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Symmetry-Protected Adiabatic Quantum Transistors

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Abstract

The standard circuit model of quantum computation differs in principle from a modern day computer chip in that computation is brought to stationary qubits in the former whereas information is routed spatially across a chip by transistors in the latter. Recently a model was proposed that addresses this key difference in implementation, it was dubbed the adiabatic quantum transistor model to emphasise its similarity to a classical transistor. Here we generalise this model to the setting of spin chains in inherently quantum phases of matter with a property called symmetry-protected order. Our generalisation is significant as it shows the computational properties of the model persist robustly throughout each symmetry-protected quantum phase of matter.

Acknowledgements

Foremost I acknowledge and thank my supervisor Stephen Bartlett for discussion and guidance throughout the project and more deeply for introducing me to the area of quantum information/computation/whatever else and its great research community. From Sydney I acknowledge useful discussion with Steve Flammia about dealing with errors in the SPAQT model and Andrew Doherty for bouncing around some ideas about traversing the phase transition at different speeds. From Vienna I acknowledge some great discussions with Benoit Descamps whose suggestion to write down the symmetry conditions as a channel yielded fruitful insights and Marek Rams for further discussion of how to traverse a phase transition in different manners. I thank Frank Verstraete for his patience as I finished this project and his support of my travels to present this work at several conferences. Also big thanks to my family and girlfriend for their support as I made the move to Vienna during the course of working on this masters. Finally I would like to wish Ross Mcphedran well in his retirement after a long and fruitful career and thank him again for introducing me to my first serious research project.

Statement of Student Contribution

The research reported on in this thesis builds upon, generalises and significantly extends the research undertaken as part of my honours project in 2012.

The idea to extend the adiabatic quantum transistor (AQT) model of Bacon et. al. to arbitrary symmetry-protected (SP) phases was my own and emerged in answer to my own conjectures that were stated at the end of my honours thesis. Specifically, the results presented in Chapter 4 of this thesis constitute the results of my investigations into extending the concept of AQTs to all SP phases and the calculations and arguments presented there are all my own. These results form the core of my masters research contribution, and are currently in preparation for publication in a refereed journal.

The background material presented in chapters 1, 2, & 3 of this thesis is originally based on material from chapters 1 & 2 of my honours thesis, with some substantive edits and the inclusion of relevant new material.

The additional background material and examples in the appendices of this thesis are taken verbatim from my honours thesis. I have also included the main results of my honours thesis in appendix B and given a brief description of these results in section 4.4 as they give an important example of a universal gate set in the symmetry-protected adiabatic quantum transistor SPAQT framework.

I certify that this report contains work carried out by myself except where otherwise acknowledged.

Signed 

Date: 18/03/2015

Contents

Chapter 1

Introduction

Computers can be built in many different ways. Early computation was done by factories of workers each manually carrying out small pieces of calculations for mathematical tables or with huge mechanical logic engines. The development of new technologies has allowed the computation of increasingly large and complex problems. A milestone in the development of modern computers was the first use of electric current to encode information which allowed simple circuits to execute elementary logic operations. This lead to order of magnitude increases in computation speed as the fundamental operations of electric logic could be carried out was so much faster than a mechanical implementation of the same logic.

The next seminal development which arguably lead to all modern day computer technology was the invention of the transistor. This advance was of a slightly different nature to the jump from mechanical to electrical logic, the importance lay not in pure speedup but the access to a robust, mass producible and miniaturizable logical element. It is this continued trend of miniaturization that underlies the increasing power of modern computers matching Moore's law. While this trend is expected to continue to the near future with the foremost obstacle being heat generated by the inefficiency of current transistors, if we extrapolate into the far distance fundamental limits will be imposed by the physical behavior of tiny objects. In this regime quantum mechanical effects become important and rather than fight these effects we imagine utilizing them to build quantum computers.

The development of quantum computers would be of a drastically different nature than previous technological jumps in that it would not only have implications for practical implementation of computation but it is speculated that it would change the class of problems that are efficiently computable in a complexity theoretic sense. This difference is attributed to the fact that quantum mechanics allows us to step outside the rules of classical logic that govern a standard computer and hence perform algorithms impossible in the classical setting.

Here we draw the distinction between the complexity based speed ups considered in theoretical computer science which are concerned only with the asymptotic scaling behavior of resources required to run an algorithm to solve a problem (or check a solution to a problem etc.) and the practical speed of computation inherent to any physical implementation of a computer intended for human use. It is the practical concerns that will be primarily of interest throughout this thesis.

With new rules come new technological challenges, since the quantum effects (such as *entanglement*) that allow us to go beyond classical computing are fragile to the jostling of a noisy environment. The destruction of carefully arranged entanglement within a

quantum computer due to its interaction with the environment is called *decoherence*. This presents a technological challenge that is not insurmountable; a quantum computer can still operate with a sufficiently small rate of decoherence, using the techniques of *quantum error correction*. The realistic goal is hence to build a *fault tolerant quantum computer*, which can operate in the noisy face of reality.

Small superconducting qubit systems are now on the verge of achieving low enough decoherence rates to satisfy the most basic requirements for fault tolerant circuit constructions. The fabrication and experimental control of such systems must be extremely precise as it involves bare qubits being manipulated by microwave pulses and frequency tunings. In contrast we will consider quantum logic circuits built up from *ground states* of many-body spin systems, which naturally possess entanglement in thermodynamic equilibrium at low temperatures. Such states can be prepared by cooling a chain of interacting spin degrees of freedom (e.g. polar molecules) to near zero temperature.

In this thesis we will consider particular phases of strongly interacting spin chains that naturally act as computational wires through which quantum information can propagate while undergoing quantum logic. Importantly the quantum gate depends only on the phase of matter which allows these models to tolerate arbitrary symmetric perturbations, without disrupting our ability to perform quantum computation within their ground states. These properties are both consequences of the chains possessing a type of non-classical order called *symmetry-protected order*. This approach is much closer to modern computer chips where information propagates spatially through a circuit as logic is performed.

The main result of this thesis presented in Chapter 4 is the generalisation of a model for quantum computation dubbed the adiabatic quantum transistor (AQT) model from particular qubit Hamiltonians to arbitrary symmetry-protected phases of matter. Most significantly, we have shown that the logical gates generated in our scheme depend only upon properties that persist throughout whole symmetry-protected (SP) phases of matter and do not depend on fine tuned parent Hamiltonians. We dub these objects symmetry-protected adiabatic quantum transistors (SPAQT) as they are inherently protected by the symmetry of a SP phase. Furthermore we have shown that any symmetric phase transition from a SP phase to a trivial phase generates a SPAQT gate.

The Chapters are arranged as follows; in Chapter 2 we will review different architectures for quantum computation, including the original adiabatic quantum transistor model.

In Chapter 3 we will identify properties of SP phases of 1D spin chains relevant for SPAQT computation and describe a particular method for understanding the action of a symmetry on a ground space which persists throughout a SP phase.

In Chapter 4 we present the main result of the thesis, a scheme to generate an adiabatic quantum transistor with any symmetry-protected phase of spin chains by adiabatic traversal of a symmetric phase transition. We will give explicit arguments to calculate the unitary transformation thus generated in terms of properties of the symmetry group that are universal to the SP phase. Finally we will assess the effect of a large class of

non-symmetric errors that may occur due to imperfect operation of the proposed scheme.

These results further the understanding of inherently robust architectures for quantum circuits that more closely resemble modern day computer chips. This brings us closer to an implementation of universal quantum computation with achievable control requirements that is inherently robust to many realistic sources of error.

1.1 Quantum Mechanical Preliminaries

In this section we will introduce the theoretical foundations on which the proceeding ideas of this thesis rest [1]. Specifically the framework of quantum mechanics, within which, all of the physics throughout the remainder of the thesis is constructed.

Within the framework of quantum mechanics, to any physical system we associate a *vector space* $\mathcal{H}$, the unit vectors of this space label every possible state that the system could be found in. This vector space is equipped the additional structure of an *inner product* $\langle\cdot|\cdot\rangle$, endowing it with a notion of geometry. We call a finite dimensional vector space $\mathcal{H} = \mathbb{C}^N$ along with its inner product, a *Hilbert space*[1] and use the *ket* notation for vectors $|\Psi\rangle \in \mathcal{H}$. Using the conventional notation we will write $\langle\Psi| : \mathcal{H} \to \mathbb{C}$ for a linear functional on $\mathcal{H}$, formed by taking an inner product of other states with $|\Psi\rangle$. Then the probability of finding a state $|\Psi\rangle$ in one of a set of basis states $\{|\psi_i\rangle\}_i$ upon making a measurement, is given by $|\langle\psi_i|\Psi\rangle|^2$. Measurements can never determine the *global phase* multiplying a state $|\Psi\rangle$, hence it is a physically unimportant quantity.

Throughout this thesis we will only consider a particular type of measurement called *projective measurement*, where one of a set of projection operators $\{|\psi_i\rangle\langle\psi_i|\}_i$ is applied to a state, corresponding to the measurement outcome ψ_i obtained from a measurement in the $\{|\psi_i\rangle\}_i$ basis.

The energetics which govern the behavior of an isolated physical system are described by a *Hamiltonian* $H : \mathcal{H} \to \mathcal{H}$ which is a *Hermitian* linear operator on the Hilbert space. The eigenvalues of H are the allowed energies that this physical system may have. The evolution of any state $|\Psi\rangle$ is governed by the *Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle$$

where $\hbar$ is the reduced Planck's constant. The evolutions which satisfy this equation are described by *unitary* linear operators $U : \mathcal{H} \to \mathcal{H}$ acting on the Hilbert space. Hence the time evolution of an isolated quantum system in state $|\Psi\rangle$ is described by the application of a unitary operator to the state $U|\Psi\rangle$.

[1]For infinite dimensional vector spaces there is the additional requirement that $\mathcal{H}$ is complete under its inner product.

Symmetries in Quantum Mechanics

Symmetries naturally form a group structure, since we can apply as many different symmetry transformations (e.g. rotation of a square by 90 degrees) as we like to a symmetric object without any apparent change. Hence if we collect all the symmetries g of an object, they will form a group G.

Since a physical system is completely described by its Hilbert space, any symmetry transformation g of the physical system must correspond to some operation $T(g)$ on the Hilbert space. Wigner's theorem [2] dictates that this operation $T(g)$ must be a unitary or anti unitary operator. Throughout this thesis we will only be treating the case of unitary operators and hence we will write $T(g) = U_g$.

To correctly describe a group of physical symmetries G by representing each symmetry transformation $g \in G$ with a unitary operator U_g, we have to enforce the following condition on the chosen operators

$$U_{g_1} U_{g_2} = U_{g_1 g_2}$$

which ensure that the operators U_g combine in the same way as the physical symmetries. Then the Hilbert space $\mathcal{H}$, along with the mapping

$$\begin{aligned} U : G &\to \mathcal{U}(\mathcal{H}) \\ g &\mapsto U_g \end{aligned}$$

is called a *unitary representation* of the group G (where $\mathcal{U}(\mathcal{H})$ are the unitary operators from $\mathcal{H} \to \mathcal{H}$). These *representations* are important in quantum mechanics, since whenever there is a group of physical symmetries of a system there must be an associated representation acting on the Hilbert space.

Many-Body Systems

Throughout this thesis we will be focusing on *many-body systems* that are made up of a large number of identical, microscopic components which are strongly interacting. In quantum mechanics such systems are described by combining the Hilbert spaces of all the microscopic components using a mathematical construction called the tensor product $\otimes$.

Here we will consider the example of a chain of N identical spin degrees of freedom, each having Hilbert space $\mathcal{S}_j$ with basis $\{|i_j\rangle\}_{i_j}$. The Hilbert space of this chain is $\mathcal{H} = \mathcal{S}_1 \otimes \mathcal{S}_2 \otimes \cdots \otimes \mathcal{S}_N = \bigotimes_{j=1}^{N} \mathcal{S}_j$, with basis $\{|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle\}_{i_1 \ldots i_N}$. An operator O acting on the mth spin is written as: $O_m = I_1 \otimes \cdots \otimes O_m \otimes \cdots \otimes I_N$. A Hamiltonian: $H_{m,m+1} = I_1 \otimes \cdots \otimes h_m \otimes h_{m+1} \otimes \cdots \otimes I_N$ on a pair of neighboring spins describes an interaction between them. A state $|\psi_i\rangle$ on each site corresponds to a *product state* $|\Psi\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle$ of the chain. A fundamental peculiarity of quantum mechanics is that a system can exist in a superposition of product states, hence a spin chain can occupy the state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(\bigotimes_{i=1}^{N} |\psi_i\rangle + \bigotimes_{j=1}^{N} |\varphi_j\rangle \right)$$

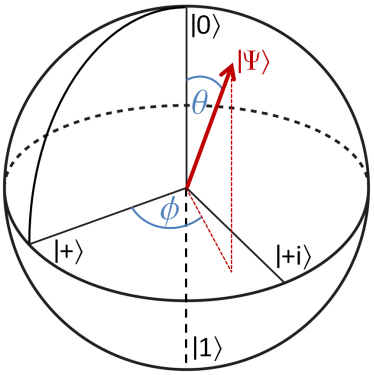

Figure 1.1: A state $|\Psi\rangle$ on the Bloch sphere.

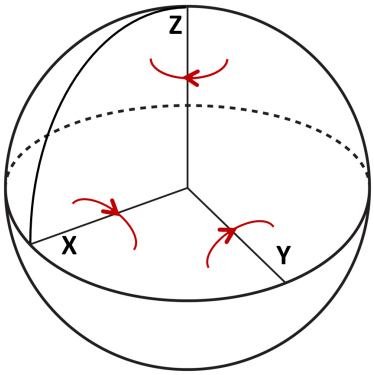

Figure 1.2: π rotations of the Bloch sphere.

Such a state is called *entangled* if it cannot be written as a product state. Entanglement is a key reason why quantum information behaves differently to classical information, since a measurement on one component of a system can destroy its entanglement with the remaining unmeasured components to reveal information about them, which is impossible in a classical description of physics.

Example: the Qubit

Here we present a simple but important example of a quantum system, the quantum bit (*qubit*) which constitutes a fundamental unit of quantum information. In this context we will give an example of a representation of the group D_2 of π-rotations about three orthogonal spatial axes.

In classical computation a single bit holds the value 0 or 1 and forms the fundamental unit of information. The qubit is a two dimensional system $\mathcal{H} = \mathbb{C}^2$, with basis $|0\rangle, |1\rangle$. The qubit can occupy the state $|0\rangle$ or $|1\rangle$ or any normalized, linear superposition of the two. A useful picture for the qubit Hilbert space is given by the *Bloch sphere* in Fig. 1.1, constructed by identifying that any state $|\Psi\rangle \in \mathbb{C}^2$ can be written

$$|\Psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle \tag{1.1.1}$$

up to an unmeasurable global phase. So where a classical bit could only store either a value 0 or 1, a qubit can store two continuous phase variables θ and ϕ which can be pictured as a vector on the Bloch sphere. A consequence of this capability is that quantum information is open to a whole class of continuous *phase errors* in the values of the encoded phases, where classical information is only open to discrete errors in the value of the bits.

Using the Bloch sphere we can construct a natural representation of the 3D rotation group $SO(3)$, where each rotation is represented by the operator formed by applying that same rotation to the Bloch sphere. To describe qubit unitary operators $\mathcal{U}(\mathbb{C}^2)$ we will make use of a convenient operator basis, given by the Pauli operators (which are

Hermitian and unitary)

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad X = \sigma^{\hat{x}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y = \sigma^{\hat{y}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z = \sigma^{\hat{z}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.1.2}$$

A vector of Pauli operators is written $\vec{\sigma} = (\sigma^{\hat{x}}, \sigma^{\hat{y}}, \sigma^{\hat{z}})$. In terms of this vector we can define an operator $\sigma^{\hat{m}} = \vec{\sigma} \cdot \hat{m} = \sin(\theta)\cos(\phi)X + \sin(\theta)\sin(\phi)Y + \cos(\theta)Z$, for any 3D axis $\hat{m} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$. Then we can write a rotation of the Bloch sphere through an angle θ about the axis $\hat{m}$ as[2] $\exp(i\theta\sigma^{\hat{m}}/2)$. Building on the intuition provided by the Bloch sphere picture, we might think of doing any single-qubit unitary transformation by applying a rotation to the Bloch sphere. This turns out to be correct, and any single-qubit unitary $U \in \mathcal{U}(\mathbb{C}^2)$ can be written as a rotation $U = \exp(i\theta\sigma^{\hat{m}}/2)$, for some angle θ and axis $\hat{m}$.

For example, a π-rotation about the $\hat{m}$ axis is simply given by the operator $i\sigma^{\hat{m}}$ itself. By combining three rotations for $\hat{m} = \hat{x}, \hat{y}, \hat{z}$ we arrive at a specific representation[3] of the group $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ of π-rotations about the spatial $\hat{x}$, $\hat{y}$ and $\hat{z}$ axes, depicted in Fig. 1.2, where each rotation is mapped to its corresponding $i\sigma^{\hat{m}}$ Pauli operator. This relatively simple example will recur throughout the thesis due to some special properties that the representation possesses.

Overview

In this chapter we have seen an elementary introduction to the basic concepts of quantum mechanics, how symmetries are described in this framework and the notion of quantum many-body systems.

In the next Chapter we will see what happens when we consider a computer built out of many quantum degrees of freedom and review several different models for the implementation of such a computer.

[2]For details on the operator exponential and generalization of the rotation operator to larger spins see Appendix E.

[3]Actually this is a projective representation which we will explore in more detail later in Sec. 3.3.2.

Chapter 2

Quantum Computation

Since we live in a universe that seems to be described by the laws of quantum mechanics, it follows that the types of information processing which are physically realizable are described by these laws. We should be able to harness these laws to build a *quantum computer*, that operates on principles of non-classical behavior such as entanglement.

The concept of a quantum computer first arose in considerations [3–6] of the fundamental, physical limits of computing. The power of these devices is apparent in the context of simulating quantum many-body systems, which is very hard, if not infeasible, using any classical computer. This can be attributed to the combination of exponential growth of the dimension of a many-body system's Hilbert space together with interfering histories. Classically the number of possible states of this many-body system will also grow exponentially as 2^N, however, to specify a classical state only N discrete on/off values along the chain are required. While a state of the quantum system requires $(2^N - 1)$ complex values to be completely specified. While classical probability distributions pose a similar scaling behavior they are restricted to be positive and hence cannot capture the full effect of the interfering paths that occur in quantum mechanics. This allows us to efficiently simulate problems involving classical probability by efficiently sampling from the relevant probability distribution using techniques such as the Monte Carlo algorithm. While quantum Monte Carlo techniques do exist, they are not universally effective as they cannot sample from all possible quantum distributions due to behavior known as the sign problem. Hence it is a somewhat subtle issue as to where exactly a quantum mechanical speedup can appear but several important instances are known.

A quantum computer built out of a many-body system of N qubits can naturally store 2^N complex values in each of its states and these values can interfere under physical unitary evolution. This makes it clear that quantum computers are natural candidates for simulating quantum many-body systems, something that is beyond the capability of modern day computers. Further research into exactly what quantum computers are capable of is still ongoing[1] but applications have been found outside of the direct simulation of other quantum systems. The most famous of these is Shor's algorithm [7], which allows a quantum computer to factorize a large number n into its prime factors using an amount of resources that increases at an exponentially slower rate than any known classical algorithm, as the size of n increases. This application is important due to its relevance to the field of cryptography, specifically any public key cryptosystem that relies on the difficulty of the factorization of large numbers would no longer be secure in the age of quantum

[1]It has not been proven that quantum computers have capabilities beyond probabilistic classical computers.

computers [1].

2.1 The Circuit Model

To theoretically classify the capabilities of a quantum computer the *circuit model* was created [8], which abstracts the computation away from any specific physical process. In this model a *quantum circuit* is constructed out of wires and logical gates. Each wire represents a qubit evolving in time from left to right, see Fig. 2.1. Logical gates on the qubits are produced by unitary evolutions, which can be single-qubit operations or multiple qubit interactions. A computation is done by initializing the qubits into their respective $|0\rangle$ states, and then applying to them a set of gates, designed to run a specific algorithm. After the gates have been applied the qubits are measured in the *computational basis* and the outcomes of these measurements are interpreted to give the result of the computation.

For a physical system to qualify as a quantum computer it must be at least as powerful, computationally, as the circuit model. That is to say, it must be able to simulate any computation possible in the circuit model in a time scaling polynomially with the number of gates in the corresponding circuit. So we have defined what constitutes a quantum computer with respect to the circuit model, but what computations can we perform with the circuit model? The answer to this question depends on what gates we can use. Of course we could allow ourselves to do any unitary transformation on the set of qubits directly, but this would require unphysical interactions of many qubits at once. In fact, if we restrict ourselves to only gates acting upon a small number of qubits at a time we can still build up any desired unitary transformation on a larger set of qubits, at least approximately [1] and possibly using an exponential amount of resources.

To make this precise we introduce the concept of a *universal gate set*. This idea derives from the situation in classical computation, in which any arbitrary function can be built up out of only logical AND, OR and NOT gates on pairs of bits. In quantum computation however this exactness is not required. Since the computations involve states that inherently lie in a continuum it is natural to consider a set of gates to be universal for quantum computation if any unitary transformation can be approximated, to within an arbitrary accuracy, by a quantum circuit built out of only those gates. A simple universal set of gates is given by either[2] the CNOT or CZ gate, along with arbitrary single-qubit gates. A result called the *Solovay-Kitaev theorem* [1] guarantees that we can efficiently simulate any single-qubit gate using only a finite set of gates. A particular choice is the Hadamard gate along with the Phase and $\pi/8$ gates.

So for a physical realization to classify as a quantum computer, we need only be able to efficiently generate a small set of gates (e.g. CZ, Hadamard, Phase, $\pi/8$) then we say that this system can be used to achieve universal quantum computation.

[2]In general this can be any nontrivial entangling gate on two sites without affecting the universality [9].

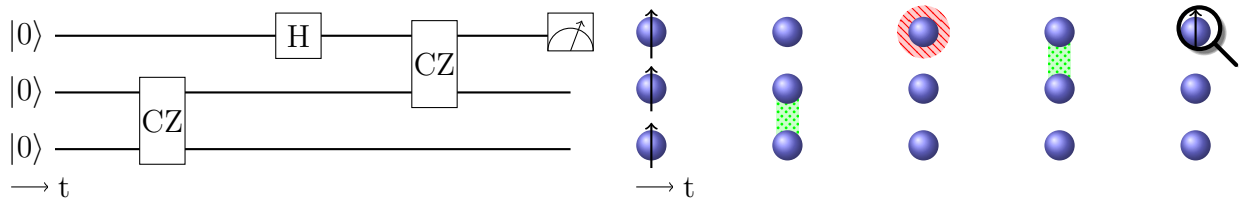

Figure 2.1: A quantum circuit and a physical realization of the circuit with spin-1/2 particles.

2.2 Different Architectures for Quantum Computation

The circuit model introduced in the previous section abstracts the behavior which defines a quantum computer, but to build such a device we need to ground this in reality. A simple physical realization of a quantum computer, depicted in Fig. 2.1, could be constructed by encoding each qubit in a separate physical degree of freedom, such as a spin-1/2 particle. Then single-qubit gates could be done by evolving individual spins under an appropriate local field for the precise length of time required to produce any desired single-qubit unitary transformation. Similarly, multiple qubit gates could be achieved by evolving groups of spins under a coupling Hamiltonian, again for a precise length of time to produce a desired multiple qubit unitary transformation. Realizing a quantum in this way is challenging because it requires a level of spatial and temporal precision that is not achievable using current technology. We consider different physical architectures whose evolutions can be interpreted as the computations of the circuit model, but are practically much easier to implement and control.

2.2.1 Topological Quantum Computation

A very appealing model for inherently robust quantum computation is based upon topological phases of matter [10]. In this scheme quantum information is encoded into the degenerate ground state of a many-body system possessing topological order. This means that the degeneracy depends upon the topology of the surface on which the many-body system lives and the information is completely robust to any perturbation that is sufficiently local in space. In such a scheme the logical operators must be inherently topological in nature, but can be implemented simply by a sequence of local operations corresponding to the creation and braiding of non-abelian anyonic quasi-particles.

2.2.2 Holonomic Quantum Computation

One realization called the *holonomic quantum computer* [11–16] can be achieved using only very specific types of evolutions that correspond to *adiabatically*[3] driving the Hamiltonian of a system around a *holonomic loop* in the space of its control parameters. This carries

[3]For a discussion of adiabatic processes in quantum mechanics see Appendix F

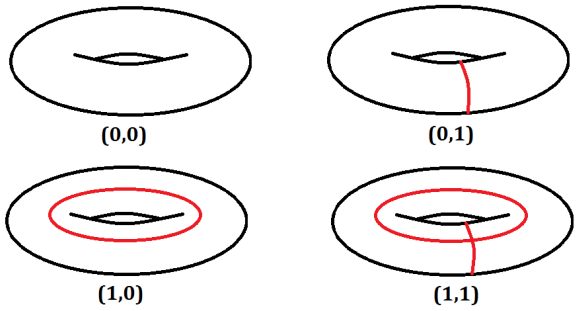

Figure 2.2: Schematic depiction of different topological ground states of the toric code.

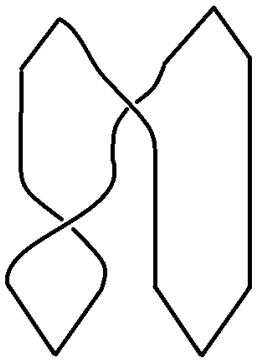

Figure 2.3: World lines of quasi particles being created, braided and then annihilated to implement a logic gate.

the advantage that it no longer requires the ability to precisely control the timing of each evolution, provided they are done adiabatically. Adiabatic evolutions have a precise definition in quantum mechanics [17] that corresponds to *slowly* changing a Hamiltonian with an *energy gap* Δ between its (possibly degenerate) ground state and first excited state, over a time period $T \sim 1/\Delta^3$.

In the holonomic model, quantum information is encoded into the degenerate ground state of the quantum system being used for computation, this carries the advantage that the logical states do not accumulate a relative dynamical phase error, so long as the ground state degeneracy persists. To perform a logical gate on this encoded information through an adiabatic evolution requires a non-zero energy gap Δ between the logical space and the first excitation of the system. This energy gap Δ must persist along any path we use, otherwise we could not perform the evolution adiabatically. This condition reveals a close relation between this model and quantum phases of matter, since these energy gap preserving paths are similar to those that lie within a quantum phase. Hence the gates employed in this architecture can be thought of as adiabatic holonomies within a quantum phase of matter [18].

We can interpret the unitary transformation of the encoded information due to an adiabatic evolution along a loop in the space of control parameters of the Hamiltonian as a geometric quantity [19–21] coming from the space itself, see Fig. 2.4. This is an instance of a generalized Berry's phase [22], induced by a non-Abelian, *Wilzcek-Zee potential*[4] [23]. This potential can be thought of as a gauge potential living on the space of control parameters, which will specify an evolution for any adiabatic path within the space. Provided that the loop is traversed slowly enough $T \sim 1/\Delta^3$ to ensure adiabaticity, the resultant logical transformation corresponds to a geometric property [24, 25], insensitive to the exact timing of the evolution. Research has shown [11, 26] that a generic instance of a holonomic quantum computer usually requires only a small set of independent loops to be capable of universal quantum computation.

[4]For a discussion of the origin and properties of the Wilczek-Zee potential and Berry's phase see Appendix F

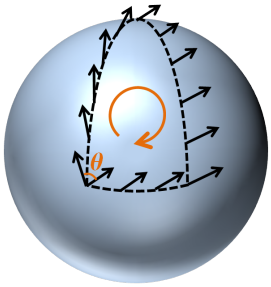

Figure 2.4: An example of a holonomy, after driving a vector around a loop it returns rotated by θ.

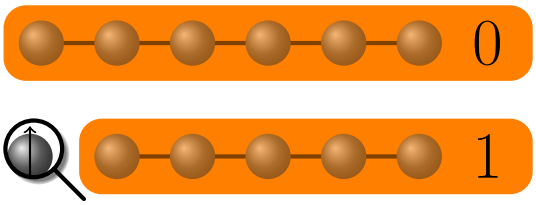

Figure 2.5: MBQC, revealing one component of
an entangled system changes its logical state.

2.2.3 Measurement-Based Quantum Computation

Another realization of a quantum computer can be achieved by *measurement-based quantum computation* (MBQC), which uses only single site measurements on a many-body system that is initialized in a highly entangled state [27]. This approach is advantageous in that it requires only single body measurements which are usually easier to achieve in practice than entangling two-body couplings.

We can view the fundamental difference between quantum and classical computing as the availability of entanglement as a computational resource. In the circuit model this is built up through the sequential application of multi-qubit entangling gates. Conversely in MBQC an initial state is prepared with a nontrivial entanglement structure, which can be consumed by single site measurements to perform quantum computation, as in Fig. 2.5.

Models for MBQC rely on encoding logical information into a subset of the states of a many-body system. Until recently only relatively simple states were considered, where the explicit analysis of the effect of a measurement is possible. There is no guarantee that any particular single site measurement will yield a unitary transformation on the encoded information, since the act of measurement is a non-unitary process. However there are many examples [28,29] where measurements in an appropriate basis yield unitary transformations on encoded information, finding such a bases can be a nontrivial problem.

An early model for MBQC called the one way quantum computer [27] was based on the 2D cluster state, a particular type of entangled resource state defined on a square lattice. It was shown that this model can be used to realize universal quantum computation using only single site measurements. A more general approach was subsequently developed [30, 31] where the information is seen as existing in the *correlation space* of the state into which it is encoded. For a more precise understanding of what this means we use a particular description for 1D states in terms of the matrix-product state (MPS) [32] formalism, which is introduced in detail in Appendix H.

Recently a significant effort [33–37] has been made to identify resources for MBQC that arise as ground states of many-body spin systems, and hence to uncover broader classes of resource states that do not rely on having simple explicit solutions for their

ground states. This research has lead to the discovery [36, 37] of a connection between a specific type of phases, possessing a property called *symmetry-protected order* (SPO), and the suitability of their ground states for MBQC, indicating that this suitability can be a property of a whole phase of ground states.

Specific examples [28, 29] of 1D phases characterized by SPO have arisen which accommodate arbitrary single-qubit operations, such models are referred to as quantum computational wires. Although these wires cannot perform universal quantum computation they serve as useful resources to store and manipulate qubits and upon coupling two such chains universality can be achieved. Inspired by these computational wire models, procedures for holonomic quantum computation have been proposed [18, 38, 39], making use of similar resources, but where the computations are done without making any measurements.

2.2.4 Adiabatic Quantum Computation

The adiabatic approach to quantum computation (sometimes labeled quantum annealing) is in some sense very different to all other methods considered, as it does not obviously involve the simulation of a standard unitary quantum circuit. Instead a Hamiltonian is adiabatically varied from an initial Hamiltonian such as $\sum X$ with a trivial ground state to a final Hamiltonian H_f whose ground state provides the solution to a desired problem.

$$H(t) := -t \sum_i X_i + (1-t)H_f \tag{2.2.1}$$

where t decreases from 1 to 0, see figure 2.6. Although not immediately obvious, it can be shown that this model is equivalent in computational power to the standard circuit model using a standard construction that encodes the history of a whole quantum circuit into the ground state of a local Hamiltonian [40].

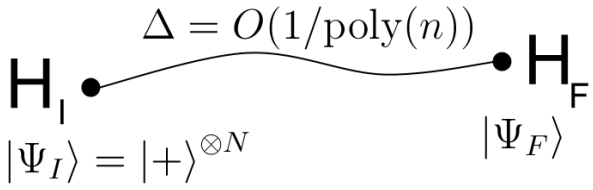

Figure 2.6: An adiabatic path connecting initial and final ground states in an adiabatic computation.

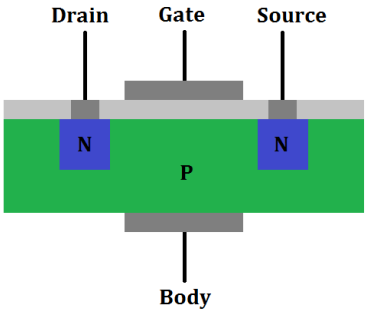

Figure 2.7: A diagram depicting a MOSFET. A voltage applied to the gate allows current to flow from source to drain.

2.2.5 Adiabatic Quantum Transistors

The idea of using an architecture which more closely mimics modern day computer chips was proposed by Bacon et al. in [41], it involves building physical circuits out of strongly coupled chains of matter that play the role of transistors. These chains were dubbed adiabatic quantum transistors as they mimic the function of a classical metal oxide field effect transistor (MOSFET) see figure 2.7. In a MOSFET, the application of a voltage to the input gate attracts electrons from the p-type semiconductor to the surface where they form a conductive layer between the two regions of n-type semiconductor at the drain and source. This opens up a channel that allows information to pass through the transistor in the form of an electric current. Analogously, when a uniform field is applied to an adiabatic quantum transistor, quantum information propagates across a chain of strongly coupled spins and undergoes some logical transformation dependent upon the system's Hamiltonian.

The Hamiltonian describing the basic adiabatic quantum transistor construction is given by

$$H_N(t) = -t\left(-Z_{N-1}X_N + \sum_{i=1}^{N-2} Z_i X_{i+1} Z_{i+2}\right) - (1-t)\sum_{i=1}^{N-1} X_i \tag{2.2.2}$$

with t decreasing from 1 to 0, the evolution is depicted in figure 2.8.

It was already pointed out in [41] that the basic model Hamiltonian 2.2.2 possesses SP order and it was speculated that this would endow the computation with some robustness. This is the model we will build upon explicitly in Chapter 4, where the speculated connection between AQTs and SP phases will be elucidated to show that all SP phases give rise to some form of AQT. Hence the SP material can be thought of as playing the role of the semiconductor in the analogy between AQTs and classical transistors.

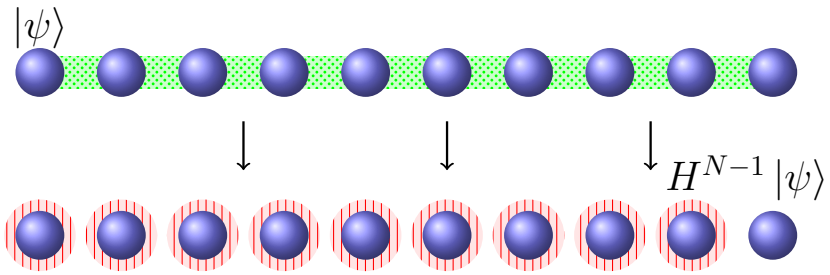

Figure 2.8: The operation of an adiabatic quantum transistor, information propagates across the chain and undergoes a logical transformation due to the application of a uniform field.

2.3 Quantum Ordered Phases: Hardware for Quantum Computing

In the previous section we saw that ground states of many-body systems are useful resources for quantum computation, holonomic, measurement-based and AQT. Recently

there has been a flurry of interest [34, 36] in relating the properties of quantum states that make them useful for quantum computation, with the ordering that defines a phase. We made explicit mention of symmetry-protected (SP) phases [42–44] above, since they are characterized by properties that make them naturally suitable for the storage and manipulation of quantum information. In this section we will reveal a little more of what symmetry-protected order means, and why such phases turn out to be useful for quantum computation.

We understand the collective behavior of strongly interacting, many-body systems in terms of *ordered phases*. Classically all ordered phases and phase transitions can be characterized by the Landau theory of spontaneous symmetry breaking [45]. We will illustrate this with the example of a ferromagnet, above the Curie temperature the magnetic domains are randomly aligned so the whole system is rotationally symmetric, below the Curie temperature the domains all align, spontaneously breaking this symmetry.

Conversely the quantum phases we will consider all exist at strictly zero temperature, defined by collections of Hamiltonians that can be connected by adiabatic evolutions. Hence a *quantum phase transition* corresponds to the closing of the energy gap such that no adiabatic path is possible through the transition point. Until relatively recently it was thought that all quantum ordered phases could also be completely described by the Landau theory. However, A number of example states [46] where such a classification did not suffice emerged around 1990, in the context of high temperature superconductivity[5]. Since then our understanding of the the reason behind the failure of the symmetry breaking theory has been founded on the entanglement of such states, which is absent in the classical case. This led to the understanding that the relation between components in a many-body system due to entanglement is fundamentally different from the purely statistical correlations between components which can arise in a classical description of physics. As a consequence quantum phases can possess long-range entanglement without showing signs of long-range statistical correlations.

A precise definition of quantum ordered phases in terms of entanglement is given in [43] and will be described in detail in Sections 3.1 and 3.2. This definition results in not only new types of phases characterized by long-range entanglement, but also new phases [47–49] characterized by more subtle patterns of short-range entanglement that exist only when all the Hamiltonians of the phase share a symmetry[6]. These symmetric, short-range entangled phases are precisely the SP phases mentioned earlier. These SP phases are beyond the Landau theory since they are made up of ground states that do not break the symmetry of their respective Hamiltonians and hence phase transitions between them would not be detected in the Landau framework. Although the entanglement of the states within these phases is only over a short scale, it is nontrivial in the sense that it cannot be removed without the system undergoing a phase transition.

[5]These states failed to describe high temperature superconductivity, although they became interesting in their own right, and have since been applied to describe other phenomena such as the fractional quantum hall effect.

[6]Throughout this section and the rest of the thesis we often refer to a 'symmetry' being either broken or respected by a state within an ordered phases, the symmetry we refer to is precisely that of the state's parent Hamiltonian.

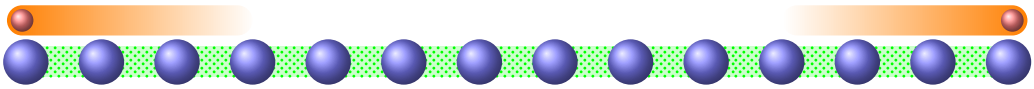

Figure 2.9: A Haldane chain of interacting spin-1 particles (blue) and two virtual spin-half particles (orange) corresponding to the chain's emergent edge modes due to its SP.

An illustrative example of a 1D SP phase is given by the Haldane phase [50] characterized by its full $SO(3)$ rotation symmetry, which was extensively studied [51–55] before the current notion of SP was formulated. Many of the properties [54–56] of the Haldane phase are just specific instances of more general properties [57–60] native to all 1D SP phases [61–63].The most important of these properties for the purposes of this thesis is the behavior of the system's ground state degeneracy. With periodic boundary conditions the system possesses a unique ground state, but with open boundaries the system takes on a four fold ground state degeneracy. This degeneracy can be attributed to the existence of a gapless, effective spin-1/2 edge excitation at each open boundary [50, 54, 56, 64] which persist throughout the phase in the presence of a symmetry. These edge modes will be studied in more detail in Sections 3.3.3 and 3.3.4.

The presence of a protected degenerate ground state is of particular relevance for applications in quantum computation, since this ground state can be used to encode quantum information that can be thought of as existing in its emergent edge modes (an instance of the Holographic principle [65]). For example, the degenerate ground space is a natural resource for holonomic quantum computation, since it persists throughout the symmetric phase.

Overview

At the beginning of this chapter we outlined the standard circuit model that describes a universal quantum computer. We then presented several alternative architectures, finally arriving at the adiabatic quantum transistor model which will be most relevant to our proposed scheme in Chapter 4. In the last section we introduced the concept of quantum computation in the ground states of SP spin systems using holonomic, AQT or measurement-based protocols. The rest of the thesis is laid out as follows; we will further our understanding of the properties of SP in Chapter 3 before giving the results of our proposed scheme to generalise AQT computation to all SP phases in Chapter 4.

Chapter 3

Topological Quantum Phases

In this chapter we will make precise the notion of a quantum order phase, introduced in Chapter 1. We will go on to present a collection of recent results on symmetry-protected (SP) phases which constitute a very new type of quantum matter that is not yet fully understood. In fact, these phases are so new that the results on them, collected in this chapter have not been brought together in this way before. The most important result covered in this chapter is the classification of 1D SP phases in terms of a discrete set of labels $H^2(G, U(1))$, determined by the symmetry groups that protect these phases.

In Sec. 3.1 we will introduce the equivalence relation between quantum states that define the phases of a quantum many-body system. We will go on to look at two particular types of inherently non-classical phases in Sec. 3.2, and see how the presence of a symmetry allows different phases to exist. Following Sec. 3.3 we restrict our interest to the case of 1D spin systems that will be studied throughout the remainder of the thesis. An example is presented in Sec. 3.3.1 which contrasts a SP phase against a symmetry breaking phase to highlight the non-classical properties inherent to such SP phases. A mathematical tool called *projective representations* is introduced in Sec. 3.3.2 to aid in understanding the symmetries of the SP ground states in terms of their emergent edge modes, discussed in Sec. 3.3.3. The analysis of these edge modes leads to the classification of 1D SP phases in terms of their *second cohomology group* $H^2(G, U(1))$, introduced in Sec. 3.3.2. Finally we will provide a characterization of the edge modes in Sec. 3.3.4 that will be used throughout the remainder of the thesis.

3.1 Quantum Phases and Local Unitary Transformations

To precisely formulate the notion of an ordered phase we follow the arguments of [43] by defining an equivalence relation between ground states of local, gapped Hamiltonians. In terms of this description we will dichotomize the possible patterns of ground state entanglement into long-range and short-range. These patterns of entanglement will later be used to define two different types of topologically ordered phases in Sec. 3.2

First we define what we mean by a *gapped, local Hamiltonian*, which plays a key role in the precise definition of a quantum phase. A Hamiltonian defined on a lattice of size N is *local* if it can be written as a sum of terms each operating on a set of sites no further than some predefined distance, $d \ll N$, apart. We call this Hamiltonian *gapped* if the difference $\Delta = \varepsilon_1 - \varepsilon_0$ between its ground state and first excitation remains greater than

some fixed, positive constant $\epsilon > 0$ in the thermodynamic limit $N \to \infty$. This leads us to consider a set of low lying energy states to be a degenerate ground state for finite N if the eigenvalues all converge to the ground state in the thermodynamic limit.

For our purposes it suffices to define a *phase transition* point for a set of Hamiltonians $H(\lambda)$ to be a point λ_0 where the gap of $H(\lambda_0)$ closes. The closing of this energy gap can be associated with the existence of long-range correlations and singular behavior of local operators, both characteristic of a phase transition point [43].

Then we say that two states $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$ are in the same phase if there is a continuous path of gapped, local Hamiltonians $H(\lambda)$, with no phase transition along the path $\lambda \in [0, 1]$, such that $|\Psi(0)\rangle$ is the ground state of $H(0)$ and $|\Psi(1)\rangle$ is the ground state of $H(1)$. This is depicted in Fig. 3.1.

Local Unitary Transformations

We can use this construction to find an adiabatic evolution that takes $|\Psi(0)\rangle \mapsto |\Psi(1)\rangle$, generated by slowly driving the Hamiltonian along the path $H(\lambda)$, $\lambda \in [0, 1]$. However, with this description there is not much, in general, that we can say about the form of the unitary evolution thus generated. This leads to the idea of *quasi adiabatic continuation* [66], which is a construction[1] that yields an explicit *local unitary transformation*[2] $U(s) = \mathbf{P}[e^{-i\int_0^s d\mu \tilde{H}(\mu)}]$ which connects the two states within the phase $|\Psi(1)\rangle = U(1)\,|\Psi(0)\rangle$. Conversely, given the unitary evolution $U(s)$, we can construct the Hamiltonian $H(s) = U(s)H(0)U^\dagger(s)$ which satisfies the adiabatic condition to relate $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$ within the same phase.

Hence the definition of a phase in terms of Hamiltonians related by adiabatic evolutions, and the definition in terms of ground states related by local unitary transformations, are equivalent.

Finally we mention a more intuitive point of view on the phase equivalence relation, by unrigorously identifying local unitary transformations with *constant depth quantum circuits*, an example is depicted in Fig. 3.2. This identification was conjectured by Wen et al. in [43] and is not rigorous but gives a clear intuition for the relation of different ground states within a phase. From this point of view a phase is a collection of ground states that can be transformed into one another using quantum circuits built out of a constant number of layers, each layer consisting of only local gates as the system size increases. This characterization will be useful to visualize how such unitaries can change the entanglement patterns within an ordered phase.

Long-Range Entanglement

With the definition of local unitary transformation established we can introduce the key concept of *long-range entanglement* which is used to define non-classical phases. *Short-*

[1] This involves the construction of a Hamiltonian $\tilde{H}(\mu)$ which is, in general, different to $H(\lambda)$.

[2] $\mathbf{P}$ denotes the path ordering of the exponential, described in Appendix F.

range entanglement is precisely any pattern of entanglement which can be removed (equivalently created) by a local unitary transformation. Intuitively we can view this entanglement as being created by a finite depth quantum circuit applied to a trivial product state. The causal cone of the circuit determines the correlation length of the resultant, short-range entangled state. Hence long-range entanglement is defined to be any pattern of entanglement which cannot be removed by any local unitary transformation.

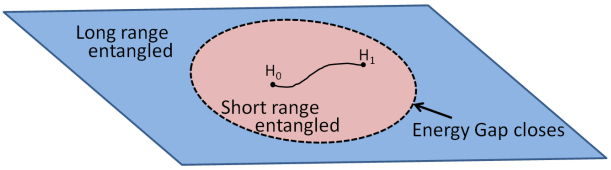

Figure 3.1: A phase diagram defined by gap preserving paths between Hamiltonians.

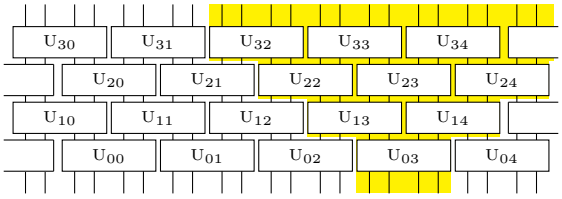

Figure 3.2: Constant depth quantum circuit and the causal cone of a site (yellow).

3.2 Topologically Ordered Phases

Equipped with the tools developed in the previous section we will give precise definition of two types of non-classical orders. The first type is called *long-range topological order*, which is defined by patterns of long-range entanglement. The second is called *symmetry-protected order* (or symmetry-protected topological order), which is characterized by subtle patterns of short-range entanglement that exist only in the presence of a symmetry that is obeyed by all Hamiltonians within a phase.

3.2.1 Long-Range Topological Order

If we consider a phase whose Hamiltonians do not share any particular symmetry, all short-range entangled states can be connected to a trivial product state (and therefore each other) by local unitary (LU) transformation. Under the local unitary classification of ordered phases this implies that all states with only short-range entanglement lie within the trivial phase. The *equivalence under LU transformations* identifies the set of all short-range entangled states with the trivial equivalence class. Under this equivalence relation any state with long-range entanglement lies in a nontrivial equivalence class made up of all the states that can be connected to it by local unitary transformation. Such a phase is defined by its long-range entanglement and is called a long-range topologically ordered phase. Distinct long-range topologically ordered phases are precisely those which cannot be related via local unitary transformations.

Long-range topologically ordered phases are referred to as such, due to the nontrivial topological properties that their ground states posses. For example, a non-zero topological entanglement entropy [67] and a ground state degeneracy that is dependent upon the topology of the lattice on which the model is defined. These phases are ideal for building hardware for quantum computation that is robust to any sufficiently small, local

perturbation to the system's Hamiltonian. An example of a long-range topologically ordered state is the toric code [10], which has a degenerate ground state when it is defined on a torus and into this subspace we can encode quantum information that is robust to sufficiently small, local perturbations. The trade off for such protection is that to perform desired manipulations on the encoded quantum information requires topologically nontrivial operators, which could be technologically challenging to produce.

3.2.2 Symmetry-Protected Topological Order

In the presence of a symmetry obeyed by all Hamiltonians within a phase, there is additional structure in the phase diagram. The long-range topological phases divide up into symmetry breaking and symmetry respecting phases that are called symmetry enriched topologically ordered phases. The short-range entangled states which were previously all within the trivial phase now divide up into symmetry breaking phases and symmetry respecting phases called *symmetry-protected* (SP) phases (also refered to as symmetry-protected topologically ordered phases in the literature). This behavior is depicted in Fig. 3.3.

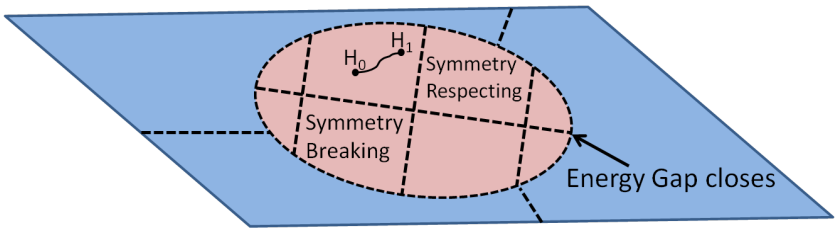

Figure 3.3: Symmetry restrictions add additional structure to the phase diagram.

To understand the new phases that arise in the presence of a symmetry we will look at the effects of this symmetry on the relations that define a phase. Due to the symmetry, there is a restriction on the adiabatic paths which can be used to connect two states within a phase, now they can only contain symmetric Hamiltonians. This restriction applies similarly to the local unitary transformations which equivalently define a phase. Hence the equivalence relation whose classes make up the ordered phases in the presence of a symmetry is precisely equivalence under *symmetry-preserving local unitary transformations*. This leads to finer structure in the phase diagram since the symmetry preserving local unitaries that connect states in a symmetric phase, are only a small subset of the possible local unitaries without any symmetry.

We classify phases in the presence of a symmetry according to whether the ground states within the phase break or respect the symmetry. We make use of a result from [1]: in the presence of a symmetry which commutes with the Hamiltonians defining a phase, the matrix representations of two Hamiltonians must take on a simultaneous block structure. Hence the symmetry has a restricted action within each degenerate eigenspace. Here we are specifically interested in the behavior of ground states which form the symmetric phases. If a nondegenerate ground state is symmetric it must be an eigenstate of the symmetry operators, whereas a degenerate ground state may be spanned by a set of

eigenstates which each individually break the symmetry. Degenerate ground spaces made up of symmetry breaking eigenstates are a defining property of what we call *symmetry breaking* phases. When there is unique symmetric ground state on closed boundary conditions that breaks into a degenerate ground space on open boundary conditions made up of eigenstates that have a symmetric bulk, this is a defining property of *symmetry-protected* phases.

Classically, the symmetry breaking phases would be sufficient to capture all possible behavior but as we have pointed out in Sec. 2.3, in quantum mechanical systems there exist phases and phase transition in which no symmetry is broken at all. These phases are precisely the symmetry-protected topological order phases, defined by subtle patterns of short-range entanglement. Such subtle, nontrivial patterns of short-range entanglement persist throughout a nontrivial SP phase since no symmetric LU can transform any state in such a phase to a product state[3]. Note that due to their short-range entanglement all SP states are considered to be trivial without any symmetry constraints on their Hamiltonian and hence only exist in the presence of a symmetry (hence the name 'symmetry-protected').

The properties which define SP phases can be harnessed to make them natural architectures for various different models of quantum computing. By using global properties of a whole SP phase we can make useful resource states for quantum computation that remain so as long as they are within the phase, this gives these states natural robustness against symmetric perturbations. This property lies at the heart of why SP phases might be advantageous as hardware for quantum computing.

3.3 1D Symmetry-Protected Topologically Ordered Phases

In this section we will focus on SP in 1D spin systems and these systems will remain the focus throughout the rest of this thesis. First we give an instructive example, in Sec. 3.3.1, which juxtaposes the qualities of symmetry breaking phases against those of symmetry-protected phases. In Sec. 3.3.2 we will introduce a mathematical tool called projective representations which play a key role in understanding the degenerate ground states present in these phases. With intuition gained from the earlier example and the tools developed in Sec.3.3.2 we will characterize[4] the ground state degeneracy of a SP phase in terms of gapless edge modes and their symmetries, described by a discrete set of labels $H^2(G, U(1))$. Finally, in Sec. 3.3.4, we will explain a useful way of looking at a single edge mode by coupling the other edge mode to a physical particle which effectively fixes that boundary degree of freedom.

[3]Provided we have two distinct SP phases the trivial product state can lie in only one phase

[4]In Appendix I we have provided the full Chen-Gu-Wen classification [61, 62] of 1D SP phases.

3.3.1 Contrasting an Ising Phase Against a Haldane Phase

Here we will contrast a simple example of a symmetry breaking phase described by the Ising model, against another example called the Haldane phase which possesses SP.

Ising Phase

The Hamiltonian which governs the ferromagnetic Ising model with a transverse field is given by

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z + K \sum_i \sigma_i^x$$

this model possesses a $\mathbb{Z}_2$ symmetry of π-rotations about the $\hat{x}$-axis. For $J > 0$, $K > 0$, the $\sigma_i^z \sigma_{i+1}^z$ term favors the alignment (or anti alignment) of neighboring spins along the $\hat{z}$-axis, while the σ_i^x field favors alignment of all spins along the $(-\hat{x})$-axis.

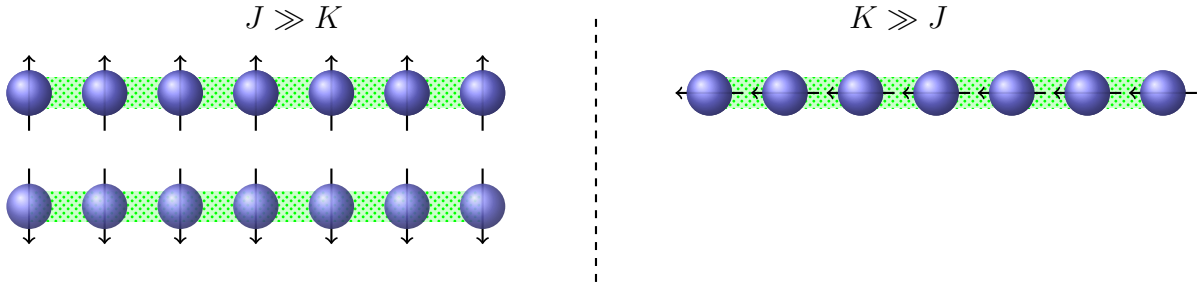

Figure 3.4: Ground states of two different phases of the Ising chain, demonstrating symmetry breaking for $J \gg K$.

There is a phase transition when the two strengths are equal, $J = K$ and for $K > J$ the phase is characterized by the ground state $|\Psi^-\rangle = |\leftarrow\leftarrow \cdots \leftarrow\rangle$, which is exact for $J = 0$, while for $K < J$ the phase is characterized by the degenerate ground states $\left|\Psi^{\uparrow}\right\rangle = |\uparrow\uparrow \ldots \uparrow\rangle$ and $\left|\Psi^{\downarrow}\right\rangle = |\downarrow\downarrow \ldots \downarrow\rangle$, exact for $K = 0$. These phases are depicted in Fig. 3.4.

Within the $K > J$ phase, the ground state does not break the $\mathbb{Z}_2$ symmetry since they can be transformed to the $|\Psi^-\rangle$ state by symmetry preserving local unitaries. In the $J > K$ phase, the degenerate ground states can be transformed to $\left|\Psi^{\uparrow}\right\rangle$ and $\left|\Psi^{\downarrow}\right\rangle$ which break the symmetry. So we can see that in this example each phase is described by the symmetry of its ground states, particularly one phase is defined by degenerate ground states that break the symmetry. This is, in essence, the type of behavior described by the Landau theory of spontaneous symmetry breaking.

This example also demonstrates the *trivial entanglement* that is characteristic of symmetry breaking phases. Notice that both the phases in our example can be transformed by a local unitary to one of the unentangled ground states, either $|\Psi^-\rangle$ or $\left|\Psi^{\uparrow}\right\rangle$ and $\left|\Psi^{\downarrow}\right\rangle$. This example gives us an intuition as to why the Landau theory works so well for classical phases, which do not possess any entanglement, while for quantum phases it fails to capture the full intricacy of the phase diagram, missing the phases defined by subtle, but nontrivial, patterns of entanglement.

Haldane Phase

We now contrast the previous example with a phase transition which is not described by the symmetry breaking of any ground state. The model is given by the Haldane phase Hamiltonian with a symmetric $\hat{z}$ field perturbation

$$H_n = J \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1} - \beta \left(\vec{S}_i \cdot \vec{S}_{i+1} \right)^2 \right) + K \sum_i (S_i^{\hat{z}})^2 \tag{3.3.1}$$

this model possesses $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry, generated by π-rotations about the $\hat{x}$ and $\hat{z}$ axes. We will consider β to lie within the range of the Haldane phase [54] under no external field $-1 < \beta < 1$ throughout our discussion, so the only phase transitions are induced by the $\hat{z}$ field term. Hence while $J \gg K$ the ground state of this Hamiltonian lies within the Haldane phase, for the particular value $\beta = -1/3$ the Haldane phase has an exact solution called the *Affleck-Kennedy-Lieb-Tasaki* (AKLT) state [53,55] which can be written exactly as a matrix-product state (see Appendix H). There is a phase transition as K increases past some critical value K_c and then for $K \gg J$ the phase is described by the product ground state $\left|S^{\hat{z}} = 0\right\rangle = |00\ldots0\rangle$ as depicted in Fig. 3.5.

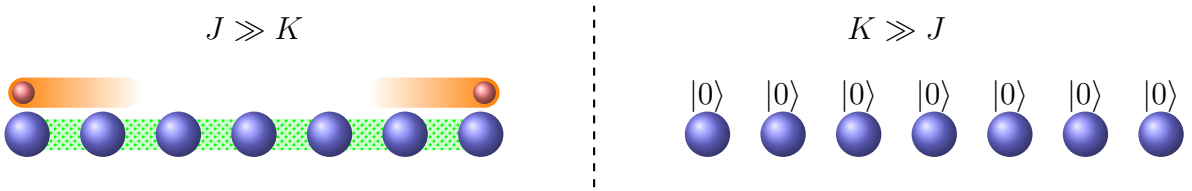

Figure 3.5: Ground states of two different phases of the Haldane chain, a nontrivial SP phase for $J \gg K$ and a trivial phase for $K \gg J$.

Once again we can classify each phase by a single ground state, which is connected to every other state in its respective phase via symmetry preserving local unitary transformations. For large K the $\left|S^{\hat{z}} = 0\right\rangle$ ground state is symmetric[5] under the D_2 symmetries formed by simultaneous π rotations of all the spins in the chain about the $\hat{x}$, $\hat{y}$ and $\hat{z}$ axes. For large J the AKLT ground state is also symmetric under the D_2 rotations, since each spin-1 rotation breaks up into a product of spin-1/2 rotations on virtual degrees of freedom at each site, and these spin-1/2 particles are in pairwise singlet states which are each invariant under these rotations (for details see Appendix H). Hence in this example we cannot differentiate the phases by any symmetry breaking of the ground states, and we have an instance of a symmetric phase transition of a symmetry-protected phase. This type of phase transition will be considered throughout chapter 4 to drive the computation of symmetry-protected adiabatic quantum transistors.

The large K phase corresponds to a trivial SP phase, since any state within the phase can be transformed to the trivial state $\left|S^{\hat{z}} = 0\right\rangle$ by a local unitary, indicating that the entanglement pattern of these states is trivial in the presence of the D_2 symmetry. While for large J, the Haldane phase contains no product state, and the entanglement pattern

[5] This is easy to see from the explicit form of the rotations given in Appendix E.

of any state within the phase is characterized by the AKLT state which posses nontrivial, short-range entanglement in the presence of the D_2 symmetry. Note that at the AKLT point, with open boundary conditions there is a fourfold degeneracy in the ground state corresponding to two virtual spin-1/2 particles at the uncoupled ends of the chain. These boundary degrees of freedom label the ground state degeneracy and persist throughout the Haldane phase [54]. They can be thought of as being spread out near the edges by the causal cone of a finite depth quantum circuit, see Fig. 3.2, that transforms the AKLT state to another state in the phase. In this way the gapless edge modes are protected by the D_2 symmetry of the phase, and this feature of the Haldane phase actually generalizes to be a defining property of all 1D SP phases [61].

In the next section we will go on to develop theoretical tools that describe how the gapless edge modes interact with the symmetries that are protecting them.

3.3.2 Projective Representations

In this section we will develop the mathematical framework to describe the symmetries of the emergent edge modes in the degenerate ground states of SP spin chains.

A unitary *projective representation* [68, 69] of a symmetry group G is a vector space $\mathcal{H}$ together with a homomorphism $V : G \to \mathcal{U}(\mathcal{H})/\mathbb{C}$ such that V_g is unitary for all $g \in G$. This definition is similar to that of non-projective representations given in Sec. 1.1, however there is an additional multiplicative degree of freedom for the projective representation, and to differentiate the two types of representation we will henceforth refer to non-projective representations as *linear representations*.

Due to the multiplicative freedom inherent to projective representations, the multiplication rule for the matrices V_g differs from that of the group elements by multiplicative factors $\omega(g_1, g_2)$ which form an object called a *factor system* $\omega : G \times G \to \mathbb{C}$. More specifically these $\omega(g_1, g_2)$ are phase factors and hence lie in $U(1) \subset \mathbb{C}$ formed by the complex numbers with norm 1. Hence $V : G \to \mathcal{U}(\mathcal{H})/U(1)$.

The multiplication rule in a projective representation can be written as

$$V_{g_1} V_{g_2} = \omega(g_1, g_2) V_{g_1 g_2} \tag{3.3.2}$$

where $\omega(g_1, g_2)$ is the phase factor coming from the factor system ω.

Since the multiplication is defined only up to a phase, here we will find how much of this freedom can be absorbed into each individual operator V_g. Due to the multiplicative freedom we consider two mappings V and V' to be the same if they differ by a 1D representation of the group. Since this corresponds to the multiplication of each matrix V_g by some phase factor $\alpha(g)$ such that $V'_g = \alpha(g) V_g$. Then

$$\begin{aligned} \omega'(g_1, g_2) &= V'_{g_1} V'_{g_2} [V'_{g_1 g_2}]^{-1} \\ &= \alpha(g_1) V_{g_1} \alpha(g_2) V_{g_2} [\alpha(g_1 g_2) V_{g_1 g_2}]^{-1} \\ &= \frac{\alpha(g_1)\alpha(g_2)}{\alpha(g_1 g_2)} \omega(g_1, g_2) \end{aligned} \tag{3.3.3}$$

We consider factor systems related by such local phase factors as *equivalent* $\omega' \sim \omega$. Upon taking the quotient of the collection of all the factor systems $\{\omega\}$ by the equivalence relation $\sim$, They are divided up into a discrete set of equivalence classes $[\omega] := \{\omega' | \omega' \sim \omega\}$. We call each equivalence class $[\omega]$ a *factor class*, and together they form an Abelian group under multiplication.

Hence any factor system which can be written in the form $\omega(g_1, g_2) = \frac{\alpha(g_1 g_2)}{\alpha(g_1)\alpha(g_2)}$ lies within the trivial class, since it can be thought of as a product of two linear representations $\alpha \times U$, where $\alpha(g)$ is a 1D representation that amounts to a g-dependent phase factor multiplying each matrix $U(g)$ representing the group element g. Hence any projective representation that can be written in this way is not truly projective, but just a product of linear representations masquerading as a projective one.

To pin down exactly what these Abelian groups are, we enforce the associativity of group multiplication and the fact that V is a homomorphism, to find

$$[V_{g_1} V_{g_2}] V_{g_3} = V_{g_1} [V_{g_2} V_{g_3}] \implies \omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\omega(g_2, g_3)$$

With this further restriction on each factor system ω we can identify the Abelian group of their equivalence classes $[\omega]$ with an object called the *second cohomology group*[6] of G, $H^2(G, U(1))$ (sometimes referred to as the *Schur multiplier*). This group can be thought of as a set of labels (usually discrete) which specify the irreducible projective representations of a particular group. Where 'irreducible' roughly means we cannot write the projective representation as a product of two smaller representations. The technical condition is that there cannot be any invariant subspace $\mathcal{W} \subset \mathcal{H}$ shared by all matrices V of the projective representation [70].

In the next section we will use these projective representations to look at the symmetries of the emergent edge modes in 1D SP ground states.

3.3.3 Symmetries of the Edge Modes

In this section we will look at the behavior of the degrees of freedom occurring at the edges of a 1D SP ground state. Specifically, we will examine the way the chain's symmetry acts on these edges using the tools developed in the previous section. We will find that the behavior of the edge modes classify all 1D SP phases in terms of the second cohomology group $H^2(G, U(1))$ (introduced above) of the symmetry group G protecting the phase.

In the example in Sec. 3.3.1 we saw that states in the Haldane phase possessed a four fold ground state degeneracy, which we associated to degrees of freedom at the edges coming from the open boundary conditions. These gapless edge modes are a defining characteristic of more general SP phases, in which the ground states do not break the symmetry of their Hamiltonians.

We consider the case of a symmetry group G which acts upon each site i via the linear representation $U^i : G \to \mathcal{H}_i$. Hence the symmetry operator on the whole chain

[6]The origin of this second cohomology group in algebraic topology is briefly described in Appendix G.

$\mathbf{U}_g = \bigotimes_i U_g^i$, depicted in Fig. 3.6, commutes with the Hamiltonian H implying that both matrices will have the same block structure. In particular, since the Hamiltonian has a degenerate ground subspace $\mathcal{W}$, the symmetry operator will have a corresponding block $\mathbf{U}_g\big|_{E_0}$ which acts on states within $\mathcal{W}$ without taking them outside of $\mathcal{W}$ (hence $\mathcal{W}$ is an invariant subspace of $\mathbf{U}$).

The degeneracy in the ground state is associated to the degrees of freedom at each boundary and hence, for a sufficiently long chain where the two ends do not interact, the Hilbert space of the degenerate ground states break up into a tensor product of Hilbert spaces for the left and right boundary modes $\mathcal{W} = L \otimes R$. Then the restriction of the symmetry operator $\mathbf{U}_g\big|_{E_0}$ to the ground space can be written as a product of symmetry operators acting upon each of the edge modes individually, $\mathbf{U}_g\big|_{E_0} = V_g^L \otimes V_g^R$ as shown in Fig. 3.7. This allows some freedom in what the V^L and V^R operators can be, the only restriction on them being that together they must form a linear representation of G, since $\mathbf{U}\big|_{E_0}$ is a linear representation of G.
Hence, $V_{g_1}^L \otimes V_{g_1}^R \cdot V_{g_2}^L \otimes V_{g_2}^R = V_{g_1 g_2}^L \otimes V_{g_1 g_2}^R$.

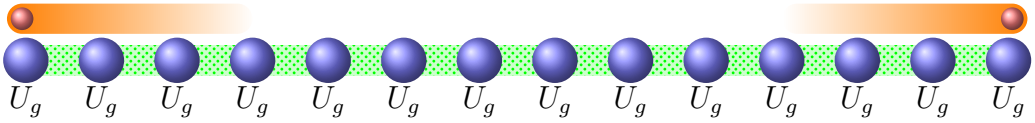

Figure 3.6: The symmetry operator $\mathbf{U}_g$ acting upon the ground state of a SP spin chain.

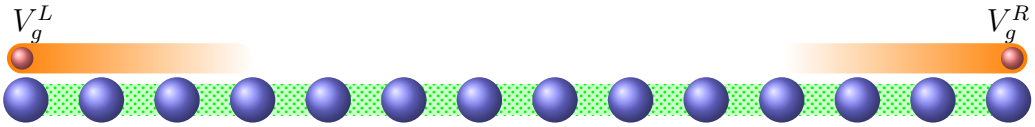

Figure 3.7: The equivalent action of the symmetry $\mathbf{U}_g$ on the edge modes.

This freedom allows each edge symmetry to obey the multiplication rule up to a phase, so long as the phases from the left and right operators cancel each other out. So we can have

$$V_{g_1}^{L/R} V_{g_2}^{L/R} = \omega_{L/R}(g_1, g_2)\ V_{g_1 g_2}^{L/R} \tag{3.3.4}$$

under the condition that $\omega_L = \omega_R^{-1}$. Hence the symmetry acts upon each edge via a projective representation and ω_L, ω_R are the factor sets of V_L and V_R respectively. Since only equivalence of the factor sets matters for labeling different projective representations we can loosen the condition $\omega_L = \omega_R^{-1}$, to $\omega_L \sim \omega_R^{-1}$.

Each projective representation has an associated vector space which corresponds to the Hilbert space of the edge mode upon which it acts. In particular, any nontrivial projective representation must have a Hilbert space that is at least two dimensional. This is easy to see by considering any one dimensional projective representation $\alpha : G \to U(1)$, then making use of the freedom to multiply each operator by a phase. Upon choosing the inverse phase $\alpha^{-1}(g)$ for each representative $\alpha(g)$, the whole representation becomes trivial. Hence for any nontrivial projective representation on an edge mode, there is an associated Hilbert space of dimension greater than two, which labels a degeneracy in the ground state associated to the open boundary.

In Sec. 3.3.2 we have seen that the different projective representations of a symmetry group G are labeled by the second cohomology group $H^2(G, U(1))$. Since the boundary modes are described by projective representations, we can similarly label each mode by some class $[\omega] \in H^2(G, U(1))$, which is a discrete, finite group in the cases we will consider. Furthermore, any symmetric perturbation to the Hamiltonian will not change the symmetry of the ground state, provided that it does not induce any phase transition. The factor class $[\omega] \in H^2(G, U(1))$ will also remain unchanged by such a transformation and hence forms a label that is invariant throughout a SP phase, and which specifies the boundary modes that characterize the phase[7], as depicted in Fig. 3.8. Hence, any 1D SP phase can be classified by the second cohomology class $H^2(G, U(1))$ of its symmetry group G. This statement has been made rigorous in the setting of matrix product states (MPS, for definition see appendix H) [71].

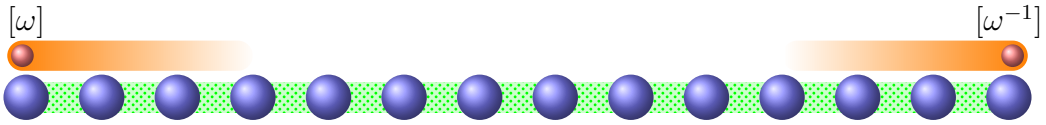

Figure 3.8: Classification of the edge modes of a SP chain in terms of their factor systems.

In this section we have seen the classification of 1D SP spin chains in terms of the second cohomology group $H^2(G, U(1))$ of their symmetry group G. In the next section we will give an interpretation of the edge modes in terms of conserved quantities of a SP chain with one boundary condition fixed by a coupling to a physical particle of the same species as one edge mode.

3.3.4 Understanding the Edge Modes in Terms of Conserved Quantities

Here we expound the relationship between the boundary modes and another projective representation of the symmetry group acting upon the whole spin chain, which is interpreted as a set of conserved quantities. To achieve this identification we introduce the idea of fixing one boundary condition by coupling to a physical particle which transforms projectively under the symmetry and combines with the boundary mode to form a linear representation of the symmetry. In this picture we will label the degenerate ground states due to one free edge mode via conserved quantities acting upon the whole chain, which form a projective representation due to the extra boundary particle. This characterization of the edge mode will turn out to be very useful in our investigations in Chapter 4.

We consider a spin chain in a ground state with nontrivial SP, protected by the on-site symmetry operators $\mathbf{U}_g = \bigotimes_i U_g^i$ which form a linear representation of the symmetry group G. In the above section we saw that such a phase is characterized by a degenerate ground space corresponding to a gapless boundary mode at each uncoupled end of the

[7] In the Appendix I we have given an alternate classification of 1D SP from [61], which more explicitly identifies the origin of the boundary modes we have used to characterize SP phases here.

chain. Furthermore these boundary modes transform projectively under the symmetry group G, and hence are labeled by a pair of factor classes $[\omega]$ and $[\omega^{-1}]$.

Earlier we argued that each boundary mode arose due to a degree of freedom which we interpreted as an open boundary condition. Keeping with this analogy we now consider fixing one boundary condition by symmetrically coupling an edge of the chain to a particle which transforms projectively under the on-site symmetry. In particular we will consider a symmetric coupling of the right boundary, corresponding to the mode labeled by $[\omega^{-1}]$, to a physical particle that transforms projectively under the on-site symmetry with factor class $[\omega]$. This physical boundary particle usually constitutes a different type of particle to those at the other sites of the chain, and its Hilbert space will be the same as the boundary mode of the chain since its projective representation has the same label $[\omega]$. The boundary freedom is effectively fixed since the product of the two projective representations coming from the emergent boundary mode and the physical edge particle combine to form a linear representation. The full degeneracy of the ground state due to the projective symmetry that was protecting the right edge state is now *lifted*, since we have *lost* the projective Hilbert space of the gapless boundary mode at that edge. For this chain, with one fixed boundary condition, there is still a reduced degeneracy in the ground state due to the left boundary that remains free.

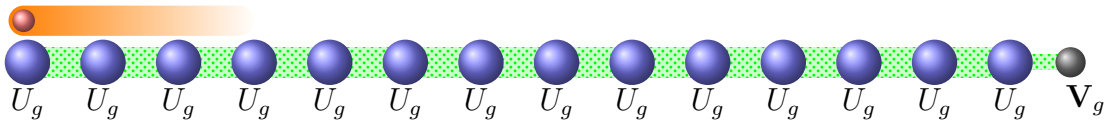

Figure 3.9: A global conserved quantity acting upon the ground state of a chain with a fixed boundary condition.

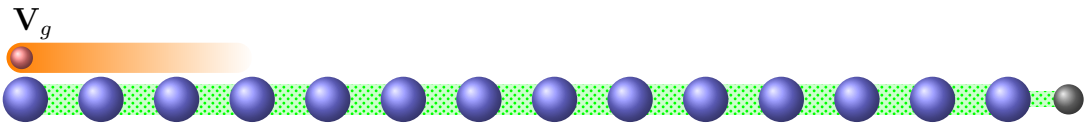

Figure 3.10: The equivalent projective representation acting on the remaining edge mode at the free boundary.

We now have conserved quantities $\mathbf{U}_g^{\mathrm{p}} = \bigotimes_i U_g^i \otimes V_g$ generated by a product of all the on-site symmetries U_g^i with the additional projective symmetry V_g that arises due to the physical particle used to fix the right boundary. We call such a product a conserved quantity because it commutes with the Hamiltonian, and hence its eigenvalues do not evolve with time under a unitary evolution. These conserved quantities form a projective representation of G, with a factor system ω coming from the projective representation of the symmetry acting on the right, physical boundary particle. This is a useful picture to have because this projective representation is labeled by $[\omega]$, the same as the left boundary mode.

Hence we can use the eigenvalues of the conserved quantities to label the degenerate ground states, and equivalently the state of the left boundary mode, using the correspondence shown in Figures 3.9 and 3.10. While this may seem somewhat artificial, since the 'projectiveness' of these quantities arise from the physical particle at the right boundary,

we will see in a later section that local manipulations at the left boundary, which can not influence the right boundary for sufficiently long chains, can cause nontrivial evolutions of the degenerate ground states by changing the eigenvalues of the conserved quantities. Hence a picture like Fig. 3.10 is justified.

Chapter Summary

In this chapter we have studied SP phases to identify the properties that make them ideal hardware for robust quantum computation. We developed a precise definition of SP in terms of the equivalence relation that defines a quantum phase in the presence of a symmetry. We went on to introduce the idea of symmetry-protected edge modes which arise in the degenerate ground states of a 1D SP phase and used them to label all 1D SP phases by their second cohomology group $H^2(G, U(1))$. We then related a single boundary mode to the symmetries of a SP chain with a fixed boundary condition. We will use these characterizations of 1D SP phases and their edge modes to develop basic symmetry-protected holonomic evolutions that will lead to full the realisation of SP adiabatic quantum transistors in the next chapter.

Chapter 4

Symmetry-Protected Adiabatic Quantum Transistor Model

In this chapter we report on the results of our investigations into quantum computation with general symmetry-protected (SP) phases in one dimension. These results are based upon our general model for encoding information into the degenerate ground space of any SP spin chain. Logical evolutions in this setting will be generated by traversing a symmetric phase transition. The key result of this thesis is our characterisation of the possible gates that persist throughout any 1D SP phase in terms of the symmetry and its physical representations. For a more concise presentation of the material in this chapter see [72].

The chapter will be laid out as follows; we will open with an explanation of the general ground state encoding used for any SP phase and its robustness properties throughout a phase. We will then move on to present the general process that will implement an elementary logic gate upon the encoded information by adiabatically decoupling a single spin under a local field. Building upon this basic gate we will explain how one can execute a nontrivial gate by adiabatically traversing a symmetric phase transition of the spin chain from a SP to trivial phase. We dub this model symmetry-protected adiabatic quantum transistor (SPAQT) computing as the spin chains closely resemble classical transistors where computation is driven by application of a uniform field that also propagates information spatially across the device. We will go on to describe the generic requirements for achieving a universal gate set with SP chains with reference to the specific example in appendix B. Finally we will discuss the robustness of our proposed scheme to a large class of realistic errors that may occur in the practical operation of SPAQT computing.

4.1 SP ground state encoding

In this section we will lay out the physical setting for our model and extend the understanding of the SP edge mode symmetry in terms of conserved quantities explained in Sec. 3.3.4 to yield an encoding of information into any SP edge that is robust throughout the SP phase.

The physical systems considered throughout this chapter are chains of spin degrees of freedom. We will focus on the ground subspace of the spin chains whose interactions are

described by a spatially local Hamiltonian that we will write as

$$H_N = \sum_{s=0}^{N-1} H_s \tag{4.1.1}$$

where the H_s terms act on a constant number of spins around site s and the energy scale is normalized s.t. $\|H_s\| \leq 1$. We only consider models where the energy gap Δ between the smallest set of quasi-degenerate eigenvalues (meaning their energy spacing shrinks exponentially as the size of the chain grows) and the next lowest eigenvalue (the first excitation) is uniformly lower bounded by a constant as N increases. This allows us to employ the framework of gapped phases described in Chapter 3. We focus on models where the Hamiltonians terms all commute with a representation of a symmetry group G. The specific representation we will consider is a tensor product of on site unitary representations U_g, hence the symmetry condition can be written as

$$\left[H_s, U_g^{\otimes N}\right], \ \forall s, \ g. \tag{4.1.2}$$

We know from our discussion in the previous chapter that such a model supports nontrivial SP phases if its second cohomology is nontrivial $H^2(G, U(1)) \neq 1$ and it is these nontrivial phases that we focus on. We will use the result from the previous chapter that any ground state in a nontrivial SP phase supports a degenerate edge mode and furthermore the symmetry acts upon this edge degeneracy via an irreducible projective representation, see eq.3.3.4.

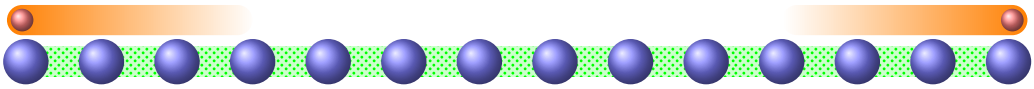

Figure 4.1: A depiction of the ground state of the symmetric Hamiltonian 4.1.1.

The essential property of SP phases that enable us to encode information into the ground space is precisely this equivalence of the global symmetry with the projective representations upon the edges. We would like to draw the analogy to the global, physical operators of a stabilizer code [73, 74] that will commute with all the stabilizer terms and perform a logical transformation on the code space.

The edges will always support either qubits or larger particles since any nontrivial projective rep must be of dimension two or larger (even for abelian groups) as all one dimensional reps are by definition trivial (in the projective setting) since they can be rephased to the identity rep.

To isolate the left edge particle for our encoding we will follow the procedure described in section 3.3.4 and terminate the right edge by a symmetric coupling h_{edge} to a physical fractional particle, transforming under a representation V_g^{phys} in the same cohomology class as the left edge mode. This coupling will be spatially local but may have to act on a number of sites up to the G-injectivity radius [75] of a MPS for the ground state such that it is possible to achieve the desired coupling on the edge mode by acting only on the physical level. The key importance of this boundary fixing is that it changes the way the

global symmetry acts as depicted in Figures 3.9 and 3.10

$$U_g^{\otimes N} \to U_g^{\otimes N} \otimes V_g^{\text{phys}}. \tag{4.1.3}$$

Hence the symmetry forms a projective representation with the same cohomology class as the remaining left edge mode. Note this necessarily makes the global symmetry projective as it allows us to directly identify the global symmetry action on the ground space with a single projective irrep V_g on the uncoupled edge mode and so their multiplication rules must match. This also has the effect of fixing the ground space degeneracy to be exact as the ground space now exactly forms a projective irrep of the symmetry group. In contrast, when there are two edge modes there can be an energy splitting that shrinks exponentially with the system size due to a weak coupling between the edges that allows the two projective irreps to couple into a direct sum of unitary irreps with slightly different energies. Avoiding this decoupling is beneficial as it allows us to avoid phase errors within the ground space due to phases of different quasi degenerate ground energy levels accumulating at different speeds under the Hamiltonian evolution.

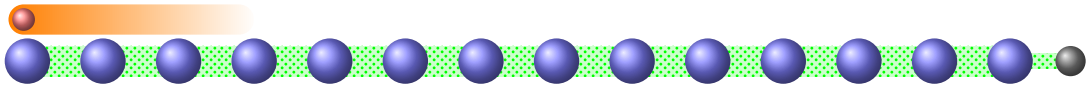

Figure 4.2: A ground state of a symmetric Hamiltonian with one boundary condition fixed by coupling to a fractional particle 4.2.1.

This identification of the global symmetries with the projective symmetry on the edge persists throughout the whole SP phase.

$$U_g^{\otimes N} \otimes V_g^{\text{phys}}\big|_{E_0} \sim V_g \tag{4.1.4}$$

We find it natural to think of these projective symmetries as 'logical' operators on the information encoded within the ground space. Since the identification in eq.4.1.4 persists throughout a SP phase any symmetry respecting, constant length, adiabatic evolution of the Hamiltonian will leave the eigenvalues of the 'logical' projective symmetry operators invariant since the global symmetries are necessarily invariant. In other words this says that it does not matter in which order we apply a projective symmetry operator and a symmetric evolution, i.e. they commute. Combining these commutation relations with the irreducibility of the projective representation within the ground space implies, by Schur's Lemma, that such an evolution must be the identity. Hence we say that the encoded information is protected by the symmetry within the SP phase.

In fact we can strengthen this statement to hold for any symmetry respecting unitary by the same argument, since it will have an action upon the ground space that commutes with the projective irrep and hence must act as the identity on the encoded information although in this case the more general symmetric unitary can cause leakage errors out of the ground space as it might not correspond to an adiabatic evolution.

We claim that edge modes without the property of irreducibility are not protected within a SP phase since for projective representations that are not irreducible we cannot make the argument that symmetric, adiabatic paths within the phase will only cause an

identity transformation on the information encoded in the ground space. In this setting Schur's lemma only ensures that the matrices generated by adiabatic paths have a tensor product form but may act arbitrarily on one of the tensor factors. Hence in a strong sense an encoding into an edge mode that supports a reducible projective representation is not a robust property of a SP phase. However, if we instead choose to encode our information only into the correct tensor subsystem of such a reducible edge mode then any adiabatic transformation within the SP phase will only operate on the other subsystem of the ground space, which is treated as a gauge and will not affect the encoded information. This is a specific instance of a decoherence free subsystem [76, 77] (whereas our normal encoding is only a decoherence free subspace) and may be useful for a more general computation scheme similar to ours.

In this section we described the physical systems we will consider for our model of computation throughout the rest of the chapter. We went on to describe how the action of the symmetry on the ground space described in section 3.3.4 can be used to encode information that is robust throughout a SP phase.

There are some subtleties not considered in the above analysis which will become important in the next section. Particularly if the evolution changes the way the symmetry acts then the arguments given above do not necessarily hold and it is precisely this fact that will allow us to perform nontrivial unitary gates upon the ground space using adiabatic evolutions. This will be described in detail in the next section.

4.2 The elementary gate

In this section we will explain how one can achieve a nontrivial unitary transformation upon information encoded into the ground state of any SP phase by adiabatically decoupling a single spin under a local symmetric field. This generalises the holonomic quantum computation scheme of Renes et al. [18] We find that the gate achieved depends only upon the symmetry of the SP phase and hence is a universal property of the phase. The characterisation of the gates inherent to any SP phase in terms of symmetry properties is the main technical result of this thesis and forms the basis of our model for SPAQT computing that is discussed throughout the chapter.

For simplicity in this section we will only consider models of SP spin chains with nearest neighbor interactions (this will hold for all 1D models considered in the previous section eq.4.1.1 after some renormalization) we will use the notation $H_{i,i+1}$ to indicate a Hamiltonian term acting on sites i and $i+1$. We explicitly single out the interaction term coupling the spins at one edge to a fractional particle h_{edge} to fix the relevant boundary condition.

$$H_N := \sum_{i=0}^{N-2} H_{i,i+1} + h_{\text{edge}} \tag{4.2.1}$$

We assume that the Hamiltonian commutes with a symmetry

$$\left[H_{i,i+1}, U_g^{\otimes N} \otimes V_g\right], \ \forall i, g \quad \& \quad \left[h_{\text{edge}}, U_g^{\otimes N} \otimes V_g\right], \ \forall g \tag{4.2.2}$$

Hence we are in the situation described in the previous section with information encoded into a single edge of a degenerate SP ground space as depicted in Figures 3.9 and 3.10.

The elementary gate is performed by adiabatically decoupling a single physical site from the chain while simultaneously applying a symmetric field to it, i.e. $[F_0, U_g]\,,\ \forall g$. The adiabatic evolutions is given by

$$H_N(t) = f(t)F_0 + g(t)H_{0,1} + \sum_{i=1}^{N-2} H_{i,i+1} + h_{\text{edge}} \tag{4.2.3}$$

with $f(0) = g(T) = 0$ and $g(0) = f(T) = 1$ and $[F_0, U_g] = 0,\ \forall g$. We require $T = \Omega\left(1/\Delta^3\right)$ [78] to ensure adiabaticity, where Δ is the minimum spectral gap of $H_N(t)$ as t is varied.

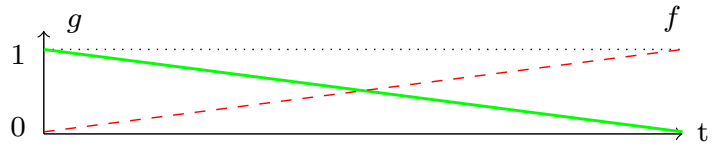

Figure 4.3: The coupling strengths f and g from eq. 4.2.3 as a function of t.

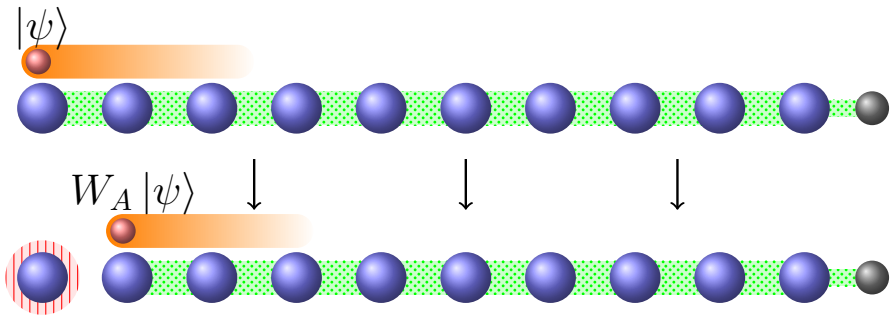

Figure 4.4: A depiction of the adiabatic decoupling evolution described by eq. 4.2.3.

Here we assume the local symmetric field F_0 has a unique ground state, otherwise the uniform field Hamiltonian

$$H_F := \sum_s F_s \tag{4.2.4}$$

that will become important later, would have an extensive ground state degeneracy. Since F_0 commutes with the symmetry U_g its ground state must be an eigenvalue of the symmetry. We label this ground state of F_0 by $|\chi\rangle$ and write the symmetry condition as $U_g\,|\chi\rangle = \chi(g)\,|\chi\rangle$ where $\chi : G \to U(1)$ is a one dimensional representation of the group and hence also a character.

We have constructed the adiabatic evolution eq.4.2.3 to commute with the symmetry and to leave the ground space invariant. Hence any ground subspace labeled by symmetry eigenvalues will retain the same labels under the evolution. We will use this property to enact a logical transformation on the SP edge mode by decoupling a spin into a symmetric state $|\chi\rangle$ in tensor product with a SP ground state on the remaining chain. This moves the encoded information spatially from the edge of the full chain to the edge of a shorter

chain while simultaneously multiplying the projective symmetry action on the new edge mode by a phase $\chi(g)$.

More precisely, if we start with an initial state

$$|\psi_0\rangle := |H_N = 0, V_g = \theta(g)\rangle \tag{4.2.5}$$

labeled by its eigenvectors under the operators H_N and $U_g^{\otimes N} \otimes V_g^{\text{phys}}$ (which acts within the ground space as V_g) respectively. Under the adiabatic evolution U_A generated by eq.4.2.3 $|\psi_0\rangle$ evolves to

$$U_A |\psi_0\rangle := |\psi_T\rangle = |\chi\rangle \otimes \left|H_{N-1} = 0, V_g' = \chi(g)^{-1}\theta(g)\right\rangle. \tag{4.2.6}$$

since the symmetry commutes with the evolution and acts upon the final state as

$$U_g^{\otimes N} \otimes V_g^{\text{phys}} |\psi_T\rangle = \chi(g) |\chi\rangle \otimes \left(U_g^{\otimes N-1} \otimes V_g^{\text{phys}}\right) \left|H_{N-1} = 0, V_g' = \chi(g)^{-1}\theta(g)\right\rangle \tag{4.2.7}$$

noting that the symmetry $U_g^{\otimes N-1} \otimes V_g^{\text{phys}}$ acts within the ground space of the $N-1$ site chain via some V_g'. Hence the symmetry action upon the edge mode of the decoupled chain at the end of the evolution is given by $V_g' = \chi^{-1}(g)V_g$.

In the Heisenberg picture the projective symmetry operators on the ground space evolve under the adiabatic decoupling eq.4.2.3 in the following way $V_g \mapsto \chi(g)V_g$. Note this evolution holds for all group elements and for all initial ground states. To state this more precisely we will write U_A for the perfect adiabatic evolution generated by eq.4.2.3 and $W_A := U_A\big|_{E_0}$ for the restriction of this adiabatic evolution to the ground space. Then we can write the evolution of the symmetry operators precisely as

$$W_A V_g W_A^\dagger = \chi(g) V_g, \ \forall g \tag{4.2.8}$$

or

$$W_A V_g = \chi(g) V_g W_A, \ \forall g. \tag{4.2.9}$$

In the case that χ is the trivial character $\chi = 1$ eq.4.2.9 together with Schur's lemma imply that $W_A = \mathbb{I}$. So we must look to nontrivial characters to generate nontrivial evolutions within the encoded subspace and in that case we cannot simply invoke Schur's lemma to calculate the evolution. Note that in all cases Schur's lemma will imply that $W_A W_A^\dagger = c\mathbb{I}$.

To gain a better handle on the general situation we can rewrite the set of conditions in eq.4.2.9 succinctly as the equivalent condition that the matrix W_A is a fixed point of the following channel

$$\Gamma_\chi(M) := \frac{1}{|G|} \sum_{g \in G} \chi(g) V_g M V_g^\dagger \tag{4.2.10}$$

To show the equivalence with the set of conditions 4.2.9 we note that if a matrix W_A

satisfies these conditions then

$$\begin{aligned}\Gamma_\chi(W_A) &= \frac{1}{|G|}\sum_g \chi(g) V_g W_A V_g^\dagger \\ &= \frac{1}{|G|}\sum_g \chi(g)\chi^{-1}(g) W_A V_g V_g^\dagger \\ &= \frac{1}{|G|}\sum_g W_A \\ &= W_A.\end{aligned}$$

Conversely, if $\Gamma_\chi(W_A) = W_A$, then $\forall g \in G$

$$\begin{aligned}V_g W_A &= V_g \frac{1}{|G|}\sum_h \chi(h) V_h W_A V_h^\dagger \\ &= \frac{1}{|G|}\sum_h \chi(h) V_{gh} W_A V_h^\dagger \\ &= \frac{1}{|G|}\sum_{h'} \chi(g^{-1}h') V_{h'} W_A V_{g^{-1}h'}^\dagger \\ &= \frac{1}{|G|}\sum_{h'} \chi(g^{-1})\chi(h') V_{h'} W_A \left(V_{g^{-1}} V_{h'}\right)^\dagger \\ &= \chi(g^{-1})\frac{1}{|G|}\sum_{h'} \chi(h') V_{h'} W_A V_{h'}^\dagger V_{g^{-1}}^\dagger \\ &= \chi^{-1}(g)\frac{1}{|G|}\sum_{h'} \chi(h') V_{h'} W_A V_{h'}^\dagger \; V_g \\ &= \chi^{-1}(g) W_A V_g\end{aligned}$$

We make note of the important fact that the fixed point of the channel Γ_χ must be unique up to a phase. To prove this we consider any two fixed point solutions $W_A,\ W_B$ and combine them to form the matrix $W_A W_B^\dagger$. It is easy to see that $W_A W_B^\dagger$ must be a fix point of the channel Γ_1 since

$$\begin{aligned}\Gamma_1\left(W_A W_B^\dagger\right) &= \frac{1}{|G|}\sum_g V_g W_A W_B^\dagger V_g^\dagger \\ &= \frac{1}{|G|}\sum_g \chi^{-1}(g) W_A V_g \left(V_g W_B\right)^\dagger \\ &= \frac{1}{|G|}\sum_g \chi^{-1}(g) W_A V_g \left(\chi^{-1}(g) W_B V_g\right)^\dagger \\ &= \frac{1}{|G|}\sum_g |\chi^{-1}(g)|^2 W_A V_g V_g^\dagger W_B^\dagger \\ &= W_A W_B^\dagger.\end{aligned}$$

Hence we must have $W_A W_B^\dagger = c\mathbb{I}$ by Schur's lemma and the irreducibility of the projective representation V_g.

A further basic property of the channel is its behavior under composition $\Gamma_\chi \circ \Gamma_\varphi = \delta_{\varphi,\chi}\Gamma_\chi$. This is easily proved

$$\begin{aligned}
\Gamma_\chi \circ \Gamma_\varphi\ (M) &= \Gamma_\varphi\left(\Gamma_\chi(M)\right) \\
&= \frac{1}{|G|}^2 \sum_g \varphi(g) V_g \left(\sum_h \chi(h) V_h M V_h^\dagger\right) V_g^\dagger \\
&= \frac{1}{|G|}^2 \sum_{g,h} \varphi(g)\chi(h) V_{gh} M V_{gh}^\dagger \\
&= \frac{1}{|G|}^2 \sum_{a,b} \varphi(b)\chi(b^{-1}a) V_a M V_a^\dagger \\
&= \frac{1}{|G|}^2 \sum_{a,b} \varphi(b)\chi(b^{-1})\chi(a) V_a M V_a^\dagger \\
&= \frac{1}{|G|}^2 \sum_b \varphi(b)\chi^*(b) \sum_a \chi(a) V_a M V_a^\dagger \\
&= \delta_{\varphi,\chi}\Gamma_\chi(M)
\end{aligned} \tag{4.2.11}$$

where we have made use of the orthonormality of distinct characters. This shows that the channels Γ_χ form a set of orthogonal projections on the space of matrices upon which it acts.

Finding the fixed points $\Gamma_\chi\left(W_A\right) = W_A$ of the channel is essentially the same problem as finding the symmetric subspace of the representation $\chi(g)V_g \otimes V_g^*$, where $*$ denotes complex conjugation. We can see that the maps $\Pi_\chi := \frac{1}{|G|}\sum_g \chi(g) V_g \otimes V_g^*$ form a set of orthogonal projections by the composition property of the channel that was proved in eq. 4.2.11. The representation $\chi(g)V_g \otimes V_g^*$ acts upon the two virtual degrees of freedom associated to any single site in a renormalisation fixed point, symmetric matrix product state (MPS, for an explanation see appendix H) representation of the ground state.

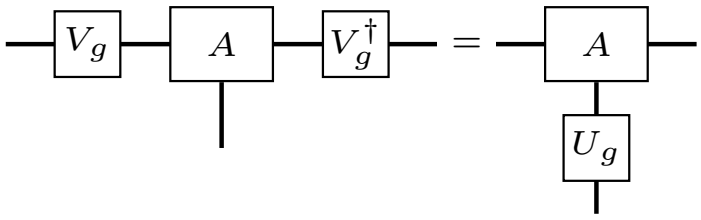

Figure 4.5: The symmetry condition satisfied by the renormalisation fixed point MPS tensor of a SP phase.

We can equivalently understand the virtual entangled states in the symmetric subspace of this representation in terms of a Clebsch-Gordon (CG) matrix coupling the two projective representations to a single representation on the physical level. We note that

the CG matrix is essentially the same object as the fixed point MPS tensor but without any projection onto a subspace of the physical level. In particular, the fixed point virtual entangled state/ fixed point adiabatic unitary $A[\chi]$ can be constructed by projecting the physical output of the Clebsch-Gordon matrix or fixed point MPS tensor onto the relevant symmetric ground state $|\chi\rangle$ of the local field F_0 as depicted in figure 4.6.

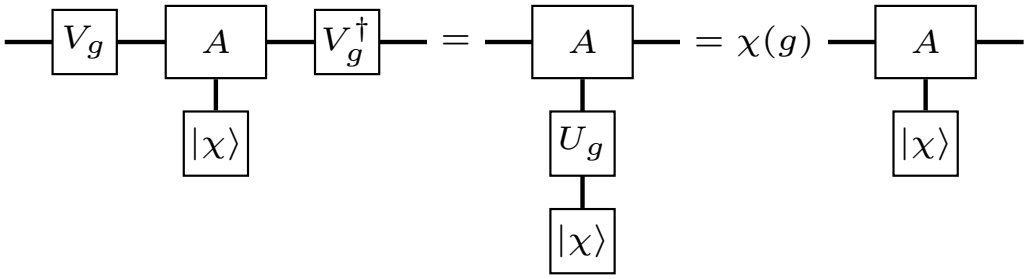

Figure 4.6: A fixed point solution $W_A = A[\chi]$ for the channel Γ_χ describing the adiabatic evolution within the ground space.

We have presented a method to turn a fixed point MPS tensor or Clebsch-Gordon matrix into a unique solution of eq.4.2.9 but we would like to go further and understand the solutions for different one dimensional reps χ. We make the definition that $A[\psi]$ is the matrix formed by projecting the physical leg of the three index MPS tensor A onto the state $|\psi\rangle$, see figure 4.6. In the case that the fixed point MPS is G-injective [75] we have that $\forall |\psi\rangle, |\phi\rangle, \exists |\lambda\rangle$, s.t. $A[\psi] A[\phi] = A[\lambda]$. For any pair of one dimensional symmetric subspaces $|\chi_1\rangle, |\chi_2\rangle$ the product of their solutions $A[\chi_1] A[\chi_2]$ is a fixed point of the channel $\Gamma_{\chi_1\chi_2}$ (this is true in more generality, see eq.4.2.12) and so must equal $A[\chi_1\chi_2]$ up to a phase since that is the unique fixed point in this case.

Hence in the G-injective fixed point MPS case there is an isomorphism between the group of one dimensional representations of G (equivalently its abelianization) and the evolutions $W_A = A[\chi]$. We can see by their multiplication rule that they form a projective representation of the abelianization of G. For an abelian group with a maximally non-commuting factor system we can explicitly construct the isomorphism between the group or equivalently its character group (note the group is trivially its own abelianization) and the representation formed by the W_A maps in terms of the original projective representation of the symmetry V_g [36, 37, 79].

We can generalise the argument of the previous paragraph by removing any reference to the particular form of the solution given by a MPS tensor. Since the fixed point of the channel Γ_χ in eq.4.2.10 is unique up to a phase, we label the solution for some choice of

phase by W_χ. We will now prove that $W_\chi W_\varphi$ is a solution to the channel $\Gamma_{\chi\varphi}$

$$\begin{aligned}\Gamma_{\chi\varphi}(W_\chi W_\varphi) &= \frac{1}{|G|}\sum_g \chi(g)\varphi(g)V_g W_\chi W_\varphi V_g^\dagger \\ &= \frac{1}{|G|}\sum_g \chi(g)\varphi(g)\chi^{-1}(g)W_\chi V_g W_\varphi V_g^\dagger \\ &= \frac{1}{|G|}\sum_g \varphi(g)\varphi(g)^{-1}(g)W_\chi W_\varphi V_g V_g^\dagger \\ &= W_\chi W_\varphi.\end{aligned} \tag{4.2.12}$$

Combining this with the uniqueness of the solution $W_{\chi\varphi}$ up to a phase we see that $W_\chi W_\varphi = \alpha(\chi,\varphi)W_{\chi\varphi}$ for some phase function $\alpha : G' \times G' \to U(1)$ where $G' := G/[G,G]$ is the abelianisation of G. The abelianisation appears since it is isomorphic to the group[1] of one dimensional representations of G. It is clear that the abelianisation is a natural object since $\Gamma_{\chi\varphi} = \Gamma_{\varphi\chi}$ and hence the multiplication of the solutions W_χ is abelian up to a phase. Hence we see that the possible solutions form a projective representation of G' and furthermore this representation is faithful since the identity is a fixed point of Γ_χ if and only if $\chi = 1$ by the orthonormality of distinct characters.

In this section we have formulated the elementary adiabatic path that can generate a nontrivial evolution on the encoded information. This involves the decoupling of a single spin under a local field, the reason for this choice was to change the action of the symmetry on the edge mode to allow the nontrivial transformation. The physical relevance of this decoupling will become clear in the next section where we consider uniform decoupling of all spins under a local field which more clearly corresponds to a phase transition. Hence we interpret the elementary adiabatic path as physically moving the boundary of a phase transition one site across the chain.

We went on to precisely describe the set of conditions the logical evolution generated by the adiabatic path must satisfy in terms of a fixed point of a particular channel Γ_χ. We then constructed explicit solutions for these conditions out of fixed point MPS tensors or CG matrices and argued that any solution is unique up to a phase. We further characterised the properties of the set of solutions for adiabatic paths with different possible symmetric fields and found that they form a faithful, projective representation of the abelianisation G' of the symmetry group G. Note that our characterisation implies that the matrices generated by all different possible logical gates of this type for a particular model must commute up to a phase. Hence these cannot form a universal gate set for quantum computation. We will address the need to change the paradigm slightly to achieve universality in section 4.4 where we will use symmetric adiabatic evolutions with respect to several different subgroups of some larger group. By not explicitly enforcing the full symmetry at every step of the evolution we will find we can side step the restricted no-go result for universality of SP gates generated by adiabatic decoupling that was described in this section.

[1]It is easy to see that the one dimensional representations of G that we are calling χ, φ etc. form an abelian group under point wise multiplication, i.e. $\chi \cdot \varphi\ (g) := \chi(g)\varphi(g)$.

4.3 Realising a transistor

In this section we will build upon the elementary gates described in the previous section to realise computation by the adiabatic application of a uniform field to a SP spin chain. This functions analogously to a classical transistor as computation is driven by a application of a uniform field which simultaneously propagates the encoded information across the chain. Our model is a generalisation of the adiabatic quantum transistor (AQT) model of Bacon et al. [41] to arbitrary SP phases in 1D which we dub the symmetry protected adiabatic quantum transistor (SPAQT) model. This yields physical insight into the computation as it can be viewed as an adiabatic traversal of a symmetric phase transition that functions throughout a SP phase. It also deepens the analogy to the transistor, with SP spin chains playing the counterpart to doped semiconductors as material that naturally facilitates robust computation. Note that the presence of a phase transition only implies the energy gap above the ground state closes in the thermodynamic limit, for any finite system size this gap generically remains open and hence it is possible to adiabatically traverse a path crossing the phase transition. The time taken to cross the phase transition adiabatically will generally scale with the system size in a way that depends on the scaling of the energy gap with the system size. We will provide some discussion of the speed at which we can pass through this phase transition adiabatically and describe different ways of applying the global field in a non-uniform manner that may increase the possible speed.

Following the ideas of Bacon et al. [41] we notice that we can apply a local field as in eq. 4.2.3 across the whole chain simultaneously without changing the symmetry arguments used to calculate the logic gates, assuming we are in the adiabatic limit. The adiabatic Hamiltonian describing this process is

$$H_N(t) = f(t)\left(\sum_{i=0}^{N-1} F_i\right) + g(t)\left(\sum_{i=0}^{N-2} H_{i,i+1} + h_{\text{edge}}\right) \tag{4.3.1}$$

where again $f(0) = g(T) = 0$ and $g(0) = f(T) = 1$ and $[F_i, U_g] = 0,\ \forall i, g$. Where $T = \Omega\left(1/\Delta^3\right)$ [78] to ensure adiabaticity, with Δ the minimum spectral gap of $H_N(t)$. This presents a more illuminating physical picture as it is now clear that the Hamiltonian is driving the system through a symmetric phase transition from a SP phase to a trivial symmetric phase. Furthermore, the unitary evolution on the ground space will be the same W_A^N no matter what point of the SP phase we start in and hence is truly a property of the whole phase.

In our arguments thus far we have assumed adiabaticity but in light of the phase transition that occurs during the Hamiltonian deformation eq.4.3.1 this is a rather large assumption to make. Due to the presence of a phase transition we expect T will have to be taken much larger than that for the single spin case as the minimum gap Δ should be approaching zero as the system size grows (although remaining nonzero for any finite size). The exact rate at which the gap closes will determine how efficient it is to simulate circuits on single or multiple qudits (dependent upon the coupling Hamiltonian) and the required time cannot increase by more than a polynomial factor for the scheme to be viable. So we will require that the gap can be bounded from below by the inverse of a polynomial in the system size $\Delta = \Omega\left(1/\operatorname{poly}(N)\right)$.

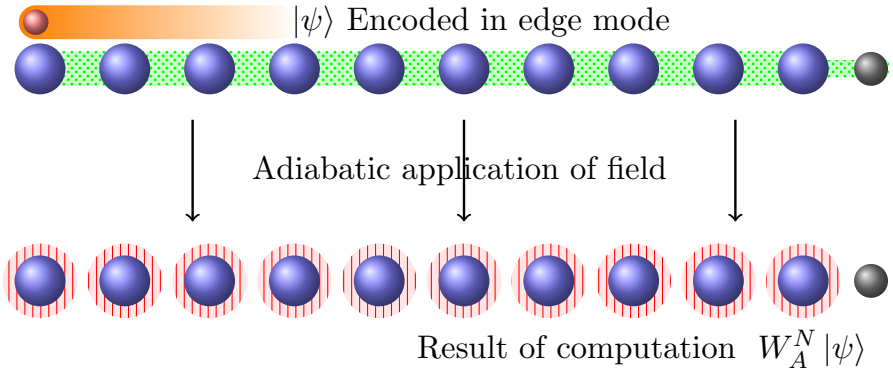

Figure 4.7: The adiabatic evolution described by eq. 4.3.1 inducing a gate W_A^N on an encoded state $|\psi\rangle$ as it propagates across the chain.

There is little hope of proving this efficiency in general for all SP phases as there are relatively few techniques for bounding the spectral gap of arbitrary Hamiltonians. However as noted by Bacon et al. a proof for a universal gate set would be an important step. Here we will not attempt to rigorously address this question as we are not focusing on any one particular model gate set. Rather we will provide the following intuitive argument that applies to all SP phases. It is easy to see that even under completely symmetric evolution any SP state can be mapped to a product state by a circuit of depth $O(N)$ [80] by taking the standard circuit to construct the MPS representation of the state and applying its inverse. Furthermore, this should be optimal as we expect one edge must communicate with the other edge to complete the disentangling map to a product state. It is conjectured by Wen et al. that equivalence classes of states making up a SP phase are connected by constant depth local circuits and we would expect a generic $O\left(\text{poly}(N)\right)$ or even $O(N)$ depth circuit to map between phases. This is supported by the findings in both [41, 80].

To provide an argumentum ad. absurdum for why we expect different phases to be connected by $O\left(\text{poly}(N)\right)$ evolutions, we point out that the equivalence class of all states connected to the product state under $O\left(\text{poly}(N)\right)$ depth circuits already contains all witness states (i.e. solutions) for problems in a complexity class called QCMA [81] (a certain quantum generalisation of NP). While this does not show that the naive interpolation used in our construction has to have a gap closing inverse polynomially slowly, it does suggest that a generic path crossing the phase transition should obey the gap condition.

Another interesting and relevant aspect of the efficiency issue is addressed in the work of Rams et al. [82, 83] concerning the adiabatic traversal of symmetry-breaking phase transitions in one dimensional quantum spin models. They compare the performance of sweeping a spatially varying field profile across a spin chain to the uniform application of the field to realise a phase transition. Their main result is that by sweeping the field sufficiently slowly, a polynomial speed up in the time required for adiabatic traversal is possible. In addition to this, the slow sweeping can lead to an exponential suppression in the density of thermal errors/excitations. We expect that similar results should apply at least in the case of an abelian symmetry group with maximally noncommutative factor system [79] since there exist locality preserving mappings between the symmetry broken

and SP phases in this case. These mappings should allow one to map the symmetry breaking phase transition to a symmetry protected phase transition while preserving the locality of all the relevant operators, and hence to carry the desirable physical properties of the symmetry breaking transition over to the SPAQT. This approach may also come more naturally when trying to implement a full circuit in practice as we may envision sweeping the field sufficiently slowly across a network of SPAQTs linked in an appropriate way to implement a desired circuit.

We also note that the adiabatic sweep can be run in reverse, which could be advantageous if—as the authors suggest in [82,83]—the thermal errors generated by the sweeping field propagate ahead of the phase transition wave front. In this situation the excitations will be swept away from the edge mode on the SP portion of the chain and into the trivial phase.

A small subtlety we have overlooked so far is the possibility of some small adiabatic deformation away from the final state (while this seems easy to suppress in practice by simply applying a stronger uniform local field) this should not cause any change in the intended logic gate so long as this deformation is symmetric. The only effect this may have is to entangle the encoded information back into some spins near the edge which may necessitate some decoding that disentangles the logical information in the fractional edge spin from the rest of the chain.

In this section we extended the model of the elementary gate introduced in section 4.2 to define SPAQTs where computation is driven by the uniform application of a field to a SP spin chain. This gave a strong physical intuition for the process of computation being caused by a symmetric phase transition. Since our arguments only referenced symmetry properties of whole SP phases the gates performed function perfectly throughout a SP phase. This deepens the analogy to the classical transistor with 1D SP phases playing the role of doped semiconductor materials in naturally generating robust computations under application of a uniform field.

We went on to discuss the issue of how efficiently we could traverse the phase transition and gave non rigorous arguments as to why we expect this to be efficient for all physically reasonable 1D SP models. We then discussed the possibility of using the approach of Rams et al. that is to sweep the field across the chain rather than apply it uniformly and we gave some reasoning as to why this may be advantageous. In the next section we will discuss requirements for generating universal sets of gates using the SPAQT model. In the final section 4.5 we will focus more closely on issues of how robust our model is to a variety of errors occurring due to nonadiabaticity and more generally due to coupling to the environment.

4.4 Universal Gate Set

In this section we will discuss the possibility of generating a universal set of gates in the SPAQT framework. This is done in a very different way to the original model of Bacon et al. [41] and MBQC models [36, 37], but is somewhat similar to the scheme of Renes et

al. [18]. We use inherently protected gates of several different SP phases and then combine chains of differing symmetry to achieve a universal set of gates. We can join these chains in a consistent way by embedding all the protecting symmetries into some larger symmetry group which also protects a SP phase. We will briefly explain an important example of a universal gate set using only 2-body interactions of spin-1 particles. This model was proved to fall within the framework of symmetry protected gates in our previous work, the full details are given in appendix B.

In sections 4.2 and 4.3 we have shown how a SP phase leads to a SPAQT that can perform a certain logical transformation by utilizing arguments about symmetry properties of the whole phase. This result is in some sense unsurprising as similar results have been achieved for MBQC within SP phases [36,37], where it is known that the identity gate can be performed perfectly throughout a phase—up to some measurement result dependent correction operators. To implement nontrivial gates in the MBQC model and similarly in the Bacon et al. AQT model, measurements or fields that do not respect the symmetry of the phase are used. We do not take this route in our approach as it is expected that these unprotected operations do not function uniformly well throughout a phase but rather only in a small vicinity around ground states of exact parent Hamiltonians such as the cluster state.

Our approach to achieving a universal gate set is to employ several different types of SP matter with different inherent symmetries that will each naturally yield a distinct, protected gate, together making up the universal set. To do this in a consistent way we propose that all symmetries used be embedded within a single larger group that also protects a SP phase. This implies that all individual SPAQT elements can be connected by chains of highly symmetric matter that allow encoded information to propagate between the transistor elements that are each protected by different symmetries within the larger group. We note the generic situation pointed out by Else et al. [79] that discrete groups protecting SP phases can be embedded in a continuum of different ways inside some larger Lie group. This can be attributed to the arbitrary choice in defining the axes of a discrete symmetry group within the Lie group of unitary transformations on physical spins. Hence we expect our approach should generically work for any choice of SPAQTs that lead to a universal gate set.

Here we will briefly explain the example given in appendix B (based upon the model of [18]) where a universal gate set of SPAQTs is generated using 2-body interactions of spin-1 particles protected by a discrete set of symmetries embedded within $SO(3)\times SO(3)$.

The parent Hamiltonian of the model is the spin-1 antiferromagnetic Haldane chain with a boundary coupling to a spin-$\frac{1}{2}$ particle

$$H_n^{Term} = J\sum_{i=1}^{n-1} \vec{S}_i \cdot \vec{S}_{i+1} + J\vec{S}_n \cdot \vec{s}_{n+1} \tag{4.4.1}$$

The Haldane chain supports spin-$\frac{1}{2}$ edge modes and hence the Hamiltonian has an exact 2-fold degenerate ground space corresponding to a single edge mode, as the other boundary condition is fixed by the coupling to a physical spin-$\frac{1}{2}$. This model has full $SO(3)$ symmetry which is equivalent to π-rotation symmetry about all axes $\hat{m}$. The symme-

try acts on the physical system via the following representation that commutes with the Hamiltonian 4.4.1

$$\Sigma_n^{\hat{m}} = \left(\bigotimes_{j=1}^{n} \exp(i\pi S_j^{\hat{m}}) \right) \otimes \exp(i\frac{\pi}{2}\sigma^{\hat{m}}) = \bigotimes_{j=1}^{n} \left(I - 2(S_j^{\hat{m}})^2 \right) \otimes i\sigma^{\hat{m}} \tag{4.4.2}$$

To implement a nontrivial gate via adiabatic decoupling under a local field we must loosen the symmetry requirement such that a nonzero symmetric field is possible. This is done by picking a set of preferred axes $\hat{x}, \hat{z}$ and only requiring the π-rotation symmetry about these axes. It is known [44] that the resulting $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry protects the same SP phase with spin-$\frac{1}{2}$ edge modes as the full $SO(3)$ rotation symmetry.

The adiabatic gate Hamiltonian—depicted in figure 4.8—is given by

$$H_n(t) = f(t)J(S_1^{\hat{z}})^2 + g(t)J\vec{S}_1 \cdot \vec{S}_2 + H_{n-1}^{Term} \tag{4.4.3}$$

where $f(0) = g(T) = 0$, $f(T) = g(0) = 1$. This results in the decoupling of a spin into the $\left|S^{\hat{z}} = 0\right\rangle$ state. By examining the conserved quantities of the Hamiltonian—as explained in appendix A—we see that the resulting logical transformation is a $\sigma^{\hat{z}}$ on the encoded qubit.

Since the choice of the $\hat{z}$ axis was arbitrary we could as well have picked the symmetry along any axis $\hat{m}$ to achieve a logical $\sigma^{\hat{m}}$ gate. Then by combining such evolutions—as described in appendix A—we can achieve an arbitrary single qubit gate.

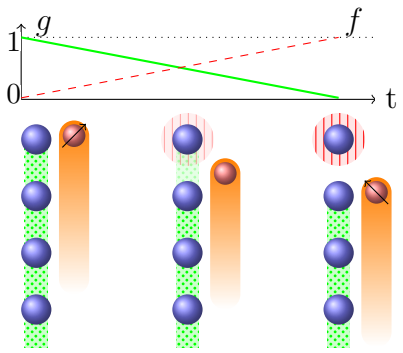

Figure 4.8: The holonomic evolution inducing a single-qubit gate and the coupling strengths throughout the process.

To extend the arbitrary single qubit gates to a set that is universal for computation we require a nontrivial entangling gate. We achieve this by generating a logical controlled-Z gate (up to single qubit gates) via the adiabatic coupling of two Haldane chains. The adiabatic evolution that generates this entangling gate upon two spin-$\frac{1}{2}$ Haldane chain edge modes is shown in 4.9 and described by

$$H_n(t) = f(t)J\ W^{AB} + g(t)J\left(\vec{S}_1^A \cdot \vec{S}_2^A + \vec{S}_1^B \cdot \vec{S}_2^B\right) + H_{n-1}^{A,Term} + H_{n-1}^{B,Term} \tag{4.4.4}$$

where $f(0) = g(T) = 0$, $f(T) = g(0) = 1$ and the coupling W^{AB} is given by

$$W^{AB} = \left[(S_1^{\hat{x}})^2 - (S_1^{\hat{y}})^2\right]_A \otimes \left[S_1^{\hat{z}}\right]_B + \left[S_1^{\hat{z}}\right]_A \otimes \left[(S_1^{\hat{x}})^2 - (S_1^{\hat{y}})^2\right]_B \tag{4.4.5}$$

The Hamiltonian 4.4.4—specifically the coupling term—possesses a certain symmetry group of simultaneous rotations on both chains. This symmetry is a particular embedding of the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \rtimes \mathbb{Z}_4$ into $SO(3) \times SO(3)$. It is shown in appendix B that this symmetry supports a nontrivial SP phase with $\frac{1}{2} \otimes \frac{1}{2}$ edge modes that protects the logical CZ transformation.

The adiabatic evolution results in a decoupling of two spin-1 particles into the state

$$|\xi\rangle = \frac{1}{2}(-|1\rangle|1\rangle + |1\rangle|-1\rangle + |-1\rangle|1\rangle + |-1\rangle|-1\rangle) \tag{4.4.6}$$

that is symmetric up to a phase under the Hamiltonian's rotation symmetry. The precise logic gate generated by the adiabatic evolution is worked out in appendix A by considering the conserved quantities that correspond to the symmetries of the Hamiltonian. The result is a $\sigma^{\hat{x}} \otimes \sigma^{\hat{x}} CZ$ gate upon the spin-$\frac{1}{2}$ edge modes of two Haldane chains.

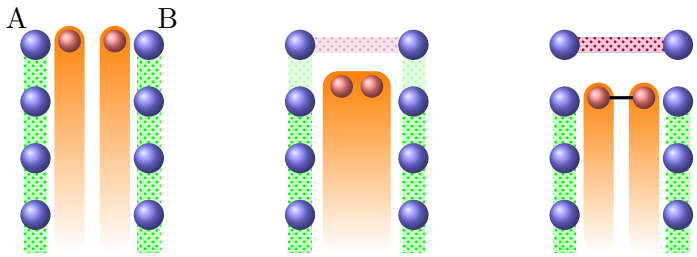

Figure 4.9: The holonomic evolution which induces a an entangling gate on two encoded qubits, the picture of the merging edge modes is justified in Chapter 4.

In appendix B it is confirmed that the universal gate set in the model of Renes. et al. that has just been described, falls into our framework of irreducible edge mode, SPAQT spin chain computation. This was done by explicitly calculating that their two qubit gate has the appropriate irreducibility and nontrivial SP edge modes required (it was already known that this holds for their single qubit gates).

The single qubit untwisted cluster gate in the Bacon et al. paper also falls into our framework, however their method for implementing two qubit gates does not posses an irreducible edge mode. This raises the possibility of looking at reducible two qubit couplings that may still generate unique logical evolutions outside of the natural gate set for the specific symmetry they possess. However, there may not be the same robustness of the edge mode encoding if the coupling Hamiltonian in such a process is varied since the edge modes are no longer irreducible and hence not protected throughout the phase as discussed in section 4.1.

We could also think of smoothly varying the embedding of the discrete symmetry within the Lie group to achieve arbitrary single qudit operations without having to use a standard gate set decomposition. This was done by Renes et al. [18] and may carry some advantages both in theory and practice. However, no satisfactory theoretical basis for the behavior of SP phases as the protecting symmetry is continuously varied has thus far been given to validate such an approach.

In this section we have described how we propose to achieve universal computation with SPAQTs by generating a universal gate set with a set of distinct symmetries embedded within a larger group. We explained how symmetries can generically be embedded into a Lie group and reference the explicit example in appendix B of a universal SPAQT gate set. In this chapter thus far we have described how we can build a universal quantum computer out of SPAQTs and execute any quantum circuit in this framework via the application of uniform fields. In the next and final section we will address the robustness of SPAQT computing in the presence of imperfect operation due to realistic errors from unavoidable coupling to the environment.

4.5 The effect of errors

In this section we will discuss the effects of relevant environmental errors on our scheme in a fashion similar to the discussion given in [41].

We start by noting that general holonomic quantum computation can be implemented in a provably fault tolerant way as shown by Oreshkov et al. [15] While their construction does not lend itself directly to our framework, at least in principle a holonomic scheme for universal quantum computation such as ours can be made fault tolerant (the question of whether we can do this while preserving the desirable physical properties of our scheme is open).

Each individual SPAQT is inherently thermodynamically protected from all symmetric errors due to the irreducibility of the edge mode representation within the ground space and the energy gap to the excited states. In this sense our encoding is essentially a decoherence free subspace [76, 77] for symmetric errors. This property can be combined with dynamical decoupling pulses [84] that implement all the global symmetries in order to symmetrize the noise operators to a certain order in perturbation theory and hence provide a thermodynamic protection from these now symmetrized errors as they must act as the identity on the ground space to the same order of approximation. However, this scheme may not suit the adiabatic implementation in practice as the dynamic decoupling requires active application of fast pulse sequences implementing global symmetries throughout the evolution.

Any errors that have the sole effect of changing the energy eigenspace of the chain during the adiabatic evolution should be equivalent to having an excited state at the end of the computation when the Hamiltonian consists of purely uniform local fields (provided the gaps between eigenspaces are sufficiently large). This implies that we can understand the error as causing one of the spins to end up in an excited eigenstate of the local symmetric field F and hence it will lie within some irrep of the symmetry. If we assume that all of these eigenspaces are also one dimensional (which should generically be true for an abelian group) and we label the excited state by $|\varphi\rangle$ then the effect of this error is precisely to implement the gate W_φ in place of the W_χ that would have occurred without the error (there may also be a global phase factor due to the W_φ matrices forming a projective representation). Note we may also measure all the final F_i fields to determine

what excitation errors have occurred during the computation and furthermore to collapse a superposition of errors into the energy eigenbasis of the F fields. With the measurement results of all the fields F it is possible to determine what errors have occurred during the computation and how to correct for these in future evolution.

For the remainder of the section we will speculate on some possible fault tolerant properties of the encoding we use at different points in a SP phase. As we described in section 4.1 the ground state encoding is associated to a gapless edge mode and it is known to be localised to the edge in the following sense; there is a renormalisation fixed point of the phase in which the information is strictly localised to a single physical site and as we follow a symmetric, adiabatic path this mode spreads out up to the point where it persists across the majority of the chain at a phase transition. This suggests that at different points of the phase diagram the encoded information will possess different inherent robustness properties to local errors. As noted by Bacon et al. [41] at an exact fixed point and at the decoupled end point of the adiabatic evolution in eq. 4.3.1 the encoded information is essentially as unprotected as a bare qubit. They propose a solution to this by scheduling the adiabatic evolution to spend a minimal amount of time at the beginning and end of the computation, where the gap is almost constant and the information is unprotected. They go a step further and conjecture that the encoded information is inherently robust to local errors during the middle of the adiabatic evolution eq.4.3.1 where it is maximally delocalised over the bulk of the chain.

We note that even for points in a SP phase a constant distance from the fixed point we could spread the encoded information over a constant number of sites that is sufficiently large to protect against errors that act independently on single physical spins. An analysis of the general case is complicated by the fact that if the parent Hamiltonian is commuting then the information lives precisely on the single physical edge spin and is unprotected. Hence any inherently robust encoding must have non-commuting Hamiltonian terms which make any analysis of the precise properties of the encoding almost impossible. We further propose that during computation when no measurements are necessary one should take advantage of the inherent robustness of points in the phase where the edge mode is spread out. In particular we consider starting and finishing the computation at such points, rather than the points with exactly localised encoded information described by eq.4.3.1.

For our scheme to be realistically implementable we have to speculate on the robustness of SP phases (whose full analysis is beyond the scope of this thesis). In particular we expect that they should exist for small non-symmetric Hamiltonian perturbations and small nonzero temperatures.

Finally we expect that the most general non symmetric error operators will need to be dealt with by software level error correction techniques [85]. This issue is complicated in our general SPAQT model since the Hamiltonian terms governing the chain do not commute and consequently there is a nonzero Lieb-Robinson group velocity [86, 87] in the system which allows local errors to propagate and spread under time evolution. We believe that it should still be possible to deal with these errors by utilizing global cooling of the system while sweeping a field to implement the computation. In this case the

errors can only propagate a certain mean path length determined by the temperature of the cooling reservoir to which the chain is coupled. Then the region which can possibly be effected by the error is of constant size in time and space and should be uncorrelated with other errors. Hence we expect that such an error should be correctable by simulating standard fault tolerant circuit constructions with the SPAQTs.

It may be possible to formalize the above analysis by treating the spin chain as weakly coupled to a bath where the open system dynamics can be described by a master equation. In some such cases it has been shown that the light cone of information spread can asymptote to a finite region [88] (although this has only been shown for dynamics with product state fixed points it may also hold for zero correlation length MPS fixed points [89]). Finally we note that some weak disorder in the system could have a similar effect in localizing the excitations caused by errors such that they can be corrected using the procedure described above.

In this section we have addressed the performance of SPAQT computation in the presence of errors due to imperfect operation and environmental noise. We argued why we expect it to be possible to operate SPAQT computing in a noisy environment by utilizing a combination of inherent robustness and standard, software based, fault tolerant circuit constructions.

4.6 Chapter Summary

We have argued that material properties of SP phases make them natural systems to use when building symmetry-protected adiabatic quantum transistors. We would like to draw the analogy of this situation to the use of semiconductor materials in building classical transistors. We have proposed an understanding of the operation of an adiabatic quantum transistor in terms of a symmetric phase transition from a symmetry-protected phase to a trivial symmetric phase. This also extends the understanding of the Hamiltonians that lead to adiabatic quantum transistor gates to SP phases of matter rather than finely tuned exact models, thereby further reducing the control requirements for the scheme. We have explained how one can understand and generalise the degenerate ground state encoding used in the AQT model in terms of the defining properties of SP phases. We then provided arguments based upon the symmetry of the phase to determine what gate is performed when the transistor is operated. In the interest of achieving computational universality we explained how different logical gates can be performed by preserving distinct subgroups of a larger symmetry group at different points in a larger network of spin chains. Finally we have addressed some issues of fault tolerance in the model and drawn the distinction between errors to which the model is inherently robust and errors which will require standard fault tolerance techniques.

We further hope that our general approach can be applied to a broad range of situations to characterise useful properties of particular fine tuned parent Hamiltonians in terms of more robust and universal properties of whole quantum phases.

Finally, we put forward the conjecture that our scheme may be adaptable to exploit

the inherent protection of 2D topologically ordered surface states of 3D topological bulk materials [90] and thus achieve inherent fault tolerance of the information encoded in the edge mode. We also suspect that it may be possible to implement a similar scheme in currently engineerable topological superconducting wires which are fermionic analogs to SP spin chains [49, 91].

Chapter 5

Discussion and Conclusion

In this thesis we have studied a quantum analog of the transistor as a basic element for quantum logic circuits. The quantum information is encoded into the ground states of a many-body system in a SP phase. Evolutions are driven by adiabatically passing through a symmetric phase transition. The logical evolutions depend only upon universal properties that persist throughout a SP phase, making them robust to any symmetric perturbation. For this reason we propose that SP matter is the natural material that should be used to build AQTs, mirroring the dependence of classical transistors on semiconductors. This presents an advantage over recent proposals for MBQC and the original AQT scheme that are only expected to operate within the small proximity of precise resource states such as the cluster state.

We proposed a scheme for encoding quantum information into the ground state of any spin chain possessing SP order. This encoding persists throughout a whole phase of SP matter and is hence robust to all symmetric perturbations. We proceeded to show how logical transformations of this information can be realised by adiabatically traversing a phase transition to a trivial symmetric phase. Importantly these transformations were completely determined by the projective representation of the symmetry group acting upon the edge mode and a character of the group that is picked out by a uniform field. Hence the evolution is a property of the phase and not a single, finely tuned Hamiltonian.

The framework of SPAQTs essentially combined and generalised the main ideas from the previous work on AQTs and SP holonomic quantum computation [18,41]. The models considered in the previous cases occur as special cases of our framework, in particular the previous example for universal SP holonomic quantum computation is an instance of universal SPAQT computing. Under some extra technical restrictions it should be possible to adapt our scheme to a measurement based implementation, but since we can essentially pick the analog of a measurement outcome with an applied field we avoid the need for assurance that all possible outcomes will give a valid computation and hence we have been able to consider a much wider class of symmetry groups.

This work answers two questions that arose in my previous work on the same topic, namely; does the property of symmetric holonomic computation extend to all 1D SP phases and is the resulting gate a geometric property of the phase. The answer to the first is yes, with the possible gates depending upon the symmetry of a given model. To the second the answer is that it is more appropriate to think of the gate as a property of the symmetric phase transition between a SP and trivial phase.

The SPAQT studied in this thesis are the basic building blocks of a quantum computer and many such devices will need to be wired together to implement interesting circuits.

This makes it tempting to consider the possibility of 2D SP phases that might be capable of simulating a full quantum circuit under application a uniform field. Unfortunately generic 2D SP phases do not seem to be the correct medium to encode many distinct qubits as the edge modes either break the symmetry or form a gapless spectrum of long range entangled states [92,93]. It is possible that a 2D SP phase together with translation symmetry in one spatial direction may have the right structure to encode a desired circuit in a similar manner to a collection of 1D SP chains each with an independent symmetry group (MBQC models of this type are known [36, 37]) but a proof of this has not been found.

A very interesting prospect to consider is 3D SP phases, since the long ranged entangled edge modes can posses topological order [90], endowing them with inherent robustness to all local errors. However in this case it is not obvious that a similarly control sequence as easy as the application of a uniform field would suffice to enact logical transformations on the information encoded into topological edge states. Another interesting direction more closely related to experiments is to consider a similar scheme in the setting of fermionic chains with SP order since it is possible to fabricate these chains, and even small circuits with current experimental technology [91]. This requires further work as the way that symmetries can act and the definition of cohomology is different in the theory of SP order for 1D fermionic chains [44,49] and hence it is not clear whether our arguments will carry over directly.

In conclusion, this thesis constitutes a step towards understanding how the properties inherent to symmetry-protected phases of matter can be harnessed to create physical architectures for robust, universal quantum computation that are similar in structure to modern day computer chips.

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Appendix A

Spin-1 Model for Holonomic Quantum Computation

In this appendix we will discuss the spin-1 Heisenberg chain proposed in [18] for holonomic quantum computation. We consider a 1D chain of spin-1s which are locally interacting via a pairwise, symmetric, antiferromagnetic coupling, favoring local anti-alignment of neighboring spins. A two-body, nearest neighbor Hamiltonian which describes such an interaction with full $SO(3)$ rotation symmetry is

$$H_n = J \sum_{i=1}^{n-1} \left(\vec{S}_i \cdot \vec{S}_{i+1} - \beta \left(\vec{S}_i \cdot \vec{S}_{i+1} \right)^2 \right) \tag{A.0.1}$$

This Hamiltonian has the well known Haldane ground state at $\beta = 0$ and the AKLT ground state at the point $\beta = -\frac{1}{3}$. Previous investigations have uncovered the emergence of spin-$\frac{1}{2}$ at the Haldane point, and the exact MPS representation of the state at the AKLT point (Which gives rise to a spin-$\frac{1}{2}$ degree of freedom in the state of the edge spins). The connection suggested by the shared boundary degree of freedom is in fact a property of the whole Haldane phase, which consists of all ground states that can be connected to the Haldane point by a path of gapped, symmetric Hamiltonians. This corresponds to the range, $-1 < \beta < 1$, in the $SO(3)$ symmetric Hamiltonian A.0.1, and the spin-$\frac{1}{2}$ boundary degrees of freedom are a defining property that persists throughout this range. This is an instance of a SPTO phase protected by the $SO(3)$ rotation symmetry, with the boundary spins described by the spin-$\frac{1}{2}$ projective representation of $SO(3)$.

The states of the spin-$\frac{1}{2}$ boundary degrees of freedom label a fourfold degeneracy (in the thermodynamic limit) in the ground states of the spin chain. To be precise we note that for any finite chain there is a small splitting between the energy eigenvalues of the set of ground states corresponding to the singlet and triplet states of the edge modes. This splitting decays exponentially as the size of the system grows, but the gap to the first excitation converges to a non-zero value. This is due to the general property that correlations decay exponentially in gapped ground states, causing the strength of the interaction between the two edge modes to decay accordingly.

As in [18] we consider coupling one boundary to a real spin-$\frac{1}{2}$ which possesses a non-trivial, projective representation of the symmetry group with a factor system, inverse to that of the boundary mode. This effectively purifies that edge mode and removes the fractional degree of freedom at that edge as discussed in Sec. 3.3.4. The product of the emergent mode Hilbert space with that of the real spin-$\frac{1}{2}$ is now equivalent to a linear representation of the symmetry. Hence, coupling at one boundary breaks the (near) four-

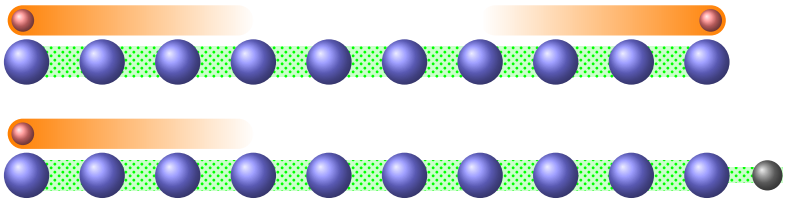


Figure A.1: A Haldane chain of spin-1 particles with open and fixed boundary conditions respectively.

fold degeneracy of the ground states down to a twofold degeneracy (which is exact, even for a finite sized system) corresponding to a single spin-$\frac{1}{2}$ boundary mode.

$$H_n^{Term} = J\sum_{i=1}^{n-1}\left(\vec{S}_i\cdot\vec{S}_{i+1} - \beta\left(\vec{S}_i\cdot\vec{S}_{i+1}\right)^2\right) + J\vec{S}_n\cdot\vec{s}_{n+1} \tag{A.0.2}$$

The purification of one boundary, effectively fixing that degree of freedom, reduces the dimension of the degenerate ground space to two. We identify the logical Pauli operators on this subspace with global conserved quantities, generated by the symmetries of the Hamiltonian

$$\Sigma_n^{\hat{m}} = \left(\bigotimes_{j=1}^{n}\exp(i\pi S_j^{\hat{m}})\right)\otimes\exp(i\frac{\pi}{2}\sigma^{\hat{m}}) = \bigotimes_{j=1}^{n}\left(I - 2(S_j^{\hat{m}})^2\right)\otimes i\sigma^{\hat{m}} \tag{A.0.3}$$

These operators form a nontrivial projective representation of $SO(3)$, which the same factor system as the 2D spin-$\frac{1}{2}$ representation an instance of the general situation discussed in Sec. 3.3.4. Then the logical Z operator is identified with $Z_L = \Sigma_n^{\hat{z}}$ and logical X with $X_L = \Sigma_n^{\hat{x}}$. The encoded spin-$\frac{1}{2}$ degree of freedom spanned by the eigenstates of these operators within the degenerate ground state is identified with the state of the gapless boundary mode. This encoding persists throughout the SPTO phase since it relies only on conserved quantities generated by the symmetries of the whole phase and the ground state degeneracy that is protected by this symmetry.

The similarity of this model to that of [29] allows us to adopt their process for initialization and readout, which can be achieved non-deterministically by measuring a single spin-1 in the $S^{\hat{z}}$ basis. The initialization procedure at the AKLT point is described at the end of Appendix H.

The process by which we generate logical gates is inspired by the measurement-based quantum computation approach of [29], in which boundary spin-1's are adiabatically decoupled from the chain and measured. We use an analogous process of adiabatic decoupling a spin, but instead of making a measurement we energetically force the decoupled spin into a ground state along the axis of an applied boundary field, maintaining unitarity throughout the evolution.

To achieve such a feat we must first loosen our symmetry constraints, since no single spin field with a preferred axis can possess full $SO(3)$ symmetry. We will make use of a well known result; the small Abelian group $D_2 = \mathbb{Z}_2\times\mathbb{Z}_2$ protects the same boundary

modes as the full $SO(3)$ symmetry (for a SPTO spin-1 chain). We can think of this group D_2 as being embedded in the natural $SO(3)$ symmetry, corresponding to a subgroup generated by π-rotations about two orthogonal, spatial axes.

The relaxation of the symmetry condition allows us to explicitly consider D_2 symmetry respecting local fields of the form $(\vec{S}^{\hat{m}})^2$ acting on single spin sites along the three spatial axes which define the embedding of $D_2 \subset SO(3)$. For an explicit embedding of D_2 generated by π-rotations about two orthogonal axes $\hat{m}$, $\hat{m}^\perp$, we can use the local fields $(\vec{S}^{\hat{m}})^2$, $(\vec{S}^{\hat{m}^\perp})^2$ and $(\vec{S}^{\hat{m}\times\hat{m}^\perp})^2$ without breaking the symmetry.

In the next section we will use these symmetry respecting fields to generate logical evolutions of the encoded qubits.

A.1 Single-Qubit Gates

In this section we will see how single-qubit Pauli rotations can be generated by adiabatically decoupling a single spin from the chain while applying a symmetry respecting field to it.

The qubit encoded in the free edge of the ground state by the X_L and Z_L operators can be manipulated by adiabatically decoupling a single spin from the end of the chain while applying a local field to it. This unitary evolution forces the decoupled spin into the ground state of the local field operator. For a field along the Z-axis this evolution is governed by the following time dependent Hamiltonian[1]

$$H_n(t) = f(t)J(S_1^{\hat{z}})^2 + g(t)J\vec{S}_1 \cdot \vec{S}_2 + H_{n-1}^{Term} \tag{A.1.1}$$

where f and g are monotonic functions with: $f(0) = g(T) = 0$ and $f(T) = g(0) = 1$. Note that the addition of the $(S_1^{\hat{z}})^2$ field fixes one axis of the embedding $D_2 \subset SO(3)$ to be the $\hat{z}$ axis. To complete a nontrivial holonomy with D_2 symmetry we apply a local field along another axis, orthogonal to $\hat{z}$. The choice of the second axis specifies the embedding of $D_2 \subset SO(3)$. The particular choice of a field along the $\hat{x}$ axis identifies $D_2 \subset SO(3)$ with the group of π-rotations about the $\hat{x}, \hat{y}, \hat{z}$ axes. The full Holonomy is then described by the Hamiltonian

$$H_n(t) = f_1(t)J(S_1^{\hat{z}})^2 + f_2(t)J(S_1^{\hat{x}})^2 + g(t)J\vec{S}_1 \cdot \vec{S}_2 + H_{n-1}^{Term} \tag{A.1.2}$$

where f_1, f_2 and g are smooth functions, piecewise-monotonic on the three time intervals: $[0, T_1]$, $(T_1, T_2]$, $(T_2, T_3]$, with: $f_1(0) = f_1(T_2) = f_1(T_3) = f_2(0) = f_2(T_1) = f_2(T_3) = g(T_1) = g(T_2) = 0$ and $f_1(T_1) = f_2(T_2) = g(0) = g(T_3) = 1$. This time varying Hamiltonian respects the D_2 symmetry throughout the coupling and hence supports the SPTO phases protected by this symmetry. Consequently the boundary modes persist so long as there is no phase transition in the path of the time dependent Hamiltonian[2].

[1] We have written this Hamiltonian for the exact Haldane state of the chain $\beta = 0$ for simplicity, but the arguments hold equally well if we allow the Hamiltonian to move throughout the Haldane phase

[2] There is strong numerical evidence in [18] that the energy gap remains finite for these evolutions and hence there is no phase transition.

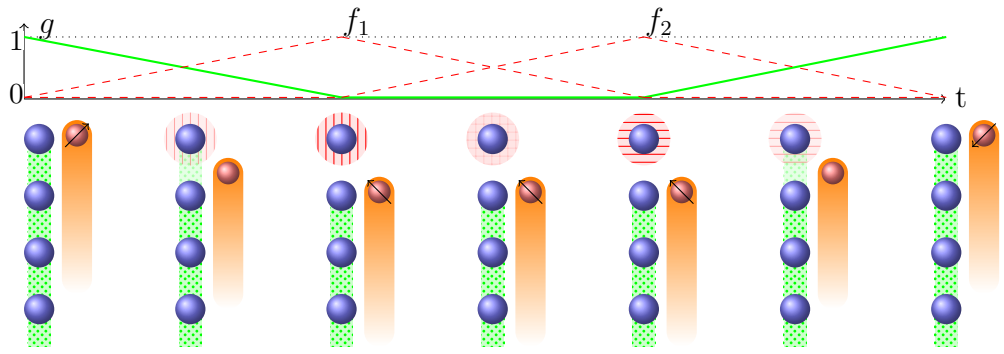


Figure A.2: The holonomic evolution inducing a single-qubit gate and the coupling strengths throughout the process.

We analyze the action of the holonomy on the encoded spin by making use of the conserved quantities $\Sigma_n^{\hat{x}}, \Sigma_n^{\hat{z}}$ generated by the on-site symmetries, which remain constant during the unitary evolution. First we consider the evolution over the interval $[0, T_1]$, as a spin-1 is decoupled from the n-chain and the encoded qubit squeezed into a shorter (n-1)-chain. This evolution results in a Pauli Z gate on the encoded information.

For an initial +1 eigenstate of $\Sigma_n^{\hat{z}}$, the $|0\rangle_L$ logical state on n spins

$$|\Psi(0)\rangle = \left|\Sigma_n^{\hat{z}} = 1, H_n^{Term} = 0\right\rangle = |0\rangle_n \tag{A.1.3}$$

after the adiabatic decoupling becomes[3]

$$|\Psi(T_1)\rangle = \left|\Sigma_n^{\hat{z}} = 1, (S_1^{\hat{z}})^2 = 0, H_{n-1}^{Term} = 0\right\rangle \tag{A.1.4}$$

up to a phase factor. This eigenstate represents a spin-1 decoupled from the remaining $(n-1)$ length chain.

To determine the results of this evolution we make use of the conserved quantities on the chain. The ground state of the decoupled spin $\left|(S_1^{\hat{z}})^2 = 0\right\rangle = \left|S_1^{\hat{z}} = 0\right\rangle$ is a $+1$ eigenstate of the rotation operator $\exp(i\pi S^{\hat{z}})$ hence the remaining symmetry operator $\Sigma_{n-1}^{\hat{z}}$ must have eigenvalue $+1$, so that the value of the total $\Sigma_n^{\hat{z}}$ is conserved. Hence the final state can be written

$$|\Psi(T_1)\rangle = \left|S^{\hat{z}} = 0\right\rangle \otimes \left|\Sigma_{n-1}^{\hat{z}} = 1, H_{n-1}^{Term} = 0\right\rangle = \left|S^{\hat{z}} = 0\right\rangle \otimes |0\rangle_{n-1} \tag{A.1.5}$$

and we see that the encoded $|0\rangle_L$ state is fixed under this evolution. Similarly an initial state $|1\rangle_n$ evolves to $\left|S^{\hat{z}} = 0\right\rangle \otimes |1\rangle_{n-1}$ up to another phase.

Since the evolution fixes the $|0\rangle_L$ and $|1\rangle_L$ states up to possibly different phase factors, it must amount to some rotation about the $\hat{z}$-axis of the Bloch sphere, corresponding to the unitary operator

$$U_L = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix} \tag{A.1.6}$$

[3]A note about notation: each Hamiltonian term contained within the kets is considered to be normalized such that its lowest eigenvalue is 0, hence a statement such as: $H_{(n-1)}^{Term} = 0$ implies that the ket lies in the ground state of the $H_{(n-1)}^{Term}$ operator.

up to an irrelevant global phase, and where $\theta \in (-\pi, \pi]$.

To calculate the rotation θ we consider the evolution of the X_L basis under the decoupling, making use of the fact that $\exp(i\pi S^{\hat{x}}) \left|S^{\hat{z}} = 0\right\rangle = -\left|S^{\hat{z}} = 0\right\rangle$. For the initial $+1$ eigenstate of $\Sigma_n^{\hat{x}}$, the $|+\rangle_L$ logical state:

$$|\Psi(0)\rangle = \left|\Sigma_n^{\hat{x}} = 1, H_n^{Term} = 0\right\rangle = |+\rangle_n \tag{A.1.7}$$

after decoupling becomes

$$\begin{aligned} |\Psi(T_1)\rangle &= \left|S^{\hat{z}} = 0\right\rangle \otimes \left|\Sigma_{n-1}^{\hat{x}} = -1, H_{n-1}^{Term} = 0\right\rangle \\ &= \left|S^{\hat{z}} = 0\right\rangle \otimes |-\rangle_{n-1} \end{aligned} \tag{A.1.8}$$

up to a phase factor $e^{i\gamma}$.

We can fully determine the evolution by comparing the two different descriptions in Eq. A.1.6 and Eq. A.1.8. This comparison implies that $U_L |+\rangle_L = |0\rangle_L + e^{i\theta} |1\rangle_L = e^{i\gamma} |-\rangle_L$ which specifies the observable phase difference since $e^{i\theta} = -1$, hence the rotation is $\theta = \pi$.

$$U_L = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

up to another irrelevant global phase. Hence the full evolution of the logical qubit described by Eq. A.1.1 has been specified to be a π-rotation about the $\hat{z}$ axis of the Bloch sphere that takes $|+\rangle_L \mapsto |-\rangle_L$ and $|-\rangle_L \mapsto |+\rangle_L$, up to an unmeasurable, global phase.

An important point is that this whole argument works just as well when we replace $\hat{z}$ and $\hat{x}$ by any pair of orthogonal axes $\hat{m}$ and $\hat{m}_\perp$ and the field $(S_1^{\hat{z}})^2 \mapsto (S_1^{\hat{m}})^2$. Which would lead to a π-rotation about the $\hat{m}$ axis, along which the local field is aligned.

Since this evolution is unitary, it can equally well be run in reverse, effectively recoupling a spin, initially in the ground state of a local field, to the chain. This increases the length of the chain and reverses the logical evolution of the decoupling process. Hence the recoupling process also causes a π-rotation about the axis along which the local field is aligned.

Equipped with a description of the coupling and decoupling processes, we can determine the full evolution described by Eq. A.1.2. We see that is corresponds to a π-rotation about the $\hat{z}$ axis as a spin is decoupled, followed by the adiabatic realignment of the local boundary field from the $\hat{z}$ axis to the $\hat{x}$ axis and finally another π-rotation about the $\hat{x}$ axis as the spin is recoupled. Hence the total evolution associated to the holonomy is just a π-rotation about the $\hat{y} = \hat{z} \times \hat{x}$ axis of the logical Bloch sphere.

A.2 Two-Qubit Gate

In this section we explain how to generate an entangling gate between the qubits encoded in two separate chain, using a similar procedure to the single-qubit evolution, but this time by coupling a pair of physical spins, one from each chain, as they are simultaneously decoupled from their respective chains.

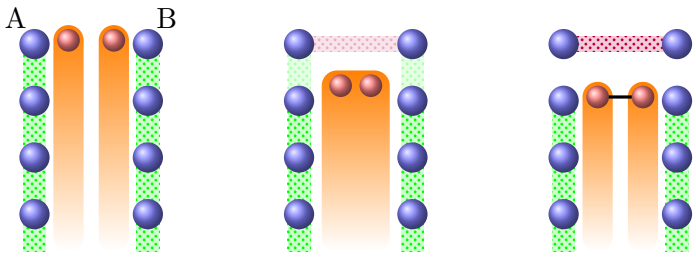


Figure A.3: The holonomic evolution which induces a an entangling gate on two encoded qubits, the picture of the merging edge modes is justified in Chapter 4.

To simulate more complicated quantum circuits involving multiple qubits we need to be able to generate entanglement between encoded qubits. We do this in a similar way to the single-qubit gates, but this time by brining together two spin chains (A and B) and applying two-body interaction terms to a pair of spins at the edge of the chains. We use the particular choice of coupling Hamiltonian W^{AB} introduced in [18] which yields a CZ gate followed by local Pauli operators (by-products) on each individual chain as we decouple the pair of end spins.

$$H_n(t) = f(t)J\ W^{AB} + g(t)J\left(\vec{S}_1^A \cdot \vec{S}_2^A + \vec{S}_1^B \cdot \vec{S}_2^B\right) + H_{n-1}^{A,Term} + H_{n-1}^{B,Term} \tag{A.2.1}$$

where $f(0) = g(T) = 0$, $f(T) = g(0) = 1$ and the symmetric coupling W^{AB} is given by

$$W^{AB} = \left[(S_1^{\hat{x}})^2 - (S_1^{\hat{y}})^2\right]_A \otimes \left[S_1^{\hat{z}}\right]_B + \left[S_1^{\hat{z}}\right]_A \otimes \left[(S_1^{\hat{x}})^2 - (S_1^{\hat{y}})^2\right]_B \tag{A.2.2}$$

To calculate the evolution of the encoded qubits under the Hamiltonian A.2.1 we make use of similar symmetry arguments to those for the single-qubit gate. For this purpose the symmetry operators[4] of the interaction term W^{AB} and the conserved operators generated by them upon the full two chains are instrumental. These symmetries are generated by the following set of rotations $\{(\sqrt{R^{\hat{z}}}, R^{\hat{x}}), (R^{\hat{u}}, R^{\hat{u}}), (R^{\hat{z}}, 1), (1, R^{\hat{z}})\}$, where $R^{\hat{m}}$ denotes the spin-1 representation of a π-rotation about the $\hat{m}$ axis, and $\hat{u} = \frac{1}{\sqrt{2}}(\hat{x}+\hat{y})$, $\hat{v} = \frac{1}{\sqrt{2}}(\hat{x}-\hat{y})$.

The state of the decoupled end spins determines the evolution of the remaining chains via the conserved quantities associated to the rotations. The W^{AB} coupling has the unique groundstate

$$|\xi\rangle = \frac{1}{2}(-|1\rangle|1\rangle + |1\rangle|-1\rangle + |-1\rangle|1\rangle + |-1\rangle|-1\rangle) \tag{A.2.3}$$

note that this groundstate is invariant under the full symmetry group of the two chain interaction and hence does not cause any symmetry breaking of the chain. The invariance of the groundstate precisely corresponds to it being an eigenstate of all the symmetry operators on the pair of decoupled spins.

[4]These are trivially symmetries of both full chains due to the full $SO(3) \times SO(3)$ symmetry of the bulk Hamiltonian.

W^{AB} Symmetry	$\|\xi\rangle$ Eigenvalue	Corresponding Conserved Quantity
$(\sqrt{R^{\hat{z}}}, R^{\hat{x}})$	i	$\sqrt{\Sigma_n^{\hat{z}}} \otimes \Sigma_n^{\hat{x}}$
$(R^{\hat{u}}, R^{\hat{u}})$	1	$\Sigma_n^{\hat{u}} \otimes \Sigma_n^{\hat{u}}$
$(R^{\hat{v}}, R^{\hat{v}})$	1	$\Sigma_n^{\hat{v}} \otimes \Sigma_n^{\hat{v}}$
$(R^{\hat{z}}, 1)$	-1	$\Sigma_n^{\hat{z}} \otimes 1$
$(1, R^{\hat{z}})$	-1	$1 \otimes \Sigma_n^{\hat{z}}$

Table A.1: The eigenvalues of $|\xi\rangle$ for various different symmetry operators.

The particular eigenvalues of $|\xi\rangle$ given in Table A.1, under symmetries[5] of W^{AB} which generate conserved quantities on the pair of chains, will determine the evolution of the encoded qubits caused by the decoupling process. The total evolution of the encoded qubits caused by the decoupling in Eq. A.2.1 turns out to be a CZ gate followed by Pauli $\sigma^{\hat{x}}$ operators on each qubit, which is a nontrivial entangling gate.

To calculate the evolution we first consider π-rotations about each $\hat{z}$ axis $(R^{\hat{z}}, 1)$ and $(1, R^{\hat{z}})$. For an initial state in the combined $S^{\hat{z}}$ product basis, $|\epsilon_1\rangle_n^A |\epsilon_2\rangle_n^B$ where $\epsilon_1, \epsilon_2 \in \{0, 1\}$, we have

$$\begin{aligned} |\Psi^{\epsilon_1\epsilon_2}(0)\rangle &= \left|\Sigma_n^{\hat{z}} \otimes 1 = (-1)^{\epsilon_1}, 1 \otimes \Sigma_n^{\hat{z}} = (-1)^{\epsilon_2}, H_n^{A,Term} = 0, H_n^{B,Term} = 0\right\rangle \\ &= |\epsilon_1\rangle_n^A |\epsilon_2\rangle_n^B \end{aligned} \tag{A.2.4}$$

which becomes, after the decoupling

$$\begin{aligned} \left|\Psi^{\epsilon_1'\epsilon_2'}(T)\right\rangle &= \left|W^{AB} = 0, \Sigma_n^{\hat{z}} \otimes 1 = (-1)^{\epsilon_1}, 1 \otimes \Sigma_n^{\hat{z}} = (-1)^{\epsilon_2}, H_{n-1}^{A,Term} = 0, H_{n-1}^{B,Term} = 0\right\rangle \\ &= |\xi\rangle \otimes \left|\Sigma_n^{\hat{z}} = -(-1)^{\epsilon_1}, H_{n-1}^{A,Term} = 0\right\rangle \otimes \left|\Sigma_n^{\hat{z}} = -(-1)^{\epsilon_2}, H_{n-1}^{B,Term} = 0\right\rangle \\ &= |\xi\rangle \otimes |\epsilon_1{+}1\rangle_{n-1}^A |\epsilon_2{+}1\rangle_{n-1}^B \end{aligned} \tag{A.2.5}$$

up to an unkown phase $\theta_{\epsilon_1\epsilon_2}$, where the addition inside these kets is mod 2. To determine the state in Eq. A.2.5 we have used the -1 eigenvalue of $|\xi\rangle$ under the $(R^{\hat{z}}, 1)$ and $(1, R^{\hat{z}})$ rotations on the decoupled spins.

Hence the logical evolution must take the form

$$U^{AB} = \begin{bmatrix} 0 & 0 & 0 & \theta_{11} \\ 0 & 0 & \theta_{10} & 0 \\ 0 & \theta_{01} & 0 & 0 \\ \theta_{00} & 0 & 0 & 0 \end{bmatrix} \tag{A.2.6}$$

for the unknown phases $\theta_{11}, \theta_{10}, \theta_{01}, \theta_{00}$ defined above.

To specify the constants $\theta_{\epsilon_1\epsilon_2}$ we consider the reducible projective representations of the conserved quantities listed in Table A.1, and match their eigenvectors to the respective eigenvalues of the irreducible projective representations shown in Table A.2. This

[5]In principle we need only know the eigenvalues of $|\xi\rangle$ under a set of symmetries which generate the full group, in practice certain symmetries prove easier to analyze and those chosen in Table A.1 are sufficient to specify the evolution.

corresponds to identifying the logical states encoded in the degenerate ground space by the operators in Eq. A.0.3 with the state of the edge mode. The action of the symmetries within the degenerate ground space is then described by the irreducible projective representation on the boundary mode.

We define a set of states in the degenerate ground space of the two chains

$$|\epsilon_u, \epsilon_v\rangle_n = \left|\Sigma_n^{\hat{u}} \otimes \Sigma_n^{\hat{u}} = (-1)^{\epsilon_u}, \Sigma_n^{\hat{v}} \otimes \Sigma_n^{\hat{v}} = (-1)^{\epsilon_v}, H_n^A = 0, H_n^B = 0\right\rangle \tag{A.2.7}$$

The +1 eigenvalue of the edge state $|\xi\rangle$ under the rotations $(R^{\hat{u}}, R^{\hat{u}})$ and $(R^{\hat{v}}, R^{\hat{v}})$ specify the evolution of the spin chain initialized in this state to be

$$|\epsilon_u, \epsilon_v\rangle_n \to |\xi\rangle \otimes |\epsilon_u, \epsilon_v\rangle_{n-1} \tag{A.2.8}$$

we have:

$$\begin{aligned}
|\epsilon_u = 1, \epsilon_v = 1\rangle &\mapsto |1\rangle|0\rangle + |0\rangle|1\rangle \\
|\epsilon_u = 1, \epsilon_v = 0\rangle &\mapsto |0\rangle|0\rangle + i|1\rangle|1\rangle \\
|\epsilon_u = 0, \epsilon_v = 1\rangle &\mapsto |0\rangle|0\rangle - i|1\rangle|1\rangle \\
|\epsilon_u = 0, \epsilon_v = 0\rangle &\mapsto |1\rangle|0\rangle - |0\rangle|1\rangle
\end{aligned}$$

To this end we will compare the action of U^{AB} on the states encoded at the boundary to the effect of the adiabatic evolution on the full spin chains, allowing us to determine the unknown constants.

The adiabatic dynamics take the set of initial states:

$$|\Psi^{\epsilon_u\epsilon_v}(0)\rangle = \left|\Sigma_n^{\hat{u}} \otimes \Sigma_n^{\hat{u}} = (-1)^{\epsilon_u}, \Sigma_n^{\hat{v}} \otimes \Sigma_n^{\hat{v}} = (-1)^{\epsilon_v}, H_n^A = 0, H_n^B = 0\right\rangle \tag{A.2.9}$$

(which are possible since the two operators $\Sigma_n^{\hat{u}} \otimes \Sigma_n^{\hat{u}}$ and $\Sigma_n^{\hat{v}} \otimes \Sigma_n^{\hat{v}}$ commute) to the final states:

$$|\Psi^{\epsilon_u\epsilon_v}(T)\rangle = |\xi\rangle \otimes \left|\Sigma_n^{\hat{u}} \otimes \Sigma_n^{\hat{u}} = (-1)^{\epsilon_u}, \Sigma_n^{\hat{v}} \otimes \Sigma_n^{\hat{v}} = (-1)^{\epsilon_v}, H_{n-1}^A = 0, H_{n-1}^B = 0\right\rangle \tag{A.2.10}$$

since $|\xi\rangle$ has the eigenvalue 1 under the rotations $(R^{\hat{u}}, R^{\hat{u}})$ and $(R^{\hat{v}}, R^{\hat{v}})$ on the decoupled spins, effectively fixing the $|\epsilon_u, \epsilon_v\rangle_L$ logical states up to a set of phase shifts $\phi_{\epsilon_u\epsilon_v}$.

We focus our attention on the (unnormalized) state $(|1\rangle|0\rangle + |0\rangle|1\rangle)$, the joint -1 eigenstate of of $i\sigma^{\hat{u}} \otimes i\sigma^{\hat{u}}$ and $i\sigma^{\hat{v}} \otimes i\sigma^{\hat{v}}$, we have

$$U^{AB}(|1\rangle|0\rangle + |0\rangle|1\rangle) = \theta_{10}|0\rangle|1\rangle + \theta_{01}|1\rangle|0\rangle \tag{A.2.11}$$

which must agree with the evolution of $|\epsilon_u = 1, \epsilon_v = 1\rangle$ that merely accumulates a phase shift ϕ_{11}. Hence after the evolution we have: $\phi_{11}(|1\rangle|0\rangle + |0\rangle|1\rangle) = (\theta_{10}|0\rangle|1\rangle + \theta_{01}|1\rangle|0\rangle)$, which requires that $\theta_{10} = \theta_{01}$.

Similarly we consider the evolution of $(|0\rangle|0\rangle + i|1\rangle|1\rangle)$, the -1 eigenstate of of $i\sigma^{\hat{u}} \otimes i\sigma^{\hat{u}}$ and +1 eigenstate of $i\sigma^{\hat{v}} \otimes i\sigma^{\hat{v}}$

$$U^{AB}(|0\rangle|0\rangle + i|1\rangle|1\rangle) = \theta_{00}|1\rangle|1\rangle + i\theta_{11}|0\rangle|0\rangle \tag{A.2.12}$$

Conserved quantity	Projective representation	eigenvectors grouped by eigenvalue
$\Sigma_n^{\hat{u}} \otimes \Sigma_n^{\hat{u}}$	$i\sigma^{\hat{u}} \otimes i\sigma^{\hat{u}}$	$\underbrace{\|0\rangle\|0\rangle + i\|1\rangle\|1\rangle,\ \|1\rangle\|0\rangle + \|0\rangle\|1\rangle}_{\text{eigenvalue: } -1},\ \underbrace{\|0\rangle\|0\rangle - i\|1\rangle\|1\rangle,\ \|1\rangle\|0\rangle - \|0\rangle\|1\rangle}_{+1}$
$\Sigma_n^{\hat{v}} \otimes \Sigma_n^{\hat{v}}$	$i\sigma^{\hat{v}} \otimes i\sigma^{\hat{v}}$	$\underbrace{\|0\rangle\|0\rangle - i\|1\rangle\|1\rangle,\ \|1\rangle\|0\rangle + \|0\rangle\|1\rangle}_{-1},\ \underbrace{\|0\rangle\|0\rangle + i\|1\rangle\|1\rangle,\ \|1\rangle\|0\rangle - \|0\rangle\|1\rangle}_{+1}$
$\sqrt{\Sigma_n^{\hat{z}}} \otimes \Sigma_n^{\hat{x}}$	$\frac{1}{\sqrt{2}}(I + i\sigma^{\hat{z}}) \otimes i\sigma^{\hat{x}}$	$\underbrace{\|1\rangle\|1\rangle - \|1\rangle\|0\rangle}_{e^{-i\frac{3\pi}{4}}},\ \underbrace{\|1\rangle\|1\rangle + \|1\rangle\|0\rangle}_{e^{i\frac{\pi}{4}}},\ \underbrace{\|0\rangle\|1\rangle - \|0\rangle\|0\rangle}_{e^{-i\frac{\pi}{4}}},\ \underbrace{\|0\rangle\|1\rangle + \|0\rangle\|0\rangle}_{e^{i\frac{3\pi}{4}}}$

Table A.2: The eigenvectors of the projective representations of various symmetry operators.

which must agree with the evolution of the state $|\epsilon_u = 1, \epsilon_v = 0\rangle_L$ encoded in the spin chains. Hence the equality: $\phi_{11}(|0\rangle|0\rangle + i|1\rangle|1\rangle) = (\theta_{00}|1\rangle|1\rangle + i\theta_{11}|0\rangle|0\rangle)$, which specifies $\theta_{00} = -\theta_{11}$.

Finally we consider the $e^{-i\frac{3\pi}{4}}$ eigenstate of $\left[\frac{1}{\sqrt{2}}(I + i\sigma^{\hat{z}}) \otimes i\sigma^{\hat{x}}\right]$, $(|1\rangle|1\rangle - |1\rangle|0\rangle)$ which evolves to:

$$U^{AB}(|1\rangle|1\rangle - |1\rangle|0\rangle) = \theta_{11}|0\rangle|0\rangle - \theta_{10}|0\rangle|1\rangle \tag{A.2.13}$$

and the corresponding encoded state:

$$|\Psi(0)\rangle = \left|\sqrt{\Sigma_n^{\hat{z}}} \otimes \Sigma_n^{\hat{x}} = e^{-i\frac{3\pi}{4}}, H_n^{A,Term} = 0, H_n^{B,Term} = 0\right\rangle \tag{A.2.14}$$

evolves to

$$|\Psi(T)\rangle = |\xi\rangle \otimes \left|\sqrt{\Sigma_{n-1}^{\hat{z}}} \otimes \Sigma_{n-1}^{\hat{x}} = e^{i\frac{3\pi}{4}}, H_{n-1}^{A,Term} = 0, H_{n-1}^{B,Term} = 0\right\rangle \tag{A.2.15}$$

up to a multiplicative phase ϕ, due to the conservation of $\sqrt{\Sigma_n^{\hat{u}}} \otimes \Sigma_n^{\hat{x}}$ and the i-eigenvalue of $|\xi\rangle$ under the symmetry operator $(\sqrt{R^{\hat{z}}}, R^{\hat{x}})$ on the decoupled spins. For these to agree requires that: $(\theta_{11}|0\rangle|0\rangle - \theta_{10}|0\rangle|1\rangle) = \phi(|0\rangle|1\rangle + |0\rangle|0\rangle)$ and hence $\theta_{11} = -\theta_{10}$ which specifies U^{AB} up to an irrelevant global phase.

$$U^{AB} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = (\sigma^{\hat{x}} \otimes \sigma^{\hat{x}})\mathrm{CZ}$$

which constitutes a CZ gate followed by a simultaneous Pauli $\sigma^{\hat{x}}$ operator on each of the encoded qubits. To complete this holonomy we could consider undoing the $\sigma^{\hat{x}}$ operators on each chain using the reverse of the evolution described in Eq. A.1.1.

We have seen, in this section and the previous one, how adiabatic holnomic evolutions of the spin chains can cause unitary logical evolutions of the qubits encoded within their degenerate ground states. In the next section we will look at the gates which we can directly generate using these evolutions withe the D_2 symmetry fixed to π-rotations about the standard $\hat{x}, \hat{y}, \hat{z}$ axes.

Appendix B

Symmetry-Protected Logical Gates

We will open this appendix with main result of my 2012 honours thesis, Theorem B.1.3, showing that universal symmetry-protected quantum computation is possible in the spin-1 model, before generalizing the arguments used to calculate symmetry-protected holonomic gates to other SP spin chains under certain constraints.

We will first present the explicit calculation of the symmetry group G_2 of the two-chain interaction Hamiltonian W^{AB}, before proving the main result of the thesis, Theorem B.1.3 in Sec. B.1.1, that this symmetry group protects a SP phase and hence also protects the two-qubit gate. Using this result, in Sec. B.2, we will give the minimal symmetry requirements for universal quantum computation with only symmetry-protected gates.

B.1 Symmetry Group of the Two-Chain Interaction

In this section we will examine the structure of the symmetry group G_2 of the two-qubit coupling Hamiltonian W^{AB} in detail. This symmetry group is important since it will determine whether or not the two-qubit gate is symmetry protected. We will determine the full set of elements within this group and use this description to identify it with one of the isomorphism classes of all size 16 groups, specifically the class labeled by $D_2 \rtimes \mathbb{Z}_4$.

The symmetries of W^{AB} that were explicitly used in the calculation of the two-qubit gate are given in Table A.1. From these we can see that the symmetry group G_2 consists of a discrete set of joint rotations of each pair of spins from the two chains. We have found that the set of symmetries listed in Table A.1 are not independent, and the group can be generated by the following π-rotations

$$(R^{\hat{u}}, R^{\hat{u}}), \quad (R^{\hat{v}}, R^{\hat{v}}), \quad (\sqrt{R^{\hat{z}}}, R^{\hat{x}}) \tag{B.1.1}$$

where $\hat{u} = \frac{1}{\sqrt{2}}(\hat{x} + \hat{y})$ and $\hat{v} = \frac{1}{\sqrt{2}}(\hat{x} - \hat{y})$. We can decompose the rotation $R^{\hat{x}} = \sqrt{R^{\hat{z}}} \cdot R^{\hat{u}}$ which leads us to conclude that the element $(R^{\hat{v}}, R^{\hat{v}})$ can be written as a product of the other two, see Table C.2 for the explicit relation. Hence the group has only two independent generators, depicted in Figures B.1 and B.2. We have written out these three redundant generators since they allow us to more easily identify this rotation group with the semidirect product[1] group $D_2 \rtimes \mathbb{Z}_4$.

As a set $D_2 \rtimes \mathbb{Z}_4$ is made up of the same elements as the direct product $D_2 \times \mathbb{Z}_4$ but possesses a different multiplication rule and hence a different group structure

$$(n_1, h_1) \times (n_2, h_2) = \left(n_1(h_1 n_2 h_1^{-1}), h_1 h_2\right), \quad n_1, n_2 \in D_2, \quad h_1, h_2 \in \mathbb{Z}_4 \tag{B.1.2}$$

[1]For a definition and details about this type of group see Appendix D.

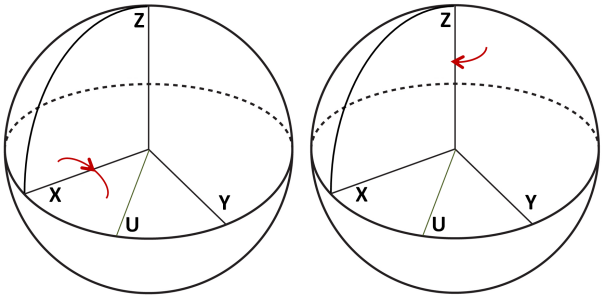


Figure B.1: The $(\sqrt{R^{\hat{z}}}, R^{\hat{x}})$ rotation generating the symmetry group.

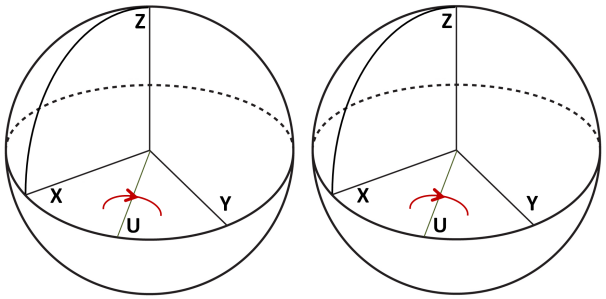


Figure B.2: The $(R^{\hat{u}}, R^{\hat{u}})$ rotation generating the symmetry group.

Under this multiplication rule, the semidirect product group is non-Abelian even though it is built from Abelian components.

To see how the symmetries of G_2 generate the $D_2 \rtimes \mathbb{Z}_4$ group structure we note that the subgroup generated by $(\sqrt{R^{\hat{z}}}, R^{\hat{x}})$ is isomorphic to $\mathbb{Z}_4$, and that generated by $(R^{\hat{u}}, R^{\hat{u}})$ is isomorphic to $\mathbb{Z}_2$. It is the interaction of these two terms via multiplication that produces $(R^{\hat{v}}, R^{\hat{v}})$ which, along with $(R^{\hat{u}}, R^{\hat{u}})$, generates D_2. It is the non-commutativity of the rotations that generates the non-Abelian structure of the semidirect product $D_2 \rtimes \mathbb{Z}_4$ upon combining the two subgroups.

A useful description of $D_2 \rtimes \mathbb{Z}_4$ is the presentation in terms of its generators and their relations

$$\left[\alpha, \beta | \alpha^4 = \beta^2 = 1, \alpha\beta = (\alpha\beta)^2\beta\alpha^3\right] \tag{B.1.3}$$

this notation means the set of all products of α and β where these two elements satisfy the relations on the right of Eq. B.1.3. Using this description we have identified $\alpha \mapsto (\sqrt{R^{\hat{z}}}, R^{\hat{x}})$ and $\beta \mapsto (R^{\hat{u}}, R^{\hat{u}})$, which specifies the correspondence for all the other group elements, explicitly listed in Table C.2, and establishes the isomorphism $G_2 \cong D_2 \rtimes \mathbb{Z}_4$.

To make this identification, we used the derived subgroup $G_2' = [G_2, G_2] := \left\{g_1 g_2 g_1^{-1} g_2^{-1} \middle| g_1, g_2 \in G_2\right\}$. G_2' is nontrivial in this case, a consequence of the group's non-Abelian structure, it has provided a useful tool since it allowed us to uniquely determine the correspondence between a nontrivial element from each of the different descriptions. This is because it consists of only two elements given by $G_2' = \{1, (\alpha\beta)^2\alpha^2\} \cong \{1, (R^{\hat{z}}, R^{\hat{z}})\}$ allowing us to identify $(\alpha\beta)^2\alpha^2 \cong (R^{\hat{z}}, R^{\hat{z}})$, which we used to pin down the exact structure given in Table C.2.

Another useful tool is the center subgroup $Z(G_2)$, consisting of elements which commute with the whole group. For G_2 it is given by $Z(G_2) = \{1, \alpha^2, (\alpha\beta)^2, (\alpha\beta)^2\alpha^2\} \cong \{1, (R^{\hat{z}}, 1), (1, R^{\hat{z}}), (R^{\hat{z}}, R^{\hat{z}})\}$. We will harness the commutative property of this subgroup later to use it as a tool in the proof that a particular projective representation of G_2 is nontrivial.

Representations of G_2

To be precise, the symmetry group $G_2 \cong D_2 \rtimes \mathbb{Z}_4$ is an embedding of the semidirect product group $D_2 \rtimes \mathbb{Z}_4$ into two copies of the 3D rotation group $G_2 \subset SO(3) \times SO(3)$. We have used this identification to construct the canonical linear representation of G_2 on a pair of spin-1 particles by mapping each rotation to its corresponding spin-1 rotation operator. Since the symmetry group G_2 is generated by the two rotations, $(\sqrt{R^{\hat{z}}}, R^{\hat{x}})$ and $(R^{\hat{u}}, R^{\hat{u}})$, we need only define the operators that represent each of these rotations to uniquely specify a representation. These are given by

$$(\sqrt{R^{\hat{z}}}, R^{\hat{x}}) \mapsto \exp(i\frac{\pi}{2}S^{\hat{z}}) \otimes \exp(i\pi S^{\hat{x}}), \qquad (R^{\hat{u}}, R^{\hat{u}}) \mapsto \exp(i\pi S^{\hat{u}}) \otimes \exp(i\pi S^{\hat{u}}) \quad \text{(B.1.4)}$$

which act upon pairs of spins from the two chains respectively.

In a similar way, we can construct *the* $\frac{1}{2} \otimes \frac{1}{2}$ *projective representation of* G_2 on a pair of spin-$\frac{1}{2}$ particles by mapping each rotation to its corresponding spin-$\frac{1}{2}$ rotation operator

$$(\sqrt{R^{\hat{z}}}, R^{\hat{x}}) \mapsto \exp(i\frac{\pi}{4}\sigma^{\hat{z}}) \otimes \exp(i\frac{\pi}{2}\sigma^{\hat{x}}), \qquad (R^{\hat{u}}, R^{\hat{u}}) \mapsto \exp(i\frac{\pi}{2}\sigma^{\hat{u}}) \otimes \exp(i\frac{\pi}{2}\sigma^{\hat{u}}) \quad \text{(B.1.5)}$$

the full representation is given in Table C.2. We will show that this projective representation is irreducible and nontrivial in the next section.

In this section we have seen the full structure of the symmetry group $G_2 = D_2 \rtimes \mathbb{Z}_4$ and two representations of it that are relevant to our model. In the next section we will use the classification of 1D SP phases to prove that this group protects the operation of the two-qubit gate.

B.1.1 The Symmetry Group Protects a nontrivial SP Phase

We introduced the classification of 1D SP phases protected by an on-site symmetry group G in terms of its second cohomology class $H^2(G, U(1))$ in Sec. 3.3.3. Each ground state is labeled by the equivalence class $[\omega]$ of the factor set ω induced by the projective representation of G acting upon its boundary modes. In this section we will show that the symmetry group G_2 of the coupling Hamiltonian W^{AB} protects a nontrivial SP phase with 4D, $\frac{1}{2} \otimes \frac{1}{2}$ boundary excitations and hence the two-qubit gate is symmetry-protected in the same sense as the single-qubit gates. Before doing so, we will introduce the arguments used in the proof, with the known example of a single chain protected by a D_2 symmetry group.

Introducing our Arguments with a D_2 Example

It is well known that the D_2 symmetry protects a nontrivial SP phase, labeled by the non-unit element of its second cohomology class $H^2(D_2, U(1)) = \mathbb{Z}_2$. This phase is characterized by the irreducible, nontrivial 2D spin-$\frac{1}{2}$ representation of D_2 that maps the non-unit group elements to Pauli operators X, Y and Z (up to multiplicative constants).

We will go through the arguments leading to the conclusion that this representation of D_2 is nontrivial and irreducible, developing tools that will be useful in making the same arguments for G_2 in the next section. To prove the irreducibility condition we will first outline a lemma which is key to the type of arguments we want to make.

Lemma B.1.1 *Any proper, nontrivial invariant subspace $\mathcal{W} \subsetneq \mathcal{H}$ of a unitary matrix $U : \mathcal{H} \to \mathcal{H}$ is spanned by the eigenvectors of the matrix $U\big|_{\mathcal{W}}$ formed by restricting U to the subspace $\mathcal{W}$.*

for a proof see [70].

Remark We have made explicit mention of this fact since it is not true for general matrices. For a general matrix M assuming that all invariant subspaces are spanned by eigenvectors of M is known as the 'diagonal fallacy'.

Example We can easily see that the Pauli projective representation of D_2 is irreducible, since all its matrices are unitary and no two Pauli operators share a joint eigenspace, see Eq. 1.1.2. Hence by Lemma B.1.1 there can be no proper subspace left invariant under all group actions in this projective representation.

To see that the factor system is nontrivial we must show that multiplication by any 2-coboundary[2] cannot take it to the trivial factor system. This is not particularly easy to show directly and so we introduce a function φ that will give us an easily calculable, sufficient condition for the nontriviality of a particular factor system, which is described in the remark to Lemma B.1.2. This function $\varphi : G \to U(1)$ is given [69] by the sum

$$\varphi_\omega(a) = \sum_{b \in G} \left(\frac{\omega(a,b)}{\omega(b,a)} \right) = \sum_{b \in G} \omega(a,b)\omega^*(b,a) \tag{B.1.6}$$

since $\omega^{-1} = \omega^*$ for a complex phase $\omega \in U(1)$.

Lemma B.1.2 *For two factor systems ω and ν, $[\varphi_\omega(a) \neq \varphi_\nu(a),\ \exists a \in Z(G)] \implies [\omega] \neq [\nu]$. Hence $[\varphi_\omega(a) \neq \varphi_\nu(a)]$ is a sufficient condition to determine whether ω and ν belong to different cohomology classes.*

Proof Suppose we had two factor systems ω and ν from the same cohomology class, then they must be equivalent $\omega \sim \nu$ under relation by 2-coboundary functions. Hence there must be some 2-coboundary $[\mu(ab)/\mu(a)\mu(b)]$ such that

$$\mu(ab)\omega(a,b) = \mu(a)\mu(b)\nu(a,b) \tag{B.1.7}$$

[2]For explanations of this terminology see Appendix G

ω	I	X	Y	Z
I	1	1	1	1
X	1	1	i	-i
Y	1	-i	1	i
Z	1	i	-i	1

Table B.1: The factor system of the Pauli projective representation of D_2.

This implies that the functions φ_ω and φ_ν, are equal for all group elements in the center[3] of the symmetry group $[a \in Z(G) \implies \varphi_\omega(a) = \varphi_\nu(a)]$, since

$$\varphi_\omega(a) = \sum_{b\in G}\left(\frac{\omega(a,b)}{\omega(b,a)}\right) = \sum_{b\in G}\left(\frac{\mu(a)\mu(b)}{\mu(b)\mu(a)}\frac{\mu(ba)}{\mu(ab)}\frac{\nu(a,b)}{\nu(b,a)}\right) = \sum_{b\in G}\left(\frac{\nu(a,b)}{\nu(b,a)}\right) = \varphi_\nu(a) \tag{B.1.8}$$

where we have used Eq. B.1.7 and the fact that $ab = ba$, $\forall\ b \in G,\ \forall\ a \in Z(G)$. ■

Remark We will make use of the fact that when ν is the trivial factor system we have $\varphi_\nu(a) = |G|$ for all group elements $a \in G$ since each $\nu(a,b) = 1$. Then for any factor system ω if we can find a group element $a \in Z(G)$ such that $\varphi_\omega(a) \neq |G|$ we will have shown ω is nontrivial. This is the sufficient condition for the nontriviality of a factor system we mentioned above.

Example For the particular example of the Pauli projective representation of D_2 we can see from Table B.1 that $\varphi_\omega(X) = \varphi_\omega(Y) = \varphi_\omega(Z) = 0$, by taking the inner product of one of these columns with its corresponding row, hence the representation is nontrivial. While we could have arrived at this conclusion simply from the fact that we have found a 2D irreducible projective representation of D_2, while all irreducible linear representations are only 1D, this is not always possible.

Applying our Arguments to the Symmetry Group G_2

We now show that the particular projective representation of G_2 given by qubit rotation operators on the 4D, $\frac{1}{2}\otimes\frac{1}{2}$ Hilbert space is irreducible and nontrivial using the arguments introduced above.

Theorem B.1.3 (Main Result) *The projective representation of G_2, constructed by mapping each pair of rotations to their corresponding rotation operators on the four dimensional $\frac{1}{2}\otimes\frac{1}{2}$ Hilbert space, is irreducible and nontrivial.*

[3]For an Abelian group $G = Z(G)$, hence $\omega \sim \nu \implies \varphi_\omega \equiv \varphi_\nu$.

Corollary B.1.4 *$H^2(G_2, U(1)) \neq 1$ and hence the symmetry group G_2 protects at least one nontrivial SP phase, labeled by the factor system ω given in Table C.1.*

Proof To prove the irreducibility of the $\frac{1}{2} \otimes \frac{1}{2}$ projective representation of the symmetry group G_2 we consider the possibilities for invariant subspaces of $\mathbb{C}^4$ under all the 4×4 matrices in this representation of G_2, explicitly listed in Table C.2. A consequence of Lemma B.1.1 is that any unitary matrix $U : \mathbb{C}^n \to \mathbb{C}^n$ which possesses an invariant subspace $\mathcal{W}$ must have a block structure, also leaving the orthogonal subspace $\mathcal{W}^\perp$ invariant under U.

For the 4D, $\frac{1}{2} \otimes \frac{1}{2}$ projective representation of G_2 we are left with the possibility of either 2 invariant 2D subspaces or a 1D and 3D invariant subspace[4]. We can immediately exclude the possibility of any 1D and 3D invariant subspaces since this would require all matrices in the projective representation to share an eigenstate due to Lemma B.1.1. This is not the case, as demonstrated in Table C.2.

We then note that any 2D invariant subspace of a matrix U in the projective representation must be the span of two eigenvectors, which is guaranteed by Lemma B.1.1. If all matrices of the projective representation are to share such an invariant subspace $\mathcal{W}$ it must be possible to write at least one eigenvector of any matrix U_1 as a linear combination of two eigenvectors of any other matrix U_2. Again this is not the case for the $\frac{1}{2} \otimes \frac{1}{2}$ projective representation of G_2 given in Table C.2. A counter example is given by considering the eigenvectors of the matrices representing the group elements $(\sqrt{R^{\hat{z}}}, R^{\hat{x}})$ and $(R^{\hat{x}}, \sqrt{R^{\hat{z}}})$. In particular, we can see from the list of eigenvectors in Table C.2 that no linear combination of any two eigenvectors of one matrix can be used to form an eigenvector of the other. Hence the $\frac{1}{2} \otimes \frac{1}{2}$ projective representation must be irreducible, since there cannot be any subspace invariant under all of its operators.

To show that the $\frac{1}{2} \otimes \frac{1}{2}$ projective representation has a nontrivial factor system ω we consider the function φ_ω defined in Eq. B.1.6. Then we can calculate φ_ω directly from the factor system given in Table C.1 to find $\varphi_\omega(R^{\hat{z}}, 1) = \varphi_\omega(1, R^{\hat{z}}) = \varphi_\omega(R^{\hat{z}}, R^{\hat{z}}) = 0$, a sufficient condition to conclude that ω cannot lie within the trivial cohomology class after invoking Lemma B.1.2.

Hence the symmetry group G_2 protects at least one nontrivial SP phase with 4D, $\frac{1}{2} \otimes \frac{1}{2}$ boundary modes. ■

In this section we have shown that the symmetry group G_2 of the two-qubit coupling Hamiltonian W^{AB} protects a nontrivial SP phase which supports two-qubit boundary modes. Hence the two-qubit gate generated under this Hamiltonian is protected against symmetric perturbations by G_2. In the next section we will use this result, along with the symmetry-protection of the single-qubit operation to find a minimal symmetry requirement for universal quantum computation using only these symmetry-protected gates.

[4]Note the invariant subspaces of dimension $d > 1$ may split up further.

B.2 Symmetry Requirements for Universal Quantum Computation

The process used to generate arbitrary single-qubit gates proposed in [18] requires a continuous, time-dependent embedding of the D_2 symmetry, that protects the single-qubit gate, within the full $SO(3)$ symmetry of the chain. For practical simplicity (and to avoid such considerations) we will now investigate the minimal set of different symmetries required for universal quantum computation with only symmetry-protected gates. Recall from chapter 2 that universal quantum computation can be achieved by generating arbitrary single-qubit gates along with a nontrivial entangling gate.

For the single-qubit gates, our argument relies on the geometric result of applying a pair of π rotations about non-orthogonal axes depicted in Fig. B.3. Applying a π rotation about the $\hat{m}$ axis, followed by a π rotation about the $\hat{m}'$ axis amounts to a total rotation through the angle $[2\ cos^{-1}(\hat{m}\cdot\hat{m}')]$ about the $\hat{m}\times\hat{m}'$ axis. In this way we can perform a smaller rotation of the encoded qubit by applying the one qubit gate described in Eq. A.1.2 twice, picking a different, but fixed, embedding of D_2 for each evolution.

To find exactly what symmetry embeddings are sufficient to simulate any single-qubit unitary transformation efficiently, we make explicit use of the Solovay-Kitaev theorem described in chapter 2. Specifically we will use a corollary of this theorem described in [1] which ensures that any single-qubit unitary can be efficiently decomposed into a product of Hadamard, Phase[5] and $\pi/8$ gates.

Hence we need only generate enough different embeddings of the D_2 symmetry to make performing these three gates possible, through the repeated application of π-rotations. The Hadamard gate can be performed using an embedding with π rotations about the axes $\hat{\mu},\hat{\nu},\hat{y}$, where $\mu = x+z$, $\nu = x-z$. The Phase gate requires a combination of two different embeddings, a suitable choice is given by the standard embedding defined by the $\hat{x},\hat{y},\hat{z}$ axes accompanied by a rotation of this embedding by $(-\pi/4)$ about the $\hat{z}$ axis. Similarly the $\pi/8$ gate can be generated with a combination of two embeddings, using the previously chosen standard embedding defined by the $\hat{x},\hat{y},\hat{z}$ axes along with another rotation of this embedding, this time by $(-\pi/8)$ about the $\hat{z}$ axis, will suffice. So we only need four independent embeddings of $D_2 \subset SO(3)$, as depicted in Fig. B.4, to generate arbitrary single-qubit gates.

Hence we have identified the ability to perform universal quantum computation in this model with the ability to generate four independent embeddings of the D_2 symmetry protecting the single-qubit gate combined with the ability to generate the nontrivial two-qubit entangling gate.

The benefits of finite embedding include ease of implementation, and the ability at each point to tolerate more general perturbations to the Hamiltonian (symmetric under a specific set of π-rotations). However, since we produce a set of gates dense in all rotations

[5]While it may seem peculiar to include the Phase gate since it is not independent, it is included for reasons to do with fault tolerance [1, 74] which relies on Clifford gates that are explicitly generated by the Hadamard and Phase.

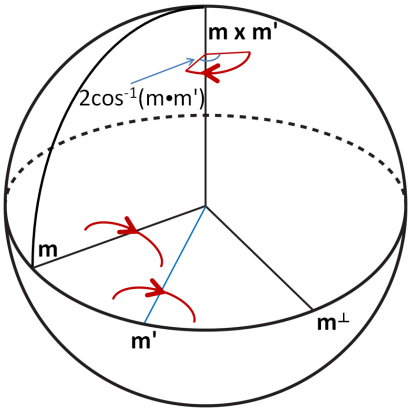


Figure B.3: Result of consecutive π-rotations about two non-orthogonal axes.

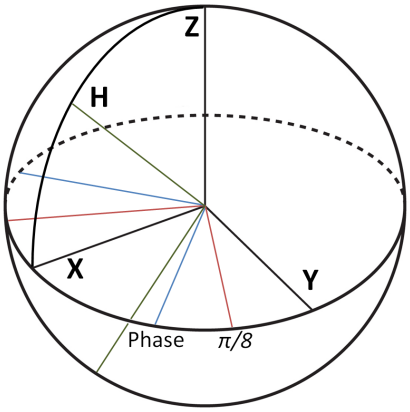


Figure B.4: Finite symmetry embedding requirement for universal quantum computation.

of the encoded qubits, the set of perturbations to the Hamiltonian that are protected against throughout all the single-qubit gates must have a rotation symmetry that is dense in all rotations, essentially the same as the full SO(3) symmetry. This restricted class of SO(3) symmetric errors could be made to correspond to the true physical errors if the computation is done in a way that symmetrically couples the SPAQT to the environment.

Appendix C

Tables of Calculations

$\omega(\downarrow,\rightarrow)$	$I\otimes I$	$\sqrt{Z}\otimes X$	$Z\otimes I$	$\sqrt{Z}^3\otimes X$	$U\otimes U$	$X\otimes\sqrt{Z}$	$V\otimes U$	$Y\otimes\sqrt{Z}$	$I\otimes Z$	$\sqrt{Z}\otimes Y$	$Z\otimes Z$	$\sqrt{Z}^3\otimes Y$	$U\otimes V$	$X\otimes\sqrt{Z}^3$	$V\otimes V$	$Y\otimes\sqrt{Z}^3$
$I\otimes I$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\sqrt{Z}\otimes X$	1	-1	1	1	-1	1	1	1	1	-1	1	1	1	-1	-1	-1
$Z\otimes I$	1	1	-1	-1	1	-1	-1	1	1	1	-1	-1	1	-1	-1	1
$\sqrt{Z}^3\otimes X$	1	1	-1	1	1	-1	1	1	1	1	-1	1	-1	1	-1	-1
$U\otimes U$	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1
$X\otimes\sqrt{Z}$	1	1	1	-1	-1	-1	1	-1	1	1	1	-1	1	1	-1	1
$V\otimes U$	1	1	1	1	1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
$Y\otimes\sqrt{Z}$	1	-1	-1	-1	1	1	1	-1	1	-1	-1	-1	-1	-1	-1	1
$I\otimes Z$	1	-1	1	-1	1	1	1	1	-1	1	-1	1	-1	-1	-1	-1
$\sqrt{Z}\otimes Y$	1	1	1	-1	1	-1	-1	-1	-1	-1	-1	1	1	-1	-1	-1
$Z\otimes Z$	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1	1	1	-1
$\sqrt{Z}^3\otimes Y$	1	-1	-1	-1	-1	1	-1	-1	-1	1	1	1	-1	1	-1	-1
$U\otimes V$	1	-1	-1	1	1	1	-1	-1	1	-1	-1	1	1	1	-1	-1
$X\otimes\sqrt{Z}^3$	1	-1	1	1	1	1	-1	1	-1	1	-1	-1	1	1	-1	1
$V\otimes V$	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1	1	-1
$Y\otimes\sqrt{Z}^3$	1	1	-1	1	-1	-1	-1	1	-1	-1	1	-1	-1	-1	-1	1

Table C.1: Factor system of the $\frac{1}{2}\otimes\frac{1}{2}$ representation of the symmetry group G_2.

Physical action	Group element	Projective representation	Eigenvectors (grouped by eigenvalue)
$(1,1)$	1	$1\otimes 1$	$\underbrace{\lvert1\rangle\lvert1\rangle,\ \lvert1\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle,\ \lvert0\rangle\lvert0\rangle}$
$(\sqrt{R^{\hat{z}}},R^{\hat{x}})$	α	$\frac{1}{\sqrt{2}}(I+i\sigma^{\hat{z}})\otimes i\sigma^{\hat{x}}$	$\lvert1\rangle\lvert1\rangle-\lvert1\rangle\lvert0\rangle,\ \lvert1\rangle\lvert1\rangle+\lvert1\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle-\lvert0\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle+\lvert0\rangle\lvert0\rangle$
$(R^{\hat{z}},1)$	α^2	$i\sigma^{\hat{z}}\otimes\mathrm{I}$	$\underbrace{\lvert1\rangle\lvert1\rangle,\ \lvert1\rangle\lvert0\rangle},\ \underbrace{\lvert0\rangle\lvert1\rangle,\ \lvert0\rangle\lvert0\rangle}$
$(\sqrt{R^{\hat{z}}}^3,R^{\hat{x}})$	α^3	$\frac{-1}{\sqrt{2}}(I-i\sigma^{\hat{z}})\otimes i\sigma^{\hat{x}}$	$\lvert1\rangle\lvert1\rangle-\lvert1\rangle\lvert0\rangle,\ \lvert1\rangle\lvert1\rangle+\lvert1\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle-\lvert0\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle+\lvert0\rangle\lvert0\rangle$
$(R^{\hat{u}},R^{\hat{u}})$	β	$i\sigma^{\hat{u}}\otimes i\sigma^{\hat{u}}$	$\underbrace{\lvert1\rangle\lvert1\rangle-i\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle+\lvert0\rangle\lvert1\rangle},\ \underbrace{\lvert1\rangle\lvert1\rangle+i\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle-\lvert0\rangle\lvert1\rangle}$
$(R^{\hat{x}},\sqrt{R^{\hat{z}}})$	$\alpha\beta$	$i\sigma^{\hat{x}}\otimes\frac{1}{\sqrt{2}}(I+i\sigma^{\hat{z}})$	$\lvert1\rangle\lvert1\rangle-\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert1\rangle+\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert0\rangle-\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle+\lvert0\rangle\lvert0\rangle$
$(R^{\hat{v}},R^{\hat{u}})$	$\alpha^2\beta$	$i\sigma^{\hat{v}}\otimes i\sigma^{\hat{u}}$	$\underbrace{\lvert1\rangle\lvert1\rangle+\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle+i\lvert0\rangle\lvert1\rangle},\ \underbrace{\lvert1\rangle\lvert1\rangle-\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle-i\lvert0\rangle\lvert1\rangle}$
$(R^{\hat{y}},\sqrt{R^{\hat{z}}})$	$\alpha^3\beta$	$i\sigma^{\hat{y}}\otimes\frac{1}{\sqrt{2}}(I+i\sigma^{\hat{z}})$	$\lvert1\rangle\lvert1\rangle-i\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert1\rangle+i\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert0\rangle+i\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle-i\lvert0\rangle\lvert0\rangle$
$(1,R^{\hat{z}})$	$(\alpha\beta)^2$	$I\otimes i\sigma^{\hat{z}}$	$\underbrace{\lvert1\rangle\lvert1\rangle,\ \lvert0\rangle\lvert1\rangle},\ \underbrace{\lvert1\rangle\lvert0\rangle,\ \lvert0\rangle\lvert0\rangle}$
$(\sqrt{R^{\hat{z}}},R^{\hat{y}})$	$(\alpha\beta)^2\alpha$	$\frac{1}{\sqrt{2}}(I+i\sigma^{z})\otimes i\sigma^{\hat{y}}$	$\lvert1\rangle\lvert1\rangle-i\lvert1\rangle\lvert0\rangle,\ \lvert1\rangle\lvert1\rangle+i\lvert1\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle-i\lvert0\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle+i\lvert0\rangle\lvert0\rangle$
$(R^{\hat{z}},R^{\hat{z}})$	$(\alpha\beta)^2\alpha^2$	$i\sigma^{\hat{z}}\otimes i\sigma^{\hat{z}}$	$\underbrace{\lvert1\rangle\lvert1\rangle,\ \lvert0\rangle\lvert0\rangle},\ \underbrace{\lvert1\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle}$
$(\sqrt{R^{\hat{z}}}^3,R^{\hat{y}})$	$(\alpha\beta)^2\alpha^3$	$\frac{-1}{\sqrt{2}}(I-i\sigma^{\hat{z}})\otimes i\sigma^{\hat{y}}$	$\lvert1\rangle\lvert1\rangle-i\lvert1\rangle\lvert0\rangle,\ \lvert1\rangle\lvert1\rangle+i\lvert1\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle-i\lvert0\rangle\lvert0\rangle,\ \lvert0\rangle\lvert1\rangle+i\lvert0\rangle\lvert0\rangle$
$(R^{\hat{u}},R^{\hat{v}})$	$(\alpha\beta)^2\beta$	$i\sigma^{\hat{u}}\otimes i\sigma^{\hat{v}}$	$\underbrace{\lvert1\rangle\lvert1\rangle+\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle-i\lvert0\rangle\lvert1\rangle},\ \underbrace{\lvert1\rangle\lvert1\rangle-\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle+i\lvert0\rangle\lvert1\rangle}$
$(R^{\hat{x}},\sqrt{R^{\hat{z}}}^3)$	$(\alpha\beta)^2\alpha\beta$	$i\sigma^{\hat{x}}\otimes\frac{-1}{\sqrt{2}}(I-i\sigma^{\hat{z}})$	$\lvert1\rangle\lvert1\rangle-\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert1\rangle+\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert0\rangle-\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle+\lvert0\rangle\lvert0\rangle$
$(R^{\hat{v}},R^{\hat{v}})$	$(\alpha\beta)^2\alpha^2\beta$	$i\sigma^{\hat{v}}\otimes i\sigma^{\hat{v}}$	$\underbrace{\lvert1\rangle\lvert1\rangle+i\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle+\lvert0\rangle\lvert1\rangle},\ \underbrace{\lvert1\rangle\lvert1\rangle-i\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle-\lvert0\rangle\lvert1\rangle}$
$(R^{\hat{y}},\sqrt{R^{\hat{z}}}^3)$	$(\alpha\beta)^2\alpha^3\beta$	$i\sigma^{\hat{y}}\otimes\frac{-1}{\sqrt{2}}(I-i\sigma^{\hat{z}})$	$\lvert1\rangle\lvert1\rangle-i\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert1\rangle+i\lvert0\rangle\lvert1\rangle,\ \lvert1\rangle\lvert0\rangle+i\lvert0\rangle\lvert0\rangle,\ \lvert1\rangle\lvert0\rangle-i\lvert0\rangle\lvert0\rangle$

Table C.2: Two descriptions of the symmetry group G_2, its $\frac{1}{2}\otimes\frac{1}{2}$ projective representation and the eigenstates thereof.

Measurement outcome	Associated unitary
$\|00\rangle$	$Z \otimes Z$
$\|1y\rangle - \|-1x\rangle$	$CZ(X \otimes X)$
$\|1y\rangle + \|-1x\rangle$	$CZ(X \otimes iY)$
$\|-1y\rangle - \|1x\rangle$	$CZ(iY \otimes X)$
$\|-1y\rangle + \|1x\rangle$	$CZ(Y \otimes Y)$
A basis for the remaining elements	Matrix action generated
$\|0-1\rangle - \|01\rangle$	$Z \otimes X$
$\|0-1\rangle + \|01\rangle$	$Z \otimes iY$
$\|-10\rangle - \|10\rangle$	$X \otimes Z$
$\|-10\rangle + \|10\rangle$	$iY \otimes Z$
A different choice for remaining basis	Matrices in $\{\|0\rangle, \|1\rangle\}^{\otimes 2}$ basis
$\|zx\rangle + \|yz\rangle$	$U_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$
$\|zx\rangle - \|yz\rangle$	$U_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$
$\|zy\rangle + \|xz\rangle$	$U_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$
$\|zy\rangle - \|xz\rangle$	$U_4 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$

Table C.3: Unitary two-qubit gates induced on the correlation space of two spin-1 chains by the measurements listed.

Appendix D

Definition: Semidirect Product

Here we will give the precise definition of the semidirect product, a standard notion from elementary group theory. We include this definition for ease of access for those unacquainted with group theory, to assist in understanding the discussion of Chapter 4.

The semidirect product of $N \lhd G$, with $H \leq G$ by the function φ is written as $N \rtimes_\varphi H$. As a set it contains the same elements as the direct product $N \times H$ but as a group it possesses a different structure, defined by the multiplication rule

$$(n_1, h_1) \times (n_2, h_2) = (n_1 \ \varphi_{h_1}(n_2), h_1 h_2)\,, \quad n_1, n_2 \in N, \quad h_1, h_2 \in H \tag{D.0.1}$$

where $\varphi : H \to Aut(N)$ ($Aut(N)$ is the group of isomorphisms from $N \to N$). This map takes $h \overset{\varphi}{\mapsto} \varphi_h : H \to H$, and is a group homomorphism (i.e. $\varphi(h_1 h_2) = \varphi(h_1)\varphi(h_2)$). The semidirect product is specified by this function φ and an important case is the one in which $\varphi : H \to Inn(N) \subset Aut(N)$ where $Inn(N)$ is the group of automorphisms of N which act via conjugation by a fixed element of N. The most common choice is $\varphi_h(g) = hgh^{-1}$, but it is important to be aware of the freedom in this definition as it has arisen in the calculations of Chapter 4.

Appendix E

Spin Operator Exponentials

The representation of the rotation group $SO(3)$ in a qubit system that was described in Sec. 1.1, extends naturally to spin systems of any magnitude. Upon identifying the qubit with a spin-1/2 system in the usual way, $|0\rangle \mapsto |\uparrow\rangle$, $|1\rangle \mapsto |\downarrow\rangle$, there is a natural connection between the angular momentum operators $S = (S^{\hat{x}}, S^{\hat{y}}, S^{\hat{z}})$ and the Pauli matrices, which is $\vec{S} = \frac{1}{2}\vec{\sigma}$. Using this relation, we can identify the standard representation of the $SO(3)$ rotation group on spin-1/2 particles by representing a rotation through θ about an axis $\hat{m}$ with the operator $\exp(i\theta S^{\hat{m}})$. This identification generalizes to give a representation of $SO(3)$ on spin systems of all magnitudes, by simply replacing the spin-1/2 angular momentum operators $\vec{S}$ with those for the larger spin.

The expansion of an exponential of a spin operator in terms of powers of said operator will be explicitly calculated below for spin-$\frac{1}{2}$ and spin-1. We make use of the cyclic nature of the sequence generated by taking integer powers of the spin operators, for spin-1/2 we have $(\sigma^{\hat{m}})^2 = 1$, while for spin-1 we have $(S^{\hat{m}})^3 = S^{\hat{m}}$. Which lead us to the relations

$$\exp(i\theta\sigma^{\hat{m}}) = \sum_{n=1}^{\infty} \frac{(i\theta\sigma^{\hat{m}})^n}{n!} \tag{E.0.1}$$

$$= \left(\sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!}\right) I + i\left(\sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!}\right)\sigma^{\hat{m}} \tag{E.0.2}$$

$$= \cos(\theta)\ I + i\sin(\theta)\ \sigma^{\hat{m}} \tag{E.0.3}$$

for spin-1/2 (where $\sigma^{\hat{m}}$ are given in terms of the Pauli matrices from Eq. 1.1.2) and

$$\exp(i\theta S^{\hat{m}}) = \sum_{n=1}^{\infty} \frac{(iS^{\hat{m}})^n}{n!} \tag{E.0.4}$$

$$= I + i\left(\sum_{n=0}^{\infty}(-1)^n \frac{\theta^{2n+1}}{(2n+1)!}\right) S^{\hat{m}} + \left(\sum_{n=1}^{\infty}(-1)^n \frac{\theta^{2n}}{(2n)!}\right)(S^{\hat{m}})^2 \tag{E.0.5}$$

$$= I + i\sin(\theta)\ S^{\hat{m}} + (\cos(\theta) - 1)(S^{\hat{m}})^2 \tag{E.0.6}$$

for spin-1, where $S^{\hat{m}}$ are given in terms of the spin 1 angular momentum operators

$$S^{\hat{x}} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \qquad S^{\hat{y}} = \begin{bmatrix} 0 & \frac{-i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & \frac{-i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{bmatrix} \qquad S^{\hat{z}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{E.0.7}$$

Appendix F

Adiabatic Theorem and Geometric Phases

F.1 The Adiabatic Theorem

Here we will present a simple version of the adiabatic theorem appropriated from [17]. We restrict our attention to evolution under time dependent Hamiltonians that vary slowly with time, the solutions of the Schrödinger equation are approximately given by the time independent eigenfunctions of the instantaneous Hamiltonian at each point in time. i.e. the solutions to the equation $i\hbar\frac{\partial}{\partial t}|\Psi_n(t)\rangle = H(t)|\Psi_n(t)\rangle$ with initial condition $|\Psi_n(0)\rangle = |n(0)\rangle$ are approximately the same as those for the continuous family of stationary Schrödinger equations $E_n(t)|n(t)\rangle = H(t)|n(t)\rangle$. To examine precisely how well this approximation holds we expand the solution

$$|\Psi_n(t)\rangle = \sum_n a_n(t) \exp\left[\frac{1}{i\hbar}\int_0^t E_n(z)dz\right]|n(t)\rangle \tag{F.1.1}$$

(note the $|n\rangle$ are orthonormal since they are instantaneously eigenvalues of a hermitian matrix and we further assume they are non-degenerate for simplicity). Substituting this expansion into the Schrödinger equation gives

$$\sum_n \left[\dot{a}_n|n(t)\rangle + a_n\frac{\partial}{\partial t}|n(t)\rangle\right]\exp\left[\frac{1}{i\hbar}\int_0^t E_n(z)dz\right] = 0 \tag{F.1.2}$$

where the condition that the states $|n(t)\rangle$ are eigenvalues of the Hamiltonian is used to cancel the terms originating from the derivative of the exponential in Eq. F.1.1 (the $\dot{()}$ denotes the time derivative). Acting upon Eq. F.1.2 with the functional $\langle k(t)|$ and dividing by $\exp[\frac{1}{i\hbar}\int_0^t E_k(z)dz]$ leads to

$$\dot{a}_k = -\sum_n a_n \left\langle k(t)\left|\frac{\partial}{\partial t}\right|n(t)\right\rangle \exp\left[\frac{1}{i\hbar}\int_0^t (E_n - E_k)dz\right] = 0 \tag{F.1.3}$$

where we have used the orthogonality of the $|n\rangle$ states. To find $\langle k|\dot{n}\rangle$ we differentiate the stationary Schrödinger equation with respect to time and act on the resulting equation with the functional $\langle k(t)|$ (for $k \neq n$)

$$\begin{aligned}\left\langle k\left|\frac{\partial E_n}{\partial t}\right|n\right\rangle + \langle k|E_n|\dot{n}\rangle &= \left\langle k\left|\frac{\partial H}{\partial t}\right|n\right\rangle + \langle k|H|\dot{n}\rangle \\ \implies \left\langle k\left|\frac{\partial H}{\partial t}\right|n\right\rangle &= (E_n - E_k)\langle k|\dot{n}\rangle, \qquad k \neq n\end{aligned} \tag{F.1.4}$$

where we have used the Hermitian property of H to obtain the E_k eigenvalue. To determine the value of the remaining case $\langle n|\dot{n}\rangle$ we note that the normalization condition $\langle n|n\rangle = 1$, implies: $\langle \dot{n}|n\rangle + \langle n|\dot{n}\rangle = 0$ i.e. $2\Re e\left(\langle n|\dot{n}\rangle\right) = 0$. Hence we can write $\langle n|\dot{n}\rangle = i\alpha(t)$ for some real function α. Since we are free to multiply each element of the orthonormal basis by a phase factor while retaining the orthonormality property we define a new basis via the relation $|n'\rangle = e^{i\gamma(t)}|n\rangle$. The new element $\langle n'|\dot{n}'\rangle = \langle n|\dot{n}\rangle + i\dot{\gamma}(t) = i(\alpha + \dot{\gamma})$ has some freedom in its value, for the particular choice $\gamma(t) = -\int_0^t \alpha(z)dz$ we set $\langle n'|\dot{n}'\rangle = 0$. Now assuming we have a basis in which the condition $\langle n|\dot{n}\rangle = 0$ holds we can rewrite Eq F.1.3 without any inner products between eigenfunctions and their derivatives

$$\dot{a}_k = \sum_{n\neq k} \frac{a_n}{\hbar\omega_{kn}} \left\langle k\left|\frac{\partial H}{\partial t}\right|n\right\rangle \exp\left[i\int_0^t \omega_{kn}(z)dz\right] \tag{F.1.5}$$

where $\hbar\omega_{kn} = E_k - E_n$. We can apply the same argument to any $\dot{a}_k$ and hence determine the solution $|\Psi\rangle$ by solving this set of differential equations for all k. This is in general just as hard as finding a solution to the Schrödinger equation itself, so we make the adiabatic approximation by assuming that the variables $(a_n, \omega_{kn}, \frac{\partial H}{\partial t}, |n(t)\rangle)$ on the right hand side of the differential equation for $\dot{a}_k$ vary so slowly with time that they can be replaced by constants (ignoring higher order terms). This allows the Eq F.1.5 to be easily integrated, and for the solution $|\Psi_m\rangle$ initially in the state $|m(0)\rangle$ we have $a_n(0) = \delta_{mn}$ and hence

$$a_k(t) \approx \frac{-i}{\hbar\omega_{km}^2} \left\langle k\left|\frac{\partial H}{\partial t}\right|m\right\rangle \left[e^{i\omega_{km}t} - 1\right], \qquad k \neq m \tag{F.1.6}$$

neglecting higher order terms in the time derivative. This shows that in our approximation the contribution of the eigenvectors that the state is not initialized in does not grow with time, but oscillates with a magnitude proportional to the change in the Hamiltonian ΔH over the Bohr period $2\pi/\omega_{km}$ divided by the energy gap $E_k - E_n$.

F.2 Geometric Berrys phase

In this section we will describe a particular type of geometric phase [19] called Berry's phase [22] that arises in the context of holonomic, adiabatic evolutions of a quantum system. Under the same assumptions we have used above for the description of the adiabatic theorem we have

$$|\Psi_n(t)\rangle = e^{i\gamma_n(t)} \exp\left[\frac{1}{i\hbar}\int_0^t E_n(\mathbf{R}(z))dz\right] |n(\mathbf{R}(t))\rangle \tag{F.2.1}$$

for a sufficiently large gap between the energy levels of the Hamiltonian compared to its rate of change, where $\mathbf{R}$ is the coordinates of a manifold of control parameters which specify the Hamiltonian H. The second phase factor corresponds to a simple dynamical phase, while the first phase factor is in general non-integrable and hence cannot be written in a closed form as a function of $\mathbf{R}$ and importantly can take different values when

continued around a closed loop in parameter space $\mathbf{R}$. The requirement that $|\Psi_n\rangle$ satisfies the Schrödinger equation implies

$$i\dot{\gamma}_n(t)|n(\mathbf{R}(t))\rangle + \frac{\partial}{\partial t}|n(\mathbf{R}(t))\rangle = 0 \tag{F.2.2}$$

$$\implies \dot{\gamma}_n(t)|n(\mathbf{R}(t))\rangle = i\nabla_{\mathbf{R}}|n(\mathbf{R}(t))\rangle \cdot \frac{\partial \mathbf{R}}{\partial t} \tag{F.2.3}$$

acting on this with the functional $\langle n(\mathbf{R}(t))|$ we obtain $\dot{\gamma}_n(t) = i\langle n(\mathbf{R}(t))|\nabla_{\mathbf{R}}|n(\mathbf{R}(t))\rangle \cdot \dot{\mathbf{R}}$. Upon tracing out a closed loop l in parameter space with $\mathbf{R}(T) = \mathbf{R}(0)$ we obtain for the evolution of the solution

$$|\Psi_n(t)\rangle = \exp\left[i\gamma_n(l)\right] \exp\left[\frac{1}{i\hbar}\int_0^T E_n(\mathbf{R}(z))dz\right]|\Psi_n(0)\rangle \tag{F.2.4}$$

where the geometric phase $\gamma_n(l) = i\oint_l \langle n(\mathbf{R})|\nabla_{\mathbf{R}}|n(\mathbf{R})\rangle \cdot d\mathbf{R}$ is dependent only on the loop l and not the parameterization of the loop (provided that it is slow enough to satisfy the requirements for the adiabatic theorem to hold). Note that the normalization $\langle n|n\rangle = 1$ implies that the quantity $\langle n|\nabla_{\mathbf{R}}|n\rangle$ must be purely imaginary by the same argument given above for the case of $\langle n|\dot{n}\rangle$ and consequently $\gamma_n(l)$ is real (as must be the case for $|\Psi_n(t)\rangle$ to remain normalized). Assuming the simplest case in which the parameter space is two dimensional we can apply Stokes's theorem to the path integral for γ_n to obtain

$$\gamma_n(l) = -\iint_l d\mathbf{A} \cdot \mathbf{V}_n(\mathbf{R}) \tag{F.2.5}$$

$$\mathbf{V}_n = \Im m \sum_{m \neq n} \frac{\langle n|\nabla H|m\rangle \times \langle m|\nabla H|n\rangle}{(E_m - E_n)^2} \tag{F.2.6}$$

note that this definition is independent of the choice of phase for the basis $|n\rangle$ hence $\mathbf{V}_n$ is independent under the transformation $|n\rangle \to e^{i\mu}|n'\rangle$ which induces $\langle n|\nabla n\rangle \to \langle n'|\nabla n'\rangle + i\nabla\mu$. The invariance of $\mathbf{V}_n$ under such a gauge transformation is elucidated in [23] in which an anti-Hermitian matrix A is defined and shown to act as a gauge potential (note A is different to $\mathbf{V}_n$). Although this argument only holds verbatim in two dimensions, it is easily generalized to an arbitrary dimensional parameter space in which the integrand is a 2-form obtained by replacing ∇ by the exterior derivative d and the cross product $\times$ by the wedge product $\wedge$ [19, 22].

F.3 Generalised non-Abelian Berry's phase and the Wilczek-Zee Potential

In his derivation Berry considered only the case of nondegenerate eigenspaces in which case the effect of the cyclic evolution is the phase of his own namesake, this case explains simple phenomena such as the sign change of a wavefunction of a real, Hermitian Hamiltonian as it is driven in a path around a degeneracy point [94]. If we generalise the situation we are considering to allow for Hamiltonians with degenerate eigenspaces (whose degeneracy

structure does not change with time) the cyclic, adiabatic evolution now leads to a unitary transformation within each subspace which can be more general than multiplication by a phase factor [23]. We consider the generic case in which we have a degenerate 0-eigenspace of a family of Hamiltonians parametrized by $\vec{\lambda}(t)$ spanned by a basis $|0_a(t)\rangle$ such that $H(\vec{\lambda}(t))\,|0_a(t)\rangle = 0$. Then we can write the solution to the Schrödinger equation with the initial condition $|\Psi_a(0)\rangle = |0_a(0)\rangle$ as $|\Psi_a(t)\rangle = \sum_b U_{ab}(t)\,|0_b(t)\rangle$ (where we have neglected higher order terms in the adiabatic approximation). The requirement that $|\Psi_a(t)\rangle$ remains normalized determines U via the equation

$$\left[U^{-1}\dot{U}\right]_{ab} = \left\langle 0_b(t)\middle|\dot{0}_a(t)\right\rangle := A_{ab} \tag{F.3.1}$$

where A is an anti Hermitian matrix (due to the normalization of $|0_a\rangle$) which acts as a gauge potential called the Wilczek-Zee potential. The operator equation Eq F.3.1 is easily solved in analogy with the case of the exponential differential equation, to give

$$U(t) = \mathbf{P}\exp\left[\int_0^t A(z)dz\right] = \lim_{N\to\infty}\left[e^{\epsilon A(t_N)}\ldots e^{\epsilon A(t_1)}e^{\epsilon A(t_0)}\right] \tag{F.3.2}$$

where $\epsilon = t/(N+1)$ and $t_i \in [i\epsilon,(i+1)\epsilon]$ and the $\mathbf{P}$ denotes path ordering of the exponential since the matrix A does not necessarily commute with itself at two different points in time. Now if we make the more general transformation via a change of basis $|\Psi'(t)\rangle = \Omega(t)\,|\Psi(t)\rangle$, then we induce the transformation $A'(t) = \dot{\Omega}\Omega^{-1} + \Omega A\Omega^{-1}$ which confirms that A transforms appropriately as a gauge potential. For the cases considered by Berry [22, 94] the A corresponds to a $U(1)$ or ($\mathbb{Z}_2$ for a real Hamiltonian) gauge field which generates the phase factor accumulated by traversing a nontrivial, closed loop. A separate generalisation of Berry's phase has also been considered in which the adiabaticity requirement is dropped [95] (in which the Berry phase is considered to be a geometric property of the projective Hilbert space rather than the parameter space) and further generalisation to paths which are not necessarily closed is given in [20].

Appendix G

Group Cohomology

The classification of projective representations according to their factor classes, given in Chapter 2, is an instance of group cohomolgy theory. This theory provides a theoretical framework to study functions from n copies of a group G to $U(1)$. This type of theory has made an appearance in theoretical physics before [96, 97], in the context of studying symmetries that define the fundamental particles and their interactions. We begin by defining the Abelian group of n-cochains $C^n(G, U(1))$ to consist of all functions from $G^n \to U(1)$. Together with a boundary map $d^n : C^n(G, U(1)) \to C^{n+1}(G, U(1))$ defined by

$$\begin{aligned}(d^n f)(g_1, g_2, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1}) \\ + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n)\end{aligned} \tag{G.0.1}$$

note in all the cases we will consider, G acts trivially on $U(1)$ hence $g_1 \cdot \equiv 1$ in the above equation. A detailed calculation yields the fact that $d^{n+1} \circ d^n = 0$ which allows us to define the sequence

$$C^1(G, U(1)) \xrightarrow{d^1} C^2(G, U(1)) \xrightarrow{d^2} \ldots \xrightarrow{d^{n-1}} C^n(G, U(1)) \xrightarrow{d^n} \ldots \tag{G.0.2}$$

then we define maps in the kernel of d^n to be n-cocycles, i.e. the group $Z^n(G, U(1)) = ker(d^n) \subset C^n(G, U(1))$. In addition we define functions in the image of d^{n-1} to be the n-coboundaries $B^n(G, U(1)) = Im(d^{n-1}) \subset C^n(G, U(1))$. Then a consequence of $d^n \circ d^{n-1} = 0$ is that $B^n(G, U(1)) \subset Z^n(G, U(1))$ and in light of this we define the nth cohomology group $H^n(G, U(1)) = B^n(G, U(1))/Z^n(G, U(1))$

The example of interest in projective representations and 1D SPTO is $H^2(g, U(1))$, to this end we calculate $(d^1\beta)(g_1, g_2) = \beta(g_2) - \beta(g_1 g_2) + \beta(g_1)$ and $(d^2\omega)(g_1, g_2, g_3) = \omega(g_2, g_3) - \omega(g_1 g_2, g_3) + \omega(g_1, g_2 g_3) - \omega(g_1, g_2)$. Once we identify the additive notation with the multiplicative notation above (since the groups are Abelian they are equivalent) we can solve the conditions for coboundaries $(d^1\beta(g_1, g_2) = \beta(g_2)\beta(g_1)\beta(g_1 g_2)^{-1}$ and for the cocycles: $(d^2\omega)(g_1, g_2, g_3) = 1 \implies \omega(g_2, g_3)\omega(g_1, g_2 g_3) = \omega(g_1, g_2)\omega(g_1 g_2, g_3)$ and so we explicitly see that elements $[\omega]$ of the cohomology group $H^2(G, U(1))$ are cosets $\{\omega' | \omega'(g_1, g_2) = \beta(g_2)\beta(g_1)\beta(g_1 g_2)^{-1}\omega(g_1, g_2)\}$ corresponding to the equivalence classes of factor systems $[\omega]$ for the projective representations.

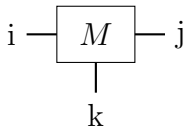
Appendix H

Matrix-Product State Representations

The matrix-product state ansatz is a useful tool for understanding the entanglement structure of 1D ground states of gapped Hamiltonians. These states first arose in the context of a numerical algorithm called the density matrix renormalization group (DMRG) [98] that proved very successful in accurately and efficiently approximating the ground state of any gapped, short-range, spin chain Hamiltonian. The success of DMRG has more recently been identified with the fact that it effectively uses a MPS ansatz for the ground state it is trying to find. Theoretical results [99] have since shown that MPS states can be used to approximate any ground state of a gapped, short-range, spin chain Hamiltonian to within an arbitrarily small error. The MPS ansatz has also been applied to the classification of quantum phases in 1D [71, 100, 101], a particular instance of this is described in Appendix I.

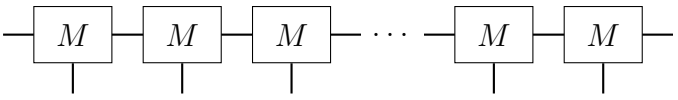
The MPS ansatz associates a set of matrices to each physical site in a spin chain, one for each basis vector in the Hilbert space $\mathcal{H}_i$ of the site. We can think of a matrix as an object M_{ij} with two indices that must be specified to produce a complex value, now a set of these matrices indexed by $k \in I$ can be written $\{M_k\}_{k\in I} = M_{ij}[k]$ which corresponds to a tensor with three indices that must be specified to produce a complex value.

We can now use Penrose's graphical notation for tensor networks to draw a simple object corresponding to the tensor $M_{ij}[k]$.



(H.0.1)

Where the vertical line corresponds to the physical index of dimension d and the two horizontal lines, the matrix indices of dimension D. We can compose such diagrams in a fashion analogous to matrix multiplication, by contracting two indices, which corresponds to summing over these indices simultaneously. Then upon repeating this process N times, we are left with N physical indices which we interpret as the state of the N spin chain.

$|\Psi\rangle =$ 

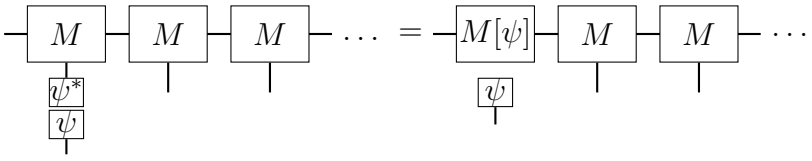
(H.0.2)

This MPS state can be written as

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N} [M_{i_1} M_{i_2} \ldots M_{i_N}] \, |i_1 i_2 \ldots i_N\rangle \tag{H.0.3}$$

In this picture notice we have two free indices of dimension D corresponding to the open boundary conditions, we can think of these as representing the state encoded in the correlation space of the ground state, due to its entanglement. In particular we can loosely interpret this degree of freedom as the edge mode of a SPTO phase (as discussed above) if the MPS represents such a state.

This pictorial representation of the state gives us a clear intuition about the evolution induced by a single site measurement, if we consider an edge state initialized in the $|0\rangle$ state, then upon measuring the first site we have

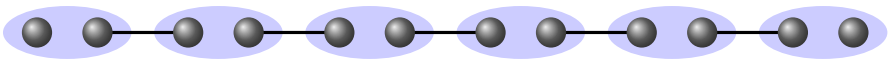


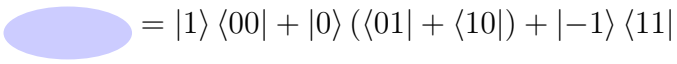
(H.0.4)

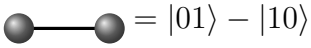
which corresponds to removing the measured spin from the chain since they are no longer entangled. Additionally we have induced a transformation in correlation space upon the initial $|0\rangle$ state determined by the result of the measurement, which is contracted with the physical index of the first tensor.

H.0.1 Example: the AKLT State

As an example we consider the AKLT state, which exists within the SPTO Haldane phase of the spin-1 chain. The AKLT state is explicitly constructed by projecting pairs of spin-$\frac{1}{2}$ particles, that are each in a singlet state with two more spin-$\frac{1}{2}$ particles, onto the triplet subspace corresponding to a spin-1 particle. This is most clearly captured by a diagram







The exact MPS representation of the AKLT state is specified by its matrices in the $|x\rangle = \frac{1}{\sqrt{2}}(|-1\rangle - |1\rangle), |y\rangle = \frac{1}{\sqrt{2}}(|-1\rangle + |1\rangle), |z\rangle = |0\rangle$ basis[1], $M[x] = X$, $M[y] = -iY$, $M[z] = Z$. It is easy to see that any measurement in this basis will correspond to a unitary evolution of the encoded qubit by a Pauli matrix.

[1] The $|x\rangle, |y\rangle, |z\rangle$ basis consists of the unique 0 eigenstates of the $S^{\hat{x}}, S^{\hat{y}}$ and $S^{\hat{z}}$ operators respectively

Furthermore, by measuring in the $S^{\hat{z}}$ basis $|1\rangle, |0\rangle, |-1\rangle$ we can induce a non-deterministic initialization of the encoded qubit [29]. Since $M[-1] = M[x+y] = |1\rangle\langle 0|$, $M[0] = Z$, $M[1] = |0\rangle\langle 1|$, a measurement result of 1 (-1) initializes the encoded qubit in the state $|0\rangle$ $(|1\rangle)$, while a result of 0 induces a unitary evolution on the encoded qubit. So after a small number of measurements the qubit has a high probability of being initialized in either the $|0\rangle$ or $|1\rangle$ state.

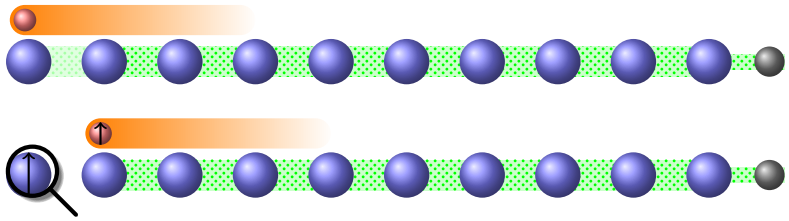


Figure H.1: Diagram of initialization of the encoded qubit by measuring a physical spin-1 in the $\hat{z}$ basis.

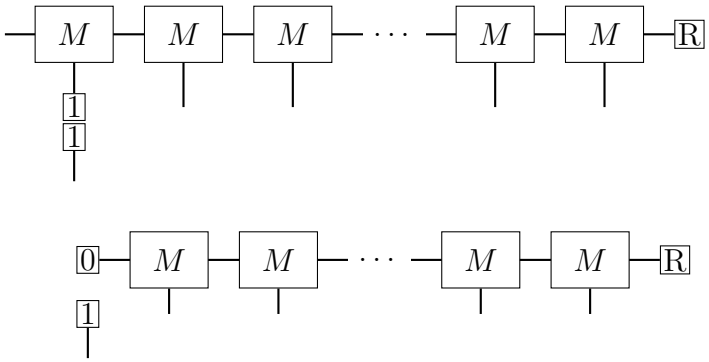


Figure H.2: The corresponding process in the MPS notation

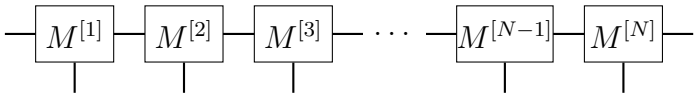
Appendix I

Chen-Gu-Wen Classification of SP Order

In this appendix we will look at an explicit classification of 1D SP ordered phases in the presence of an on-site symmetry [61]. This classification makes use of a renormalization procedure which causes any state within a SP phase to flow to a fixed point state, which can be used to characterize the phase. The arguments used to classify the ground states rely heavily on the matrix-product state (MPS) ansatz, which produces good approximations to short-range, gapped ground states.

Solving for the exact ground state of strongly interacting many-body systems is a hard task, the MPS formalism allows us to efficiently approximate gapped 1D spin systems to a desired accuracy. This description of the ground states will be used here to classify the SP ordered phases. An MPS ground state can be written

$$|\Psi\rangle = \sum_{i_1,i_2,\ldots,i_N} \left[M^{[1]}_{i_1} M^{[2]}_{i_2} \ldots M^{[N]}_{i_N} \right] |i_1 i_2 \ldots i_N\rangle \tag{I.0.1}$$

$|\Psi\rangle =$ 

(I.0.2)

where i_j indexes the d different physical states on-site $[j]$. For a fixed k, $M^{[j]}_k$ is a $D \times D$ matrix, where D is called the inner dimension, which is used to describe the entanglement between site $[j]$ and the rest of the state. To study the structure of this entanglement we can use the quantity called the double tensor $\mathbb{E}^{[j]}$, formed by tracing out the physical dimension, i_j, of the site:

$$\mathbb{E}^{[j]} = \sum_{i_j} M^{[j]}_{i_j} \otimes \left(M^{[j]}_{i_j} \right)^* \tag{I.0.3}$$

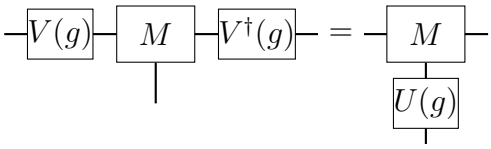
It is important to note that $\mathbb{E}^{[j]}$ uniquely determines $M^{[j]}_{i_j}$ up to a change of basis on the site. So we have a unitary freedom upon decomposing $\mathbb{E}^{[j]}$ to recover the tensor describing a site in the MPS. We will now use this freedom, along with the blocking of sites in the MPS to form a renormalization flow that takes all the states within a SP phase to a fixed point that possesses a simple entanglement structure which defines the phase.

The renormalization procedure is simply done by taking n sites which we will label 1 to n for convenience. Note these sites do not correspond to the physical sites $i_1, i_2, \ldots, i_n$ and hence we label them differently $l_1, \ldots, l_n$ to highlight this. The double tensor from each site is then combined to form a single double tensor $\tilde{\mathbb{E}} = \mathbb{E}_1 \ldots \mathbb{E}_n$. We then harness the

unitary freedom in decomposing the double tensor to remove all the entanglement within the block and in doing so we totally mix up the physical labels $l_1, \ldots, l_n$ for the sites. So we adopt a combined label $\tilde{l}$ which runs over all the physical states on the n spins. Finally we discard all the states within $\tilde{l}$ which are irrelevant in describing the entanglement of the block, these can be thought of as internal degrees of freedom which we combine to reduce the dimension of the block without changing its entanglement structure. This fully describes a single step in the renormalization procedure, for a more precise description see [61].

This renormalization procedure is easiest to analyze for an infinite chain of spins. In that case we can think of the renormalization procedure as taking the matrices on each site $\left(M_{i_j}^{[j]}\right)_0$ and blocking n sites together to form an effective site described by the renormalized matrices $\left(M_{i_j}^{[j]}\right)_1$. Repeating this induces a renormalization flow on all the states within the phase, which could have a fix point $\left(M_{i_j}^{[j]}\right)_\infty$ that can then be used to characterize the entanglement structure which characterizes the entire phase. Recall that this state can be transformed to any other within the SP phase by symmetry preserving, local unitary transformations which do not alter the underlying entanglement structure. As an aside, note that the authors of [61] use this procedure to show that there can be no long-range topological order in 1D since the fix point states can always be disentangled into a product state by local unitaries if there is no symmetry present.

Now in the presence of a symmetry the renormalization procedure works analogously, but with the additional constraint that the symmetry is enforced at each step, for details see [61]. The fixed point state found by this renormalization flow has MPS tensors which can be shown to satisfy a symmetry condition which is most easily represented graphically



(I.0.4)

and correspondingly the physical sites can be thought of as consisting of pairs of virtual spins identified with the correlation space of the MPS tensors, which are each entangled with another virtual spin on a neighboring site. Again this structure is most easily captured with a diagram, see Fig. I.1. Then the Linear representation of the symmetry $U(g)$

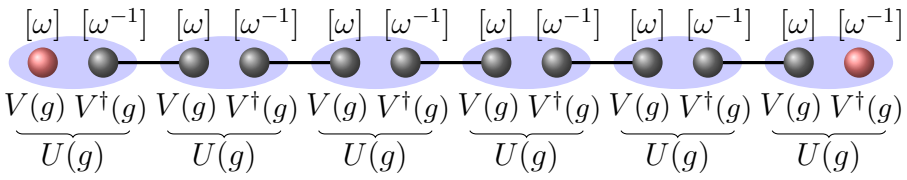


Figure I.1: The structure of the fix point state which characterizes a SP phase, with gapless edge degrees of freedom (red).

on each site splits up into a product of projective representations $V(g)$ and $V^\dagger(g)$ acting

on the virtual spins with opposite factor classes $[\omega]$ and $[\omega^{-1}]$. Hence these virtual degrees of freedom are labeled by the second cohomology group $H^2(G, U(1))$.

In the presence of open boundary conditions this state then has two virtual spin degrees of freedom, one depicted in red at each boundary, which are not in any entangled pair. These degrees of freedom correspond to the degeneracy in the ground subspace of the fix point state projectively and transform projectively under the symmetry group G with opposite factor classes $[\omega]$ and $[\omega^{-1}]$ and hence they can be labeled by the second cohomology group $H^2(G, U(1))$. Upon transforming this fix point state to any other state in the phase by a local unitary, these edge modes can be spread out over only a finite distance, corresponding to the causal cone of the edge in the finite depth quantum circuit corresponding to this local unitary, depicted in Fig. 3.2. Hence the edge modes, and the corresponding ground state degeneracy, are protected by the symmetry of the phase, and will persist so long as perturbations to the Hamiltonian respect the symmetry and do not cause a phase transition. We label the edge modes and the different 1D SP phases by the second cohomology group $H^2(G, U(1))$.

A simple example of this type of fix point state is the AKLT point which can be thought of as possessing the entanglement structure that defines the SP order of the Haldane phase.

Here we have seen a direct derivation of the edge modes, how they correspond to a ground state degeneracy and why they are protected by the symmetry of the phase.