Advancement of Fractionally Differenced Gegenbauer Processes with Long Memory

Gnanadarsha Sanjaya Dissanayake

A thesis submitted in fulfillment of
the requirements for the degree of
Doctor of Philosophy

School of Mathematics & Statistics
University of Sydney
Australia

THE UNIVERSITY OF SYDNEY

June 2015

Copyrights

The copyrights statement for this thesis is provided below.

Quotations

Famous quotes linked with research and knowledge are provided below.

"Do not believe anything you see, read or hear from others, whether of authority, teachers or texts" - Lord Buddha.

"A question really is: What's coming next? where are we going? what are the interesting things in the future?" - Robert Engle (Nobel Co-Laureate-Economics, 2003).

Abstract

The class of long memory time series models involving Gegenbauer processes is investigated in detail in terms of formulation, parameter estimation, prediction and testing. Corresponding truncated AR (autoregressive) and MA (moving average) approximations driven by Gaussian white noise are analysed through state space modelling and Kalman filtering to assess the viability of estimating techniques . The optimal approximation option is employed to proceed with the estimation of model parameters. The resulting mean square errors are validated by the predictive accuracy to establish an optimal lag order through a large scale simulation study. It is shown that the use of this newly established lag order for a real data application provides benchmarks which are comparable and mostly better than a number of existing results in the literature. It is followed by an execution of this technique to extract and assess seasonal models through a Monte Carlo experiment. Thereafter empirical applications are provided.

The above approach has been extended to model fractionally differenced Gegenbauer processes with conditional heteroskedastic errors and models with seasonality. Potential applications are provided. In addition, quasi-likelihood type ratio tests have been developed for testing unit roots, stationarity versus non-stationarity and Gegenbauer long memory versus standard long memory.

Publications

All chapters of this thesis are based on results produced in the following publications and working papers.

1. "State Space Modeling of Gegenbauer Processes with Long Memory", G.S. Dissanayake, M. S. Peiris and T. Proietti. (Accepted for publication in the Computational Statistics and Data Analysis (CSDA) Special Issue: Annals of Computational and Financial Econometrics) DOI: http://dx.doi.org/10.1016/j.csda.2014.09.014.(To appear in 2015).

2. "Estimation of Generalized Fractionally Differenced Processes with Conditionally Heteroskedastic Errors", G.S. Dissanayake, M. S. Peiris and T. Proietti. (Published in the ISI cited refereed Proceedings of the International Time Series Work Research Conference sponsored by IEEE Chapter and Universidad De Granada, Spain, 2014).

3. "Generalized Fractional Processes with Conditional Heteroskedasticity", G.S. Dissanayake and M. S. Peiris, Sri Lankan Journal of Applied Statistics (2011, Volume 12;

pages 1-11, ISSN:1391-4987).

4. "Fractionally Differenced Gegenbauer Processes with Long Memory A Review", G.S. Dissanayake and M.S. Peiris (Under Review).

5. "Modelling Generalized Persistence and Conditional Variance: An application to Financial Economics", G. S. Dissanayake, M. S. Peiris and T. Proietti (Under Review).

6. "State Space Modeling of Seasonal Gegenbauer Processes with Long Memory", G. S. Dissanayake, M. S. Peiris and T. Proietti. Working Paper, School of Mathematics and Statistics, University of Sydney.

7. "A Nearly Efficient Unit Root Test for a Gegenbauer Process", G. S. Dissanayake, M. S. Peiris, T. Proietti and Q. Wang. Working Paper, School of Mathematics and Statistics, University of Sydney.

Note: Approximately 80-85 percent of the work with respect to the publications and working papers were done by the first author (thesis writer). Furthermore approximately 90 percent of each thesis chapter comprises of published and submitted work and results from working papers.

Presentations

All chapters of this thesis are based on results and information imparted through invited and contributed presentations at the following international / local conferences and seminars.

1. "Generalized Fractional Differencing with Conditional Heteroskedasticity", International Statistics Research Conference held in Colombo, Sri Lanka (December, 2011) jointly organized by Institute of Applied Statistics Sri Lanka, University of Sydney, Australia and the University of Colombo, Sri Lanka.

2. "State Space Modeling of Gegenbauer Processes with Long Memory - An initial Assessment", Postgraduate Statistics Seminar held at the School of Mathematics and Statistics, University of Sydney, Australia in February, 2013.

3. "State Space Modeling of Gegenbauer Processes with Long Memory Initial Results", Statistical Society of Australia (SSAI) Young Statisticians Conference held at the University of Melbourne, Australia in February 2013.

4. "State Space Modeling of Gegenbauer Processes with Long Memory Final Assessment", Postgraduate Statistics Seminar held at the School of Mathematics and Statistics, University of Sydney, Australia in November, 2013.

5. "State Space Modeling of Gegenbauer Processes with Long Memory", Computational Statistics and Financial Econometrics International Research Conference, (Sponsored by London School of Economics, Queen Mary College University of London and the Journal of Computational Statistics and Data Analysis (CSDA), University of London, UK) held at the University of London, UK in December 2013.

6. "Gegenbauer Processes with Long Memory", Postgraduate Departmental Seminar held at the School of Mathematics and Statistics, University of Sydney, Australia in March 2014.

7. "Gegenbauer Processes with Long Memory Extended Analysis", Postgraduate Departmental Seminar held at the School of Mathematics and Statistics, University of New South Wales, Australia in March 2014. **(Invited Talk)**.

8. "Gegenbauer Processes with Long Memory Final Analysis", Postgraduate Departmental Seminar held at the Department of Statistics, Macquarie University, NSW, Australia in June 2014. **(Invited Talk)**.

9. "Efficient Estimation of Generalized Fractionally Differenced Processes with Conditionally Heteroskedastic Errors" International Time Series Work Research Conference (Sponsored by IEEE Chapter, Spain and Universidad De Granada, Granada, Spain) held in Granada, Spain in June 2014.

10. "Optimal Estimation of Generalized Fractionally Differenced Processes with GARCH Errors" Statistical Society of Australia (SSAI) and Institute of Mathematical Statistics (IMS), USA International Research Conference (Sponsored by SSAI & IMS, Sydney, Australia) held in Sydney, Australia in July 2014.

11. "State Space Modeling of Seasonal Gegenbauer Processes with Long Memory", Computational Statistics and Financial Econometrics International Research Conference (Sponsored by London School of Economics, Queen Mary College University of London and the Journal of Computational Statistics and Data Analysis (CSDA), University of London, UK) held at the University of Pisa, Italy in December 2014.

12. "A Nearly Efficient Unit Root Test and Asymptotics of a Long Memory Gegenbauer Process", Statistical Society of Australia (SSAI) Young Statisticians Conference held at the University of Adelaide, Australia in February 2015.

Declaration

I declare that this thesis represents my own work, except where due acknowledgement is made, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any other institution for a degree, diploma or other qualifications.

Signed: *G. S. Dissanayake*

Gnanadarsha Sanjaya Dissanayake

Acknowledgements

The Doctoral candidacy years at the School of Mathematics & Statistics, University of Sydney was a pleasent and a memorable period in my life. It gave me a taste of cutting edge original research through a well drafted research plan laid out by my supervisory panel. The end result of it is this thesis symbolizing excellent guidance, supervision and mentoring of my main supervisor Professor Shelton Peiris and co-supervisors Professors Tommaso Proietti & Qiying Wang. Working with three eminent scientists was my honour and privilege & a special appreciation is extended, not withstanding the fact that I'll be in debt to them throughout my life.

Professor Peiris shared his immense knowledge on fractional differencing and long memory time series, provided more than adequate discussion time, taught excellent writing skills in addition to being extremely patient & tolerant, while extending his kindness and generosity. The time I spent with Professor Proietti in Sydney, London, Pisa & Rome provided new insights into our research endeavour. I still remember the days when he gave me time without any reservation in sharing his extraordinary knowledge on state space modelling and research paper writing - spicing it up with kindness & humour. Professor Wang provided me with his acumen on unit root testing through fruitful discussions without any hindrance and taught the virtue of patience in constructing mathematical proofs. He was also extremely generous in providing travel support to various national and international research conferences. It still lingers in my mind the great times we had together at various bars, cafes & restaurants in Sydney, Colombo, London, Pisa and Rome. For sure it relieved the stress to a great extent in doing research & reach new heights in terms of academic achievements. Their endless care, understanding, encouragement and meticulous attention to detail made this thesis possible. Thank you Professors Peiris, Proietti & Wang for listening when things were not rosy, giving advise, supporting and introducing great ideas into my work.

I would also like to extend my appreciation to everyone in the School of Mathematics & Statistics. My special thanks go to our head - Professor Neville Weber and Postgraduate Coordinator Prof Samuel Mueller for going the extra mile in helping me obtain an Australian Postgraduate Award Scholarship, Kopystop International Research Conference Scholarship and funding through the Postgraduate Research Support Scheme, Statistics Research Fund, the Statistical Society of Australia and tutoring assignments. Many thanks to Sonia Morr, Susan Liddell, Ivana Crossley, Caroline Dyson, Chamreun Cheen, Paul Szabo, John Mcqueen & Lucy Kennedy for always helping with administration and technical issues. Special thanks go to my friends and colleagues at the University, who touched my life in numerous ways. Especially Kaushala Jayawardena who began her PhD journey together coming from the same motherland of Sri Lanka. Garth Tarr, Inga Samonenko, Andrew Stephan and Victor Vera who happened to be my close buddies. The friendship and camaraderie during these years created a very special environment & in that context a special thanks is extended to colleagues Justin Wishart, Thomas Fung, Joanna Wang, Emi Tanaka, Alice Dong, Matthew Fitzpatrick, Ellis Patrick, Matthew Chan, Joshua Ching, Gemie Nitithumbundit, Shila Ghazanfar , Michael Barwick, Wei Wu, Chong You, Louis Bhim and Thomas Porter. I will always remember the fun during statistics postgraduate hours relieving academic stress to a great extent. Two other special people from my motherland at the School - namely Dr's Howard D'Abrera and Krishnaswamy Muraleedharan motivated me to a great extent during my candidacy and will always be appreciated.

Finally, I need to thank Dr Ganes Ganesalingam (formerly of Massey University, New Zealand) and Jackie Nicholas (Mathematics Learning Centre, University of Sydney) for proof reading the thesis, enabling me to improve and enhance it's quality and readability.

This thesis is dedicated to my late father, ailing mother and to my lovely wife Rasika for providing all the love, care & support, while treading along her own PhD journey.

CONTENTS

List of Figures

List of Tables

Acronyms

acf autocorrelation function

mle maximum likelihood estimation

pacf partial autocorrelation function

AR Autoregressive

ARCH Autoregressive Conditionally Heteroskedastic

ARFIMA Autoregressive Fractionally Integrated Moving Average

ARFISMA Autoregressive Fractionally Integrated Seasonal Moving Average

ARIMA Autoregressive Integrated Moving Average

ARMA Autoregressive Moving Average

ADF Augmented Dickey Fuller

CARMA Continuous Autoregressive Moving Average

CSS Conditional Sum of Squares

E.MSE Estimator Mean Square Error Sum for GARSMA model

F.MSE Forecast Mean Square Error for GARSMA model

E-MSE Estimator Mean Square Error Sum for GARMA model

F-MSE Forecast Mean Square Error for GARMA model

FARIMA Fractional Autoregressive Integrated Moving Average

FDWN Fractionally Differenced White Noise

FI Fractionally Integrated

FI-GARCH Fractionally Integrated Generalized Autoregressive Conditionally Heteroskedastic

GAR Generalized Autoregressive

GARCH Generalized Autoregressive Conditionally Heteroskedastic

GARMA Gegenbauer Autoregressive Moving Average

GARSMA Gegenbauer Autoregressive Seasonal Moving Average

GEXP Generalized Exponential

I Integrated

IID Independent and Identically Distributed

KF Kalman Filter

LM Lagrange Multiplier

LR Likelihood Ratio

MA Moving Average

MAE Mean Absolute Error

MLE Maximum Likelihood Estimator/Estimate

MSE Mean Square Error

NID Normally and Independently Distributed

QLR Quasi Likelihood Ratio

QML Quasi Maximum Likelihood

QMLE Quasi Maximum Likelihood Estimator/Estimate

RMSE Root Mean Square Error

SARFIMA Seasonal Autoregressive Fractionally Integrated Moving Average

SD Standard Deviation / Standard Error Estimate

SDF Standard Dickey Fuller

SE Standard Error

TTP-MSE Total Trace Parameter Mean Square Errors

TTP-RMSE Total Trace Parameter Root Mean Square Errors

WN White Noise

Nomenclature

$\mathcal{T}$ Index Set

$\mathcal{Z}$ Set of Integers

$\mathcal{Z}^+$ Set of Positive Integers

γ Autocovariance Function

ρ Autocorrelation Function

ϵ White Noise Term

ε GARCH Error Term

$f(\bullet)$ Generic Spectral Density Function

$|\bullet|$ Generic Absolute Value

$[\bullet]$ Generic Integer Part of Given Value

max Maximum Value

Σ Covariance Matrix

$||\bullet||$ Generic Norm

sup Supremum

inf Infimum

$a.s.$ Almost Surely

$\rightarrow$ Approaches (Tends to)

lim Limit

tr Trace of Matrix

A^T Transpose of Matrix A

$\mathbb{R}^2$ Two-Dimensional Real Space

$I(\bullet)$ Generic Sample Spectrum

$f_x(\bullet)$ Generic True Spectrum

$\mathcal{R}^+$ Set of Positive Real Numbers

exp Exponential Function

$\xrightarrow{p}$ Converges in Probability

$\xrightarrow{d}$ Converges in Distribution

δ A Small Number

$l_{\sigma^2}(\bullet, \bullet)$ Generic Profile Likelihood Function

$\Gamma(\bullet)$ Generic Gamma Function

CHAPTER 1

Introduction

1.1. Preliminaries

The subject area of time series analysis is used in many disciplines including finance, economics, astronomy, environmental science, medicine, physics, engineering and hydrology. It is mainly used to infer properties of a system by the analysis of a measured time record referred to as data. This is done by fitting the best possible model to the data aiming to discover the underlying structure to an acceptable degree of accuracy. Conventional time series analysis is based on the assumptions of linearity and stationarity. But in recent times, there has been a growing interest in nonlinear and non-stationary time series models in many practical applications. The first and the simplest reason for it is that many real world problems do not adhere to the assumptions of linearity and / or stationarity.

In general time series analysis, it is established that there are a large number of nonlinear features such as cycles, asymmetries, bursts, jumps, chaos, thresholds, heteroscedasticity and combinations of them that should be taken into consideration. For example, the analysis of financial markets suggests that there is a greater need to explain behaviours that are far from being even approximately linear. Therefore, the enhancement of the theory and applications for nonlinear models is essential. To consider such attributes from an analytical perspective certain basic definitions and notation on time series analysis will be useful to comprehend the material in all the chapters of this thesis. As the analysis of time series becomes a subclass of stochastic processes, we begin with the following definition:

Definition 1.1

A stochastic process is a family of random variables $\{X_t\}$, indexed by a parameter t, where t belongs to some index set $\mathcal{T}$.

In terms of stochastic processes the concept of *stationarity* plays an important role in many applications.

Definition 1.2

A stochastic process $\{X_t; t \in \mathcal{T}\}$ is said to be strictly stationary if the probability distribution of the process is invariant under translation of the index, i.e., the joint probability distributions of $(X_{t_1}, ..., X_{t_n})$ is identical to that of $(X_{t_1+k}, ..., X_{t_n+k})$, for all $n \in \mathcal{Z}^+$(Set of positive integers), $(t_1, ..., t_n) \in \mathcal{T}$, $k \in \mathcal{Z}(set \quad of \quad integers)$. i.e.

$$F(x_1, ..., x_n; t_1, ..., t_n) = F(x_1, ..., x_n; t_{1+k}, ..., t_{n+k}), \tag{1.1}$$

Definition 1.3

A stochastic process $\{X_t\}$ is said to be a Gaussian process if and only if the probability distribution associated with any set of time points is multivariate normal.

In particular, if the multivariate moments $E(X_{t_1}^{s_1}...X_{t_n}^{s_n})$ depend only on the time differences, the process is called stationary up to order s, when $s \leq s_1 + ... + s_n$.

Note that, the second order stationarity is obtained by setting $s = 2$ and this weak stationarity asserts that the mean μ is a constant (i.e. independent of t) and the covariance function $\gamma_{t\tau}$ is dependent only on the time difference. That is,

$E(X_t) = \mu$, for all t

and

$Cov(X_t, X_\tau) = E[(X_t - \mu)(X_\tau - \mu)] = \gamma_{|t-\tau|}$, for all t, τ.

Time difference $k = |t - \tau|$ is called the *lag*. The corresponding autocovariance function is denoted by γ_k.

Definition 1.4

The acf (autocorrelation function) of a stationary process $\{X_t\}$ is a function whose value at lag k is

$$\rho_k = \gamma_k/\gamma_0 = Corr(X_t, X_{t+k}), \quad for \quad all \quad t, k \in \mathcal{Z}, \tag{1.2}$$

Definition 1.5

The pacf (partial autocorrelation function) at lag k of a stationary process $\{X_t\}$ is the additional correlation between X_t and X_{t+k} when the linear dependence of X_{t+1} through to X_{t+k-1} is removed.

Definition 1.6

The process $\{\epsilon_t\}$ is said to be white noise with mean 0 and variance σ^2 if and only if $\{\epsilon_t\}$ is a sequence of uncorrelated random variables (not necessarily independent) with zero mean and autocovariance function

$$\gamma_k = \begin{cases} \sigma^2, & if \quad k = 0, \\ 0, & if \quad k \neq 0. \end{cases} \tag{1.3}$$

This is written as

$$\{\epsilon_t\} \sim WN(0, \sigma^2) \tag{1.4}$$

If the random variables ϵ_t are independently and identically distributed (iid) with mean 0 and variance σ^2, then it is written

$$\{\epsilon_t\} \sim IID(0, \sigma^2). \tag{1.5}$$

Note: In this case ϵ_t can be considered as strict WN.

Definition 1.7

A time series is a set of observations on X_t, each being recorded at a specific time t, where $t \in (0, \infty)$.

Remark: A wide class of discrete stationary time series models can be generated by using white noise as forcing terms in a set of linear difference equations.

The background information provided in this section forms the basis in conducting analytical time series research and it's evolution. In lieu of it next discussion compare and contrast different memory types of time series.

Let $\{X_t\}$ be a stationary time series with autocovariance at lag k, $\gamma_k = Cov(X_t, X_{t+k})$, ρ_k=Corr(X_t,X_{t+k}) and the normalized spectrum or spectral density function (sdf) is $f(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\rho_k e^{-i\omega k}$; $-\pi < \omega < \pi$, where ω is the Fourier frequency. There are two main types of time series uniquely identified by the behaviour of ρ_k and $f(\omega)$. They are classified as $memory$ types and a feature summary of them are provided in table 1.1 as follows:

TABLE 1.1. Characteristics of memory types.

Short Memory Series	**Long Memory Series**
Stationary	Stationary
Exponential decay of ρ_k	Hyperbolic decay of ρ_k
$\rho_k \sim a^k$ for $\|a\| < 1$	$\rho_k \sim k^{-d}$ for $d > 0$
$\sum \|\rho_k\| < \infty$	$\sum \|\rho_k\| = \infty$
$lim_{\omega->0}f(\omega)$ exists & bounded	$lim_{\omega->0}f(\omega)$ does not exist or $f(\omega)$ unbounded

Note: Graphical illustrations of the acf's of short and long memory processes are provided below as Figures 1.1 and 1.2.

In her paper Guegan (2005) provides a number of alternative characteristics of long memory processes. Interestingly, Wang et.al (2006) introduced a characteristic-based clustering method to capture the characteristic of long-range dependence (self-similarity).

Note: Processes in which the decay of ρ_k takes a shape in between exponential and hyperbolic arcs are known as $intermediate\ memory$.

Remark: Also, pacf of each memory type will provide corresponding shapes related to that of the acf.

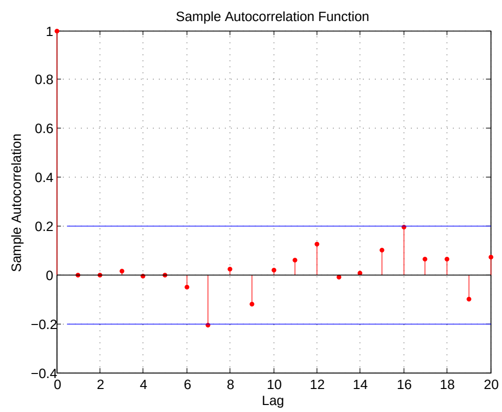


FIGURE 1.1. Autocorrelation function of Standard Short Memory Process

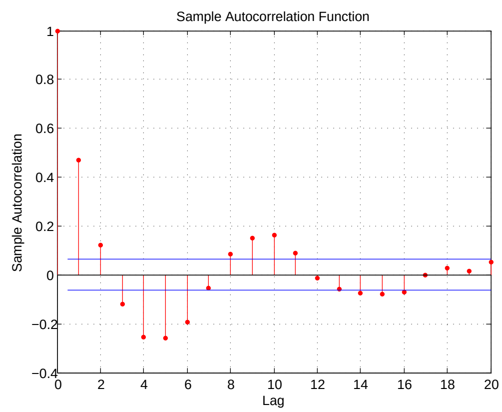


FIGURE 1.2. Autocorrelation function of Standard Long Memory Process

Note: A review of the literature linked with the above preliminaries in formulating a research question is provided in the next section.

1.2. Literature Review

The stochastic analysis of time series began with the introduction of Autoregressive Moving Average (ARMA) model by Whittle (1951), popularization by Box and Jenkins (1970) and subsequent developments of a number of path breaking research endeavours. In particular, in the early 1980's the introduction of long memory processes became an extensive practice among time series specialists and econometricians. In their papers Granger and Joyeux (1980) and Hosking (1981) proposed the class of fractionally integrated autoregressive moving average (ARFIMA or FARIMA) processes, extending the traditional autoregressive integrated moving average (ARIMA) series with a fractional degree of differencing. A hyperbolic decay of the acf, and an unbounded spectral density peak at or near the origin are two special characteristics of the ARFIMA family in contrast to exponential decay of the acf and a bounded spectrum at the origin in the traditional ARMA family. In addition to the mle (maximum likelihood estimation) approach, Geweke and Porter-Hudak (1983) have considered the estimation of parameters of ARFIMA using the frequency domain approach. Chen et. al (1994), Reisen (1994) and Reisen et. al (2001) have considered the estimation of ARFIMA parameters using the smoothed periodogram approach. Additional expositions presented in Andel (1986), Brockwell and Davis (1991), Beran (1994), Rangarajan and Ding (2003), Chan and Palma (2006), Teyssiere and Kirman (2007), Giraitis et. al (2012), Beran et. al (2013) and references provide a comprehensive discussion about long memory series estimation.

Utilizing fractional differencing of Hosking (1981), Gray et. al (1989) developed another class of time series known as Gegenbauer ARMA abbreviated as GARMA using the theory of Gegenbauer polynomials. This generalized class can be used to represent long memory depicting multiple unbounded spectral peaks away from the origin unlike in the ARFIMA case of Hosking (1981) at the origin (Figures 1.3 and 1.4 provides a visual illustration). A detailed analysis of the long memory features of GARMA time series is illustrated through the unbounded spectral density around the Gegenbauer frequency by Chung (1996). The existing literature suggests that the ARFIMA model of Hosking (1981), and the generalized fractional Gegenbauer autoregressive moving-average (GARMA) model of Gray et.al

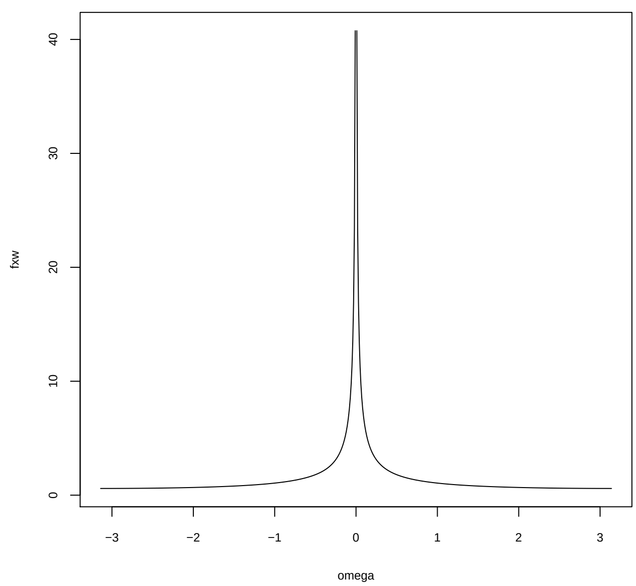


FIGURE 1.3. Spectral Density of Standard Long Memory Process with peak at the origin

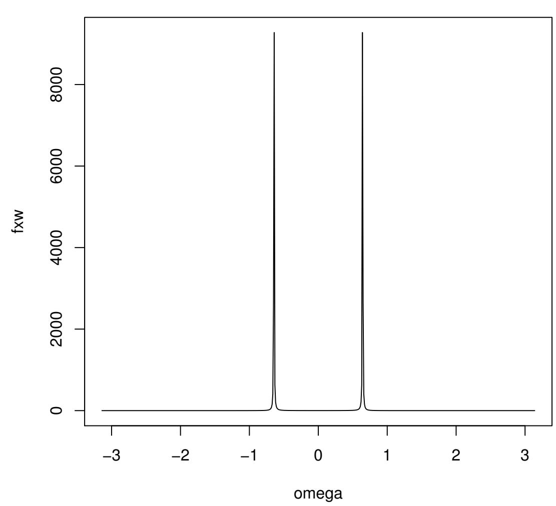


FIGURE 1.4. Spectral Density of Gegenbauer Long Memory Process with peak away from the origin

(1989) have unique time series characteristics of their own. In yet another development, fractionally differenced long memory model parameters were estimated using maximum likelihood and least squares with their convergence rates, limiting distribution and strong consistency by Yajima (1985). But the utilization of state space methodology following the work of Anderson and Moore (1979), Pearlman (1980), Harvey (1989), Aoki (1990), Brockwell and Davis (1996) Chan and Palma (1998), Durbin and Koopman (2001), Harvey and Proietti (2005), Palma (2007) and Grassi and De Magistris (2014) in estimating parameters, establishing predictive accuracy and an optimal lag order is not evident in the literature.
Therefore, state space modelling coupled with Kalman Filter (KF) is deemed suitable as an alternative technique to estimate & forecast GARMA time series. In developing such methodology assessing a generic representation becomes a useful topic. It is presented in the next subsection.

1.2.1. State Space Representation of a generic time series - An overview. In the literature, there are various state space configurations proposed by different authors. Each of them consists of two fundamental equations for a process $\{X_t\}$. Suppose that an observed vector series $\{X_t\}$ can be written in terms of an observed state vector $\{\alpha_t\}$ (of dimension $m \times 1$). This first equation is known as the *observation (measurement) equation* and is given by

$$X_t = Z_t \alpha_t + \epsilon_t, \quad t = 1, 2, ..., \tag{1.6}$$

where $\{\epsilon_t\} \sim WN(0, \sigma_{\epsilon^2})$ and $\{Z_t\}$ is a sequence of $1 \times m$ matrices with m being the lag cut off point.

The second equation known as *state (transition) equation* determines the evolution of state α_t at time $'t'$ in terms of the previous state α_{t-1} and the noise term ϵ_t. It is given by

$$\alpha_t = T_t \alpha_{t-1} + \epsilon_t, \quad t = 2, 3, ..., \tag{1.7}$$

where T_t is a sequence of $m \times m$ matrices and $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$.
It is assumed that the initial state α_1 is uncorrelated with all the noise terms. In general, $\alpha_t = f_t(\alpha_1, \epsilon_1, ..., \epsilon_{t-1})$
and

$X_t = g_t(\alpha_1, \epsilon_1, ..., \epsilon_{t-1}, \epsilon_t)$

Note: The state space representation given by (1.6) and (1.7) is not unique, but useful in developing important results in linear and non-linear time series models utilizing *recursive estimation methods*. It becomes the topic of discussion in the next subsection.

1.2.2. Recursive Estimation. Recursive estimation revolves around the process known as $Recursion$. In a mathematical and computational sense it is defined as the mechanism of a function calling itself within an embedded structure. Estimators of this type are aimed at tracking time varying parameters. Therefore it is desirable to make calculations recursively to save computation time. In recursive estimation a parameter estimate defined as P_t (a vector of the parameter at time t) could be obtained as a function of the previous estimate P_{t-1} and of the current (or new) measurements.

Some standard recursive algorithms are utilized in practice to perform the tracking of time series parameters through estimation and forecasting. Of them the most popular algorithm utilized by state space modelling specialists for estimation and prediction is known as the KF, and becomes the focal point of the next subsection.

1.2.3. KF and Estimation Process. A set of recursions known as the KF was introduced by Kalman (1961) and Kalman and Bucy (1961) to provide estimates of parameters in a state space model of a time series or a linear dynamic system disturbed by Gaussian white noise. Approximate maximum likelihood estimation and prediction of time series parameters can be executed by adopting a state space approach coupled with the KF.

Due to the presence of stochastic elements in the system, it uses a series of measurements observed over time containing random variations (noise) to return innovations of the model in creating the log-likelihood and profile likelihood functions. This gives the optimal maximum likelihood estimates (MLE's) of the model parameters through the $Kalman\ gain$ (which shows the effect of estimate of the previous state to the estimate of the current state), with a minimal prediction error variance. Furthermore, a discussion about the KF formulation of the likelihood function can be found in Jones (1980). The methodology presented in this section could probably be employed to model a seasonal GARMA series and becomes the next discussion topic. The same methodology could be employed to analyze seasonality in long memory Gegenbauer series as an extension to address the prevalent issue.

It will be important in terms of modelling another extended class of Gegenbauer processes since in practice, researchers have noticed that many time series consist of seasonality in both ARFIMA and GARMA with long memory. This phenomenon had been observed in monetary aggregates of Porter-Hudak (1990), revenue series of Ray (1993), inflation rates of Hassler and Wolters (1995), quarterly gross national product and shipping data of Ooms (1995) and monthly Nile river flows of Montanari et.al. (2000). Several statistical modelling methodologies for such processes have been developed. The fractional Gaussian noise process of Abrahams and Dempster (1979), seasonal fractionally integrated autoregressive moving average (SARFIMA) model of Porter-Hudak (1990), flexible seasonal fractionally integrated process (flexible ARFISMA) of Hassler (1994), k-GARMA process of Woodward et.al.(1998), seasonal long range dependent process of Palma and Chan (2005), seasonal fractionally integrated process of Reisen et. al (2006) and the KF based SARFIMA process of Bisognin and Lopes (2009) are some examples available in the literature. Properties of these models have been investigated by Giraitis and Leipus (1995), Chung (1996), Arteche and Robinson (2000), Velasco and Robinson (2000), Giraitis et.al. (2001), Palma (2007), Arteche (2007), Koopman et.al. (2007), Bisognin and Lopes (2009), Hsu and Tsai (2009) and Arteche (2012).

Similarly, in many financial time series modelling problems, it is known that heteroscedasticity plays an important role. Such models in common use are the autoregressive conditionally heteroskedastic (ARCH) model of Engle (1982) and its generalized ARCH (GARCH) model due to Bollerslev (1986). An extension of it by Baillie, et. al (1996a) resulted in a fractionally integrated GARCH (FI-GARCH) to model the conditional variance. Incorporating heteroscedasticity in ARFIMA models with GARCH errors has been studied by Ling and Li (1997). An attempt to capture and blend some of these established GARCH features in introducing conceptual properties of a new class of models with conditionally heteroskedastic errors is missing in the literature. In such a context it could be based on the generalized fractional operator that was used by Anh et. al (1999) and later developed by Peiris (2003),

& Shitan and Peiris (2008, 2013). This scaled down new operator with further applications was employed by Peiris et. al (2005) and Peiris and Thavaneswaran (2007) in long memory models driven by heteroskedastic GARCH errors. Employing the state space methodology to estimate the new class of models with GARCH errors could provide an additonal contribution to the literature.

Several studies in the literature report evidence of long memory in empirical volatility returns as illustrated by Robinson (1991), Shephard (1996), Lobato and Savin (1998) and Baillie (1996). McAleer and Medeiros (2008) proposed a flexible model to describe nonlinearities and long memory in time series dynamics for purposes of forecasting volatility. Additionally, Lieberman and Phillips (2008) have provided some analytical explanations for long memory behavior that has been observed in realized volatility. Furthermore, a simple additive cascade approximate long memory model of realized volatility was proposed by Corsi (2009). In a recent development Rossi and De Magistris (2014) illustrated if instantaneous volatility is a long memory process, then the integrated variance is characterized by the same long range dependence. Furthermore, Bos et. al (2014) provides evidence that properties of time series such as US inflation are unstable over time through the model-based analysis of a specified long memory ARFIMA process with variance modelled by stochastic volatility. Most econometric models dealing with long memory and heteroskedastic behaviour are nonlinear in the sense that the noise sequence is not necessarily independent. Such models based on conditional variance are primarily driven by a GARCH process.

Furthermore, it has become a customary practice in applied time series analysis to conduct tests to assess whether a time series is stationary or to be integrated at a suitable degree of differencing. Such testing to check if a variable is integrated of order one or stationary without being differenced using null hypotheses was illustrated by Phillips and Xiao (1998). This procedure became very popular among econometricians and was made famous under the theme of *"unit root hypothesis"* (See for example: Phillips and Xiao (1998), Dolado et.al. (2002)). Such hypotheses of stationary series with respect to unit roots could be extended to fractional processes with long memory. A unit root test for fractionally integrated processes has been proposed in Dolado et. al (2002) and asymptotic results of a similar test is presented in Wang et. al (2003). Additionally, Taylor (2005) introduced a set of new tests to assess constant trend stationarity against the change in persistence from trend stationarity to difference stationarity or vice versa. Furthermore, Ohanissian et. al (2008) proposed a statistical test to distinguish between true and spurious long memory.

Hypothesis testing of linear or nonlinear constraints on the parameters of econometric models could be performed using one of three methods. They are "Likelihood Ratio (LR), Wald and Lagrange Multiplier (LM) tests. All are asymptotically equivalent, but in most finite samples the results will differ. In many ways the LR test is conceptually the easiest to use and can be employed for any model estimated by maximum or quasi-maximum likelihood. Finding an appropriate test for a specific time series generally becomes a challenge. For a long memory fractionally differenced Gegenbauer process and related parameter estimation through state space modeling and KF the estimation process employs a quasi maximum likelihood function. Therefore an LR test becomes the most feasible assessing tool in such a context. Generating results of such a test with an acceptable power may add another creative component to the existing body of knowledge available in the literature.

Certain preliminary concepts and results gathered from the perused literature is provided next in order to develop a research question.

Facts provided in terms of the background information, literature review, preliminaries, fundamental concepts on fractional differencing and long memory discussed in this chapter & from the current body of knowledge, formed the basis in establishing new insights and the motivation to create a research problem.

Note: The literature review of this chapter highlights certain gaps linked with fractionally differenced long memory Gegenbauer processes. It creates a huge void in the existing body of knowledge with respect to a new class of models. Motivation to deal with such a void could stem out of the belief that a hassle free, alternative estimation / prediction mechanism could be introduced to offset the complex issue of multiple discontinuities of the spectrum away from the origin. Which leads towards the formulation of the *research problem* introduced in the next section.

1.3. Research Problem

A research problem could be defined as a gap (vacuum or lacuna) that exists in the current literature of a specific subject area between the actual state and the desired ideal state as defined by Sekaran (2006). Therefore in reviewing the existing literature on fractionally differenced Gegenbauer processes with long memory, it is apparent that there exists a vacuum in terms of model formulation, optimal parameter estimation, prediction and testing. In a precise and broader sense, the research problem segmented in phases could be presented as the following:

- Is it possible to establish a state space model based alternative estimation method for a GARMA series to deliver an optimal lag order validated by predictive accuracy as a creative component?
- Can the methodology be extended to newly created original members of the same time series family such as generalized fractionally differenced conditionally heteroskedastic & seasonal GARMA (GARSMA) processes in formulating and establishing a new class of fractionally differenced long memory models and related results?
- Can some and/or all of the proposed members of the new class be assessed through an appropriate nearly efficient unit root test based on the proposed state space methodology as an additional contribution?

This multi faceted holistic problem will be presented as segmented topics of interest in this thesis, and addressed in a series of chapters as a contribution to the current body of knowledge.

Brief details in addressing segmented issues of the research problem in terms of the chapters are provided next in the thesis outline.

1.4. Thesis Outline

Consideration of this thesis will be given to a class of Gegenbauer processes generated by Gaussian white noise and GARCH errors since most of the other models are nested within this new family. In terms of this new class of models the thesis will be organized as follows. Chapter 2 will be a theoretical & analytical assessment of two topics relevant to the class known as "*fractional differencing*" and "*long memory*". It will provide the conceptual and methodological framework to address the observed research question. In lieu of it, state space modeling of Gegenbauer processes with long memory becomes the topic of Chapter 3. It will be the core chapter of the thesis with multiple contributions. They include a state space configuration of a GARMA process, a comparative assessment of a couple of estimation techniques, establishing an optimal lag order, introducing an asymptotic variance estimate of the long memory parameter (corroborating the theory), a positive meta analytical study (in terms of other similar models in the literature) & a real application as creative components. It will be followed by Chapter 4 on modelling of persistence and conditional variance through the extension of the methodology presented in the core chapter to a series of the same class with a different error distribution. The extended state space configuration, advent of related theory, outperforming of a similar model in the literature and a real application forms the contributory factors of the chapter. Theory and results on another extension through the state space modelling of seasonal Gegenbauer processes will be introduced in Chapter 5. Thereafter a model testing phase will be presented in Chapter 6 by extending an already established appropriate unit root test in the literature resulting in a original contribution in terms of asymptotics and the power. Chapter 7 will provide a discussion with respect to the entire research endeavour and an assessment on further related research.

An in-depth discussion on the special class of long memory processes titled as "fractionally differenced Gegenbauer processes with long memory" begins in the next chapter in line with the thesis outline.

CHAPTER 2

Fractionally Differenced Gegenbauer Processes with Long Memory

---

It is clear that standard long memory processes do not have well defined spectral properties near the origin. Fortunately, a class of long memory processes can be transformed to short memory processes using a suitable fractional differencing filter. These fractional filters are widely used in long memory modelling contrary to integer differencing filters in ARIMA series.

2.1. Fractional Differencing

Suppose that $\{X_t\}$ is a stationary long memory time series. It is well known that the time series X_t can be transformed to a short memory series Y_t through a fractional filter of the form

$$Y_t = (I - B)^d X_t,\ d \in (0, 0.5),$$

where B is defined as the backshift operator.
See for example, Granger and Joyeux (1980) and Hosking (1981).
When $Y_t = \epsilon_t$, $\{X_t\}$ is said to be fractionally differenced white noise or FDWN.
The above fractional differencing is used in long memory time series modelling and analysis.
This is used as building blocks for long memory ARFIMA(p,d,q) modelling given by

$$\phi(B)(1 - B)^d X_t = \theta(B)\epsilon_t, \tag{2.1}$$

where $\phi(B)$ and $\theta(B)$ are stationary AR(p) and invertible MA(q) operators with no common zeros and that $\phi(B) = 1 - \phi_1 B - ... - \phi_p B^p$, $\theta(B) = 1 + \theta_1 B + ... + \theta_q B^q$ have zeros outside the unit circle. Hualde and Robinson (2011) argued that the pseudo-maximum likelihood estimator depicts consistency and asymptotic normality properties even when $d \in [-\frac{3}{2}, \frac{3}{2}]$.

A general scheme of an alternative fractional differencing filter is presented in the next sub section.

2.1.1. Generalized Fractional Difference Operator and its Properties. This section considers the first and second order generalized fractional operators and their applications to time series.

• The first-order case:

Consider the process generated by

$$(I - \alpha B)^d X_t = \epsilon_t;\ -1 < \alpha < 1;\ d > 0 \quad and \quad \{\epsilon_t\} \sim WN(0, \sigma^2). \tag{2.2}$$

Clearly, this covers the standard first order autoregressive [AR(1)] family when $d = 1$, and FDWN when $\alpha = 1$, $0 < d < 1/2$.

• The second-order case:

A second order model is often a natural extension of (2.2) and is given by:

$$(1 - \alpha_1 B - \alpha_2 B^2)^d X_t = \epsilon_t, \tag{2.3}$$

where $1 - \alpha_1 z - \alpha_2 z^2 \neq 0$ for all $|z| \leq 1,\ d > 0$ and $\{\epsilon_t\} \sim WN(0, \sigma^2)$.

In (2.3), conditions $\alpha_2 + \alpha_1 < 1,\ \alpha_2 - \alpha_1 < 1$ and $-1 < \alpha_2 < 1$ must be satisfied by α_1 and α_2 to ensure stationary solutions. An interesting case from (2.3) arises when the stationary assumption fails. That is, at least one of the above three inequalities does not hold. This will be considered in subsection 2.2.1 with details.

By writing the model (2.3) as,

$$[(1 - \xi_1 B)(1 - \xi_2 B)]^d X_t = \epsilon_t,$$

where $\xi_1 + \xi_2 = \alpha_1$ and $\xi_1\xi_2 = -\alpha_2$.

It can be shown that following Shitan and Peiris (2008) the solution for (2.3) is given by,

$$X_t = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\Gamma(k+d)\Gamma(l+\delta)\xi_1^k\xi_2^l}{\Gamma(k+1)\Gamma(l+1)\Gamma^2(d)}\epsilon_{t-k-l},$$

where $\Gamma(\cdot)$ is the gamma function satisfying $\Gamma(k) = (k-1)!$ for integers $k \geq 1$.

The model in (2.3) is called the generalized second order autoregression with index d. For simplicity, we call it Generalized Autoregressive of order 2 [GAR(2)] model. It is clear that the class generated by (2.3) is more flexible than the standard AR(2) model. Although the autocorrelation function of the GAR(2) model can be obtained, it is not mathematically tractable as there is no closed form solution. The spectral density $f(\omega)$ is given by,

$$f(\omega) = \frac{\sigma^2}{2\pi}[(1+\alpha_1^2+\alpha_2^2) - 2\alpha_1(1-\alpha_2)\cos\omega - 2\alpha_2\cos 2\omega]^{-d}.$$

See Shitan and Peiris (2008) for details.

As mentioned before, when $\alpha_2 = -1$ in (2.3), the roots of $1 - \alpha_1 Z - \alpha_2 Z^2 = 0$ do not lie in the stationary region. This leads to Gegenbauer processes which extends to a mathematically elegant and interesting class of time series models that seem to have very useful applications. Now we look at this class of models in the next section.

2.2. Gegenbauer Processes

These processes evolve around Gegenbauer polynomials and it will be considered in the next subsection.

2.2.1. Gegenbauer Polynomials and their Properties. Gegenbauer polynomials are a special case of Jacobi polynomials that generalize Legendre and Chebyshev polynomials. They are named after German mathematician Leopold Gegenbauer and are classified as *orthogonal polynomials* within a support interval $[-1, 1]$.

Gegenbauer or *ultraspherical polynomials* $C_j^d(u),\ |u| \leq 1;\ d > 0$ are defined in terms of their generating functions as follows:

$$\frac{1}{(1-2ut+t^2)^d} = \sum_{j=0}^{\infty} C_j^d(u)t^j, \tag{2.4}$$

where

$$C_j^d(u) = \sum_{k=0}^{[j/2]} (-1)^k \frac{\Gamma(j-k+d)}{\Gamma(d)\Gamma(j+1)\Gamma(j-2k+1)} (2u)^{j-2k},$$

and $[j/2]$ stands for the integer part of $\frac{j}{2}$ (See: Erdélyi et.al, 1953, 10.9).

It is easy to verify that

- $C_0^d(u) = 1$
- $C_1^d(u) = 2du$
- $C_2^d(u) = -d + 2d(1+d)u^2$
- $C_3^d(u) = -2d(1+d)u + \frac{4}{3}d(1+d)(2+d)u^3.$

Further it can be shown that $C_j^d(u)$ satisfies the recursion

$$C_j^d(u) = \frac{1}{j}[2u(j+d-1)C_{j-1}^d(u) - (j+2d-2)C_{j-2}^d(u)];\ j \geq 2,$$

with initial values $C_0^d(u)$ & $C_1^d(u)$ as given above.

See Rainville (1960), Gould (1974) and Gradshteyn and Ryzhik (1980) for details.

The coefficients $C_j^d(u)$ or C_j^d (for convenience) for large j can be approximated by:

$$C_j^d \sim \frac{cos[(j+d)\omega_0 - d\pi/2]}{\Gamma(d)sin^d(\omega_0)(\frac{2}{j})^{1-d}}, \tag{2.5}$$

where $\omega_0 = cos^{-1}(u)$ (see Chung (1996)).

However, for $|u| < 1$ the polynomial $1 - 2ut + t^2$ has complex zeros which are on the unit circle. Therefore, we need an additional restriction on d such that $d < 1/2$ to ensure the square summability of C_j^d or $\sum_{j=0}^{\infty}\{C_j^d\}^2 < \infty$. Note that when $|u| = 1$, the real zeros are on the unit circle and d must satisfy, $d < \frac{1}{4}$ for square summability of coefficients. The process generated by $(1-2uB+B^2)^d X_t = \epsilon_t$ is called a Gegenbauer process of order (0,d,0) or GARMA(0,d,0) and is discussed in the next subsection.

2.2.2. Fractional Gegenbauer(0,d,0) Process and its Properties. Consider the process generated by

$$(1-2uB+B^2)^d X_t = \epsilon_t, \tag{2.6}$$

where $|u| \leq 1$, $0 < d < 1$ and $\{\epsilon_t\} \sim WN(0, \sigma^2)$.

As shown by Chung(1996), when $|u| < 1$ and $0 < d < 1/2$, the acf of $\{X_t\}$ in (2.6) can be approximated by:

$$\rho_j \sim K \cos(j\omega_0) j^{2d-1} \quad as \quad j \to \infty, \tag{2.7}$$

K a constant that depends on d and ω_0.

It is clear that X_t has the following $MA(\infty)$ representation to the Gegenbauer process given in (2.6):

$$X_t = (1-2uB+B^2)^{-d} \epsilon_t, \tag{2.8}$$

or

$$X_t = \sum_{j=0}^{\infty} C_j \epsilon_{t-j}, \tag{2.9}$$

A general class to include AR & MA components is called GARMA(p,d,q) and will be discussed in 2.2.3.

2.2.3. The GARMA(p,d,q) Process. The process $\{X_t\}$ generated by the following equation is called Gegenbauer ARMA(p,d,q) or GARMA(p,d,q):

$$\phi(B)(1-2uB+B^2)^d X_t = \theta(B)\epsilon_t, \tag{2.10}$$

where $\phi(B)$, $\theta(B)$ and other parameters are as defined in (2.1) & (2.6) respectively.

Now consideration is given to the properties of a GARMA(0,d,0) process with long memory.

2.2.4. Stationary Gegenbauer Processes with Long Memory. Firstly, the following theorem due to Gray et.al.(1989) is stated as:

Theorem 2.1:

(i) A fractional Gegenbauer(0,d,0) process is stationary if:

(a) $|u| < 1$ and $d < 1/2$, or (b) $u = \pm 1$ and $d < 1/4$.

(ii) A fractional Gegenbauer(0,d,0) process is invertible if:

(a') $|u| < 1$ and $d > -1/2$, or (b') $u = \pm 1$ and $d > -1/4$.

Refer Gray et.al.(1989) for the proof of Theorem 2.1.

The spectrum of (2.6) is:

$$f(\omega) = |1 - 2u\exp(i\omega) + \exp(2i\omega)|^{-2d}\sigma^2/\pi. \tag{2.11}$$

This can be simplified to

$$f(\omega) = \frac{\sigma^2}{2\pi}\{2|\cos(\omega) - cos(\omega_0)|\}^{-2d} = \frac{\sigma^2}{2\pi}|4\sin(\frac{\omega+\omega_0}{2})sin(\frac{\omega-\omega_0}{2})|^{-2d}, \tag{2.12}$$

where $\omega_0 = cos^{-1}(u)$.

Moreover, it could be written as;

$$f(\omega) = \frac{\sigma^2}{2\pi}4(\cos\omega - u)^{2-d}, \quad -\pi < \omega < \pi \tag{2.13}$$

implying an unbounded spectral density at $\omega_0 = cos^{-1}(u)$. Therefore a stationary Gegenbauer process is long memory when $0 < d < \frac{1}{2}$. The frequency ω_0 is called Gegenbauer or G frequency.

The above results are summarized for later reference. Furthermore, theorem 2.2 due to Gray et.al.(1989) states,

Theorem 2.2:

A stationary Gegenbauer process is long memory if $|u| < 1$ and $0 < d < 1/2$ or if $|u| = 1$ and $0 < d < 1/4$.

Refer Gray et.al.(1989) for the proof of Theorem 2.2.

Given the spectral density $f(\omega)$, we can compute the autocovariance γ_j at lag j of a stationary long memory Gegenbauer process for the case $|u| < 1$ following Gray et.al (1989) as:

$$\gamma_j = \frac{\sigma^2}{2\sqrt{(\pi)}}\Gamma(1-2d)[2sin(\omega_0)]^{1/2-2d}[P_{j-1/2}^{2d-1/2}(u) + (-1)^j P_{j-1/2}^{2d-1/2}(-u)], \tag{2.14}$$

where $P_a^b(x)$ are associated Legendre functions that can be calculated using the recursion:

$$P_a^b(x) = \frac{2a-1}{a-b}xP_{a-1}^b(x) - \frac{a+b-1}{a-b}P_{a-2}^b(x) \tag{2.15}$$

with initial terms:

$$P_{-1/2}^{2d-1/2}(u) = [(1+u)/(1-u)]^{d-1/4}[1/\Gamma(3/2-2d]F(1/2,1/2;3/2-2d;(1-u)/2), \tag{2.16}$$

$$P_{1/2}^{2d-1/2}(u) = [(1+u)/(1-u)]^{d-1/4}[1/\Gamma(3/2-2d]F(-1/2,1/2;3/2-2d;(1-u)/2), \tag{2.17}$$

and $F(a,b;c;w) = \sum_{n=1}^{\infty}[\Gamma(c)\Gamma(a+n)\Gamma(b+n)/\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)]w^n$ being hypergeometric functions. It leads towards the following approximation on autocovariances:

$$\gamma_j = \frac{2^{1-2d}\sigma^2}{\pi}sin^{-2d}(\omega_0)sin(d\pi)\Gamma(1-2d)cos(j\omega_0)\frac{\Gamma(j+2d)}{\Gamma(j+1)} \tag{2.18}$$

From Figure 2.1 it is apparent that the illustrated Gegenbauer process driven by white noise has long memory due to the hyperbolic decay of acf and pacf functions, and the infinite peaks of the spectrum. Subsection 2.2.5 provides an extension of the process, while for a visual comparison acf and pacf graphs of Gegenbauer(0,d,q) and Gegenbauer(p,d,0) processes are provided as Figures 2.2-2.5.

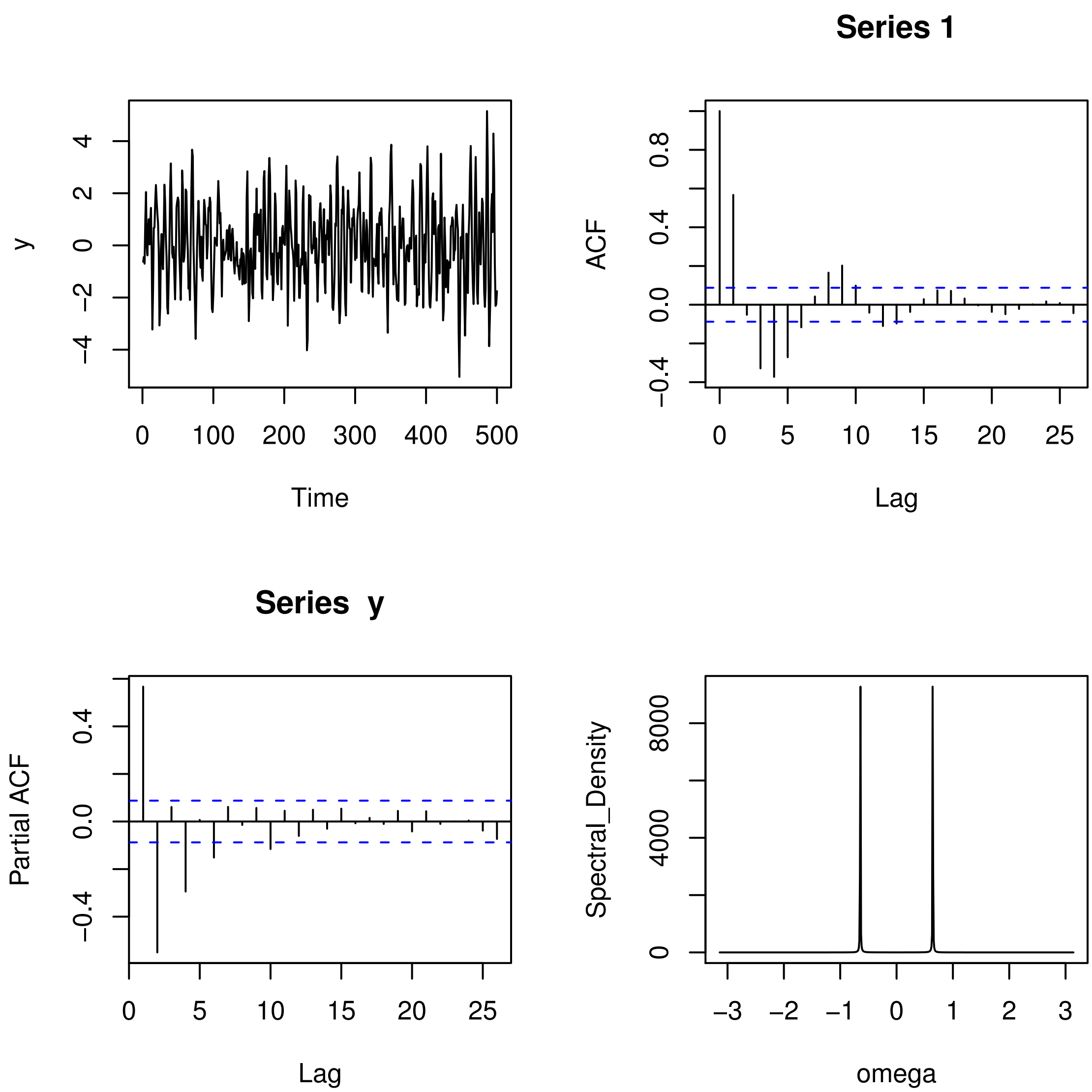


FIGURE 2.1. Gegenbauer$(0, d, 0)$ model with $d = 0.05$, $u = 0.8$

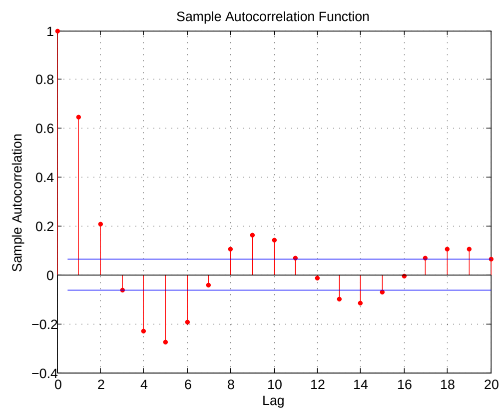


FIGURE 2.2. Autocorrelation function of Gegenbauer(0,d,q) Process

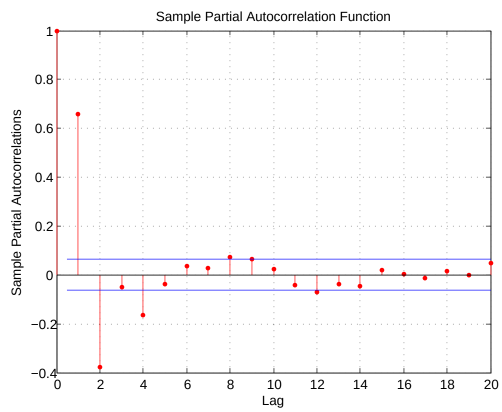


FIGURE 2.3. Partial Autocorrelation function of Gegenbauer(0,d,q) Process

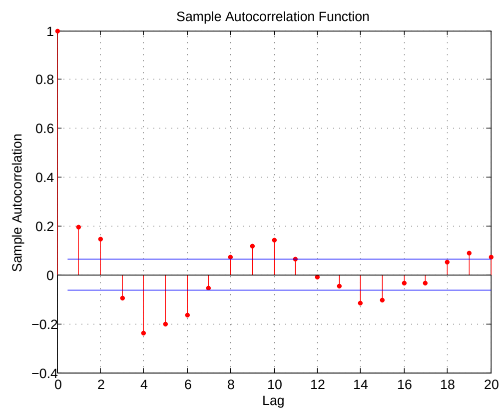


FIGURE 2.4. Autocorrelation function of Gegenbauer(p,d,0) Process

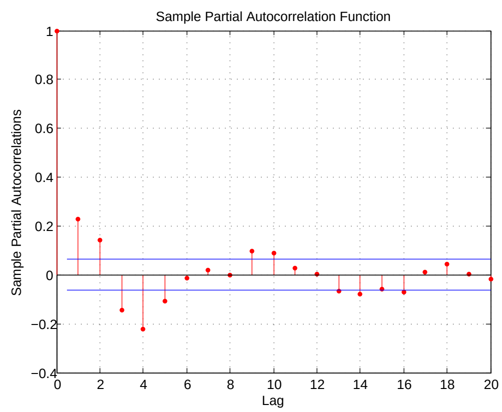


FIGURE 2.5. Partial Autocorrelation function of Gegenbauer(p,d,0) Process

2.2.5. K-factor Gegenbauer Processes. A k-factor Gegenbauer process as introduced by Woodward et.al (1998) is given by:

$$\prod_{i=1}^{k}(1-2u_iB+B^2)^{d_i}X_t=\epsilon_t, \tag{2.19}$$

illustrating properties of it by defining the spectrum as:

$$f(\omega)=\sigma^2/\pi|\prod_{j=1}^{k}[1-2u_jexp(-i\omega)+exp(-2i\omega)]^{-d_j}|^2 \tag{2.20}$$

for $0<\omega<\pi$.

As in (2.13) this reduces (2.20) to:

$$f(\omega)=\sigma^2/\pi\prod_{j=1}^{k}[4\{cos(\omega)-cos(\omega_{0,j})\}^2]^{-d_j} \tag{2.21}$$

Clearly, the spectrum $f(\omega)$ has k unbounded peaks at frequencies $f(\omega_{0,j})=\omega_{0,j}/2\pi$, around $j=1,...,k$ called G frequencies.

Certain k-factor Gegenbauer process properties are highlighted as follows:

- The k-factor Gegenbauer process of (2.19) is stationary if the u_i are distinct and $d_i<1/2$ whenever $|u_i|<1$ and $d_i<1/4$ when $|u_i|=1$, and
- Long memory in each case if $0<d_i<1/2$, $i=1,...,k$.

Woodward et.al (1998) pointed out that the true autocorrelations of a k-factor Gegenbauer process in (2.19) could be obtained through the Fourier transform of its spectrum $f(\omega)$. Further information on estimation and forecasting of k-factor Gegenbauer process is available in Ferrara and Guegan (2001) and the evolution of it from a k-factor $GARMA(0,d,0)$ process in Guegan (2005). Estimation of the time-varying long memory parameter of a locally stationary k-factor Gegenbauer process utilizing the wavelet method is presented in Lu and Guegan (2011).

Remark: If a GARMA(0,d,0) process is further differenced by convoluting it with a seasonal filter $(1 - B^{s'})$, with s' as the seasonal period it results in a seasonal fractionally differenced Gegenbauer process. It becomes the topic of interest in the next subsection.

2.2.6. Seasonal Fractionally Differenced Gegenbauer Processes - An overview. Let s' be the period of a seasonal time series and $0 < D_{s'} < \frac{1}{2}$. Then

$$(1 - B^{s'})^{D_{s'}}(1 - 2uB + B^2)^d X_t = \epsilon_t. \tag{2.22}$$

represents a seasonal Gegenbauer process with long memory if $0 < d < \frac{1}{2}$, $|u| < 1$. This is an extension of the standard seasonal long memory introduced by Robinson (1994).

In general, for any s', it can be shown that

$$(1 - B^{s'}) = (1 - B).(1 + B + B^2 + B^3 + ... + B^{s'-1}), \tag{2.23}$$

The seasonal operator $(1 - B^{s'})^{D_{s'}}$ can be written as a product of suitable Gegenbauer polynomials in order to simplify calculations. For example:

When $s' = 3$ (4 annual or trimester data), it results in

$$(1 - B^3)^{D_{s'}} = (1 - B)^{D_{s'}}.(1 + B + B^2)^{D_{s'}}, \tag{2.24}$$

and

when $s' = 2$ (bi-annual data) the result is

$$(1 - B^2)^{D_{s'}} = (1 - 2Bcos(\omega_0) + B^2)^{D_{s'}/2}.(1 - 2Bcos(\omega_1) + B^2)^{D_{s'}/2}, \tag{2.25}$$

with $\omega_0 = 0$, $\omega_1 = \pi$ and

in addition for quarterly data (ie. $s' = 4$) the derivation becomes

$$(1 - B^4)^{D_{s'}} = (1 - 2Bcos(\omega_0) + B^2)^{D_{s'}/2}.(1 - 2Bcos(\omega_1) + B^2)^{D_{s'}/2}.(1 - 2Bcos(\omega_2) + B^2)^{D_{s'}}, \tag{2.26}$$

where $\omega_0 = 0$, $\omega_1 = \pi$ and $\omega_2 = \pi/2$. (See Giraitis and Leipus (1995) and Reisen et. al (2006) for details).

The expressions (2.22)-(2.26) could then be incorporated to form a class of fractionally differenced seasonal Gegenbauer processes with long memory.

The state space mechanism introduced in chapter 1 could be employed to assess processes of the form defined by equation (2.22). Further extended details with respect to the derivations of the seasonal operator and related state space modelling will be provided in chapter 6 of this thesis.

Adhering to the thesis outline, chapter 3 develops the theory related to "State space modeling of Gegenbauer processes with long memory".

CHAPTER 3

State Space Modeling of Gegenbauer Processes with Long Memory

3.1. Introduction

Modelling of long memory processes have become increasingly popular in the recent past in financial econometrics and mathematical statistics. In that context certain unique characteristics of long memory models were introduced by Andel (1986), Beran (1994), Robinson (2003) & Palma (2007) following Granger and Joyeux (1980) & Hosking (1981). They provided a detailed account of long memory models in terms of statistical and econometric interpretations. At present a variety of applications in macroeconomics and finance depict the relevance of fractional integration and long memory. A list of such examples are: Diebold et al. (1991) for exchange rates, Andersen et al. (2001a,b) for financial volatility series, and Baillie et al.(1996) for inflation data. Initial expositions of the estimation of long range dependent models are provided in Fox and Taqqu (1986), Dahlhaus (1989), Sowell (1992), Chen et al. (1994) and Robinson (1995). Leipus and Viano (2000) added a class of generalized fractional filters and associated theoretical aspects in modelling long memory time series with finite or infinite variance to the literature. A review of this area is given in Chan and Palma (2006), Palma (2007), Giraitis et al. (2012) and Beran et al. (2013).

Consider the stationary long memory GARMA(0,d,0) model introduced in (2.6):

$$(1-2uB+B^2)^d X_t = \epsilon_t, \tag{3.1}$$

with the power spectrum

$$f_X(\omega) = c[4(\cos\omega - u)^2]^{-d},\ -\pi < \omega < \pi, \tag{3.2}$$

where c is a suitable constant.

As a contribution to the existing knowledge in terms of long memory models, truncated state space representations and Kalman Filter are considered to estimate the parameters of a GARMA time series. Further investigation of the equivalent AR and MA approximations to (2.1) by extending the state space modeling techniques introduced by Chan and Palma (1998) enabled to propose a procedure to find an optimal lag of the truncation through a comparative assessment. As a special case of interest and for simplicity, we consider the stationary long memory GARMA(0,d,0) model given in (3.1)

This chapter is organized as follows. Section 3.2 introduces the MA and AR approximations and corresponding state space representations of a truncated GARMA(0,d,0) process. In section 3.3 the Kalman filter theory and fundamental concepts involved in quasi maximum likelihood estimation (QMLE) are discussed. Section 3.4 provides an illustrative example. In section 3.5, a Monte Carlo Simulation experiment based on approximation techniques is presented to assess state space methodology and parameter estimation. The results of the suggested experiment in this section are used to establish asymptotic and other statistical properties of the estimates in extending the discussion. This is followed by section 3.6, where a comparative assessment of AR and MA approximations of a truncated GARMA process is presented for the purpose of choice and selection. Section 3.7 presents empirical evidence with corraborating real applications on two different data sets comprising Wolfer's and Tong's annual sunspots. Finally, concluding remarks are provided in section 3.8.

3.2. MA and AR approximations and their state space representations

The Wold representation of the Gaussian Gegenbauer or GARMA(0,d,0) process given in (3.1) with $\epsilon_t \sim \mathrm{NID}(0, \sigma^2)$ is

$$X_t = C(B)\epsilon_t = \sum_{j=0}^{\infty} C_j \epsilon_{t-j}, \tag{3.3}$$

where $\sum_{j=0}^{\infty} C_j B^j = (1 - 2uB + B^2)^{-d}$, $C_0 = 1$ and the coefficients C_j are functionally dependent on d and u as given in (2.4).

The m-th order MA approximation arises from truncating the Wold representation of (3.3) at lag m, i.e.

$$X_{t,m} = \sum_{j=0}^{m} C_j \epsilon_{t-j}. \tag{3.4}$$

$X_{t,m}$ will be referred to as a truncated Gegenbauer process.
A state space representation of the above MA(m) model in (3.4) is

$$\begin{aligned} X_{t,m} &= Z\alpha_t + \epsilon_t, \quad and \\ \alpha_{t+1} &= T\alpha_t + H\epsilon_t, \end{aligned} \tag{3.5}$$

where α_{t+1} is the $m \times 1$ state vector has elements - $\alpha_{j,t+1} = E(X_{t+j,m}|\mathcal{F}_{t,m}), \quad \mathcal{F}_{t,m} = \{X_{t,m}, X_{t-1,m}, \ldots,\}$, and the system matrices:

$$Z = [1, 0, \ldots, 0], \quad T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad H = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ \vdots \\ C_m \end{bmatrix}.$$

Z, T and H are having dimensions $1 \times m$, $m \times m$ and $m \times 1$. See, for example, Chan and Palma (1998) for details.

The specification is completed by the initial state vector distribution, $\alpha_1 \sim \mathrm{N}(a_1, P_1)$, where $a_1 = 0$ and P_1 is the Toeplitz matrix with elements $p_{hk} = \sum_j C_j C_{j+|h-k|}$. As an alternative, the corresponding AR(m) approximation could be derived by truncating the AR(∞) representation $\pi(B)X_t = \epsilon_t, \pi(B) = (1 - 2uB + B^2)^d$. See Chan and Palma (1998) and Grassi and De Magistris (2014) for a comparison of the two approximations in the fractionally integrated case.

This leads to AR(m) and MA(m) approximations that can be estimated using the KF (see Harvey (1989) and Harvey and Proietti (2005) for information) as presented in the next section.

3.3. KF and QMLE

Given a sample time series $\{x_t, t = 1, \ldots, n\}$, the likelihood function of the approximating MA(m) model given in (3.4) is evaluated using the KF (see Harvey (1989) and Durbin and Koopman (2001)), using the following set of recursions:

$$\begin{aligned} \nu_t &= x_t - Za_t, & f_t &= ZP_tZ', \\ & & K_t &= (TP_tZ')/f_t, \\ a_{t+1} &= Ta_t + K_t\nu_t, & P_{t+1} &= TP_tT' + HH' - K_tK_t'/f_t. \end{aligned} \tag{3.6}$$

with $t = 1, \ldots, n$, and P_1 is as given before.

The KF returns pseudo-innovations ν_t, such that if the MA(m) approximation were the true model, $\nu_t \sim \text{NID}(0, \sigma^2 f_t)$ so that the log-likelihood of (d, u, σ^2) is (apart from a constant term)

$$\ell(d, u, \sigma^2) = -\frac{1}{2}\left(n \ln \sigma^2 + \sum_{t=1}^{n} \ln f_t + \frac{1}{\sigma^2}\sum_{t=1}^{n}\frac{\nu_t^2}{f_t}\right). \tag{3.7}$$

The scale parameter σ^2 can be concentrated out of the likelihood function, so that

$$\hat{\sigma}^2 = \sum_t \frac{\nu_t^2}{f_t},$$

and the profile likelihood is

$$\ell_{\sigma^2}(d, u) = -\frac{1}{2}\left[n(\ln \hat{\sigma}^2 + 1) + \sum_{t=1}^{n} \ln f_t\right]. \tag{3.8}$$

The maximisation of (3.8) can be performed by a quasi-Newton algorithm, after a reparameterization which constrains d and u in the subset of $\mathbb{R}^2$ $[0, 0.5) \times [0, 1)$. This can be done in practice using the reparameterization: $\theta_1 = \exp(2d)/(1 + \exp(2d))$ and $\theta_2 = \exp(u)/(1 + \exp(u))$. Asymptotic standard errors for the QMLE of the parameters d and u, can be obtained from the numerical second derivatives evaluated with respect to the transformed parameters θ_1 and θ_2, by using the delta method. For the properties of the QMLE see theorem 3.1 given at the end of section 3.5.

Note: The general GARMA(p,d,q) case can be treated similarly through the use of a finite order (m) autoregressive/moving average polynomial to be computed and cast in a suitable form of state space.

3.4. An Illustrative example

The above state space approach is illustrated through the fitting of a Gaussian Gegenbauer process as follows:

The noise $\epsilon_t \sim NID(0,1)$ is filtered to construct the series $\{X_t\}$; $t = 1, 2, ..., T$ $(eg : T = 1000)$ from (3.1) with $u = 0.8$, $d = 0.4$, $\sigma^2 = 1$ and to calculate the coefficients $C_j, j = 1, \ldots, m$ $(eg : m = 1000)$ such that theoretically m satisfies the condition $|C_m| < 0.0001$. The last 516 values of the simulated series through the corresponding MA filter in (3.3) is chosen for fitting. Hence, in practice the series is a realisation of the stochastic process $X_{t,m}$ given by (3.3), with m sufficiently large. The simulated series is plotted in Figure 3.1 along with its sample autocorrelation function. Figure 3.2 shows the sample spectrum,

$$I(\omega_j) = \frac{1}{2\pi}\left[\hat{\gamma}(0) + 2\sum_{k=1}^{n-1}\hat{\gamma}(k)\cos(\omega_j t)\right],$$

where $\hat{\gamma}(k)$ is the sample autocovariance function at lag k of the series, and $\omega_j = 2\pi j/n, j = 0, \ldots, [n/2]$, are the Fourier frequencies. Note that the true spectrum $f_X(\omega)$ at $\omega = cos^{-1}(u) = cos^{-1}(0.8)$ is infinite. The corresponding period is about 9.7 observations.

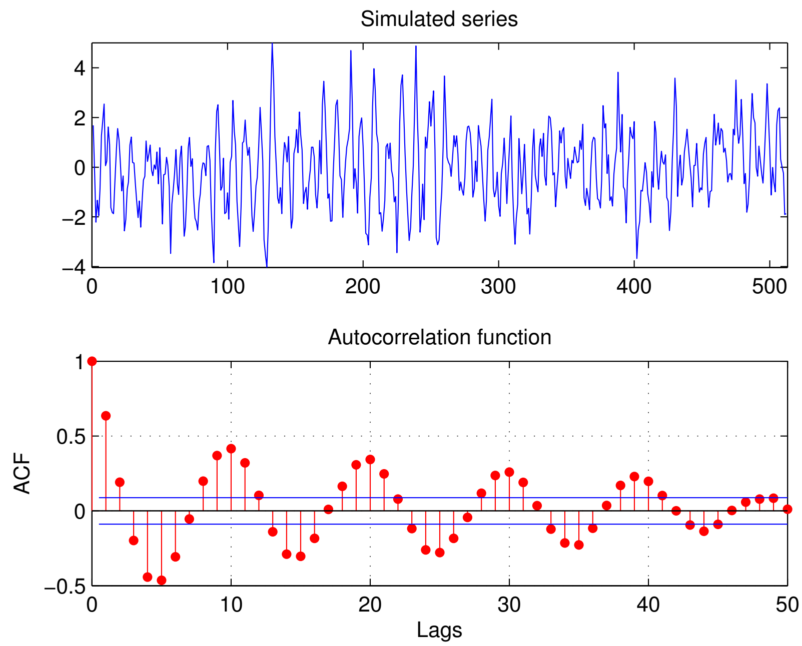
Various MA(m) approximating models have been fitted to the series by the approximate maximum likelihood estimation (MLE) method outlined in the previous section. The QMLEs of d, u and σ^2 are reported in Table 3.1 with a single replication using fast Fourier transform technique.

TABLE 3.1. QMLE Results.

m	1	5	15	25	50
$\hat{d}$	0.32	0.41	0.50	0.44	0.43
$\hat{u}$	0.98	0.80	0.75	0.78	0.78
$\hat{\sigma}^2$	1.31	1.06	0.97	0.95	0.96

The likelihood is monotonically increasing with m. However, it changes very slowly. Estimation standard error in this context will not depend on d in large samples and will add

FIGURE 3.1. Simulated time series of length $n = 500$ from the Gaussian Gegenbauer process with $d = 0.4, u = 0.8, \sigma^2 = 1$, $(1 - 1.6B + B^2)^{0.4} X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0, 1)$: plot of the series and its sample autocorrelation function.



a bias into the estimation results if the sample size was small (<100) coupled with the Monte Carlo simulation error. It was evident through the distortions in lag order convergence of small sample experiments and will not be a feasible option. Figure 3.3 displays the implied spectral density of $X_{t,m}$; corresponding to the above parameter estimates. For $m > 1$ they are characterised by a spectral peak around the frequency $cos^{-1}(\hat{u})$ and are side lobes due to the truncation of the MA filter. It results in a Fourier series oscillation overshoot near a discontinuity that does not die out with increasing frequency but approaches a finite limit known as *Gibbs phenomenon*. The autoregressive estimates do not suffer from the Gibbs phenomenon. This is illustrated in Figure 3.4.

FIGURE 3.2. Simulated time series of length $n = 500$ from the Gaussian Gegenbauer process with $d = 0.4, u = 0.8, \sigma^2 = 1$, $(1 - 1.6B + B^2)^{0.4}X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0, 1)$: sample spectrum (periodogram) and true spectral density function.

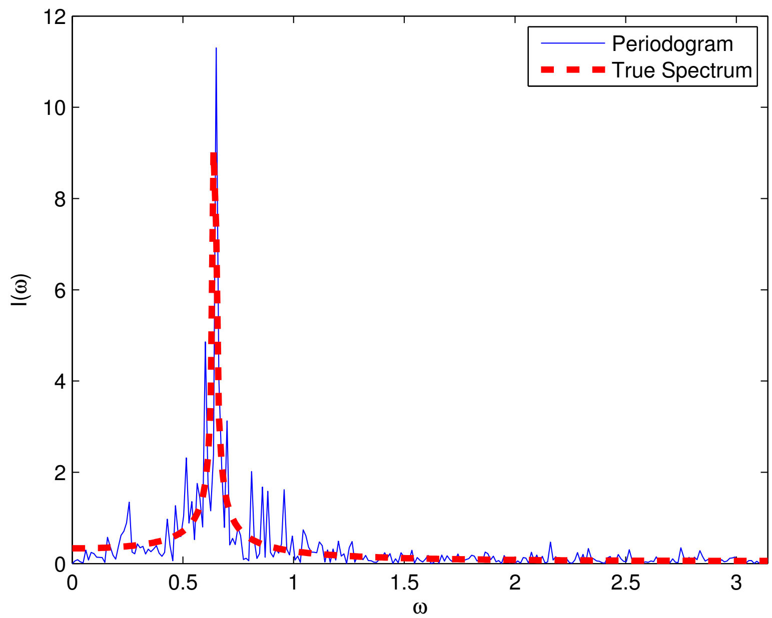


FIGURE 3.3. Simulated time series of length $n = 500$ from the Gaussian Gegenbauer process with $d = 0.4, u = 0.8, \sigma^2 = 1$, $(1 - 1.6B + B^2)^{0.4}X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0, 1)$: spectral densities estimated by fitting an MA approximating model of order m, with $m = 1, 5, 15, 25, 50$.

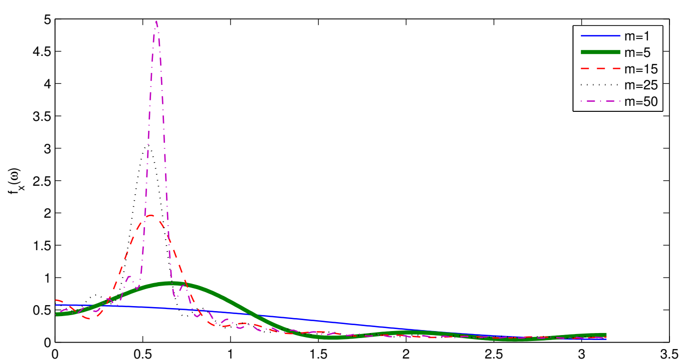
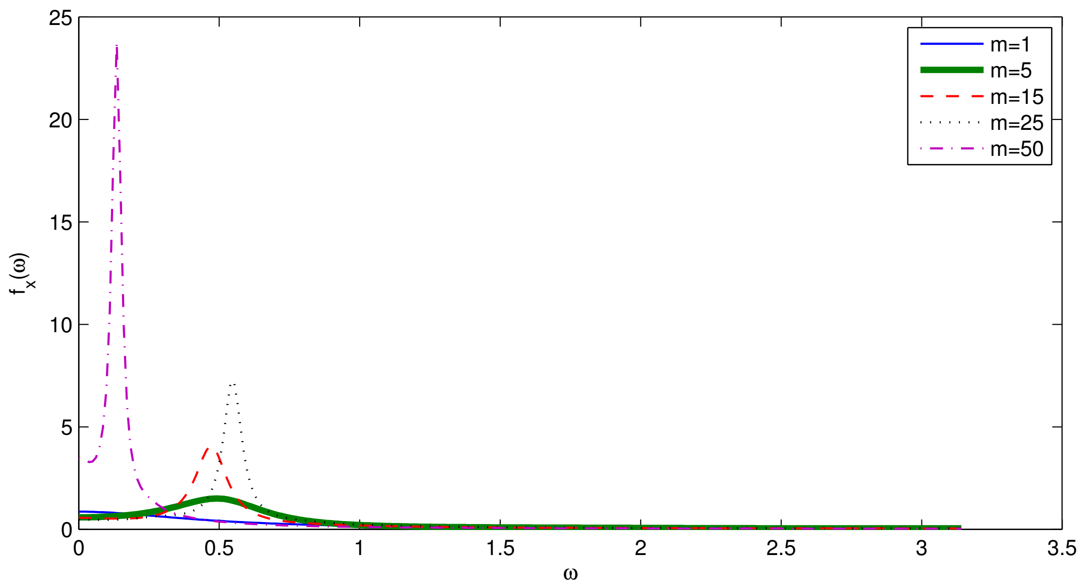


FIGURE 3.4. Simulated time series of length $n = 500$ from the Gaussian Gegenbauer process with $d = 0.4, u = 0.8, \sigma^2 = 1$, $(1 - 1.6B + B^2)^{0.4}X_t = \epsilon_t, \epsilon_t \sim$ IID N$(0, 1)$: spectral densities estimated by fitting an AR approximating model of order m, with $m = 1, 5, 15, 25, 50$.



Properties highlighted in the illustrative example were assessed in a large scale simulation study. It established an optimal lag order and assessed the viability of the approximations in terms of parameter estimation and prediction. Results are presented next in Section 3.5.

3.5. Monte Carlo Experiments

Apart from the estimation of model parameters the KF provides a mechanism to compute one step ahead and multi step ahead forecasts by generating the state predictor matrix. The matrix of one step ahead prediction forecasts is generated through the product of the state space matrix Z and the state predictor matrix. Similarly, the multi step ahead forecasts are found through the product of certain numeric quantities from the state space matrix T and the state predictor matrix through a recursive mechanism. The dual features of estimation and forecasting of the KF was utilized in a series of Monte Carlo experiments to validate the optimal estimation truncation point with the optimal truncation point for forecasting. To further assess the viability of the two approximation techniques an iterative experiment of both techniques with minimal Monte Carlo error was conducted on a simulated GARMA(0,d,0) series with lengths (n) of 100, 200, 500, 1000 and 2000 with parameters $d = 0.1, 0.2, 0.3, 0.4, 0.45$, for $u = 0.8$. The experiments with 1000 iterations were executed using the 32 bit version of the MATLAB-R 2011b package on 64 bit machines that were parallelized on 9 different servers with capacities ranging from 16-48 gigabytes. As a result of the iterative exercise, all 10 experiments resulted in providing a unique optimal value of m in terms of the total mean square error of estimators and the in- sample forecast mean square error. In each experiment it was intended to find the total sum of all estimator mean square errors (E-MSE) of the trace estimates of the estimator mean square error matrix and validate it by the forecasting mean square error (F-MSE) of the one-step ahead forecast. F-MSE is also known as the mean squared forecast error. The Tables 3.2-3.11 below provide the results of the experiment. In terms of notation the Monte Carlo results tables below denote standard errors of estimates as SD (•), and mean square errors as MSE(•), where • depicts an estimator. The tables report the average of the estimated d, u and σ, denoted $\hat{d}, \hat{u}$ and $\hat{\sigma}$, computed across the 1,000 simulations, along with the standard error, e.g. SD($\hat{d}$) $= \sqrt{\sum_{r=1}^{R}(\hat{d}_r - \hat{d})^2/R}$, where $\hat{d}_r$ is the QMLE of d for the r-th replication and R denotes the number of replications, as well as the estimation mean square error, e.g. MSE($\hat{d}$) $= \sum_{r=1}^{R}(\hat{d}_r - d)^2/R$.

Note: Rarely the optimal lag order was between *5 lag differences* (eg: 29 between 25 and 30) in MA approximation (see Tables 3.2-3.6) and *2 lag differences* (eg: 10 between 9 and 11) in AR approximation (see Tables 3.7-3.11). Exact optimal approximation lag orders are in Tables 3.12-3.13.

From the simulation results of the 2 approximations given in Tables 3.2-3.11 for each length of the considered time series the smallest value of E-MSE is validated by the minimal value of F-MSE. It results in an optimal truncation point (Optimal m) introduced as an original concept that is independent of the series length n. Another interesting result

TABLE 3.2. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the MA approximation. The true generating process is $(1 - 2uB + B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0, \sigma^2)$, with $d = 0.1, u = 0.8, \sigma = 1$. The results are based on 1,000 Monte Carlo replications.

m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
$n = 100$									
20	0.132	0.064	0.005	0.612	0.351	0.158	0.990	0.144	0.021
25	0.132	0.064	0.005	0.605	0.359	0.166	0.989	0.143	0.020
30	0.132	0.064	0.005	0.582	0.389	0.199	0.989	0.144	0.020
35	0.133	0.064	0.005	0.588	0.367	0.179	0.990	0.144	0.021
40	0.132	0.064	0.005	0.575	0.382	0.196	0.990	0.144	0.021
$n = 200$									
20	0.117	0.046	0.002	0.705	0.270	0.081	0.997	0.097	0.009
25	0.118	0.046	0.002	0.691	0.280	0.090	0.997	0.097	0.009
30	0.118	0.046	0.002	0.677	0.298	0.103	0.997	0.097	0.009
35	0.119	0.046	0.002	0.673	0.277	0.093	0.997	0.097	0.009
40	0.119	0.046	0.002	0.677	0.283	0.095	0.997	0.097	0.009
$n = 500$									
20	0.105	0.029	0.0008	0.788	0.165	0.027	0.995	0.063	0.004
25	0.106	0.029	0.0009	0.777	0.179	0.032	0.995	0.063	0.004
30	0.107	0.029	0.0009	0.765	0.188	0.036	0.996	0.063	0.004
35	0.107	0.030	0.0009	0.764	0.185	0.035	0.996	0.063	0.004
40	0.107	0.031	0.0010	0.751	0.197	0.041	0.996	0.063	0.004
$n = 1000$									
20	0.101	0.019	0.0003	0.841	0.074	0.007	1.002	0.045	0.002
25	0.102	0.019	0.0003	0.831	0.084	0.008	1.002	0.045	0.002
30	0.102	0.019	0.0003	0.828	0.086	0.008	1.002	0.045	0.002
35	0.102	0.019	0.0003	0.828	0.085	0.008	1.002	0.045	0.002
40	0.103	0.019	0.0004	0.820	0.099	0.010	1.002	0.045	0.002
$n = 2000$									
20	0.100	0.013	0.0001	0.849	0.037	0.003	1.001	0.031	0.001
25	0.100	0.013	0.0001	0.846	0.043	0.004	1.001	0.032	0.001
30	0.100	0.013	0.0001	0.845	0.040	0.003	1.001	0.032	0.001
35	0.101	0.013	0.0001	0.843	0.045	0.004	1.001	0.031	0.001
40	0.101	0.013	0.0001	0.841	0.042	0.003	1.001	0.031	0.001

TABLE 3.3. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the MA approximation. The true generating process is $(1 - 2uB + B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0, \sigma^2)$, with $d = 0.2, u = 0.8, \sigma = 1$. The results are based on 1,000 Monte Carlo replications.

$n = 100$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.207	0.058	0.003	0.831	0.149	0.023	0.991	0.146	0.021
25	0.207	0.057	0.003	0.826	0.160	0.026	0.991	0.145	0.021
30	0.206	0.057	0.003	0.829	0.172	0.030	0.991	0.145	0.021
35	0.205	0.055	0.003	0.834	0.148	0.023	0.991	0.145	0.021
40	0.205	0.055	0.003	0.833	0.154	0.024	0.991	0.145	0.021

$n = 200$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.202	0.042	0.001	0.854	0.087	0.010	0.998	0.101	0.010
25	0.202	0.042	0.001	0.852	0.090	0.011	0.997	0.100	0.010
30	0.201	0.042	0.001	0.856	0.097	0.012	0.998	0.101	0.010
35	0.201	0.041	0.001	0.853	0.096	0.012	0.998	0.101	0.010
40	0.201	0.041	0.001	0.855	0.098	0.012	0.998	0.100	0.010

$n = 500$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.202	0.028	0.0007	0.859	0.038	0.005	1.005	0.065	0.004
25	0.202	0.028	0.0008	0.858	0.039	0.005	1.004	0.065	0.004
30	0.202	0.028	0.0007	0.858	0.041	0.005	1.004	0.065	0.004
35	0.202	0.028	0.0008	0.855	0.043	0.005	1.004	0.064	0.004
40	0.202	0.028	0.0007	0.859	0.049	0.006	1.004	0.064	0.004

$n = 1000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.203	0.019	0.0003	0.856	0.020	0.003	1.012	0.046	0.002
25	0.204	0.019	0.0003	0.852	0.019	0.003	1.011	0.046	0.002
30	0.204	0.019	0.0004	0.850	0.021	0.003	1.010	0.046	0.002
35	0.204	0.020	0.0004	0.849	0.025	0.003	1.010	0.046	0.002
40	0.204	0.019	0.0004	0.850	0.030	0.003	1.010	0.046	0.002

$n = 2000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.203	0.014	0.0002	0.855	0.012	0.003	1.011	0.033	0.001
25	0.204	0.014	0.0002	0.851	0.011	0.002	1.010	0.032	0.001
30	0.204	0.014	0.0002	0.848	0.011	0.002	1.010	0.032	0.001
35	0.205	0.014	0.0002	0.846	0.012	0.002	1.010	0.032	0.001
40	0.205	0.014	0.0002	0.844	0.012	0.002	1.010	0.032	0.001

TABLE 3.4. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the MA approximation. The true generating process is $(1 - 2uB + B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID N$(0, \sigma^2)$, with $d = 0.3, u = 0.8, \sigma = 1$. The results are based on 1,000 Monte Carlo replications.

			$n = 100$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.306	0.063	0.004	0.876	0.068	0.010	1.064	0.174	0.034
25	0.304	0.063	0.004	0.881	0.073	0.012	1.074	0.184	0.039
30	0.304	0.062	0.004	0.879	0.073	0.011	1.073	0.185	0.039
35	0.302	0.062	0.003	0.887	0.077	0.013	1.086	0.199	0.047
40	0.302	0.062	0.003	0.887	0.080	0.014	1.092	0.211	0.053
			$n = 200$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.307	0.046	0.002	0.873	0.050	0.007	1.068	0.131	0.022
25	0.307	0.046	0.002	0.873	0.054	0.008	1.072	0.138	0.024
30	0.305	0.046	0.002	0.877	0.058	0.009	1.081	0.148	0.028
35	0.303	0.046	0.002	0.886	0.066	0.011	1.098	0.159	0.035
40	0.302	0.047	0.002	0.885	0.066	0.011	1.097	0.165	0.036
			$n = 500$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.308	0.029	0.0009	0.857	0.030	0.004	1.058	0.085	0.010
25	0.309	0.030	0.0010	0.861	0.029	0.004	1.057	0.077	0.009
30	0.306	0.030	0.0009	0.860	0.040	0.005	1.064	0.095	0.013
35	0.308	0.031	0.0011	0.865	0.052	0.006	1.073	0.107	0.016
40	0.305	0.032	0.0011	0.860	0.047	0.005	1.069	0.106	0.016
			$n = 1000$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.312	0.022	0.0006	0.857	0.014	0.003	1.051	0.051	0.005
25	0.310	0.021	0.0005	0.853	0.011	0.003	1.049	0.051	0.005
30	0.309	0.021	0.0005	0.852	0.022	0.003	1.052	0.064	0.006
35	0.313	0.023	0.0007	0.852	0.032	0.003	1.055	0.075	0.008
40	0.307	0.024	0.0006	0.848	0.027	0.003	1.053	0.070	0.007
			$n = 2000$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.312	0.016	0.0004	0.857	0.008	0.003	1.052	0.036	0.004
25	0.311	0.015	0.0003	0.852	0.008	0.002	1.050	0.038	0.004
30	0.310	0.015	0.0003	0.849	0.009	0.002	1.049	0.040	0.004
35	0.314	0.016	0.0004	0.847	0.017	0.002	1.049	0.046	0.004
40	0.305	0.019	0.0004	0.844	0.011	0.002	1.048	0.043	0.004

TABLE 3.5. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the MA approximation. The true generating process is $(1-2uB+B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0,\sigma^2)$, with $d=0.4, u=0.8, \sigma=1$. The results are based on 1,000 Monte Carlo replications.

m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
				$n=100$					
20	0.419	0.061	0.004	0.873	0.053	0.008	1.087	0.208	0.051
25	0.417	0.063	0.004	0.884	0.064	0.011	1.094	0.211	0.053
30	0.416	0.062	0.004	0.876	0.060	0.009	1.083	0.197	0.046
35	0.415	0.063	0.004	0.880	0.065	0.010	1.085	0.195	0.045
40	0.416	0.063	0.004	0.883	0.068	0.011	1.084	0.194	0.045
				$n=200$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.421	0.049	0.002	0.867	0.041	0.006	1.089	0.149	0.030
25	0.424	0.050	0.003	0.869	0.050	0.007	1.090	0.151	0.031
30	0.422	0.051	0.003	0.866	0.049	0.006	1.082	0.139	0.026
35	0.418	0.051	0.003	0.878	0.064	0.010	1.100	0.165	0.037
40	0.422	0.052	0.003	0.875	0.062	0.009	1.091	0.150	0.030
				$n=500$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.422	0.033	0.001	0.860	0.022	0.004	1.092	0.108	0.020
25	0.425	0.034	0.001	0.857	0.029	0.004	1.086	0.100	0.017
30	0.425	0.036	0.001	0.852	0.026	0.003	1.080	0.094	0.015
35	0.419	0.035	0.001	0.867	0.056	0.007	1.102	0.137	0.029
40	0.426	0.035	0.002	0.855	0.043	0.005	1.084	0.104	0.018
				$n=1000$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.424	0.027	0.001	0.858	0.014	0.003	1.091	0.083	0.015
25	0.429	0.028	0.001	0.853	0.017	0.003	1.084	0.079	0.013
30	0.429	0.029	0.001	0.847	0.011	0.002	1.076	0.066	0.010
35	0.421	0.029	0.001	0.856	0.043	0.005	1.092	0.109	0.020
40	0.431	0.028	0.001	0.844	0.023	0.002	1.073	0.067	0.010
				$n=2000$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.426	0.022	0.001	0.857	0.007	0.003	1.092	0.059	0.012
25	0.431	0.024	0.001	0.851	0.010	0.002	1.084	0.056	0.010
30	0.432	0.023	0.001	0.846	0.006	0.002	1.078	0.052	0.008
35	0.422	0.024	0.001	0.852	0.036	0.004	1.090	0.090	0.016
40	0.433	0.023	0.001	0.841	0.012	0.001	1.072	0.050	0.007

TABLE 3.6. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the MA approximation. The true generating process is $(1 - 2uB + B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID N$(0, \sigma^2)$, with $d = 0.45, u = 0.8, \sigma = 1$. The results are based on 1,000 Monte Carlo replications.

$n = 100$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.459	0.049	0.002	0.874	0.050	0.008	1.113	0.217	0.060
25	0.458	0.051	0.002	0.882	0.061	0.010	1.116	0.217	0.060
30	0.457	0.050	0.002	0.878	0.060	0.009	1.109	0.209	0.055
35	0.457	0.051	0.002	0.875	0.061	0.009	1.104	0.210	0.055
40	0.457	0.051	0.002	0.882	0.066	0.011	1.109	0.211	0.056
$n = 200$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.463	0.039	0.001	0.870	0.039	0.006	1.125	0.186	0.050
25	0.464	0.040	0.001	0.875	0.052	0.008	1.124	0.181	0.048
30	0.463	0.040	0.001	0.867	0.047	0.006	1.109	0.156	0.036
35	0.462	0.041	0.001	0.873	0.058	0.008	1.124	0.189	0.051
40	0.463	0.040	0.001	0.875	0.061	0.009	1.123	0.180	0.047
$n = 500$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.470	0.028	0.001	0.863	0.026	0.004	1.123	0.136	0.034
25	0.472	0.028	0.001	0.865	0.040	0.005	1.120	0.132	0.032
30	0.473	0.028	0.001	0.854	0.027	0.003	1.102	0.107	0.021
35	0.470	0.030	0.001	0.864	0.051	0.006	1.125	0.159	0.041
40	0.473	0.028	0.001	0.861	0.050	0.006	1.115	0.134	0.031
$n = 1000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.474	0.022	0.001	0.860	0.018	0.004	1.123	0.105	0.026
25	0.477	0.021	0.001	0.858	0.029	0.004	1.117	0.102	0.024
30	0.478	0.021	0.001	0.849	0.013	0.002	1.100	0.076	0.016
35	0.477	0.022	0.001	0.858	0.045	0.005	1.122	0.139	0.034
40	0.478	0.021	0.001	0.851	0.038	0.004	1.107	0.103	0.022
$n = 2000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
20	0.479	0.018	0.001	0.858	0.009	0.003	1.127	0.074	0.021
25	0.482	0.017	0.001	0.855	0.022	0.003	1.120	0.085	0.021
30	0.483	0.017	0.001	0.847	0.008	0.002	1.106	0.062	0.015
35	0.482	0.017	0.001	0.851	0.035	0.003	1.118	0.113	0.026
40	0.484	0.017	0.001	0.843	0.021	0.002	1.101	0.070	0.015

TABLE 3.7. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the AR approximation. The true generating process is $(1-2uB+B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID N$(0, \sigma^2)$, with $d = 0.1, u = 0.8, \sigma = 1$. The results are based on 1,000 Monte Carlo replications.

m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
$n = 100$									
7	0.149	0.086	0.009	0.512	0.394	0.238	0.995	0.147	0.021
9	0.150	0.086	0.010	0.497	0.385	0.240	0.996	0.148	0.021
11	0.149	0.083	0.009	0.487	0.377	0.239	0.995	0.147	0.021
13	0.151	0.084	0.009	0.480	0.368	0.237	0.996	0.147	0.021
15	0.150	0.083	0.009	0.464	0.363	0.244	0.996	0.147	0.021
$n = 200$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.128	0.059	0.004	0.574	0.334	0.162	1.002	0.102	0.010
9	0.130	0.060	0.004	0.558	0.326	0.164	1.003	0.102	0.010
11	0.131	0.059	0.004	0.539	0.321	0.171	1.003	0.102	0.010
13	0.132	0.058	0.004	0.532	0.302	0.163	1.003	0.103	0.010
15	0.133	0.058	0.004	0.517	0.302	0.171	1.004	0.102	0.010
$n = 500$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.116	0.042	0.002	0.636	0.243	0.086	1.007	0.066	0.004
9	0.118	0.042	0.002	0.612	0.246	0.095	1.008	0.066	0.004
11	0.119	0.041	0.002	0.599	0.239	0.097	1.008	0.066	0.004
13	0.120	0.041	0.002	0.589	0.236	0.100	1.008	0.066	0.004
15	0.120	0.041	0.002	0.578	0.240	0.107	1.009	0.066	0.004
$n = 1000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.110	0.031	0.001	0.657	0.195	0.058	1.009	0.047	0.002
9	0.111	0.032	0.001	0.647	0.205	0.065	1.009	0.047	0.002
11	0.112	0.031	0.001	0.632	0.194	0.065	1.010	0.047	0.002
13	0.113	0.031	0.001	0.624	0.192	0.067	1.010	0.047	0.002
15	0.114	0.030	0.001	0.612	0.187	0.070	1.010	0.047	0.002
$n = 2000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.105	0.025	0.0006	0.678	0.165	0.042	1.011	0.033	0.001
9	0.106	0.025	0.0006	0.669	0.175	0.047	1.012	0.033	0.001
11	0.107	0.024	0.0006	0.655	0.165	0.048	1.012	0.033	0.001
13	0.107	0.024	0.0006	0.654	0.164	0.048	1.012	0.033	0.001
15	0.108	0.023	0.0006	0.644	0.159	0.049	1.013	0.033	0.001

TABLE 3.8. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the AR approximation. The true generating process is $(1-2uB+B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0,\sigma^2)$, with $d=0.2, u=0.8, \sigma=1$. The results are based on 1,000 Monte Carlo replications.

m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
				$n=100$					
7	0.234	0.092	0.009	0.637	0.271	0.100	1.028	0.152	0.024
9	0.233	0.092	0.009	0.637	0.268	0.098	1.030	0.153	0.024
11	0.231	0.088	0.008	0.636	0.262	0.095	1.031	0.152	0.024
13	0.231	0.087	0.008	0.630	0.257	0.094	1.031	0.153	0.024
15	0.231	0.087	0.008	0.623	0.258	0.098	1.031	0.152	0.024
				$n=200$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.217	0.069	0.005	0.661	0.223	0.068	1.038	0.107	0.013
9	0.219	0.069	0.005	0.649	0.231	0.075	1.040	0.107	0.013
11	0.218	0.068	0.005	0.649	0.223	0.072	1.040	0.107	0.013
13	0.220	0.066	0.004	0.635	0.218	0.074	1.041	0.107	0.013
15	0.220	0.064	0.004	0.632	0.214	0.074	1.041	0.106	0.013
				$n=500$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.207	0.049	0.002	0.684	0.167	0.041	1.044	0.067	0.006
9	0.206	0.049	0.002	0.682	0.176	0.045	1.045	0.067	0.006
11	0.209	0.048	0.002	0.667	0.166	0.045	1.047	0.067	0.006
13	0.210	0.048	0.002	0.662	0.170	0.047	1.047	0.068	0.006
15	0.210	0.047	0.002	0.659	0.166	0.047	1.047	0.068	0.006
				$n=1000$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.201	0.036	0.001	0.690	0.130	0.028	1.047	0.048	0.004
9	0.196	0.036	0.001	0.706	0.139	0.028	1.048	0.049	0.004
11	0.201	0.037	0.001	0.685	0.137	0.031	1.050	0.049	0.004
13	0.201	0.037	0.001	0.685	0.140	0.032	1.050	0.049	0.005
15	0.203	0.036	0.001	0.676	0.134	0.033	1.051	0.049	0.005
				$n=2000$					
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.198	0.027	0.0007	0.693	0.100	0.021	1.048	0.033	0.003
9	0.191	0.026	0.0007	0.717	0.107	0.018	1.049	0.033	0.003
11	0.197	0.028	0.0008	0.692	0.111	0.024	1.050	0.033	0.003
13	0.196	0.029	0.0008	0.696	0.114	0.023	1.051	0.034	0.003
15	0.198	0.028	0.0008	0.686	0.110	0.025	1.051	0.034	0.003

TABLE 3.9. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the AR approximation. The true generating process is $(1-2uB+B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID N$(0, \sigma^2)$, with $d = 0.3, u = 0.8, \sigma = 1$. The results are based on 1,000 Monte Carlo replications.

$n = 100$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.298	0.091	0.008	0.716	0.214	0.053	1.084	0.172	0.036
9	0.296	0.088	0.007	0.720	0.221	0.055	1.088	0.175	0.038
11	0.291	0.089	0.008	0.732	0.222	0.054	1.089	0.173	0.038
13	0.289	0.086	0.007	0.732	0.221	0.053	1.092	0.174	0.038
15	0.288	0.085	0.007	0.733	0.220	0.052	1.091	0.172	0.038
$n = 200$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.288	0.070	0.005	0.725	0.176	0.036	1.093	0.121	0.023
9	0.284	0.068	0.005	0.733	0.182	0.037	1.095	0.122	0.024
11	0.282	0.068	0.005	0.736	0.183	0.037	1.097	0.122	0.024
13	0.280	0.067	0.004	0.741	0.181	0.036	1.099	0.123	0.024
15	0.279	0.065	0.004	0.743	0.179	0.035	1.099	0.122	0.024
$n = 500$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.287	0.051	0.002	0.714	0.130	0.024	1.099	0.076	0.015
9	0.278	0.048	0.002	0.737	0.132	0.021	1.100	0.076	0.015
11	0.279	0.051	0.003	0.737	0.141	0.023	1.102	0.076	0.016
13	0.279	0.050	0.002	0.735	0.132	0.021	1.102	0.076	0.016
15	0.278	0.048	0.002	0.736	0.134	0.022	1.104	0.077	0.016
$n = 1000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.285	0.039	0.001	0.710	0.103	0.018	1.101	0.055	0.013
9	0.276	0.034	0.001	0.736	0.099	0.014	1.102	0.055	0.013
11	0.281	0.040	0.002	0.721	0.114	0.019	1.105	0.055	0.014
13	0.278	0.036	0.001	0.728	0.099	0.015	1.104	0.055	0.014
15	0.280	0.036	0.001	0.724	0.103	0.016	1.105	0.055	0.014
$n = 2000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.282	0.029	0.001	0.712	0.079	0.014	1.103	0.038	0.012
9	0.273	0.025	0.001	0.740	0.073	0.008	1.104	0.038	0.012
11	0.280	0.032	0.001	0.717	0.091	0.015	1.107	0.038	0.013
13	0.277	0.027	0.001	0.727	0.074	0.010	1.107	0.038	0.012
15	0.279	0.030	0.001	0.722	0.087	0.013	1.108	0.038	0.013

TABLE 3.10. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the AR approximation. The true generating process is $(1 - 2uB + B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0, \sigma^2)$, with $d = 0.4, u = 0.8, \sigma = 1$. The results are based on 1,000 Monte Carlo replications.

m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
$n = 100$									
7	0.363	0.091	0.009	0.762	0.192	0.038	1.164	0.181	0.059
9	0.364	0.085	0.008	0.766	0.190	0.037	1.167	0.183	0.061
11	0.361	0.084	0.008	0.772	0.189	0.036	1.172	0.185	0.064
13	0.356	0.083	0.008	0.782	0.190	0.036	1.177	0.190	0.067
15	0.351	0.082	0.009	0.792	0.188	0.035	1.178	0.189	0.067
$n = 200$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.363	0.077	0.007	0.753	0.163	0.028	1.175	0.130	0.047
9	0.358	0.070	0.006	0.769	0.156	0.025	1.176	0.129	0.047
11	0.354	0.072	0.007	0.779	0.160	0.026	1.182	0.130	0.050
13	0.347	0.069	0.007	0.794	0.162	0.026	1.188	0.134	0.053
15	0.340	0.071	0.008	0.809	0.162	0.026	1.190	0.133	0.054
$n = 500$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.365	0.058	0.004	0.740	0.120	0.018	1.178	0.081	0.038
9	0.356	0.050	0.004	0.765	0.108	0.013	1.180	0.081	0.039
11	0.355	0.056	0.005	0.768	0.123	0.016	1.184	0.082	0.040
13	0.351	0.055	0.005	0.780	0.124	0.015	1.187	0.084	0.042
15	0.348	0.058	0.006	0.788	0.132	0.017	1.191	0.085	0.043
$n = 1000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.363	0.040	0.003	0.735	0.089	0.012	1.179	0.059	0.035
9	0.355	0.035	0.003	0.762	0.079	0.007	1.181	0.059	0.036
11	0.356	0.044	0.003	0.759	0.102	0.012	1.185	0.059	0.038
13	0.352	0.041	0.004	0.769	0.096	0.010	1.187	0.061	0.038
15	0.350	0.046	0.004	0.776	0.107	0.012	1.189	0.060	0.039
$n = 2000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.359	0.028	0.002	0.740	0.067	0.008	1.178	0.040	0.033
9	0.357	0.025	0.002	0.755	0.056	0.005	1.182	0.040	0.034
11	0.357	0.035	0.003	0.751	0.082	0.009	1.185	0.041	0.036
13	0.358	0.028	0.002	0.753	0.066	0.006	1.185	0.041	0.036
15	0.357	0.034	0.003	0.756	0.081	0.008	1.187	0.041	0.036

TABLE 3.11. Sampling properties of the QMLE estimates of the parameters d and u of a Gaussian Gegenbauer process using the AR approximation. The true generating process is $(1-2uB+B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID $\mathrm{N}(0,\sigma^2)$, with $d=0.45, u=0.8, \sigma=1$. The results are based on 1,000 Monte Carlo replications.

$n=100$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.388	0.083	0.010	0.783	0.172	0.029	1.208	0.203	0.084
9	0.385	0.079	0.010	0.804	0.167	0.028	1.220	0.210	0.092
11	0.378	0.078	0.011	0.816	0.173	0.030	1.227	0.211	0.096
13	0.374	0.079	0.012	0.824	0.172	0.030	1.233	0.213	0.099
15	0.369	0.078	0.012	0.833	0.170	0.030	1.236	0.209	0.099
$n=200$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.388	0.072	0.008	0.778	0.145	0.021	1.216	0.142	0.067
9	0.385	0.067	0.008	0.797	0.134	0.018	1.222	0.148	0.071
11	0.377	0.069	0.010	0.814	0.143	0.020	1.232	0.150	0.076
13	0.370	0.068	0.011	0.831	0.144	0.021	1.241	0.157	0.082
15	0.365	0.068	0.011	0.841	0.144	0.022	1.244	0.155	0.083
$n=500$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.394	0.053	0.005	0.761	0.107	0.013	1.221	0.089	0.057
9	0.391	0.049	0.005	0.779	0.099	0.010	1.224	0.089	0.058
11	0.386	0.056	0.007	0.791	0.120	0.014	1.232	0.092	0.062
13	0.381	0.056	0.007	0.804	0.121	0.014	1.240	0.098	0.067
15	0.376	0.058	0.008	0.816	0.124	0.015	1.244	0.098	0.069
$n=1000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.394	0.039	0.004	0.757	0.084	0.008	1.221	0.065	0.053
9	0.394	0.036	0.004	0.767	0.072	0.006	1.225	0.064	0.055
11	0.390	0.045	0.005	0.775	0.099	0.010	1.232	0.066	0.058
13	0.388	0.045	0.005	0.783	0.100	0.010	1.236	0.072	0.061
15	0.384	0.049	0.006	0.792	0.108	0.011	1.240	0.072	0.063
$n=2000$									
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)
7	0.390	0.027	0.004	0.761	0.062	0.005	1.222	0.044	0.051
9	0.396	0.028	0.003	0.761	0.055	0.004	1.227	0.044	0.053
11	0.394	0.035	0.004	0.763	0.077	0.007	1.230	0.045	0.055
13	0.395	0.035	0.004	0.766	0.075	0.006	1.233	0.048	0.056
15	0.393	0.037	0.004	0.768	0.081	0.007	1.234	0.048	0.057

that stems from the MA approximation tabulations above is the fact that asymptotic variance of the long memory parameter estimate $\hat{d}$ of the GARMA series under consideration is approximately equal to $\frac{\pi^2}{24n}$, independent of n relating to the following theorem 3.1 due to Chan and Palma (1998). It could be assessed by calculating asymptotic variance for cases $n = 100, 200, 500, 1000, 2000$ and comparing with corresponding generated values.

Furthermore, the likelihood function is a monotonically increasing function of m, although the change of the function is tiny as m gets close to the optimal m reported in the table. The latter refers either to estimation of the long memory parameter, for which as m increases the typical bias-variance trade-off is observed, the bias decreasing and the variance increasing with m, or to the out-of-sample performance of the pseudo-true predictor arising from the MA approximation. The simulation experiment shows that the optimal m (i.e. minimising the mean square estimation and prediction error for the purpose of estimating d and prediction) is rather insensitive to the lag order of MA approximation. The estimation standard error should not depend on d in large samples. As a matter of fact, for the MA approximation, the standard error does not vary relevantly and the variation is due to the Monte Carlo simulation error. The AR estimator is much more unreliable and unstable, and this justifies the variation observed: as d increases the reliability of the AR approximation decreases. In terms of asymptotic properties of an approximate MLE relating to the GARMA (0,d,0) model considered above, Chan and Palma (1998) show results as theorems 3.1-3.3 for a specific member of the GARMA (p,d,q) family. Due to their relevance, these results are established in theorem 3.1 below.

Theorem 3.1: Let n be the length of a GARMA(0,d,0) series, m the optimal lag order, $k \in \mathcal{R}^+$ and $m = n^k$. If $\hat{\Theta}_{KF}$ is the estimator due to the Kalman Filter, then under the assumptions that $0 < d < 1/2$, and $\epsilon_t \sim N(0, \sigma^2)$, the following results hold:

(a) $Consistency$: $\hat{\Theta}_{KF} \xrightarrow{p} \Theta_0$ as $n \to \infty$, with $k > 0$, where $\Theta_0 = (d, u, \sigma)' = (\Theta_{01}, \Theta_{02}, \Theta_{03})'$ is the true vector of parameters.

(b) $Asymptotic\ Normality$: $\sqrt{n}(\hat{\Theta}_{KF} - \Theta_0) \xrightarrow{d} N(0, \Sigma(\Theta_0))$, as $n \to \infty$ with $k > 1/2$,

where $\Sigma^{-1}(\Theta_0) = (\Sigma_{ij}^{-1}(\Theta_0))$ such that

$\Sigma_{ij}^{-1}(\Theta_0) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [\frac{\partial log f_X(\omega)}{\partial \Theta_{0i}}][\frac{\partial log f_X(\omega)}{\partial \Theta_{0j}}] d\omega$ with $f_X(\omega)$ defined as per equation (2.2),

(c) Efficiency: $\hat{\Theta}_{KF}$ is an efficient estimator for $k > 1/2$.

Proof: A detailed proof of theorem 3.1 can be found in Chan and Palma (1998) and is applicable to the truncated GARMA(0,d,0) process considered in section 3.5.

The evidence reported in this section is the outcome of a larger experiment where the parameter u varied in the admissible range, $u \in (-1, 1)$ and the GARMA(1,d,0) process $(1 - \phi B)(1 - 2uB + B^2)^d X_t = \epsilon_t, \epsilon_t \sim$ IID N$(0, 1)$ was also considered in the data generating process, with $\phi \in (-1, 1)$. In the second case ϕ is estimated assuming knowledge of the true model, i.e. the MA approximation is formed from truncating the Wold polynomial $C_j(B) = (1 - \phi B)^{-1}(1 - 2uB + B^2)^{-d}$, at truncation lag m, with the coefficients C_j computed by convoluting the lag polynomials $(1 - \phi B)^{-1}$ and $(1 - 2uB + B^2)^{-d}$. The main results concerning the optimal choice of m for the estimation of the memory parameter d were rather insensitive to both the true value of u and ϕ. So for clarity and synthesis only the results referring to the GARMA(0,d,0) process with u fixed at 0.8 are reported. The size of the Monte Carlo errors affecting the results was checked and it was decided that 1,000 replications struck a very good balance between the experiment reliability and its computational complexity.

3.6. Comparative Assessment of Approximations

MA and AR approximations of a truncated GARMA process utilized by employing the KF on simulated data yields QMLE estimates as well as one and multi step ahead forecasts of model parameters. Rapid convergence towards optimal likelihood value with a truncated state space employing the KF defined by the Gaussian likelihood equation and mean square errors (MSE's) of estimators will provide a benchmark assessment to choose a better approximation option. In assessing the two approximations, lag truncation values were set at $m = [5, 10, 15, 20, 25, 30, 35, 40, 45]$ for MA and $m = [4, 7, 9, 11, 13, 15, 17]$ for AR approximations. It was noted that the AR approximation did converge faster towards the optimal likelihood value with an approximate difference of 20 lags. However, the MA approximation provided much smaller MSE values for the estimators. Figure 3.5 (shown later) provides a visual illustration for the case $d = 0.4$, $u = 0.8$, with Replications = 1000. Furthermore, due to the easy implementation of KF recursions and simplicity of the analysis of theoretical properties of the MLE's (as mentioned in Chan and Palma (2006) and Palma (2007) for long memory time series), the MA representation became the approximation of choice over its competing rival. Another reason for choosing the MA approximation was the advent of a smaller error variance than for the AR approximation in terms of a differenced long memory Gegenbauer time series. By summarizing the results in Tables 3.2-3.11 above for both the approximations, the optimal value of the truncation point for each length of the same series is shown in Tables 3.12-3.13.

TABLE 3.12. Optimal values of m with $u = 0.8$ using MA approximation and 1000 replications

n	d=0.1	d=0.2	d=0.3	d=0.4	d=0.45
100	35	35	30	29	35
200	35	35	33	30	30
500	35	30	35	30	30
1000	35	30	25	30	30
2000	30	30	30	30	30

Note: MA approximation - Optimal lag order interval: $[29, 35]$.

TABLE 3.13. Optimal values of m with $u = 0.8$ using AR approximation and 1000 replications

n	d=0.1	d=0.2	d=0.3	d=0.4	d=0.45
100	13	13	12	9	9
200	13	11	9	9	9
500	13	10	9	9	9
1000	11	10	9	13	9
2000	12	13	9	13	9

Note: AR approximation - Optimal lag order interval: $[9, 13]$.

Remark: It is evident that the approximations presented in this section are appropriate for a Wold type linear series driven by Gaussian white noise that could be fitted with a long memory Gegenbauer process depicting hyperbolically decaying autocorrelation / partial autocorrelation functions, and an unbounded spectral density away from the origin. The implementation efficiency in terms of processing time is achieved up to approximately $m = 70$ lags.

Optimal truncation lag order m and total parameter estimator mean square errors for approximations as a function of the sample size n, resulting from 1000 Monte Carlo samples from a Gaussian Gegenbauer process with $d = 0.4$ and $u = 0.8$.

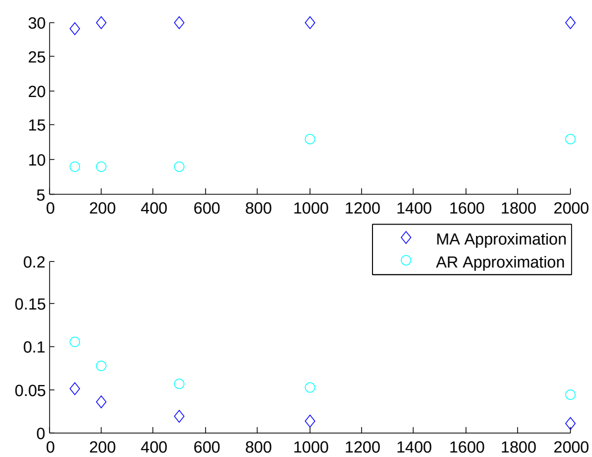


FIGURE 3.5. Comparison of Approximations - The top graph depicts optimal truncation lag orders, while the bottom graph depicts total parameter estimator mean square errors with respect to the sample size n

.

Note: Taking into consideration the overall attributes of the two approximations from sections 3.5 and 3.6, the better performing MA approximation became the chosen option. Therefore the majority of the results presented below are based on the MA approximation. Further results based on the AR approximation will be clearly labelled adjacent to the presented information. As confirmatory examples of both the approximations, results of the case with $d = 0.4$, $u = 0.8$ and $Replications = 1000$ for varying lengths of n are provided in Tables 3.14-3.15 below.

TABLE 3.14. MA Approximation, $d = 0.4, u = 0.8$, Replications = 1000

m	n=100/F-MSE	n=200/F-MSE	n=500/F-MSE	n=1000/F-MSE	n=2000/F-MSE
20	1.2487	1.2062	1.0424	1.0479	1.1374
25	1.2497	1.1913	1.0421	1.0453	1.1319
30	1.2482	1.1909	1.0370	1.0416	1.1274
35	1.2395	1.1952	1.0395	1.0453	1.1293
40	1.2505	1.1868	1.0383	1.0443	1.1234

TABLE 3.15. AR Approximation, $d = 0.4, u = 0.8$, Replications = 1000

m	n=100/F-MSE	n=200/F-MSE	n=500/F-MSE	n=1000/F-MSE	n=2000/F-MSE
7	1.1726	1.1438	1.1582	1.1769	1.1810
9	1.1435	1.0997	1.1529	1.1764	1.1869
11	1.1801	1.1372	1.1547	1.1751	1.1878
13	1.1792	1.1207	1.1515	1.1736	1.1854
15	1.1912	1.0999	1.1548	1.1743	1.1872

By investigating the results of Tables 3.12 and 3.14 for the MA approximation, it is clear that the optimal truncation point for the long memory Gegenbauer series under consideration will lie in the interval [29,35] for any value of n. In addition to the experimental results given above for comparison purposes, several other Monte Carlo experiments with MA approximation were conducted to estimate the long memory and Gegenbauer frequency parameters of the desired GARMA model. This was executed using combinations of the sets of values $d = [0.1, 0.2, 0.3, 0.4, 0.45]$, $m = [6, 14]$, $n = [1000, 5000, 10000]$

and $Replications = [100, 200, 500, 1000]$. Some results are provided in table 3.16 for series length $n = 10000$. In one such experiment the estimated value and the standard error for the long memory model parameter $d = 0.4$ of a series with $n = 1000$ after 100 replications was far superior to a similar result from a 2 factor GARMA model with same length and 1000 replications presented in Bisaglia et.al.(2003). Furthermore, estimates comparable with the results given by Chan and Palma (1998) for the ARFIMA model were obtained through additional Monte Carlo experiments. All the simulation estimates were comparable with similar GARMA model parameter estimates shown in Gray et.al.(1989), Chung (1996) and Beaumont and Ramachandran (2001). As the lags increase in the proposed model, until the optimal order is met the typical bias/variance tradeoff is observed minimizing the prediction error. Through the employment of a rolling window in the forecast the prediction error variance is further minimized. Yet due to the effect of the Monte Carlo simulation error, the prediction error variance may still increase marginally prior to and beyond $n = 2000$ and subsequently decrease, taper and converge at a higher time point.

TABLE 3.16. MA Approximation - GARMA QMLE's with 1000 replications.

sample size $(n) = 10000$, *lag order*$(m) = 6$, $\sigma = 1$							
d	$\hat{d}$	SD($\hat{d}$)	u	$\hat{u}$	SD($\hat{u}$)	$\hat{\sigma}$	SD($\hat{\sigma}$)
0.1	0.0999	0.0071	0.8	0.8037	0.0339	1.0001	0.0137
0.2	0.1998	0.0070	0.8	0.8051	0.0169	1.0025	0.0138
0.3	0.3005	0.0069	0.8	0.8047	0.0107	1.0084	0.0148
0.4	0.4032	0.0070	0.8	0.8010	0.0076	1.0195	0.0152
0.45	0.4555	0.0070	0.8	0.7977	0.0064	1.0281	0.0155

The histogram, frequency polygon and kernel plots of the probability density function (pdf) based on the much more feasible MA approximation for estimators $\hat{d}$, $\hat{u}$ and $\hat{\sigma}$, when $d = 0.45$, $u = 0.8$ and $\sigma = 1$ with length $n = 10000$ were created. They were based on the Monte Carlo experiment result of table 3.16 with 1000 iterations and are shown in figures 3.6-3.8. They depict that the sampling distributions of the estimators are approximately normal in accordance with the asymptotic normality theory of theorem 3.1(b) as per the central limit theorem.

FIGURE 3.6. Histogram, Frequency Polygon and Probability density function kernel plot of $\hat{d}$ for a series length $n = 10000$, resulting from 1000 Monte Carlo samples of a Gaussian Gegenbauer process with $d = 0.45$, $u = 0.8$ and $\sigma = 1$.

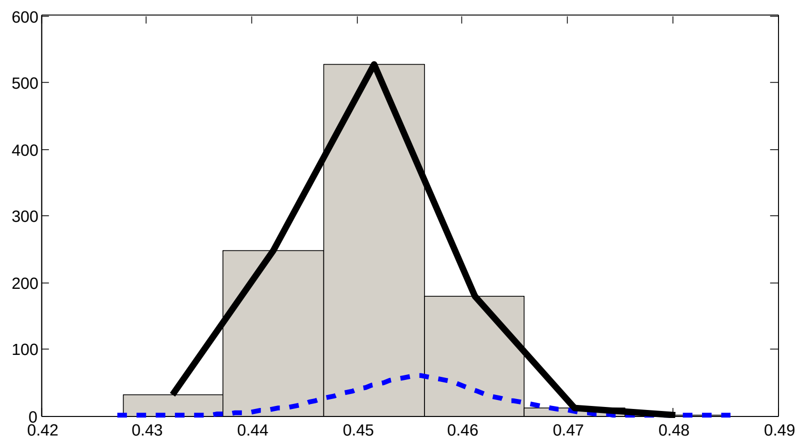


FIGURE 3.7. Histogram, Frequency Polygon and Probability density function kernel plot of $\hat{u}$ for a series length $n = 10000$, resulting from 1000 Monte Carlo samples of a Gaussian Gegenbauer process with $d = 0.45$, $u = 0.8$ and $\sigma = 1$.

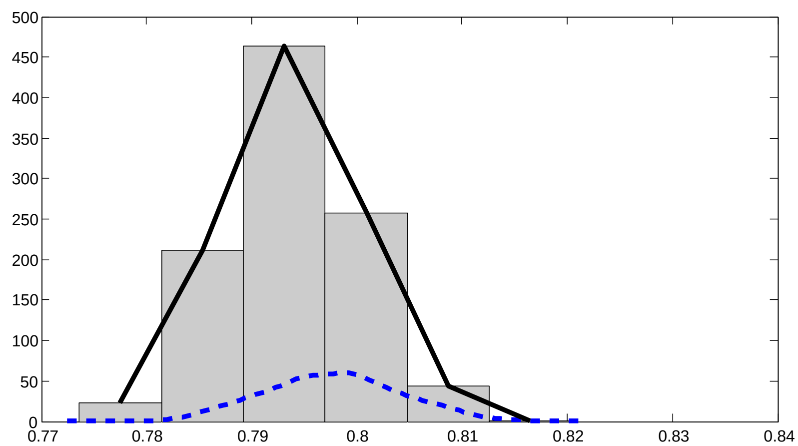
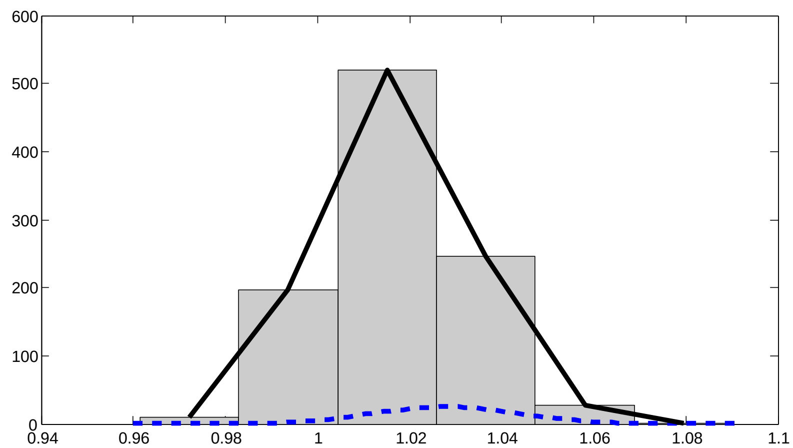


FIGURE 3.8. Histogram, Frequency Polygon and Probability density function kernel plot of $\hat{\sigma}$ for a series length $n = 10000$, resulting from 1000 Monte Carlo samples of a Gaussian Gegenbauer process with $d = 0.45$, $u = 0.8$ and $\sigma = 1$.



Since the truncated state space approach with the MA approximation has been identified and assessed as a feasible procedure creating robust estimators validated through an optimal lag truncation value, it was incorporated into two segments of a real data application consisting of sunspots illustrated by Wolfer and Tong. The sunspots were chosen as the desired empirical data source, since they had been utilized for applications of various GARMA models in the past based on the work of Gray et al.(1989), Chung (1996) and Beaumont and Ramachandran (2001). The analytical results of the empirical applications are the focal point of the next section.

3.7. Results of Empirical Applications

The truncated state space approach incorporating the MA approximation after being classified as a viable and feasible technique in estimating and forecasting with respect to Gegenbauer long memory time series models was tested with a real application from the popular versions of sunspot time series. In this section, for comparative purposes in certain instances the AR approximation results are included as well. The mean square error (MSE) and the mean absolute error (MAE) were used as benchmark measures in the comparison tables given below, where

$$MSE = (1/N)\sum_{i=1}^{N} e_i^2, \tag{3.9}$$

$$MAE = (1/N)\sum_{i=1}^{N} |e_i|, \tag{3.10}$$

with e_i defined as the forecast error based on the difference between actual and predicted series values, and N the length of the series of forecast errors. A long memory Gegenbauer model is fitted to the Wolfer's series with length 176 [Source: Waldmeier(1961)] and another series with length 289 [Source: Tong (1990)] comprising of sunspots data. The results are given in tables 3.17-3.19 below. Time series plots of the two sunspots series are shown in Figure 3.9.

FIGURE 3.9. Sunspots Realizations.

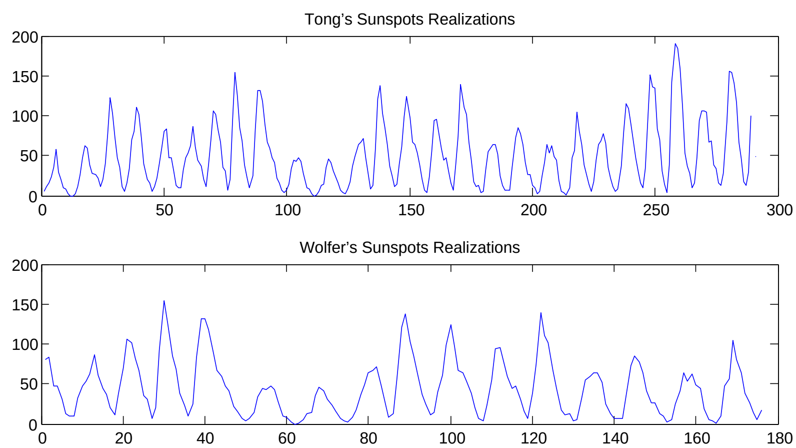


TABLE 3.17. Estimation Results for Wolfer's / Tong's Sunspot series (Standard Errors are given within brackets)

Series	$\hat{d}$(MA)	$\hat{u}$(MA)	$\hat{d}$(AR)	$\hat{u}$(AR)
Wolfer's Sunspots	0.49 (0.002)	0.85 (0.003)	0.49 (0.002)	0.89 (0.006)
Tong's Sunspots	0.49 (0.002)	0.84 (0.002)	0.49 (0.002)	0.89 (0.005)

Note: The estimated values $\hat{d}$ for both the sunspot series were less than 0.5 with $\hat{u} < 1$ up to 4 decimal places illustrating stationarity. Under the AR approximation both sunspot series correspond to periods of around 13.2 observations. But the MA approximation delivers periods of 11.3 and 10.9 observations respectively for Wolfer's and Tong's sunspot series. Therefore Tong's series with a larger length provides a better estimate under the MA approximation that is closer to the approximate theoretical sunspot cycle periodicity of 10 observations. Yet again the MA approximation outperforms its AR rival through empirical evidence.

In order to make a forecast assessment of Wolfer's sunspot data series a rolling window of 100 data points was utilized with 8 step ahead forecast predictions at each iteration to achieve predicted values for the entire series. It yielded mean square forecast errors that were better up to the five step ahead forecast than the values given in Morris (1977) for the same series using a similar forecasting technique. In Table 3.18 the comparative results of the MSE values beginning at the third step ahead forecast beyond the minimum year of a sunspot cycle is recorded, since Morris (1977) begins the assessment at the same evaluation point. To further illustrate the forecasting efficiency of the method proposed in this paper, a comparative evaluation was made with respect to the results provided in Zhang (2003) and Bijari and Khashei (2011) with respect to Wolfer's sunspots by utilizing the same data and are shown in Table 3.19.

TABLE 3.18. Comparative assessment of forecast errors for Wolfer's sunspot series with Morris's model

Year after minimum	3	4	5
MSE for Morris's AR model	580.5	537.9	345.5
MSE for proposed model	117.1	174.1	268.6

TABLE 3.19. Comparative assessment of forecast errors for Wolfer's sunspot series with optimal Autoregressive integrated moving average (ARIMA) models

Model	67 time points ahead - MAE
AR(9)-ARIMA	13.033
Zhang's hybrid ARIMA	12.780
Bijari-Khashei hybrid ARIMA	11.446
Proposed model (MA Approximation) at $m = 30$	12.877

From the results of table 3.18 it is clear that the forecast model presented in this paper performs better than the AR model introduced in Morris (1977) in terms of the forecast mean square errors ranging from 3 to 5 step ahead predictions. In his paper it is acknowledged that the utilized traditional AR model up to 30 lags does not yield satisfactory forecast MSE's. He compensates for it by mixing it with an outburst regression model through a weighting mechanism. Also he does not mention an optimal lag truncation value. In the proposed model as the lags increase until an optimal order of $m = 30$ the typical bias-variance tradeoff is observed. Thereby, it minimizes the prediction error to a great extent resulting in MSE's significantly smaller than corresponding MSE's of Morris's AR model. Furthermore, from the comparison results of table 3.19 it is apparent that the proposed model performs better than the traditional AR(9)-ARIMA time series model given in the literature of Zhang (2003) and Bijari and Khashei (2011) with a lag order of $m = 30$ further corroborating the optimal lag interval of $[29, 35]$ established by the preceding simulations. Furthermore, in a forecasting sense the Gegenbauer model performs as efficiently as the Zhang (2003) and Bijari and Khashei (2011) hybrid models without the propelling of any neural networks. It provides the implication that a hybrid Gegenbauer long memory model with neural networks may outperform the recently established modern hybrid models. Therefore it will clearly match or outperform all the three ARIMA models given in Table 3.19 with respect to the 67 time points ahead prediction (67 has been used, since it had been used as the benchmark in Zhang (2003) and Bijari and Khashei (2011) for comparison) within the proposed optimal lag truncation range of $[29, 35]$ at $m = 30$ illustrating and corroborating the long memory property. The investigative results arrived at in the preceding sections are summarized and become the conclusion in the final section.

3.8. Conclusion

A truncated state space model entailing a KF was utilized to derive an efficient estimation/prediction framework for a long memory GARMA time series. It was used to generate QMLE estimates of the long memory parameter, noise and the Gegenbauer frequency index coefficient of various simulated series and applications belonging to the GARMA (0,d,0) process family. Within the same conceptual and methodological environment, multi-step ahead forecasts were applied as diagnostic, prediction and validating measures.

This exercise provides a truncated GARMA state space framework, an alternative estimation approach, a comparison of AR and MA approximation techniques in estimating parameters of a GARMA process, and most importantly the introduction of an optimal lag truncation value (m) for estimation and forecasting as original contributions. Furthermore, the model presented in this chapter outperforms traditional time series forecasting models and closely matches extremely advanced modern hybrid models with neural networks. The utilization of the given conceptual paradigms could be extended to a model of the same family driven by a different error distribution.

In such a context based on the thesis outline, Chapter 4 begins next titled "Modelling of persistence and conditional variance".

CHAPTER 4

Modelling of Persistence and Conditional Variance

Heteroskedasticity in long memory processes have attracted the focus of many economists and statisticians due to its close link with financial markets. In extending the work of Engle (1982) and Bollerslev (1986), the contributions of Ling and Li (1997), Peiris et.al. (2005), Peiris and Thavaneswaran (2007) and Palma (2007) were considered in developing the results of this chapter.

Extending the work of Gray et.al. (1989), Ling and Li (1997), Peiris et.al.(2005) and Peiris and Thavaneswaran (2007), Dissanayake and Peiris (2011) introduced a class of generalized fractionally differenced Gegenbauer processes with conditionally heteroskedastic errors.

This chapter considers the modelling of GARMA processes driven by GARCH errors incorporating the state space methodology introduced in chapters 1 and 3 of this thesis. A truncated state space representation and the KF in estimating the long memory and other parameters will be considered. It further investigates the significance of AR and MA approximations following Chan and Palma (1998) and Dissanayake et.al. (2014a).

Structure of this chapter is organized as follows: Section 4.1 introduces notation and preliminaries and then section 4.2 develops the related KF theory and fundamental concepts in computing QMLE's. Section 4.3 presents Monte Carlo Simulation evidence to assess the proposed methodology. Section 4.4 considers applications utilizing the chosen approximate QMLE procedure to highlight the importance of the approach. Concluding remarks are provided in Section 4.5.

Notation and preliminaries will be provided next to begin the discussion of the chapter.

4.1. Notation and Preliminaries

4.1.1. The Stationary Case: Consider a GARMA(0,d,0)-GARCH(r,s) process defined as:

$$(1 - 2uB + B^2)^d X_t = \varepsilon_t, \tag{4.1}$$

where $|u| < 1$ and $0 < d < 1/2$, $\varepsilon_t | F_{t-1} \sim N(0, h_t)$ (F_{t-1} is the history of the process), the conditional variance h_t satisfies the GARCH(r,s) process
such that,

$$\varepsilon_t = \epsilon_t \sqrt{h}_t, \quad \epsilon_t \sim NID(0, 1), \tag{4.2}$$

and

$$h_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{s} \beta_j h_{t-j}, \tag{4.3}$$

where $\alpha_0 > 0$, all $\alpha_j \geq 0$, all $\beta_j \geq 0$.

The theoretical (true) acf of the process (4.1) could be found as a closed form solution utilizing Gegenbauer coefficients through the expression

$$E(X_k X_{k+t}) = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} C_j C_i [E(\varepsilon_{t-j})(\varepsilon_{k+t-i})]. \tag{4.4}$$

Simulation graphic grids depicting 400 length realizations (top left), sample acf (top right), true acf (middle left), pacf (middle right) and sdf (bottom) for examples of the introduced class with differing sets of statistical parameters are shown in figures 4.1-4.4. They were developed in R by extending the ideas of Shumway and Stoffer (2010) to provide a visual illustration of properties such as volatility, long memory and conditional heteroskedasticity.

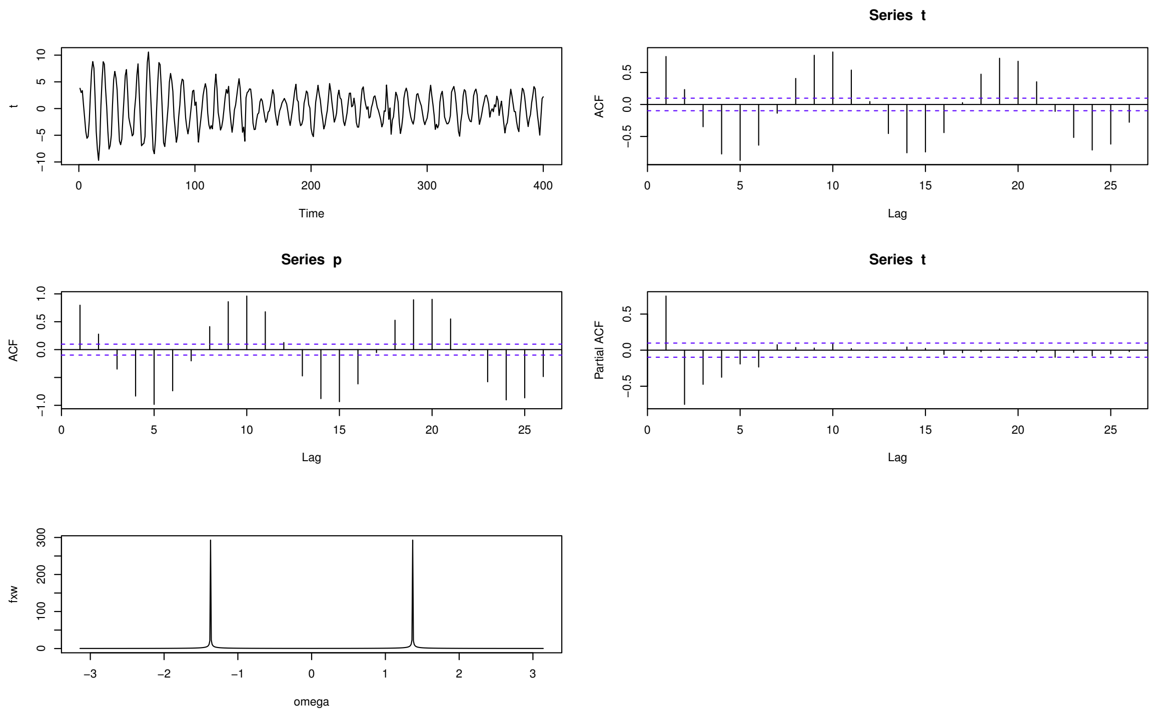


FIGURE 4.1. $u = 0.8$, $d = 0.45$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

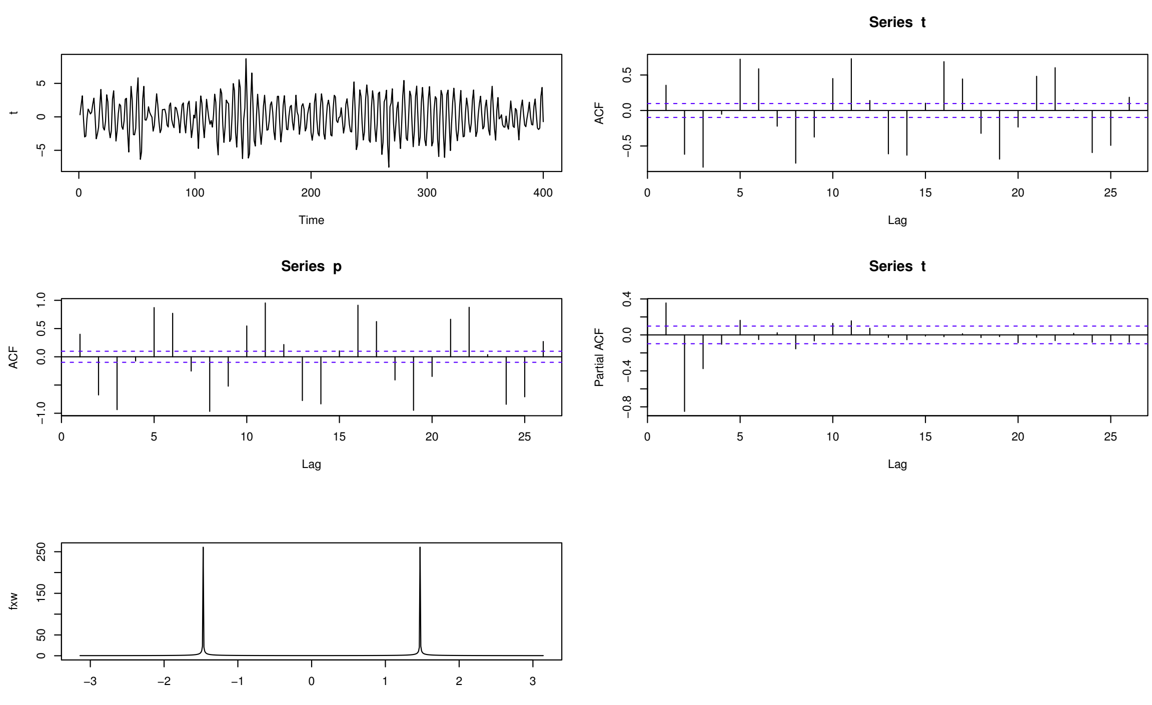


FIGURE 4.2. $u = 0.4$, $d = 0.45$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

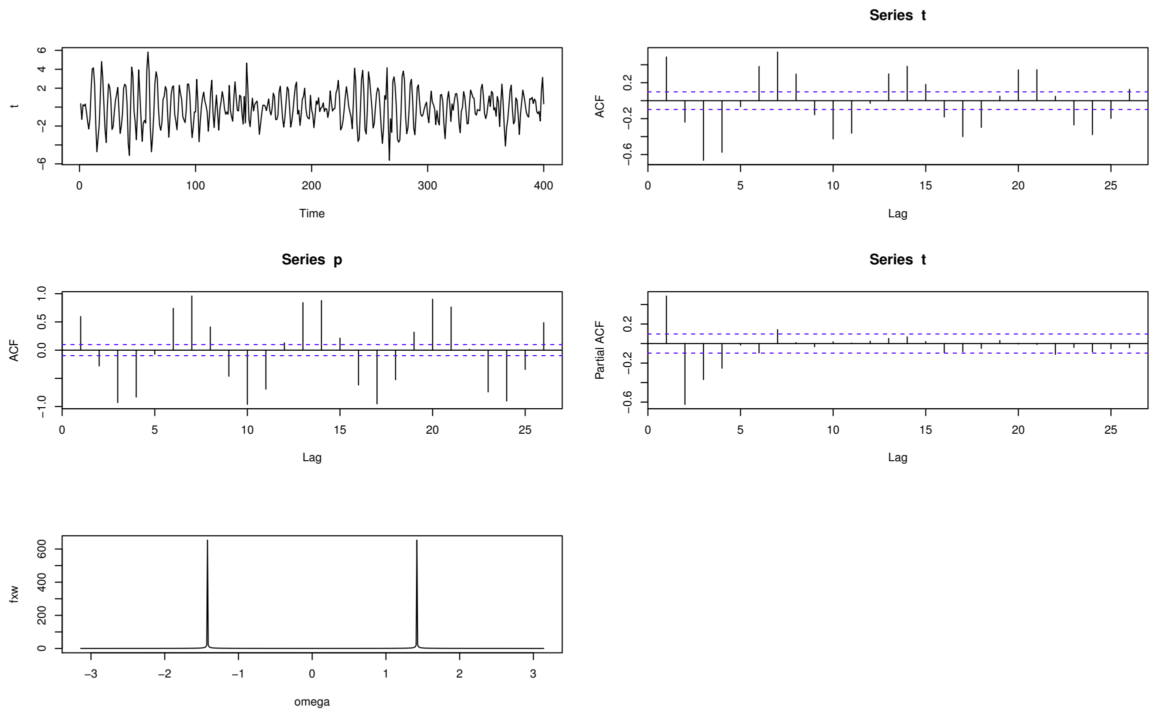


FIGURE 4.3. $u = 0.6$, $d = 0.45$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

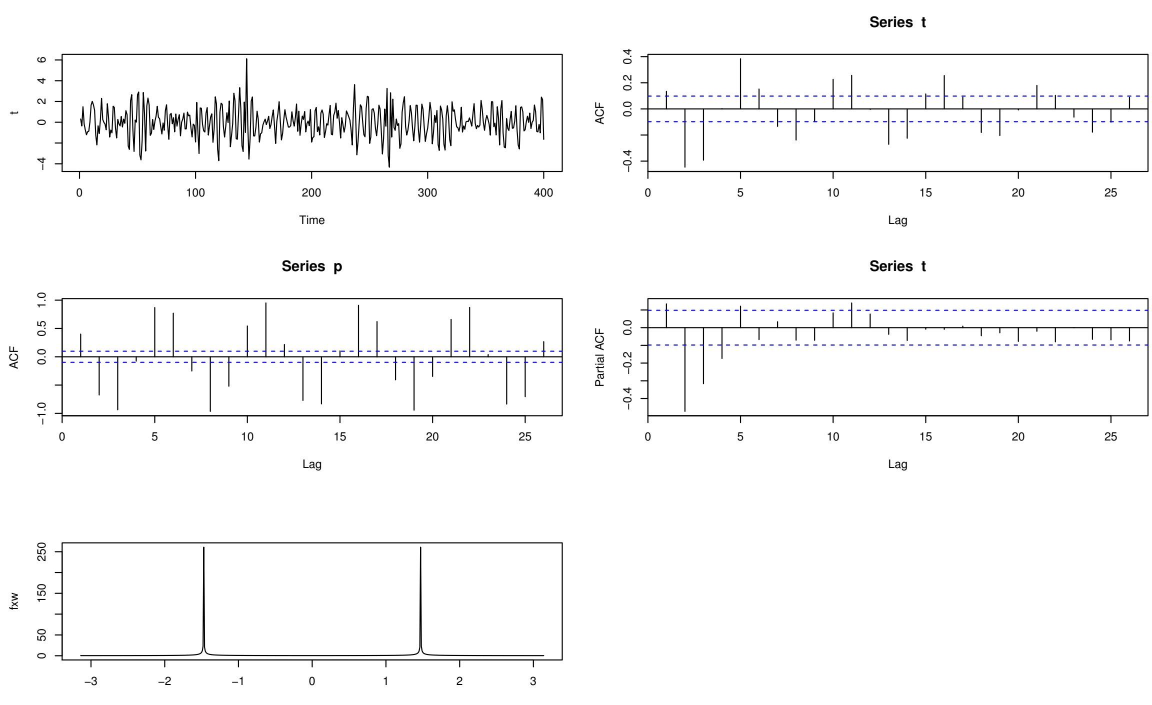


FIGURE 4.4. $u = 0.4$, $d = 0.10$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$

By inspecting figures 4.1-4.4 it is evident that these plots show a very high degree of volatility and conditional heteroskedasticity. By inspecting the sample and theoretical autocorrelations it is clear that they are similar. Furthermore, it is clear that all the partial autocorrelation functions above decay slowly at a hyperbolic rate characterizing long memory. It is corroborated by the spectral density function with an unbounded high peak away from the origin symbolizing long memory.

The following theorem provides some basic properties of the process introduced in (4.1)-(4.3).

Theorem 4.1

Suppose $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$, $\sum_{i=1}^{r} \alpha_i + \sum_{j=1}^{s} \beta_j < 1$, $|u| = 1$ and $d < 1/4$. Then for the process (4.1)-(4.3), there exists a F_t-measurable second-order stationary solution $\{\varepsilon_t, X_t\}$ and it is the only second-order stationary solution given the ϵ_t's. The solution $\{\varepsilon_t, X_t\}$ has the following causal representations:

$$\varepsilon_t = \epsilon_t \{\alpha_0 + \sum_{j=1}^{\infty} \delta^T (\prod_{i=1}^{j} A_{t-i}) \mathcal{E}_{t-j}\}^{1/2} \quad a.s., \tag{4.5}$$

and

$$X_t = \sum_{j=0}^{[M/2]} C_j \varepsilon_{t-j} \quad a.s., \tag{4.6}$$

where

- $\mathcal{E}_t = (\alpha_0 \epsilon_t^2, 0, ...0, \alpha_0, 0, ..., 0)^T_{(r+s)\times 1}$,
- the first component is $\alpha_0 \epsilon_t^2$ and the (r+1)th component is α_0, $\{\epsilon_t\}$ are independently normally distributed with mean 0 and variance 1, $\delta = (\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s)^T$, and

$$A_t = \left[\begin{array}{ccc|ccc} \alpha_1\epsilon_t^2 & \cdots & \alpha_r\epsilon_t^2 & \beta_1\epsilon_t^2 & \cdots & \beta_s\epsilon_t^2 \\ & I_{(r-1)\times(r-1)} & O_{(r-1)\times 1} & & O_{(r-1)\times s} & \\ \hline \alpha_1 & \cdots & \alpha_r & \beta_1 & \cdots & \beta_s \\ & O_{(s-1)\times r} & & I_{(s-1)\times(s-1)} & O_{(s-1)\times 1} & \end{array}\right],$$

where

- $\mathbf{I_{r\times r}}$ is the $r \times r$ identity matrix.

Proof: For the proof of Theorem 4.1 reference could be made to the proof of Theorem 2.1 given in the appendix of Ling and Li (1997) to a similar specific member that is applicable to the generalized class presented in this paper.

Lemma 4.1 $\{\varepsilon_t\}$ and $\{X_t\}$ in (4.1) are stationary.

The next theorem shows some additional properties of the GARMA(0,d,0)-GARCH(r,s) model introduced in (4.1)-(4.3).

Theorem 4.2

Let $\{X_t\}$ be generated by (4.1)-(4.3). Assume roots of associated autoregressive and moving average operators lie outside the unit circle, and $\sum_{i=1}^{r}\alpha_i + \sum_{j=1}^{s}\beta_j < 1$:

(a.) If $d < 1/2$, then $\{X_t\}$ is second-order stationary with the generic MA approximation representation:

$$X_t = \sum_{j=0}^{[M/2]} C_j \varepsilon_{t-j}, \tag{4.7}$$

where ε_t has representation (4.5). Hence $\{X_t\}$ is stationary.

(b.) If $d > -1/2$, then $\{X_t\}$ is invertible; that is ε_t can be written as the generic AR approximation representation:

$$\varepsilon_t = \sum_{j=0}^{[M/2]} \pi_j X_{t-j}, \tag{4.8}$$

where $\pi(B) = (1 - 2uB + B^2)^d$.

Proof: For part (a.) Let $\eta_1(Z) = (1 - 2uZ + Z^2)^{-d}$. The closed form solution of the recursive definition of $(1 - 2uZ + Z^2)^{-d}$ converge for $|Z| \leq 1$. Hence $\{X_t\}$ exists with representation

(4.7). Refer Theorem (4.1) and proof of Theorem (2.2) of Ling and Li (1997), that establishes $\{\varepsilon_t\}$ is second-order stationary. By utilizing the same results of Ling and Li (1997) it could be shown that $\{X_t\}$ is also second-order stationary. From representation (4.5), ε_t is a measurable function of iid random variable Z_t's and hence so is $\{X_t\}$. Therefore $\{X_t\}$ is stationary. For part (b.) Let $\eta_2(Z) = (1 - 2uZ + Z^2)^d$. Similar to η_1, the function η_2 converges, when $|Z| \leq 1$ and (4.8) holds true.

Note: On certain occasions the non-stationary version of the model (4.1) becomes useful from a meta-analytical perspective and becomes the next discussion topic.

4.1.2. The Non-stationary Case: The model in (4.1) could be reduced to a non-stationary version of a GARMA(0,d,0)-GARCH(r,s) series defined by:

$$(1 - 2uB + B)^{d_1}(1 - 2uB + B)^m X_t = \varepsilon_t, \tag{4.9}$$

where $|u| < 1$, $|d_1| < 1/2$, $m \in \mathcal{Z}^+$, ε_t represents GARCH errors.

4.1.3. MA and AR approximations: The Wold representation of the GARMA(0,d,0)-GARCH(r,s) process in (4.1) is:

$$X_t = \sum_{j=0}^{\infty} C_j \varepsilon_{t-j}, \tag{4.10}$$

The m-th order moving average approximation of the GARMA(0,d,0)-GARCH(r,s) process is

$$X_{t,m} = \sum_{j=0}^{m} C_j \varepsilon_{t-j}. \tag{4.11}$$

$X_{t,m}$ will be referred to as a truncated Gegenbauer process; the coefficients C_j are functionally dependent on d and u. For simplicity we replace $X_{t,m}$ by X_t hereafter.

The corresponding state space representation of the MA(m) model is

$$\begin{aligned} X_t &= Z\alpha_t + \varepsilon_t, \\ \alpha_{t+1} &= T\alpha_t + H\varepsilon_t, \end{aligned} \tag{4.12}$$

where the system matrices are:

$$Z = [1, 0, \ldots, 0], \quad T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad H = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ \vdots \\ C_m \end{bmatrix}$$

for suitably chosen matrices Z, α_t, T, H with dimensions 1 x m, m x 1, m x m and m x 1. The vector $\alpha_t = [X(t|t-1), X(t+1|t-1), X(t+2|t-1), ..., X(t+m-1|t-1)]'$ consists of observations conditioned upon state at $t-1$. These system matrices typically depend on hyperparameters.The first and second equations of (4.12) are the *Measurement (or Observation)* and *Transition (or State)* equations of the process.

We may also derive the AR(m) approximation by truncating the AR(∞) representation $\pi(B)X_t = \varepsilon_t, \pi(B) = (1-2uB+B^2)^d$. We compare these two approximations with Chan and Palma (1998) and Grassi and De Magistris (2014) for the autoregressive fractionally integrated moving average (ARFIMA) case when $u = 1$. The relevant estimation methodology is provided in the next section.

4.2. Quasi Maximum Likelihood Estimation Methodology

Given a sample time series $\{x_t, t = 1, \ldots, n\}$, the likelihood function of the approximating MA(m) model represented in (4.11) is evaluated with the support of the KF, which is the following set of recursions ($t = 1, \ldots, n$),

$$\begin{aligned} \nu_t &= x_t - Za_t, & f_t &= ZP_tZ', \\ & & K_t &= (TP_tZ')/f_t, \\ a_{t+1} &= Ta_t + K_t\nu_t, & P_{t+1} &= TP_tT' + HH' - K_tK_t'/f_t. \end{aligned} \tag{4.13}$$

The KF returns the pseudo-innovations ν_t, such that if the MA(m) approximation were the true model, $\nu_t \sim$ NID$(0, \varepsilon f_t)$, so that the log-likelihood of (d, u, ε) is (apart from a constant term)

$$\ell(d, u, \varepsilon) = -\frac{1}{2}\left(n \ln \varepsilon + \sum_{t=1}^{n} \ln f_t + \frac{1}{\varepsilon}\sum_{t=1}^{n}\frac{\nu_t^2}{f_t}\right). \tag{4.14}$$

The scale parameter ε can be concentrated out of the likelihood function, so that

$$\hat{\varepsilon} = \sum_t \frac{\nu_t^2}{f_t},$$

and the profile likelihood is

$$\ell_\varepsilon(d, u) = -\frac{1}{2}\left[n(\ln \hat{\varepsilon} + 1) + \sum_{t=1}^{n} \ln f_t\right]. \tag{4.15}$$

Chan and Palma (1998) states MA and AR approximations within a state space configuration delivers QMLE estimates of parameters due to an approximate likelihood function. Restricting parameters α_0, α_1, β within a state space configuration becomes a cumbersome exercise and affects overall accuracy. Therefore a build-in GARCH error fitting function known as "$garchfit$" in MATLAB, which returns QMLE estimates due to the conditional variance of a GARCH model as described by Ho and Houmani (2010), was incorporated with the state space modelling configuration. Monte Carlo evidence based on the introduced model for simulated data is given in the next section to highlight optimal and positive attributes of the considered time series.

4.3. Monte Carlo Evidence

An extensive Monte Carlo experiment on simulated data was conducted to estimate the parameters d, u, α_0, α_1, and β. A GARMA(0,d,0)-GARCH(1,1) process was considered in all the experiments for practical convenience. The experiment was later extended to establish the optimal estimation order of the process, distinguish the better approximation technique and make a comparative assessment with other available results of similar methods in the literature.

In presenting the experiment results let v be the number of iterations (in the tables below as well), n the series length and θ a generic term for d, u, α_0, α_1, and β. The following computations were carried out from the simulation study. Approximations of each measure is depicted as a subscript or included in brackets as "MA" or "AR" hereafter in all the tables.

(1) $Mean\ \hat{\theta} = \frac{1}{v}\sum_{i=1}^{v}\theta$.

(2) $Standard\ Error\ (SE) = (\frac{1}{v-1}\sum_{i=1}^{v}(\theta - \hat{\theta})^2)^{1/2}$.

(3)$Mean\ Square\ Error\ MSE$=$\frac{1}{v}\sum_{i=1}^{v}(\hat{\theta} - \theta)^2$.

Note: The benchmark for the model MSE was the Total Trace Parameter MSE (TTP-MSE) defined as the sum of the MSE's of all trace parameters of a estimator variance-covariance matrix. It is denoted as:

(4) $TTP - MSE = MSE(d) + MSE(u) + MSE(\alpha_0) + MSE(\alpha_1) + MSE(\beta)$.

(5) $RMSE = \sqrt{MSE}$.

Note: The benchmark for the model RMSE was the Total Trace Parameter RMSE (TTP-RMSE) defined as the sum of the RMSE's of all trace parameters of a estimator variance-covariance matrix. It is denoted as:

(6) $TTP - RMSE = RMSE(d) + RMSE(\alpha_0) + RMSE(\alpha_1) + RMSE(\beta)$ [Used in the analysis below of the Special Case (*) ARFIMA(0,d_*,0)-GARCH(r,s) process, when $u = 1$ in model of (4.1) with $d_* = 2d$].

TABLE 4.1. QMLE Estimates for stationary case with d=0.1, u=0.8, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 1000$

$n = 100$										
Estimator	$\hat{d}_{MA}$	$\hat{u}_{MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{AR}$	$\hat{u}_{AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	0.146	0.520	0.423	0.277	0.297	0.141	0.527	0.428	0.272	0.293
SE	0.086	0.497	0.149	0.182	0.097	0.080	0.490	0.152	0.181	0.096
MSE	0.009	0.325	0.022	0.033	0.009	0.008	0.314	0.024	0.033	0.009
$n = 200$										
Estimator	$\hat{d}_{MA}$	$\hat{u}_{MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{AR}$	$\hat{u}_{AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	0.119	0.659	0.422	0.276	0.297	0.122	0.643	0.416	0.282	0.296
SE	0.058	0.358	0.136	0.165	0.090	0.060	0.364	0.133	0.162	0.083
MSE	0.003	0.148	0.019	0.027	0.008	0.004	0.157	0.018	0.026	0.007
$n = 500$										
Estimator	$\hat{d}_{MA}$	$\hat{u}_{MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{AR}$	$\hat{u}_{AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	0.107	0.757	0.415	0.281	0.299	0.109	0.756	0.417	0.283	0.297
SE	0.039	0.213	0.103	0.123	0.063	0.038	0.200	0.103	0.123	0.064
MSE	0.001	0.047	0.010	0.015	0.004	0.001	0.042	0.011	0.015	0.004
$n = 1000$										
Estimator	$\hat{d}_{MA}$	$\hat{u}_{MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{AR}$	$\hat{u}_{AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	0.102	0.790	0.406	0.292	0.301	0.102	0.784	0.408	0.290	0.301
SE	0.028	0.117	0.067	0.083	0.045	0.028	0.127	0.072	0.090	0.045
MSE	0.0008	0.013	0.004	0.007	0.002	0.0007	0.016	0.005	0.008	0.002
$n = 2000$										
Estimator	$\hat{d}_{MA}$	$\hat{u}_{MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{AR}$	$\hat{u}_{AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	0.100	0.795	0.403	0.295	0.300	0.100	0.797	0.404	0.296	0.298
SE	0.021	0.085	0.052	0.065	0.034	0.019	0.076	0.052	0.063	0.033
MSE	0.0003	0.006	0.002	0.004	0.001	0.0003	0.005	0.002	0.004	0.001

TABLE 4.2. QMLE Estimates for stationary case with d=0.3, u=0.8, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 1000$

Estimator	$\hat{d}_{MA}$	$\hat{u}_{MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{AR}$	$\hat{u}_{AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
$n = 100$										
Mean	0.306	0.788	0.430	0.264	0.296	0.305	0.792	0.424	0.273	0.293
SE	0.082	0.130	0.149	0.180	0.098	0.081	0.126	0.148	0.184	0.096
MSE	0.006	0.017	0.023	0.033	0.009	0.006	0.016	0.022	0.034	0.009
$n = 200$										
Mean	0.301	0.810	0.422	0.276	0.297	0.299	0.800	0.424	0.277	0.292
SE	0.057	0.091	0.136	0.165	0.090	0.060	0.084	0.135	0.161	0.086
MSE	0.003	0.008	0.019	0.027	0.008	0.003	0.007	0.018	0.026	0.007
$n = 500$										
Mean	0.297	0.814	0.415	0.281	0.299	0.300	0.800	0.415	0.281	0.299
SE	0.037	0.060	0.103	0.123	0.063	0.037	0.051	0.103	0.123	0.063
MSE	0.001	0.003	0.010	0.015	0.004	0.001	0.002	0.010	0.015	0.004
$n = 1000$										
Mean	0.296	0.814	0.403	0.296	0.300	0.299	0.804	0.406	0.291	0.299
SE	0.026	0.040	0.068	0.085	0.045	0.027	0.033	0.069	0.087	0.045
MSE	0.0007	0.001	0.004	0.007	0.002	0.0007	0.001	0.004	0.007	0.002
$n = 2000$										
Mean	0.299	0.804	0.403	0.296	0.300	0.299	0.804	0.403	0.296	0.299
SE	0.018	0.022	0.050	0.063	0.034	0.020	0.028	0.052	0.065	0.034
MSE	0.0004	0.0008	0.002	0.004	0.001	0.0003	0.0005	0.002	0.004	0.001

TABLE 4.3. QMLE Estimates for stationary case with d=0.45, u=0.8, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 1000$

Estimator	$\hat{d}_{MA}$	$\hat{u}_{MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{AR}$	$\hat{u}_{AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
$n = 100$										
Mean	0.441	0.820	0.422	0.277	0.295	0.436	0.809	0.427	0.268	0.300
SE	0.058	0.083	0.152	0.183	0.098	0.061	0.087	0.149	0.184	0.100
MSE	0.003	0.007	0.023	0.034	0.009	0.004	0.007	0.023	0.034	0.01
$n = 200$										
Mean	0.442	0.827	0.422	0.276	0.297	0.444	0.811	0.422	0.276	0.297
SE	0.047	0.057	0.136	0.165	0.090	0.046	0.056	0.136	0.165	0.090
MSE	0.002	0.004	0.019	0.027	0.008	0.002	0.003	0.019	0.027	0.008
$n = 500$										
Mean	0.453	0.809	0.408	0.290	0.300	0.449	0.808	0.421	0.276	0.301
SE	0.036	0.053	0.089	0.110	0.057	0.033	0.029	0.099	0.123	0.063
MSE	0.001	0.002	0.008	0.012	0.003	0.001	0.0009	0.010	0.015	0.004
$n = 1000$										
Mean	0.443	0.831	0.406	0.292	0.301	0.449	0.809	0.405	0.293	0.300
SE	0.025	0.022	0.067	0.083	0.045	0.024	0.020	0.068	0.085	0.045
MSE	0.0006	0.001	0.004	0.007	0.002	0.0006	0.0004	0.004	0.007	0.002
$n = 2000$										
Mean	0.454	0.797	0.403	0.295	0.300	0.450	0.808	0.401	0.299	0.298
SE	0.020	0.018	0.052	0.065	0.034	0.018	0.013	0.052	0.064	0.035
MSE	0.0004	0.00003	0.002	0.004	0.001	0.0003	0.0002	0.002	0.004	0.001

The estimates of tables 4.1-4.3 above were arrived at by taking into consideration the TTP-MSE and the validating maximum likelihood value ($Likelihood$) as depicted by confirmatory examples in table 4.4 below. The $Likelihood$ is an output of the KF, while the TTP-MSE is a sum of mse values with respect to parameters d, u, α_0, α_1, and β. Therefore it makes the validation process extremely plausible and feasible. In table 4.4 if the TTP-MSE values are equal upto 3 decimal places then the value corresponding to the lag order (m) in the middle will have the smallest magnitude upto 4 decimal places. The results of table 4.4 is used to establish the optimal lag orders given in table 4.5 below.

Note: Another interesting result from the MA approximation results given above is that the asymptotic variance of the long memory parameter (d) estimates of any stationary Gegenbauer series is approximately equal to $\frac{\pi^2}{12n}$ for any value of n.

Remark: Tables 4.6-4.8 below are based on a non-stationary $ARFIMA(0, d_*, 0)-GARCH(r, s)$ model. It is a special case of a $GARMA(0,\text{d}, 0) - GARCH(r, s)$ model and was taken into consideration, since estimation results of the considered exact generalized model is missing in the literature. The exercise was adequate for a comparative meta analysis assessment of the newly introduced state space approximation techniques of this chapter.

$Note$: Results of tables 4.1-4.5 and implementation of Wold expansion and KF recursions show that the MA approximation technique is better than it's AR counterpart in terms of ease of implementation and optimal lag order. Additionally processing speeds of 9 and 11 minutes approximately for MA and AR techniques per 100 iterations makes the former the better estimating option.

Furthermore, table 4.8 depicts that the model RMSE's of the estimation techniques presented in this paper outperform similar benchmarks due to the traditional maximum likelihood estimation (mle) technique given in Ling and Li (1997) for a $ARFIMA(0, d_*, 0) - GARCH(r, s)$ series.

TABLE 4.4. Approximation estimation validation results for d=0.3, u=0.8, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 1000$

$n = 100$					
m(MA)	$TTP - MSE(MA)$	$Likelihood(MA)$	m(AR)	$TTP - MSE(AR)$	$Likelihood(AR)$
6	0.091	-140.917	9	0.090	-122.205
7	0.090	-140.851	10	0.089	-122.184
8	0.093	-140.858	11	0.091	-122.364
$n = 200$					
m(MA)	$TTP - MSE(MA)$	$Likelihood(MA)$	m(AR)	$TTP - MSE(AR)$	$Likelihood(AR)$
6	0.067	-300.327	10	0.067	-248.425
7	0.066	-299.843	11	0.063	-248.208
8	0.069	-300.805	12	0.065	-248.543
$n = 500$					
m(MA)	$TTP - MSE(MA)$	$Likelihood(MA)$	m(AR)	$TTP - MSE(AR)$	$Likelihood(AR)$
6	0.037	-762.011	9	0.035	-759.909
7	0.035	-758.499	10	0.034	-757.427
8	0.037	-858.307	11	0.034	-758.236
$n = 1000$					
m(MA)	$TTP - MSE(MA)$	$Likelihood(MA)$	m(AR)	$TTP - MSE(AR)$	$Likelihood(AR)$
6	0.017	-1484.006	10	0.016	-1406.632
7	0.016	-1478.803	11	0.016	-1406.320
8	0.017	-1479.803	12	0.016	-1406.493
$n = 2000$					
m(MA)	$TTP - MSE(MA)$	$Likelihood(MA)$	m(AR)	$TTP - MSE(AR)$	$Likelihood(AR)$
5	0.010	-2900.998	10	0.009	-2890.806
6	0.009	-2894.747	11	0.008	-2885.443
7	0.009	-2895.531	12	0.009	-2886.006

TABLE 4.5. Optimal values of m for varying d and u using both approximations with 1000 replications

n	d=0.1(MA)	d=0.3(MA)	d=0.45(MA)	d=0.1(AR)	d=0.3(AR)	d=0.45(AR)
100	6	7	7	9	10	10
200	7	7	7	9	11	11
500	8	7	6	9	10	11
1000	8	7	7	11	11	11
2000	9	6	6	12	11	11

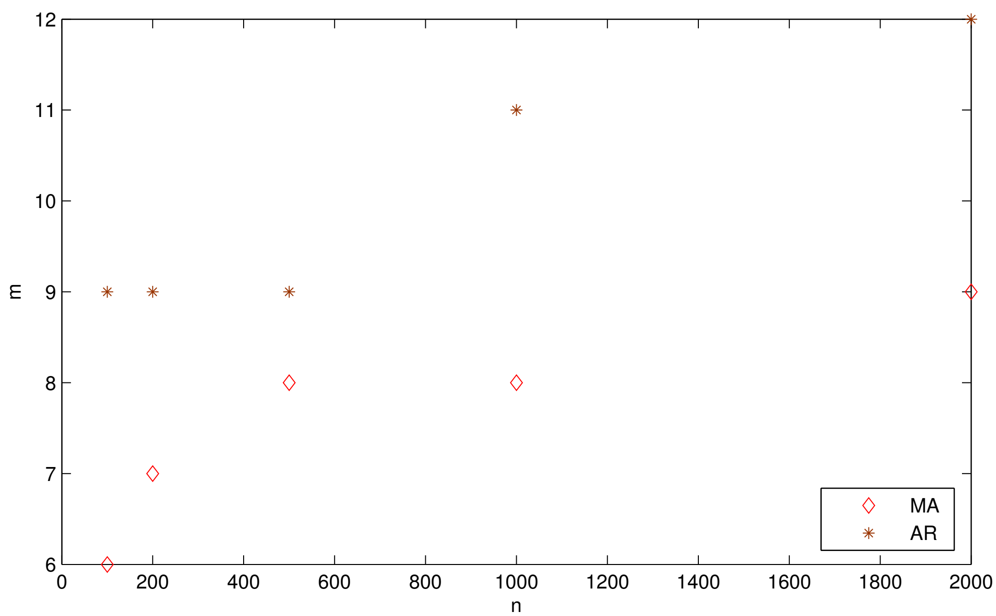


FIGURE 4.5. Optimal Estimation Lag Orders with $d = 0.1$, $u = 0.8$, $\alpha_0 = 0.4$, $\alpha_1 = 0.3$, $\beta = 0.3$ for both approximations

Remark: Estimation results of a non-stationary ARFIMA(0,d_*,0)-GARCH(1,1) series is continued in the next table.

TABLE 4.6. QMLE Estimates for Non-Stationary ARFIMA(0,d_*,0)-GARCH(1,1) Series

Estimator	$\hat{d}_{*MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{*AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
$n = 200$, $d_* = 0.7$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$								
Mean	0.7009	0.3216	0.2667	0.2980	0.7067	0.3120	0.2805	0.3005
SE	0.0631	0.1130	0.1823	0.0980	0.0041	0.1200	0.1819	0.0988
MSE	0.0040	0.0132	0.0343	0.0096	0.000060893	0.0148	0.0335	0.0098
$n = 400$, $d_* = 0.7$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$								
Mean	0.7070	0.3120	0.2841	0.2972	0.7068	0.3120	0.2805	0.3005
SE	0.0012	0.0942	0.1501	0.0774	0.0025	0.0886	0.1458	0.0763
MSE	0.000050056	0.0090	0.0227	0.0060	0.000052426	0.0080	0.0216	0.0058
$n = 200$, $d_* = 1.0$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$								
Mean	0.9947	0.2009	0.2022	0.1905	0.9951	0.2009	0.2022	0.1905
SE	0.0250	0.0696	0.2164	0.0817	0.0241	0.0696	0.2164	0.0817
MSE	0.0006498	0.0048	0.0468	0.0067	0.00060478	0.0048	0.0468	0.0067
$n = 400$, $d_* = 1.0$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$								
Mean	0.9986	0.1981	0.2108	0.1940	0.9984	0.1981	0.2108	0.1940
SE	0.0124	0.0615	0.1949	0.0687	0.0131	0.0615	0.1949	0.0687
MSE	0.00015565	0.0038	0.0380	0.0048	0.00017448	0.0038	0.0380	0.0048
$n = 200$, $d_* = 1.2$, $\alpha_0 = 0.25$, $\alpha_1 = 0.25$, $\beta = 0.25$, $v = 500$								
Mean	1.2045	0.2617	0.2294	0.2419	1.2030	0.2617	0.2294	0.2419
SE	0.0644	0.0912	0.1956	0.0914	0.0659	0.0912	0.1956	0.0914
MSE	0.0042	0.0084	0.0386	0.0084	0.0043	0.0084	0.0386	0.0084
$n = 400$, $d_* = 1.2$, $\alpha_0 = 0.25$, $\alpha_1 = 0.25$, $\beta = 0.25$, $v = 500$								
Mean	1.2184	0.2547	0.2444	0.2437	1.2175	0.2547	0.2444	0.2437
SE	0.0395	0.0791	0.1752	0.0749	0.0426	0.0791	0.1752	0.0749
MSE	0.0019	0.0063	0.0307	0.0056	0.0021	0.0063	0.0307	0.0056

TABLE 4.7. QMLE Estimates for Non-Stationary ARFIMA(0,d_*,0)-GARCH(1,1) Series

$n = 200$, $d_* = 1.4$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$								
Estimator	$\hat{d}_{*MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{*AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	1.3757	0.3209	0.2769	0.2899	1.3654	0.3117	0.2808	0.2997
SE	0.0934	0.1200	0.1819	0.0988	0.0987	0.1063	0.1807	0.0992
MSE	0.0093	0.0148	0.0335	0.0098	0.0109	0.0114	0.0330	0.0098
$n = 400$, $d_* = 1.4$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$								
Estimator	$\hat{d}_{*MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{*AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	1.4032	0.3120	0.2841	0.2972	1.4038	0.3202	0.2693	0.2992
SE	0.0542	0.0942	0.1501	0.0774	0.0493	0.0973	0.1548	0.0788
MSE	0.0029	0.0090	0.0227	0.0060	0.0024	0.0099	0.0249	0.0062
$n = 200$, $d_* = 2.2$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$								
Estimator	$\hat{d}_{*MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{*AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	2.2288	0.2001	0.2034	0.1943	2.2202	0.2009	0.2022	0.1905
SE	0.0458	0.0680	0.2162	0.0886	0.0541	0.0696	0.2164	0.0817
MSE	0.0029	0.0046	0.0467	0.0079	0.0033	0.0048	0.0468	0.0067
$n = 400$, $d_* = 2.2$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$								
Estimator	$\hat{d}_{*MA}$	$\hat{\alpha}_{0MA}$	$\hat{\alpha}_{1MA}$	$\hat{\beta}_{MA}$	$\hat{d}_{*AR}$	$\hat{\alpha}_{0AR}$	$\hat{\alpha}_{1AR}$	$\hat{\beta}_{AR}$
Mean	2.2342	0.1981	0.2108	0.1940	2.2326	0.1981	0.2108	0.1940
SE	0.0152	0.0615	0.1949	0.0687	0.0263	0.0615	0.1949	0.0687
MSE	0.0014	0.0038	0.0380	0.0048	0.0018	0.0038	0.0380	0.0048

TABLE 4.8. TTP-RMSE Estimate Comparison Table for Non-Stationary ARFIMA(0,d_*,0)-GARCH(1,1) Series

$n = 200$, $d_* = 0.7$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.498	0.4613	0.4114
$n = 400$, $d_* = 0.7$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.400	0.3300	0.3198
$n = 200$, $d_* = 1.0$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.469	0.3929	0.3920
$n = 400$, $d_* = 1.0$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.414	0.3383	0.3390
$n = 200$, $d_* = 1.2$, $\alpha_0 = 0.25$, $\alpha_1 = 0.25$, $\beta = 0.25$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.490	0.4445	0.4453
$n = 400$, $d_* = 1.2$, $\alpha_0 = 0.25$, $\alpha_1 = 0.25$, $\beta = 0.25$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.433	0.3730	0.3752
$n = 200$, $d_* = 1.4$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.551	0.5001	0.4918
$n = 400$, $d_* = 1.4$, $\alpha_0 = 0.3$, $\alpha_1 = 0.3$, $\beta = 0.3$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.417	0.3768	0.3850
$n = 200$, $d_* = 2.2$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.501	0.4266	0.4249
$n = 400$, $d_* = 2.2$, $\alpha_0 = 0.2$, $\alpha_1 = 0.2$, $\beta = 0.2$, $v = 500$		
Li and Ling - MLE	QMLE(MA Approximation)	QMLE(AR Approximation)
0.422	0.3632	0.3682

Due to the positive features of the GARMA(0,d,0)-GARCH(1,1) model introduced in this chapter real applications governed by it are provided in the next section.

4.4. Results of Applications

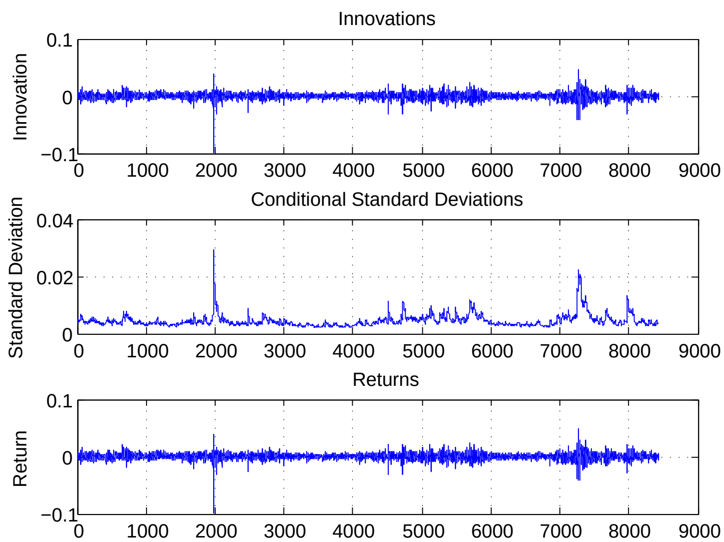
Application 1- S & P 500 Daily Stock index: The log of the daily S and P 500 index with 8435 observations from 2nd January, 1980 to 10th June 2013 [Source: https://au.finance.yahoo.com] was used. A large time series of the Daily S and P 500 index was chosen as a real application of the introduced process anticipating the persistence of conditional heteroskedasticity.The acf, pacf and sdf (shown in figures 4.7-4.9) of the series suggests that it has generalized persistence, since the acf and the pacf depict hyperbolic decays and the sdf consists of unbounded peaks away from the origin. In lieu of it a GARMA(0,d,0)-GARCH(1,1) model was suggested & fitted using the better option of the MA approximation.

Graphical illustrations of the fitted series properties are shown below in figures 4.6-4.10. The conditional standard deviations and the innovations due to employing a state space configuration of the process are depicted in figure 4.6. Figures 4.7-4.9 below clearly illustrate long memory through hyperbolically decaying partial/total sample acf's and unbounded peaks away from the origin of the sdf. It shows that even during a time period spanning more than two decades the initial effect of the stock market will have an impact on the final indices. The application yielded the following results at an optimal truncation lag order $m = 6$ (within optimal lag order interval) as follows: $\hat{d}_{MA} = 0.1042$, $\hat{u}_{MA} = 0.9151$, $\hat{\alpha}_{0MA} = 2.6422 \times 10^{-7}$, $\hat{\alpha}_{1MA} = 0.078939$, $\hat{\beta}_{MA} = 0.91106$.

Therefore the fitted $GARMA(0\text{,d}, 0) - GARCH(r, s)$ model is:
$(1 - 2 \times 0.9151B + B^2)^{0.1042} X_t = \varepsilon_t$,
where the standard errors of $\hat{d}_{MA}$, $\hat{u}_{MA}$, $\hat{\alpha}_{0MA}$, $\hat{\alpha}_{1MA}$, $\hat{\beta}_{MA}$ are 2.3131×10^{-7}, 0.0763, 2.3398×10^{-8}, 0.0018024, and 0.0029494 respectively.

An in-sample rolling forecast was performed to assess the final 200 observations of the data set resulting in a one-step ahead forecast MSE of 2.4723×10^{-5}. Original & forecast values of the final 200 observations are provided in figure 4.8. It depicts that they are reasonably close proving that the utilized model is feasible in terms of forecast accuracy.

FIGURE 4.6. Realization of S and P 500 Returns, Innovations and Conditional Standard Deviations



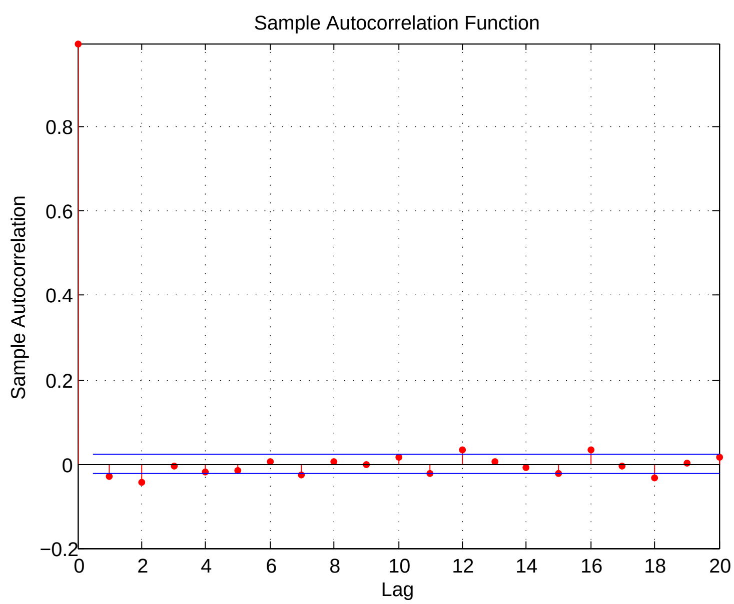


FIGURE 4.7. S&P 500 Application Sample Autocorrelation Function

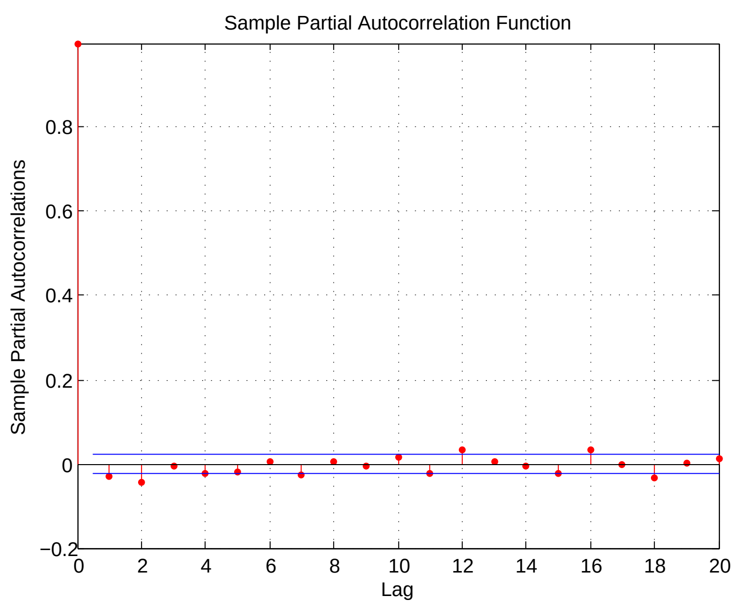


FIGURE 4.8. S&P 500 Application Sample Partial Autocorrelation Function

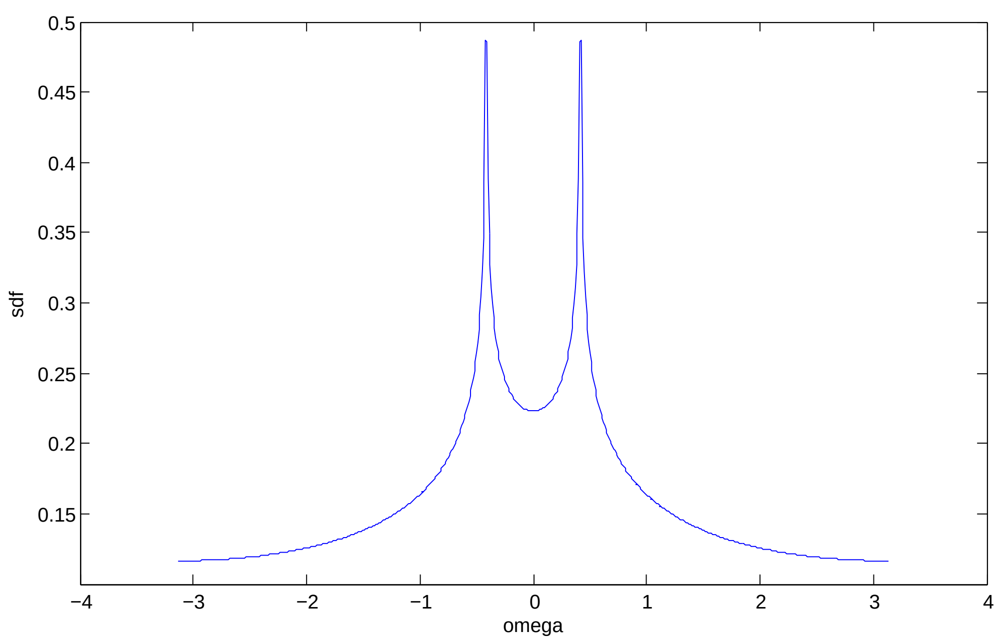
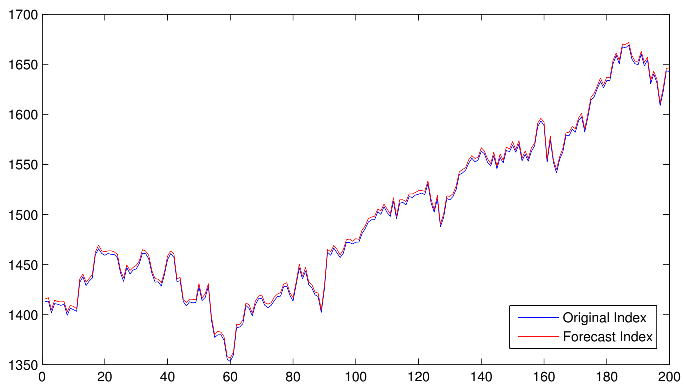


FIGURE 4.9. S&P 500 Application Spectral Density Function of Returns

FIGURE 4.10. S and P 500 - Final 200 original observations and their in-sample one-step ahead predictions



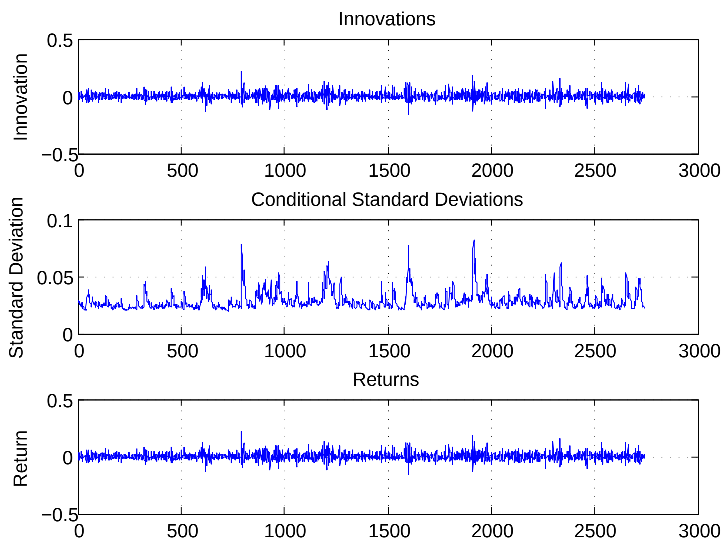
Application 2- Chicago Board Options Exchange (CBOI) market expectation daily measurement volatility index: As a second application, the log of the daily CBOI index with 2744 observations from 2nd January 2004 to 10th November 2014 [Source: http://www.cboe.com] was used in fitting a GARMA(0,d,0)-GARCH(1,1) model using the better option of the MA approximation, since the acf, pacf and the sdf suggest generalized persistence quite similar to the corresponding functions of the S & P daily stock index of application 1. Graphical illustrations of the fitted series properties are shown in figure 4.8.

The application yielded the following results at an optimal truncation lag order $m = 6$ (within optimal lag order interval) as follows: $\hat{d}_{MA} = 0.1196$, $\hat{u}_{MA} = 0.7932$, $\hat{\alpha_0}_{MA} = 7.6094 \times 10^{-5}$, $\hat{\alpha_1}_{MA} = 0.11957$, $\hat{\beta}_{MA} = 0.79317$.

Therefore the fitted $GARMA(0,\text{d},0) - GARCH(r,s)$ model is:
$(1 - 2 \times 0.7932B + B^2)^{0.1196} X_t = \varepsilon_t$,
where the standard errors of $\hat{d}_{MA}$, $\hat{u}_{MA}$, $\hat{\alpha_0}_{MA}$, $\hat{\alpha_1}_{MA}$, $\hat{\beta}_{MA}$ are 7.6089×10^{-5}, 0.0719, 8.4106×10^{-6}, 0.011141, and 0.017216 respectively.

In a theoretical sense for both applications the values of $\hat{d}_{MA}$, $\hat{u}_{MA}$, $\hat{\alpha_0}_{MA}$, $\hat{\alpha_1}_{MA}$, $\hat{\beta}_{MA}$ fall within the stipulated bounds of a $GARMA(0,\text{d},0) - GARCH(r,s)$ series as per the model definitions of (4.2)-(4.4). Therefore the chosen daily S and P 500 & CBOI index series theoretically illustrate the properties of heteroskedasticity and persistence. It results in

FIGURE 4.11. Realization of CBOI Returns, Innovations and Conditional Standard Deviations



$GARMA(0,\text{d},0) - GARCH(1,1)$ models with a high degree of volatility due to conditional heteroskedasticity.

Based on the positive attributes of the introduced model and contributions of this chapter illustrated in sections 4.1-4.4 concluding remarks are provided in the next section.

4.5. Concluding Remarks

A truncated state space model entailing the KF of a simple Gegenbauer process is introduced to arrive at two types of parameter estimates. These two types are constructed using AR and MA approximations of the series to introduce optimal lag order intervals as a novel contribution. A comparative assessment of the two approximations based on Monte Carlo evidence is provided as an additional creative component. As a novel contributory link, it also proves that in the non-stationary form of the model the two introduced approximate QMLE techniques are superior to a traditional MLE mechanism in the literature. Finally the better approximation method is applied to two real applications within the established optimal lag order interval. It proves the persistence of volatility as per the introduced model within a long duration interval. Therefore the introduced process could significantly impact the global economy if a large volume of data indices behave in line with it.

Furthermore, the original GARMA process of (3.1) with the addition of a seasonal filter could create a seasonal Gegenbauer process with a wide variety of real applications. In line with the thesis outline of chapter 1 it will be the discussion topic of chapter 5.

CHAPTER 5

State Space Modeling of Seasonal Gegenbauer Processes with Long Memory

5.1. Introduction

It is known that certain time series exhibit the properties of seasonal patterns and long range dependence simultaneously. Examples of such behaviour can be found in physics, economics and hydrology. Further references in this context have been provided in sub section 1.2.3 of chapter 1 including Porter-Hudak (1990), Ray (1993), Hassler and Wolters (1995), Ooms (1995), Montanari et.al. (2000) and Palma (2007).

Initially a statistical methodology to include seasonal components was proposed by Mandelbrot and Van Ness (1968). Later Palma and Chan (2005) provided asymptotic properties of exact maximum likelihood estimates (MLE) for a general class of the Gaussian seasonal long range dependent process. The estimation of seasonal long memory processes was proposed by Reisen et.al. (2006). In addition, the estimation of seasonal fractionally integrated models using log periodogram regression has been established by Reisen et.al. (2006). A semiparametric estimation technique for seasonal long memory time series utilizing generalized exponential (GEXP) models was illustrated by Hsu and Tsai (2009). Furthermore, a good review of estimating seasonal long memory models is presented in Palma (2007).

In this chapter, theoretical properties and estimation tools used in ARFIMA model with seasonal long memory modelling will be extended to GARMA models with seasonal and long memory components. The estimation tools coupled with additional prediction mechanisms are created through state space modeling entailing the Kalman filter based on the work of Harvey (1989), Durbin and Koopman (2001) and Harvey and Proietti (2005).

The structure of this chapter is organized as follows. Section 2 introduces a discussion of the seasonal operator. It entails an introduction to the factorization of the seasonal lag polynomial. Certain theoretical aspects of the GARSMA model is introduced in section 3. It includes the creation of related Wold expressions for MA and AR approximations. State space modelling and Kalman filter theory related to the process is provided in section 4. Finite small sample performance of the estimators is produced in section 5 through Monte Carlo evidence inclusive of a meta analytical comparative assessment of the newly established optimal lag order. Section 6 is devoted to the analysis of real data applications. Finally, concluding remarks are provided in Section 7.

5.2. Seasonal Operator

Let s' be the seasonal period. Then the seasonal operator introduced in chapter 2 is considered for the corresponding theoretical developments. Details of the derived factorizations are provided next in the discussion.

Factorization of the seasonal operator

Consider the seasonal lag polynomial $(1 - B^{s'})$, with B the backshift operator. It is known that the s'-th roots of unity are given by

$$\zeta_k = cos(\omega_k) + isin(\omega_k), \quad k = 0, 1, 2, ..., s' - 1, \tag{5.1}$$

where $\omega_k = 2k\pi/s'$.

This implies that we can write:

$$(1 - B^{s'}) = \varrho_{s'} \prod_{k=0}^{s'-1} (\zeta_k - B), \tag{5.2}$$

for some constant $\varrho_{s'}$.

Comparing the coefficients of $B^{s'}$, it is clear that:

$$\varrho_{s'} = \begin{cases} 1, & \text{if } s' \text{ is odd} \\ -1, & \text{if } s' \text{ is even.} \end{cases}$$

For every $k > 0$, we have $\zeta_{s'-k} = cos(2\pi - \omega_k) + isin(2\pi - \omega_k) = cos(\omega_k - isin(\omega_k)$, and that $(\zeta_k - B)(\zeta_{s'-k} - B) = cos^2(\omega_k) + sin^2(\omega_k) - 2cos(\omega_k)B + B^2 = 1 - 2cos(\omega_k)B + B^2$.

Case 1: s' is odd

If s' is odd, pairing off is done for all factors except when $k = 0$. It is clear that, $1 - B^{s'} = \prod_{k=0}^{s'-1}(\zeta_k - B) = (\zeta_0 - B)\prod_{k=1}^{(s'-1)/2}[(\zeta_k - B)(\zeta_{s'-k} - B)]$
$=(1 - B)\prod_{k=1}^{(s'-1)/2}(1 - 2cos(\omega_k)B + B^2)$.

It is easy to verify that
$1 - B = (1 - 2B + B^2)^{1/2} = (1 - 2cos(0)B + B^2)^{1/2} = (1 - 2cos(\omega_0)B + B^2)^{1/2}$, and hence

$$1 - B^{s'} = (1 - 2cos(\omega_0)B + B^2)^{1/2} \prod_{k=1}^{(s'-1)/2} (1 - 2cos(\omega_k)B + B^2). \tag{5.3}$$

Case 2: s' is even

If s' is even, pairing off is done for all factors except $k = 0$ and $k = s'/2$. Noting that $\zeta_0 = 1$ and $\zeta_{s'/2} = -1$ it is clear that,
$1 - B^{s'} = \prod_{k=0}^{s'-1}(\zeta_k - B) = -(\zeta_0 - B)(\zeta_{s'/2} - B)\prod_{k=1}^{(s'/2)-1}[(\zeta_k - B)(\zeta_{s'-k} - B)]$
$=-(1 - B)(-1 - B)\prod_{k=1}^{(s'/2)-1}(1 - 2cos(\omega_k)B + B^2)$
$=(1 - B)(1 + B)\prod_{k=1}^{(s'/2)-1}(1 - 2cos(\omega_k)B + B^2)$.

As before
$1 + B = (1 + 2B + B^2)^{1/2} = (1 - 2cos(\pi)B + B^2)^{1/2} = (1 - 2cos(\omega_{s'/2})B + B^2)^{1/2}$, and hence

$$1-B^{s'} = (1-2cos(\omega_0)B+B^2)^{1/2}(1-2cos(\omega_{s'/2})B+B^2)^{1/2} \prod_{k=1}^{(s'/2)-1} (1-2cos(\omega_k)B+B^2). \tag{5.4}$$

The next section considers the use of (5.3) and (5.4) in seasonal modelling.

5.3. The GARSMA(0,d,0)x(0,$D_{s'}$,0) model

Suppose that $\{Y_t\}$ is a seasonal time series with period s' that can be transformed into

$$X_t = (1 - B^{s'})^{D_{s'}} Y_t$$

to remove any seasonal components, where $0 < D_{s'} < \frac{1}{2}$.

Then GARMA(0,d,0) series with seasonality can be written as

$$(1 - 2uB + B^2)^d (1 - B^{s'})^{D_{s'}} X_t = \epsilon_t, \quad 0 < d < \frac{1}{2} \quad and \quad |u| < 1. \tag{5.5}$$

Extending the notion of an ARFISMA series (5.5) can be defined as a GARSMA(0,d,0)x(0,$D_{s'}$,0) model.

Let

$$(1 - 2uB + B^2)^d (1 - B^{s'})^{D_{s'}} = \sum_{j=1}^{\infty} \pi_j' B^{s'j}, \tag{5.6}$$

$$(1 - 2uB + B^2)^{-d} (1 - B^{s'})^{-D_{s'}} = \sum_{j=1}^{\infty} \psi_j' B^{s'j}, \tag{5.7}$$

where ψ_j', π_j' are the corresponding coefficients of each

$$(1 - B^{s'})^{-D_{s'}} = \sum_{j=1}^{\infty} C_j' B^{s'j}$$

expansion. The following lemma is useful for later reference.

Lemma 5.1: A GARSMA(0,d,0)x(0,$D_{s'}$,0) process of (5.5) is stationary and long memory when $|u| < 1$, $0 < d < 1/2$ and $0 < D_{s'} < 1/2$.

Properties of GARSMA(0,d,0)x(0,$D_{s'}$,0) model

A Wold representation of a MA approximation based GARSMA(0,d,0)x(0,$D_{s'}$,0) model is

$$X_t = \psi^{'}(B)\epsilon_t = \sum_{j=0}^{\infty} \psi^{'}_j \epsilon_{t-j}, \tag{5.8}$$

Therefore $\psi^{'}_j$ will be a convolution of coefficients $C^{'}_j$ and C_j such that
$C^{'}_j = \Gamma(s' \times j + d)/\Gamma(d)\Gamma(s' \times j + 1)$,
and C_j are Gegenbauer coefficients.

Where $\Gamma(\bullet)$ is the Gamma function.

Refer Palma (2007) for details.

Now we consider the state space representation of a GARSMA(0,d,0)x(0,$D_{s'}$,0) process

Extending state space modelling to assess the properties of a GARSMA(0,d,0)x(0,$D_{s'}$,0) process remains a viable exercise and will be the discussion topic of the next section.

5.4. State Space Representation of a GARSMA(0,d,0)x(0,$D_{s'}$,0) Process

As in the non-seasonal case, the m-th order truncated MA approximation or the Wold representation of (5.8) at lag m is

$$X_{t,m} = \sum_{j=0}^{m} \psi^{'}_j \epsilon_{t-j} \tag{5.9}$$

Following Chan and Palma (1998) and Dissanayake et.al (2014a) the state space representation of the MA(m) model is given by Measurement/Observation and Transition/State equations as:

$$\begin{aligned} X_{t,m} &= Z\alpha_t + \epsilon_t, \\ \alpha_{t+1} &= T\alpha_t + H\epsilon_t, \end{aligned} \tag{5.10}$$

where α_{t+1} is the $m\times 1$ state vector with elements $\alpha_{j,t+1} = E(X_{t+j,m}|\mathcal{F}_t), \mathcal{F}_t = \{X_{t,m}, X_{t-1,m}, \ldots\}$.

Following Chan and Palma (1998) it can be shown that the system matrices of the constructed State Space configuration are:

$$Z = [1, 0, \ldots, 0], \quad T = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \cdots & \cdots & 0 & 1 \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \quad H = \begin{bmatrix} \psi_1^{'} \\ \psi_2^{'} \\ \vdots \\ \vdots \\ \psi_m^{'} \end{bmatrix}$$

H is a column vector of convoluted seasonal Gegenbauer coefficients and $G = [1]$.

It is possible to derive the corresponding AR(m) approximation by truncating the AR(∞) representation $\pi^{'}(B)X_t = \epsilon_t, \pi^{'}(B) = (1-B^{s'})^{D_{s'}}(1-2uB+B^2)^d$ following Chan and Palma (1998) and Grassi and De Magistris (2014).

An estimation procedure based on the state space models originally developed by Kalman (1960) and used in chapter 3 for a GARMA(0,d,0) model is extended to the system given in (5.10) using KF recursions.

QML Estimation through KF Recursion For a time series $\{x_t, t = 1, \ldots, n\}$, the approximate likelihood function of an MA(m) model evaluated using the KF is given below:

$$\begin{aligned} \nu_t &= x_t - Za_t, & f_t &= ZP_tZ', \\ & & K_t &= (TP_tZ')/f_t, \\ a_{t+1} &= Ta_t + K_t\nu_t, & P_{t+1} &= TP_tT' + HH' - K_tK_t'/f_t. \end{aligned} \tag{5.11}$$

The KF returns pseudo-innovations ν_t, creating a log-likelihood of $(d, u, D_{s'}, \sigma^2)$ (apart from constant term)

$$\ell(d, u, D_{s'}, \sigma^2) = -\frac{1}{2}\left(n \ln \sigma^2 + \sum_{t=1}^{n} \ln f_t + \frac{1}{\sigma^2}\sum_{t=1}^{n}\frac{\nu_t^2}{f_t}\right) \tag{5.12}$$

and the profile likelihood of

$$\ell_{\sigma^2}(d, u, D_{s'}) = -\frac{1}{2}\left[n(\ln\hat{\sigma}^2 + 1) + \sum_{t=1}^{n} \ln f_t\right] \tag{5.13}$$

Remark: Adopting the methodology introduced in sections 5.2 and 5.3 of this chapter in order to assess finite sample performance of QML estimates in terms of estimation and forecasting a number of Monte Carlo simulations were performed. In that context a GARSMA(0,d,0)x(0,$D_{s'}$,0) process having a monthly periodicity of $s' = 12$ is considered throughout this chapter to appropriately assess the model. The chosen special case GARSMA(0,d,0)x(0,$D_{s'}$,0) model is,

$$(1 - 2uB + B^2)^d(1 - B^{12})^{D_{s'}} X_t = \epsilon_t. \tag{5.14}$$

Monte Carlo evidence for small lengths ($n = 100, 200$) of the desired model was performed in lieu of it and the results are presented in the next section.

5.5. Monte Carlo Evidence

Monte Carlo experiments were executed to assess and corroborate the developed theory of the desired GARSMA(0,d,0)x(0,$D_{s'}$,0) model. The results are provided in tables 5.1-5.12. In order to arive at an optimal lag order (m) the total estimator mean square error (E.MSE) was validated by a rolling forecast one step ahead prediction mean square error (F.MSE), where $E.MSE = MSE(\hat{d}) + MSE(\hat{u}) + MSE(\hat{D_{s'}}) + MSE(\hat{\sigma})$. $d = [0.1, 0.3, 0.45]$, $D_{s'} = [0.1, 0.3, 0.45]$ and $n = [100, 200]$

TABLE 5.1. MA approximation with $d = 0.1$, $u = 0.8$, $D_{s'} = 0.45$, Replications = 1000

$n = 100$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
40	0.080	0.063	0.004	0.697	0.364	0.153
45	0.087	0.063	0.004	0.687	0.346	0.145
50	0.087	0.063	0.004	0.696	0.348	0.145

$n = 100$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D}_{s'}$	SD($\hat{D}_{s'}$)	MSE($\hat{D}_{s'}$)	E. MSE	F. MSE
40	0.995	0.149	0.022	0.443	0.016	0.003	0.182	1.0805
45	0.991	0.150	0.022	0.428	0.017	0.004	0.175	1.0804
50	0.991	0.150	0.022	0.431	0.015	0.007	0.178	1.1171

TABLE 5.2. AR approximation with $d = 0.1$, $u = 0.8$, $D_{s'} = 0.45$, Replications = 1000

$n = 100$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
30	0.083	0.075	0.007	0.686	0.157	0.121
35	0.083	0.072	0.007	0.676	0.158	0.119
40	0.081	0.067	0.007	0.678	0.149	0.123

$n = 100$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D}_{s'}$	SD($\hat{D}_{s'}$)	MSE($\hat{D}_{s'}$)	E. MSE	F. MSE
30	1.020	0.151	0.023	0.386	0.074	0.009	0.160	0.991
35	1.021	0.150	0.023	0.384	0.074	0.009	0.158	0.988
40	1.023	0.152	0.023	0.385	0.072	0.009	0.162	0.992

TABLE 5.3. MA approximation with $d = 0.1$, $u = 0.8$, $D_{s'} = 0.45$, Replications = 1000

$n = 200$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
45	0.079	0.051	0.003	0.638	0.240	0.085
50	0.079	0.048	0.002	0.669	0.245	0.072
55	0.072	0.052	0.003	0.562	0.243	0.080

$n = 200$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
45	1.008	0.107	0.011	0.389	0.015	0.002	0.101	0.974
50	1.007	0.107	0.011	0.382	0.015	0.002	0.087	0.971
55	1.008	0.110	0.012	0.369	0.014	0.005	0.103	0.977

TABLE 5.4. AR approximation with $d = 0.1$, $u = 0.8$, $D_{s'} = 0.45$, Replications = 1000

$n = 200$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
35	0.061	0.052	0.006	0.707	0.109	0.083
40	0.061	0.050	0.006	0.694	0.107	0.001
45	0.060	0.048	0.006	0.696	0.105	0.036

$n = 200$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
35	1.027	0.104	0.011	0.422	0.012	0.006	0.106	1.0066
40	1.028	0.104	0.011	0.419	0.012	0.008	0.026	1.0065
45	1.028	0.105	0.011	0.418	0.012	0.008	0.061	1.0066

TABLE 5.5. MA approximation with $d = 0.3$, $u = 0.8$, $D_{s'} = 0.3$, Replications = 1000

$n = 100$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
50	0.305	0.052	0.002	0.856	0.091	0.014
55	0.311	0.050	0.002	0.845	0.096	0.011
60	0.304	0.051	0.002	0.854	0.109	0.015

Table continued on next page

$n = 100$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
50	1.14	0.133	0.037	0.307	0.024	0.003	0.056	1.0805
55	1.12	0.114	0.028	0.321	0.060	0.004	0.045	1.0804
60	1.14	0.133	0.037	0.313	0.054	0.003	0.057	1.1171

TABLE 5.6. AR approximation with $d = 0.3$, $u = 0.8$, $D_{s'} = 0.3$, Replications = 1000

$n = 100$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
40	0.281	0.095	0.009	0.804	0.273	0.074
45	0.286	0.086	0.007	0.818	0.207	0.043
50	0.258	0.112	0.014	0.729	0.470	0.226

$n = 100$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
40	1.077	0.245	0.066	0.369	0.163	0.003	0.152	0.997
45	1.078	0.244	0.066	0.380	0.157	0.003	0.119	0.993
50	1.116	0.283	0.094	0.391	0.151	0.003	0.337	0.995

TABLE 5.7. MA approximation with $d = 0.3$, $u = 0.8$, $D_{s'} = 0.3$, Replications = 1000

$n = 200$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
40	0.267	0.029	0.002	0.969	0.069	0.003
45	0.307	0.028	0.0008	0.844	0.072	0.007
50	0.309	0.026	0.0008	0.857	0.075	0.009

$n = 200$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
40	1.284	0.112	0.033	0.362	0.019	0.001	0.039	1.110
45	1.114	0.101	0.023	0.371	0.051	0.003	0.033	1.108
50	1.130	0.119	0.031	0.384	0.046	0.003	0.043	1.114

TABLE 5.8. AR approximation with $d = 0.3$, $u = 0.8$, $D_{s'} = 0.3$, Replications = 1000

$n = 200$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
40	0.265	0.058	0.004	0.951	0.131	0.040
45	0.316	0.070	0.005	0.820	0.106	0.011
50	0.266	0.066	0.005	0.936	0.106	0.027

$n = 200$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
40	1.240	0.215	0.042	0.323	0.117	0.001	0.087	1.146
45	1.104	0.178	0.042	0.389	0.149	0.002	0.060	0.917
50	1.219	0.220	0.090	0.392	0.150	0.002	0.124	1.150

TABLE 5.9. MA approximation with $d = 0.45$, $u = 0.8$, $D_{s'} = 0.1$, Replications = 1000

$n = 100$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
40	0.442	0.041	0.002	0.869	0.061	0.008
45	0.457	0.040	0.002	0.818	0.058	0.007
50	0.460	0.040	0.003	0.910	0.062	0.007

$n = 100$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
40	1.66	0.265	0.081	0.107	0.038	0.003	0.094	1.011
45	1.71	0.216	0.077	0.121	0.023	0.004	0.090	0.932
50	1.60	0.298	0.078	0.113	0.016	0.003	0.091	0.988

TABLE 5.10. AR approximation with $d = 0.45$, $u = 0.8$, $D_{s'} = 0.1$, iterations = 1000

$n = 100$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
30	0.458	0.096	0.013	0.649	0.354	0.062
35	0.455	0.099	0.013	0.634	0.348	0.060
40	0.457	0.095	0.013	0.654	0.314	0.066

$n = 100$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
30	1.38	0.355	0.099	0.104	0.089	0.017	0.191	0.977
35	1.40	0.355	0.080	0.100	0.089	0.018	0.171	0.960
40	1.41	0.361	0.088	0.103	0.082	0.018	0.185	0.974

TABLE 5.11. MA approximation with $d = 0.45$, $u = 0.8$, $D_{s'} = 0.1$, iterations = 1000

$n = 200$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
45	0.459	0.037	0.001	0.935	0.060	0.006
50	0.456	0.030	0.001	0.956	0.058	0.006
55	0.474	0.032	0.001	0.706	0.060	0.007

$n = 200$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
45	1.77	0.195	0.055	0.151	0.012	0.001	0.063	0.912
50	1.80	0.124	0.051	0.096	0.016	0.001	0.059	0.908
55	1.51	0.197	0.057	0.142	0.015	0.001	0.066	1.117

TABLE 5.12. AR approximation with $d = 0.45$, $u = 0.8$, $D_{s'} = 0.1$, iterations = 1000

$n = 200$						
m	$\hat{d}$	SD($\hat{d}$)	MSE($\hat{d}$)	$\hat{u}$	SD($\hat{u}$)	MSE($\hat{u}$)
30	0.479	0.062	0.008	0.645	0.158	0.039
35	0.477	0.068	0.007	0.621	0.154	0.040
40	0.471	0.087	0.008	0.644	0.161	0.035

$n = 200$								
m	$\hat{\sigma}$	SD($\hat{\sigma}$)	MSE($\hat{\sigma}$)	$\hat{D_{s'}}$	SD($\hat{D_{s'}}$)	MSE($\hat{D_{s'}}$)	E. MSE	F. MSE
30	1.38	0.123	0.075	0.119	0.051	0.007	0.129	0.990
35	1.39	0.118	0.065	0.118	0.053	0.007	0.119	0.974
40	1.41	0.125	0.077	0.118	0.083	0.008	0.128	0.986

Note: By carefully inspecting tables 5.1-5.12 it could be observed that the E.MSE value decrease and subsequently increase with a turning point (first lowest dip) in terms of the lag order. It is also evident that a similar behaviour takes place with respect to the F.MSE at the same lag order. Therefore the total model estimator mean square error is validated by the predictive accuracy resulting in an optimal lag order (m). The exercise is similar to what was done for a GARMA(0,d,0) model in chapter 3 and Dissanayake et.al.(2014a).

Note: In the next segment a comparative meta analysis in terms of optimal lag order of the above results in terms of the two approximations and corresponding results of a GARMA model are provided.

Comparison of AR and MA approximations

TABLE 5.13. Optimal values of m with $u = 0.8$ using MA approximation and 1000 replications

n	d=0.1/$D_{s'}$=0.45	d=0.3/$D_{s'}$=0.3	d=0.45/$D_{s'}$=0.1
100	45	55	45
200	50	45	50

TABLE 5.14. Optimal values of m with $u = 0.8$ using AR approximation and 1000 replications

n	d=0.1/$D_{s'}$=0.45	d=0.3/$D_{s'}$=0.3	d=0.45/$D_{s'}$=0.1
100	35	45	35
200	40	45	35

Optimal lag order using MA approximation - $[45, 55]$

Optimal lag order using AR approximation - $[35, 45]$

Remark: Next a comparison of lag orders with GARMA(0,d,0) model is provided.

A meta analysis with GARMA(0,d,0) model results

TABLE 5.15. Comparative assessment of optimal lag orders

Series	Optimal Lag Order-MA	Optimal Lag Order-AR
GARMA(0,d,0)	[29,35]	[9,13]
GARSMA(0,d,0)x(0,$D_{s'}$,0)	[45,55]	[35,45]

From the above results it is clear that the introduction of the seasonal filter has an impact in extending the optimal lag order.

Since the difference in the lag order intervals was not great and due to the fact that the MA approximation was returning smaller MSE's for the estimators it was chosen as the better representation. Therefore the MA approximation within the established optimal lag order interval was utilized to assess two real applications.

5.6. Empirical Applications

5.6.1. El'Nino Data: Literature on Climatology records El'Nino series observations based on seasonal monthly drought outlook. In that context a current El' Nino series was chosen as the first real application to fit the GARSMA(0,d,0)x(0,$D_{s'}$,0) model introduced in this chapter, since Besse et.al(2000) and others had used an El'Nino series to make forecasts about some functional climatic variations using an autoregressive process. The following table provides the estimation results with the standard error estimates shown within brackets.

TABLE 5.16. MA approximation Estimates for El' Nino series with $n = 1656$.

Series	m	$\hat{d}$(MA)	$\hat{u}$(MA)	$\hat{D_{s'}}$(MA)
El' Nino	55	0.269(0.0009)	0.971(0.005)	0.324(0.0005)

Note: The optimal series model will be $(1 - 2 \times 0.971B + B^2)^{0.269}(1 - B^{12})^{0.324}X_t = \epsilon_t$. The standard error estimates of both the memory parameters are small and reasonably close in value. The estimates fall within the bounds of the stipulated values of the introduced model depicting seasonality and long memory. The optimal forecasts are achieved at comparatively large lag orders raising questions about further improving and enhancing the model in terms of cost, processing speed, affordability and efficiency.

5.6.2. Sunspots Data: This application uses the sunspots data series considered by Tong (1990). It displays periodicity with peaks and troughs as explained in Chapter 3 illustrating seasonal cycles. The fitted optimal series model at $m = 45$ is $(1 - 2 \times 0.956B + B^2)^{0.43}(1 - B^{12})^{0.308}X_t = \epsilon_t$, with the standard errors for $\hat{d}$(MA), $\hat{u}$(MA) and $\hat{D_{s'}}$(MA) being 0.0004, 0.0018 and 0.0004, where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$.

Remark: Therefore from the empirical evidence it is evident that one could only include toy applications for a real data case that depict periodicity.

5.7. Concluding Remarks

A comprehensive seasonal operator factorization and state space configuration for a GARSMA process are introduced. The methodology is employed to conduct Monte Carlo experiments for a small sample GARSMA model with a monthly seasonal periodicity. It returns an optimal lag order based on KF based QML estimates validated by predictive accuracy. The AR approximation returns smaller lag order and MA representation provides better total MSE's for model profiles making it more feasible. A minor meta analysis with GARMA model results of chapter 3 was performed and the better approximating option was applied to a real application El′ Nino series from environmental science. Probing the parent family of the introduced model in terms of persistence and stationarity through a unit root assessment becomes a worthwhile proposition. In line with the thesis outline it is introduced in the next chapter titled as "Nearly efficient unit root tests for GARMA(0,d,0) processes with long memory".

CHAPTER 6

Nearly Efficient Unit Root Tests for GARMA(0,d,0) Processes with Long Memory

---

6.1. Introduction

Linear stochastic processes are said to have a unit root if 1 is a root of the process's characteristic equation. Such a process is non-stationary. If the other roots of the characteristic equation lie inside the unit circle (i.e. have a modulus or absolute value less than 1) then the first difference of the process will be stationary. Therefore, a *unit root* is a feature of processes that evolve through time, which can cause problems in statistical inference involving time series models.

Fundamentally, in a unit root testing sense, a time series X_t is said to be integrated of order 1, if it becomes stationary after being differenced once. Then we may write the null hypothesis of X_t as $I(1)$. On the contrary a series that is stationary without being differenced is said to be integrated of order zero possessing a null hypothesis denoted as $I(0)$. Therefore a series that becomes stationary after being differenced d times is said to be integrated of order d with a null hypothesis $I(d)$. Proceeding in this manner it is often found that both the null hypotheses are rejected suggesting many series are not well represented by $I(0)$ and $I(1)$. In view of it the class of fractionally integrated processes denoted as $FI(d)$, where order of integration d is extended to any real number has been introduced.

In a generic sense unit root tests are consistent with a fairly low power if the alternative hypothesis is a $FI(d)$ process according to Diebold and Rudebusch (1991) and Lee and Schmidt (1996). This lack of power in general unit root tests has inspired and prompted researchers to create and adopt new testing procedures by taking the $FI(d)$ alternative into consideration. Therefore, unit root testing of GARMA(0,d,0) processes that fall into the extended $FI(d)$ class has aroused much interest.

In such a context Dolado et.al. (2002) extended a standard Dickey Fuller (SDF) test to an augmented Dickey Fuller (ADF) test for an ARFIMA model by employing the $FI(d)$ class. But extending the methodology to assess a much more generalized GARMA model involves a multivariate dimension. Therefore unit root testing of GARMA(0,d,0) processes for stationarity and long memory properties could be based on 2 parameters of the model - either the long memory parameter or the degree of differencing d and/or polynomial index parameter u. Chung (1996) executed such a test for d and u using the conditional sum of squares (CSS) estimation method utilizing its concentrated profile likelihood function.

However, an extension of the unit root test based on state space methodology and KF estimates for a GARMA model is seemingly absent in the literature. Since the KF delivers QML estimates an LR type test becomes the most feasible assessing option. Jansson and Nielsen (2012) proposed a nearly efficient unit root test that is applicable in such a context and assessing it's power becomes a worthwhile exercise.

The work presented in this chapter will comprise of 2 different tests using a nearly efficient variant of an LR type test (briefly introduced in chapter 1). The first proposed test will revolve around u to check if the desired series is a long memory GARMA(0,d,0) process or depicts standard long memory. Test to follow in this section is based on d to assess stationarity against non-stationarity.

Both these tests are based on state space modelling of long memory GARMA processes. They are done to assess properties of a GARMA(0,d,0) model revolving around QMLE estimators of u and d. In such a context the QMLE estimators of u and d are calculated by maximizing ℓ_{σ^2} of (3.8). This method suggests that the development of the asymptotic theory shall hinge on the normality assumption. Therefore an appropriate quasi-likelihood ratio unit root test based on Gaussian likelihood could be considered for a long memory GARMA(0,d,0) model. Jansson and Nielsen (2012) suggests the practical importance in exploring the power properties of their quasi-likelihood ratio test utilizing models with nuisance parameters and/or serial correlation. In that context as an extension to the established work of Jansson and Nielsen (2012) the construction of Jansson-Nielsen unit root tests in terms of u and d based on quasi likelihood ratios (QLR's) are presented in the next section.

6.2. Construction of Jansson-Nielsen type tests for u and d

Consider the GARMA(0,d,0) process defined in terms of (2.10) with ϵ_t as white noise.

It satisfies the criteria of Jansson-Nielsen type tests in the sense that it could be written in the form:

$(1-\nu B)\tau(B)X_t = \epsilon_t$,

where $1-\nu B = \frac{\phi(B)}{\theta(B)} = 1$ and $\tau(B) = (1-2uB+B^2)^d$,

in accordance with equation (4) of Jansson and Nielsen (2012).

Let $(1-2uB+B^2)^d = [2(u-1)(1-B)+(1-B)^2-2(u-1)]^d = [2(u-1)\Delta+\Delta^2-2(u-1)]^d$, where $\Delta = (1-B)$.

Hence for a fixed d at $0.25 \leq d < 0.5$;

it becomes;

$[2(u-1)(\Delta-1)+\Delta^2]^d X_t = \epsilon_t$.

Testing for u:

In order to test a null hypothesis of $H_0 : u = 1$ against an alternative hypothesis of $H_1 : u < 1$, we let $u = \phi + 1$ with $|u| < 1$ implying $\phi \in (-2, 0)$, to obtain the model:

$$[\Delta^2 + 2\phi(\Delta - 1)]^d X_t = \epsilon_t, \tag{6.1}$$

Under the null hypothesis, of $\phi = 0$ or $u = 1$ we get:

$$\Delta^{2d} X_t = \epsilon_t, \tag{6.2}$$

corresponding to X_t being a $FI(2d)$ process.

If $\phi < 0$, or $u < 1$ we get:

$$[\Delta^2 - 2\phi B]^d X_t = \epsilon_t. \tag{6.3}$$

Therefore in order to formulate a test statistic in terms of the standard QLR testing of $H_0 : u = 1$ against $H_1 : u < 1$ is equivalent to the test:

$$H_0 : \phi = 0. \tag{6.4}$$

$$H_1 : \phi < 0. \tag{6.5}$$

Note: This test will assess if the model depicts standard long memory or a GARMA process with long memory.

Since a QLR test has been formulated in terms of the parameter u, a quasi-likelihood ratio-type test statistic based in line with the result of Jansson and Nielsen (2012) will have the form:

$$LR_n^u = \max_{\bar{u} \leq 1} [\ell_{\sigma^2}(d, \bar{u})] - \max[\ell_{\sigma^2}(d, 1)], \tag{6.6}$$

where $\bar{u}$ is the mean of u within the estimation profile domain, and $\ell_{\sigma^2}(\bullet, \bullet)$ is from (3.8).

Testing for d:

Consider the GARMA(0,d,0) process model defined in terms of (2.10).

It's Gegenbauer polynomial raised to the power d could be written as:

$$(1 - 2uB + B^2)^d = (1 - 2uB + B^2)^{d' - 1/2}, \tag{6.7}$$

with conditions $|u| < 1$, $d' \in (0.5, 1.0)$ and $d = d' - 1/2$.

Therefore in order to formulate a test statistic in terms of a standard QLR test following hypotheses could be proposed:

Null hypothesis

$$H_0 : d' = 1, \tag{6.8}$$

against the alternative of

$$H_1 : d' < 1. \tag{6.9}$$

Since a QLR test has been formulated in terms of the parameter d' a quasi-likelihood ratio-type test statistic based in line with the result of Jansson and Nielsen (2012) will have the form:

$$LR_n^{d'} = \max_{\bar{d'} \leq 1} [\ell_{\sigma^2}(\bar{d'} - 1/2, u)] - \max[\ell_{\sigma^2}(1, u)], \tag{6.10}$$

where $\bar{d}'$ is the mean of d' within the estimation profile domain, and $\ell_{\sigma^2}(\bullet,\bullet)$ is from (3.8).

This test will assess if the model is non-stationary or stationary for a given parameter profile. Asymptotics related to the state space modelling based estimation of long memory Gegenbauer processes in creating the Jansson-Nielsen QLR tests are presented in the next section based on the work of Chan and Palma (1998).

6.3. Asymptotic Properties of State Space based QMLE Estimation

Initially, prior to stating the main theorems, we introduce some definitions, regularity conditions, and notation. Let $\hat{\Theta}_{n,m} = (\hat{d},\hat{u},\hat{\sigma})'$ be the QMLE estimator that maximizes the approximate quasi log-likelihood of a truncated GARMA(0,d,0) model such that $\Theta = (d,u,\sigma_\epsilon)' = (\Theta_{01},\Theta_{02},\Theta_{03})'$ is a 3 dimensional true parameter estimate vector. Assume that the regularity conditions in Dahlhaus (1989) hold.

Let the partial derivatives be defined as:

$\nabla f(\Theta) = (\frac{\partial}{\partial\Theta_j} f(\Theta))_{j=1,\dots,r}$

and

$\nabla^2 f(\Theta) = (\frac{\partial^2}{\partial\Theta_j\partial\Theta_k} f(\Theta))_{j,k=1,\dots,r}$,

and the matrices for $i,j = 1,...,r$ are defined as:

$T_{\partial i} = T(\frac{\partial}{\partial\Theta_i} f_\Theta)$, $\quad T_{\partial i\partial j} = T(\frac{\partial^2}{\partial\Theta_i\partial\Theta_j} f_\Theta)$,

$A_i^{(1)} = T^{-1}T_{\partial i}T^{-1}$, $\quad A^{(1)} = T^{-1}T_\nabla T^{-1}$,

$\hat{A}^{(1)} = \hat{T}^{-1}\hat{T}_\nabla \hat{T}^{-1}$, $\quad \hat{A}_i^{(1)} = \hat{T}^{-1}\hat{T}_{\partial i}\hat{T}^{-1}$,

$A_{ij}^{(2)} = T^{-1}T_{\partial i}T^{-1}T_{\partial j}T^{-1}$, $\quad A^{(2)} = T^{-1}T_\nabla T^{-1}T_\nabla T^{-1}$,

$\hat{A}^{(2)} = \hat{T}^{-1}\hat{T}_\nabla \hat{T}^{-1}\hat{T}_\nabla \hat{T}^{-1}$, $\quad \hat{A}_{ij}^{(2)} = \hat{T}^{-1}\hat{T}_{\partial i}\hat{T}^{-1}\hat{T}_{\partial j}\hat{T}^{-1}$,

$A_{ij}^{(3)} = T^{-1}T_{\partial i\partial j}T^{-1}$, $\quad A^{(3)} = T^{-1}T_{\nabla^2}T^{-1}$,

$\hat{A}^{(3)} = \hat{T}^{-1}\hat{T}_{\nabla^2}\hat{T}^{-1}$, $\quad \hat{A}_{ij}^{(3)} = \hat{T}^{-1}\hat{T}_{\partial i\partial j}\hat{T}^{-1}$,

with $T_n(\Theta)$ being the covariance matrix of $(X_{1,m}, X_{2,m}, ..., X_{n,m})'$ for series length (n) such that $T = T_n(f_\Theta)$, $T_\nabla = T_n(\nabla f_\Theta)$, $T_{\nabla^2} = T_n(\nabla^2 f_\Theta)$, $\hat{T} = T_n(\hat{f}_{\Theta,m})$, $\hat{T}_\nabla = T_n(\nabla \hat{f}_{\Theta,m})$, $\hat{T}_{\nabla^2} = T_n(\nabla^2 \hat{f}_{\Theta,m})$, and $f(\bullet)$ denoting the spectrum defined in (3.2).

The three results shown next establish consistency, asymptotic normality and efficiency of truncated QMLE's in terms of GARMA(0,d,0) model parameters.

Theorem 6.1 (Consistency):

Assume that $m = n^\beta$ with $\beta > 0$, then as $n \to \infty$,

$\hat{\Theta}_{n,m} \to \Theta$ in probability.

Theorem 6.2 (Central Limit Theorem):

Suppose that $m = n^\beta$ with $\beta > 1/2$, then as $n \to \infty$,

$\sqrt(n)(\hat{\Theta}_{n,m} - \Theta) \to_{\mathcal{L}} N(0, \Sigma(\Theta))$,

where "$\to_{\mathcal{L}}$" denotes convergence in distribution and $\Sigma^{-1}(\Theta) = (\Sigma_{ij}^{-1}(\Theta))$ with

$$\Sigma_{ij}^{-1}(\Theta) = \frac{1}{4\pi}\int_{-\pi}^{\pi}\{\frac{\partial logk(\omega,\Theta)}{\partial\Theta_i}\}\{\frac{\partial logk(\omega,\Theta)}{\partial\Theta_j}\}d\omega,$$

and

$k(\omega, \Theta) = |\sum_{j=0}^{\infty}\psi_j(\Theta)e^{ij\omega}|^2$,

where ω is defined as *prediction error covariance estimate*.

Theorem 6.3 (Efficiency):

Assume that $m = n^\beta$ with $\beta > 1/2$, then $\hat{\Theta}_{n,m}$ is an *efficient estimator* of Θ_0.

Note: Next the proofs of Theorems 6.1-6.3 are provided.

Proofs of Theorems 6.1-6.3

Prior to proving theorems 6.1-6.3 certain auxiliary lemmas need to be established. It is done next with the help of a generic constant K.

Lemma 6.1 Let C_j be the coefficients given in (3.4), then for j large and $u > 0$,

$$|C_j(\Theta)| \le Kj^{d-2}.$$

Proof:$C(z) = (\vartheta(z)/\Phi(z))(1-z)^{-d+1} = \sum_{j=0}^{\infty}C_jz^j$. Define $\varphi(z) = (\vartheta(z)/\Phi(z)) = \sum_{k=0}^{\infty}\varphi_kz^k$ and $\zeta(z) = (1-z)^{-d+1} = \sum_{k=0}^{\infty}\zeta_kz^k$. The coefficients φ_k can be written [See page 92 of Brockwell and Davis (1991)] as:

$$\varphi_k = \sum_{i=1}^{m}\sum_{l=0}^{r_i-1}\alpha_{il}k^l\varepsilon_i^k, \quad k \ge \max(p, q+1) - p,$$

where m is the number of distinct roots of $\Phi(z)$,$|\varepsilon_i| < 1$ is the inverse of the ith root of

$\Phi(z)$ with multiplicity r_i and α_{il} are constants. Let $L \geq \max(p, q+1) - p$, then

$|C_j| = |\sum_{k=0}^{L} \varphi_k \zeta_{j-k} + \sum_{k=L+1}^{\infty} \varphi_k \zeta_{j-k}|$

$= |\sum_{k=0}^{L} \varphi_k \zeta_{j-k} + \sum_{k=L+1}^{\infty} [\sum_{i=1}^{m} \sum_{l=0}^{r_i-1} \alpha_{il} k^l \varepsilon_i^k] \zeta_{j-k}|$

$\leq \sum_{k=0}^{L} |\varphi_k||\zeta_{j-k}| + \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} |\alpha_{il}| [\sum_{k=L+1}^{\infty} k^l |\varepsilon_i|^k] |\zeta_{j-k}|]$.

Note: For $k \geq L$, there exists constants $c_l \geq 0$ and $0 < a_i < 1$ such that $k^l|\varepsilon_i|^k \leq c_l a_i^k$. This can be seen as follows. Let $a_i = |\varepsilon_i| + u_i$ where $u_i > 0$. Then $0 < a_i < 1$ (since $|\varepsilon_i| < 1$, it is always possible to find such an ϵ_i). For $l \geq 0$, $k^l(|\varepsilon_i|/a_i)^k \to 0$ as $k \to \infty$. Thus, for $k \geq L$, there is a constant c_l such that $k^l(|\varepsilon_i|/a_i)^k \leq c_l$. Which implies $k^l|\varepsilon_i|^k \leq c_l a_i^k$. Therefore,

$|C_j| \leq \sum_{k=0}^{L} |\varphi_k||\zeta_{j-k}| + \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} |\alpha_{il}| [\sum_{k=L+1}^{\infty} c_l a_i^k |\zeta_{j-k}|]$

$\leq \sum_{k=0}^{L} |\varphi_k||\zeta_{j-k}| + \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} |\alpha_{il}| c_l [\sum_{k=L+1}^{\infty} a_i^k |\zeta_{j-k}|]$

$= \sum_{k=0}^{L} |\varphi_k||\zeta_{j-k}| + |\zeta_j| \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} |\alpha_{il}| c_l [\sum_{k=L+1}^{\infty} a_i^k |P(k,j,d)]$,

where $\quad P(k,j,d) = \frac{\zeta_{j-k}}{\zeta_j} = \frac{j!(j-k+d-2)!}{(j-k)!(j+d-2)!}$,

since $\zeta_{j-k} = \frac{(j-k+d-2)!}{(j-k)!(d-2)!}$ and $\zeta_j = \frac{(j+d-2)!}{j!(d-2)!}$.

Thus,

$|C_j| \leq \sum_{k=0}^{L} |\varphi_k||\zeta_{j-k}| + |\zeta_j| \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} |\alpha_{il}| c_l F(1, -j, 2-d-j, a_i)$,

where F is the hypergeometric function [see Hosking (1981)]. Moreover, $|\zeta_{j-k}| \leq |\zeta_{j-L}|$, for $0 \leq k \leq L$. This implies

$|C_j| \leq |\zeta_{j-L}| \sum_{k=0}^{L} |\varphi_k| + |\zeta_j| \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} |\alpha_{il}| c_l F(1, -j, 2-d-j, a_i).$

As $j \to \infty$, $F(1, -j, 2-d-j, a_i) \to (1-a_i)^{-1}$ [see Hosking (1981)], and for j large, $|\zeta_j| \sim K j^{d-2}$ and $|\zeta_{j-L}| \sim K j^{d-2}$ (for fixed L). Therefore, for large j,

$|C_j| \leq K j^{d-2} [\sum_{k=0}^{L} |\varphi_k| + \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} \frac{|\alpha_{il}|}{1-a_i}].$

Since $[\sum_{k=0}^{L} |\varphi_k| + \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} \frac{|\alpha_{il}|}{1-a_i}]$ is a rational function of the parameter $\Theta \in \vartheta$, it is continuous. Given that the parameter space ϑ is compact, there is a constant K such that

$\sum_{k=0}^{L} |\varphi_k| + \sum_{i=1}^{m} \sum_{l=0}^{r_i-1} \frac{|\alpha_{il}|}{1-a_i} \leq K.$

Thus,

$|C_j| \leq K j^{d-2},$

and it completes the proof of Lemma 6.1.

Lemma 6.2 If $\{a_k\}$ is a sequence of numbers, then for large n,

$\frac{\sum_{k=n}^{\infty} k^{2d-4}}{n^{2d-3}} \leq K.$

Proof: Let $\beta = 2d - 3 < 0$. Then

$\lim_{n\to\infty} \frac{\sum_{k=n}^{\infty} k^{\beta-1}}{n^{\beta}} = \lim_{n\to\infty} \sum_{k=n}^{\infty} (k/n)^{\beta-1} \frac{1}{n}.$

$= \lim_{n\to\infty} \sum_{j=0}^{\infty} (j/n+1)^{\beta-1} \frac{1}{n}.$

By Polya and Szego [(1992), page 53], the last sum equals $\int_0^\infty (x+1)^{\beta-1}dx = 1/|\beta|$ and it completes the proof.

Lemma 6.3 Let $m = n^\beta$ with $\beta > 0$. Then as $n \to \infty$ uniformly in Θ, we have the following:

(i) For $\beta > 0, ||T^{-1} - \hat{T}^{-1}|| \to 0$,
(ii)for $\beta > 1/2, \sqrt(n)||A_i^{(1)} - \hat{A}_i^{(1)}|| \to 0, \quad$ for $\quad i = 1, ..., r$,
(iii)for $\beta > 1/2, ||A_{ij}^{(2)} - \hat{A}_{ij}^{(2)}|| \to 0, \quad$ for $\quad i, j = 1, ..., r$,
(iv)for $\beta > 1/2, ||A_{ij}^{(3)} - \hat{A}_{ij}^{(3)}|| \to 0, \quad$ for $\quad i, j = 1, ..., r$.

Proof: For (i) the matrix T satisfies $x'Tx \geq Kx'x$ uniformly in Θ, where K is a constant. From Lemma 6.1 it is evident that for $u > 0$, there exists n_0, independent of Θ, such that for $n \geq n_0$, $||T - \hat{T}|| \leq u$, where $||A||^2 = \sup_{x \in R^n}((x'AA'x)/x'x)$ is the square of the spectral norm of the matrix A. Thus, $|x'Tx - x'\hat{T}x| \leq ux'x$, and

$$x'\hat{T}x \geq x'Tx - ux'x \geq (k-u)x'x.$$

It follows that $||\hat{T}^{-1}|| \leq K$ uniformly in Θ. Using the property $||AB|| \leq ||A||||B||$ [see Dahlhaus (1989)], $||T^{-1} - \hat{T}^{-1}|| \leq ||T^{-1}||||\hat{T}^{-1}||||T - \hat{T}|| \leq K||T - \hat{T}||$. By using the theory in Lemma 6.1, it is evident that the last term tends to zero uniformly in Θ and hence completes the proof of (i).

For (ii) observe that

$$\sqrt(n)||A_i^{(1)} - \hat{A}_i^{(1)}|| = \sqrt(n)||T^{-1}T_{\partial i}T^{-1} - \hat{T}^{-1}\hat{T}_{\partial i}\hat{T}^{-1}||$$

$$\leq \sqrt(n)||T^{-1}T_{\partial i}T^{-1} - T^{-1}T_{\partial i}\hat{T}^{-1}||$$

$+\sqrt(n)||T^{-1}T_{\partial i}\hat{T}^{-1} - \hat{T}^{-1}T_{\partial i}\hat{T}^{-1}||$

$+\sqrt(n)||\hat{T}^{-1}(T_{\partial i} - \hat{T}_{\partial i})\hat{T}^{-1}||$

$\leq \sqrt(n)||T^{-1}T_{\partial i}||||T^{-1} - \hat{T}^{-1}||$

$+\sqrt(n)||\hat{T}^{-1}T_{\partial i}||||T^{-1} - \hat{T}^{-1}|| + \sqrt(n)||\hat{T}^{-1}||^2||T_{\partial i} - \hat{T}_{\partial i}||$

$\leq K\sqrt(n)||T^{-1} - \hat{T}^{-1}|| + K\sqrt(n)||T_{\partial i} - \hat{T}_{\partial i}||$

$\leq Kn^{(2d-2)\beta+1/2}$.

Therefore for $0 < d < 1/2$ and $\beta \geq 1/2$, uniform convergence is obtained completing the proof for (ii). The proofs of parts (iii) and (iv) are analogous to the proof of (ii).

Lemma 6.4 For $\beta \geq 1/2$, as $n \to \infty$, uniformly in Θ,

(i) $\frac{1}{\sqrt(n)}|tr[T^{-1}T_{\partial i} - \hat{T}^{-1}\hat{T}_{\partial i}]| \to 0$ for $i = 1, ..., r$,

(ii) $\frac{1}{n}|tr[T^{-1}T_{\partial i}T^{-1}T_{\partial j} - \hat{T}^{-1}\hat{T}_{\partial i}\hat{T}^{-1}\hat{T}_{\partial j}]| \to 0$ for $i, j = 1, ..., r$,

(iii) $\frac{1}{\sqrt(n)}|tr[T^{-1}T_{\partial i\partial j} - \hat{T}^{-1}\hat{T}_{\partial i\partial j}]| \to 0$ for $i, j = 1, ..., r$.

Proof: For (i) observe that

$\frac{1}{\sqrt(n)}|tr[T^{-1}T_{\partial i} - \hat{T}^{-1}\hat{T}_{\partial i}]| \leq \frac{1}{\sqrt(n)}|tr[T^{-1}T_{\partial i} - T^{-1}\hat{T}_{\partial i}]|$

$+\frac{1}{\sqrt(n)}|tr[(T^{-1} - \hat{T}^{-1})\hat{T}_{\partial i}]|$

$\leq \frac{1}{\sqrt(n)}||T^{-1}||||T_{\partial i} - \hat{T}_{\partial i}|| + \frac{1}{\sqrt(n)}||T^{-1} - \hat{T}^{-1}||||\hat{T}_{\partial i}||$

$\leq K\sqrt(n)||T_{\partial i} - \hat{T}_{\partial i}]|| + K\sqrt(n)||T^{-1} - \hat{T}^{-1}||$

$\leq Kn^{(2d-2)\beta+1/2}$.

Therefore from an extension of Lemma 6.3(i) for $0 < d < 1/2$ and $\beta \geq 1/2$, Lemma 6.4(i) holds. The proofs of parts 6.4(ii) and 6.4(iii) are analogous to proof of 6.4(i).

Lemma 6.5 Let $m = n^{\beta}$ with $\beta > 0$. Then uniformly in Θ,

$\lim_{n\to\infty} \frac{1}{n} log[det\{\hat{T}T^{-1}\}] = 0$

Proof:

$\frac{1}{n} log[det\{\hat{T}T^{-1}\}] \leq log\{\frac{1}{n} tr\{TT^{-1}\}\} = log\{\frac{1}{n} tr\{T^{-1}[\hat{T} - T]\} + 1\}$.

Since $T^{-1} \leq KI_n$ uniformly in Θ,

$|\{\frac{1}{n} tr\{T^{-1}[\hat{T} - T]\}\}| \leq K|\{\frac{1}{n} tr\{\hat{T} - T\}\}| = K\sum_{k=m+1}^{\infty} C_k^2(\Theta)$.

From Lemma 6.1, $\sum_{k=m+1}^{\infty} k^{2d-4} \leq Km^{2d-3}$. Therefore, $\sum_{k=m+1}^{\infty} C_k^2(\Theta) \leq Km^{2d-3} = Kn^{\beta(2d-3)}$. Since $2d - 3 < 0$ and $\beta > 0$, $n^{\beta(2d-3)} \to 0$ as $n \to \infty$. Thus it completes the proof.

Lemma 6.6 Let $m = n^{\beta}$ with $\beta > 0$. Then as $n \to \infty$,

$\sup_{\Theta} |\hat{L}_n(\Theta) - L_n(\Theta)| \to 0 \quad$ a.s.

Proof:

Observe that

$|\hat{L}_n(\Theta) - L_n(\Theta)| \leq \frac{1}{2n} log[det\{\hat{T}T^{-1}\}] + \frac{1}{2n} Z_n' Z_n ||\hat{T}^{-1} - T^{-1}||.$

From Lemma 6.5, $\frac{1}{2n} log[det\{\hat{T}T^{-1}\}] \to 0$ and from Hannan(1979, page 133), $\frac{1}{2n} Z_n' Z_n \to \gamma_0/2$ a.s. as $n \to \infty$ where γ_0 is the variance of the process $\{Z_t\}$. From Lemma 6.3(i), $||\hat{T}^{-1} - T^{-1}|| \to 0$ uniformly in Θ as $n \to \infty$. It completes the proof of our result.

Proof of Theorem 6.1 (Consistency): From Lemma 6.6, $\sup_\Theta |\hat{L}_n(\Theta) - L_n(\Theta)| \to 0$ a.s., as $n \to \infty$. Therefore, clearly

$-\hat{L}_n(\Theta) \leq |\hat{L}_n(\Theta) - L_n(\Theta)| - L_n(\Theta).$

Then,

$\sup_\Theta \{-\hat{L}_n(\Theta)\} \leq \sup_\Theta |\hat{L}_n(\Theta) - L_n(\Theta)| + \sup_\Theta \{-L_n(\Theta)\}.$

Equivalently,

$-\inf_\Theta \hat{L}_n(\Theta) \leq \sup_\Theta |\hat{L}_n(\Theta) - L_n(\Theta)| - \inf_\Theta L_n(\Theta),$

or

$\inf_\Theta L_n(\Theta) - \inf_\Theta \hat{L}_n(\Theta) \leq \sup_\Theta |\hat{L}_n(\Theta) - L_n(\Theta)|.$

Similarly,

$\inf_\Theta \hat{L}_n(\Theta) - \inf_\Theta L_n(\Theta) \leq \sup_\Theta |\hat{L}_n(\Theta) - L_n(\Theta)|$.

Thus,

$|\inf_\Theta \hat{L}_n(\Theta) - \inf_\Theta L_n(\Theta)| \leq \sup_\Theta |\hat{L}_n(\Theta) - L_n(\Theta)|$.

Therefore,

$$|\inf_\Theta \hat{L}_n(\Theta) - \inf_\Theta L_n(\Theta)| \to 0 \quad a.s., \quad as \quad n \to \infty. \tag{6.11}$$

Let $U(\Theta_0)$ be a neighborhood of Θ_0 with radius d_0. Using the triangle inequality we get,

$|\inf_\Theta \hat{L}_n(\Theta) - \inf_{U(\Theta_0)} L_n(\Theta)| \leq |\inf_\Theta \hat{L}_n(\Theta) - \inf_\Theta L_n(\Theta)| + |\inf_\Theta L_n(\Theta) - \inf_{U(\Theta_0)} L_n(\Theta)|$

Based on equation (6.11), first term on the right hand side converges to zero a.s., and hence in probability. On the other hand $|\inf_\Theta L_n(\Theta) - \inf_{U(\Theta_0)} L_n(\Theta)| \to 0$, in probability, since the QMLE, $\hat{\Theta}_n \to \Theta_0$ in probability, as per Theorem 3.1 of Dahlhaus (1989). Thus,

$|\inf_\Theta \hat{L}_n(\Theta) - \inf_{U(\Theta_0)} L_n(\Theta)| \to 0$ in probability,

implying $\Theta_{n,m} \to \Theta_0$ in probability as required.

Proof of Theorem 6.2 (Central Limit Theorem): It suffices to prove that:

(i) $\sup_{\Theta\in\vartheta} \sqrt(n)|\nabla L_n(\Theta) - \nabla\hat{L}_n(\Theta)| \to 0$ a.s.,

(ii) $\sup_{\Theta\in\vartheta} |\nabla^2 L_n(\Theta) - \nabla^2 \hat{L}_n(\Theta)| \to 0$ a.s.,

(i) The gradient of $L_n(\Theta)$ can be written as:

$\nabla L_n(\Theta) = \frac{1}{2n} tr[T^{-1}T_\nabla] - \frac{1}{2n} Z_n' A^{(1)} Z_n,$

and

$\nabla \hat{L}_n(\Theta) = \frac{1}{2n} tr[\hat{T}^{-1}\hat{T}_\nabla] - \frac{1}{2n} Z_n' \hat{A}^{(1)} Z_n.$

Hence,

$\sqrt(n)|\nabla L_n(\Theta) - \nabla \hat{L}_n(\Theta)| \le \frac{1}{2\sqrt(n)} |tr[T^{-1}T_\nabla - \hat{T}^{-1}\hat{T}_\nabla]| + \frac{Z_n' Z_n}{2n} \sqrt(n)||A^{(1)} - \hat{A}^{(1)}||.$

Observe that

$\frac{1}{2\sqrt(n)} |tr[T^{-1}T_\nabla - \hat{T}^{-1}\hat{T}_\nabla]| = \frac{1}{2\sqrt(n)} (\sum_{i=1}^{r} |tr[T^{-1}T_{\partial i} - \hat{T}^{-1}\hat{T}_{\partial i}]|^2)^{1/2}.$

From Lemma 6.4(i), $\frac{1}{2\sqrt(n)} |tr[T^{-1}T_{\partial i}] - tr[\hat{T}^{-1}\hat{T}_{\partial i}]|$ goes to zero uniformly in Θ, as $n \to \infty$. Hence, the same result holds for $\frac{1}{2\sqrt(n)} |tr[T^{-1}T_\nabla - tr\hat{T}^{-1}\hat{T}_\nabla]|$. On the other hand, $\frac{Z_n' Z_n}{2n}$ is asymptotically equal to $\gamma_0/2$ and

$\sqrt(n)||A^{(1)} - \hat{A}^{(1)}|| = \sqrt(n)(\sum_{i=1}^{r} ||A_i^{(1)} - \hat{A}_i^{(1)}||^2)^{1/2}.$

By Lemma 6.3(ii), for $i = 1, ..., r$, $\sqrt(n)||A_i^{(1)} - \hat{A}_i^{(1)}|| \to 0$ as $n \to \infty$ uniformly in Θ. Thus (i) of Theorem 6.2 is established.

(ii) The second derivatives of $L_n(\Theta)$ can be written as

$\nabla^2 L_n(\Theta) = -\frac{1}{2n} tr[T^{-1} T_{\nabla} T^{-1} T_{\nabla}] + \frac{1}{2n} tr[T^{-1} T_{\nabla^2}]$

$+\frac{1}{2n} X_n' A^{(2)} X_n - \frac{1}{2n} Z_n' A^{(3)} Z_n$

and

$\nabla^2 \hat{L}_n(\Theta) = -\frac{1}{2n} tr[\hat{T}^{-1} \hat{T}_{\nabla} \hat{T}^{-1} \hat{T}_{\nabla}] + \frac{1}{2n} tr[\hat{T}^{-1} \hat{T}_{\nabla^2}]$

$+\frac{1}{2n} X_n' \hat{A}^{(2)} X_n - \frac{1}{2n} Z_n' \hat{A}^{(3)} Z_n$

Hence

$|\nabla^2 L(\Theta) - \nabla^2 \hat{L}(\Theta)| \leq \frac{1}{2n} |tr\{T^{-1} T_{\nabla} T^{-1} T_{\nabla} - \hat{T}^{-1} \hat{T}_{\nabla} \hat{T}^{-1} \hat{T}_{\nabla}\}|$

$+\frac{1}{2n} |tr\{T^{-1} T_{\nabla^2} - \hat{T}^{-1} \hat{T}_{\nabla^2}\}|$

$+\frac{1}{2n} X_n' X_n ||A^{(2)} - \hat{A}^{(2)}|| + \frac{1}{2n} Z_n' Z_n ||A^{(3)} - \hat{A}^{(3)}||$

$= I + II + III, \quad say.$

Now

$I = \frac{1}{2} (\sum_{i=1}^{r} [\frac{1}{n} |tr\{T^{-1} T_{\partial i} T^{-1} T_{\partial i} - \hat{T}^{-1} \hat{T}_{\partial i} \hat{T}^{-1} \hat{T}_{\partial i}\}|]^2)^{1/2}.$

By Lemma 6.4(ii), this term converges to zero for $\beta \geq 1/2$ uniformly in Θ. Also,

$II = \frac{1}{2n} |\sum_{i=1}^{r} tr\{T^{-1} T_{\partial i \partial i} - \hat{T}^{-1} \hat{T}_{\partial i \partial i}\}|$

$\leq \frac{1}{2}\sum_{i=1}^{r}\frac{1}{n}|tr\{T^{-1}T_{\partial i\partial i}-\hat{T}^{-1}\hat{T}_{\partial i\partial i}\}|$,

and by virtue of Lemma 6.4(iii), the right hand side converges to zero for $\beta \geq 1/2$ uniformly in Θ. Finally, since $Z_n'Z_n/2n \sim \gamma_0/2$, $||A^{(2)}-\hat{A}^{(2)}|| = (\sum_{i,j=1}^{r}||A_{ij}^{(2)}-\hat{A}_{ij}^{(2)}||^2)^{1/2} \to 0$ as $n\to\infty$ from Lemma 6.3(iii), and $||A^{(3)}-\hat{A}^{(3)}|| = (\sum_{i,j=1}^{r}||A_{ij}^{(3)}-\hat{A}_{ij}^{(3)}||^2)^{1/2} \to 0$ according to Lemma 6.3(iv), $III \to 0$ and hence (ii) is established completing the proof.

Proof of Theorem 6.3 (Efficiency): The proof is based on the Fisher information matrices $I_n(\Theta_0)$ and $\hat{I}_n(\Theta_0)$ of the exact and truncated log likelihood models evaluated at Θ_0 respectively.

Therefore,

$$(1/n)I_n(\theta_0)-(1/n)\hat{I}_n(\theta_0) = ntr\{T^{-1}T_{\nabla}T^{-1}T_{\nabla}-\hat{T}^{-1}\hat{T}_{\nabla}\hat{T}^{-1}\hat{T}_{\nabla}\}$$

.

Where

$$T = T_n(f)$$

and

$$T_{\nabla} = T_n(\nabla f)$$

with f being the spectral density of the process.

By applying Lemma 6.4(ii) yields

$$\lim_{n\to\infty}\{(1/n)I_n(\theta_0-(1/n)\hat{I}_n(\theta_0\} = 0$$

.

Finally, by applying Theorem 5.1 of Dahlhaus (1989),

$$\lim_{n\to\infty}(1/n)\hat{I}_n(\theta_0) = \Gamma(\theta_0)$$

,

completing the proof.

Assessing the strength of the above tests in terms of their power is a worthwhile exercise. Especially, since it could provide a remedy to the perception that general unit root tests with an $FI(d)$ alternative hypothesis lacks power. It becomes the topic of interest in the next section.

6.4. Power of Tests

Monte Carlo experiments were done to assess the $power$ of the tests introduced for varying combinations of parameters. The results are illustrated in tables 6.1-6.2. Comparative assessments of the power with respect to the series lengths are provided in figures 6.1-6.2.

TABLE 6.1. The power of Testing for u with 1000 replications for H_0 : $u = 1$, $H_1 : u < 1$

$Power$					
n	$(d = 0.1)$	$(d = 0.2)$	$(d = 0.3)$	$(d = 0.4)$	$(d = 0.45)$
100	71.9%	74.5%	79.1%	79.7%	79.9%
200	81.3%	84.8%	89.2%	89.3%	89.8%

TABLE 6.2. The power of Testing for d with 1000 replications for H_0 : $d^{'} = 1$, $H_1 : d^{'} < 1$

$Power$					
n	$(d^{'} = 0.6)$	$(d^{'} = 0.7)$	$(d^{'} = 0.8)$	$(d^{'} = 0.9)$	$(d^{'} = 0.95)$
100	50.8%	69.5%	73.8%	77.2%	79.7%
200	70.8%	74.3%	82.6%	83.1%	83.9%

Remark: From the two figures 6.1 and 6.2 it is evident that in both the tests an increase in the series length results in a corresponding increase of the power.

Note: It is clear that the notion of a general unit root test with $FI(d)$ alternative hypothesis lacking power as mentioned by Hassler and Wolters (1994), Diebold et.al (1991) and

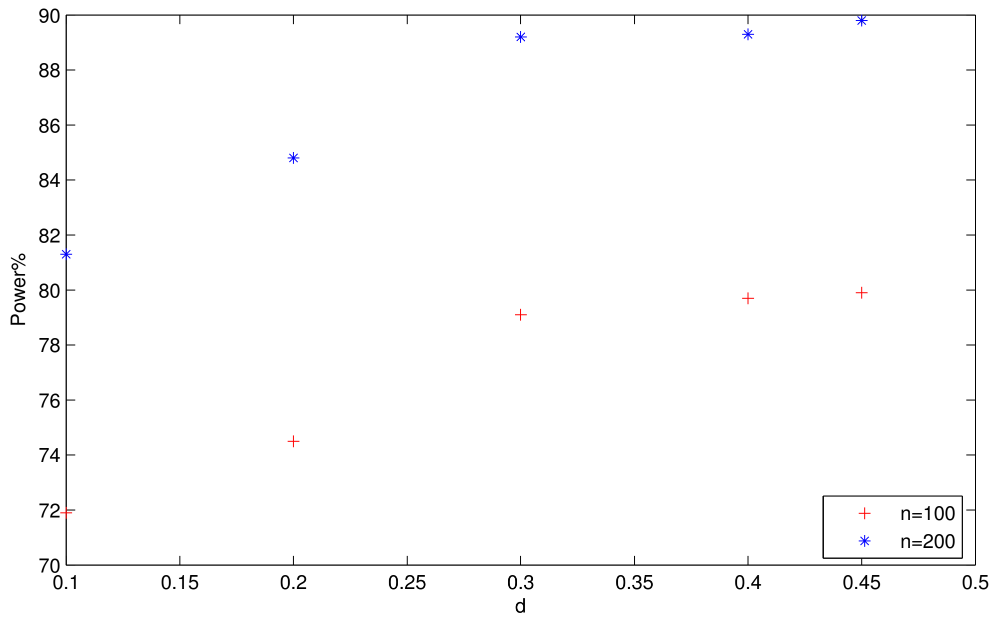


FIGURE 6.1. Power comparison in testing u for $n = 100$ & $n = 200$ with 1000 replications

Lee and Schmidt (1996) is addressed to a great extent by the tests discussed in this chapter with minimal hassle. The values obtained for the power of the tests are reasonable and comparable with similar results for the same model in Beaumont and Ramachandran (2001). Furthermore in a practical sense adequately powerful tests introduced in this chapter will be appropriate and feasible tools to employ on potential applications such as "international interest rates". In such a context a GARMA(0,d,0) model would be a best fit as shown by Beaumont and Ramachandran (2001). For testing the fractional order of seasonal and non-seasonal unit roots of long memory processes see Ferrara et. al (2010).

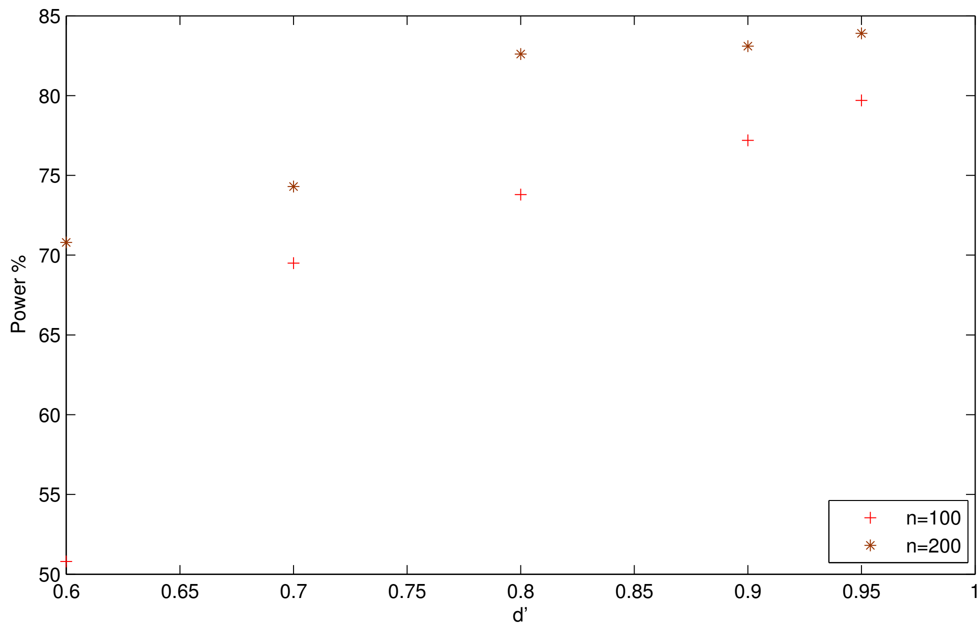


FIGURE 6.2. Power comparison in testing d for $n = 100$ & $n = 200$ with 1000 replications

By considering all the facts given in the chapter concluding remarks are provided in the next section.

6.5. Concluding Remarks

A "nearly efficient unit root testing procedure" for QMLE's derived from a state space model of a long memory Gegenbauer process is introduced. It is utilized as a dual purpose mechanism to assess stationarity versus non-stationarity and standard long memory versus long memory Gegenbauer process attributes. Asymptotic properties of the QMLE estimator of a long memory GARMA(0,d,0) process is provided as an additional contribution. The efficiency of the test is corroborated by the Monte Carlo evidence of its power, in refuting an established notion of generic unit root tests with fractionally integrated alternative hypotheses are consistent with a fairly low power.

In line with the thesis outline a culminating essence of chapters 1-6 is provided in chapter 7 titled as "Discussion and Further Research".

CHAPTER 7

Discussion and Further Research

7.1. Introduction

It is evident from the material presented in chapters 1-6 that there are significant contributions to time series literature. Furthermore, the entire research endeavour covered in chapters 1-6 provide useful insights into the research problem and paves the way for further research that may yield fruitful results in future. In lieu of it, this chapter is segmented into three sections as presented below.

7.2. Original Contributions

All creative components of this research endeavour resulting in this thesis is a cumulative product of a series of refereed and submitted publications and working papers. The results of which could be found in Dissanayake and Peiris (2011), Dissanayake et.al. (2014a), Dissanayake et.al.(2014b), Dissanayake and Peiris (2015), Dissanayake, Peiris and Proietti (2015a), Dissanayake, Peiris and Proietti (2015b) and Dissanayake, Peiris, Proietti and Wang (2015).

The contributions of this research endeavour are outlined as follows:

- Develop the theory of estimation for the family of fractionally differenced GARMA models with long memory.
- Create a state space configuration of a GARMA(0,d,0) model.
- Establish an optimal lag order for a GARMA(0,d,0) model.
- AR & MA approximations of GARMA(0,d.0) model.
- Establish the asymptotic variance for d, the long memory parameter of a GARMA(0,d,0)

series through MA representation.

- Meta-analytical comparative analysis of the proposed GARMA(0,d,0) model estimators with other similar traditional/hybrid time series estimates in the literature. Proving that the proposed methodology is superior to conventional estimation and prediction mechanisms and comparable with hybrid model estimators propelled by neural networks.
- Formulate of a GARMA(0,d,0)-GARCH(1,1) model and related theoretical aspects.
- Extend the state space mechanism developed for the GARMA(0,d,0) model to the GARMA(0,d,0)-GARCH(1,1) series.
- Introduce an optimal lag order for estimation of a GARMA(0,d,0)-GARCH(1,1) model.
- Establish an asymptotic variance estimate for long memory parameter d of a GARMA(0,d,0)-GARCH(1,1) series through QMLE estimation using MA representation.
- Establish the QMLE estimates of a special case non-stationary GARMA(0,d,0)-GARCH(1,1) model to show they outperform traditional MLE estimates of a similar model available in the literature in terms of RMSE's.
- Provide empirical evidence to prove that the GARMA(0,d,0)-GARCH(1,1) series and the related state space methodology is feasible for forecasting in terms of forecast MSE's.
- Introduce nearly efficient unit root tests for state space based long memory GARMA(0,d,0) series with an acceptable power refuting the claim in the literature of serially correlated time series returning a low power.
- Extend the introduced state space methodology to GARSMA(0,d,0)x(0,$D_{s'}$,0) model with a seasonal filter in establishing an optimal lag order for small sample series.

7.3. Observations

In executing the research, the following observations were made and addressed:

- It was observed that the likelihood function was a monotonically increasing function of m, and that the change of it was tiny as m got closer to the optimal m.
- In estimating the long memory parameter d, as m increased a typical bias-variance trade-off was observed (with the bias decreasing and the variance increasing with m).
- The simulation experiments of chapter 3 show that the optimal m (i.e. minimising the

mean square estimation and prediction error for the purpose of estimating d and prediction) is rather insensitive to the lag order of MA approximation.

- For the MA approximation (with reference to chapter 3) , it can be observed that the standard error does not vary relevantly and the variation is thereby due to the Monte Carlo simulation error.
- In the results of chapter 3, the AR estimator seems much more unreliable and unstable justifying the variation observed, since as d increases the reliability of the AR approximation decreases.
- In all Monte Carlo experiments for GARMA(0,d,0), GARSMA(0,d,0)x(0,$D_{s'}$,0) and GARMA(0,d,0)-GARCH(1,1) models the optimal m of the AR approximation was reached with a smaller number of lags as opposed to the MA approximation. But in all experiments a smaller total estimator MSE was returned by the MA approximation making it more feasible.
- When the sample size of a GARMA(0,d,0) series was increased it resulted in a greater power of the QLR tests.
- Adding the seasonal filter to a GARMA(0,d,0) series to create a GARSMA(0,d,0)x(0,$D_{s'}$,0) model by extending the state space methodology resulted in the retardation of the processing speed of the simulations.

7.4. Further Research

The research done in developing the class of GARMA processes together with the estimation, prediction and testing of the parameters utilizing state space modelling and the KF as an alternative mechanism could be extended to other models such as continuous ARMA (CARMA) series. In addition, further research with respect to large samples of the GARSMA(0,d,0(x(0,$D_{s'}$,0) model will no doubt return rich results in the future. Additionally, the simulations could be extended to quarterly, bi-annual, trimester and annual seasonal periods to obtain comprehensive results. It will enable time series researchers to compare time reliant MA and AR representation based QML estimators with frequency based counterparts such as Whittle's and Smooth Periodogram estimators. The nearly efficient unit root tests introduced in this thesis could also be applied to GARSMA(0,d,0)x(0,$D_{s'}$,0) and GARMA(0,d,0)-GARCH(1,1) series in making an assessment of the power. A comparative assessment of the new class and other models in terms of the methodology introduced in this thesis will be another interesting endeavour. Additionally, the conceptual paradigms proposed could be utilized to assess a variety of real applications that occur in nature and society.

Bibliography

Abrahams, M and Dempster, A. (1979) Research on seasonal analysis. progress report on the asa/census project on seasonal adjustment. Technical report. Department of Statistics, Harvard University, Boston, MA.

Andel, J. (1986). Long memory time series models. *Kybernetika*, **22(2)**, 105-123.

Andersen, T. G., Bollerslev, T., Diebold, F. X. and Ebens, H. (2001a). The distribution of realized stock return volatility. *Journal of Financial Economics*, **61**, 43-76.

Andersen, T. G., Bollerslev, T., Diebold, F. X. and Labys, P. (2001b). The distribution of realized exchange rate volatility. *Journal of American Statistical Association*, **96**, 42-55.

Anderson, B. D. O. and Moore, J. B. (1979). *Optimal Filtering*. Prentice-Hall, New York.

Anh, V. V., Angulo, J. M. and Ruiz-Medina, M. D. (1999). Possible long-range dependence in fractional random fields. *Journal of Statistical Planning and Inference*, **80(1-2)**, 95-110.

Aoki, M. (1990). *State Space Modeling of Time Series.* Springer, Berlin.

Arteche, J. and Robinson, P. M. (2000). Semiparametric inference in seasonal and cyclical long memory processes. *Journal of Time Series Analysis*, **21**, 1-25.

Arteche, J. (2007). The Analysis of Seasonal Long Memory: The Case of Spanish Inflation.

Oxford Bulletin of Economics and Statistics, **69**, 749-772.

Arteche, J. (2012). Standard and Seasonal Long Memory in Volatility: An application to Spanish Inflation. *Empirical Economics*, **42**, 693-712.

Baillie, R. T. (1996). Long Memory Processes and Fractional Integration in Econometrics. *Journal of Econometrics*, **73**, 5-59.

Baillie, R. T., Bollerslev, T. and Mikkelsen, H. O. (1996a). Fractionally Integrated Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, **74(1)**, 3-30.

Baillie, R. T.,Chung, C. F. and Tieslau, M. A. (1996). Analysing inflation by the fractionally integrated ARFIMA-GARCH model. *Journal of Applied Econometrics*, **11**, 23-40.

Beaumont, P. and Ramachandran, R. (2001). Robust Estimation of GARMA Model Parameters with an Application to Cointegration among Interest Rates of Industrialized Countries. *Computational Economics*, **17**, 179-201.

Beran, J. (1994). *Statistics for Long-Memory Processes.* Chapman and Hall,New York.

Beran, J., Feng, Y., Ghosh, S. and Kulik, R. (2013). *Long-Memory Processes Probabilistic Properties and Statistical Methods,* Springer-Verlag Berlin Heidelberg.

Besse, P.C., Cardot, H. and Stephenson, D. B. (2000). Autoregressive Forecasting of Some Functional Climatic Variations. *Scandinavian Journal of Statistics*, **27(4)**, 673-687.

Bijari, M. and Khashei, M. (2011). A novel hybridization of artificial neural networks and ARIMA models for time series forecasting. *Applied Soft Computing*, **11 (2011)**, 2664-2675.

Bisaglia, L., Bordignon, S. and Lisi, F. (2003). K-Factor GARMA models for intraday volatility forecasting. *Applied Economics Letters*, **10(4)**, 256-264.

Bisognin, C. and Lopes, S. R. C. (2009). Properties of Seasonal Long Memory Processes. *Mathematical and Computer Modelling*, **49**, 1837-1851.

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*, **31**, 307-327.

Bos, C. S., Koopman, S. J. and Ooms, M. (2014). Long Memory with Stochastic Variance Model: A Recursive Analysis for US Inflation. *Computational Statistics and Data Analysis - The Annals of Computational and Financial Econometrics - Issue 2*, **76**, 144-157.

Box, G. E. P. and Jenkins, G. M. (1970). *Time Series Analysis: Forecasting and Control.* San Francisco: Holden-Day.

Brockwell, P.J. and Davis, R.A. Time Series: Theory and Methods. Springer-Verlag: New York, 1991.

Brockwell, P.J. and Davis, R.A. Introduction to Time Series and Forecasting. Springer: New York, 1996.

Chan, N. H. and Palma, W. (1998). State Space Modeling of Long-Memory Processes. *The Annals of Statistics*, **26(2)**, 719-740.

Chan, N. H. and Palma, W. (2006). Estimation of Long-Memory Time Series Models: a Survey of Different Likelihood-Based Methods. *Advances in Econometrics*, **20(2)**, 89-121.

Chen, G., Abraham, B. and Peiris, M. S. (1994). Lag Window Estimation of the degree of differencing in fractionally integrated time series models. *Journal of Time Series Analysis*, **15(5)**, 473-487.

Chung, C. -F. (1996). A Generalized Fractionally Integrated Autoregressive Moving-Average Process. *Journal of Time Series Analysis*, **17(2)**, 111-140.

Corsi, F. (2009). A Simple Approximate Long-Memory model of Realized Volatility. *Journal of Financial Econometrics*, **7(2)**, 174-196.

Dahlhaus, R. (1989). Efficient Parameter Estimation for Self Similar Processes. *The Annals of Statistics*, **17(4)**, 1749-1766.

Diebold, F.X.,Husted, S. and Rush, M. (1991). Real exchange rates under the gold standard. *Journal of Political Economy*, **99**, 1252-1271.

Diebold, F. X. and Rudebusch, G. D. (1991). On the Power of the Dickey-Fuller Tests against Fractional Alternatives. *Economic Letters*, **35**, 155-160.

Dissanayake, G. S. and Peiris, M. S. (2011). Generalized Fractional Processes with Conditional Heteroskedasticity. *Sri Lankan Journal of Applied Statistics*, **12**, 1-12.

Dissanayake, G.S. and Peiris, M.S. (2015). Fractionally Differenced Gegenbauer Processes with Long Memory-A Review. Working Paper, School of Mathematics and Statistics, University of Sydney, Australia.

Dissanayake, G. S., Peiris, M. S. and Proietti. T. (2014a). State Space Modeling of Gegenbauer Processes with Long Memory. Computational Statistics and Data Analysis- *The Annals of Computational and Financial Econometrics*,
http://dx.doi.org/10.1016/j.csda.2014.09.014

Dissanayake G.S., Peiris, M.S. and Proietti, T. (2014b) Estimation of Generalized Fractionally Differenced Processes with Conditionally Heteroskedastic Errors. International Work Conference on Time Series, Proceedings ITISE 2014, Ignacio Rojas Ruiz and Gonzalo Ruiz Garcia (ed.) , Copicentro Granada S L, ISBN 978-84-15814-97-9, 871âĂŞ890.

Dissanayake G.S., Peiris, M.S. and Proietti, T. (2015a) Modelling Generalized Persistence and Conditional Variance: An application to Financial Economics. Working Paper, School of Mathematics and Statistics, University of Sydney, Australia.

Dissanayake G.S., Peiris, M.S. and Proietti, T. (2015b) State Space Modeling of Seasonal Gegenbauer Processes with Long Memory. Working Paper, School of Mathematics and Statistics, University of Sydney, Australia.

Dissanayake, G. S., Peiris, M. S., Proietti, T. and Wang, Q. (2015) Optimal Testing and Asymptotics of a Long Memory Gegenbauer Process. Working Paper, School of Mathematics and Statistics, University of Sydney, Australia.

Dolado, J. J., Gonzalo, J. and Mayoral, L. (2002). A Fractional Dickey-Fuller Test for Unit Roots. *Econometrica*, **70(5)**, 1963-2006.

Durbin, J. and Koopman, S. J.(2001). *Time Series Analysis by State Space Methods.* Number 24 in Oxford Statistical Science Series. Oxford: Oxford University Press.

Engle, R.F. (1982) Autoregressive Conditional Heteroskedasticity with Estimates of Variance of U.K. Inflation. *Econometrica*, **50**, 987-1008.

Erdelyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1953). Higher Transcendental Functions, Vol II, Bateman Manuscript Project, McGraw and Hill.

Ferrara, L. and Guegan, D. (2001). Forecasting with k-factor Gegenbauer Processes: Theory and Applications. *Journal of Forecasting*, **20(8)**, 581-601.

Ferrara, L., Guegan, D. and Lu, Z. (2010). Testing Fractional Order of Long Memory Processes: A Monte Carlo Study. *Communications in Statistics - Simulation and Computation*, **39(4)**, 795-806.

Fox, R. and Taqqu, M. S.(1986). Large Sample Properties of Parameter Estimates for Strongly Dependent Stationary Gaussian Time Series. *The Annals of Statistics*, **14(2)**, 517-532.

Geweke, J. and Porter-Hudak, S. (1983). The estimation and application of long Memory time series models. *Journal of Time Series Analysis*, **4(4)**, 221-238.

Giraitis, L., Hidalgo, J. and Robinson, P. M. (2001). Gaussian estimation of parametric spectral density with unknown pole. *The Annals of Statistics*, **29**, 987-1023.

Giraitis, L. and Leipus, R. (1995) A Generalized Fractionally Differencing approach in Long Memory modeling. *Lithuanian Mathematical Journal*, **35(1)**, 65-81.

Giraitis, L., Koul, H. L. and Surgailis, D. (2012). *Large Sample Inference for Long Memory Processes.* London: Imperial College Press.

Gould, H. W. (1974) Coefficient Identities for Powers of Taylor and Dirichlet Series. *The American Mathematical Monthly*, **81(1)**, 3-14.

Gradshteyn, I. S., and Ryzhik, I. M. (1980) *Tables of Integrals Series and Products.* New York: Academic Press.

Granger, C. W. J. and Joyeux, R. (1980). An introduction to long-memory time series models and fractional differencing. *Journal of Time Series Analysis*, **1**, 15-29.

Grassi, S., Santucci de Magistris, P. (2014). When long memory meets the Kalman filter: A comparative Study. *Computational Statistics and Data Analysis*, **76**, 301-319.

Gray, H. L., Woodward, W. A. and Zhang, N. F. (1989). On generalized fractional processes. *Journal of Time Series Analysis*, **10**, 233-257.

Guegan, D. (2005). How can we define the Concept of Long Memory? An Econometric Survey. *Econometric Reviews*, **24(2)**, 113-149.

Hannan, E.J. (1979) The Central Limit Theorem for Time Series Regression.*Stochastic Processes and Their Applications*, 9, 281-289.

Harvey, A. C.(1989). *Forecasting, Structural Time Series Models and the Kalman Filter.* Cambridge: Cambridge University Press.

Harvey, A. C.and Proietti, T.(2005). *Readings in Unobserved Components Models.* Oxford: Oxford University Press.

Hassler, U. and Wolters, J. (1994). On the Power of Unit Root Tests against Fractional Alternatives. *Economics Letters*, **45**, 1-5.

Hassler, U. and Wolters, J. (1995). Long memory in inflation rates: International evidence. *Journal of Business and Economic Statistics*, **13**, 37-45.

Ho, K. W. and Houmani, H. (2010). *Investigation of GARCH models for the Estimation Power and Normality*. Master's Thesis. Lund University.

Hosking, J.R.M. (1981). Fractional Differencing. *Biometrika*, **68**, 165-176.

Hsu, N-J. and Tsai, H. (2009) Semiparametric Estimation for Seasonal Long Memory Time Series using Generalized Exponential Models. *Journal of Statistical Planning and Inference*, **139(6)**, 1992-2009.

Hualde, J. and Robinson, P. M. (2011). Gaussian Pseudo-Maximum Likelihood Estimation of Fractional Time Series Models. *Annals of Statistics*, **39(6)**, 3152-3181.

Jansson, M. and Nielsen, M. O., (2012). Nearly Efficient Likelihood Ratio Tests of the Unit Root Hypothesis. *Econometrica*, **80(5)**, 2321-2332.

Jones, R. H. (1980) Maximum Likelihood fitting of ARMA models to time series with missing observations. *Technometrics*, **22**, 389-395.

Kalman, R. E. (1961). A new approach to linear filtering and prediction problems. *Transactions of the American Society of Mechanical Engineers*, **83D**, 35-45.

Kalman, R. E. and Bucy, R. S. (1961). New results in linear filtering and prediction theory. *Transactions of the American Society of Mechanical Engineers*, **83**, 95-108.

Koopman, S. J., Ooms, M. and Carnero, M. A. (2007). Periodic Seasonal reg-arfima-garch models for daily electricity spot prices. *Journal of the American Statistical Association*, **102(477)**, 16-27.

Lee, D. and Schmidt, P. (1996). On the Power of the KPSS Test of Stationarity against Fractionally Integrated Alternatives. *Journal of Econometrics*, **73**, 285-302.

Ling, S. and Li, W. K. (1997) On Fractionally Integrated Autoregressive Moving-Average Time Series Models with Conditional Heteroscedasticity. *Journal of the American Statistical Association*, **92(439)**, 1184-1194.

Lieberman, O., Phillips, P. C. B. (2008). Refined inference on long memory in realized volatility. *Econometric Reviews*, **27**, 254-267.

Leipus, R. and Viano, M. (2000). Modelling Long-memory Time Series with Finite or Infinite Variance: a General Approach. *Journal of Time Series Analysis*, **21(1)**, 61-74.

Lobato, I. N. and Savin, N. E. (1998).Real and spurious long-memory properties of stock-market data. *Journal of Business and Economic Statistics*, **16**, 261-283.

Lu, Z. and Guegan, D. (2011). Estimation of Time-Varying Long Memory Parameter Using Wavelet Method. *Communications in Statistics - Simulation and Computation*, **40(4)**, 596-613.

Mandelbrot, B. B. and Van Ness, J. W.(1968). Fractional Brownian motions, fractional noises and applications.*SIAM Review*, 10, 423-437.

McAleer, M. and Medeiros, M. C. (2008). A Multiple regime smooth transition Heterogeneous Autoregressive model for long memory and asymmetries. *Journal of Econometrics*, **147(1)**, 104-119.

Montanari, A., Rosso, R. and Taqqu, M. S. (2000). A seasonal fractional ARIMA modelapplied to Nile River monthly flows at Aswan. *Water Resources Research*, **36**, 1249-1259.

Morris, M. J. (1977). Forecasting the Sunspot Cycle. *Journal of the Royal Statistical Society*, **140(4)**, 437-468.

Ohanissian, A., Russell, J. R. and Tsay, R. S. (2008). True or Spurious Long Memory? A New Test. *Journal of Business and Economic Statistics*, **26(2)**, 161-175.

Ooms, M. (1995). Flexible seasonal long memory and economic time series. Technical Report EI-9515/A. Econometric Institute, Erasmus University, Rotterdam.

Palma, W. (2007). *Long Memory Time Series Theory and Methods.* John Wiley and Sons, New Jersey.

Palma, W. and Chan, N. H. (2005). Efficient Estimation of Seasonal Long-Range-Dependent Processes. *Journal of Time Series Analysis*, **26(6)**, 863-892.

Pearlman, J. G., (1980). An Algorithm for the Exact Likelihood of a high-order autoregressive-moving average process. *Biometrika*, **67(1)**, 232-233.

Peiris, M. S. (2003) Improving the Quality of Forecasting using Generalized AR models: An application to Statistical Quality Control. *Statistical Methods*, **5(2)**, 156-171.

Peiris, S., Allen, D. and Peiris, U. Generalized Autoregressive Models with Conditional Heteroscedasticity: an application to financial time series modeling. *Proceedings of the Workshop on Research Methods: Statistics and Finance*, 75-83, 2005.

Peiris, S. and Thavaneswaran, A. (2007) An introduction to volatility models with indices. *Appl. Math. Lett.*, **20**, 177-182.

Phillips, P. C. B. and Xiao, Z. (1998). A Primer on Unit Root Testing. *Journal of Economic Surveys*, **12(5)**, 423-470.

Polya, G. and Szego, G. (1992).*Problems and Theorems in Analysis I.* Springer, New York.

Porter-Hudak, S. (1990). An application of the seasonal fractionally differenced model to the monetary aggregates. *Journal of the American Statistical Association, Applic. Case Studies*, **85**, 338-344.

Rainville, E.D. (1960). *Special functions.* Macmillan, New York.

Rangarajan, G. and Ding, M, editors. (2003). *Processes with Long-Range Correlations.* Springer, Berlin.

Ray, B. K. (1993). Modeling long-memory processes for optimal long-range prediction. *Journal of Time Series Analysis*, **14**, 511-525.

Reisen, V. (1994). Estimation of the fractional difference parameter in the ARIMA(p,d,q) model using the smoothed periodogram. *Journal of Time Series Analysis*, **15**, 335-315.

Reisen, V., Abraham, B. and Lopes, S. (2001). Estimation of parameters in ARFIMA processes: A simulation study. *Communications in Statistics - Simulation and Computation*, **30(4)**, 787-803.

Reisen, V.A, Rodrigues, A.L. and Palma, W. (2006). Estimation of Seasonal fractionally integrated processes. *Computational Statistics and Data Analysis*, **50(2)**, 568-582.

Robinson, P. M. (1991). Testing for strong serial correlation and dynamic conditional heteroskedasticity in multiple regression. *Journal of Econometrics*, **47**, 67-84.

Robinson, P. M. (1994). Efficient Tests of Non-stationary Hypotheses. *Journal of the American Statistical Association* **89**, 1420-1457.

Robinson, P. M. (1995). Log-periodogram regression of time series with long range dependence. *The Annals of Statistics* **23**, 1048-1072.

Robinson, P. M. (2003). *Long Memory Time Series,* Oxford University Press, Oxford.

Rossi, E., and Santucci de Magistris, P.(2014). Estimation of Long Memory in Integrated Variance. *Econometric Reviews* **33(7)**, 785-814.

Sekaran, U. (2006). *Research Methods for Business: A Skill Building Approach.* John Wiley and Sons, Inc.

Shephard, N. (1996). Statistical aspects of ARCH and stochastic volatility. In D.R. Cox, D.B. Hinkley, and O.E. Barndorff-Nielsen, editors, *Time Series Models: In Econometrics, Finance and Other Fields*. Chapman Hall, London.

Shitan, M. and Peiris, S. (2008). Generalised Autoregressive (GAR) model: a comparison of maximum likelihood and Whittle estimation procedures using a simulation study. *Commun. Statist. Simul. Comput.*, **37(3)**, 560-570.

Shitan, M. and Peiris, S. (2013). Approximate Asymptotic Variance-Covariance Matrix for the Whittle Estimators of GAR(1) Parameters. *Communications in Statistics - Theory and Methods*, **42(5)**, 756-770.

Shumway, R.H. and Stoffer, D.S. (2010). *Time Series Analysis and It's Applications with R Examples.* Springer Science + Business Media ,New York.

Sowell, F. (1992). Maximum likelihood estimation of stationary univariate fractionally integrated time series models. *Journal of Econometrics*, **53**, 165-188.

Taylor, A. M. R. (2005). Fluctuation Tests for a Change in Persistence. *Oxford Bulletin of Economics and Statistics*, **67(2)**, 207-230.

Teyssiere, G. and Kirman, A., editors. (2007). *Long Memory in Economics.* Springer, Berlin.

Tong, H. (1990). *Non-Linear Time Series: A Dynamical System Approach*, Oxford: Oxford University Press.

Velasco, C. and Robinson, P. M. (2000). Whittle pseudo-maximum likelihood estimation for nonstationary time series. *Journal of the American Statistical Association*, **95**, 1229-1243.

Waldmeier, M.(1961). *The Sunspot Activity in the Years 1610-1960,* Zurich: Zurich Schulthess.

Wang, Q., Lin, Y. and Gulati, C. M.(2003). Asymptotics for General Fractionally Integrated Processes with Applications to Unit Root Tests. *Econometric Theory*, **19(1)**, 143-164.

Wang, X., Smith, K. and Hyndman, R. J. (2006). Characteristic-Based Clustering for Time Series Data. *Data Mining and Knowledge Discovery*, **13**, 335-364.

Whittle, P. (1951). *Hypothesis Testing in Time Series.* Uppsala: Almqvist and Wiksells.

Woodward, W. A., Chen, Q. C. and Gray, H. L. (1998). A k-factor GARMA long-memory model. *Journal of Time Series Analysis*, **19**, 485-504.

Yajima, Y. (1985). On Estimation of Long-Memory Time Series Models. *Australian and New Zealand Journal of Statistics*, **27(3)**, 303-320.

Zhang, G. P. (2003). Time Series forecasting using a hybrid ARIMA and neural network model. *Neurocomputing*, **50**, 159-175.

Appendix

Some of the original computer source code created during the research project in R & MATLAB are provided.

1. R Computer Program Source Code used to create realizations, acf, pacf & sdf of GARMA(0,d,0) series driven by WN errors.

```
> set.seed(1)
> z=rnorm(500)
> coeff=numeric(499)
> t=numeric(500)
> y=numeric(500)
> for(n in 3:499)
+ l=0.1
+ x=0.8
+ coeff[1]=1
+ coeff[2]=0.16
+ coeff[n]=(((2*x)*(n+l-1)*coeff[n-1])-((n+(2*l)-2)*coeff[n-2]))/n
+
> t=c(1,coeff[1:499])
> M=NULL
> N=length(t)
> for(i in 1:N)
+ M=rbind(M,c(t[i:1],rep(0,N-i)))
+
```

> y=M

> ts.plot(y)

> acf=acf(y)

> pacf=acf(y,type="partial")

> omega=-314:314

> omega=omega/100

> u=0.8

> $sigma2 = (sd(z)^2)$

> $fxw = (sigma2/(2 * pi)) * ((4 * (cos(omega) - u)^2)^(- 1))$

> plot(omega,fxw,'l')

2. R Computer Program Source Code used to create realizations, acf, pacf & sdf of GARMA(0,d,0)-GARCH(1,1) series

> set.seed(1)

> require(TSA)

Loading required package: TSA

Loading required package: leaps

Loading required package: locfit

Loading required package: akima

Loading required package: lattice

locfit 1.5-6 2010-01-20

Loading required package: mgcv

This is mgcv 1.7-5. For overview type 'help("mgcv-package")'.

Loading required package: tseries

Loading required package: quadprog

Loading required package: zoo

Attaching package: 'zoo'

The following object(s) are masked from 'package:base':

as.Date

```
'tseries' version: 0.10-26
'tseries' is a package for time series analysis and computational finance.
See 'library(help="tseries")' for details.

Attaching package: 'TSA'
The following object(s) are masked from 'package:stats':
acf, arima
The following object(s) are masked from 'package:utils':
tar
> t=numeric(400)
> x=garch.sim(alpha=c(0.4,0.3),beta=c(0.3),n=1000)
> y=x[201:1000]
> p=numeric(400)
> coeff=numeric(800)
> g=numeric(400)
> for(j in 3:800)
+ l=0.45
+ v=0.8
+ coeff[1]=1
+ coeff[2]=0.72
+ coeff[j]=(((2*v)*(j+l-1)*coeff[j-1])-((j+(2*l)-2)*coeff[j-2]))/j
+
> cone=coeff[1:400]
> ctwo=coeff[401:800]
> k=1:400
> for(i in 401:800)
+ g[i]=sum(cone[k]*y[i-k])
+
> t=g[401:800]
```

```
> for(n in 1:400)
+ p[n]=sum(cone[k]*ctwo[n])
+
> Realization=ts.plot(t)
> sample - acf = acf(t)
> True - acf = acf(p)
> omega=-314:314
> omega=omega/100
> sigma2 = (sd(y)^2)
> fxw = (sigma2 * ((4 * cos(omega) - v)^2)^( - 0.45))
> spectrum=plot(omega,fxw,"l")
> postscript(file = "Colombo - plot1.eps", width = 12, height = 7)
> par(mfrow=c(3,2))
> ts.plot(t)
> acf(t)
> acf(p)
> acf(t,type="partial")
> plot(omega,fxw,"l")
> dev.off()
X11cairo
2
```

Source Code Program Segments 3-10 are based on the MA approximation.

3. MATLAB Computer Program Source Code used to create Simulated Series of GARMA(0,d,0) series driven by WN for State Space Modelling & Kalman Filtering.

```
Nrepl = 2000;
n = 2000;
drop = n;
d-true = 0.45;
eta-true = 0.8;
```

```
phi-true = 0;
theta-true = 0;
s-true=1;
l=acos(eta-true);
vphi = [1; -2*eta-true; 1];
vm = 5:5:75;
vhc = [];
ga=PowerMAcoeff(vphi, -d-true, n+drop-1);
for i=1:Nrepl
vy=filter(ga,1,randn(n+drop,1));
vy=vy(drop+1:end);
vy = vy - mean(vy);
vhc = horzcat(vhc,vy);
disp(i);
end
disp(vhc(1:10,:));
plot(vhc);
autocorr(vhc(:,1));
save('datafile');
```

4. MATLAB Main Computer Program Source Code used for State Space Modelling & Kalman Filtering for GARMA(0,d,0) model driven by WN.

```
clc;
clear all;
d-true = 0.45;
eta-true = 1.0;
phi-true = 0;
theta-true = 0;
s-true=1;
my = load('SimulatedSeries');
```

```
my100=my.vhc(1:500,:);
vm = [29];
m-values = length(vm);
[ , Nrepl] = size(my100);
Nrepl = 1000;
Results = zeros(4, m-values,Nrepl);
for i = 1:Nrepl
vy = my100(:,i);
for j = 1:m-values
vP = [0.3; 1.0];
m = vm(j);
[Results(:, j,i),mZ,mT, mStatePred, mP] = Estimate(vy, vP, m);
d = Results(2, j);
u = Results(3, j);
sigma2 = Results(4, j);
vphi = [1; -2*u; 1];
ga = PowerMAcoeff(vphi, -d, m);
end
disp(i);
end
disp(Results);
d-sum=0;
u-sum=0;
s-sum=0;
for p=1:1000
d-sum=d-sum+sum(Results(2,:,p));
u-sum=u-sum+sum(Results(3,:,p));
s-sum=s-sum+sum(Results(4,:,p));
end
```

d-hat=d-sum/1000;

u-hat=u-sum/1000;

s-hat=s-sum/1000;

d-est=Results(2,:,:);

u-est=Results(3,:,:);

s-est=Results(4,:,:);

SD-d=std(d-est);

SD-u=std(u-est);

SD-s=std(s-est);

disp(d-hat);

disp(SD-d);

disp(u-hat);

disp(SD-u);

disp(s-hat);

disp(SD-s);

osaf=(mZ*mStatePred)';

osaf=osaf(1:end-1);

l-f=vy(2:end);

$mse - osaf = mean((osaf - (l - f)).^2)$;

$mse - d = mean((d - est - (d - true)).^2)$;

$mse - u = mean((u - est - (eta - true)).^2)$;

$mse - s = mean((s - est - (s - true)).^2)$;

$mse - tot = mse - d + mse - u + mse - s$;

disp(mse-osaf);

disp(mse-d);

disp(mse-u);

disp(mse-s);

disp(mse-tot);

disp(['Likelihood: ', num2str(-1.0*fval)]);

5. MATLAB Computer Function Source Code used for State Space Modelling & Kalman Filtering Estimation of GARMA(0,d,0) series.

```
function [Results, mZ, mT, mStatePred, vP] = Estimate(vy, vP, k)
f = @(vP)SsfLogLikConc(vP, vy, k);
[vP, fval, exitflag, output, grad, hessian] = fminunc(f, vP);
[mZ, mG, mT, mH, va, mP] = Statespacemodel-MA(vP,k);
[ , , mStatePred, , , dSigma2, , ] = KalmanFilter(vy, mZ, mG, mT, mH, va, mP);
d = 0.5 / (1+exp(-vP(1)));
lambda = pi / (1+exp(-vP(2)));
eta = cos(lambda);
Results = [k; d; eta; dSigma2];
end
```

6. MATLAB Computer Function Source Code used for creating the State Space Model for GARMA(0,d,0) series.

```
function [ mZ, mG, mT, mH, va, mP ] = Statespacemodel-MA( vP, k )
l = 0.5 / (1+exp(-vP(1)));
lambda = pi / (1+exp(-vP(2)));
v = cos(lambda);
vphi = [1; -2*v; 1];
ga = PowerMAcoeff(vphi, -l, k);
phi-zero=0;
theta-one=0;
ga = ga(2:end);
cm = k;
```

$mZ = [1, zeros(1, k-1)];$

$mG = 1;$

```
mT = [zeros(k-1,1),eye(k-1);zeros(1,k)];
mH = ga;
va = zeros(cm,1);
T = zeros(k^2, k^2);
Q = zeros(k^2, k^2);
T = kron(mT, mT);
Q = eye(k * k) - T;
W = mH * mH';
V = W(:);
P = Q V;
mP = reshape(P, k, k);
end
```

7. MATLAB Computer Program Source Code used to create Simulated GARMA(0,d,0)-GARCH(1,1) Series for State Space Modelling & Kalman Filtering.

```
Nrepl = 2000;
n = 2000;
drop = n;
d-true = 0.3;
eta-true = 0.8;
phi-true = 0;
theta-true = 0;
kappa = 0.4;
alpha = 0.3;
bet = 0.3;
vm = 5:5:60;
vhc = [];
vphi = [1; -2*eta-true; 1];
ga=PowerMAcoeff(vphi, -d-true, n+drop-1);
spec=garchset('k',0.4,'GARCH',0.3,'ARCH',0.3);
```

```
for i=1:Nrepl
x=ugarchsim(kappa, alpha, bet, n+drop);
vy=filter(ga,1,x);
vy=vy(drop+1:end);
vy = vy - mean(vy);
vhc = horzcat(vhc,vy);
disp(i);
end
disp(vhc(1:10,:));
plot(vhc);
autocorr(vhc(:,1));
save('datafile-new');
```

8. MATLAB Main Computer Program Source Code used for State Space Modelling & Kalman Filtering for GARMA(0,d,0)-GARCH(1,1) model.

```
clc;
clear all;
my = load('required source file');
my100 = my.vhc(1:500,:);
d-true = 0.3;
eta-true = 0.8;
phi-true = 0;
theta-true = 0;
kappa = 0.4;
alpha = 0.3;
bet = 0.3;
vm = 12;
m-values = size(vm, 2);
[ , Nrepl] = size(my100);
Nrepl = 1000;
```

```
Results = zeros(3, m-values, Nrepl);
Gresults = zeros(3,m-values,Nrepl);
Estarray = zeros(3,m-values,Nrepl);
spec=garchset('k',0.4,'GARCH',0.3,'ARCH',0.3);
for i = 1:Nrepl
vy = my100(:,i);
for j = 1:m-values
vP = [0.3; 1.0];
m = vm(j);
[Results(:, j, i),mZ,mT, mStatePred, vP,fval] = Estimate-GARCH(vy, vP, m);
d = Results(2, j);
u = Results(3, j);
vphi = [1; -2*u; 1];
ga = PowerMAcoeff(vphi, -d, m);
[Coeff]=garchfit(spec,my100(:));
D=[Coeff];
Gresults=struct2cell(D);
Estarray(:,j,i)=cell2mat(Gresults(7:9));
end
disp(i);
end
disp(Results);
d-sum=0;
u-sum=0;
for p=1:1000
d-sum=d-sum+sum(Results(2,:,p));
u-sum=u-sum+sum(Results(3,:,p));
end
d-hat=d-sum/1000;
```

u-hat=u-sum/1000;

d-est=Results(2,:,:);

u-est=Results(3,:,:);

alphan-est=Estarray(1,:,:);

alphao-est=Estarray(2,:,:);

beta-est=Estarray(3,:,:);

SD-d=std(d-est);

SD-u=std(u-est);

alphan-hat=mean(Estarray(1,:,:));

SD-alphan=std(alphan-est);

alphao-hat=mean(Estarray(2,:,:));

SD-alphao=std(alphao-est);

beta-hat=mean(Estarray(3,:,:));

SD-beta=std(beta-est);

osaf=(mZ*mStatePred)';

osaf=osaf(1:end-1);

l-f=vy(2:end);

$mse - osaf = mean((osaf - (l - f)).^2)$;

$mse - d = mean((d - est - (d - true)).^2)$;

$mse - u = mean((u - est - (eta - true)).^2)$;

$mse - alphan = mean((alphan - est - (kappa)).^2)$;

$mse - alphao = mean((alphao - est - (alpha)).^2)$;

$mse - beta = mean((beta - est - (bet)).^2)$;

$mse - tot = mse - d + mse - u + mse - alphan + mse - alphao + mse - beta$;

disp(d-hat);

disp(SD-d);

disp(mse-d);

disp(u-hat);

disp(SD-u);

disp(mse-u);

disp(alphan-hat);

disp(SD-alphan);

disp(mse-alphan);

disp(alphao-hat);

disp(SD-alphao);

disp(mse-alphao);

disp(beta-hat);

disp(SD-beta);

disp(mse-beta);

disp(mse-osaf);

disp(mse-tot);

disp(['Likelihood: ', num2str(-1.0*fval)]);

9. MATLAB Computer Program Source Code used to create GARSMA(0,d,0)x(0,$D_{s'}$,0) Simulated Series driven by WN for State Space Modelling & Kalman Filtering.

Nrepl = 2000;

n = 2000;

drop = n;

d-true = 0.1;

D-true = 0.45;

eta-true = 0.8;

phi-true = 0;

theta-true = 0;

s-true=1;

vphi = [1; -2*eta-true; 1];

vm = 5:5:75;

vhc = [];

$ga = PowerMAcoeff_S(vphi, -d - true, -D - true, n + drop - 1);$

for i=1:Nrepl

```
vy=filter(ga,1,randn(n+drop,1));
vy=vy(drop+1:end);
vy = vy - mean(vy);
vhc = horzcat(vhc,vy);
disp(i);
end
disp(vhc(1:10,:));
plot(vhc);
autocorr(vhc(:,1));
save('Simulated-Seasonal-Series.mat', 'vhc');
```

10. MATLAB Main Computer Program Source Code used for State Space Modelling & Kalman Filtering for GARSMA(0,d,0)x(0,$D_{s'}$,0) model.

```
clc;
clear all;
d-true = 0.1;
eta-true = 0.8;
D-true = 0.45;
phi-true = 0;
theta-true = 0;
s-true=1;
my = load('Simulated-Seasonal-Series.mat');
my100=my.vhc(1:100,:);
vm = [60];
m-values = length(vm);
[ , Nrepl] = size(my100);
Nrepl = 1000;
Results = zeros(5, m-values,Nrepl);
for i = 1:Nrepl
vy = my100(:,i);
```

```
for j = 1:m-values
vP = [0.3; 0.5 ;0.3];
m = vm(j);
[Results(:, j,i),mZ,mT, mStatePred, mP] = Estimate-S(vy, vP, m);
d = Results(2, j);
u = Results(3, j);
sigma2 = Results(4, j);
D = Results(5, j);
vphi = [1; -2*u; 1];
ga=PowerMAcoeff-S(vphi,-d, -D,m);
end
disp(i);
end
disp(Results);
d-sum=0;
u-sum=0;
s-sum=0;
D-sum=0;
for p=1:1000
d-sum=d-sum+sum(Results(2,:,p));
u-sum=u-sum+sum(Results(3,:,p));
s-sum=s-sum+sum(Results(4,:,p));
D-sum=D-sum+sum(Results(5,:,p));
end
d-hat=d-sum/1000;
u-hat=u-sum/1000;
s-hat=s-sum/1000;
D-hat=D-sum/1000;
d-est=Results(2,:,:);
```

u-est=Results(3,:,:);

s-est=Results(4,:,:);

D-est=Results(5,:,:);

SD-d=std(d-est);

SD-u=std(u-est);

SD-s=std(s-est);

SD-D=std(D-est);

disp(d-hat);

disp(SD-d);

disp(u-hat);

disp(SD-u);

disp(s-hat);

disp(SD-s);

disp(D-hat);

disp(SD-D);

osaf=(mZ*mStatePred)′;

osaf=osaf(1:end-1);

l-f=vy(2:end);

$mse - osaf = mean((osaf - (l - f)).^2)$;

$mse - d = mean((d - est - (d - true)).^2)$;

$mse - u = mean((u - est - (eta - true)).^2)$;

$mse - s = mean((s - est - (s - true)).^2)$;

$mse - D = mean((D - est - (D - true)).^2)$;

$mse - tot = mse - d + mse - u + mse - s + mse - D$;

disp(mse-osaf);

disp(mse-d);

disp(mse-u);

disp(mse-s);

disp(mse-D);

disp(mse-tot);

11. MATLAB Computer Function Source Code used for State Space Modelling & Kalman Filtering Estimation of GARSMA(0,d,0)x(0,$D_{s'}$,0) series.

```
function [Results, mZ, mT, mStatePred, vP] = Estimate-S(vy, vP, k)
f = @(vP)SsfLogLikConc(vP, vy, k);
[vP, fval, exitflag, output, grad, hessian] = fminunc(f, vP);
[mZ, mG, mT, mH, va, mP] = Statespacemodel-MA-S(vP,k);
[ , , mStatePred, , , dSigma2, , ] = KalmanFilter(vy, mZ, mG, mT, mH, va, mP);
d = 0.5 / (1+exp(-vP(1)));
lambda = pi / (1+exp(-vP(2)));
eta = cos(lambda);
D = 0.5 / (1+exp(-vP(3)));
Results = [k; d; eta; dSigma2; D];
end
```

12. MATLAB Computer Function Source Code used for creating the State Space Model for GARSMA(0,d,0)x(0,$D_{s'}$,0) series.

```
function [ mZ, mG, mT, mH, va, mP ] = Statespacemodel-MA-S( vP, k)
l = 0.5 / (1+exp(-vP(1)));
lambda = pi / (1+exp(-vP(2)));
L = 0.5 / (1+exp(-vP(3)));
v = cos(lambda);
vphi = [1; -2*v; 1];
ga = PowerMAcoeff-S(vphi,-l,-L,k);
ga = ga(2:end);
cm = k;
Measurement equation
mZ = [1,zeros(1,k-1)];
mG = 1;
Transition equation
```

mT = [zeros(k-1,1),eye(k-1);zeros(1,k)];

mH = ga;

va = zeros(cm,1);

$T = zeros(k^2, k^2)$;

$Q = zeros(k^2, k^2)$;

$T = kron(mT, mT)$;

$Q = eye(k * k) - T$;

$W = mH * mH'$;

$V = W(:)$;

$P = Q\ V$;

mP = reshape(P,k,k);

end

13. MATLAB Computer Function Source Code used to create GARSMA(0,d,0)x(0,$D_{s'}$,0) coefficients.

function ga = PowerMAcoeff-S(vpsi, dp, D, cq)

cK = length(vpsi);

ga = [1; zeros(cq, 1)];

dat = [1; zeros(cq, 1)];

vpsi-q = [vpsi; zeros(cq+1-cK, 1)];

for j = 1: cq;

dc = 0;

ds(1) = 1;

k = 1;

s=12;

for k = 1:j

s=12;

dc = dc + (k * (dp + 1) - j) * vpsi-q(k+1) * dat(j -k+1);

ds-t(k) = exp(gammaln((s*k)+D)-(gammaln(s*k)+1));

ds(k+1)=ds-t(k)/gamma(D);

```
end
dat(j+1) = dc/j;
ga = conv(ds,dat);
ga=ga(1:cq+1); end
ga=ga(1:cq+1);
end
```

Source Code Program Segments 16-18 are based on the AR approximation.

14. MATLAB Computer Program Source Code to used to create Simulated Series driven by WN for State Space Modelling & Kalman Filtering of GARMA(0,d,0) series.

```
Nrepl = 2000;
n = 2000;
drop = n;
d-true = 0.45;
eta-true = 0.8;
phi-true = 0;
theta-true = 0;
s-true=1;
l=acos(eta-true);
vphi = [1; 2*eta-true; 1];
vm = 5:5:75;
vhc = [];
ga=PowerMAcoeff(vphi, d-true, n+drop-1);
for i=1:Nrepl
vy=filter(ga,1,randn(n+drop,1));
vy=vy(drop+1:end);
vy = vy - mean(vy);
vhc = horzcat(vhc,vy);
disp(i);
end
```

```
disp(vhc(1:10,:));
plot(vhc);
autocorr(vhc(:,1));
save('new data file');
```

15. MATLAB Main Computer Program Source Code used for State Space Modelling & Kalman Filtering for GARMA(0,d,0) model driven by WN.

```
clc;
clear all;
my = load('required simulated data');
my100 = my.vhc(1:500,:);
d-true = 0.3;
eta-true = 0.6;
s-true=1;
vm = 8;
m-values = size(vm, 2);
[ , Nrepl] = size(my100);
Nrepl = 1000;
Results = zeros(4, m-values,Nrepl);
for i = 1:Nrepl
vy = my100(:,i);
for j = 1:m-values
vP = [0.3; 0.5];
m = vm(j);
[Results(:, j,i),mZ,mT, mStatePred, vP] = Estimate-AR(vy, vP, m);
d = Results(2, j);
u = Results(3, j);
```

```
sigma2 = Results(4, j);
vphi = [1; 2*u; 1];
ga = PowerMAcoeff(vphi, d, m);
end
disp(i);
end
disp(Results);
d-sum=0;
u-sum=0;
s-sum=0;
for p=1:1000
d-sum=d-sum+sum(Results(2,:,p));
u-sum=u-sum+sum(Results(3,:,p));
s-sum=s-sum+sum(Results(4,:,p));
end
d-hat=d-sum/1000;
u-hat=u-sum/1000;
s-hat=s-sum/1000;
d-est=Results(2,:,:);
u-est=Results(3,:,:);
s-est=Results(4,:,:);
SD-d=std(d-est);
SD-u=std(u-est);
SD-s=std(s-est);
disp(d-hat);
disp(SD-d);
disp(u-hat);
disp(SD-u);
disp(s-hat);
```

disp(SD-s);

$osaf = (mZ * mStatePred)'$;

$osaf = osaf(1 : end - 1)$;

$l - f = vy(2 : end)$;

$mse - osaf = mean((osaf - (l - f)).^2)$;

$mse - d = mean((d - est - (d - true)).^2)$;

$mse - u = mean((u - est - (eta - true)).^2)$;

$mse - s = mean((s - est - (s - true)).^2)$;

$mse - tot = mse - d + mse - u + mse - s$;

disp(mse-osaf);

disp(mse-d);

disp(mse-u);

disp(mse-s);

disp(mse-tot);

16. MATLAB Computer Function Source Code used for State Space Modelling & Kalman Filtering Estimation of GARMA(0,d,0) series.

```
function [Results, mZ, mT, mStatePred, vP] = Estimate-AR(vy, vP, k)
f = @(vP)SsfLogLikConc(vP, vy, k);
[vP, fval, exitflag, output, grad, hessian] = fminunc(f, vP);
[mZ, mG, mT, mH, va, mP] = Statespacemodel-AR(vP,k);
[ , , mStatePred, , , dSigma2, , ] = KalmanFilter(vy, mZ, mG, mT, mH, va, mP);
d = 0.5 / (1+exp(-vP(1)));
lambda = pi / (1+exp(-vP(2)));
eta = cos(lambda);
side-label=['m:';'d:';'u:'; 's2'];
Results = [k; d; eta; dSigma2];
end
```

17. MATLAB Computer Function Source Code used for creating the State Space Model for GARMA(0,d,0) series.

```
function [ mZ, mG, mT, mH, va, mP ] = Statespacemodel-AR( vP, k )
l = 0.5 / (1+exp(-vP(1)));
lambda = pi / (1+exp(-vP(2)));
v = cos(lambda);
vphi = [1; -2*v; 1];
c = PowerMAcoeff(vphi, l, k);
c = -c(2:end);
cm = k;
mZ = [1,zeros(1,k-1)];
mG = 1;
mat-int = [eye(k-1);zeros(1,k-1)];
mT = [c, mat-int];
mH = c(1:k);
va = zeros(cm,1);
T = kron(mT,mT);
Q = eye(k*k)-T;
R = inv(Q);
W = mH*mH';
V = W(:);
P = R*V;
mP = reshape(P,k,k);
end
```

18. MATLAB Computer Program Source Code used to create Simulated GARMA(0,d,0)-GARCH(1,1) Series for State Space Modelling & Kalman Filtering.

```
Nrepl = 2000;
n = 2000;
drop = n;
d-true = 0.3;
eta-true = 0.8;
```

```
phi-true = 0;
theta-true = 0;
kappa = 0.4;
alpha = 0.3;
bet = 0.3;
vm = 5:5:60;
vhc = [];
vphi = [1; -2*eta-true; 1];
ga=PowerMAcoeff(vphi, d-true, n+drop-1);
spec=garchset('k',0.4,'GARCH',0.3,'ARCH',0.3);
for i=1:Nrepl
x=ugarchsim(kappa, alpha, bet, n+drop);
vy=filter(ga,1,x);
vy=vy(drop+1:end);
vy = vy - mean(vy);
vhc = horzcat(vhc,vy);
disp(i);
end
disp(vhc(1:10,:));
plot(vhc);
autocorr(vhc(:,1));
save('simulated data source');
```

19. MATLAB Main Computer Program Source Code used for State Space Modelling & Kalman Filtering for GARMA(0,d,0)-GARCH(1,1) model.

```
clc;
clear all;
my = load('required data source');
my100 = my.vhc(1:500,:);
```

```
d-true = 0.3;
eta-true = 0.8;
phi-true = 0;
theta-true = 0;
kappa = 0.4;
alpha = 0.3;
bet = 0.3;
vm = 12;
m-values = size(vm, 2);
[ , Nrepl] = size(my100);
Nrepl = 1000;
Results = zeros(3, m-values, Nrepl);
Gresults = zeros(3,m-values,Nrepl);
Estarray = zeros(3,m-values,Nrepl);
spec=garchset('k',0.4,'GARCH',0.3,'ARCH',0.3);
for i = 1:Nrepl
vy = my100(:,i);
for j = 1:m-values
vP = [0.3; 1.0];
m = vm(j);
[Results(:, j, i),mZ,mT, mStatePred, vP,fval] = Estimate-GARCH-AR(vy, vP, m);
d = Results(2, j);
u = Results(3, j);
vphi = [1; -2*u; 1];
ga = PowerMAcoeff(vphi, d, m);
[Coeff]=garchfit(spec,my100(:));
D=[Coeff];
Gresults=struct2cell(D);
Estarray(:,j,i)=cell2mat(Gresults(7:9));
```

```
end
disp(i);
end
disp(Results);
d-sum=0;
u-sum=0;
for p=1:1000
d-sum=d-sum+sum(Results(2,:,p));
u-sum=u-sum+sum(Results(3,:,p));
end
d-hat=d-sum/1000;
u-hat=u-sum/1000;
d-est=Results(2,:,:);
u-est=Results(3,:,:);
alphan-est=Estarray(1,:,:);
alphao-est=Estarray(2,:,:);
beta-est=Estarray(3,:,:);
SD-d=std(d-est);
SD-u=std(u-est);
alphan-hat=mean(Estarray(1,:,:));
SD-alphan=std(alphan-est);
alphao-hat=mean(Estarray(2,:,:));
SD-alphao=std(alphao-est);
beta-hat=mean(Estarray(3,:,:));
SD-beta=std(beta-est);
osaf=(mZ*mStatePred)′;
osaf=osaf(1:end-1);
l-f=vy(2:end);
```

$mse - osaf = mean((osaf - (l - f)).^2)$;

$mse - d = mean((d - est - (d - true)).^2)$;

$mse - u = mean((u - est - (eta - true)).^2)$;

$mse - alphan = mean((alphan - est - (kappa)).^2)$;

$mse - alphao = mean((alphao - est - (alpha)).^2)$;

$mse - beta = mean((beta - est - (bet)).^2)$;

$mse - tot = mse - d + mse - u + mse - alphan + mse - alphao + mse - beta$;

disp(d-hat);

disp(SD-d);

disp(mse-d);

disp(u-hat);

disp(SD-u);

disp(mse-u);

disp(alphan-hat);

disp(SD-alphan);

disp(mse-alphan);

disp(alphao-hat);

disp(SD-alphao);

disp(mse-alphao);

disp(beta-hat);

disp(SD-beta);

disp(mse-beta);

disp(mse-osaf);

disp(mse-tot);

disp(['Likelihood: ', num2str(-1.0*fval)]);

20. MATLAB Computer Program Source Code used to create GARSMA(0,d,0)x(0,$D_{s'}$,0) Simulated Series driven by WN for State Space Modelling & Kalman Filtering.

Nrepl = 2000;

n = 2000;

```
drop = n;
d-true = 0.1;
D-true = 0.45;
eta-true = 0.8;
phi-true = 0;
theta-true = 0;
s-true=1;
vphi = [1; -2*eta-true; 1];
vm = 5:5:75;
vhc = [];
```

$ga = PowerMAcoeff_S(vphi, d - true, D - true, n + drop - 1);$

```
for i=1:Nrepl
vy=filter(ga,1,randn(n+drop,1));
vy=vy(drop+1:end);
vy = vy - mean(vy);
vhc = horzcat(vhc,vy);
disp(i);
end
disp(vhc(1:10,:));
plot(vhc);
autocorr(vhc(:,1));
save('generated data');
```

21. MATLAB Main Computer Program Source Code used for State Space Modelling & Kalman Filtering for GARSMA(0,d,0)x(0,$D_{s'}$,0) model.

```
clc;
clear all;
d-true = 0.1;
eta-true = 0.8;
D-true = 0.45;
```

```
phi-true = 0;
theta-true = 0;
s-true=1;
my = load('newly created data');
my100=my.vhc(1:100,:);
vm = [60];
m-values = length(vm);
[ , Nrepl] = size(my100);
Nrepl = 1000;
Results = zeros(5, m-values,Nrepl);
for i = 1:Nrepl
vy = my100(:,i);
for j = 1:m-values
vP = [0.3; 0.5 ;0.3];
m = vm(j);
[Results(:, j,i),mZ,mT, mStatePred, mP] = Estimate-S-AR(vy, vP, m);
d = Results(2, j);
u = Results(3, j);
sigma2 = Results(4, j);
D = Results(5, j);
vphi = [1; -2*u; 1];
ga=PowerMAcoeff-S-AR(vphi,d, D,m);
end
disp(i);
end
disp(Results);
d-sum=0;
u-sum=0;
s-sum=0;
```

```
D-sum=0;
for p=1:1000
d-sum=d-sum+sum(Results(2,:,p));
u-sum=u-sum+sum(Results(3,:,p));
s-sum=s-sum+sum(Results(4,:,p));
D-sum=D-sum+sum(Results(5,:,p));
end
d-hat=d-sum/1000;
u-hat=u-sum/1000;
s-hat=s-sum/1000;
D-hat=D-sum/1000;
d-est=Results(2,:,:);
u-est=Results(3,:,:);
s-est=Results(4,:,:);
D-est=Results(5,:,:);
SD-d=std(d-est);
SD-u=std(u-est);
SD-s=std(s-est);
SD-D=std(D-est);
disp(d-hat);
disp(SD-d);
disp(u-hat);
disp(SD-u);
disp(s-hat);
disp(SD-s);
disp(D-hat);
disp(SD-D);
osaf=(mZ*mStatePred)';
osaf=osaf(1:end-1);
```

l-f=vy(2:end);

$mse - osaf = mean((osaf - (l - f)).^2)$;

$mse - d = mean((d - est - (d - true)).^2)$;

$mse - u = mean((u - est - (eta - true)).^2)$;

$mse - s = mean((s - est - (s - true)).^2)$;

$mse - D = mean((D - est - (D - true)).^2)$;

$mse - tot = mse - d + mse - u + mse - s + mse - D$;

disp(mse-osaf);

disp(mse-d);

disp(mse-u);

disp(mse-s);

disp(mse-D);

disp(mse-tot);

22. MATLAB Computer Function Source Code used for State Space Modelling & Kalman Filtering Estimation of GARSMA(0,d,0)x(0,$D_{s'}$,0) series.

function [Results, mZ, mT, mStatePred, vP] = Estimate-S-AR(vy, vP, k)

f = @(vP)SsfLogLikConc(vP, vy, k);

[vP, fval, exitflag, output, grad, hessian] = fminunc(f, vP);

[mZ, mG, mT, mH, va, mP] = Statespacemodel-AR-S(vP,k);

[, , mStatePred, , , dSigma2, ,] = KalmanFilter(vy, mZ, mG, mT, mH, va, mP);

d = 0.5 / (1+exp(-vP(1)));

lambda = pi / (1+exp(-vP(2)));

eta = cos(lambda);

D = 0.5 / (1+exp(-vP(3)));

Results = [k; d; eta; dSigma2; D];

end

23. MATLAB Computer Function Source Code used for creating the State Space Model for GARSMA(0,d,0)x(0,$D_{s'}$,0) series.

function [mZ, mG, mT, mH, va, mP] = Statespacemodel-AR-S(vP, k) l = 0.5 / (1+exp(-vP(1)));

lambda = pi / (1+exp(-vP(2)));

L = 0.5 / (1+exp(-vP(3)));

v = cos(lambda);

vphi = [1; -2*v; 1];

ga = PowerMAcoeff-S(vphi,l,L,k);

ga = ga(2:end);

cm = k;

Measurement equation

mZ = [1,zeros(1,k-1)];

mG = 1;

Transition equation

mT = [zeros(k-1,1),eye(k-1);zeros(1,k)];

mH = ga;

va = zeros(cm,1);

$T = zeros(k^2, k^2)$;

$Q = zeros(k^2, k^2)$;

$T = kron(mT, mT)$;

$Q = eye(k * k) - T$;

$W = mH * mH'$;

$V = W(:)$;

$P = Q\,V$;

mP = reshape(P,k,k);

end

The Utilized Real Data Sets

Tong's Sunspots.

5 11 16 23 36 58 29 20 10 8 3 0 0 2 11 27 47 63 60 39 28 26 22 11 21 40 78 122 103 73 47 35 11 5 16 34 70 81 111 101 73 40 20 16 5 11 22 40 60 80.9 83.4 47.7 47.8 30.7 12.2 9.6 10.2 32.4 47.6 54 62.9 85.9 61.2 45.1 36.4 20.9 11.4 37.8 69.8 106.1 100.8 81.6 66.5 34.8 30.6 7 19.8 92.5 154.4 125.9

84.8 68.1 38.5 22.8 10.2 24.1 82.9 132 130.9 118.1 89.9 66.6 60 46.9 41 21.3 16 6.4 4.1 6.8 14.5 34 45 43.1 47.5 42.2 28.1 10.1 8.1 2.5 0 1.4 5 12.2 13.9 35.4 45.8 41.1 30.1 23.9 15.6 6.6 4 1.8 8.5 16.6 36.3 49.6 64.2 67 70.9 47.8 27.5 8.5 13.2 56.9 121.5 138.3 103.2 85.7 64.6 36.7 24.2 10.7 15 40.1 61.5 98.5 124.7 96.3 66.6 64.5 54.1 39 20.6 6.7 4.3 22.7 54.8 93.8 95.8 77.2 59.1 44 47 30.5 16.3 7.3 37.6 74 139 111.2 101.6 66.2 44.7 17 11.3 12.4 3.4 6 32.3 54.3 59.7 63.7 63.5 52.2 25.4 13.1 6.8 6.3 7.1 35.6 73 85.1 78 64 41.8 26.2 26.7 12.1 9.5 2.7 5 24.4 42 63.5 53.8 62 48.5 43.9 18.6 5.7 3.6 1.4 9.6 47.4 57.1 103.9 80.6 63.6 37.6 26.1 14.2 5.8 16.7 44.3 63.9 69 77.8 64.9 35.7 21.2 11.1 5.7 8.7 36.1 79.7 114.4 109.6 88.8 67.8 47.5 30.6 16.3 9.6 33.2 92.6 151.6 136.3 134.7 83.9 69.4 31.5 13.9 4.4 38 141.7 190.2 184.8 159 112.3 53.9 37.5 27.9 10.2 15.1 47 93.8 105.9 105.5 104.5 66.6 68.9 38 34.5 15.5 12.6 27.5 92.5 155.4 154.7 140.5 115.9 66.6 45.9 17.9 13.4 29.2 100.2

Wolfer's Sunspots.

80.9 83.4 47.7 47.8 30.7 12.2 9.6 10.2 32.4 47.6 54 62.9 85.9 61.2 45.1 36.4 20.9 11.4 37.8 69.8 106.1 100.8 81.6 66.5 34.8 30.6 7 19.8 92.5 154.4 125.9 84.8 68.1 38.5 22.8 10.2 24.1 82.9 132 130.9 118.1 89.9 66.6 60 46.9 41 21.3 16 6.4 4.1 6.8 14.5 34 45 43.1 47.5 42.2 28.1 10.1 8.1 2.5 0 1.4 5 12.2 13.9 35.4 45.8 41.1 30.1 23.9 15.6 6.6 4 1.8 8.5 16.6 36.3 49.6 64.2 67 70.9 47.8 27.5 8.5 13.2 56.9 121.5 138.3 103.2 85.7 64.6 36.7 24.2 10.7 15 40.1 61.5 98.5 124.7 96.3 66.6 64.5 54.1 39 20.6 6.7 4.3 22.7 54.8 93.8 95.8 77.2 59.1 44 47 30.5 16.3 7.3 37.6 74 139 111.2 101.6 66.2 44.7 17 11.3 12.4 3.4 6 32.3 54.3 59.7 63.7 63.5 52.2 25.4 13.1 6.8 6.3 7.1 35.6 73 85.1 78 64 41.8 26.2 26.7 12.1 9.5 2.7 5 24.4 42 63.5 53.8 62 48.5 43.9 18.6 5.7 3.6 1.4 9.6 47.4 57.1 103.9 80.6 63.6 37.6 26.1 14.2 5.8 16.7

El'Nino Data.

11.3 11 0.2 9.4 6.8 17.2 -5.6 12.3 10.5 -8 -2.7 -3 -9.7 -6.5 -4.7 -9.6 3.6 -16.8 -10.2 -8.2 -17.2 -16 -12.6 -12.6 -8.7 -21.1 -15.5 -8.8 2.1 -3.1 15.9 13 17.7 10.9 15.1 17.9 12.7 14.3 13.2 12.7 2.1 16.4 21.8 22.6 18.9 15.2 9.8 -5.5 10.8 7.7 14.3 5.3 12.3 9.1 1.6 14.3 8.1 4.8 7.2 -1.9 -7.3 -5.5 1.8 0.3 -4.3 -4.7 -5.6 -11.4 -13.6 -23.9 7.2 9.8 -6.8 -1.3 5.1 1.2 6.8 -12 -21.3 -25.6 -14.8 -2.5 2.6 10.3 6 9.1 -25.3 14.4 13.9 3.4 -10.2 1.4 -8.2 4.8 5.2 -15.2 -12.5 -5 9.4 -15.4 1.3 9.1 -3 -5 -7 4.2 -1.4 -12.6 -16.3 1.6 5.1 -0.5 -4.3 -14.4 -5 -9.5 -4 -17.8 -15.9 5.2 -0.6 1.6 2.9 4.5 6 5 7.4 13.6 13.5 13.4 10.5 14.4 12.2 11 10 9.4 -4.3 5 4.8 4.6 5.1 4.8 -5.3 5.2 -3 -2.2 -11.7 -23.6 -9.8 -16 -16.7 -8.9 -9.4 -14.7 -12.6 -2.4 -25.9 -1.7 -27.5 -0.5 -1.9 22 1.6 2.1 11.1 4.2 23 22 20.8 11 14.3 6.9 3.6 5.8 -2.3 -3.1 9.3 3.6 2.6 0.6 15.6 -3.6 -9.5 4.5 -0.3 -1.5 -6.3 -8.9 -10.6 0.6 -4.7 -4.5 2.7 -10.2 11.1 6.9 10 19.6 7.4 5.9 6.3 8.5 -0.7 3.7 11.3 7.7

-1.4 1.2 -3.5 10.7 14 7.8 5.7 7.9 2.6 1.6 17.5 10 5.6 -3 -5.1 -1.5 -2.3 -5.7 -1.6 1.8 7.2 0.1 5.6 3 -0.3
-7.1 -8.2 -4.7 -0.4 -6.3 -4 -5.6 -8.6 -3.5 1.3 4.9 -6.3 -8.8 -42.2 -30.6 -20.6 -22.4 -19 -19 -11.9 -14.2
-12.5 -7.4 -16.6 -17.8 -16.9 0.2 -2.3 0.8 0.2 1.8 -8 10.3 7 6.3 19.2 11.1 -1.9 -2.3 6.1 2.1 3.2 -0.7 -2.7
-0.4 13.2 9.1 13.8 4.5 -7.4 -10.4 -5.6 -10.1 -1.6 6.1 15.8 -3 -7.3 -6.5 -25.3 -18.7 -7.4 26.1 10 7.8 -16.6
-17.2 -6 -5.5 -0.1 3 9.4 4.5 -0.3 19.6 14.6 9.8 -16 -22.1 -8.6 -1.9 17 -2.2 11.6 7.8 7.6 2.6 1.6 -8.9 -17.8
-7.4 -3.4 -3 -9.2 -10.2 17.6 17.7 7.6 -0.6 6.1 0.1 8.7 4.2 1.3 15.9 14.1 16.2 9.4 31.7 9.2 -7.1 -8.9 0.8
0.2 1.2 -17.2 2.6 -9.2 -16.8 -30.2 -42.6 -37.4 -31.4 -21.3 -7.6 -7 -5.6 -17.9 -13.1 -3.5 -7.4 -5.2 -8.8 1.3
-3.9 6.8 15.5 18.3 9.1 21.7 4.7 5.1 1.6 -0.3 4.5 10 8.3 -4.3 -8.2 0.2 0.6 -2 8.8 -10.6 7.7 0.2 16.8 -1.1
-2.3 2.2 5.3 17.7 7.9 2.6 -5.5 -2.5 -3.2 -0.3 -14.5 2.1 22.8 10.7 9.8 0.8 4.2 9.2 4.7 5.6 15.2 12.7 5.3
0.5 22 20.5 9.8 15.3 10.3 19.7 15.9 3.2 1.6 3.5 2 -8.2 -12 -12.8 -12.1 -8.8 -11.7 -7.3 -1.4 -9.7 -17.3
-9 -21.1 -13 -6.3 -0.4 -7.6 -4 -8 2.6 -8 -3.5 -5 1.3 -6.3 -8.2 -3.9 -1.7 -7.6 -9.4 -9.2 -11.9 -7 -5.4 2 9.4
-14.5 -0.3 -16.8 -18 -17.2 -12.4 -8.6 -11.9 -1.4 -21.6 -2.2 -20.4 -17.8 -12.2 6.6 14 7.2 7.5 2.4 -14.6
9.8 5.6 -3.6 -6.3 -0.5 6.8 9.1 25.7 16.2 4.5 6.1 9.8 15.4 5.1 10 18.1 21.8 21.8 21.2 28.3 34.8 29.7 15.2
21 22.5 14.6 16.6 -2 16.8 10 -4.7 -14.1 -4.4 -8.2 -5 1.3 -8 -14.9 -11.2 -12.8 -3 -7.4 -10.4 -8.9 -6.9 -5.8
-10.5 -11.3 -9.1 1.8 -1.7 -4.1 0.3 -2.7 6.6 9.4 5.3 5.1 -4.3 -0.1 9.8 10.8 6.7 8.9 -7.1 2.1 22 2.9 -6.9 5.1
9.7 8.5 8.2 8 9.1 5.6 -5.5 -5.1 5.8 2.2 -1.2 5.1 6.1 8.5 11.8 5.6 4.4 8.9 8.6 2.1 1 -11.5 -18.5 -14.8 -6.2
-12.6 2.1 -5.4 1.1 2.4 -15.4 11.5 8.3 7.4 10.4 8.1 7.9 11.8 5.2 5.6 13.8 14.9 14.4 -1.1 -4.7 -13.4 -10.8
-6.4 -12.9 -9.3 -7 -5.4 -14.5 -13.3 -7.1 -2.7 -7.1 -1 -7.6 1.4 4.2 1.3 6.2 5.1 1.1 18.1 6.9 6 8.3 6.1 -5
-0.4 -4.3 -8 7.7 -10.1 10.5 13.8 11.9 -2.7 -7.9 -0.4 9.8 8.1 9.1 2.6 11.8 16 18 5.1 4.5 -12.2 1 1.6 0.1
-0.4 7.9 11.1 5.7 12.7 7.7 1.8 -3.8 2.1 -5.5 -4.3 -1.8 -7 3.6 1.9 -1.4 7 -14.9 5.6 8.6 13.1 18.8 9.4 0.1
5.1 -12.9 -4.7 4.7 1.8 -3.6 -2.5 -2.1 2.8 -4.7 -5 -6.9 -8.8 -4.3 -4.7 3.2 -11.1 4.9 -2 3.6 6 -3.9 3.5 -0.5 2
3.6 7.2 8.2 6.5 0.1 0.2 6.1 -7.4 10.7 2.9 -22.4 -6.4 4.2 13.1 -2.4 6.5 -4.6 12.2 2.8 -6.6 -2.3 -0.4 2.1 6.3
7.3 3.9 -4 -2 0.6 1.8 22.6 4.4 -1.5 4.2 -8.9 2.6 -0.1 -13.9 0.6 9.4 -5 6.2 2 -0.3 3.4 -5.6 3.3 0.8 -2.5 -2
6.7 7.5 3.4 -3.6 3.6 13.1 18 18.5 13 7.5 12.8 1.9 13.8 17 7.7 11.6 9.4 -1.1 -1.5 8.1 -0.5 -9.4 -14.7 -8
-8.6 -0.1 -4.1 -10.6 -9.6 -14.5 -19.3 -15.4 -18.5 -19.6 -18.4 -6.7 -29.4 -9.7 -15.4 -10.6 -11.2 -6.6 -14.4
-20.6 -19.1 -8.2 -20.2 -9.3 -8.6 -13 -3.6 -5.8 -5.5 5.2 8.3 -1 4 8.7 8.5 -4 13.8 9.4 10.5 4 13.5 2.8 -7.9
2.9 7.8 5.7 9.1 3.9 -8.6 -8.2 3.9 5.6 -5.5 -1.1 -3.9 -8.9 3.3 2.6 -8.6 -6.7 4.2 5.1 6.3 13.2 -7.1 -0.3 8.3
3.5 11.7 8.7 2.4 -3.4 6.7 -2.5 4.4 -2 -9.6 -11.4 -9.6 -10.2 -4.4 -16 -12.3 -1.4 -5.5 -4.9 -4.1 11.6 -4.6
-13.7 2.6 9.4 7.2 11.7 -1.9 9.2 5.2 -3 -2.7 -4.1 2.8 3.6 -4.7 0.9 -4.4 -7.6 6.1 4.6 -5.5 -7.3 2 5.6 1.2 -5.8

-12 -1.7 -4.4 2 5.4 -6 7.7 5.1 17.6 17.6 16.8 7.6 26.9 21.1 12.3 6.9 17.1 12.5 23 16.5 9.6 -1.4 -1.3 -6.6
5 -8.2 -0.5 -7 -8 -3.4 -3 -9.2 -7.9 0.2 -8.8 6 7.4 3.5 -3.7 -3.4 1.8 -0.7 -12.6 2.2 -6 -5.8 -0.5 -31.9 -2.3
-1 -17.2 -13 -0.1 -2 -4 6 -3.6 -0.9 6.9 4.4 -1.5 4.2 10.4 4.5 1.8 3.9 12.8 -5.4 15.2 2.9 -3 13.1 16.4 19.2
14.9 14.1 15.2 15.1 9.3 11.3 12.4 9.4 11.1 17.9 12.3 12.6 11 0.2 18.3 1.9 10.3 5.6 -2.2 -0.9 1.2 -12.2
-2.3 0.9 -9.5 -10.6 -1.3 -11.9 -3.5 -16.8 -6.9 -1.4 1.2 -8.2 0.2 2.2 7.8 -3.4 -1.9 -4.7 -6.5 -8.7 -14 8.4
3.6 2.8 -6.3 -5 -5 0.2 4.2 11.1 8.2 0.3 -2.2 5.6 7.8 5.2 -2.3 4.8 6.6 6.9 -0.7 7.2 6.7 -2.5 6.3 -20.9 9.4
1.3 -3.1 2.2 0.1 0.8 -5 7.2 13.8 17 5.3 -1.4 1.2 12.3 5 -0.4 4.6 5.1 10.3 5.2 0.6 9.4 3 7.3 6.1 2.8 -9.6 -1
-2.4 -5.2 -12.9 -9.3 -11.6 -4 -0.3 8.4 13.5 2.8 7.4 6.8 14.3 14.1 12.8 2.6 -3 -4 1.6 2.9 -12.9 -0.3 -12.8
-22.6 -11.4 -14.2 -11.1 -17.9 1.6 -12 -4.1 -13.9 -7.1 -9 1 -1 4 -2.2 -2.5 -0.1 -4 14.6 12.9 7.8 -3 -3.5 6.6
1.6 5.9 5.1 -0.1 -4 -5.5 4.1 9.6 -3 -3 14.7 12.3 7.4 0.1 -2.8 -1.9 -3.4 2.1 -13.5 -6.9 1.8 -8.8 -6.6 -0.6
-6.9 -4.4 -10.6 -11.7 -0.1 3.7 -10.1 -10.7 1.8 -4.6 2.1 9.9 -5.6 4 12.9 10.3 19.7 17.4 2.7 15.7 19.2 22.6
9.2 2.6 1.6 14.9 15.9 17.7 7.2 2.1 3.7 8.2 2.4 -5.5 -16.1 -12 -18.6 -8.9 -14.8 -11.1 -3.4 -12.1 -3 -13.5
0.8 -2.1 2.8 12.3 6.1 12.3 13.5 9.7 31.6 16.9 20.8 16.2 20.3 11.1 10.7 2.6 12 6.6 12.3 8.5 -1.4 -0.9
-4.9 5.3 11.6 14.4 6 15.5 21.1 20.7 22.5 17.7 13.8 19.5 11.8 12.9 13.2 1.2 2.1 0.2 -12.8 -12.1 -13 3
9.8 -3 -4 7.7 -9.5 -9.6 -11.4 -17.7 -14.7 -12.1 -9.4 -12.9 -14.6 -10.6 -3 -24.4 -5.8 -7.9 16.3 5.8 6.1 1.4
0.8 -6.2 -2 -0.9 -4 6.7 -3 -5.5 3.6 5.8 -8.2 -5 1.4 -2.5 -4.7 -7.5 3.2 1.1 -8.5 -12.9 -3.5 -4.7 -1.7 1.4 -5.2
-1.9 -3.4 -0.9 2.7 -3.2 -16.6 -5.5 7.6 11.5 9.4 5.9 7.5 -5 2.6 4.7 9.4 0.6 2.4 -3.8 -8.2 -20.1 -19.3 -23.6
-21.4 -20.2 -31.1 -21.3 -30.6 -33.3 -28 -17 6 -3.1 -7.6 0.1 9.9 4.2 -0.7 0.1 1.3 5.8 -5.8 2 -0.3 -8.7 2.2
2.7 2 -5 3.9 -1.4 -3.5 6.7 -2 14.4 2.8 -9.6 -2.3 8.5 0.2 -5.6 -1.4 2.1 8 -10.7 0.8 1.2 -6.6 10.7 2.2 -7.6
-5.2 6.1 -13.9 -13.6 -6.3 -12.6 -16.6 -24.4 -21.6 -20.1 -18.6 -14 -11.2 -5.6 -1.4 -4.5 -1.1 -5 2.4 -1.3
10 -3.9 11.3 14.9 20.1 14.6 21 10.8 13.2 9.1 6.7 21 14.7 7.4 9.4 -6.3 5.7 7.3 -2 -5 -1.1 -17.3 -8.5 -0.5
13.1 1 5.5 -5 -7.6 1.8 -5.3 -2.4 5.1 0.6 -10.6 -12.9 -19.3 -5.5 -1.7 -7.6 -16.6 -12.9 -7.3 -16.7 -25.4 -9.3
-24.2 -18.7 0.5 -12.8 -6.9 1.4 0.8 -17.2 -7.3 -5.5 -8.2 -7.9 -8.5 -21.1 -8.2 -16 -10.8 -14 -7.6 -13.5 0.6
1.6 -1.6 0.6 -10.6 -22.8 -13 -10.4 -18 -17.2 -17.2 -14.1 -7.3 -11.6 -4 -2.7 3.5 -16.2 -9 -1.5 4.2 0.8 3.2
-1.3 1.3 -5.5 8.4 1.1 6.2 7.8 1.3 13.9 6.8 4.6 6.9 4.2 -0.1 7.2 4.1 13.3 -8.5 -16.2 -22.4 -24.1 -9.5 -19.8
-14.8 -17.8 -15.2 -9.1 -23.5 -19.2 -28.5 -24.4 0.5 9.9 14.6 9.8 11.1 10.9 12.5 13.3 15.6 8.6 8.9 18.5
1.3 1 4.8 2.1 -0.4 9.1 13.1 12.8 5.1 12.9 9.4 16.8 3.6 -5.5 -3.7 5.3 9.9 9.7 22.4 7.7 8.9 11.9 6.7 0.3 -9
1.8 -3 -8.9 1.4 -1.9 7.2 -9.1 2.7 7.7 -5.2 -3.8 -14.5 -6.3 -7.6 -14.6 -7.6 -7.4 -6 -10.6 -2 -7.4 -6.8 -5.5
-7.4 -12 2.9 -1.8 -2.2 -1.9 -3.4 9.8 -11.6 8.6 0.2 -15.4 13.1 -14.4 -6.9 -7.6 -2.8 -3.7 -9.3 -8 1.8 -29.1

0.2 -11.2 -14.5 2.6 0.9 -6.9 3.9 10.9 -2.7 0.6 12.7 0.1 13.8 15.2 -9.8 -5.5 -8.9 -15.9 -5.1 -15.3 -1.4 -3 -7.3 -2.7 -1.4 -3 -2.7 5 -4.3 2.7 1.5 5.4 9.8 14.4 14.1 21.3 12.2 4.5 -4.3 5 2.2 9.1 14.1 13.4 17.1 13.3 9.4 14.8 0.2 8.6 -5.1 -2.3 1.6 -5 3.9 -14.7 -6.7 -7 -10.1 -14.5 -10.6 15.2 10 1.8 20.5 18.8 25 18.3 16.4 27.1 19.9 22.3 21.4 25.1 2.1 0.2 10.7 2.1 11.7 7.3 13.8 23 9.4 2.5 2.9 -7.1 -2.7 -10.4 -1.7 -5 2.7 2.4 3.9 -6 -1.1 -3.6 11.1 0.3 8.4 13.9 8.1 -0.5 3.9 -1.9 9.2 0.6

Standard and Poor 500 Data.

105.76 105.22 106.52 106.81 108.95 109.05 109.89 109.92 110.38 111.14 111.05 110.70 111.07 112.10 111.51 113.44 113.70 113.61 114.85 114.07 115.20 114.16 115.12 114.37 114.66 115.72 116.28 117.95 117.12 117.90 118.44 116.72 115.41 114.60 116.47 115.28 115.04 113.33 113.98 112.38 112.35 113.66 112.50 112.78 111.13 108.65 106.90 106.51 107.78 106.87 105.62 105.43 102.26 104.10 104.31 103.12 102.31 99.28 99.19 98.68 98.22 100.68 102.09 102.18 102.68 102.15 100.19 101.20 103.11 104.08 103.79 102.84 102.63 101.54 101.05 100.55 99.80 103.43 103.73 104.40 105.16 105.64 105.86 106.29 105.46 105.58 106.38 106.25 107.18 106.13 104.72 104.78 106.30 106.85 106.99 107.35 107.67 107.62 107.72 109.01 110.62 111.40 112.06 110.27 111.24 110.76 110.51 112.61 112.78 113.20 113.71 114.66 116.02 115.52 115.81 116.09 116.03 116.26 114.66 114.06 114.51 115.14 116.72 116.19 116.00 114.24 114.93 115.68 117.46 118.29 117.84 117.98 116.95 117.84 120.01 119.30 119.63 121.44 122.04 122.51 122.19 121.93 121.79 120.78 121.43 122.40 122.23 121.67 121.21 120.98 120.74 121.55 123.30 123.61 124.78 123.79 123.28 125.25 125.72 123.39 122.60 123.77 125.46 126.02 125.16 124.84 123.52 122.08 122.38 123.73 126.12 125.42 124.88 123.31 124.07 124.81 125.66 125.54 125.67 126.74 128.87 128.40 129.25 130.40 129.43 130.37 128.72 126.35 123.54 125.46 127.13 128.09 129.33 131.73 131.00 131.65 131.04 130.29 132.03 132.02 133.70 132.22 131.52 132.61 131.84 131.92 129.53 129.85 127.88 128.05 127.91 126.29 127.47 129.04 131.33 128.91 129.18 129.48 131.26 134.59 136.49 137.15 137.75 139.70 139.06 140.40 139.11 138.31 139.33 140.17 140.52 137.21 136.97 136.71 136.48 134.03 130.61 130.48 128.26 127.36 129.23 129.45 130.60 132.89 133.00 133.70 135.78 135.30 135.88 136.57 135.03 135.33 135.76 136.34 137.97 138.12 135.08 133.06 133.48 133.52 133.29 133.47 134.22 134.77 134.37 131.65 131.36 130.26 130.23 129.84 131.12 130.34 130.24 129.55 126.91 128.46 128.59 129.63 130.60 129.27 129.24 128.24 127.48 126.98 127.81 128.48 126.61 126.58 127.35 127.39 128.52 130.10 131.27 132.01 130.56 130.86 129.93 129.85 131.12 130.46

129.95	133.19	133.11	134.68	133.92	134.22	133.46	134.08	135.69	134.67	137.11	136.27	134.65
134.28	136.00	136.57	136.32	135.49	133.93	133.91	134.31	134.67	134.51	133.15	132.68	134.17
134.70	135.45	134.23	134.14	133.94	135.14	135.48	134.33	133.05	132.81	132.72	130.67	130.32
130.78	131.67	131.66	129.71	130.72	130.55	131.28	132.17	132.54	132.09	132.00	131.75	131.33
132.77	133.77	133.45	132.59	132.41	130.62	130.71	130.96	132.22	132.24	131.97	132.32	133.75
133.49	133.61	132.15	133.32	131.64	132.27	131.95	133.35	132.66	132.81	132.56	131.89	131.21
129.77	128.64	127.37	128.24	128.32	129.30	129.37	129.64	129.65	130.23	130.34	130.76	128.72
128.34	127.13	127.40	128.46	129.90	129.14	129.16	130.01	130.92	130.48	131.18	132.67	132.64
131.75	132.54	133.85	133.40	133.51	132.49	131.22	130.11	130.49	130.69	129.23	125.50	125.13
124.96	123.51	124.08	122.79	123.02	123.49	121.24	120.07	117.98	118.40	120.14	121.61	120.66
119.77	118.87	117.15	116.26	117.24	116.68	115.65	115.01	112.77	115.53	115.94	116.18	117.08
119.36	119.51	119.39	121.31	122.31	121.45	121.21	120.78	118.80	119.71	119.19	118.98	120.28
120.10	119.64	118.60	118.16	119.29	119.45	119.06	121.89	124.20	124.80	124.74	123.54	122.67
123.29	122.70	122.92	123.19	121.67	120.24	121.15	120.26	120.71	121.71	121.60	123.51	124.05
125.09	126.35	126.10	124.69	125.12	126.26	125.19	124.82	125.48	125.71	124.93	122.78	122.99
122.42	123.12	124.00	123.34	122.88	122.31	122.54	122.27	121.67	122.30	122.55	122.74	120.05
119.18	118.93	119.55	116.78	116.30	114.88	115.54	116.33	117.22	115.97	115.27	115.75	115.38
115.41	115.19	115.74	118.92	120.40	117.78	118.01	116.48	116.42	117.26	114.63	113.68	114.66
114.43	114.38	114.06	113.69	113.82	113.22	111.59	111.51	113.47	113.21	113.11	113.31	112.68
110.92	109.88	109.34	107.34	108.83	109.41	109.36	108.61	109.45	109.28	109.08	110.30	110.61
112.77	113.55	112.97	113.21	111.94	112.30	112.27	111.96	113.79	115.12	114.73	115.36	115.46
116.22	116.00	115.99	115.83	116.35	116.81	116.70	115.44	115.72	117.19	118.64	119.26	118.00
117.26	116.13	116.44	116.81	117.46	117.67	118.68	119.47	118.38	119.42	119.17	118.22	118.01
116.71	115.84	114.89	114.59	114.89	114.79	114.40	113.11	112.66	111.88	111.68	112.04	111.86
110.09	110.12	109.63	108.99	109.61	111.24	109.96	109.69	108.87	107.60	107.28	107.20	108.30
110.14	109.83	109.14	110.26	110.21	109.61	108.71	107.65	107.29	107.22	107.53	108.83	109.57
109.45	110.44	110.47	111.07	110.73	111.54	111.42	111.47	111.17	110.36	109.43	107.73	107.72
107.09	108.98	107.83	106.14	105.16	103.71	103.08	102.84	102.60	102.42	103.85	104.09	109.04
108.53	109.16	113.02	116.11	115.34	117.58	118.55	117.11	117.66	119.51	118.25	120.28	122.68

121.37	122.20	121.97	120.97	122.24	123.10	124.29	123.77	122.55	122.51	124.88	123.99	123.81
123.32	123.62	123.24	121.63	120.42	121.97	121.51	121.98	125.97	128.80	131.05	134.47	134.44
136.71	134.57	133.57	136.73	136.58	139.23	139.06	138.83	133.32	134.48	135.28	133.59	133.71
135.47	137.49	142.87	141.85	142.16	140.44	143.02	141.16	141.76	139.53	137.03	135.42	137.93
138.34	137.02	134.22	132.93	133.88	134.88	134.20	138.54	138.72	138.82	138.69	141.77	142.72
141.81	140.00	139.57	139.95	137.39	135.24	135.30	137.49	136.26	138.61	138.83	139.72	142.18
140.77	141.24	140.33	140.64	138.34	141.35	141.96	145.27	145.18	146.78	145.78	146.69	145.72
146.65	146.71	146.40	145.27	146.29	143.85	139.97	141.75	141.54	144.27	144.51	145.30	142.96
143.23	144.26	146.14	146.93	145.70	145.00	147.50	147.63	148.92	148.30	147.43	147.44	148.00
145.48	146.79	149.60	149.74	148.06	150.88	152.30	153.48	153.67	153.67	151.26	152.87	151.80
151.24	150.83	151.36	149.80	149.59	149.90	151.19	150.66	152.81	153.37	152.67	151.85	151.59
153.39	152.96	153.02	151.89	151.04	151.76	152.85	155.14	155.82	156.77	158.11	158.75	159.74
158.71	160.71	160.05	160.42	158.81	161.81	161.44	162.95	164.42	162.11	162.34	163.31	164.28
166.10	165.81	165.95	164.96	164.25	164.90	163.40	163.71	163.27	161.99	162.14	163.43	165.54
166.21	165.48	164.46	162.39	162.55	163.98	164.42	164.83	162.77	161.36	161.83	162.68	164.84
165.53	167.12	169.14	169.13	169.02	170.53	170.99	170.57	170.40	168.46	165.68	166.64	168.11
168.91	166.60	168.48	167.56	167.08	168.11	165.53	165.46	166.01	164.29	163.95	164.83	169.29
169.06	168.89	169.53	170.35	167.59	165.04	162.56	162.04	162.01	163.44	161.33	161.74	159.18
160.13	161.54	161.55	162.16	163.71	163.41	165.29	163.55	163.98	164.34	162.77	161.25	160.84
162.14	162.25	162.58	164.40	164.23	165.00	167.89	167.96	167.77	166.92	165.48	164.80	165.36
164.38	166.24	167.62	169.25	168.41	169.76	169.51	170.07	168.43	168.01	167.23	166.07	165.80
166.27	167.74	170.28	170.80	172.65	170.34	169.63	169.88	169.86	170.43	167.81	166.73	166.98
165.95	165.99	166.47	165.38	164.84	163.37	163.55	163.64	164.84	163.45	162.44	161.91	161.76
163.97	164.41	166.29	166.58	165.36	166.08	166.13	165.09	166.05	166.84	166.96	167.18	166.54
167.91	166.40	166.49	165.44	165.77	165.47	165.91	165.20	165.09	165.62	164.93	163.33	161.67
162.39	162.32	162.00	163.56	163.27	163.22	164.76	165.34	164.86	164.93	164.04	166.78	168.81
169.28	168.90	167.95	167.79	167.75	167.02	167.18	167.83	167.55	167.04	166.21	164.87	165.94
164.84	164.24	163.94	162.87	163.41	162.74	163.36	160.91	158.08	158.74	155.85	155.42	156.30
154.95	156.61	156.25	156.13	155.74	154.64	154.31	154.29	157.51	159.30	156.82	157.06	158.19

159.24	157.89	156.25	154.57	155.19	154.35	156.34	156.78	156.77	157.41	159.27	157.78	158.86
158.66	156.69	156.86	156.67	157.30	159.88	159.52	159.18	157.98	157.66	157.54	155.04	155.48
155.45	155.87	155.00	157.73	157.31	158.32	158.97	157.90	158.02	156.80	158.07	158.65	160.30
159.89	160.05	161.68	161.90	161.20	159.11	159.47	160.52	160.11	160.00	158.49	157.50	158.00
157.99	156.57	155.78	154.73	153.88	153.15	151.23	151.62	150.29	150.35	150.55	153.24	154.34
153.65	155.01	154.92	155.17	153.06	152.19	152.13	150.39	149.03	151.73	152.61	154.84	154.51
154.46	153.97	152.71	151.64	152.84	153.18	153.20	153.70	152.76	152.24	153.36	152.89	150.56
150.03	150.88	151.60	152.38	151.40	150.37	149.55	148.95	147.82	148.83	150.08	151.19	150.19
150.66	154.08	157.99	162.35	162.60	162.71	161.75	165.54	165.42	165.43	164.42	162.80	163.77
164.14	164.94	167.83	167.06	167.12	167.51	166.44	167.40	167.10	166.60	166.68	164.88	164.29
165.65	164.37	164.26	164.45	164.68	167.94	168.78	168.87	167.65	166.94	167.47	165.67	165.28
165.62	166.28	166.96	166.10	164.62	163.59	162.44	162.92	162.68	162.13	161.67	162.11	162.78
164.18	165.77	164.78	164.14	168.10	167.96	167.36	167.09	167.20	166.31	165.29	164.78	166.84
166.09	167.49	167.42	168.58	170.41	169.17	168.68	167.60	167.36	165.97	165.99	165.89	164.10
163.10	164.18	164.52	166.92	165.55	166.29	165.02	163.91	163.58	162.82	163.38	162.10	162.76
162.26	162.83	163.07	162.63	161.81	162.69	163.61	168.11	167.16	166.38	165.51	166.76	166.47
165.75	166.26	167.24	165.37	164.57	163.68	164.24	163.99	165.18	168.31	167.91	170.51	170.81
171.19	170.73	171.32	175.23	175.48	177.30	176.71	177.35	177.40	179.18	179.39	179.63	178.63
180.35	180.61	180.43	181.82	182.19	180.51	180.56	183.35	182.41	181.60	181.33	181.18	180.19
179.36	179.23	181.17	180.71	181.18	183.23	182.06	182.23	180.65	179.51	179.10	178.79	179.66
178.19	177.84	176.53	176.88	179.54	179.08	179.35	179.04	177.97	178.43	179.54	179.54	180.66
181.27	180.53	179.11	179.03	178.03	178.21	179.42	180.19	180.54	180.92	181.20	181.68	180.84
181.11	180.70	181.88	182.26	183.43	182.18	180.63	179.83	178.37	179.01	180.08	179.99	180.76
180.62	181.92	184.28	184.61	183.87	184.54	185.66	187.42	189.72	189.64	188.56	187.60	188.29
187.86	187.68	187.75	189.55	189.32	190.04	190.16	191.06	189.68	189.51	189.04	187.61	185.33
187.10	186.53	187.34	186.63	186.73	189.61	189.15	189.74	190.06	191.23	191.85	192.43	192.01
191.45	192.52	191.93	191.05	192.37	192.94	193.29	192.72	194.72	195.65	194.38	195.13	194.35
192.55	191.58	192.06	192.40	189.60	189.93	190.92	192.11	191.48	190.62	187.93	187.68	188.95
188.32	187.63	187.30	187.41	187.26	186.10	186.38	188.08	189.16	187.36	187.17	187.31	188.10

188.83	188.93	188.63	187.91	187.37	187.27	188.24	188.25	186.90	185.03	183.69	182.91	182.88
181.36	181.71	183.39	182.05	184.30	182.62	180.66	181.29	181.30	182.08	185.07	184.06	184.36
183.22	181.87	181.87	182.52	182.78	184.28	186.37	186.08	187.98	187.66	187.04	186.96	188.04
189.09	188.50	187.52	187.76	189.23	190.07	189.82	191.53	191.25	192.37	192.76	192.62	193.72
197.28	198.08	197.10	199.06	198.11	198.71	198.67	198.99	201.41	201.52	200.35	200.67	202.54
202.17	200.46	200.86	204.23	203.88	202.99	204.25	204.39	206.31	206.73	209.94	212.02	210.65
209.81	210.02	210.94	208.57	207.14	207.65	209.61	210.68	211.28	209.59	210.88	210.65	213.80
207.97	206.11	205.96	206.72	206.64	208.26	209.17	208.43	207.53	205.79	203.49	204.25	206.43
207.39	209.81	210.29	209.33	211.78	213.96	212.79	212.96	213.47	214.56	216.24	215.92	215.97
217.40	219.76	222.45	219.76	222.22	224.62	224.34	223.79	224.04	226.77	226.92	225.42	224.38
224.34	225.13	225.57	226.58	231.69	232.54	233.19	236.55	234.67	235.78	235.60	236.54	233.34
235.33	234.72	237.30	238.97	238.90	235.14	235.71	232.47	228.69	228.63	233.52	233.75	236.44
235.97	237.28	237.73	242.22	243.03	242.38	244.74	242.42	241.75	242.02	242.29	243.08	240.51
235.52	235.16	234.79	237.73	237.24	236.08	237.13	237.85	237.58	236.41	237.54	234.43	232.76
233.20	236.11	235.45	240.12	241.35	244.75	246.63	247.98	247.35	245.04	245.51	243.94	245.65
245.67	239.96	239.58	241.13	241.49	245.73	246.13	244.35	244.99	244.06	247.58	245.26	247.03
248.93	248.74	249.60	250.84	252.04	252.70	251.79	244.05	241.59	242.82	243.01	242.22	238.11
233.66	235.01	236.07	236.36	236.24	238.18	238.67	237.95	240.22	236.01	234.55	236.59	236.12
234.91	235.99	237.03	236.84	237.04	236.88	240.68	243.34	245.67	246.25	247.15	247.38	246.51
249.77	249.67	250.19	247.81	252.84	253.30	252.84	252.93	248.52	250.08	253.83	250.47	248.14
247.67	247.06	235.18	230.67	231.94	231.72	231.68	232.31	232.21	234.93	235.67	236.28	231.83
232.23	229.91	231.32	233.60	233.92	233.71	234.78	234.41	236.68	235.85	235.48	235.91	235.37
238.80	239.53	238.84	235.97	235.88	236.26	239.28	238.26	238.77	239.26	240.94	243.71	243.98
245.80	246.20	246.58	245.87	245.77	246.13	247.08	246.64	243.02	244.50	243.21	236.78	237.66
242.05	245.86	247.45	248.17	248.77	249.22	249.05	254.00	253.85	253.04	251.17	251.16	249.28
250.96	248.17	247.35	248.21	250.04	247.56	246.78	249.73	248.75	246.34	246.75	246.92	244.67
243.37	242.17	246.45	252.19	252.78	255.33	257.29	258.73	260.30	259.95	262.64	265.49	266.28
269.34	269.04	267.84	273.91	270.10	269.61	273.75	275.40	274.24	274.08	276.45	275.99	279.64
281.16	280.04	278.16	275.07	277.54	275.62	279.70	285.49	285.42	285.57	285.48	282.38	282.88

284.00	282.96	284.20	283.00	284.12	288.62	290.52	290.66	288.30	290.86	290.31	291.22	289.89
288.23	292.47	292.78	294.08	298.17	301.16	301.64	300.38	300.93	296.13	289.20	291.70	292.39
293.63	300.41	301.95	296.69	297.26	292.86	292.49	285.62	279.16	284.44	286.91	286.09	293.07
287.19	286.82	281.52	281.83	282.51	284.57	288.36	288.03	289.36	295.34	295.47	294.71	293.37
291.57	293.30	293.98	294.24	287.43	286.65	279.62	278.21	280.17	282.16	289.11	288.73	290.76
290.10	289.83	288.46	293.47	295.09	293.45	296.72	297.28	297.47	298.73	301.62	303.14	304.76
304.81	305.69	306.97	309.65	308.43	306.86	308.96	307.16	307.90	304.00	302.94	305.63	304.92
307.40	308.29	307.52	308.37	307.63	310.68	310.42	312.70	314.59	311.39	308.55	308.47	307.81
309.27	310.65	312.33	315.65	318.05	318.66	317.57	316.23	318.45	322.09	323.00	328.00	333.33
332.39	334.65	333.99	334.11	329.25	329.83	334.84	335.90	333.33	336.77	334.57	331.38	327.04
329.80	323.40	321.68	320.21	316.70	313.56	313.92	317.13	321.98	323.08	317.74	314.86	314.93
314.86	310.54	319.50	321.19	319.72	320.16	323.20	321.69	321.83	327.33	328.07	328.08	319.22
318.54	314.16	311.07	309.39	314.52	305.23	298.08	282.70	224.84	236.83	258.38	248.25	248.22
227.67	233.19	233.28	244.77	251.79	255.75	250.82	248.96	254.48	250.41	243.17	239.00	241.90
248.52	245.64	246.76	243.04	245.55	240.05	242.00	242.99	246.39	244.10	240.34	230.30	232.00
233.45	225.21	223.92	228.76	234.91	238.89	233.57	235.32	242.19	242.80	248.08	242.98	249.16
249.54	249.95	253.16	252.02	245.57	244.59	247.86	247.08	255.94	258.63	258.89	261.07	243.40
247.49	245.42	245.81	245.88	252.05	251.88	249.32	242.63	243.14	246.50	252.17	249.57	249.38
253.29	257.07	255.04	255.57	252.21	252.21	250.96	249.10	251.72	256.66	255.95	257.63	259.83
259.21	257.91	261.61	265.64	265.02	264.43	261.58	262.46	267.82	267.22	267.98	267.88	267.30
267.38	269.43	269.06	263.84	264.94	266.37	266.13	268.65	271.22	271.12	268.74	268.84	268.91
263.35	258.51	258.06	260.07	258.07	258.89	256.09	258.51	265.49	266.16	269.43	270.16	271.37
271.57	259.75	259.77	259.21	257.92	256.13	256.42	260.14	262.46	263.93	263.80	262.61	261.33
261.56	263.00	260.32	258.79	257.48	256.54	257.62	253.31	253.85	256.78	258.71	255.39	251.35
252.57	253.02	250.83	253.51	253.76	254.63	253.42	262.16	266.69	265.33	266.45	267.05	265.17
271.52	270.20	271.26	271.43	274.30	274.45	269.77	270.68	268.94	271.67	275.66	274.82	273.78
269.06	272.31	270.98	273.50	271.78	275.81	272.02	271.78	270.02	270.55	267.85	269.32	270.26
272.05	270.51	268.47	270.00	266.66	263.50	264.68	265.19	262.50	266.02	272.02	272.21	272.06
272.98	271.93	271.15	269.98	266.49	261.90	262.75	262.55	258.69	260.56	260.77	261.03	260.24

256.98	257.09	261.13	259.18	259.68	262.33	262.51	261.52	258.35	264.48	265.59	265.87	265.88
266.84	266.47	267.43	269.31	268.13	270.65	268.82	269.73	270.16	269.18	269.76	268.88	268.26
269.08	272.59	271.91	271.38	270.62	271.86	272.39	278.07	278.24	277.93	273.98	275.22	275.50
276.41	279.38	276.97	282.88	283.66	282.28	282.38	281.38	277.28	278.53	278.97	279.06	279.06
279.20	276.31	273.93	275.15	273.33	273.69	267.92	267.72	268.34	263.82	264.60	266.47	266.22
267.21	269.00	267.23	268.64	270.91	273.70	272.49	271.81	274.93	277.59	278.13	276.57	277.03
276.52	276.31	275.31	274.28	276.29	278.91	277.47	277.38	276.87	277.87	276.83	277.08	279.40
277.72	275.31	279.43	280.01	280.67	280.98	280.38	282.01	283.17	283.87	284.14	283.55	286.53
286.90	286.63	284.50	288.49	289.14	291.69	293.82	294.99	297.47	297.09	296.84	296.97	296.04
299.63	298.65	296.06	292.02	292.54	291.81	294.24	294.81	296.76	295.98	290.91	292.05	287.13
287.82	288.86	287.11	289.95	291.18	294.81	293.87	294.08	293.93	292.88	295.32	295.14	296.67
299.44	292.69	289.92	291.33	290.49	288.98	290.57	291.59	292.35	292.52	294.87	296.39	295.31
296.24	295.29	297.16	297.11	298.49	298.99	296.40	301.36	301.72	306.02	307.15	306.19	309.61
308.69	306.75	306.93	309.58	309.64	309.12	308.12	308.16	307.77	307.61	306.00	305.19	305.80
306.95	313.84	316.16	315.28	317.48	317.97	321.24	321.98	318.32	319.14	319.17	321.59	319.05
320.52	321.97	325.52	322.03	324.24	326.95	326.75	326.69	326.24	323.91	323.83	320.08	321.35
321.89	321.25	320.48	322.32	328.00	326.60	328.44	325.81	319.68	317.98	319.23	320.64	321.55
324.91	327.07	328.78	329.81	329.95	331.84	332.44	331.35	335.73	333.51	335.90	333.67	333.88
338.05	341.99	342.15	346.08	343.75	344.34	344.74	343.92	349.41	349.35	346.94	348.25	344.74
343.06	344.71	345.66	344.45	346.03	340.67	341.19	344.70	351.52	350.52	352.09	349.84	350.65
351.45	353.73	352.56	349.24	348.35	348.76	347.66	348.70	345.46	343.16	345.06	346.73	346.55
346.47	345.70	347.05	344.23	344.33	345.10	348.60	349.15	350.87	354.71	356.94	356.97	358.78
359.80	359.13	356.99	355.39	333.63	342.85	341.16	341.76	347.13	347.16	344.83	343.70	342.50
337.93	335.06	335.07	340.36	341.20	338.48	337.62	332.61	334.81	338.15	336.57	339.10	339.55
337.99	340.54	340.58	341.61	339.35	339.59	341.91	343.97	345.61	345.77	343.60	345.99	350.63
351.41	349.58	348.55	347.59	348.69	348.56	351.73	352.75	350.93	350.06	343.69	342.46	342.84
344.78	347.42	346.81	348.81	350.67	353.40	359.69	358.76	355.67	352.20	353.79	349.62	347.31
348.53	339.93	337.00	340.75	337.40	338.19	339.15	330.38	331.61	330.26	326.08	325.80	325.20
322.98	329.08	328.79	330.92	331.85	329.66	333.75	332.96	333.62	330.08	331.02	332.01	334.89

332.72	327.99	327.67	325.70	324.15	328.67	330.26	331.89	332.74	335.54	333.74	337.93	336.95
340.27	337.93	338.67	336.00	336.87	338.07	341.91	343.53	341.57	339.74	335.69	337.22	337.63
341.50	342.00	340.79	339.94	338.70	343.64	341.09	340.73	340.08	341.37	342.07	341.92	344.34
344.74	344.68	340.72	338.09	335.12	331.05	330.36	332.03	332.92	329.11	330.80	332.25	334.48
335.57	338.39	340.53	342.01	342.86	343.82	352.00	354.75	354.28	354.00	354.47	354.64	358.00
358.43	359.29	358.41	354.58	360.65	360.86	361.23	363.16	367.40	366.64	364.96	363.15	358.71
361.63	366.25	364.90	362.90	362.91	356.88	358.47	359.10	360.47	355.43	352.31	352.06	355.14
357.63	358.02	359.54	360.16	355.68	358.42	359.52	356.49	361.23	365.44	367.31	368.95	367.52
364.22	365.32	361.61	355.31	355.79	357.09	355.91	353.44	355.55	356.15	355.52	351.48	344.86
334.43	334.83	338.35	339.94	335.52	338.84	339.39	340.06	332.39	327.83	328.51	321.86	316.55
307.06	311.51	321.44	321.34	324.19	318.71	322.56	323.09	324.39	320.46	323.40	321.63	321.04
322.54	318.65	316.83	317.77	318.60	316.60	311.48	311.32	304.59	308.26	305.06	300.97	306.05
314.94	315.21	311.40	312.69	311.50	313.48	305.10	300.39	295.46	300.03	303.23	298.92	298.76
305.74	312.48	314.76	312.36	312.60	310.17	304.71	301.88	304.06	304.00	307.02	311.85	314.59
311.62	306.01	307.61	313.74	319.48	317.67	320.40	317.02	317.12	319.34	315.31	316.03	315.10
316.51	318.10	317.95	316.42	322.22	324.10	326.35	329.92	329.07	327.75	328.89	326.44	330.19
329.34	326.82	326.02	330.05	330.20	330.12	331.75	329.90	330.85	328.29	328.72	330.22	326.45
321.91	321.00	315.44	314.90	311.49	314.53	315.23	312.49	313.73	316.17	327.97	332.23	331.06
328.31	330.21	334.78	336.07	336.03	335.84	340.91	343.93	343.05	348.34	351.26	358.07	356.52
359.35	368.58	365.50	369.02	364.22	369.06	369.39	365.14	364.97	365.65	367.26	362.81	367.74
367.07	370.47	369.33	376.72	376.17	375.91	374.95	372.96	370.03	374.57	373.50	373.59	372.11
366.59	367.92	366.58	367.48	369.83	376.30	375.35	375.22	371.30	379.50	378.94	379.77	375.36
378.66	373.56	373.15	377.63	380.40	381.19	387.62	390.45	388.46	384.20	380.95	381.76	382.76
379.25	379.02	373.66	375.35	380.29	380.52	380.80	380.08	377.32	378.51	383.25	375.74	376.76
371.62	368.57	372.19	372.39	372.28	375.35	376.19	374.97	377.49	381.94	382.79	386.96	389.83
388.06	387.74	385.09	383.63	379.43	378.57	381.05	376.65	377.63	382.29	380.13	378.59	375.09
375.42	377.75	370.94	370.65	371.59	374.40	371.16	377.92	377.47	373.33	374.08	377.94	376.11
375.74	376.97	380.25	382.39	381.54	381.18	385.37	384.22	382.88	379.42	378.64	380.96	380.93
383.15	386.69	387.81	387.12	387.18	385.06	390.62	390.56	389.32	387.12	388.02	389.62	389.90

389.33	385.58	376.47	379.43	390.59	391.33	394.17	393.85	393.06	396.64	396.47	395.43	392.15
389.97	389.14	389.10	388.57	384.56	385.09	387.34	383.59	385.78	385.50	386.94	387.56	387.92
385.92	387.71	386.88	386.49	385.90	387.86	389.20	388.26	384.47	381.24	379.50	380.67	376.80
380.55	381.45	386.47	391.01	392.80	391.92	392.50	390.02	387.83	387.94	385.07	384.20	389.52
391.48	392.96	392.46	391.32	390.28	388.71	389.97	393.72	392.89	393.12	396.74	397.41	397.15
382.62	385.24	379.42	378.53	380.06	376.14	375.34	377.96	376.55	375.22	381.40	380.96	380.07
377.40	379.09	378.26	377.90	377.70	381.55	384.47	384.46	382.74	383.48	382.52	387.04	396.82
399.33	404.84	406.46	415.14	417.09	417.26	419.34	417.96	417.40	418.10	417.61	415.10	414.34
420.44	420.77	418.21	418.86	416.36	412.64	418.13	414.96	415.48	414.99	414.96	410.34	411.63
408.79	409.53	413.85	413.84	413.82	411.09	413.77	413.76	417.13	413.69	412.48	407.38	408.26
413.90	411.46	412.27	410.45	415.35	413.86	412.70	412.45	412.85	409.33	406.51	404.44	405.21
406.89	404.03	403.89	405.84	406.39	409.58	409.15	409.80	411.30	409.91	408.88	407.52	407.86
403.50	403.00	403.69	404.23	400.50	401.55	405.59	398.06	394.50	400.64	404.29	406.08	412.39
416.28	416.05	410.16	410.26	409.81	411.60	409.02	408.45	409.11	412.02	414.95	412.53	416.91
416.84	416.79	415.85	416.05	418.49	416.29	416.45	413.14	410.09	412.81	416.37	415.39	412.60
414.02	411.41	412.17	416.74	415.35	417.30	413.50	414.59	413.26	413.48	413.36	410.06	407.25
409.05	409.76	410.29	408.32	402.26	400.96	403.67	403.40	404.04	403.83	403.12	403.45	408.94
408.14	412.88	411.77	413.84	409.16	410.28	414.23	414.62	414.87	417.68	417.10	417.54	415.62
413.75	413.76	410.93	412.08	411.60	411.54	417.52	422.23	423.92	424.21	425.09	424.36	422.19
420.59	418.88	419.42	418.90	417.78	417.73	419.91	420.74	421.34	418.18	418.26	414.85	410.72
411.61	413.51	413.53	414.84	414.03	416.07	417.98	417.98	417.08	414.44	416.36	419.95	419.58
425.27	419.77	419.92	419.93	422.93	422.14	417.14	417.44	418.47	414.35	416.62	416.80	417.80
416.29	410.47	407.57	407.18	404.25	407.75	402.66	407.44	409.30	409.37	409.60	411.73	414.98
415.48	415.67	414.90	414.10	418.16	418.49	420.13	420.86	418.68	422.75	419.92	417.11	418.34
417.58	418.59	418.62	422.20	422.87	422.43	420.68	419.27	422.85	423.61	426.65	425.12	427.59
429.19	430.16	431.35	430.78	429.89	429.91	432.06	435.31	436.99	435.65	434.64	433.73	432.84
432.57	431.52	435.43	441.28	440.70	440.31	439.03	439.77	439.15	437.98	438.82	435.71	435.38
434.34	434.52	430.73	429.05	430.95	431.04	433.03	435.94	437.15	436.84	435.13	433.37	435.49
436.11	440.01	439.95	438.11	438.66	438.78	442.52	442.56	447.20	449.56	448.93	447.85	445.33

446.23	447.66	444.58	433.91	433.30	431.90	434.22	435.24	434.80	440.87	442.34	443.38	442.01
447.90	449.26	447.34	446.11	454.71	454.40	456.33	453.72	449.83	451.43	451.37	448.31	451.89
450.18	448.88	448.76	448.07	450.88	447.78	450.77	451.97	451.67	450.30	441.39	442.29	441.16
442.73	441.84	448.37	449.22	448.66	448.40	448.94	447.46	445.10	443.63	439.46	437.03	433.54
438.01	438.02	438.89	440.19	442.46	444.05	444.52	443.26	442.31	442.80	444.36	444.80	439.23
439.56	440.37	440.32	447.57	450.59	445.84	448.00	448.85	453.44	452.41	450.19	453.83	453.85
452.49	450.06	447.69	444.71	445.78	445.38	447.26	447.71	446.27	447.43	448.54	443.68	446.22
445.93	443.19	446.62	447.60	451.85	450.69	450.53	449.02	445.84	441.43	442.83	448.64	448.13
448.98	448.09	450.08	449.22	445.75	446.03	447.31	447.18	444.51	447.10	449.09	448.24	447.19
450.24	448.13	450.15	449.27	448.54	448.13	448.68	450.72	449.45	450.46	448.96	450.14	452.38
453.13	456.04	456.43	456.16	455.23	459.77	460.13	461.04	460.54	461.90	463.56	463.15	461.30
461.34	458.52	456.65	457.50	461.72	462.06	459.90	461.60	459.43	458.83	455.05	452.95	456.20
457.74	457.63	461.80	461.53	460.11	458.93	461.29	461.34	461.20	460.74	459.18	460.31	460.88
461.12	461.49	466.83	469.50	468.45	466.21	466.07	465.36	463.27	464.20	464.30	464.61	467.73
467.83	469.10	468.44	463.02	457.49	459.57	460.21	460.33	463.72	462.64	465.39	463.75	466.74
464.81	463.62	462.60	459.13	461.03	462.36	463.06	461.90	461.79	461.89	463.11	464.89	466.43
466.76	466.29	464.18	463.93	465.70	463.06	461.84	463.34	466.38	465.85	465.30	467.32	467.38
470.54	470.94	470.58	468.64	466.45	465.44	466.89	467.55	467.12	469.90	475.27	474.13	474.17
472.47	474.91	473.30	474.25	474.30	474.98	474.72	471.97	470.92	473.20	477.05	478.70	481.61
479.62	482.00	480.71	469.81	471.76	471.05	472.77	468.93	470.18	470.23	472.52	472.79	470.34
467.69	471.46	470.69	464.26	466.07	467.14	464.44	464.81	463.01	464.74	466.91	465.88	467.06
463.90	466.44	467.39	467.01	469.42	470.90	471.06	468.54	468.80	468.54	464.35	460.58	460.00
452.48	445.55	445.77	438.92	448.29	448.05	450.88	447.10	449.87	447.57	446.26	446.38	446.18
442.46	442.54	441.96	448.73	447.63	452.71	451.87	449.10	450.91	453.02	453.03	451.72	451.38
447.82	442.32	446.01	441.49	443.75	444.14	444.49	449.37	453.69	456.48	454.92	453.20	454.81
456.34	457.06	457.33	456.50	457.63	457.65	460.13	458.88	458.21	457.06	457.86	458.67	459.10
462.37	460.61	461.93	458.45	455.48	451.34	453.09	449.63	442.80	447.31	446.07	447.63	444.27
446.20	446.37	446.13	448.38	449.55	448.06	447.95	448.73	453.41	454.16	455.22	453.86	451.60
452.61	453.11	454.25	453.36	452.57	454.23	458.26	461.01	460.56	461.45	458.40	457.09	457.89

457.93	460.30	458.88	461.94	461.23	465.01	465.17	463.17	463.68	462.32	464.51	469.03	468.08
473.80	474.59	476.07	475.49	473.17	470.99	471.86	470.96	473.14	468.18	466.21	467.51	468.80
474.81	471.19	470.85	463.36	461.46	461.27	459.67	460.82	462.05	464.84	462.24	462.69	461.74
454.59	453.52	452.36	455.10	459.04	465.79	465.47	467.77	469.10	468.96	467.66	470.28	466.85
464.89	460.83	461.53	462.62	465.85	473.77	472.35	468.42	466.50	467.91	462.28	463.07	465.65
465.40	464.37	462.35	466.04	465.03	465.60	463.56	461.47	458.30	450.09	449.93	452.29	454.16
455.17	453.69	448.92	453.30	453.32	453.11	451.23	445.45	446.96	449.47	450.15	454.97	455.34
458.80	457.91	457.10	459.61	459.67	459.83	462.47	460.86	461.16	459.27	459.11	460.71	460.34
460.68	460.83	461.68	461.66	461.64	465.97	469.38	470.05	469.72	466.95	464.78	465.81	465.86
467.44	468.32	470.39	468.51	470.42	470.40	472.78	478.64	481.14	480.81	481.19	480.19	481.46
481.65	482.55	484.54	485.22	481.97	482.74	485.07	486.91	488.11	483.81	487.39	485.65	485.13
485.42	485.63	482.12	483.14	483.16	489.57	490.05	492.89	491.88	495.41	495.52	496.15	495.07
495.67	495.95	500.97	503.20	503.90	503.12	502.22	500.71	501.85	505.24	505.57	506.08	506.42
507.01	505.53	507.17	509.23	506.13	505.37	504.92	505.29	508.49	512.89	512.10	512.66	513.55
514.71	514.26	514.86	520.48	520.54	520.12	523.96	523.56	524.36	524.37	525.55	527.74	528.19
527.07	519.58	519.19	523.65	528.59	528.61	528.59	523.65	523.58	533.40	533.49	532.51	535.60
535.55	533.13	532.35	527.94	530.88	536.05	536.47	537.12	539.83	545.22	544.98	543.98	551.07
549.71	544.13	542.43	544.73	543.87	544.75	547.09	547.26	553.99	556.37	557.19	554.78	560.89
561.00	559.89	562.72	558.46	550.98	553.54	553.62	556.63	561.10	561.61	565.22	562.93	562.06
559.64	558.80	558.75	558.94	560.03	560.39	559.71	557.45	555.11	559.74	558.57	559.97	559.04
559.21	558.11	559.52	557.14	557.46	560.10	559.05	560.00	560.92	561.88	563.84	569.17	570.17
570.29	572.68	573.91	576.51	578.77	583.61	583.35	582.77	584.20	586.77	583.00	581.73	581.81
581.41	581.04	585.87	584.41	581.72	582.34	581.47	582.63	582.49	578.37	577.52	579.46	583.10
584.50	583.03	586.78	587.44	590.65	587.46	585.06	586.54	582.47	576.72	579.70	583.25	581.50
584.22	589.72	590.57	588.46	586.32	591.71	593.26	592.72	592.30	589.29	593.96	597.34	600.07
596.85	600.24	598.40	599.97	601.32	606.45	607.64	605.37	606.98	613.68	617.68	620.18	616.17
617.48	619.52	618.78	621.69	616.92	616.34	606.81	611.93	605.94	610.49	611.96	614.30	614.53
614.12	615.93	620.73	621.32	617.70	616.71	618.46	609.45	598.48	602.69	601.81	599.82	608.44
606.37	608.24	611.83	613.40	612.79	619.96	617.03	621.62	624.22	630.15	636.02	638.46	635.84

641.43	646.33	649.93	656.07	656.37	661.45	660.51	655.58	651.32	647.98	640.65	648.10	658.86
659.08	650.46	647.24	644.75	640.43	644.37	650.81	655.79	652.00	653.65	633.50	640.02	637.09
638.55	640.87	641.43	652.65	651.69	649.98	649.19	650.62	650.04	652.97	648.91	648.94	645.50
653.73	655.26	655.88	655.86	644.24	642.19	633.50	631.18	636.71	642.49	645.00	641.61	643.61
645.07	647.89	651.58	650.17	652.87	653.46	654.16	654.17	654.58	643.38	641.63	640.81	638.26
644.77	645.44	652.09	661.51	665.60	665.42	664.85	668.91	673.15	672.76	678.42	676.00	678.51
672.23	667.93	671.70	669.12	667.68	672.56	678.44	673.03	673.31	672.16	670.97	669.04	667.92
665.85	665.16	662.06	661.96	662.10	666.84	668.85	668.48	664.39	668.55	670.63	675.88	673.61
672.40	657.44	652.54	654.75	656.06	645.67	646.19	629.80	628.37	634.07	643.56	638.73	633.77
626.87	626.65	631.17	635.90	630.91	635.26	639.95	650.02	662.49	660.23	662.38	664.16	662.59
662.10	665.77	660.20	662.05	662.28	665.21	666.58	665.69	665.07	670.68	667.03	663.88	666.40
664.81	657.40	651.99	654.72	655.61	649.44	655.68	663.76	663.81	667.28	671.15	680.54	683.98
682.94	681.47	683.00	687.03	686.48	685.61	685.83	685.86	686.19	687.31	689.08	694.01	692.78
701.46	703.34	700.64	696.74	694.61	700.66	703.54	702.57	704.41	706.99	710.82	709.85	706.57
707.27	702.29	700.92	697.26	701.50	700.90	705.27	703.77	706.73	714.14	724.59	727.65	730.82
731.87	729.56	731.13	735.88	737.62	737.02	742.16	743.95	742.75	748.73	757.03	755.96	755.00
757.02	756.56	748.28	745.10	744.38	739.60	749.81	747.54	740.73	729.33	728.64	720.98	726.04
731.54	745.76	748.87	746.92	751.03	755.82	756.79	753.85	740.74	737.01	748.03	747.65	753.23
748.41	754.85	759.50	759.51	768.86	767.20	769.75	776.17	776.70	782.72	786.23	777.56	770.52
765.02	765.02	772.50	784.17	786.16	786.73	789.26	778.28	780.15	789.56	785.43	789.59	802.77
811.82	808.48	816.29	812.49	802.80	801.77	810.28	812.10	805.68	795.07	790.82	795.31	790.95
801.99	798.56	804.97	813.65	811.34	804.26	789.56	793.17	795.71	789.66	785.77	782.65	784.10
790.89	789.07	790.50	773.88	757.12	759.64	750.11	750.32	757.90	762.13	766.12	760.60	758.34
737.65	743.73	754.72	763.53	761.77	766.34	760.37	774.61	773.64	771.18	765.37	772.96	794.05
801.34	798.53	812.97	830.24	827.76	815.62	820.26	824.78	837.66	833.13	836.04	841.88	829.75
833.27	841.66	839.35	835.66	847.03	849.71	847.21	844.08	848.28	846.36	845.48	840.11	843.43
858.01	862.91	865.27	869.57	883.48	893.27	893.90	894.42	889.06	897.99	898.70	878.62	896.34
888.99	883.68	887.30	885.14	891.03	904.03	916.92	912.20	918.75	907.54	913.78	916.68	918.38
925.76	936.59	931.61	915.30	912.94	933.98	936.56	940.30	938.79	936.45	942.29	952.29	954.29

947.14 950.30 952.37 960.32 951.19 933.54 937.00 926.53 922.02 924.77 900.81 912.49 926.01
939.35 925.05 923.55 920.16 913.02 913.70 903.67 899.47 927.58 927.86 930.87 929.05 931.20
933.62 919.03 912.59 923.91 919.77 945.64 943.00 947.29 950.51 955.43 951.93 944.48 937.91
945.22 953.34 947.28 955.41 960.46 965.03 972.69 983.12 973.84 970.62 966.98 968.10 970.28
965.72 955.23 944.16 955.61 972.28 968.49 950.69 941.64 876.99 921.85 919.16 903.68 914.62
938.99 940.76 942.76 938.03 927.51 921.13 923.78 905.96 916.66 928.35 946.20 938.23 944.59
958.98 963.09 946.67 950.82 951.64 955.40 974.78 971.68 976.77 973.10 983.79 982.37 975.78
969.79 954.94 953.39 963.39 968.04 965.54 955.30 946.78 953.70 939.13 932.70 936.46 953.35
970.84 970.43 975.04 977.07 966.58 964.00 956.05 927.69 939.21 952.12 957.94 950.73 961.51
978.60 970.81 963.04 957.59 956.95 969.02 977.46 985.49 980.28 1001.27 1006.00 1006.90 1003.54
1012.46 1010.74 1019.01 1020.01 1024.14 1020.09 1022.76 1032.08 1028.28 1034.21 1038.14 1030.56
1042.90 1048.67 1049.34 1047.70 1052.02 1047.33 1035.05 1055.69 1052.31 1064.25 1068.47 1069.92
1068.61 1079.27 1080.45 1085.52 1089.74 1099.16 1095.55 1105.65 1101.93 1100.80 1095.44 1093.55
1101.75 1108.15 1120.01 1122.70 1121.38 1109.55 1101.65 1110.67 1109.69 1115.75 1119.32 1108.17
1122.72 1123.65 1126.67 1130.54 1119.58 1107.90 1086.54 1085.11 1094.63 1111.75 1121.00 1122.07
1115.50 1104.92 1095.14 1108.14 1106.64 1115.79 1118.86 1117.37 1108.73 1105.82 1109.52 1119.06
1114.64 1110.47 1094.02 1092.23 1097.60 1090.82 1090.98 1093.22 1082.73 1094.83 1113.86 1115.72
1118.41 1112.28 1094.58 1098.84 1077.01 1087.59 1107.11 1106.37 1100.65 1103.21 1119.49 1132.88
1129.28 1133.20 1138.49 1133.84 1148.56 1146.42 1157.33 1154.66 1166.38 1158.56 1164.33 1165.19
1177.58 1174.81 1183.99 1186.75 1184.10 1165.07 1164.08 1139.75 1140.80 1147.27 1130.24 1125.21
1142.95 1120.67 1112.44 1072.12 1081.43 1089.63 1089.45 1083.14 1068.98 1084.22 1074.91 1062.75
1083.67 1101.20 1098.06 1091.60 1081.24 1088.14 1092.85 1084.19 1042.59 1027.14 957.28 994.26
990.47 982.26 973.89 1023.46 1006.20 980.19 1009.06 1029.72 1037.68 1045.48 1018.87 1020.09
1023.89 1029.63 1066.09 1042.72 1044.75 1048.69 1049.02 1017.01 986.39 1002.60 988.56 984.59
970.68 959.44 984.39 997.71 994.80 1005.53 1047.49 1056.42 1062.39 1063.93 1069.92 1078.48
1070.67 1072.32 1065.34 1068.09 1085.93 1098.67 1111.60 1110.84 1118.67 1133.85 1141.01 1130.20
1128.26 1120.97 1117.69 1125.72 1135.87 1139.32 1144.48 1152.61 1163.55 1188.21 1182.99 1186.87
1192.33 1163.63 1175.28 1171.25 1150.14 1176.74 1187.70 1181.38 1183.49 1165.02 1166.46 1141.20
1162.83 1161.94 1179.98 1188.03 1202.84 1203.57 1228.54 1226.27 1225.49 1241.81 1231.93 1229.23

1228.10 1244.78 1272.34 1269.73 1275.09 1263.88 1239.51 1234.40 1212.19 1243.26 1252.00 1256.62
1235.16 1225.19 1233.98 1252.31 1243.17 1265.37 1279.64 1273.00 1261.99 1272.07 1248.49 1239.40
1243.77 1216.14 1223.55 1254.04 1230.13 1241.87 1224.03 1237.28 1239.22 1272.14 1271.18 1253.41
1245.02 1238.33 1236.16 1225.50 1227.70 1246.64 1275.47 1282.73 1279.84 1286.84 1297.68 1294.59
1307.26 1306.38 1297.82 1316.55 1299.29 1297.01 1262.14 1268.59 1289.99 1282.80 1310.17 1300.75
1286.37 1293.72 1321.12 1317.89 1326.89 1343.98 1348.35 1358.64 1349.82 1328.44 1322.86 1319.00
1289.48 1306.17 1336.12 1358.82 1356.85 1360.04 1362.80 1350.91 1342.83 1335.18 1354.63 1332.00
1347.31 1332.05 1345.00 1340.30 1355.61 1364.00 1367.56 1337.80 1339.49 1333.32 1344.23 1338.83
1330.29 1306.65 1284.40 1304.76 1281.41 1301.84 1294.26 1294.81 1299.54 1327.75 1334.52 1317.33
1318.64 1302.82 1293.64 1294.00 1301.16 1330.41 1339.90 1342.84 1349.00 1335.88 1333.06 1315.78
1315.31 1331.35 1351.45 1372.71 1380.96 1391.22 1388.12 1395.86 1394.42 1403.28 1399.10 1393.56
1398.17 1409.62 1418.78 1407.65 1377.10 1379.29 1360.97 1356.94 1347.76 1362.84 1365.40 1341.03
1328.72 1328.05 1322.18 1305.33 1313.71 1300.29 1297.80 1281.43 1301.93 1298.16 1327.68 1330.77
1344.16 1332.84 1323.59 1336.61 1360.22 1363.50 1381.79 1362.01 1348.27 1324.02 1320.41 1331.07
1319.11 1357.24 1350.45 1344.15 1347.66 1351.66 1344.13 1336.29 1317.97 1318.48 1335.42 1335.53
1307.58 1310.51 1280.41 1277.36 1283.31 1282.20 1268.37 1282.71 1282.81 1304.60 1301.35 1325.40
1317.64 1336.02 1335.21 1313.04 1285.55 1283.42 1247.41 1254.13 1261.32 1289.43 1283.61 1301.65
1293.63 1281.91 1296.71 1342.44 1362.93 1354.12 1347.74 1354.93 1362.64 1370.23 1377.01 1365.28
1373.46 1381.46 1396.06 1394.39 1420.07 1410.71 1424.94 1422.00 1420.94 1404.64 1417.08 1416.62
1407.83 1388.91 1397.72 1409.04 1433.30 1423.33 1409.17 1403.88 1408.11 1417.04 1415.22 1403.17
1413.32 1418.78 1421.03 1418.09 1433.43 1436.13 1458.34 1457.10 1457.66 1463.46 1464.47 1469.25
1455.22 1399.42 1402.11 1403.45 1441.47 1457.60 1438.56 1432.25 1449.68 1465.15 1455.14 1455.90
1445.57 1441.36 1401.53 1410.03 1404.09 1398.56 1360.16 1394.46 1409.28 1409.12 1424.97 1424.37
1424.24 1441.72 1411.70 1416.83 1387.12 1389.94 1402.05 1387.67 1388.26 1346.09 1352.17 1360.69
1353.43 1333.36 1348.05 1366.42 1379.19 1381.76 1409.17 1391.28 1355.62 1366.70 1401.69 1395.07
1383.62 1359.15 1392.15 1458.47 1464.47 1456.63 1493.87 1500.64 1527.35 1527.46 1523.86 1507.73
1508.52 1487.92 1498.58 1505.97 1494.73 1487.37 1501.34 1516.35 1504.46 1500.59 1467.17 1440.51
1356.56 1401.44 1441.61 1427.47 1434.54 1429.86 1477.44 1460.99 1464.92 1452.43 1468.25 1446.29
1415.10 1409.57 1432.63 1424.17 1412.14 1383.05 1407.81 1420.96 1452.36 1466.04 1447.80 1437.21

1406.95 1400.72 1373.86 1399.05 1381.52 1378.02 1422.45 1420.60 1448.81 1477.26 1467.63 1457.84
1471.36 1461.67 1456.95 1446.00 1469.44 1470.54 1478.73 1464.46 1486.00 1475.95 1479.13 1452.18
1441.48 1455.31 1450.55 1454.82 1442.39 1454.60 1469.54 1446.23 1456.67 1478.90 1475.62 1480.88
1492.92 1495.84 1509.98 1510.49 1493.74 1481.96 1495.57 1480.19 1464.29 1474.47 1452.42 1449.62
1419.89 1430.83 1438.10 1438.70 1452.56 1462.93 1479.32 1482.80 1472.87 1460.25 1471.84 1491.56
1484.43 1479.85 1496.07 1491.72 1499.48 1498.13 1505.97 1508.31 1506.45 1514.09 1509.84 1502.59
1517.68 1520.77 1507.08 1492.25 1502.51 1494.50 1489.26 1481.99 1484.91 1480.87 1465.81 1444.51
1459.90 1451.34 1449.05 1448.72 1439.03 1427.21 1426.57 1458.29 1436.51 1436.23 1426.46 1434.32
1436.28 1408.99 1402.03 1387.02 1364.59 1329.78 1374.17 1374.62 1349.97 1342.13 1388.76 1396.93
1395.78 1398.13 1364.90 1364.44 1379.58 1398.66 1429.40 1421.22 1428.32 1426.69 1432.19 1431.87
1409.28 1400.14 1365.98 1351.26 1382.95 1389.81 1372.32 1367.72 1342.62 1347.35 1322.36 1341.77
1348.97 1336.09 1341.91 1314.95 1315.23 1324.97 1376.54 1351.46 1343.55 1369.89 1380.20 1371.18
1359.99 1340.93 1312.15 1322.74 1305.60 1264.74 1274.86 1305.97 1315.19 1328.92 1334.22 1320.28
1283.27 1347.56 1333.34 1298.35 1295.86 1300.80 1313.27 1326.82 1318.32 1326.65 1329.47 1347.97
1342.54 1342.90 1360.40 1364.30 1357.51 1354.95 1364.17 1373.73 1366.01 1373.47 1349.47 1354.31
1352.26 1340.89 1332.53 1314.76 1330.31 1318.80 1315.92 1326.61 1301.53 1278.94 1255.27 1252.82
1245.86 1267.65 1257.94 1239.94 1241.23 1234.18 1241.41 1253.80 1261.89 1264.74 1233.42 1180.16
1197.66 1166.71 1173.56 1150.53 1170.81 1142.62 1122.14 1117.58 1139.83 1152.69 1182.17 1153.29
1147.95 1160.33 1145.87 1106.46 1103.25 1151.44 1128.43 1137.59 1168.38 1165.89 1183.50 1179.68
1191.81 1238.16 1253.70 1242.98 1224.36 1209.47 1228.75 1234.52 1253.05 1249.46 1266.44 1267.43
1248.58 1266.61 1263.51 1261.20 1255.54 1255.18 1245.67 1248.92 1249.44 1284.99 1288.49 1291.96
1312.83 1309.38 1289.05 1293.17 1277.89 1267.93 1248.08 1255.82 1260.67 1267.11 1283.57 1270.03
1276.96 1264.96 1254.39 1255.85 1241.60 1219.87 1214.36 1208.43 1212.58 1223.14 1237.04 1225.35
1218.60 1216.76 1211.07 1226.20 1224.42 1236.71 1234.45 1219.24 1190.59 1198.78 1181.52 1180.18
1208.14 1215.68 1202.45 1214.44 1207.71 1215.02 1210.85 1191.03 1171.65 1190.49 1202.93 1205.82
1204.52 1211.23 1215.93 1220.75 1214.35 1200.48 1204.40 1183.53 1183.43 1190.16 1191.29 1186.73
1178.02 1181.66 1161.97 1171.41 1157.26 1165.31 1162.09 1184.93 1179.21 1161.51 1148.60 1129.03
1133.58 1132.94 1131.74 1106.40 1085.78 1092.54 1038.77 1032.74 1016.10 984.54 965.80 1003.45
1012.27 1007.04 1018.61 1040.94 1038.55 1051.33 1072.28 1069.63 1071.38 1062.44 1056.75 1080.99

1097.43 1091.65 1089.98 1097.54 1077.09 1068.61 1073.48 1089.90 1084.78 1085.20 1100.09 1104.61
1078.30 1059.79 1059.78 1084.10 1087.20 1102.84 1118.86 1115.80 1118.54 1120.31 1118.33 1139.09
1141.21 1142.24 1138.65 1151.06 1142.66 1137.03 1150.34 1157.42 1149.50 1128.52 1140.20 1139.45
1129.90 1144.80 1170.35 1167.10 1158.31 1139.93 1136.76 1137.07 1119.38 1123.09 1134.36 1142.92
1149.56 1139.93 1144.89 1144.65 1149.37 1157.13 1161.02 1148.08 1154.67 1165.27 1172.51 1164.89
1160.71 1155.14 1156.55 1145.60 1138.41 1146.19 1127.57 1138.88 1127.58 1119.31 1128.18 1132.15
1133.28 1133.06 1100.64 1113.57 1130.20 1122.20 1094.44 1090.02 1083.51 1080.17 1096.22 1111.94
1107.50 1118.51 1116.48 1104.18 1083.34 1097.98 1080.95 1089.84 1109.43 1109.38 1109.89 1106.73
1131.78 1153.84 1146.14 1162.77 1157.54 1164.31 1168.26 1165.58 1154.09 1153.04 1166.16 1165.55
1170.29 1151.85 1153.59 1148.70 1131.87 1138.49 1144.58 1147.39 1146.54 1136.76 1125.40 1126.34
1122.73 1125.29 1117.80 1130.47 1103.69 1111.01 1102.55 1128.37 1126.07 1124.47 1125.17 1107.83
1100.96 1093.14 1091.48 1076.32 1065.45 1076.92 1086.46 1084.56 1073.43 1052.67 1049.49 1088.85
1073.01 1054.99 1074.56 1097.28 1091.07 1098.23 1106.59 1091.88 1079.88 1086.02 1097.08 1083.82
1074.55 1067.66 1064.66 1067.14 1040.68 1040.69 1049.90 1029.15 1027.53 1030.74 1013.60 1020.26
1009.56 1007.27 1036.17 1037.14 1019.99 1006.29 989.14 992.72 976.14 973.53 990.64 989.81
968.65 948.09 953.99 989.03 976.98 952.83 920.47 927.37 921.39 917.93 901.05 906.04 881.56
847.76 819.85 797.70 843.42 838.68 852.84 898.96 902.78 911.62 884.66 864.24 834.60 859.57
876.77 905.46 908.64 903.80 884.21 919.62 930.25 928.77 950.72 937.43 949.36 962.70 940.86
947.95 934.82 917.87 917.80 916.07 878.02 893.40 879.15 893.92 902.96 909.58 909.45 886.91
889.81 891.10 873.52 869.46 843.32 845.39 833.70 819.29 839.66 854.95 827.37 815.28 847.91
827.91 818.95 800.58 785.28 798.55 776.76 803.92 835.32 841.44 881.27 860.02 879.20 884.39
899.72 890.16 896.14 882.50 897.65 890.23 882.15 890.71 885.76 900.96 908.35 915.39 923.76
902.65 894.74 876.19 882.95 882.53 904.27 909.83 900.36 896.74 914.15 933.76 930.55 932.88
913.31 938.87 936.31 934.53 920.75 917.57 906.55 912.23 892.00 904.45 904.96 901.59 889.48
910.40 902.99 891.12 884.25 895.75 897.38 892.47 889.66 875.40 879.39 879.82 909.03 908.59
929.01 922.93 909.93 927.58 927.57 926.26 931.66 918.22 914.60 901.78 887.62 878.36 887.34
861.40 847.48 858.54 864.36 844.61 855.70 860.32 848.20 843.59 838.15 829.69 835.97 829.20
818.68 817.37 834.89 851.17 845.13 837.10 848.17 832.58 838.57 827.55 837.28 841.15 834.81
821.99 829.85 822.10 828.89 807.48 800.73 804.19 831.90 833.27 862.79 866.45 874.02 875.67

895.79 864.23 874.74 869.95 868.52 863.50 848.18 858.48 880.90 876.45 878.85 879.93 878.29
865.99 871.58 868.30 885.23 890.81 879.91 893.58 892.01 911.37 919.02 911.43 898.81 914.84
917.84 916.92 916.30 930.08 926.55 934.39 929.62 920.27 933.41 945.11 942.30 939.28 946.67
944.30 920.77 919.73 923.42 931.87 933.22 951.48 953.22 949.64 963.59 967.00 971.56 986.24
990.14 987.76 975.93 984.84 997.48 998.51 988.61 1010.74 1011.66 1010.09 994.70 995.69 981.64
983.45 975.32 985.82 976.22 974.50 982.32 993.75 985.70 1004.42 1007.84 1002.21 988.70 998.14
1003.86 1000.42 994.00 981.73 993.32 978.80 988.11 988.61 981.60 998.68 996.52 989.28 987.49
990.31 980.15 982.82 965.46 967.08 974.12 977.59 980.59 990.35 984.03 990.51 990.67 999.74
1002.35 1000.30 1003.27 993.06 993.71 996.73 996.79 1002.84 1008.01 1021.99 1026.27 1027.97
1021.39 1031.64 1023.17 1010.92 1016.42 1018.63 1014.81 1029.32 1025.97 1039.58 1036.30 1022.82
1029.03 1009.38 1003.27 996.85 1006.58 995.97 1018.22 1020.24 1029.85 1034.35 1039.25 1033.78
1038.73 1038.06 1045.35 1049.48 1046.76 1050.07 1039.32 1044.68 1046.03 1030.36 1033.77 1028.91
1031.13 1046.79 1048.11 1046.94 1050.71 1059.02 1053.25 1051.81 1058.05 1053.21 1047.11 1046.57
1058.56 1058.41 1050.35 1043.63 1034.15 1042.44 1033.65 1035.28 1052.08 1053.89 1058.45 1058.20
1070.12 1066.62 1064.73 1069.72 1061.50 1069.30 1060.18 1059.05 1071.21 1074.14 1068.04 1075.13
1076.48 1089.18 1088.67 1092.94 1096.02 1094.04 1095.89 1109.48 1109.64 1111.92 1108.48 1122.22
1123.67 1126.33 1131.92 1121.86 1127.23 1121.22 1130.52 1132.05 1139.83 1138.77 1147.62 1143.94
1141.55 1155.37 1144.05 1128.48 1134.11 1131.13 1135.26 1136.03 1126.52 1128.59 1142.76 1139.81
1145.54 1157.76 1152.11 1145.81 1156.99 1151.82 1147.06 1144.11 1140.99 1139.09 1143.67 1144.91
1144.94 1155.96 1149.10 1151.04 1154.88 1156.87 1147.21 1140.58 1123.89 1106.78 1120.57 1104.49
1110.70 1123.75 1122.32 1109.78 1095.40 1093.95 1091.33 1109.19 1108.06 1122.47 1127.00 1126.21
1132.17 1141.81 1150.57 1148.16 1140.53 1139.32 1145.20 1129.44 1128.17 1128.84 1134.61 1135.82
1118.15 1124.09 1139.93 1140.60 1135.53 1138.11 1122.41 1113.89 1107.30 1117.49 1119.55 1121.53
1113.99 1098.70 1087.12 1095.45 1097.28 1096.44 1095.70 1084.10 1091.49 1088.68 1089.19 1093.56
1095.41 1113.05 1114.94 1121.28 1120.68 1121.20 1124.99 1116.64 1122.50 1140.42 1142.18 1131.33
1136.47 1125.29 1132.01 1133.56 1132.05 1135.02 1130.30 1134.41 1144.06 1140.65 1134.43 1133.35
1136.20 1140.84 1128.94 1125.38 1116.21 1118.33 1109.11 1112.81 1114.35 1115.14 1111.47 1106.69
1101.39 1100.90 1108.67 1093.88 1096.84 1086.20 1084.07 1094.83 1095.42 1100.43 1101.72 1106.62
1099.69 1098.63 1080.70 1063.97 1065.22 1079.04 1075.79 1063.23 1064.80 1079.34 1081.71 1095.17

1091.23	1098.35	1095.68	1096.19	1104.96	1105.09	1107.77	1099.15	1104.24	1105.91	1118.31	1113.63
1121.30	1116.27	1118.38	1123.92	1125.82	1128.33	1120.37	1123.50	1128.55	1122.20	1129.30	1113.56
1108.36	1110.11	1103.52	1110.06	1114.80	1114.58	1131.50	1135.17	1134.48	1142.05	1130.65	1122.14
1124.39	1121.84	1113.65	1103.29	1108.20	1114.02	1103.23	1103.66	1106.49	1095.74	1094.81	1111.09
1125.40	1127.44	1130.20	1130.51	1130.54	1143.20	1161.67	1166.17	1164.89	1164.08	1162.91	1173.48
1184.17	1183.81	1175.43	1181.94	1183.55	1170.34	1177.24	1176.94	1181.76	1182.65	1178.57	1173.82
1191.37	1190.33	1191.17	1190.25	1177.07	1182.81	1189.24	1188.00	1198.68	1203.38	1205.72	1203.21
1194.22	1194.65	1205.45	1209.57	1210.13	1204.92	1213.54	1213.45	1213.55	1211.92	1202.08	1188.05
1183.74	1187.89	1186.19	1190.25	1182.99	1187.70	1177.45	1184.52	1195.98	1184.63	1175.41	1167.87
1163.75	1168.41	1174.07	1174.55	1171.36	1181.27	1189.41	1193.19	1189.89	1203.03	1201.72	1202.30
1191.99	1197.01	1205.30	1206.14	1210.12	1210.34	1200.75	1201.59	1184.16	1190.80	1200.20	1211.37
1203.60	1210.41	1210.08	1210.47	1222.12	1225.31	1219.43	1207.01	1209.25	1200.08	1206.83	1197.75
1188.07	1190.21	1189.65	1183.78	1171.71	1172.53	1171.42	1174.28	1165.36	1181.41	1180.59	1172.92
1176.12	1181.39	1184.07	1191.14	1181.20	1181.21	1187.76	1173.79	1162.05	1142.62	1145.98	1152.78
1137.50	1159.95	1152.12	1162.10	1151.74	1156.38	1143.22	1156.85	1162.16	1161.17	1175.65	1172.63
1171.35	1178.84	1166.22	1171.11	1159.36	1154.05	1165.69	1173.80	1185.56	1191.08	1189.28	1193.86
1194.07	1190.01	1197.62	1198.78	1191.50	1202.27	1204.29	1196.02	1197.51	1197.26	1194.67	1200.93
1198.11	1200.82	1203.91	1206.58	1210.96	1216.96	1216.10	1213.61	1213.88	1200.73	1191.57	1190.69
1201.57	1199.85	1191.33	1194.44	1204.99	1194.94	1197.87	1211.86	1219.44	1222.21	1223.29	1226.50
1227.92	1221.13	1229.35	1235.20	1227.04	1233.68	1229.03	1231.16	1236.79	1243.72	1234.18	1235.35
1244.12	1245.04	1235.86	1226.42	1223.13	1231.38	1229.13	1237.81	1230.39	1233.87	1219.34	1220.24
1219.02	1219.71	1221.73	1217.57	1209.59	1212.39	1205.10	1212.28	1208.41	1220.33	1221.59	1218.02
1233.39	1236.36	1231.67	1241.48	1240.56	1231.20	1227.16	1227.73	1237.91	1231.02	1221.34	1210.20
1214.62	1215.29	1215.63	1215.66	1216.89	1227.68	1228.81	1226.70	1214.47	1196.39	1191.49	1195.90
1187.33	1184.87	1177.68	1176.84	1186.57	1190.10	1178.14	1195.76	1177.80	1179.59	1199.38	1196.54
1191.38	1178.90	1198.41	1207.01	1202.76	1214.76	1219.94	1220.14	1222.81	1218.59	1220.65	1230.96
1234.72	1233.76	1229.01	1231.21	1242.80	1248.27	1254.85	1261.23	1265.61	1268.25	1257.46	1257.48
1249.48	1264.67	1265.08	1262.09	1263.70	1257.37	1255.84	1259.37	1260.43	1267.43	1272.74	1270.94
1267.32	1259.92	1259.62	1262.79	1268.12	1268.66	1256.54	1258.17	1254.42	1248.29	1268.80	1273.46

1273.48	1285.45	1290.15	1289.69	1294.18	1286.06	1287.61	1282.93	1277.93	1285.04	1261.49	1263.82
1266.86	1264.68	1273.83	1283.72	1285.20	1280.08	1282.46	1270.84	1264.03	1265.02	1254.78	1265.65
1263.78	1266.99	1262.86	1275.53	1280.00	1289.38	1287.24	1283.03	1292.67	1287.79	1289.43	1294.12
1280.66	1291.24	1289.14	1287.23	1278.26	1275.88	1278.47	1272.23	1281.58	1284.13	1297.48	1303.02
1305.33	1307.25	1305.08	1297.23	1305.04	1301.67	1302.95	1301.61	1293.23	1302.89	1300.25	1294.83
1297.81	1305.93	1311.56	1309.04	1295.50	1296.60	1286.57	1288.12	1289.12	1285.33	1307.65	1309.93
1311.46	1311.28	1308.11	1301.74	1305.41	1309.72	1310.61	1305.19	1313.21	1307.85	1312.25	1325.76
1324.66	1325.14	1322.85	1305.92	1291.24	1294.50	1292.08	1270.32	1261.81	1267.03	1262.07	1256.58
1258.57	1272.88	1280.16	1259.84	1270.09	1285.71	1288.22	1265.29	1263.85	1256.15	1257.93	1252.30
1236.40	1223.69	1230.04	1256.16	1251.54	1240.14	1240.12	1252.20	1245.60	1244.50	1250.56	1239.20
1246.00	1272.87	1270.20	1280.19	1270.91	1274.08	1265.48	1267.34	1272.52	1258.60	1242.29	1236.20
1234.49	1236.86	1259.81	1249.13	1240.29	1260.91	1268.88	1268.40	1263.20	1278.55	1276.66	1270.92
1278.55	1280.27	1279.36	1275.77	1271.48	1265.95	1271.81	1266.74	1268.21	1285.58	1295.43	1297.48
1302.30	1297.52	1298.82	1292.99	1296.06	1295.09	1301.78	1304.28	1304.27	1303.82	1311.01	1313.25
1300.26	1294.02	1298.92	1299.54	1313.11	1318.07	1316.28	1319.87	1321.18	1318.31	1325.18	1318.03
1314.78	1326.37	1336.34	1336.59	1339.15	1335.85	1331.32	1334.11	1350.22	1353.22	1349.58	1350.66
1353.42	1349.95	1362.83	1365.62	1369.05	1364.05	1365.96	1366.96	1368.60	1377.02	1377.38	1382.22
1389.08	1377.34	1377.93	1377.94	1367.81	1367.34	1364.30	1379.78	1382.84	1385.72	1378.33	1380.90
1384.42	1393.22	1396.57	1399.76	1401.20	1400.50	1402.81	1406.09	1400.95	1381.90	1386.72	1399.48
1400.63	1396.71	1409.12	1414.76	1412.90	1407.29	1409.84	1413.04	1411.56	1413.21	1425.49	1427.09
1422.48	1425.55	1423.53	1418.30	1410.76	1416.90	1426.84	1424.73	1418.30	1416.60	1418.34	1409.71
1412.84	1412.11	1414.85	1423.82	1430.73	1431.90	1430.62	1426.37	1430.50	1422.95	1427.99	1440.13
1423.90	1422.18	1420.62	1428.82	1438.24	1445.94	1448.39	1446.99	1448.00	1450.02	1448.31	1438.06
1433.37	1444.26	1455.30	1456.81	1455.54	1459.68	1457.63	1456.38	1451.19	1449.37	1399.04	1406.82
1403.17	1387.17	1374.12	1395.41	1391.97	1401.89	1402.85	1406.60	1377.95	1387.17	1392.28	1386.95
1402.06	1410.94	1435.04	1434.54	1436.11	1437.50	1428.61	1417.23	1422.53	1420.86	1424.55	1437.77
1439.37	1443.76	1444.61	1448.39	1438.87	1447.80	1452.85	1468.47	1471.48	1472.50	1470.73	1484.35
1480.93	1480.41	1495.42	1494.25	1494.07	1482.37	1486.30	1495.92	1502.39	1505.62	1509.48	1507.72
1512.58	1491.47	1505.85	1503.15	1501.19	1514.14	1512.75	1522.75	1525.10	1524.12	1522.28	1507.51

1515.73 1518.11 1530.23 1530.62 1536.34 1539.18 1530.95 1517.38 1490.72 1507.67 1509.12 1493.00
1515.67 1522.97 1532.91 1531.05 1533.70 1512.84 1522.19 1502.56 1497.74 1492.89 1506.34 1505.71
1503.35 1519.43 1524.87 1525.40 1530.44 1531.85 1510.12 1518.76 1547.70 1552.50 1549.52 1549.37
1546.17 1553.08 1534.10 1541.57 1511.04 1518.09 1482.66 1458.95 1473.91 1455.27 1465.81 1472.20
1433.06 1467.67 1476.71 1497.49 1453.09 1453.64 1452.92 1426.54 1406.70 1411.27 1445.94 1445.55
1447.12 1464.07 1462.50 1479.37 1466.79 1432.36 1463.76 1457.64 1473.99 1489.42 1472.29 1478.55
1453.55 1451.70 1471.49 1471.56 1483.95 1484.25 1476.65 1519.78 1529.03 1518.75 1525.75 1517.73
1517.21 1525.42 1531.38 1526.75 1547.04 1546.63 1539.59 1542.84 1557.59 1552.58 1565.15 1562.47
1554.41 1561.80 1548.71 1538.53 1541.24 1540.08 1500.63 1506.33 1519.59 1515.88 1514.40 1535.28
1540.98 1531.02 1549.38 1508.44 1509.65 1502.17 1520.27 1475.62 1474.77 1453.70 1439.18 1481.05
1470.58 1451.15 1458.74 1433.27 1439.70 1416.77 1440.70 1407.22 1428.23 1469.02 1469.72 1481.14
1472.42 1462.79 1485.01 1507.34 1504.66 1515.96 1477.65 1486.59 1488.41 1467.95 1445.90 1454.98
1453.00 1460.12 1484.46 1496.45 1497.66 1476.37 1478.49 1468.36 1447.16 1447.16 1411.63 1416.18
1390.19 1409.13 1420.33 1401.02 1416.25 1380.95 1373.20 1333.25 1325.19 1310.50 1338.60 1352.07
1330.61 1353.97 1362.30 1355.81 1378.55 1395.42 1380.82 1336.64 1326.45 1336.91 1331.29 1339.13
1348.86 1367.21 1348.86 1349.99 1348.78 1360.03 1342.53 1353.11 1371.80 1381.29 1380.02 1367.68
1330.63 1331.34 1326.75 1333.70 1304.34 1293.37 1273.37 1320.65 1308.77 1315.48 1288.14 1276.60
1330.74 1298.42 1329.51 1349.88 1352.99 1341.13 1325.76 1315.22 1322.70 1370.18 1367.53 1369.31
1370.40 1372.54 1365.54 1354.49 1360.55 1332.83 1328.32 1334.43 1364.71 1365.56 1390.33 1388.17
1375.94 1379.93 1388.82 1397.84 1396.37 1390.94 1385.59 1409.34 1413.90 1407.49 1418.26 1392.57
1397.68 1388.28 1403.58 1403.04 1408.66 1423.57 1425.35 1426.63 1413.40 1390.71 1394.35 1375.93
1385.35 1390.84 1398.26 1400.38 1385.67 1377.65 1377.20 1404.05 1360.68 1361.76 1358.44 1335.49
1339.87 1360.03 1360.14 1350.93 1337.81 1342.83 1317.93 1318.00 1314.29 1321.97 1283.15 1278.38
1280.00 1284.91 1261.52 1262.90 1252.31 1273.70 1244.69 1253.39 1239.49 1228.30 1214.91 1245.36
1260.32 1260.68 1260.00 1277.00 1282.19 1252.54 1257.76 1234.37 1263.20 1284.26 1267.38 1260.31
1249.01 1284.88 1289.19 1266.07 1296.32 1305.32 1289.59 1285.83 1292.93 1298.20 1278.60 1266.69
1274.54 1277.72 1292.20 1266.84 1271.51 1281.66 1300.68 1282.83 1277.58 1274.98 1236.83 1242.31
1267.79 1224.51 1232.04 1249.05 1251.70 1192.70 1213.59 1156.39 1206.51 1255.08 1207.09 1188.22
1185.87 1209.18 1213.01 1106.39 1166.36 1161.06 1114.28 1099.23 1056.89 996.23 984.94 909.92

899.22 1003.35 998.01 907.84 946.43 940.55 985.40 955.05 896.78 908.11 876.77 848.92 940.51
930.09 954.09 968.75 966.30 1005.75 952.77 904.88 930.99 919.21 898.95 852.30 911.29 873.29
850.75 859.12 806.58 752.44 800.03 851.81 857.39 887.68 896.24 816.21 848.81 870.74 845.22
876.07 909.70 888.67 899.24 873.59 879.73 868.57 913.18 904.42 885.28 887.88 871.63 863.16
868.15 872.80 869.42 890.64 903.25 931.80 927.45 934.70 906.65 909.73 890.35 870.26 871.79
842.62 843.74 850.12 805.22 840.24 827.50 831.95 836.57 845.71 874.09 845.14 825.88 825.44
838.51 832.23 845.85 868.60 869.89 827.16 833.74 835.19 826.84 789.17 788.42 778.94 770.05
743.33 773.14 764.90 752.83 735.09 700.82 696.33 712.87 682.55 683.38 676.53 719.60 721.36
750.74 756.55 753.89 778.12 794.35 784.04 768.54 822.92 806.12 813.88 832.86 815.94 787.53
797.87 811.08 834.38 842.50 835.48 815.55 825.16 856.56 858.73 841.50 852.06 865.30 869.60
832.39 850.08 843.55 851.92 866.23 857.51 855.16 873.64 872.81 877.52 907.24 903.80 919.53
907.39 929.23 909.24 908.35 883.92 893.07 882.88 909.71 908.13 903.47 888.33 887.00 910.33
893.06 906.83 919.14 942.87 944.74 931.76 942.46 940.09 939.14 942.43 939.15 944.89 946.21
923.72 911.97 910.71 918.37 921.23 893.04 895.10 900.94 920.26 918.90 927.23 919.32 923.33
896.42 898.72 881.03 879.56 882.68 879.13 901.05 905.84 932.68 940.74 940.38 951.13 954.58
954.07 976.29 979.26 982.18 979.62 975.15 986.75 987.48 1002.63 1005.65 1002.72 997.08 1010.48
1007.10 994.35 1005.81 1012.73 1004.09 979.73 989.67 996.46 1007.37 1026.13 1025.57 1028.00
1028.12 1030.98 1028.93 1020.62 998.04 994.75 1003.24 1016.40 1025.39 1033.37 1044.14 1042.73
1049.34 1052.63 1068.76 1065.49 1068.30 1064.66 1071.66 1060.87 1050.78 1044.38 1062.98 1060.61
1057.08 1029.85 1025.21 1040.46 1054.72 1057.58 1065.48 1071.49 1076.19 1073.19 1092.02 1096.56
1087.68 1097.91 1091.06 1081.40 1092.91 1079.60 1066.95 1063.41 1042.63 1066.11 1036.19 1042.88
1045.41 1046.50 1066.63 1069.30 1093.08 1093.01 1098.51 1087.24 1093.48 1109.30 1110.32 1109.80
1094.90 1091.38 1106.24 1105.65 1110.63 1091.49 1095.63 1108.86 1109.24 1099.92 1105.98 1103.25
1091.94 1095.95 1102.35 1106.41 1114.11 1107.93 1109.18 1096.08 1102.47 1114.05 1118.02 1120.59
1126.48 1127.78 1126.20 1126.42 1115.10 1132.99 1136.52 1137.14 1141.69 1144.98 1146.98 1136.22
1145.68 1148.46 1136.03 1150.23 1138.04 1116.48 1091.76 1096.78 1092.17 1097.50 1084.53 1073.87
1089.19 1103.32 1097.28 1063.11 1066.19 1056.74 1070.52 1068.13 1078.47 1075.51 1094.87 1099.51
1106.75 1109.17 1108.01 1094.60 1105.24 1102.94 1104.49 1115.71 1118.31 1118.79 1122.97 1138.70
1138.50 1140.45 1145.61 1150.24 1149.99 1150.51 1159.46 1166.21 1165.83 1159.90 1165.81 1174.17

1167.72 1165.73 1166.59 1173.22 1173.27 1169.43 1178.10 1187.44 1189.44 1182.45 1186.44 1194.37
1196.48 1197.30 1210.65 1211.67 1192.13 1197.52 1207.17 1205.94 1208.67 1217.28 1212.05 1183.71
1191.36 1206.78 1186.69 1202.26 1173.60 1165.90 1128.15 1110.88 1159.73 1155.79 1171.67 1157.44
1135.68 1136.94 1120.80 1115.05 1071.59 1087.69 1073.65 1074.03 1067.95 1103.06 1089.41 1070.71
1098.38 1102.83 1064.88 1050.47 1062.00 1055.69 1086.84 1091.60 1089.63 1115.23 1114.61 1116.04
1117.51 1113.20 1095.31 1092.04 1073.69 1076.76 1074.57 1041.24 1030.71 1027.37 1022.58 1028.06
1060.27 1070.25 1077.96 1078.75 1095.34 1095.17 1096.48 1064.88 1071.25 1083.48 1069.59 1093.67
1102.66 1115.01 1113.84 1106.13 1101.53 1101.60 1125.86 1120.46 1127.24 1125.81 1121.64 1127.79
1121.06 1089.47 1083.61 1079.25 1079.38 1092.54 1094.16 1075.63 1071.69 1067.36 1051.87 1055.33
1047.22 1064.59 1048.92 1049.33 1080.29 1090.10 1104.51 1091.84 1098.87 1104.18 1109.55 1121.90
1121.10 1125.07 1124.66 1125.59 1142.71 1139.78 1134.28 1124.83 1148.67 1142.16 1147.70 1144.73
1141.20 1146.24 1137.03 1160.75 1159.97 1158.06 1165.15 1165.32 1169.77 1178.10 1173.81 1176.19
1184.71 1165.90 1178.17 1180.26 1183.08 1185.62 1185.64 1182.45 1183.78 1183.26 1184.38 1193.57
1197.96 1221.06 1225.85 1223.25 1213.40 1218.71 1213.54 1199.21 1197.75 1178.34 1178.59 1196.69
1199.73 1197.84 1180.73 1198.35 1189.40 1187.76 1180.55 1206.07 1221.53 1224.71 1223.12 1223.75
1228.28 1233.00 1240.40 1240.46 1241.59 1235.23 1242.87 1243.91 1247.08 1254.60 1258.84 1256.77
1257.54 1258.51 1259.78 1257.88 1257.64 1271.87 1270.20 1276.56 1273.85 1271.50 1269.75 1274.48
1285.96 1283.76 1293.24 1295.02 1281.92 1280.26 1283.35 1290.84 1291.18 1296.63 1299.54 1276.34
1286.12 1307.59 1304.03 1307.10 1310.87 1319.05 1324.57 1320.88 1321.87 1329.15 1332.32 1328.01
1336.32 1340.43 1343.01 1315.44 1307.40 1306.10 1319.88 1327.22 1306.33 1308.44 1330.97 1321.15
1310.13 1321.82 1320.02 1295.11 1304.28 1296.39 1281.87 1256.88 1273.72 1279.20 1298.38 1293.77
1297.54 1309.66 1313.80 1310.19 1319.44 1328.26 1325.83 1332.41 1332.87 1332.63 1335.54 1333.51
1328.17 1324.46 1314.16 1314.41 1314.52 1319.68 1305.14 1312.62 1330.36 1337.38 1335.25 1347.24
1355.66 1360.48 1363.61 1361.22 1356.62 1347.32 1335.10 1340.20 1346.29 1357.16 1342.08 1348.65
1337.77 1329.47 1328.98 1340.68 1343.60 1333.27 1317.37 1316.28 1320.47 1325.69 1331.10 1345.20
1314.55 1312.94 1300.16 1286.17 1284.94 1279.56 1289.00 1270.98 1271.83 1287.87 1265.42 1267.64
1271.50 1278.36 1295.52 1287.14 1283.50 1268.45 1280.10 1296.67 1307.41 1320.64 1339.67 1337.88
1339.22 1353.22 1343.80 1319.49 1313.64 1317.72 1308.87 1316.14 1305.44 1326.73 1325.84 1343.80
1345.02 1337.43 1331.94 1304.89 1300.67 1292.28 1286.94 1254.05 1260.34 1200.07 1199.38 1119.46

1172.53 1120.76 1172.64 1178.81 1204.49 1192.76 1193.89 1140.65 1123.53 1123.82 1162.35 1177.60

1159.27 1176.80 1210.08 1212.92 1218.89 1204.42 1173.97 1165.24 1198.62 1185.90 1154.23 1162.27

1172.87 1188.68 1209.11 1216.01 1204.09 1202.09 1166.76 1129.56 1136.43 1162.95 1175.38 1151.06

1160.40 1131.42 1099.23 1123.95 1144.03 1164.97 1155.46 1194.89 1195.54 1207.25 1203.66 1224.58

1200.86 1225.38 1209.88 1215.39 1238.25 1254.19 1229.05 1242.00 1284.59 1285.09 1253.30 1218.28

1237.90 1261.15 1253.23 1261.12 1275.92 1229.10 1239.69 1263.85 1251.78 1257.81 1236.91 1216.13

1215.65 1192.98 1188.04 1161.79 1158.67 1192.55 1195.19 1246.96 1244.58 1244.28 1257.08 1258.47

1261.01 1234.35 1255.19 1236.47 1225.73 1211.82 1215.75 1219.66 1205.35 1241.30 1243.72 1254.00

1265.33 1265.43 1249.64 1263.02 1257.60 1277.06 1277.30 1281.06 1277.81 1280.70 1292.08 1292.48

1295.50 1289.09 1293.67 1308.04 1314.50 1315.38 1316.00 1314.65 1326.05 1318.43 1316.33 1313.01

1312.41 1324.09 1325.54 1344.90 1344.33 1347.05 1349.96 1351.95 1342.64 1351.77 1350.50 1343.23

1358.04 1361.23 1362.21 1357.66 1363.46 1365.74 1367.59 1372.18 1365.68 1374.09 1369.63 1364.33

1343.36 1352.63 1365.91 1370.87 1371.09 1395.95 1394.28 1402.60 1404.17 1409.75 1405.52 1402.89

1392.78 1397.11 1416.51 1412.52 1405.54 1403.28 1408.47 1419.04 1413.38 1398.96 1398.08 1382.20

1358.59 1368.71 1387.57 1370.26 1369.57 1390.78 1385.14 1376.92 1378.53 1366.94 1371.97 1390.69

1399.98 1403.36 1397.91 1405.82 1402.31 1391.57 1369.10 1369.58 1363.72 1354.58 1357.99 1353.39

1338.35 1330.66 1324.80 1304.86 1295.22 1315.99 1316.63 1318.86 1320.68 1317.82 1332.42 1313.32

1310.33 1278.04 1278.18 1285.50 1315.13 1314.99 1325.66 1308.93 1324.18 1314.88 1329.10 1342.84

1344.78 1357.98 1355.69 1325.51 1335.02 1313.72 1319.99 1331.85 1329.04 1362.16 1365.51 1374.02

1367.58 1354.68 1352.46 1341.47 1341.45 1334.76 1356.78 1353.64 1363.67 1372.78 1376.51 1362.66

1350.52 1338.31 1337.89 1360.02 1385.97 1385.30 1379.32 1375.14 1365.00 1390.99 1394.23 1401.35

1402.22 1402.80 1405.87 1404.11 1403.93 1405.53 1415.51 1418.16 1418.13 1413.17 1413.49 1402.08

1411.13 1410.44 1409.30 1410.49 1399.48 1406.58 1404.94 1403.44 1432.12 1437.92 1429.08 1433.56

1436.56 1459.99 1465.77 1461.19 1459.32 1461.05 1460.26 1460.15 1456.89 1441.59 1433.32 1447.15

1440.67 1444.49 1445.75 1450.99 1461.40 1460.93 1455.88 1441.48 1432.56 1432.84 1428.59 1440.13

1454.92 1460.91 1457.34 1433.19 1433.82 1413.11 1408.75 1412.97 1411.94 1412.16 1427.59 1414.20

1417.26 1428.39 1394.53 1377.51 1379.85 1380.03 1374.53 1355.49 1353.33 1359.88 1386.89 1387.81

1391.03 1409.15 1406.29 1398.94 1409.93 1415.95 1416.18 1409.46 1407.05 1409.28 1413.94 1418.07

1418.55 1427.84 1428.48 1419.45 1413.58 1430.36 1446.79 1435.81 1443.69 1430.15 1426.66 1419.83

1418.10 1402.43 1426.19 1462.42 1459.37 1466.47 1461.89 1457.15 1461.02 1472.12 1472.05 1470.68 1472.34 1472.63 1480.94 1485.98 1492.56 1494.81 1494.82 1502.96 1500.18 1507.84 1501.96 1498.11 1513.17 1495.71 1511.29 1512.12 1509.39 1517.93 1517.01 1519.43 1520.33 1521.38 1519.79 1530.94 1511.95 1502.42 1515.60 1487.85 1496.94 1515.99 1514.68 1518.20 1525.20 1539.79 1541.46 1544.26 1551.18 1556.22 1552.48 1554.52 1563.23 1560.70 1552.10 1548.34 1558.71 1545.80 1556.89 1551.69 1563.77 1562.85 1569.19 1562.17 1570.25 1553.69 1559.98 1553.28 1563.07 1568.61 1587.73 1593.37 1588.85 1552.36 1574.57 1552.01 1541.61 1555.25 1562.50 1578.78 1578.79 1585.16 1582.24 1593.61 1597.57 1582.70 1597.59 1614.42 1617.50 1625.96 1632.69 1626.67 1633.70 1633.77 1650.34 1658.78 1650.47 1667.47 1666.29 1669.16 1655.35 1650.51 1649.60 1660.06 1648.36 1654.41 1630.74 1640.42 1631.38 1608.90 1622.56 1643.38 1642.81

CBOI Index Data.

-0.017758563 -0.019293869 -0.033164243 0.003071205 0.030611908 0.00181118 0.030410542 -0.032221722 -0.032005219 -0.015918334 0.006037955 -0.025580063 0.011063521 0.003821228 -0.008570908 0.023245386 0.038683577 0.009218861 -0.013118568 0.01235776 0.005799084 0.013075459 -0.003905991 -0.044098579 0.010458971 -0.012090637 -0.015249697 -0.002263429 0.007592263 -0.005046733 0.005325394 0.005810972 0.006547277 0.00671672 -0.01052396 -0.027337317 -0.002918657 -0.008278158 -0.0032958 0.012451616 -0.009155816 -0.004500501 0.00240607 0.037613568 0.021725958 0.05103623 0.044196159 -0.052889387 0.062448407 -0.016548548 -0.050432498 0.009956969 0.014293359 0.051882662 -0.018710964 -0.018456001 -0.044516961 -0.013568952 -0.021314619 -0.005829544 0.012101053 -0.002341216 -0.027177489 -0.019014948 0.010036965 0.012297448 0.013564328 -0.026997187 0.052917437 -0.043359762 0.003323698 -0.022654131 0.013733776 0.033851226 -0.028811001 -0.028474382 -0.018212081 0.02294236 0.008732757 0.033807832 0.008187004 0.015167789 -0.014644857 -0.001833021 -0.020966305 0.03389269 0.026673421 0.037608865 -0.027194766 -0.010174621 0.016904406 -0.009074793 0.033693642 -0.013928683 -0.00908124 -0.006006296 -0.004207407 -0.009738485 -0.054165539 0.000272029 -0.019181562 0.006208344 0.021855906 -0.00590156 0.024928604 -0.006422691 -0.037553337 -0.010857928 0.010857928 -0.009990784 0.028768041 -0.028479377 -0.007568326 0.010444459 -0.004611 0.007752901 -0.0279149 -0.010132462 0.025047887

0.011002715 0.024458103 -0.016525563 -0.032941162 0.025294437 -0.003442246 0.032452024
-0.011921495 0.010583145 -0.011408016 -0.023175405 -0.014763301 -0.021549859 0.028994239
-0.011063521 0.024436429 -0.029615731 0.063738731 -0.017828023 0.020203386 0.020562159
-0.019248105 -0.010625471 -0.012826468 -0.010087293 0.001415102 0.018259655 0.004849492
0.053142454 0.023531 -0.010224512 -0.033939053 0.013943628 0.024341837 -0.025788683
-0.010017926 -0.013812206 -0.020641036 0.019107328 -0.025305865 -0.003269485 -0.015308343
-0.010030341 -0.00203417 -0.005864971 0.021034623 -0.004239811 -0.010929842 -0.018749436
-0.011401077 0.004966967 -0.000308777 -0.001547185 -0.007819701 -0.019032659 0.012673915
0.033281387 -0.007480283 -0.011003123 0.01220866 -0.023815632 0.033046784 0.001764232
-0.015533508 0.010219165 -0.024125193 -0.019919362 0.004253012 -0.019645645 0.021918593
0.01714543 -0.021376133 0.038169927 0.016168498 -0.009923827 0.009923827 0.010547874
0.02755319 -0.038389727 -0.009635164 0.012226255 -0.008112474 -0.009162047 0.021558948
0.035461172 -0.005005573 -0.018126412 -0.009213922 0.024148933 0 -0.002409036
-0.061611409 -0.002170702 -0.004060316 -0.001257004 -0.006020961 -0.017250381
-0.001330153 0.009552558 0.001625964 -0.005553296 0 -0.007628125 0.017059076 -0.017393792
-0.010163361 0.001710496 0.002043743 0.017320787 -0.001963656 -0.008948009 0.000334716
-0.000669691 0.007639794 0.015523719 -0.015523719 -0.010328933 -0.004065189 -0.007553138
0.006530867 -0.013161446 -0.002822395 -0.011476657 -0.004383161 -0.01040276 -0.003776498
-0.00842573 0.03383893 -0.005037441 -0.013975118 0.033783511 0.024535342 0.025077674
-0.003095483 0.003403822 -0.016011223 -0.00288782 -0.008452105 -0.001315049 -0.021255156
0.009575384 -0.014093895 0.001395325 0.024048957 0.02090677 0.01633226 0.008683185
-0.017852304 -0.019586052 -0.006511284 0 -0.01399996 -0.027622398 -0.013567077
0.004815255 -0.021908193 0.0196924 -0.004840023 0.014723257 -0.018105922 -0.003029093
0.003406249 -0.009528563 -0.006600937 0.025453484 -0.022334659 0.070153562 -0.025524059
-0.029737947 -0.00301333 0.021746906 -0.001440447 0.016283526 0.014688512 -0.034594198
0.011486143 0.004931215 0.010382036 -0.007241283 0.010647531 -0.014138663 0.025854446
0.011086197 -0.006486969 -0.004929616 0.01526276 0.020565848 -0.006438652 -0.020232805
0.010550182 0.022765687 -0.026254015 0.011933643 0.002162979 0.000616021 -0.013440916
-0.017160345 -0.027962676 0.010096278 -0.022602537 -0.025378375 0.071099612 0.038087559

0.086688001 -0.029893283 -0.044128739 0.053468765 -0.069736378 0.028292355 -0.022008963
0.008530271 -0.001166675 0.054546602 -0.04188238 -0.0054234 -0.017286177 -0.020815841
0.004057398 0.002169153 -0.009373626 0.035174945 -0.013609796 0.04749719 0.005355117
-0.017374096 -0.031886507 -0.028963696 -0.009991631 -0.00590886 -0.006325597 -0.008808146
-0.003780981 -0.011899223 -0.00320514 0.038948703 -0.03150651 -0.018666768 0.011224576
0.004622089 0.00387294 0.010732415 -0.021736787 -0.004335755 -0.011405254 0.00518788
-0.012329187 -0.01190975 0.012667021 -0.00037847 -0.015023658 -0.001177482 0.040498523
0.001786487 0.011957041 -0.033895769 0.007067903 0.009850024 -0.023721636 0.010537991
0.02140172 0.007717876 -0.037756952 -0.006496387 -0.01289498 -0.004384837 -0.001203588
-0.019725372 0.018115382 -0.013099413 -0.009240657 0.030330994 -0.018190888 0.023307239
-0.004325286 -0.025637937 0.006655984 0.041317619 0.018733575 -0.012029068 0.002946878
0.024619584 -0.001389744 0.024688232 -0.027481132 -0.00070104 0.001400951 0.011047832
-0.016678958 0.042486221 -0.007125051 0.003900875 0 0 -0.002596686 0.026214021
-0.013699313 -0.000316426 -0.00637742 0.00415596 -0.034762106 0.018555208 0.013654095
-0.020981323 -0.013994196 0.013994196 -0.033141707 -0.012130893 0.026745381 0.017854936
-0.014363804 -0.046569581 0.03422583 0.017528387 0.037817192 -0.014734117 -0.012225148
0.00267259 -0.009426917 -0.004447324 -0.013621933 -0.011505162 0.019241787 0.025055889
0.042289062 0.0120686 -0.010876302 0.027675101 0.002228585 0.016091872 0.006642749
-0.044382631 -0.005880855 0.019112041 -0.055208386 0.076761772 0.000538827 -0.039136884
-0.006231869 0.001789678 0.04060722 -0.050847647 0.031443901 -0.013532312 -0.042036561
-0.01574654 0.005642423 -0.002314479 -0.000663552 -0.009397774 -0.031663008 -0.009967247
0.020067574 0.001779169 0.001064013 -0.037337948 -0.005047735 -0.011877526 -0.008921396
0.014504689 -0.003181659 0.036722807 0.001830152 0.006165453 -0.030580997 -0.008978992
0.02267067 -0.00300551 0.024194809 0.001068376 -0.018901153 -0.008251093 -0.013849359
-0.025352776 0.010238439 -0.002028469 0.027571009 -0.007312176 -0.015004393 -0.021410319
-0.000844931 0.051762915 -0.008337497 0.009836358 0.01687505 -0.034822079 0.008875274
-0.00229786 -0.01206992 0.005102479 -0.010665339 0.003187497 0.010200701 0.001161734
0.025532005 0.012224327 -0.009679271 0.084704557 -0.019210259 -0.019773061 -0.014599509
-0.015456951 -0.016027445 0.014977156 0.019198462 -0.020251298 0.029541373 -0.008954843

0.00267259 0.017941865 -0.0249928 0.009707179 -0.008355288 0.015902719 -0.037345177
0.002121964 -0.030316165 0.019601119 0.014228774 -0.018955341 -0.000365722 -0.015266101
0.004898818 0.027231724 -0.029109351 0.006721803 0.008803568 0.027438248 -0.002735722
-0.011822998 0.012508546 -0.029400903 -0.017957886 -0.024756183 0.02399158 0.023460957
0.005045802 -0.011988815 -0.006307677 -0.015600515 -0.001552439 0.000776913 0.010354531
0.004523942 -0.02429444 0.02391924 -0.006809635 0.006809635 -0.016448168 -0.000390027
0.012310322 0.029684984 -0.002486765 0.027939647 -0.008092678 -0.01313003 0.006959996
-0.04277579 -0.003058424 0.012106553 -0.001869544 0.005954431 0 0.000369455 0.002944381
-0.009268266 0.034214101 -0.019478353 0 -0.004734494 -0.008878561 0.013975118
-0.000362063 -0.007673893 0.025417148 0.055419957 -0.019402548 -0.007098582 0.085639276
0.019072838 0.004829781 0.013440558 0.013037056 -0.021951052 -0.049218023 -0.036212173
0.116792114 -0.055009826 -0.053935197 -0.006023598 0.06547122 0.017634855 0.011370909
0.013216066 -0.005477875 0.063233085 0.055368117 -0.045129678 -0.130232593 0.035391975
0.014362244 -0.028695006 -0.03156462 0.009958781 0.000273399 -0.007442868 0.021162819
-0.016461718 -0.083437714 0.001663328 -0.000997232 0.035145928 -0.015623788 0.010063755
0.001551608 -0.028152648 0.04247302 0.089107563 0.006301258 0.013968702 -0.021492293
-0.057223222 0.018052621 0.030766233 -0.065037435 -0.00378536 -0.006779081 0.009403225
-0.018104407 0.018395002 0.002895307 -0.020987349 0.003619142 -0.003619142 0.026150752 0
-0.000856315 -0.021675295 -0.004832255 -0.001216512 -0.02636701 -0.033980734 -0.005990364
-0.021828437 0.021118226 -0.0010675 0.00741798 0 -0.003163632 -0.004610765 0.003551079
-0.002127161 0.003186847 -0.012526873 0.023672171 0.036583382 0.004402733 -0.023133577
-0.005646738 -0.037332896 -0.027834452 0.014140181 0.007825338 0.000737969 0.007311528
-0.021933094 0.031612365 0.011889641 -0.01652311 -0.021673313 0.001879252 0.005219052
0.009529206 0.02087846 -0.01155386 -0.013696729 0.004372129 -0.015498984 0.004485009
-0.005990364 0.003753649 -0.020274582 -0.013523082 0.013523082 0.024366466 -0.014684958
-0.017186557 -0.010893233 0.018006496 -0.011921 -0.004861556 -0.004093286 0.009759837
0.015794267 -0.003895044 0.015752345 -0.00340922 -0.010001909 0 -0.002732648 -0.013523082
0.010378855 -0.008765874 0.002808381 -0.014640526 -0.007930634 -0.006364957 -0.004727646
-0.003470903 -0.003059964 0.01040276 0.024561767 0.059305389 -0.024698983 -0.030577671

0.003196294 0.0288738 -0.016318794 0.00154416 0.002305994 0.048546705 -0.021069345
-0.051917799 -0.002439863 -0.01960183 -0.00905262 0.003470903 0.023139804 -0.012468641
-0.001689864 0.01128101 0.03294996 -0.003839941 -0.024596763 0.014056064 0.021960142
0.017668653 -0.019551163 0.023143363 -0.005037441 -0.003269485 -0.016348344 -0.023333874
-0.029763502 0.024538239 -0.006108321 0.010533778 -0.018396399 0.015182364 -0.017695165
-0.019324247 0.054796565 -0.003497693 0.012310322 -0.018994933 -0.021942835 -0.004609054
-0.009798133 0.019791928 0.004097148 -0.01366991 0.005020801 0.02662248 0.019509241
-0.050311681 -0.004644905 -0.000424738 -0.008583174 0.009432235 -0.001699785 -0.000852394
0.01673789 0.0227892 0.215413477 -0.074603971 0.011122105 0.070539894 0.023173926
-0.089887413 -0.02004792 -0.027952738 -0.006121236 -0.003093279 0.11258009 -0.021105467
-0.021654774 0.009413133 -0.060995404 -0.041184369 -0.036867217 0.025594819 0.000671244
0.006986121 0.010434003 0.045821921 0.004614062 -0.014584798 -0.003275462 -0.033220554
-0.007157075 -0.000328141 -0.002964479 -0.015476112 0.026892696 -0.025866399 -0.01778572
-0.007903013 0.005761869 0.009902909 0.004175941 -0.016590266 0.033570321 0.002656244
0.002968983 -0.014032273 -0.011701193 0.057730245 -0.022244247 -0.014047605 0.000331903
-0.006013404 0.007999511 0.001977065 -0.010986955 0.023623045 -0.02126914 0.03261565
0.001552717 -0.016104367 0.000321581 -0.024804675 0.018000967 -0.007908464 0.005944808
0.02671467 -0.023446825 0.006141967 -0.02307114 0.007383855 -0.009079658 0.016994127
0.010970875 0.037815113 0.059668058 -0.060545126 -0.003821228 0.054322927 -0.053732853
-0.033388377 0.009448403 -0.016510258 -0.018849388 0.057526986 -0.013836036 0.04468648
0.02413368 0.05481772 -0.085060502 0.000279559 0.018867505 -0.022797799 -0.013751898
0.016002133 -0.021863146 0.012791391 0.06407256 -0.02361844 -0.029702307 -0.011038382
0.012433482 0.001112863 0.010161005 -0.021420079 0.046469799 -0.003601989 0.042776201
-0.010665339 0.059130177 0.066467898 -0.063754201 0.051914868 0.002760941 -0.047452878
0.073965257 -0.040117223 -0.026944657 -0.00222146 0.091490684 0.028868455 -0.027394881
0.017774531 0.04454769 0.002259749 -0.011997059 -0.056525607 -0.018189477 -0.04261559
-0.005153192 -0.03810285 0.040018576 0.063547421 -0.043196353 0.022221671 -0.03013656
-0.011290787 0.033028159 -0.010551631 0.038768043 0.018635153 -0.034828202 -0.005360661
-0.003493941 0.002797398 0.026369943 -0.114353567 -0.006883464 0.009012363 -0.031939711

0.00837602 -0.017616677 -0.023260632 -0.015803391 0.024823584 -0.003877655 0.015542061
0.007220938 -0.008396933 -0.037617309 0.013900632 -0.034679202 0.014570562 0.05406639
-0.027293254 0.035721998 0.017033339 -0.033354343 -0.000938001 0.093800155 -0.025714627
-0.025414252 0.00822033 0.007657523 -0.034352008 0.006829017 0.025466668 -0.055789116
0.097799721 -0.003758522 0.02386834 -0.055574174 0.092871174 -0.005444219 0.03720712
0.037775862 -0.110603679 0.031952929 0.034117695 -0.041717831 0.008770517 -0.019289976
0.032932135 -0.020372933 0.052638513 -0.041422731 -0.037428151 -0.002529176 -0.020401869
0.013829822 0.003298455 -0.02363325 -0.031369913 -0.002285219 -0.002297307 0.055919189
-0.021124868 0.001736023 0.013457288 0.022724083 -0.034644043 -0.018817145 -0.022613907
-0.046978475 0.003046049 0.001398695 0.035727802 0.010169311 0.035373766 0.012743516
-0.012936578 0.027134691 -0.002729704 0.028951918 -0.022969057 -0.012234456 0.004238851
-0.014546216 0.008265369 0.01893285 0.067201194 -0.019985443 0.057252314 -0.028804358
-0.018965167 0.019862161 -0.019862161 -0.007251546 0.004742979 -0.022922383 -0.037728288
0.034233276 0.036058413 0.011083803 -0.020096319 0.005460933 -0.006404027 -0.020458223
-0.024600483 0.011370517 -0.008933588 0.009782981 -0.02068046 0.012629809 -0.018724012
-0.019001685 -0.021849823 0.015390381 0.015787431 0.052278991 -0.004275558 -0.012744691
-0.015945563 0.049186496 -0.000946863 0.028877052 -0.047106386 0.013942715 0.001115415
0.057593913 0.014797584 -0.096943691 0.063347477 -0.049590768 -0.014768265 -0.000168822
0.006036623 -0.003343315 -0.002862195 -0.001692498 -0.052766528 0.014129238 -0.004097148
-0.014458795 -0.000580737 -0.001163809 0.008653486 -0.016097597 0.02830032 0.006613749
-0.019388037 -0.04516477 -0.00339792 -0.005147254 0.007910086 0.007768588 -0.012883008
-0.004308512 -0.010296493 0.001107047 0.013069025 0.011643981 -0.041852499 -0.016408111
0.016867925 -0.016151858 0.034817139 -0.007325355 0.000223805 -0.037849587 0.004613739
-0.007798988 -0.034803095 0.004505995 0.014010714 0.014314557 0.024260519 -0.012802184
0.034669555 0.001994722 -0.01279079 -0.02171341 -0.00748594 0.046171371 0.008887794
0.011852827 -0.04785048 0.101962431 -0.008187456 0.001125602 0.017263872 -0.014462565
-0.041169359 -0.005561352 0.00371547 0.022235286 -0.013083343 0.025214724 -0.004389742
-0.004240814 -0.025530625 0.053837716 -0.008985091 0.00934791 -0.005474373 0.039803852
-0.019533695 0.017181611 -0.046731918 0.037366255 0.006153035 0.031104454 0.015365245

0.000913984 -0.055780247 -0.00156003 -0.016998611 -0.018444151 -0.036744974 0.002657494
0.041374158 -0.009932518 0.024328325 -0.041338917 -0.016473829 0.034052745 -0.007061854
0.017351458 -0.045778034 -0.0191091 0.019314489 -0.010180055 -0.011502341 0.022092882
0.007726416 -0.025096326 -0.016538261 0.029992796 0.00616614 -0.017915886 -0.012952088
-0.022714855 0.047209635 -0.010056472 -0.015755018 -0.00731414 0.026447255 0.027305173
-0.011203058 0.0497316 -0.017894468 -0.00798288 0.051152522 -0.016508479 -0.002308666
0.022044852 0.09180261 -0.019616634 0.077505817 -0.039120452 -0.013729034 0.023459713
0.023352777 -0.006492183 -0.03028069 0.024691237 0.12867302 -0.074116853 0.004606197
0.055722372 -0.001152995 0.061859179 0.013391773 0.030081867 0.045742392 0.039150953
-0.104503999 0.001104273 0.099031785 -0.010408842 0.017129681 -0.123110644 0.001146328
0.117744819 -0.011691427 0.067111472 0.00507442 -0.077600141 0.019034356 -0.046199155
-0.021296332 -0.047541806 -0.051021073 0.058082927 0.067128693 -0.055040193 0.0290436
0.010444746 0.03410913 -0.045641334 0.044660025 0.018213157 -0.009588586 0.040551346
0.036978792 -0.046378577 -0.05045088 -0.026286988 -0.044886764 0.002837506 0.093185933
-0.036551311 -0.015870893 0.020398408 -0.026085893 -0.010562656 0.003107401 -0.02409998
0.000389467 -0.011838671 0.019402548 -0.034959813 -0.02150454 -0.02234978 -0.022691835
-0.003591236 0.004460308 -0.007884976 -0.008230967 0.005174973 -0.023058109 -0.01734642
-0.008884728 -0.001220708 -0.00581753 0.051252625 -0.008388031 0.002645044 0.029597946
-0.025057714 0.055248257 0.016135024 -0.043775054 0.089404792 -0.086494778 0.008064178
-0.000183712 -0.014764444 -0.033994445 -0.027474003 0.031362623 0.021950271 0.006536649
-0.024128228 0.007895572 -0.00119012 -0.003590055 0.00269532 0.029153059 -0.020385106
-0.033228742 0.017336936 0.054411216 -0.001788695 -0.012546956 0.020010465 0.028303924
-0.063234907 -0.007899984 -9.72337E-05 0.01613102 0.055348637 -0.014424699 -0.02973183
0.023202255 -0.007332986 0.003070472 -0.049092158 -0.007503342 -0.024899744 0.012269609
0.013922827 -0.030218616 -0.007949218 0.037571685 0.021435428 -0.025932822 -0.003024347
-0.006934175 -0.019875556 0.007256195 0.045185674 -0.013560697 -0.01869735 -0.002472271
-0.024872201 0.013251238 -0.005767891 -0.01688283 -0.026741351 0.014957005 -0.001611057
-0.017647111 -0.00458681 -0.02304986 0.062352594 -0.023222532 0.011083076 -0.010966158
-0.003875034 0.017341716 -0.00421372 -0.021945251 0.005026336 -0.014518159 -0.009578127

-0.014970536 -0.012011341 0.013051568 -0.018438235 0.01097167 -0.014372584 0.024557949
-0.03047055 0.023575809 -0.039508541 -0.021189299 0.003454548 0.033390509 0.017379529
-0.027611887 0.024003327 -0.00936045 -0.039449775 0.01650164 -0.005968277 0.019910142
-0.011922558 -0.008134181 0.00219378 -0.022453026 0.002909087 -0.00537405 0.000617554
0.039213299 0.02559035 -0.015420359 -0.021306357 -0.030552434 0.046733904 -0.008299321
-0.022291344 -0.042200731 -0.007142889 -0.009824553 0.016802656 -0.002147932 0.027749125
0.016016186 0.02685717 0.006289169 -0.021619644 -0.011227285 -0.04257656 -0.021833543
0.014844745 -0.007956504 -0.018854972 0.001069252 -0.009537407 -0.007339329 -0.000740801
-0.006348356 0.021824749 0.012865009 0.010295887 -0.003575862 0.008801281 -0.006074148
-0.011535953 0.00017445 0.013226522 -0.015675228 0.004015616 0.017040023 -0.009118493
-0.012815041 -0.007802969 0.060378738 -0.027478872 0.00132508 -0.01979406 -0.00138697
0.002251582 -0.003817235 0.000522512 -0.004725395 0.001405485 0.021389712 0.049498207
-0.003740716 -0.027928552 -0.030535945 0.006145779 -0.022615555 -0.013972659 0.010926224
-0.005246696 -0.008083549 0.00497817 -0.000733916 0.00493003 0.002534448 -0.018059819
0.007647212 0.026187533 0.011339029 -0.012559202 0.005377791 0.007181411 0.04291623
0.006253339 -0.028796635 -0.018849388 -0.017587968 -0.008888859 -0.019468467 -0.002071211
-0.000377647 -0.002462745 -0.022216405 -0.00583765 0.001214244 -0.012090129 0.026597768
-0.030983564 0.031959726 0.038064742 0.009191761 0.050783117 -0.052009196 0.093246248
-0.013072195 -0.014381435 -0.016750032 -0.037446866 -0.021710492 -0.019084873
-0.005854896 0.003786375 0.022050141 -0.016059777 -0.008827046 -0.009203936 -0.015385337
0.019628035 -0.008527252 -0.020641639 -0.014397821 0.00021211 0.082069743 -0.004056384
-0.048502761 -0.016146636 0.026715838 -0.024050818 0.017033339 0.030172787 -0.019305155
-0.006565715 -0.014441548 -0.008942271 0.006926044 -0.019635976 0.039775056 -0.016316217
-0.024517319 -0.020617399 0.003762065 -0.005320673 0.010141347 0.00173979 -0.001086552
0.035898741 -0.034161561 -0.015216748 -0.004285465 -0.002272608 -0.021725092 -0.014120683
0.016985748 -0.009624648 -0.005385908 0.006843274 -0.008076715 0.026358001 0.076343345
0.088601483 -0.031317035 -0.014953169 -0.025688142 0.010934384 0.01599031 -0.037371818
-0.021881954 0.002419474 0.081853836 0.000499285 0.006602856 -0.00843638 -0.010139631
-0.025346903 -0.022887378 -0.00926942 -0.010470194 -0.022360593 -0.013035155 -0.001738919

0.030079368 -0.022950773 -0.003657691 -0.013161446 -0.005378329 -0.004533386 -0.005272576
-0.002544476 -0.031257694 0.009127797 0.003162057 0.015473898 -0.012094158 -0.011698875
0.010253634 -0.007544672 -0.019584225 -0.007512588 0.009050823 -0.00256676 -0.013597326
0.030759364 0.020540702 -0.015130395 -0.004421588 -0.011508476 0.011508476 -0.002972934
-0.011333349 -0.020641036 0.0103125 -0.003673812 -0.009053677 -0.015336077 0.016947561
-0.016668899 0.008277782 0.06274878 -0.024823584 -0.042320371 0.015991432 0.003973445
0.00393742 0.021661886 0.11583238 -0.034254679 -0.05810969 0.077647677 -0.038272275
0.072169932 0.019067476 0.119500116 0.096380062 -0.15225865 -0.007902007 -0.045212579
0.019305155 0.0685252 -0.005596656 0.036578155 0.02232817 0.112749949 -0.057626271
-0.019718872 -0.044223901 0.005114542 -0.071852245 0.033635063 0.044618464 -0.071142113
-0.010342568 0.080750869 0.013141349 -0.035495059 0.00038644 -0.042720902 -0.026053774
-0.00317944 -0.043265796 0.000838568 -0.014827267 -0.019502212 0.016544858 0.036316893
-0.002253572 0.043427266 -0.018039193 0.007096226 0.07073829 0.005186045 -0.021654774
-0.037812592 -0.00683027 -0.043242186 -0.018680435 -0.012509643 -0.009668967 0.002304895
0.005796534 0.004340377 0.018764034 -0.004657358 -0.035529251 0.029975322 -0.017453609
-0.020951322 -0.013913654 0.008701313 0.019410994 -0.002154421 -0.01148946 -0.02844782
0.012064511 -0.008135995 -0.002156285 -0.007132734 0.007918077 0.004488368 0.054996717
0.005777085 0.008524044 -0.002323323 -0.030498398 0.00461642 0.031502922 -0.015891615
0.002886817 0.029443881 -0.012189272 0.010763536 -0.048995934 0.046449678 -0.018920814
-0.037591578 -0.012915401 -0.036717299 0.047993507 -0.010154 -0.008297672 -0.015900056
-0.015684561 0.007108088 0.010743517 -0.007532453 0.005760222 -0.010181583 0.016839068
0.003097968 0.025476924 -0.041192596 0.016294088 0.001154527 0.012314518 0.008325389
-0.022565828 0.019439409 -0.033963036 -0.005422476 0.001412341 -0.017468658 -0.038341766
-0.000687719 0.003200079 0.018065687 -0.018977592 0.00136714 0.0336933 -0.018053434
-0.01156408 -0.011186127 0.024064923 0.00802064 0.010398948 0.003550395 0.006605367
0.012717875 -0.005203591 -0.042481295 -0.023727868 -0.006140209 0.000712932 0.018364665
-0.014111475 0.003979013 0.043632084 -0.008726622 0.048372568 -0.016065047 -0.064657619
-0.016764739 0.007872623 0.050390072 -0.023130378 0.055375204 -0.013700025 0.038762429
-0.04220521 -0.042023439 -0.032064096 0.000241074 -0.000723623 -0.006077548 -0.012164516

0.008970257 -0.001482235 0.001482235 0.008063083 -0.013522857 -0.033204042 0.008013041
0.002112075 -0.028292172 0.027765115 0.03054295 -0.003702448 -0.005990364 0.005990364
0.005664256 -0.003439001 -0.005709584 -0.009090216 0.009589693 -0.006538431 0.010018771
-0.016399939 -0.017043625 0.003992929 -0.025369464 0.011367437 0.037720141 0.016734095
0.011435732 -0.019722186 -0.00147887 -0.024112518 -0.012980795 0.093725191 -0.011195474
-0.044449931 -0.008206209 -0.015589766 -0.020240561 0.009438625 -0.012722531 0.001645057
0.005979117 -0.010933101 0.007137744 0.011287992 0.009187594 -0.003389887 -0.004208823
0.102425772 0.026918079 -0.016194214 -0.045033817 -0.020117315 0.058789984 -0.006455707
-0.046457401 0.010609952 0.035007421 -0.018026667 0.008677501 0.034266166 -0.037283609
0.022135789 0.061064074 0.08238376 -0.047237201 -0.033008928 -0.07402321 -0.008511678
-0.022944201 -0.027349608 -0.002176919 0.035600675 -0.029580416 -0.010897283 0.000735054
-0.008404367 0.0024888 -0.006248949 -0.008902395 0.005363305 0.018874543 -0.032278166
0.012895677 -0.004341704 -0.017012806 -0.026128788 0.044167083 -0.029944933 -0.021367663
-0.011091457 0.030809898 -0.004150664 -0.00757265 -0.021161007 0.003844648 0.035056443
0.018868007 0.009771395 0.027583522 0.004746435 -0.030300539 -0.032847104 0.027499523
-0.02423618 0.027299999 0.028791313 -0.016747713 -0.033958601 -0.019426803 0.05040567
0.02044116 -0.010830848 -0.018674179 -0.025677477 -0.002979269 -0.014648291 0.073522606
-0.005012523 -0.003374114 0.012872458 -0.009978759 0.016968628 -0.024239352 0.025854261
0.016935905 -0.03097682 0.067286427 0.027812235 -0.017147994 -0.038638647 -0.025271106
-0.007900706 0.017691245 0.038950228 -0.011259345 -0.030400997 -0.045329775 -0.019282295
-0.017433116 0.005168614 0.007506511 -0.010491365 0 0.061821042 0.033616138 0.000873393
0.018992075 -0.027361092 0.030481784 -0.037656662 -0.002721436 -0.036281417 -0.00099041
0.043146868 0.019314913 0.055354142 0.01413069 0.026780668 -0.028246642 0.020261786
-0.02543202 0.131666404 0.004639068 0.176091259 -0.136429326 0.088555533 -0.042302838
-0.030440732 -0.057241811 0.01315331 -0.017123248 0.130710517 0.003850513 -0.006197781
-0.06822782 -0.004453107 0.044351927 -0.048118388 -0.042394462 0.008130347 -0.017102008
0.00273831 0.027755668 0.03774588 -0.044715392 0.012060947 0.050138999 0.0007885
-0.019330733 -0.028067946 -0.034333463 -0.013661223 0.023864592 0.001721554 0.055274076
0.044533879 -0.001051561 -0.024136688 -0.014830733 0.037173903 -0.024351214 0.043785051

0.024469615 -0.046660887 -0.033266323 -0.018059122 -0.000838985 -0.039931502 -0.00210951
-0.021678585 -0.007850598 -0.036273683 0.072751727 -0.024479425 0.037926148 0.004266435
-0.045507824 -0.029547432 0.041851214 -0.033035733 -0.069231404 -0.016160851 0.086844261
0.064662882 -0.026126016 -0.030778836 -0.004868502 -0.004487002 -0.035927607 0.119211693
-0.042222191 -0.038306302 0.015479192 0.001253778 0.030741529 0.012770531 -0.032794981
0.012177904 -0.012585246 0.026480738 0.006217909 -0.030530558 -0.020621965 -0.042243965
-0.00613576 0.001739394 0.005020801 0.004500501 0.008257961 0.028151784 -0.064304686
-0.011848923 -0.004421204 0.010636315 -0.015794267 -0.014419198 0.011120523 -0.030685823
-0.034840044 -0.005506508 -0.008916361 0.024043075 0.03079494 -0.016369111 0.014147651
-0.008054862 -0.014416941 -0.014709778 -0.017535049 0.009165308 -0.007904045 0.00749161
-0.012134258 0.00923619 0.025998942 -0.026414534 -0.021740573 -0.036221676 0.009168127
0.005547211 -0.014003185 0.006123559 -0.000936484 0.019926311 0.000894531 -0.020352347
-0.013554227 -0.021793577 0.016446851 -0.002698252 0.012371134 0.011097011 0.047641634
-0.038187545 0.011257615 0.034180424 -0.0413516 -0.033821627 0.009900942 0 -0.034523417
0.012987786 0.021535631 -0.005526367 0.011219003 -0.028484544 0.000754202 0.018682213
0.063045243 -0.039171756 -0.02628624 -0.020814443 -0.039013261 -0.023975033 0.014713475
0.003109183 -0.027615843 0.016779305 0.015319617 -0.012728525 0.012449685 -0.021440409
-0.016728678 0.03872659 -0.003355801 0.000280643 0.000560742 0.003905051 0.000555009
0.021110055 0.006814658 0.051672324 0.03502843 -0.007953153 -0.065935626 0.055618315 0
-0.024915065 0.004214211 -0.006573231 -0.022326196 0.03652085 -0.020388756 -0.031852583
-0.015239967 0.00213413 0.02154397 -0.014156036 0.007264354 0.017152069 0.037870993
-0.00501553 0.002515005 0.022868728 -0.027913388 0.023784463 0.041214 0.001981274
0.00589016 0.041268568 0.010684936 -0.057053679 0.009176264 -0.002907584 -0.015643024
0.004413192 -0.014819618 0.059897993 -0.001441643 0.044564522 -0.008886972 -0.024628017
-0.046775399 -0.008709935 -0.009909827 0.045225292 -0.02797957 0.04087406 -0.049010498
-0.011571045 -0.061562764 0.001420038 -0.027808246 0.066226447 -0.044845258 0.051285729
-0.014297269 -0.005987305 0.005767019 -0.062198758 -0.007178585 -0.003634285 0.021363052
-0.010041938 0.021793577 0.017516157 -0.018241391 0.009098012 -0.039407011 0.009494556
-0.016292802 -0.008515851 -0.019512873 0.022459069 0.058592124 0.041138166 -0.024661373

-0.042674554 -0.021065445 0.033279256 0.021154887 0.000687719 -0.033066572 -0.050535013
0.008523939 0.001087776 -0.018589698 -0.001135411 -0.015625871 -0.031776916 0.035005886
-0.006482128 -0.010212097 -0.026309944 0.018025729 0.029921919 0.002594532 0.023768423
-0.021761115 0.032245985 0.003702899 0.014758371 0.019172316 -0.008858438 0.012496782
-0.005836072 -0.055829017 -0.035365712 0.053895515 0.00345418 -0.016451494 -0.050980763
0.013991088 0.002387879 -0.012379061 -0.009286765 0.005904631 -0.002786926 0.005249268
0.037609486 0.037201787 -0.054133812 0.025294822 0.015991432 -0.016543969 -0.007810259
-0.025502933 -0.006616803 0.023018274 0.034784215 -0.002127595 -0.019074969 0.015057415
-0.024064493 -0.001424385 -0.0043014 -0.001154272 0.055020046 -0.011348007 0.054219301
-0.011687855 -0.005004272 -0.007494274 0.018849025 -0.047056608 0.022809503 0.020023786
-0.020270755 0.0355595 -0.013641459 0.002809462 -0.047550327 -0.000781808 0.031923767
0.001693158 -0.039922582 -0.032123614 -0.004583625 0.006573849 -0.004849316 0.010205823
0.011611365 -0.011331266 -0.012786826 0.022751955 0.020576395 0.012350438 -0.017073929
0.003154695 -0.018187402 0.004077912 -0.013186424 0.010472075 0.016299645 0.011388589
-0.017196869 -0.02096344 0.047261108 0.007686829 0.004158301 0 0.038194103 -0.000223001
0.067042376 -0.10065354 -0.089028731 -0.003564681 -0.022339195 -0.001257914 -0.005387159
0.006016571 -0.010181729 -0.004205492 0.005170233 0.000962604 -0.004186779 0.004827332
-0.037061805 -0.001046914 0.001046914 0.00794358 0.006791295 0.02232693 -0.008401792
0.031764962 -0.001214811 -0.044138497 0.055840404 -0.029076002 -0.009925334 0.002904991
-0.015722784 -0.002676708 -0.010187202 0.011527619 -0.010840987 -0.006915663 -0.005259989
0.076468003 0.015688597 -0.031044802 0.127155114 -0.051409882 -0.058912336 0.022409051
-0.004220582 -0.03995308 -0.016748243 0.001607904 -0.01535462 -0.015917447 -0.037067896
0.025886729 -0.015859818 -0.019906301 0 0.072728015 0.032254336 -0.055284179 0.0430411
-0.013237867 0.005406885 -0.031795835 0.012734856 -0.015122032 0.029096049 -0.026368916
0.046063224 -0.009891832 0.00093699 -0.02339444 -0.011679772 -0.016546553 -0.004237053
-0.00643411 0.15594503 -0.092406919 0.072861655 0.026777438 -0.069302711 -0.017161006
-0.028370902 0.004168233 0.000318982 -0.000318982 0.00317933 -0.006060763 0.030091694
-0.027848929 -0.024316329 -0.006469422 0.005792951 -0.005792951 0.01583102 -0.018238996
-0.001382004 0.007547171 0.001358232 0.008726458 -0.021106236 0.019441633 0.011520423

0.014376636 0.007786054 -0.002476383 0.014950847 0.010372589 -0.008875537 0.04992199
-0.000533204 -0.000266848 0.031650496 -0.022145799 -0.040766374 0.008521421 0.043586225
0.037045869 -0.054170809 0.019155543 -0.008954843 -0.004939649 0.00078369 0.090388636
-0.035080154 0.026950266 -0.036945175 -0.030686025 -0.0089233 0 -0.012808891 0.001853134
-0.006386799 -0.036620317 -0.003220264 -0.012822533 -0.004257823 -0.006155943
-0.005302045 -0.001571824 0.019400994 -0.019716043 -0.000315277 -0.040636404 -0.008745654
0.012881823 0.017481705 -0.006975434 -0.008452865 0.022293466 0 0.001941707 -0.016788008
-0.033477458 -0.005105116 0.031135409 0.008787581 -0.008446289 0.022600374 -0.019879648
-0.017291077 0.025019538 0.052925155 -0.010745979 0.021520179 -0.005499304 0.029010674
-0.03340196 -0.023579186 0.030294461 0.04873143 -0.007312407 0.008347058 0.0051366
-0.010334681 -0.019519134 -0.003018805 0.002197573 -0.006070289 -0.031693364 -0.021757571
0.014524186 -0.003968975 0.006695633 0.004506728 -0.029046158 -0.013963567 -0.001322054
0.037705799 -0.007036979 -0.00216452 0.001547185 0.041224169 0.030898598 -0.028657074
0.028657074 0.027128461 -0.023481096 0.064270082 0.020325413 -0.016094877 -0.075298864
-0.020504666 0.009563335 0.064895763 -0.103298967 -0.037922781 -0.014412301 0.003978298
0.00557426 0.002922366 -0.007178585 -0.003634285 0.007238409 0.003250722 0.007703874
0.003170047 -0.015104623 -0.01159955 0.011272398 -0.020094308 0.040550515 -0.03273742
-0.012638639 0.009936954 -0.010283696 -0.005234629 -0.006365994 0.03126759 0.009509281
0.000324221 -0.024671093 -0.013943235 0.018380074 0.000678585 0.005725563 0.023445875
0.016484333 0.009658093 0.004454341 0.011084007 -0.038837075 -0.009552317 0.01331518
0.044757244 0.003366641 0.006105199 0.007377309 0.004849492 -0.069903928 0.010877353
-0.011192174 -0.024286675 -0.019063006 -0.005251509 0.004554966 0.036741647 0.005094422
0.015850789 -0.014586466 -0.006679139 -0.020676782 -0.001683967 0.00067437 -0.026034231
0.038979388 -0.033999708 0 0.008752704 -0.003130691 0.014758167 -0.001013523 0.030368917
0.119703342 -0.017589132 -0.042391064 0.040642392 -0.001504486 0.027258795 0.066170993
-0.049964094 0.018682213 -0.063657623 -0.051877792 -0.000852952 -0.021887121 -0.006331375
-0.004886628 -0.017869562 0.009496613 0.048255237 -0.020363524 -0.003242118 -0.013521155
-0.017436386 0.021083387 -0.009484793 -0.001239072 0.057991947 -0.05490087 -0.006516867
0.009891832 -0.003067064 0.002761331 0.017973371 -0.009793184 0.049582319 0.04085685

-0.056670951 -0.032270132 0.017585175 -0.017585175 0.014124643 0.002597981 -0.031941226 0.027311794 -0.009112435 -0.006283392 -0.016274515 -0.02511817 -0.000331649 0.009191761 0.018754011 0.047403194 -0.019393915 -0.032386655 0.060615854 0.030090751 -0.024119108 -0.013692637 -0.041726672 -0.025869773 -0.00359058 -0.001971083 0.002626127 0.001633302 0.023481096 -0.002788915 -0.008158951 -0.009608677 -0.0052129 -0.011289636 0.012598739 0.016354105 -0.012774288 0.000971214 -0.016813499 -0.023836057 -0.003565656 0.001429777 0.034295197 -0.024765395 -0.000698785 0.018483406 -0.03669324 0.004353866 -0.024887296 0.005696992 0.006367519 -0.004109484 -0.006428508 0.006803708 0.01074216 0.007616215 -0.014624092 -0.036843121 0.016675145 -0.006277175 0.023460297 0.03453165 -0.013342351 0.016443237 -0.020743218 -0.055631924 0.000409133 0.009305221 0.005172602 0.043258461 -0.019777365 0.001496279 -0.014041324 0.011794968 -0.016058492 -0.013047607 -0.020547563 0.040550213 0.024226908 -0.012130893 0.033699805 -0.017958796 -0.009449458 0.005113703 -0.036338494 0.121171721 -0.081217099 0.026201822 -0.019767712 -0.026328939 0.011899223 0.03010992 -0.004471983 0.024208436 0.001632074 0.104339553 0.002044945 -0.051662857 0.047563291 -0.013066403 0.007626318 -0.023843304 -0.044626793 -0.003062738 -0.039552452 -0.016468114 0.024804157 -0.028315045 -0.003895044 -0.015570373 -0.000737969 -0.010843904 0.008622444 -0.002606147 0.005565576 0.009841756 -0.002530229 0.009679271 0.003882382 0.009728603 -0.019320773 0.020007405 0.027900063 -0.020417905 -0.002705893 0.016968086 0.025656641 -0.045006293 -0.002737878 -0.021824898 0.002878516 0.053259305 0.03765636 -0.051188885 0.071365826 -0.022510295 0.031850321 0.008877186 0.010522489 -0.014535093 -0.045578363 0.026346496 0.046318957 -0.056263983 0.09396837 0.053921678 0.064486191 -0.033896378 0.061384983 -0.017728767 -0.059175311 -0.073413326 -0.062525859 0.045838508 -0.033851699 -0.011177313 -0.001891176 -0.04714357 0.022351839 -0.018446016 -0.014908945 0.021145076 0.004691951 -0.021524848 -0.015601336 -0.01783468 -0.01515722 0.008485899 0.003348471 0.024953282 -0.015386211 0.021639659 -0.004054484 0.003122188 -0.011985648 -0.02231006 -0.009530355 -0.012923266