back testing, Re-CARE type model

Table 6: Counts of 1% ES rejections with UC, CC, DQ, VQR tests for different models on 6 indices and 3 assets.

Model	UC	CC	DQ4	VQ	Boot	Total
GARCH-t	3	3	6	3	2	7
CARE	2	1	4	1	0	5
RG-RV-GG	6	6	6	2	2	**8**
RG-RV-tG	2	1	2	1	0	3
RC-Ra	1	1	3	0	0	3
RC-RaO	0	0	2	0	0	$\boxed{2}$
RC-RV	1	1	2	1	1	$\boxed{2}$
RC-RR	2	0	2	1	1	4
RC-ScRV	0	0	3	0	0	3
RC-ScRR	1	0	2	0	0	3
RC-SubRV	0	0	2	0	1	3
RC-SubRR	1	0	2	0	1	4
Mean	0	0	3	0	0	3
Median	0	0	3	0	0	3
Min	2	1	0	1	2	3
Max	9	9	8	6	3	**9**

Note:A box indicates the model with least number of rejections, whilst bold indicates that with the highest number os rejections. All tests are conducted at 5% significance level.

are also less likely to be rejected than their counterparts: Re-CARE with RaO is rejected least for VaR forecasting, and Re-CARE models with RaO, RV are rejected least for ES forecasting. The Re-CARE type models with range, range considering overnight jump and realized range should be considered for financial applications when forecasting volatility or tail risk, and should allow financial institutions to more accurately allocate capital under the Basel Capital Accord to protect their investments from extreme market movements. This work can be extended by testing asymmetric and non-linear Re-CARE specification, and using alternative frequencies of observation for the realized measures.

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