# Alternating quiver Hecke algebras

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# Contents

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Chapter 1

# Introduction

Of all classical topics in algebra, there is none of greater ubiquity, beauty and elegance than the theory of the symmetric groups. Symmetric groups are of utmost importance to mathematics because any finite group embeds inside some symmetric group. The symmetric group also lies at the crossroads of Lie theory, combinatorics, and mathematical physics. It is of course a prevailing and intelligent paradigm to study finite groups via their *actions* on vector spaces – this leads to group representations, and establishes the mathematical discipline of representation theory, to which we humbly submit some contributions in this thesis.

Representations of symmetric groups have been studied for over a century. In the early 1900s, Young [**107**] classified the irreducible representations of symmetric groups over the complex numbers and provided elegant combinatorial rules for restriction and induction; his combinatorics of partitions and tableaux are still used extensively today. Describing the modular representations of symmetric groups, that is, their representations over fields whose characteristic divides the rank of the symmetric group, is a much harder problem. Although considerable progress has been made in the past forty years since a landmark paper of James [**48**], culminating with Kleshchev's modular branching rules for symmetric groups [**55**, **56**, **57**, **58**], many important questions in this area remain unanswered, including long-sought combinatorial formulas for dimensions of simple modules.

Later developments led to the introduction of Iwahori-Hecke algebras, a family of $\mathbb{C}$-algebras dependent on a parameter $q$ which can be specialised to take values in $\mathbb{C}$, such that specialising to $q = 1$ recovers the ordinary group algebra of the symmetric group. Although originally motivated from geometry [**46**], these algebras were found to have profound connections with the representation theory of symmetric groups. The precise nature of this connection was conjectured by James [50] – in 2013 however his conjecture was disproven by Williamson [105]. The exact relationship between symmetric groups and their Iwahori-Hecke algebras therefore remains a tantalising mystery.

Meanwhile, Iwahori-Hecke algebras were being further generalised into cyclotomic Hecke algebras, a larger family of algebras which include and generalise complex reflection group algebras. The study of the structure and representation theory of these algebras has been extensive and much is known about them [78], although there are several important unanswered questions as well, again including ever elusive formulas for simple modules at certain parameters and over certain fields.

In 2006, Khovanov and Lauda [53, 54], and independently Rouquier [95], introduced a new family of algebras, defined in terms of braid-like diagrams (though Rouquier's definition is algebraic), which, as they demonstrated, categorify the positive part of the quantum groups of Kac-Moody algebras. Even more remarkably, several years later, Brundan and Kleshchev [17] demonstrated that certain Khovanov-Lauda-Rouquier algebras are actually isomorphic to cyclotomic Hecke algebras of type $A$. Since the former algebras were equipped with a $\mathbb{Z}$-grading in a natural way, this gave a way to construct gradings on cyclotomic Hecke algebras; in particular on Iwahori-Hecke algebras, and on group algebras of symmetric groups in positive characteristic. With the presence of a grading, interesting structural questions become more tractable, and other new questions of graded representation theory can be posed. There has been considerable progress in this area in the last several years [18, 20, 41, 42, 43, 62, 79].

The symmetric group contains a subgroup of index two called the alternating group. It will be the alternating group and its generalisations which attract the focus of this thesis. The ordinary representation theory of the alternating group comes as a relatively straightforward consequence of Young's description for symmetric groups [39, 102], [51, §2.5]. Their modular representation theory is also straightforward once the results are known for the symmetric group, at least in odd characteristic [29], assuming the existence and combinatorial description of an involution on partitions known as the Mullineux map [21, 85]. Mullineux's conjecture was proven in 1996 by Ford and Kleshchev [30], giving a satisfactory description of the modular representation theory of alternating groups for the first time. The problem of classifying the modular representations of alternating groups over fields of characteristic two was solved in 1988 by Benson [10] without reference to the Mullineux map. We almost entirely avoid discussing the even characteristic case in this thesis.

In 2001, Mitsuhashi [83] defined a new family of algebras which are $q$-analogues of alternating group algebras; by analogy with the symmetric group case, he called these alternating Hecke algebras. Mitsuhashi provided a description of their semisimple representation theory, and also gave a presentation by generators and relations. There is a conceptually neater way to view alternating Hecke algebras however, by considering them as fixed-point subalgebras of a certain involution. This approach more naturally leads to generalisation and one can obtain families of alternating cyclotomic Hecke algebras in "higher levels"; for level 2 these correspond precisely with the second family of alternating Hecke algebras of type $B_n$ introduced by Mitsuhashi a few years later [84].

Our main aim in this thesis is to provide a study of alternating cyclotomic Hecke algebras. After a brief summary of important ideas from graded representation theory and the theory of cellular algebras [33] in Chapter 2, which we will use in later chapters, we discuss in Chapter 3 cyclotomic Hecke algebras in full generality and determine which types of cyclotomic Hecke algebras allow for the definition of the involution we need to pass to their alternating subalgebras, pausing to incorporate Mitsuhashi's algebras into our framework. We then define alternating cyclotomic Hecke algebras for arbitrary level, and give a classification of semisimple representations and a dimension theorem.

In Chapter 4 we discuss quiver Hecke algebras, which is our terminology for the broad class of algebras originally defined by Khovanov-Lauda and Rouquier (these have also been referred to as KLR algebras in the literature), and their cyclotomic quotients. The main result which we need is the celebrated Brundan-Kleshchev isomorphism theorem [**17**, Main Theorem], which links cyclotomic quiver Hecke algebras with the classical theory of cyclotomic Hecke algebras.

In Chapter 5 we discuss alternating quiver Hecke algebras. We start by constructing a Clifford system for these algebras in general using the graded sign automorphism [**62**], which gives us the formal mechanism for many proofs. We then give a basis theorem for the full (or affine) alternating quiver Hecke algebras, in the spirit of Khovanov and Lauda, and give a presentation by generators and relations which is strikingly similar to the presentation for the quiver Hecke algebras. We then consider cyclotomic quotients of these algebras; these so-called alternating cyclotomic quiver Hecke algebras are the primary contribution of this thesis, as they simultaneously generalise alternating groups as well as Mitsuhashi's type A and B alternating Hecke algebras.

In Chapter 6 we prove the main result of this thesis, Theorem 6.2.41, which is a Brundan-Kleshchev style isomorphism theorem in level 1 between alternating cyclotomic quiver Hecke algebras and the alternating cyclotomic Hecke algebras defined by the hash involution. Our proof uses machinery from Hu and Mathas' graded seminormal form framework [**43**], in particular their alternative approach [**43**, Theorem A] to Brundan and Kleshchev's isomorphism theorem. Given some technical requirements on the rings involved, our main corollary is that alternating group algebras and alternating Hecke algebras of type $A$ are now $\mathbb{Z}$-graded.

Finally, in Chapter 7 we discuss some representation-theoretic consequences of our results. Using the graded cellular bases of Hu and Mathas [**41**] we can construct a homogeneous basis for alternating cyclotomic quiver Hecke algebras and obtain a graded dimension formula, in arbitrary level. We also obtain Specht modules for these algebras in the semisimple case in level 1. In the non-semisimple case, Specht modules are much more difficult to define. We give some examples in this direction. We also give a classification of graded simple modules for alternating cyclotomic quiver Hecke algebras in arbitrary level, and finish with a discussion of their graded decomposition numbers; formulas are given for these graded decomposition numbers in certain cases assuming knowledge of the corresponding numbers for cyclotomic quiver Hecke algebras, which can in some cases be explicitly computed.

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**Declaration of originality**

The work presented in this thesis is original except where stated otherwise. No part of this thesis has been submitted for the award of any other degree or diploma at this or any other university. Some of the material in Chapter 3 was obtained jointly with the author's supervisor and will appear in [**13**]. Theorem 6.2.41 and a special case of Theorem 5.4.9 were obtained jointly in [**13**].

Chapter 2

# Graded representation theory

Graded representation theory is the study of actions on graded vector spaces. The extra structure afforded by a grading can often give a remarkable amount of new information – problems which seem intractable in an ungraded setting can become significantly more transparent in the presence of a grading, and questions about the new structure can be posed. In later chapters we will explore and exploit gradings and their structure and discuss graded representations of particular algebras; here we give an elementary exposition of graded representation theory from first principles. We assume only the fundamental definitions and concepts from the theory of modules, rings and algebras, such as one might find in the excellent book by Lam [**65**]. We will also use the elegant language of category theory; the strokes of its creator's brush are on exhibit in [**73**].

## 2.1. Graded algebras

Let us work over a general unital integral domain $\mathcal{Z}$ and consider a $\mathcal{Z}$-module $A$.

**Definition 2.1.1.** If $(G, +)$ is an abelian group then $A$ is a *$G$-graded $\mathcal{Z}$-module* if there is a family $\{A_g \mid g \in G\}$ of $\mathcal{Z}$-submodules of $A$ with a $\mathcal{Z}$-submodule decomposition such that $A = \bigoplus_{g \in G} A_g$, and such that $A_g$ is of finite rank for all $g \in G$. A nonzero element $x \in A_g$ is said to be *homogeneous of degree $g$* and this therefore defines a *degree function* on nonzero homogeneous elements with $\deg x = g$, if $x \in A_g$. Given a $G$-graded $\mathcal{Z}$-module $A$, we write $\underline{A}$ for the $\mathcal{Z}$-module obtained by forgetting the grading on $A$. We can also speak of *graded $\mathcal{Z}$-submodules* of a $\mathcal{Z}$-module $A$; these are just $\mathcal{Z}$-submodules of $\underline{A}$ which are graded $\mathcal{Z}$-modules in their own right.

**Example 2.1.2** (Laurent polynomials)**.** Consider the ring $\mathcal{Z}[x, x^{-1}]$ of Laurent polynomials over $\mathcal{Z}$ in an indeterminate $x$. Then $\mathcal{Z}[x, x^{-1}]$ is a $\mathbb{Z}$-graded $\mathcal{Z}$-module with degree function given by the usual degree of Laurent polynomials $\deg(x^n) = n$, that is $\mathcal{Z}[x, x^{-1}]_n = \mathcal{Z}x^n$, for $n \in \mathbb{Z}$.

We now define a $G$-graded $\mathcal{Z}$-algebra for an abelian group $G$. In this thesis, all algebras of interest will turn out to be unital, associative and free as $\mathcal{Z}$-modules.

**Definition 2.1.3** (Graded algebra)**.** A $G$-graded $\mathcal{Z}$-algebra is a $\mathcal{Z}$-algebra $A$ which is a $G$-graded $\mathcal{Z}$-module such that

$$A_g A_h \subseteq A_{g+h}$$

for all $g, h \in G$.

**Example 2.1.4** (Matrix rings)**.** Let $A = M_n(\mathcal{Z})$ be the ring of $n \times n$ matrices with entries in $\mathcal{Z}$ and recall matrix units $\{e_{ij}\}_{i,j=1}^n$ multiply according to the rule $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Choose $n$ integers $\{d_1, d_2, \ldots, d_n\}$ and define

$$\deg(e_{ij}) = d_i - d_j.$$

Then each such choice gives rise to a $\mathbb{Z}$-graded $\mathcal{Z}$-algebra $A^{\{d_1,\ldots,d_n\}}$ with

$$A_d^{\{d_1,\ldots,d_n\}} = \mathcal{Z}\langle e_{ij} \mid d_i - d_j = d\rangle.$$

**Example 2.1.5** (Polynomial algebras)**.** If $\mathcal{Z} = F$ is a field, the algebra $F[x]$ of polynomials in an indeterminate $x$ is $\mathbb{Z}$-graded with degree function $\deg x^n = n$. One can obtain gradings on finite-dimensional polynomial algebras $F[x]/(x^n)$ by taking quotients of this algebra.

**Example 2.1.6** (Superalgebras)**.** A common example of graded algebras in the literature are *superalgebras* (see [21]), which are $\mathbb{Z}_2$-graded algebras $A$. $A_0$ is often referred to as the *even* part of the algebra and written $A_+$, and $A_1$ the *odd* part, written $A_-$.

**Definition 2.1.7.** Let $A$ be a $G$-graded $\mathcal{Z}$-algebra. A *graded (left) $A$-module* is a graded $\mathcal{Z}$-module $M$ which is a (left) $A$-module (in the usual sense) and whose (left) action satisfies

$$A_g M_h \subseteq M_{g+h}$$

for all $g, h \in G$. Similarly we can define graded right $A$-modules. We write $\underline{M}$ for the $\underline{A}$-module obtained from $M$ by forgetting the grading.

In this thesis we will mainly be concerned with $\mathbb{Z}$-graded algebras and their graded modules.

**Definition 2.1.8** (Degree shift)**.** Given a $G$-graded $\mathcal{Z}$-module $M$, for $g \in G$ let $M\langle g\rangle$ be the graded $\mathcal{Z}$-module obtained by shifting the grading on $M$ up by $g$; that is $M\langle g\rangle_h = M_{h-g}$.

If $\mathcal{Z} = F$ is a field and $G = \mathbb{Z}$ is the group of integers under addition, we have the notion of *graded dimension* for a finite-dimensional $\mathbb{Z}$-graded $k$-vector space:

$$\mathrm{qdim}_F M = \sum_{n \in \mathbb{Z}} (\dim_F M_n) q^n,$$

where $q$ is an indeterminate (this sum is well-defined since each $\dim_F M_n$ is finite). Notice that in general $\mathrm{qdim}_F M \in \mathbb{N}[q, q^{-1}]$; a module $M$ for which $\mathrm{qdim} M \in \mathbb{N}[q]$ is called *positively graded*. The (Laurent) polynomial $\mathrm{qdim}_F M$ is often referred to as the *Poincaré* or *Hilbert series* for $M$. The q in the notation for graded dimension comes from the usual indeterminate $q$ used for graded dimensions; even if we use a different indeterminate we still write qdim.

**Example 2.1.9.** If $\mathcal{Z} = F$ is a field and $F[x]$ is the ring of polynomials with coefficients in $F$, then

$$\mathrm{qdim}_F F[x] = 1 + q + q^2 + \ldots = \frac{1}{1-q}.$$

**Remark 2.1.10.** It is easy to see that if $M$ is finite-dimensional as a $F$-vector space, then replacing $q$ with 1 in its graded dimension gives its $F$-dimension, i.e. $\mathrm{qdim}_F M(1) = \dim_F M$.

## 2.2. The category of graded modules

In this section, let us fix a graded $\mathcal{Z}$-algebra $A$ and consider $\mathbb{Z}$-graded $A$-modules.

**Definition 2.2.1** (Category of graded modules)**.** The category $A$-**Mod** is the abelian category whose objects are finitely-generated $\mathbb{Z}$-*graded* $A$-modules and whose maps are degree-preserving $\underline{A}$-module homomorphisms. More precisely, for $M, N \in \mathrm{Obj}(A\text{-}\mathbf{Mod})$, we have

$$\mathrm{Mor}(M, N) := \mathrm{Hom}_A(M, N) = \{f \in \mathrm{Hom}_{\underline{A}}(\underline{M}, \underline{N}) \mid f(M_d) \subseteq N_d \text{ for all } d \in \mathbb{Z}\}.$$

The maps in $\mathrm{Hom}_A(M, N)$ are called *homogeneous maps of degree zero*; more generally we define

$$\mathrm{Hom}_A(M, N)_d = \mathrm{Hom}_A(M\langle d\rangle, N) \cong \mathrm{Hom}_A(M, N\langle -d\rangle),$$

which are the homogeneous maps $M \to N$ of degree $d$, for $d \in \mathbb{Z}$. The last isomorphism follows since $f(M\langle d\rangle_e) \subseteq N_e$ if and only if $f(M_e) \subseteq N_{e-d} = N\langle -d\rangle_e$. This allows us to define the set

$$\mathrm{HOM}_A(M, N) = \bigoplus_{d\in\mathbb{Z}} \mathrm{Hom}_A(M, N)_d \tag{2.2.2}$$

as in [**41**, p601]. Notice that $\mathrm{HOM}_A(M, N) \cong \mathrm{Hom}_{\underline{A}}(\underline{M}, \underline{N})$ as $\mathcal{Z}$-modules.

**Definition 2.2.3.** We say an ungraded $\underline{A}$-module $\underline{M}$ has a *graded lift* if there exists a graded $A$-module $N$ such that $\underline{N} \cong \underline{M}$ as $\underline{A}$-modules.

**Lemma 2.2.4.** [**9**, Lemma 2.5.3] *Let $\mathcal{Z} = F$ be a field and $A$ a graded $F$-algebra, and suppose $\underline{M}$ is a finite-dimensional $\underline{A}$-module. If $\underline{M}$ is an indecomposable $\underline{A}$-module with graded lifts $M$ and $M'$, then $M \cong M'\langle d\rangle$ for some $d \in \mathbb{Z}$.*

**Proof.** By (2.2.2) any map $\phi$ between two $\underline{A}$-modules can be written in a unique way as a sum of homogeneous $A$-module homomorphisms $\phi_d$; take the isomorphism $\theta$ between $\underline{M}$ and $\underline{M'}$ and its inverse $\theta^{-1}$ between $\underline{M'}$ and $\underline{M}$. Then for each $m, n$ we have $\theta_n \circ (\theta^{-1})_m \in \mathrm{End}_{\underline{A}}(\underline{M'})$. By Fitting's Lemma, since $\theta$ and

$\theta^{-1}$ are isomorphisms there must exist some $m, n$ for which $\theta_n \circ (\theta^{-1})_m$ is not nilpotent. Since $\mathrm{End}_{\underline{A}}(\underline{M'})$ is local $\theta_n \circ (\theta^{-1})_m$ is an isomorphism and hence of degree zero; whence $n = -m$ and so $M \cong M'\langle m\rangle$ since $\deg \theta_n(a) = \deg a + n$. $\square$

**Remark 2.2.5.** Modules do not in general have graded lifts, and graded lifts are not in general unique.

**Example 2.2.6.** It is easy to construct examples of modules without graded lifts. For example, if $A = \mathbb{R}[x]$ with degree function as in Example 2.1.5 and $\underline{M}$ is the *ungraded* $\underline{A}$-module $\mathbb{R}^2$ with basis $v, w$ on which $x$ acts as the linear transformation $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, then $\underline{M}$ cannot have a graded lift because $x^2$ acts as the identity on $M$ whereas $x$, having positive degree, must act nilpotently on any finite-dimensional graded $A$-module.

In this thesis, we will be interested in categories of graded modules for some particular families of algebras. In order to discuss these categories in more detail we need to introduce the concept of graded simple modules. Recall that for an ungraded algebra $\underline{A}$, an $\underline{A}$-module is *simple* if it has no nontrivial proper submodules.

**Definition 2.2.7.** A graded $A$-module is *simple* (or *irreducible*) if it has no nontrivial proper graded submodules.

We need to introduce some constraints on our algebras and modules which will allow us to discuss elementary representation theory.

**For the remainder of this chapter, suppose $A$ is an algebra, over a field, which satisfies the descending and ascending chain conditions on ideals, and that all $A$-modules satisfy the descending and ascending chain conditions on submodules.**

These assumptions are readily satisfied by the algebras whose representation theory is of interest to us in this thesis. Additionally, for the remainder of this thesis, **except for in Section 5.3, where representation theory will not be of concern, we will be only interested in algebras graded by the group of integers $\mathbb{Z}$ under addition**. Hence for the rest of this chapter, a graded $\mathcal{Z}$-algebra $A$ is a $\mathbb{Z}$-graded algebra.

The next important result allows us to reduce many questions about the module category $A$-**Mod** to a study of its simple objects.

**Definition 2.2.8.** A *graded composition series* for a graded $A$-module $M$ is a series

$$M = M_n \supset M_{n-1} \supset M_{n-2} \supset \cdots \supset M_1 \supset M_0 = 0$$

of graded submodules of $M$ with each successive quotient $M_i/M_{i-1}$ being simple.

The proofs of the following results are entirely analogous to proofs of the ungraded versions, which are well-known (and can be found for example in [25, Theorems 13.4 and 13.7]).

**Lemma 2.2.9.** *Let $F$ be a field and $A$ a graded $F$-algebra. Let $M$ be a graded $A$-module. Then $M$ has a graded composition series.*

**Theorem 2.2.10** (Graded Jordan-Hölder theorem)**.** *Let $F$ be a field and $A$ a graded $F$-algebra. Let $M$ be a graded $A$-module and let*

$$M = M_n \supset M_{n-1} \supset M_{n-2} \supset \cdots \supset M_1 \supset M_0 = 0 \quad \textit{and}$$
$$M = M'_m \supset M'_{m-1} \supset M'_{m-2} \supset \cdots \supset M'_1 \supset M'_0 = 0$$

*be two graded composition series for $M$ with $M_i/M_{i-1} \cong D_i$ for $i = 1, 2, \ldots, n$ and $M'_j/M'_{j-1} \cong D'_j$ for $j = 1, 2, \ldots, m$. Then $m = n$ and $D_i \cong D'_{\pi(i)}$ for some permutation $\pi \in \mathfrak{S}_n$ , for all $i = 1, 2, \ldots, n$.*

**Definition 2.2.11.** For an ungraded algebra $\underline{A}$, let $\mathrm{Irr}(\underline{A})$ be the collection of (isomorphism classes of) simple $\underline{A}$-modules. For a graded algebra $A$, let $\mathrm{Irr}(A)$ be the collection of (isomorphism classes of) *graded* simple $A$-modules.

**Theorem 2.2.12.** [88, Theorem 9.6.8] *Let $F$ be a field and $A$ a graded $F$-algebra. Then the collection*

$$\operatorname{Irr}(A) = \{D\langle d\rangle \mid D \in \operatorname{Irr}(\underline{A}),\ d \in \mathbb{Z}\}$$

*gives a complete list of pairwise nonisomorphic irreducible graded $A$-modules.*

The above results allow us to work in the graded Grothendieck group of the category $A$-**Mod**, which we now define.

**Definition 2.2.13.** The *graded Grothendieck group* $K_0(A) = K_0(A\text{-}\mathbf{Mod})$ of the $\mathbb{Z}$-graded $\mathcal{Z}$-algebra $A$ is the abelian group generated by isomorphism classes $[D]$ of simple $A$-modules subject to relations

$$[D] = [E] + [F]$$

whenever there is a short exact sequence $0 \to E \to D \to F \to 0$. Define a $\mathbb{Z}[q, q^{-1}]$-module structure on $K_0(A)$ by

$$q^d \cdot [D] = [D\langle d\rangle]$$

for $d \in \mathbb{Z}$.

Combining Lemma 2.2.9, Theorem 2.2.10 and Theorem 2.2.12 gives the following basis theorem for graded Groethendieck groups.

**Proposition 2.2.14.** *Let $F$ be a field and $A$ a $\mathbb{Z}$-graded $F$-algebra. Then $\{[D] \mid D \in \operatorname{Irr}(\underline{A})\}$ is a $\mathbb{Z}[q, q^{-1}]$-basis for $K_0(A)$.*

## 2.3. Graded cellular algebras

Cellular algebras were introduced by Graham and Lehrer [33] to give a neat way of generalising and systematising the representation theory of a number of algebras, including symmetric group algebras, Hecke algebras, and different families of diagram algebras [35].

The theory of cellular algebras provides a framework for studying algebras with different parameters simultaneously, especially algebras which are deformations of generically semisimple algebras. This gives an elegant way in which information about the non-semisimple representation theory can be obtained from the semisimple theory. Hu and Mathas [41] have extended the theory of cellular algebras to graded algebras.

**Definition 2.3.1.** [33, 41] (Graded cellular algebra)**.** Let $A$ be a $\mathcal{Z}$-algebra which is free and of finite rank as a $\mathcal{Z}$-module. A *cell datum* for $A$ is an ordered triple $(\mathcal{P}, \mathcal{T}, C)$ where $(\mathcal{P}, \rhd)$ is the *weight poset*, $\mathcal{T}(\lambda)$ is a finite set for all $\lambda \in \mathcal{P}$, and

$$C : \bigsqcup_{\lambda \in \mathcal{P}} T(\lambda) \times T(\lambda) \to A \qquad (\mathtt{s}, \mathtt{t}) \mapsto c_{\mathtt{st}}$$

is an injective function such that

(i) $\{c_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathcal{T}(\lambda) \text{ for } \lambda \in \mathcal{P}_n\}$ is a $\mathcal{Z}$-basis of $A$;

(ii) if $\mathtt{s}, \mathtt{t} \in \mathcal{T}(\lambda)$ for some $\lambda \in \mathcal{P}_n$, then for all $a \in A$ there exist scalars $r_{\mathtt{vs}}(a)$ which do not depend on $\mathtt{t}$ such that

$$ac_{\mathtt{st}} \equiv \sum_{\mathtt{v} \in \mathcal{T}(\lambda)} r_{\mathtt{sv}}(a) c_{\mathtt{vt}} \mod A^{\rhd\lambda},$$

where $A^{\rhd\lambda}$ is the $\mathcal{Z}$-submodule of $A$ spanned by $\{c_{\mathtt{ab}} \mid \mu \rhd \lambda \text{ and } \mathtt{a}, \mathtt{b} \in \mathcal{T}(\mu)\}$, and

(iii) the $\mathcal{Z}$-linear map $* : A \to A$ determined by $(c_{\mathtt{st}})^* = c_{\mathtt{ts}}$ for all $\lambda \in \mathcal{P}_n$ and $\mathtt{s}, \mathtt{t} \in \mathcal{T}(\lambda)$ is an anti-isomorphism of $A$.

A *cellular algebra* is a $\mathcal{Z}$-algebra $A$ which has a cell datum $(\mathcal{P}_n, \mathcal{T}, C)$, and the basis $\{c_{\mathtt{st}} \mid \lambda \in \mathcal{P}_n \text{ and } \mathtt{s}, \mathtt{t} \in \mathcal{T}(\lambda)\}$ is called a *cellular basis* of $A$. If in addition $A$ is a $\mathbb{Z}$-graded algebra then a *graded cell datum* is an ordered 4-tuple $(\mathcal{P}_n, \mathcal{T}, C, \deg)$ where the *degree function*

$$\deg : \bigsqcup_{\lambda \in \mathcal{P}_n} \mathcal{T}(\lambda) \to \mathbb{Z}$$

satisfies

(iv) the basis vector $c_{\mathtt{st}}$ is homogeneous of degree $\deg c_{\mathtt{st}} = \deg(\mathtt{s}) + \deg(\mathtt{t})$ for all $\lambda \in \mathcal{P}_n$ and $\mathtt{s}, \mathtt{t} \in \mathcal{T}(\lambda)$.

We say $A$ is a *graded cellular algebra* and that $\{c_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathcal{T}(\lambda) \text{ for } \lambda \in \mathcal{P}_n\}$ is a *graded cellular basis* for $A$.

The rather opaque definition above is best understood through a series of examples, which show that the idea of a cellular algebra is a natural simultaneous generalisation of a number of important algebras. It is worth mentioning that König and Xi have given an alternative definition of cellular algebras from a more abstract point of view [**64**]; Graham and Lehrer's original definition is more suited to our needs in this thesis.

**Example 2.3.2.**

(i) If $\mathcal{Z} = F$ is a field, for $r \in \mathbb{N}$ the truncated polynomial algebra $A = F[x]/(x^r)$ is cellular. Take $\mathcal{P}_n = \{0, 1, \dots, n-1\}$ with the natural order, and for all $k \in \mathcal{P}_n$ let $\mathcal{T}(k) = \{1\}$. Set $c_{11}^k = x^k$ – this gives a cellular basis $\{c_{11}^k \mid 0 \leq k < n\} = \{x^k \mid 0 \leq k < n\}$ for $A$, with the anti-automorphism $*$ being the identity map. We can also equip this algebra with the degree function $\deg : \sqcup_{k=0}^{n-1} T(k) \to \mathbb{Z}$ given by $k \mapsto k$; this then gives a *graded* cellular algebra with $\deg x = 1$.

(ii) Let $A$ be the algebra of $n \times n$ matrices with real entries. Let $\mathcal{P}_n = \{n\}$ and let $\mathcal{T}(n) = \{1, 2, \dots, n\}$. Then letting $c_{ij} = e_{ij}$ be the $(i,j)$-th matrix unit gives a cellular basis for $A$ – the map $*$ is just transposition in this case. It is not hard to see that all the degree functions given in Example 2.1.4 give rise to (non-isomorphic) graded cellular algebras.

(iii) If $A$ is a split semisimple $F$-algebra, where $F$ is a field, then the Wedderburn decomposition gives that

$$A \cong \bigoplus_{j=1}^{\ell} \mathrm{Mat}_{k_j}(k)$$

for some $k_1, \ldots, k_\ell \in \mathbb{N}$ for some $\ell \in \mathbb{N}$. Since it is straightforward to prove that direct sums of cellular algebras are cellular, this decomposition together with (ii) shows that all finite-dimensional split semisimple algebras over a field are cellular.

(iv) Let $TL_n(\mathcal{Z}, \delta)$ be the $n$th *Temperley-Lieb algebra* (see [**34**] for the definition) over $\mathcal{Z}$ with parameter $\delta \in \mathcal{Z}$. That is, $TL_n$ is the $\mathcal{Z}$-algebra with basis given by planar Brauer diagrams on $2n$ points with multiplication given by concatenation, where interior circuits are removed and replaced by premultiplication by a power of $\delta$ according to the number of removed circuits. See [**33**] for a detailed example of how this multiplication is defined. $TL_n$ is an (ungraded) cellular algebra with a cellular basis given by certain involution diagrams as defined in [**33**]. Plaza and Ryom-Hansen [**91**] have since constructed a graded cellular basis for Temperley-Lieb algebras.

(v) A graded version of the Wedderburn decomposition in (iii) (see [**88**, Theorem 2.10.10]) shows that any semisimple graded algebra is isomorphic to a direct sum of the graded matrix rings in Example 2.1.4.

(vi) Let $\mathfrak{S}_n$ be the symmetric group on $n$ letters and let $F$ be a field of characteristic zero. Let $\mathcal{P}_n$ be the set of partitions of $n$ (see §3.2) and for each $\mu \in \mathcal{P}_n$ define $m_\mu = \sum_{\sigma \in \mathfrak{S}_\mu} \sigma$, where $\mathfrak{S}_\mu$ is the Young subgroup corresponding to $\sigma$. One can define [**75**, Chapter 3] distinguised coset representatives $d(s)$ and $d(t)$ for this Young subgroup for each $s, t \in \mathcal{T}(\mu)$, the set of standard tableaux of shape $\mu$, and defining $c_{st} = d(s) m_\mu d(t)^{-1}$ gives a cellular basis for the group algebra $F\mathfrak{S}_n$ [**75**, Theorem 3.20].

(vii) The approach given in (vi) generalises to the Iwahori-Hecke algebras, which can be seen as "deformations" of the group algebra $F\mathfrak{S}_n$ by some parameter $q$. This basis is called the Murphy basis [**26**], [**75**, Chapter 3] and we will discuss this in §3.3.

(viii) Hu and Mathas [**41**] have defined a cellular basis for the Khovanov-Lauda-Rouquier algebra in type $A$. We discuss these bases at length in Chapter 4. Indeed in [**41**] it is shown that the Hu-Mathas basis is a *graded* cellular basis, and in particular all algebras from (vi)-(viii) are graded cellular algebras.

For graded cellular algebras, we are able to define the graded notion of the dual of a module.

**Definition 2.3.3.** Let $M$ be a graded $A$-module, where $A$ is a $\mathcal{Z}$-algebra. The *contragredient (graded) dual* of $M$ is the graded $A$-module

$$M^{\circledast} = \mathrm{HOM}_{\mathcal{Z}}(M, \mathcal{Z}) = \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{Z}}(M\langle d \rangle, \mathcal{Z}).$$

with $A$-action determined by

$$(a \cdot f)(m) = f(a^* m)$$

for all $f \in M^{\circledast}$, $a \in A$ and $m \in M$.

**Remark 2.3.4.** Notice that as a $\mathcal{Z}$-module, $(M^{\circledast})_d = \mathrm{Hom}_{\mathcal{Z}}(M_{-d}, \mathcal{Z})$, so the graded dimension of $M^{\circledast}$ is $\mathrm{qdim} M^{\circledast}(q) = \mathrm{qdim} M(q^{-1})$.

## 2.4. Graded cell modules, simple modules and projective modules

The theory of cellular algebras which we have just introduced gives a very convenient way to discuss the representation theory of these algebras. Specifically, once a graded cellular basis is constructed, one can define a family of distinguished modules called *graded cell modules*; from these modules one can in theory construct all simple modules and develop a theory of *graded decomposition numbers*, which are graded composition multiplicities of simple modules in cell modules.

**Definition 2.4.1** (Cell modules)**.** Let $A$ be a a cellular $\mathcal{Z}$-algebra. For each $\lambda \in \mathcal{P}$, define the (left) *cell module* $S_\lambda$ to be the free $\mathcal{Z}$-module with basis $\{c_{\mathtt{s}} \mid \mathtt{s} \in \mathcal{T}(\lambda)\}$ and $A$-action given by

$$a \cdot c_{\mathtt{s}} = \sum_{\mathtt{t} \in \mathcal{T}(\lambda)} r_{\mathtt{st}}(a) c_{\mathtt{t}}$$

where $r_{\mathtt{st}}(a)$ are the structure constants appearing in Definition 2.3.1 of a cellular algebra. Note that this action is well-defined since by definition the scalars $r_{\mathtt{st}}(a)$ do not depend on $\mathtt{t}$. Similarly one can also define *right* cell modules.

**Example 2.4.2.**

(i) Let $\mathcal{Z} = F$ be a field and $x$ an indeterminate over $F$. For $1 \leq n < r$, the $F[x]/(x^r)$-module $S_n$ is the one-dimensional module with basis $v_n$ and trivial action

$$x^m \cdot v_n = \delta_{m0} v_n.$$

Notice that $\underline{S_k} \cong \underline{S_j}$ for all $j, k$ and that, for all $k$, $\underline{S_k} \cong \underline{D}$, the unique one-dimensional simple $F[x]/(x^r)$-module. As graded modules, it can be shown that $S_k \cong S_0\langle k\rangle$.

(ii) For matrix algebras $M_n(F)$ over a field, the (ungraded) cell modules are the usual irreducible column-space modules.

(iii) When $A = F\mathfrak{S}_n$, the symmetric group algebra, the cell modules are the well-known *Specht modules*, as in James [**49**]. A study of these modules and generalisations of them to other algebras will occupy a large portion of Chapter 6 of this thesis, including their graded analogues which were constructed by Brundan, Kleshchev and Wang [**20**] and Hu and Mathas [**41**].

**Proposition 2.4.3.** [**33**, Theorem 3.4] *If $A$ is a semisimple cellular algebra,*

$$\operatorname{Irr}(\underline{A}) = \{S^\lambda \mid \lambda \in \mathcal{P}_n\}$$

$$\operatorname{Irr}(A) = \{S^\lambda\langle d\rangle \mid \lambda \in \mathcal{P}_n,\ d \in \mathbb{Z}\}.$$

**Proposition 2.4.4.** [**41**, Corollary 2.5] *For a graded cellular algebra $A$, the cell module $S^\lambda$ has graded dimension*

$$\operatorname{qdim} S^\lambda = \sum_{\mathtt{s} \in \mathcal{T}(\lambda)} q^{\deg \mathtt{s}}.$$

*Moreover, the graded dimension of $A$ is*

$$\operatorname{qdim} A = \sum_{\lambda \in \mathcal{P}_n} \sum_{\mathtt{s},\mathtt{t} \in \mathcal{T}(\lambda)} q^{\deg \mathtt{s} + \deg \mathtt{t}}.$$

For a general (not necessarily semisimple) cellular algebra $A$, one can define a bilinear form $\langle \cdot, \cdot \rangle_\lambda$ on each cell module $S_\lambda$ by the formula

$$c_{\mathtt{st}} c_{\mathtt{uv}} \equiv \langle c_{\mathtt{t}}, c_{\mathtt{u}} \rangle c_{\mathtt{sv}} \mod A^{\rhd \lambda}.$$

It is not hard to show [41, Lemma 2.7] that for each $\lambda \in \mathcal{P}_n$, this form is homogeneous and so the *radical* $\operatorname{rad}(S_\lambda)$ of this form is a *graded* submodule of $S_\lambda$. This allows us to make the following definition.

**Definition 2.4.5.** For each $\lambda$, the module $D_\lambda$ is defined to be the quotient

$$D_\lambda = S_\lambda / \operatorname{rad}(S_\lambda)$$

of the cell module by the radical of this form.

**Remark 2.4.6.** Note that since the form $\langle \cdot, \cdot \rangle$ is homogeneous, the quotient module $D_\lambda$ is a *graded* $A$-module.

**Theorem 2.4.7.** [41, Theorem 2.10(c)] *The collection*

$$\{D_\lambda \langle k \rangle \mid D_\lambda \neq 0 \text{ and } k \in \mathbb{Z}\}$$

*is a complete set of pairwise non-isomorphic irreducible graded $A$-modules.*

**Definition 2.4.8.** $\mathcal{P}_n^0$ is the set $\{\lambda \in \mathcal{P}_n \mid D_\lambda \neq 0\}$.

**Corollary 2.4.9.** $\{\underline{D_\lambda} \mid D_\lambda \neq 0\}$ *is a complete set of irreducible $\underline{A}$-modules.*

**Example 2.4.10.** If $\mathbb{F}_3$ is a field of characteristic 3, the algebra $\mathbb{F}_3[x]/(x^3)$ is cellular as in Example 2.1.5 with graded cellular basis

$$c_{11}^0 = 1, \quad c_{11}^1 = x, \quad c_{11}^2 = x^2$$

and degree function $\deg c_{11}^i = i$. The cell modules are

$$S_0 = \mathbb{F}_3 v, \qquad S_1 \cong S_0\langle 1\rangle, \qquad S_2 \cong S_0\langle 2\rangle$$

where $\deg v = 0$. Moreover, one can check that $\mathrm{rad}(S_1) = S_1$ and $\mathrm{rad}(S_2) = S_2$ and so the unique (up to shift) graded simple module is $D \cong S_0$.

Graded cellular theory as we have developed it allows us to discuss *graded* decomposition numbers of algebras. If $M$ is a graded $A$-module and $D$ is a simple $A$-module, let $[M : D\langle k\rangle]$ be the multiplicity of the simple module $D\langle k\rangle$ as a graded composition factor of $M$; these numbers are well-defined by Theorem 2.2.10. This is an obvious generalisation of the notation $[\underline{M} : \underline{D}]$ for ungraded composition multiplicities.

**Definition 2.4.11.** Let $A$ be a graded cellular algebra. Then the *graded decomposition matrix* of $A$ is the matrix $(d_{\lambda\mu}(q))_{\lambda\in\mathcal{P}_n,\mu\in\mathcal{P}_n^0}$ where

$$[S^\lambda : D^\mu]_q = d_{\lambda\mu}(q) = \sum_{k\in\mathbb{Z}} [S^\lambda : D_\mu\langle k\rangle] q^k.$$

**Example 2.4.12.** Continuing with Example 2.4.10, the graded decomposition matrix for $\mathbb{F}_3[x]/(x^3)$ is

| | $D$ |
|---|---|
| $S_0$ | $1$ |
| $S_1$ | $q$ |
| $S_2$ | $q^2$ |

We will see this matrix appearing again in Chapter 7.

CHAPTER 3

# Alternating cyclotomic Hecke algebras

In this chapter we study the semisimple representation theory of alternating cyclotomic Hecke algebras in detail. We start by considering cyclotomic Hecke algebras, which grew via generalisation from the symmetric and hyperoctahedral groups, as well as their Iwahori-Hecke algebras. We also discuss the seminormal form theory of these algebras, which dates back to Young [107] in the symmetric group case, and is due to Hoefsmit [40] and Ariki-Koike [6] in general for cyclotomic Hecke algebras – we follow the approach of Hu and Mathas [43]. We then define alternating cyclotomic Hecke algebras for arbitrary level as fixed-point subalgebras of cyclotomic Hecke algebras under the hash involution, pausing to incorporate Mitsuhashi's alternating Hecke algebra [83] into our framework. In the final section we give a classification of irreducible representations for semisimple alternating cyclotomic Hecke algebras, including a dimension formula. These algebras comprise one side of a two-sided framework which has remarkably been linked together in recent years; we study the other side in Chapter 4.

## 3.1. Cyclotomic Hecke algebras

To give the definition of cyclotomic Hecke algebras as abstract algebras with a presentation by generators and relations, we need some additional notation. As in the previous chapter, let $\mathcal{Z}$ be a unital integral domain. If $k \in \mathbb{Z}$ and $\xi \in \mathcal{Z}^\times$, define the *quantum integer* $[k]_\xi$ by

$$[k]_\xi = \begin{cases} (1 + \xi + \cdots + \xi^{k-1}), & \text{if } k \geq 0, \\ -(\xi^{-1} + \xi^{-2} + \cdots + \xi^{k}), & \text{if } k < 0. \end{cases}$$

We write $[k]$ for $[k]_\xi$ when there is no confusion over the quantum parameter; by the geometric series formula, $[k] = \frac{\xi^k - 1}{\xi - 1}$ provided $\xi \neq 1$ (if $\xi = 1$, $[k] = k$).

**Definition 3.1.1** (Cyclotomic Hecke algebras [6])**.** Let $\mathcal{Z}$ be a unital integral domain, let $\xi \in \mathcal{Z}^\times$ and let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_\ell)$ be an $\ell$-tuple of elements in $\mathbb{Z}$. The *cyclotomic Hecke algebra* $\mathscr{H}_n = \mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa})$ is the unital associative $\mathcal{Z}$-algebra generated by $L_1, L_2, \ldots, L_n, T_1, T_2, \ldots, T_{n-1}$ subject to the relations

$$\begin{aligned}
\prod_{i=1}^{\ell}(L_1 - [\kappa_i]_\xi) &= 0 \\
(T_r + 1)(T_r - \xi) &= 0 \\
L_r L_s &= L_s L_r \\
T_r T_s &= T_s T_r, \quad \text{if } |r - s| > 1 \\
T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1} \\
T_r L_s &= L_s T_r, \quad \text{if } s \neq r, r+1 \\
L_{r+1}(T_r - \xi + 1) &= T_r L_r + 1.
\end{aligned} \tag{3.1.2}$$

The $\ell$-tuple $(\kappa_1, \ldots, \kappa_\ell) \in \mathbb{Z}^\ell$ is called the *multicharge* of $\mathscr{H}_n$ and is closely related to a *dominant weight* $\Lambda = \Lambda(\boldsymbol{\kappa})$ of a Kac-Moody algebra (see (4.1.5)). The quantum integers $[\kappa_i]$ for $1 \leq i \leq \ell$ are the *cyclotomic parameters*. The elements $L_1, L_2, \ldots, L_n$ are called *Jucys-Murphy elements*.

**Definition 3.1.3.** For a cyclotomic Hecke algebra $\mathscr{H}_n$, let $e$ be the *quantum characteristic* of $\xi$ in $\mathcal{Z}$, that is,

$$e = \min\{k > 0 \mid 1 + \xi + \xi^2 + \ldots + \xi^{n-1} = 0\},$$

or $e = \infty$ if no such $k$ exists. Note that if $\xi = 1$ and $\mathcal{Z} = F$ is a field of positive characteristic, $e = \operatorname{char}(F)$. If $\xi \neq 1$ and $\xi$ is a root of unity then $e$ is the multiplicative order of $\xi$.

**Example 3.1.4.** The family of algebras defined above contains many well-known examples, some of which we will study further. We note that these algebras are often referred to in the literature as *Ariki-Koike algebras*, as they were first studied in full generality by Ariki and Koike [**6**] (see Remark 3.1.5(iii)).

(i) Let $\ell = 1$, $\xi = 1$ and $\kappa_1 = 0$. Then $\mathscr{H}_{n,1}(\mathcal{Z}, 1, 0) \cong \mathcal{Z}\mathfrak{S}_n$ is the group algebra of the symmetric group $\mathfrak{S}_n$ on $n$ letters.

(ii) Let $\ell = 1$, $\mathcal{Z} = \mathbb{C}$, $\xi = \zeta_e$ be an $e$th root of unity and $\kappa_1 = 0$. Then $\mathscr{H}_{n,1}(\mathbb{C}, \zeta_e, 0) \cong \mathscr{H}_{\zeta_e,\mathbb{C}}(\mathfrak{S}_n)$ is the Iwahori-Hecke algebra of $\mathfrak{S}_n$ at an $e$th complex root of unity.

**Remark 3.1.5.**

(i) The collection of generators in Definition 3.1.1 contains many superfluous elements; one may start with only $L_1, T_1, \ldots, T_{n-1}$ and *define* the Jucys-Murphy elements $L_2, \ldots, L_n$ inductively using the final relation in (3.1.2), showing they satisfy the remaining relations. We include them in our list of generators as they are of utmost importance, and make for a more transparent presentation.

(ii) Provided $\xi \neq 1$ (in [**17**] this is called the non-degenerate case), the cyclotomic Hecke algebra is a quotient of the *affine* Hecke algebra $\widehat{H}_\xi(n)$ by the first relation in (3.1.2). The affine Hecke algebra can be defined as a tensor product $\mathscr{H}_\xi(\mathfrak{S}_n) \otimes_{\mathcal{Z}} \mathcal{Z}[L_1^{\pm 1}, \ldots, L_n^{\pm 1}]$ of $\mathcal{Z}$-modules, with the action of $T_i$ and $L_i^{\pm 1}$ twisted by the relations above, where $\mathscr{H}_\xi(\mathfrak{S}_n)$ is the ordinary Iwahori-Hecke algebra of $\mathfrak{S}_n$ [**75**, Chapters 1 and 3]. In particular the cyclotomic Hecke algebra *contains* a copy of $\mathscr{H}_\xi(\mathfrak{S}_n)$ as the subalgebra generated by $T_1, \ldots, T_{n-1}$.

(iii) The algebras we have defined only comprise a subset of the algebras normally defined as cyclotomic Hecke algebras or Ariki-Koike algebras [**6**], [**14**], [**78**], which do not have the requirement that each cyclotomic parameter be equal to some (quantum) integer. Our algebras are sometimes referred to in the literature as *integral* cyclotomic Hecke algebras because of our restriction on the cyclotomic parameter multiset; an important theorem of Dipper and Mathas [**27**] implies that these algebras determine all the others up to Morita equivalence.

The following basis theorem of Ariki-Koike gives the first indication that cyclotomic Hecke algebras are interesting objects of study: their rank depends only on $\ell$ and $n$ (i.e. it is independent of the choices of parameters). Importantly, this rank does not depend on the choice of $\xi$ or its quantum characteristic.

**Theorem 3.1.6** (Ariki-Koike [6])**.** *The cyclotomic Hecke algebra $\mathscr{H}_{n,\ell}(\mathcal{Z},\xi,\boldsymbol{\kappa})$ is a free $\mathcal{Z}$-module with basis*

$$\{L_1^{\gamma_1}L_2^{\gamma_2}\cdots L_n^{\gamma_n}T_\omega \mid 0\leq\gamma_i<\ell \text{ and } \omega\in\mathfrak{S}_n\},$$

*where $T_\omega = T_{i_1}T_{i_2}\cdots T_{i_k}$ if $s_{i_1}s_{i_2}\cdots s_{i_k}$ is any reduced expression for $\omega$. In particular its rank as a $\mathcal{Z}$-module is*

$$\mathrm{rk}_{\mathcal{Z}}(\mathscr{H}_{n,\ell}(\mathcal{Z},\xi,\boldsymbol{\kappa})) = \ell^n n!.$$

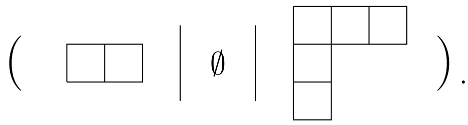
## 3.2. Combinatorics

In order to discuss the representation theory, semisimple and otherwise, of cyclotomic Hecke algebras, we need to introduce the combinatorial framework of (multi-) partitions and (multi-) tableaux.

**Definition 3.2.1.** A *partition* of $n\geq 0$ is a weakly decreasing sequence $\lambda = (\lambda_1,\lambda_2,\ldots)$ of non-negative integers which sum to $n$. An *$\ell$-multipartition* of $n$ is an $\ell$-tuple $\boldsymbol{\lambda} = (\lambda^{(1)},\lambda^{(2)},\ldots,\lambda^{(\ell)})$ of partitions whose total sum is $n$. When writing multipartitions, we omit trailing zeroes, group repeated integers with exponents and separate components with bars. We write $\mathcal{P}_n$ for the set of partitions of $n$; $\lambda\vdash n$ means $\lambda\in\mathcal{P}_n$. We write $\mathcal{P}_n^\ell$ for the set of $\ell$-multipartitions of $n$; $\boldsymbol{\lambda}\vdash_\ell n$ means $\boldsymbol{\lambda}\in\mathcal{P}_n^\ell$.

**Example 3.2.2.** $\lambda = (6,6,4,3,2,2,2,1,1,0,0,\ldots)$ is a partition of 27 which we write as $\lambda = (6^2,4,3,2^3,1^2)$. An example of a 3-multipartition of 7 is $\boldsymbol{\mu} = (2 \mid \emptyset \mid 3,1^2)$.

We usually visualise a multipartition using its *diagram*; we abuse notation frequently and identify multipartitions with their diagrams. The diagram of a partition is the set of left-aligned square boxes, called *nodes*, starting with $\lambda_1$ boxes, then $\lambda_2$ boxes underneath, and so on. The diagram of a multipartition is its sequence of constituent diagrams. For example, below is the diagram of the aforementioned 3-multipartition $(2 \mid \emptyset \mid 3, 1^2)$:



Note we can refer to a node $A$ in $\boldsymbol{\lambda}$ by a triple $(r, c, \ell)$ of the row $r$, column $c$ and constituent part $\boldsymbol{\lambda}^{(l)}$ of $\boldsymbol{\lambda}$ in which it appears.

**Definition 3.2.3.** For a multipartition $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, a $\boldsymbol{\lambda}$-*tableau* is a bijective filling of the boxes of $\boldsymbol{\lambda}$ with the numbers $1, 2, \ldots, n$. A $\boldsymbol{\lambda}$-tableau is *standard* if the entries increase along rows and down columns within each constituent diagram. The collection of standard tableaux with $n$ boxes is written $\mathrm{Std}(\mathcal{P}_n^\ell)$; the collection of those of *shape* $\boldsymbol{\lambda}$ (i.e. those tableaux such that deleting all the numbers from the diagram recovers the diagram of the multipartition $\boldsymbol{\lambda}$) is written $\mathrm{Std}(\boldsymbol{\lambda})$.

**Definition 3.2.4.** We define the *dominance order* on $\mathcal{P}_n^\ell$ by writing $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$, read as $\boldsymbol{\lambda}$ dominates $\boldsymbol{\mu}$, if

$$\sum_{k=1}^{r-1} |\lambda^{(k)}| + \sum_{j=1}^{i} \lambda_j^{(r)} \geq \sum_{k=1}^{r-1} |\mu^{(k)}| + \sum_{j=1}^{i} \mu_j^{(r)}$$

for all $1 \leq r \leq \ell$ and $i \geq 1$. This is a partial order which gives $(\mathcal{P}_n^\ell, \trianglerighteq)$ the structure of a poset. We can extend the dominance ordering to the set $\mathrm{Std}(\mathcal{P}_n^\ell)$ by defining $\mathtt{s} \trianglerighteq \mathtt{t}$ if $\mathrm{sh}(\mathtt{s}\downarrow_m) \trianglerighteq \mathrm{sh}(\mathtt{t}\downarrow_m)$ for all $1 \leq m \leq n$, where by $\mathrm{sh}(\mathtt{u})$ we mean the shape of the tableau $\mathtt{u}$, and by $\mathtt{u}\downarrow_k$ we mean the tableau with $k$ boxes obtained from $\mathtt{u}$ by deleting entries $k+1, k+2, \ldots, n$. At several points in this thesis we will also want to use the *reverse dominance order*, which corresponds to the poset $(\mathcal{P}_n^\ell, \trianglelefteq)$. There is also the *lexicographic* total ordering on partitions, written $\lambda > \mu$ if there exists an integer $j \geq 1$ such that $\lambda_i = \mu_i$ for $1 \leq i < j$ and $\lambda_i > \mu_j$; we extend this to multipartitions by writing $\boldsymbol{\lambda} > \boldsymbol{\mu}$ if there is some integer $k > 1$ such that $\boldsymbol{\lambda}^{(i)} = \boldsymbol{\mu}^{(i)}$ for $1 \leq i < k$ and $\boldsymbol{\lambda}^{(k)} > \boldsymbol{\mu}^{(k)}$. This is a refinement of the dominance partial order in the sense that $\boldsymbol{\lambda} \rhd \boldsymbol{\mu}$ implies $\boldsymbol{\lambda} > \boldsymbol{\mu}$ (see [72, (1.10)] for a proof in the partition case; it is straightforward to generalise this argument to the multipartition case).

There are two special tableaux for each multipartition $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ which are used frequently: the *initial* $\boldsymbol{\lambda}$-tableau $\mathtt{t}^{\boldsymbol{\lambda}}$ which contains the entries $1, 2, 3, \ldots, n$ increasing along rows starting from $\lambda^{(1)}$, and the *final* $\boldsymbol{\lambda}$-tableau $\mathtt{t}_{\boldsymbol{\lambda}}$ which contains the same entries increasing down columns, starting from $\lambda^{(\ell)}$.

**Definition 3.2.5.** For $\lambda \in \mathcal{P}_n$, the *conjugate* partition $\lambda'$ is the partition with

$$\lambda'_j = \#\{i \geq 1 \mid \lambda_i \geq j\}.$$

In terms of diagrams, $\lambda'$ is the diagram of $\lambda$ with rows and columns swapped. The *conjugate* $\boldsymbol{\lambda}'$ of a multipartition $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \in \mathcal{P}_n^\ell$ is

$$\boldsymbol{\lambda}' = (\lambda^{(\ell)\,\prime}, \ldots, \lambda^{(1)\,\prime}).$$

For tableaux, conjugation $\mathtt{t} \mapsto \mathtt{t}'$ is defined by interchanging rows and columns; the *conjugate* of the multitableau $\mathtt{t} = (\mathtt{t}^{(1)}, \ldots, \mathtt{t}^{(r)})$ is $\mathtt{t}' = (\mathtt{t}^{(\ell)\,\prime}, \ldots, \mathtt{t}^{(1)\,\prime})$. Notice that

$$(\mathtt{t}^{\boldsymbol{\lambda}})' = \mathtt{t}_{\boldsymbol{\lambda}'}. \tag{3.2.6}$$

The following lemma gives the important relationship between the dominance ordering and the conjugation involution.

**Lemma 3.2.7.** [72, (1.11)] *Conjugation reverses the dominance order on multipartitions and multitableaux, that is, $\boldsymbol{\lambda} \trianglerighteq \boldsymbol{\mu}$ if and only if $\boldsymbol{\mu}' \trianglerighteq \boldsymbol{\lambda}'$ and $\mathtt{s} \trianglerighteq \mathtt{t}$ if and only if $\mathtt{t}' \trianglerighteq \mathtt{s}'$.*

**Definition 3.2.8.** For $\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$, where $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, the *content* of $k$ in $\mathtt{t}$, for $1 \leq k \leq n$, is the number

$$c_{\mathtt{t}}(k) = \kappa_l + c - r,$$

where $k$ appears in component $l$ in column $c$ and row $r$ of $\mathtt{t}$; the *content sequence* of $\mathtt{t}$ is the $n$-tuple

$$c(\mathtt{t}) = (c_{\mathtt{t}}(1), c_{\mathtt{t}}(2), \dots, c_{\mathtt{t}}(n)).$$

The *e-residue* of $k$ in $\mathtt{t}$ is the residue of $c_{\mathtt{t}}(k)$ modulo $e$:

$$\mathrm{res}_{\mathtt{t}}(k) = c_{\mathtt{t}}(k) + e\mathbb{Z} \in \mathbb{Z}/e\mathbb{Z}$$

and the *residue sequence* of $\mathtt{t}$ is the $n$-tuple

$$\mathbf{i}_{\mathtt{t}} = (\mathrm{res}_{\mathtt{t}}(1), \mathrm{res}_{\mathtt{t}}(2), \dots, \mathrm{res}_{\mathtt{t}}(n)) \in (\mathbb{Z}/e\mathbb{Z})^n.$$

Finally, for $\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n$, $\mathrm{Std}(\mathbf{i})$ is the set $\{\mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell) \mid \mathbf{i}_{\mathtt{t}} = \mathbf{i}\}$.

### 3.3. Idempotents and the Murphy bases

We now have the combinatorial language to discuss the semisimple representation theory of cyclotomic Hecke algebras. We will do this by exhibiting a cellular basis for these algebras; when $\ell = 1$ this basis is to due to Murphy [87], and the result for higher levels was proved by Dipper, James and Mathas [26] and Ariki, Mathas and Rui [7].

**Definition 3.3.1.** Let $*$ be the unique involutive anti-automorphism of $\mathscr{H}_n$ which fixes the generators $L_1, T_1, \dots, T_{n-1}$.

For each $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, define the elements

$$u_{\boldsymbol{\lambda}} = \prod_{1 \le r \le \ell} \prod_{j=1}^{|\lambda^{(1)}|+\dots+|\lambda^{(r)}|} \frac{1}{1 + (\xi - \xi^{-1})[\kappa_r]} (L_j - [\kappa_{r+1}])$$

and

$$x_{\boldsymbol{\lambda}} = \sum_{\omega \in \mathfrak{S}_{\boldsymbol{\lambda}}} T_\omega,$$

where $\mathfrak{S}_{\boldsymbol{\lambda}}$ is the *row-stabiliser* or *Young* subgroup corresponding to $\boldsymbol{\lambda}$; concretely it is the direct product group

$$\mathfrak{S}_{\boldsymbol{\lambda}} = \prod_{r=1}^{\ell} \prod_{j \ge 1} \mathfrak{S}_{\lambda_j^{(r)}}$$

which embeds into $\mathfrak{S}_n$ in the obvious way. The symmetric group acts on the set of tableaux from the left by permuting entries; for each standard tableau $\mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, we define permutations $d(\mathtt{t})$ and $d'(\mathtt{t})$ by

$$\mathtt{t} = d(\mathtt{t}) \cdot \mathtt{t}^{\boldsymbol{\lambda}} \quad \text{and} \quad \mathtt{t} = d'(\mathtt{t}) \cdot \mathtt{t}_{\boldsymbol{\lambda}}. \tag{3.3.2}$$

**Definition 3.3.3** (The Murphy basis)**.** For each $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, for a pair $(\mathtt{s}, \mathtt{t})$ of standard $\boldsymbol{\lambda}$-tableaux define

$$m_{\mathtt{st}} = T_{d(\mathtt{s})^{-1}} u_{\boldsymbol{\lambda}} x_{\boldsymbol{\lambda}} T_{d(\mathtt{t})}.$$

**Theorem 3.3.4** (Dipper, James and Mathas [**26**])**.** *The cyclotomic Hecke algebra $\mathscr{H}_n$ is a free $\mathcal{Z}$-algebra with cellular basis $\{m_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ with respect to the poset $(\mathcal{P}_n^r, \trianglerighteq)$ and the involution $*$.*

We will see another basis for these semisimple algebras, which is better adapted to studying their non-semisimple representation theory, later in this chapter.

Using the Jucys-Murphy elements, we may produce a full set of mutually orthogonal idempotents for the Hecke algebras. For $\mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, define

$$F_{\mathtt{t}} = \prod_{k=1}^{n} \prod_{\substack{\mathtt{s} \in \mathrm{Std}(\boldsymbol{\lambda}) \\ \mathrm{res}_{\mathtt{s}}(k) \neq \mathrm{res}_{\mathtt{t}}(k)}} \frac{L_k - [\mathrm{res}_{\mathtt{s}}(k)]_\xi}{[\mathrm{res}_{\mathtt{t}}(k)]_\xi - [\mathrm{res}_{\mathtt{s}}(k)]_\xi}. \tag{3.3.5}$$

**Remark 3.3.6.** By (3.1.2), the Jucys-Murphy elements commute so there is no ambiguity in the order of factors in (3.3.5).

**Proposition 3.3.7** (Ariki's semisimplicity criterion [**3**])**.** *Let $\mathscr{H}_n = \mathscr{H}_{n,\ell}(F, \xi, \boldsymbol{\kappa})$ be a cyclotomic Hecke algebra with $e > 2$, where $F$ is a field. Then $\mathscr{H}_n$ is a semisimple $F$-algebra if and only if the element*

$$P_{\mathscr{H}} = P_{\mathscr{H}}(F, \xi, \boldsymbol{\kappa}) = [1]_\xi [2]_\xi \cdots [n]_\xi \prod_{1 \le r < s \le \ell} \prod_{-n < d < n} [\kappa_r + d - \kappa_s]_\xi \tag{3.3.8}$$

*is nonzero. Moreover, if $P_{\mathscr{H}}$ is nonzero then the collection $\{F_{\mathtt{t}} \mid \mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)\}$ is a complete set of pairwise orthogonal idempotents for $\mathscr{H}_n$.*

**Example 3.3.9.** Let us see how the above Proposition works in an explicit example. Suppose that $e = \infty$ and $n = 3$ and consider $\mathscr{H}_{3,1}(\mathbb{C}, q, 0)$, where $q$ is not an $e$th root of unity for any $e > 0$. It is well-known [**12**] that this is a semisimple algebra, isomorphic to the symmetric group algebra $\mathbb{C}\mathfrak{S}_3$. As there are four standard tableaux with three boxes $\mathtt{s}, \mathtt{t}, \mathtt{u}$ and $\mathtt{v}$, which we denote as

$$\boxed{1}\boxed{2}\boxed{3} \qquad \begin{array}{l}\boxed{1}\boxed{2}\\ \boxed{3}\end{array} \qquad \begin{array}{l}\boxed{1}\boxed{3}\\ \boxed{2}\end{array} \qquad \begin{array}{l}\boxed{1}\\ \boxed{2}\\ \boxed{3}\end{array}$$

respectively, we expect four idempotents, which we can easily compute from (3.3.5) as

$$\begin{aligned}
F_{\mathtt{s}} &= \frac{(L_2-[-1])^2}{([1]-[-1])^2}\frac{(L_3-[-1])(L_3-[1])(L_3-[-2])}{([2]-[-1])([2]-[1])([2]-[-2])}\\
F_{\mathtt{t}} &= \frac{(L_2-[-1])^2}{([1]-[-1])^2}\frac{(L_3-[2])(L_3-[1])(L_3-[-2])}{([-1]-[2])([-1]-[1])([-1]-[-2])}\\
F_{\mathtt{u}} &= \frac{(L_2-[1])^2}{([-1]-[1])^2}\frac{(L_3-[2])(L_3-[-1])(L_3-[-2])}{([1]-[2])([1]-[-1])([1]-[-2])}\\
F_{\mathtt{v}} &= \frac{(L_2-[1])^2}{([-1]-[1])^2}\frac{(L_3-[2])(L_3-[-1])(L_3-[1])}{([-2]-[2])([-2]-[-1])([-2]-[1])}
\end{aligned}$$

It is now a tedious calculation to use the relations in (3.1.2) to show that each of these elements is indeed an idempotent.

## 3.4. The seminormal form

Young [**107**] introduced the seminormal form for symmetric group algebras to give a particularly elegant description of their ordinary representation theory. We will see the seminormal form arise as a basis of simultaneous eigenvectors for the Jucys-Murphy elements. As we will see, this basis, which is intimately linked to the semisimple representation theory of cyclotomic Hecke algebras, is particularly well-adapted to computations.

The following lemma states the well-known result that tableaux are uniquely determined by their content sequences (which are the same as their residue sequences for $e > n$) in the semisimple case.

**Lemma 3.4.1.** [75, Lemma 3.34] *Suppose that* $\mathtt{s}, \mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)$ *and that* $P_{\mathscr{H}}$ *is nonzero. Then* $\mathtt{s} = \mathtt{t}$ *if and only if* $[c_r(\mathtt{s})] = [c_r(\mathtt{t})]$, *for* $1 \le r \le n$. *In particular* $\mathtt{s} = \mathtt{t}$ *if and only if* $\mathbf{i}_\mathtt{s} = \mathbf{i}_\mathtt{t}$.

Recall that if $\mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)$ and $1 \le r \le n$ then $c_r(\mathtt{t}) \in \mathbb{Z}$ is the content of $r$ in $\mathtt{t}$.

**Definition 3.4.2.** Define the integer $\rho_r(\mathtt{t}) \in \mathbb{Z}$ by

$$\rho_r(\mathtt{t}) = c_r(\mathtt{t}) - c_{r+1}(\mathtt{t}). \tag{3.4.3}$$

$\rho_r(\mathtt{t})$ is called the *axial distance* from $r+1$ to $r$ in $\mathtt{t}$.

**Definition 3.4.4** (Hu-Mathas [43, §3]). A $*$-*seminormal coefficient system* is a set of scalars $\boldsymbol{\alpha} = \{\alpha_r(\mathtt{s}) \mid 1 \le r < n \text{ and } \mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^\ell)\}$ in $\mathcal{Z}$ such that

(i) for $\mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)$ we have

$$\alpha_k(\mathtt{t})\alpha_m(s_k \cdot \mathtt{t}) = \alpha_m(\mathtt{t})\alpha_k(s_m \cdot \mathtt{t}) \tag{3.4.5}$$

for $1 \le k, m \le n$ if $|k - m| > 1$

(ii) for $\mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)$ and $1 \le r \le n-2$ we have

$$\alpha_r(s_{r+1}s_r\mathtt{t})\alpha_{r+1}(s_r\mathtt{t})\alpha_r(\mathtt{t}) = \alpha_{r+1}(s_{r+1}s_r\mathtt{t})\alpha_r(s_{r+1}\mathtt{t})\alpha_{r+1}(\mathtt{t}) \tag{3.4.6}$$

(iii) for $\mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)$ and $1 \le r < n$ and $\mathtt{v} = (r, r+1)\mathtt{t}$ then $\alpha_r(\mathtt{s}) = 0$ if $\mathtt{v} \notin \operatorname{Std}(\mathcal{P}_n^\ell)$ and otherwise

$$\alpha_r(\mathtt{t})\alpha_r(\mathtt{v}) = \frac{[1+\rho_r(\mathtt{t})][1+\rho_r(\mathtt{v})]}{[\rho_r(\mathtt{t})][\rho_r(\mathtt{v})]}. \tag{3.4.7}$$

As noted in [43, §3], examples of seminormal coefficient systems for the symmetric groups date back to Young in 1901 [107]. For example, we can take $\alpha_r(\mathtt{t}) = \frac{[1+\rho_r(\mathtt{t})]}{[\rho_r(\mathtt{t})]}$, whenever $\mathtt{t}, s_r \cdot \mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)$.

**Remark 3.4.8.** We will see in Chapter 6 that in order to discuss alternating cyclotomic quiver Hecke algebras and their representations, it is necessary to take a particular choice of seminormal coefficient system. Our choice will be motivated by the following example.

**Example 3.4.9.** Suppose that $\mathcal{Z} = \mathcal{K}$ is a field which contains $\xi \in \mathcal{K}^\times$, $\sqrt{\xi}$ and square roots $\sqrt{-1}$ and $\sqrt{[h]_\xi}$, for $1 \le |h| \le n$, such that $\sqrt{[-h]_\xi} = \sqrt{-1}(\sqrt{\xi})^h\sqrt{[h]_\xi}$, for $1 < h \le n$. Define

$$\alpha_r(\mathtt{t}) = \begin{cases} \frac{\sqrt{-1}\sqrt{[1+\rho_r(\mathtt{t})]}\sqrt{[1-\rho_r(\mathtt{t})]}}{[\rho_r(\mathtt{t})]} & \text{if } s_r \cdot \mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell) \text{ and } c_r(\mathtt{t}) > 0 \\ \frac{-\sqrt{-1}\sqrt{[1+\rho_r(\mathtt{t})]}\sqrt{[1-\rho_r(\mathtt{t})]}}{[\rho_r(\mathtt{t})]} & \text{if } s_r \cdot \mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell) \text{ and } c_r(\mathtt{t}) < 0 \end{cases}$$

and $\alpha_r(\mathtt{t}) = 0$ if $s_r \cdot \mathtt{t}$ is not standard. One can easily check that these scalars satisfy the requirements of a seminormal coefficient system.

**Definition 3.4.10.** A basis $\{f_{\mathtt{st}} \mid \mathtt{s},\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ of $\mathscr{H}_n = \mathscr{H}_{n,\ell}(\mathcal{Z},\xi,\boldsymbol{\kappa})$ is a *seminormal basis* if

$$L_k f_{\mathtt{st}} = [c_k(\mathtt{s})] f_{\mathtt{st}} \quad \text{and} \quad f_{\mathtt{st}} L_k = [c_k(\mathtt{t})] f_{\mathtt{st}},$$

for all $\mathtt{s},\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $1 \le k \le n$. The above basis is a $*$-*seminormal basis* if in addition $f_{\mathtt{st}}^* = f_{\mathtt{ts}}$, for all $\mathtt{s},\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$.

**Theorem 3.4.11** (The Seminormal Form [43, Theorem 3.14])**.** *Suppose that $\mathcal{K}$ is a field in which $P_{\mathscr{H}}$ is nonzero and which contains a seminormal coefficient system $\boldsymbol{\alpha}$ for $\mathscr{H}_n(\mathcal{K})$. Then*

(i) *$\mathscr{H}_n(\mathcal{K})$ has a $*$-seminormal basis $\{f_{\mathtt{st}} \mid \mathtt{s},\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$, such that*

$$f_{\mathtt{st}}^* = f_{\mathtt{ts}}, \quad L_k f_{\mathtt{st}} = [c_k(\mathtt{s})] f_{\mathtt{st}} \quad \text{and} \quad T_r f_{\mathtt{st}} = \alpha_r(\mathtt{s}) f_{\mathtt{ut}} - \frac{1}{[\rho_r(\mathtt{s})]} f_{\mathtt{st}},$$

*where $\mathtt{u} = s_r \cdot \mathtt{s}$ (and $f_{\mathtt{ut}} = 0$ if $\mathtt{u}$ is not standard);*

(ii) *there exist non-zero scalars $\gamma_{\mathtt{t}} \in \mathcal{K}$, for $\mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)$ such that*

$$f_{\mathtt{st}} f_{\mathtt{uv}} = \delta_{\mathtt{tu}} \gamma_{\mathtt{t}} f_{\mathtt{sv}};$$

(iii) *$\{\frac{1}{\gamma_{\mathtt{t}}} f_{\mathtt{tt}} \mid \mathtt{t} \in \operatorname{Std}(\mathcal{P}_n^\ell)\}$ is a complete set of pairwise orthogonal primitive idempotents for $\mathscr{H}_n(\mathcal{K})$, and*

(iv) *the* $*$*-seminormal basis* $\{f_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ *for* $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ *is uniquely determined by the* $*$*-seminormal coefficient system* $\boldsymbol{\alpha}$ *together with the scalars* $\{\gamma_{\mathtt{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$.

Finally, we will need the following identity relating the $\alpha$ and $\gamma$ coefficients.

**Corollary 3.4.12.** [43, Corollary 3.17] *Suppose that* $\mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ *is such that* $\mathtt{u} = s_r \cdot \mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$*, where* $1 \leq r < n$*. Then* $\alpha_r(\mathtt{u})\gamma_{\mathtt{t}} = \alpha_r(\mathtt{t})\gamma_{\mathtt{u}}$.

By Theorem 3.4.11, if $\mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)$ then $F_{\mathtt{t}} = \frac{1}{\gamma_{\mathtt{t}}} f_{\mathtt{tt}}$ is a primitive idempotent in $\mathscr{H}_n$. As we saw in (3.3.5), there is an explicit formula for this idempotent which in particular is independent of the choice of seminormal basis. Again by Theorem 3.4.11, as an $(\mathscr{L}, \mathscr{L})$-bimodule $\mathscr{H}_n$ decomposes as

$$\mathscr{H}_n = \bigoplus_{\substack{\boldsymbol{\lambda} \in \mathcal{P}_n^\ell \\ \mathtt{s},\mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})}} H_{\mathtt{st}}, \tag{3.4.13}$$

where $H_{\mathtt{st}} = \mathcal{K} f_{\mathtt{st}}$, and where $\mathscr{L}$ is the commutative subalgebra of $\mathscr{H}_n$ generated by the Jucys-Murphy elements. Equivalently,

$$H_{\mathtt{st}} = \{h \in \mathscr{H}_n \mid L_k h = [c_k(\mathtt{s})]h \text{ and } hL_k = [c_k(\mathtt{t})]h \text{ for } 1 \leq k \leq n\},$$

for $\mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$.

The following easy corollary of Theorem 3.4.11 is the mechanism by which we perform many of the computations in this thesis. It allows us to prove identities in cyclotomic Hecke algebras by comparing coefficients of their actions on the seminormal basis. Importantly, we can also use it prove identities in the *non-semisimple* versions of thse algebras.

**Corollary 3.4.14.** *In the cyclotomic Hecke algebra* $\mathscr{H}_n(\mathcal{K})$*, the identity element* $1 \in \mathscr{H}_n$ *decomposes as*

$$1 = \sum_{\mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)} \frac{1}{\gamma_{\mathtt{t}}} f_{\mathtt{tt}}.$$

## 3.5. Cyclotomic Hecke algebras with symmetric multicharges

In this section we define the particular subfamily of cyclotomic Hecke algebras in which we will be interested throughout this thesis. This subfamily allows for the definition of an involution whose fixed-point subalgebra is our main topic.

Recall the Jucys-Murphy elements from (3.1.2). For $k = 1, 2, \ldots, n$, let

$$\widetilde{L_k} = \xi^{1-k} T_{k-1} T_{k-2} \cdots T_2 T_1 L_1 T_1 T_2 \cdots T_{k-2} T_{k-1}.$$

These may be referred to as the *affine* Jucys-Murphy elements in $\mathscr{H}_n$. If $\xi \neq 1$, it follows by induction on $k$ that the two different definitions are related by

$$L_k = \frac{\widetilde{L}_k - 1}{\xi - 1}. \tag{3.5.1}$$

**Definition 3.5.2.** For a multicharge $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \ldots, \kappa_\ell)$, the multicharge $\boldsymbol{\kappa}'$ is defined by $(-\kappa_\ell, -\kappa_{\ell-1}, \ldots, -\kappa_1)$.

**Proposition 3.5.3.** *Let $\mathcal{Z}$ be a unital integral domain and let $\mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa})$ be a cyclotomic Hecke algebra. Then there is a unique algebra homomorphism*

$$\# : \mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa}) \to \mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa}')$$

*satisfying*

$$T_i \mapsto -\xi T_i^{-1} \quad \textit{for } i = 1, 2, \ldots, n-1.$$

*and*

$$\begin{aligned} L_1 &\mapsto -L_1, \quad \textit{if } \xi = 1 \\ \widetilde{L_1} &\mapsto \widetilde{L_1}^{-1}, \quad \textit{if } \xi \neq 1 \end{aligned}$$

**Proof.** We prove the result for the algebra $\widetilde{\mathscr{H}_n} = \mathscr{H}_{n,\ell}(\overline{\mathbb{Z}}, \xi, \boldsymbol{\kappa})$, where $\overline{\mathbb{Z}} = \mathbb{Z}[\xi, \xi^{-1}, \boldsymbol{\kappa}]$; it is clear by Definition 3.1.1 that we can then base-change to our arbitrary integral domain $\mathcal{O}$ by $\mathscr{H}_n(\mathcal{O}) \cong \mathscr{H}_n(\overline{\mathbb{Z}}) \otimes_{\overline{\mathbb{Z}}} \mathcal{O}$ to obtain the result in general.

Note that $\widetilde{L_1}, T_1, T_2, \ldots, T_{n-1}$ is also a generating set for the algebra $\widetilde{\mathscr{H}_n}$ when $\xi \neq 1$. Indeed, Ariki-Koike give such a presentation for $\mathscr{H}_n$ in [6, Definition

3.1] which, for now ignoring the first relation, has all the same relations as in Definition 3.1.2 but with the final relation replaced with the relation

$$T_1\widetilde{L_1}T_1\widetilde{L_1} = \widetilde{L_1}T_1\widetilde{L_1}T_1.$$

Using this presentation, the fact that # preserves all relations in (3.1.2) except the $L_1$ eigenvalue relation are quick checks that we leave to the reader. For the remaining relation we split into two cases. First suppose $\xi = 1$. Then, since $L_0 = T_1$,

$$\Big(\prod_{i=1}^{\ell}(L_1 - \kappa_i)\Big)^{\#} = \prod_{i=1}^{\ell}(-L_1 - \kappa_i) = 0$$

since if $\kappa_i$ appears in $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$, so does $-\kappa_i$. On the other hand, if $\xi \neq 1$, let $\mathcal{K}$ be the field of fractions of $\overline{\mathbb{Z}}_{(\xi)}$ and let $\{f_{\mathtt{st}} \mid \mathtt{s},\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}\}$ be a seminormal basis for the semisimple $\mathcal{K}$-algebra $\widetilde{\mathscr{H}}_n(\mathcal{K})$. Then $\widetilde{L_1} = (\xi - 1)L_1 + 1$ and so for a standard tableau $\mathtt{t}$, $\widetilde{L_1}f_{\mathtt{tt}} = \xi^{c_1(\mathtt{t})}f_{\mathtt{tt}} = \xi^{\kappa_i}f_{\mathtt{tt}}$ if 1 appears in component $i$ of $\mathtt{t}$. So, working in $\widetilde{\mathscr{H}}_n(\mathcal{K})$,

$$\begin{aligned}
\Big(\prod_{i=1}^{\ell}(L_1 - [\kappa_i])\Big)^{\#} f_{\mathtt{tt}} &= \Big[\prod_{i=1}^{\ell}\Big(\frac{\widetilde{L_1} - 1}{\xi - 1} - [\kappa_i]\Big)\Big]^{\#} f_{\mathtt{tt}} \\
&= \frac{1}{(\xi - 1)^{\ell}}\Big(\prod_{i=1}^{\ell}(\widetilde{L_1} - \xi^{\kappa_i})\Big)^{\#} f_{\mathtt{tt}} \\
&= \frac{1}{(\xi - 1)^{\ell}}\prod_{i=1}^{\ell}(\widetilde{L_1}^{-1} - \xi^{\kappa_i}) f_{\mathtt{tt}} \\
&= 0
\end{aligned}$$

since $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$ and $\widetilde{L_1}^{-1} f_{\mathtt{tt}} = \xi^{-\kappa_i} f_{\mathtt{tt}}$ if 1 appears in component $i$ of $\mathtt{t}$. The result now follows from Corollary 3.4.14 since $\mathscr{H}_n(\overline{\mathbb{Z}}) \hookrightarrow \widetilde{\mathscr{H}}_n(\mathcal{K})$. □

**Definition 3.5.4.** The map $\# : \mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa}) \to \mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa}')$ from Proposition 3.5.3 is called the *hash map.*

**Remark 3.5.5.** The hash map was originally defined by Goldman [**46**, Theorem 5.4] in level one; we have extended the definition to higher levels.

In order to proceed in this section, we will need to perform a number of calculations, computing the images of various elements we have defined so far under the hash involution from Definition 3.5.4.

**Lemma 3.5.6.** *For $1 \leq k \leq n$ we have $\widetilde{L_k}^{\#} = \widetilde{L_k}^{-1}$.*

**Proof.** Using the final relation from (3.1.2) and noting $\xi T_r^{-1} = T_r - \xi + 1$ we compute that $L_{r+1} = \xi^{-1} T_r L_r T_r + \xi^{-1} T_r$ and so we obtain an inductive formula for $\widetilde{L_{r+1}}$ as

$$\widetilde{L_{r+1}} = \xi^{-1} T_r \widetilde{L_r} T_r.$$

Hence

$$\begin{aligned}\widetilde{L_k}^{\#} &= (\xi^{-1} T_{k-1} \widetilde{L_{k-1}} T_{k-1})^{\#} \\ &= \xi^{-1} \xi^2 T_{k-1}^{-1} \widetilde{L_{k-1}}^{-1} T_{k-1}^{-1} \\ &= \widetilde{L_k}^{-1}\end{aligned}$$

by induction. □

For the remainder of this section we need to be more careful with our choices of rings and fields. The following definition gives us the freedom to use the seminormal form to make meaningful statements for our algebras in general.

**Definition 3.5.7.** [**43**, Definition 4.1] Let $\mathcal{K}$ be a field in which $P_{\mathscr{H}}$ is nonzero. Let $\mathcal{O}$ be a subring of $\mathcal{K}$ and $t \in \mathcal{O}^\times$. Then $(\mathcal{O}, t)$ is an *$e$-idempotent subring* of $\mathcal{K}$ if the following hold:

(i) $[k]_t$ is invertible in $\mathcal{O}$ whenever $k \not\equiv 0 \pmod{e}$ for $k \in \mathbb{Z}$; and

(ii) $[k]_t \in \mathcal{J}(\mathcal{O})$ whenever $k \in e\mathbb{Z}$,

where $\mathcal{J}(\mathcal{O})$ is the Jacobson radical of $\mathcal{O}$, i.e. the intersection of all its maximal ideals.

Let $F$ be an arbitrary field with $\xi \in F^\times$ such that the quantum characteristic of $\xi$ in $F$ is $e > 2$, and let $(\mathcal{O}, t)$ be an $e$-idempotent subring of a field $\mathcal{K}$ such that

(i) $\mathcal{K}$ contains a seminormal coefficient system for $\mathscr{H}_n$; and

(ii) $F = \mathcal{O}/\mathfrak{m}$ for some maximimal ideal $\mathfrak{m}$ of $\mathcal{O}$ and $\xi = t + \mathfrak{m}$.

**Remark 3.5.8.** It is shown in [**41**, Example 4.2] that $e$-idempotent subrings exist. We will make a particular choice of $F, \mathcal{O}$ and $\mathcal{K}$ in Chapter 6 that is suited to our needs. Note that by (ii) above, $\mathscr{H}_n(F) \cong \mathscr{H}_n(\mathcal{O}) \otimes_{\mathcal{O}} F$ and, since $\mathcal{O}$ is a subring of $\mathcal{K}$, $\mathscr{H}_n(\mathcal{K}) \cong \mathscr{H}_n(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{K}$.

For the rest of this section, let us fix the notation above, together with a seminormal basis $\{f_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ for $\mathscr{H}_n(\mathcal{K})$ and a seminormal coefficient system $\boldsymbol{\alpha}$. In particular, for the remainder of this section, $e > 2$.

We now perform a number of calculations in the semisimple cyclotomic Hecke algebra $\mathscr{H}_n(\mathcal{K})$. It is clear from Proposition 3.5.3 that if $\mathscr{H}_n$ is a cyclotomic Hecke algebra with multicharge $\boldsymbol{\kappa}$ such that $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$, then $\#$ is an *involution* of $\mathscr{H}_n$. We use this assumption implicitly in many calculations below.

**Lemma 3.5.9.** *Suppose that $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$, that $1 \le k \le n$ and $\mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^\ell)$. Then*

$$L_k^\# f_{\mathtt{ss}} = [c_k(\mathtt{s}')] f_{\mathtt{ss}}.$$

**Proof.** We give a proof if $\xi \neq 1$; the proof when $\xi = 1$ is easier and we leave it to the reader. By Theorem 3.4.11, $L_k f_{\mathtt{st}} = [c_k(\mathtt{s})] f_{\mathtt{ss}}$, so $\widetilde{L}_k f_{\mathtt{ss}} = \xi^{c_k(\mathtt{s})} f_{\mathtt{ss}}$ by (3.5.1). By Definition 3.5.4 then, $\widetilde{L}_k^\# = \widetilde{L}_k^{-1}$. Therefore,

$$\begin{aligned} L_k^\# f_{\mathtt{ss}} &= \frac{\widetilde{L}_k^\# - 1}{\xi - 1} f_{\mathtt{ss}} \\ &= \frac{\widetilde{L}_k^{-1} - 1}{\xi - 1} f_{\mathtt{ss}} \\ &= \frac{\xi^{-c_k(\mathtt{s})} - 1}{\xi - 1} f_{\mathtt{ss}} \\ &= [c_k(\mathtt{s}')] f_{\mathtt{ss}}, \end{aligned}$$

where the last equality follows because $c_k(\mathtt{s}') = -c_k(\mathtt{s})$ by Definitions 3.2.5 and 3.2.8. $\square$

**Lemma 3.5.10.** *Suppose that* $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$ *and* $\mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^\ell)$. *Then* $F_\mathtt{s}^\# = F_{\mathtt{s}'}$.

**Proof.** Since $F_\mathtt{s} = \frac{1}{\gamma_\mathtt{s}} f_{\mathtt{ss}}$, applying Lemma 3.5.9 gives

$$L_k F_\mathtt{s}^\# = (L_k^\# F_\mathtt{s})^\# = ([c_k(\mathtt{s}')F_\mathtt{s})^\# = [c_k(\mathtt{s}')]F_\mathtt{s}^\#$$

since the $\#$ map is an automorphism. Similarly $F_\mathtt{s}^\# L_k = [c_k(\mathtt{s}')]L_k$. Therefore, $F_\mathtt{s}^\# \in H_{\mathtt{s}'\mathtt{s}'}$ in the decomposition of (3.4.13). As $F_\mathtt{s}$ is an idempotent, and $\#$ is an algebra isomorphism, it follows that $F_\mathtt{s}^\# = F_{\mathtt{s}'}$ since this is the unique idempotent in $H_{\mathtt{s}'\mathtt{s}'} = \mathcal{K}F_{\mathtt{s}'}$. $\square$

**Corollary 3.5.11.** *Suppose that* $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$ *and* $\mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^\ell)$. *Then*

$$f_{\mathtt{ss}}^\# = \frac{\gamma_\mathtt{s}}{\gamma_{\mathtt{s}'}} f_{\mathtt{s}'\mathtt{s}'}.$$

**Proof.** Using Theorem 3.4.11 and Lemma 3.5.10, $f_{\mathtt{ss}}^\# = \frac{1}{\gamma_\mathtt{s}} F_\mathtt{s}^\# = \frac{1}{\gamma_\mathtt{s}} F_{\mathtt{s}'} = \frac{\gamma_{\mathtt{s}'}}{\gamma_\mathtt{s}} f_{\mathtt{s}'\mathtt{s}'}$. $\square$

**Definition 3.5.12** (Cyclotomic Hecke algebra with symmetric multicharge)**.** We say the cyclotomic Hecke algebra $\mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa})$ has *symmetric multicharge* if $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$.

The symmetry of the multicharge which allowed us to perform the calculations above also gives us the following useful combinatorial lemma.

**Lemma 3.5.13.** *For a cyclotomic Hecke algebra* $\mathscr{H}_n$ *with symmetric multicharge,* $\operatorname{res}_k(\mathtt{t}) \equiv -\operatorname{res}_k(\mathtt{t}')$ *mod* $e$ *for all* $\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ *for* $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ *and* $1 \leq k \leq n$.

**Proof.** Let $\boldsymbol{\lambda} \in \mathcal{P}_n^r$. Then for $\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ and $1 \leq k \leq n$, we have

$$\begin{aligned}
\operatorname{res}_k(\mathtt{t}) &= \kappa_\ell + r_k(\mathtt{t}) - c_k(\mathtt{t}) \\
&\equiv -\kappa_{r-\ell+1} - r_k(\mathtt{t}') + c_k(\mathtt{t}') \\
&= -(\kappa_{r-\ell+1} + r_k(\mathtt{t}') - c_k(\mathtt{t}'))
\end{aligned}$$

$$= -\operatorname{res}_k(\mathtt{t}').$$

as required. □

**Remark 3.5.14.**

(i) Although our algebras $\mathscr{H}$ do not themselves depend on the choice of multicharge (only on the residues modulo $e$), the choice of seminormal coefficient system, and therefore the algebra $\mathscr{H}^{\mathcal{O}}$, does depends on this choice. Our notation reflects this dependence on $\kappa$.

(ii) The reader may notice that for the proof of Proposition 3.5.3, we could have made the slightly weaker assumption that $\kappa_i \in \boldsymbol{\kappa}$ implies $-\kappa_i \in \boldsymbol{\kappa}$; it is for Lemma 3.5.13 that we need the multicharge to be symmetric in this particular way. This leads to the slightly uncomfortable reality that $\boldsymbol{\kappa} = (0,1,2)$ is not a symmetric 3-multicharge, but $\boldsymbol{\kappa} = (1,0,2)$ is.

We want to study the subalgebra of a cyclotomic Hecke algebra $\mathscr{H}_n$ with a symmetric multicharge consisting of elements fixed by the hash involution; the study of these algebras will occupy the majority of this thesis.

**Definition 3.5.15** (Alternating cyclotomic Hecke algebras)**.** Let $\mathcal{Z}$ be a unital integral domain and let $\xi \neq -1$ and suppose 2 is invertible in $\mathcal{Z}$. The *alternating cyclotomic Hecke algebra* $\mathscr{H}_n^\# = \mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa})^\#$ of type $(\ell, n)$ with Hecke parameter $\xi$ and symmetric cyclotomic parameters $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$ is the fixed-point subalgebra of $\mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa})$ under the $\#$ involution.

Alternating cyclotomic Hecke algebras were first studied for $\ell = 1$ by Mitsuhashi [**83**], who called them *alternating Iwahori-Hecke algebras* by analogy with existing terminology for the $\ell = 1$ case. Mitsuhashi studied the basic structure and representation theory of these algebras over the field of complex numbers when the parameter $q$ was not equal to a root of unity. However, Mitsuhashi made no mention of the hash involution in [**83**] so it is worth pausing for a moment to incorporate his algebras and the theorems from his paper into our framework.

For $q \in \mathbb{C}^\times$ with $q \neq -1$, [83, Definition 4.1] defines a subalgebra of the Iwahori-Hecke algebra $\mathscr{H}_q(\mathfrak{S}_n) = \mathscr{H}_{n,1}(\mathbb{C}, q, 0)$, denoted by $\mathscr{H}_q(\mathfrak{A}_n)$ and called the *alternating Hecke algebra*, which satisfies $\mathscr{H}_1(\mathfrak{A}_n) \cong \mathbb{C}\mathfrak{A}_n$. A description of its representation theory is obtained for generic $q$ [83, Theorem 5.5]. Specifically for each $T_i$, for $q \neq -1$, define an element

$$U_i = \frac{2T_i - (q-1)}{q+1} \tag{3.5.16}$$

and then set $X_i = U_1 U_{i+1}$. Then $\mathscr{H}_q(\mathfrak{A}_n)$ is defined to be the subalgebra of $\mathscr{H}_q(\mathfrak{S}_n)$ generated by $X_1, X_2, \ldots, X_{n-2}$. The presentation below is reminiscent of a familiar presentation of the alternating group $\mathfrak{A}_n$ by elements $x_i = s_1 s_i$.

**Proposition 3.5.17.** [83, §4] *Let $q \in \mathbb{C}$ and suppose $q \neq -1$. The alternating cyclotomic Hecke algebra $\mathscr{H}_{n,1}(\mathbb{C}, q, 0)^\#$ is isomorphic to Mitsuhashi's alternating Hecke algebra. In particular, $\mathscr{H}_{n,1}(\mathbb{C}, q, 0)^\#$ is generated by $X_1, X_2, \ldots, X_{n-2}$ subject to the relations*

$$\begin{aligned}
X_1^3 &= 1 - \left(\frac{q-1}{q+1}\right)^2 (X_1^2 - X_1) && \\
X_i^2 &= 1 && \text{for } i = 2, 3, \ldots, n-2 \\
(X_{i-1}X_i)^3 &= 1 - \left(\frac{q-1}{q+1}\right)^2 [(X_{i-1}X_i)^2 - X_{i-1}X_i] && \text{for } i = 2, 3, \ldots, n-2 \\
(X_i X_j)^2 &= 1 && \text{if } |i-j| > 1
\end{aligned}$$

*Moreover, $\mathscr{H}_{n,1}(\mathbb{C}, q, 0)^\#$ is semisimple if and only if $\mathscr{H}_{n,1}(\mathbb{C}, q, 0)$ is semisimple, which occurs precisely when $q$ is not a pth root of unity for $2 \leq p \leq n$, by Proposition 3.3.8. Moreover, $\mathscr{H}_{n,1}(\mathbb{C}, q, 0)^\#$ has $\mathbb{C}$-dimension $\dfrac{n!}{2}$.*

Mitsuhashi also studied the case $\ell = 2$ when $\boldsymbol{\kappa} = (1, e-1)$ [84] (using a Clifford theory approach similar to our methods in Chapter 5); we will not discuss this case but his algebra is isomorphic to our algebra $\mathscr{H}_{n,2}(\mathbb{C}, q, (1, -1))^\#$ so his results give explicit generators and relations for this family of alternating cyclotomic Hecke algebras [84, Proposition 3.2].

Our next goal is to determine the rank of $\mathscr{H}_n^\#$ as an $\mathcal{O}$-algebra, where $\mathcal{O}$ is the $e$-idempotent subring from Definition 3.5.7. First we need some lemmas and some additional notation. For $\mathbf{i} \in I^n$, define

$$f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\substack{\mathtt{t}\in \mathrm{Std}(\mathcal{P}_n^\ell) \\ \mathrm{res}(\mathtt{t})=\mathbf{i}}} F_{\mathtt{t}}. \tag{3.5.18}$$

These idempotents will be very important in later chapters, as they appear in a different guise when we study these algebras through a different looking glass. The following standard result shows that, despite ostensibly belonging to $\mathscr{H}_n(\mathcal{K})$, the idempotents $f_{\mathbf{i}}^{\mathcal{O}}$ actually belong to the $\mathcal{O}$-form of the algebra, justifying their notation.

**Lemma 3.5.19.** [43, Lemma 4.5], [86] *For $\mathbf{i} \in I^n$, $f_{\mathbf{i}}^{\mathcal{O}} \in \mathscr{H}_n(\mathcal{O})$.*

These elements generalise Proposition 3.3.7 in the following sense.

**Proposition 3.5.20.** [36, 76] *If the element $P_{\mathscr{H}}(F, \xi, \boldsymbol{\kappa})$ is zero, then $\mathscr{H}_n(F)$ is a non-semisimple $F$-algebra and the collection $\{f_{\mathbf{i}}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_F \mid \mathbf{i} \in I^n\}$ is a complete set of pairwise orthogonal idempotents for $\mathscr{H}_n(F)$.*

**Example 3.5.21.** Suppose $n = e = 3$ and suppose $\mathbb{F}$ is a field of characteristic 3 and consider $\mathscr{H}_{3,1}(\mathbb{F}, 1, 0) = \mathbb{F}\mathfrak{S}_3$, the 3-modular group algebra of the symmetric group $\mathfrak{S}_3$. This algebra is not semisimple and if we compute the set $\{F_{\mathtt{t}} \mid \mathtt{t} \in \mathrm{Std}(\mathcal{P}_3)\}$ of four elements by (3.3.5) in this case, using the definition of the Jucys-Murphy elements to write everything in terms of the group basis, using the notation from Example 3.3.9, we obtain the elements

$$\begin{aligned}
F_{\mathtt{s}} &= \frac{1}{6}(1 + s_1 + s_2 + s_1s_2 + s_2s_1 + s_1s_2s_1)\\
F_{\mathtt{t}} &= \frac{1}{6}(2 + 2s_1 - s_2 - s_1s_2 - s_2s_1 - s_1s_2s_1)\\
F_{\mathtt{u}} &= \frac{1}{6}(2 - 2s_1 + s_2 - s_1s_2 - s_2s_1 + s_1s_2s_1)\\
F_{\mathtt{v}} &= \frac{1}{6}(1 - s_1 - s_2 + s_1s_2 + s_2s_1 - s_1s_2s_1)
\end{aligned}$$

which are no longer idempotents. However, their *sums* grouped by 3-residue sequences (012) and (021) can be calculated as

$$e(012) = F_{\mathsf{s}} + F_{\mathsf{t}} = \frac{1}{2}(1 + s_1)$$
$$e(021) = F_{\mathsf{u}} + F_{\mathsf{v}} = \frac{1}{2}(1 - s_1),$$

which are easily verified to be idempotents.

We now compute the image of the idempotent $f_{\mathbf{i}}^{\mathcal{O}}$ under the hash map. Given $\mathbf{i} \in I^n$, define $-\mathbf{i} \in I^n$ by

$$-\mathbf{i} = (-i_1, -i_2, \ldots, -i_n). \tag{3.5.22}$$

**Lemma 3.5.23.** *Suppose that* $\mathbf{i} \in I^n$. *Then* $(f_{\mathbf{i}}^{\mathcal{O}})^{\#} = f_{-\mathbf{i}}^{\mathcal{O}}$ *in* $\mathscr{H}_n(\mathcal{O})$.

**Proof.** First observe that $\mathsf{s} \in \mathrm{Std}(\mathbf{i})$ if and only if $\mathsf{s}' \in \mathrm{Std}(-\mathbf{i})$ by Lemma 3.5.13. Then by Lemma 3.5.10,

$$(f_{\mathbf{i}}^{\mathcal{O}})^{\#} = \sum_{\mathsf{s}\in\mathrm{Std}(\mathbf{i})} F_{\mathsf{s}}^{\#} = \sum_{\mathsf{s}\in\mathrm{Std}(\mathbf{i})} F_{\mathsf{s}'} = f_{-\mathbf{i}}^{\mathcal{O}}$$

as claimed. □

We want to work with equivalence classes of $I^n$ under the involution on residue sequences defined in (3.5.22). More precisely, for $\mathbf{i}, \mathbf{j} \in I^n$ let $\sim$ be the equivalence relation on $I^n$ generated by $\mathbf{i} \sim \mathbf{j}$ if $\mathbf{i} = -\mathbf{j}$. From this we obtain a partition of the set $I^n$ into equivalence classes of size 1 or 2; we denote the equivalence class containing a sequence $\mathbf{i}$ by $[\mathbf{i}]$, noting that, since $e > 2$,

$$[\mathbf{i}] = \begin{cases} \{\mathbf{i}\}, & \text{if } \mathbf{i} = \underbrace{(0, 0, \ldots, 0)}_{n \text{ zeroes}} \\ \{\mathbf{i}, -\mathbf{i}\}, & \text{otherwise} \end{cases}$$

We denote the set of equivalence classes by $I^n_{\sim}$ and in each equivalence class we choose a representative $\mathbf{i}^+ \in [\mathbf{i}]$.

We now give a dimension formula for alternating cyclotomic Hecke algebras. The condition on the multicharge will reappear in Chapter 5, as will the above equivalence relation and the element $\varepsilon$ which appears in the proof below.

**Theorem 3.5.24.** *Let 2 be invertible in $\mathcal{O}$ and suppose that $\boldsymbol{\kappa} = \boldsymbol{\kappa}'$ is such that*

$$|\{j \mid \kappa_j \equiv 0 \ mod\ e\}| < n.$$

*Then the alternating cyclotomic Hecke algebra $\mathscr{H}_n^{\#}$ has $\mathcal{O}$-rank $\dfrac{r^n n!}{2}$.*

**Proof.** Define an element

$$\varepsilon = \sum_{\substack{[\mathbf{i}] \in I^n_\sim \\ |[\mathbf{i}]|=2}} \left( f^{\mathcal{O}}_{\mathbf{i}^+} - f^{\mathcal{O}}_{-\mathbf{i}^+} \right)$$

and note that, since $e > 2$ and since the condition on $\boldsymbol{\kappa}$ disallows the sequence $(0, 0, \dots, 0)$, $|[\mathbf{i}]| = 2$ for all $[\mathbf{i}] \in I^n_\sim$. Then $\mathscr{H}_n^{\#} \cong \varepsilon\mathscr{H}_n^{\#}$ as $\mathcal{O}$-modules, since $\varepsilon^2 = 1$ and $\varepsilon^{\#} = -\varepsilon$ by Lemma 3.5.23. Writing $f^{\mathcal{O}}_{[\mathbf{i}]}$ for $f^{\mathcal{O}}_{\mathbf{i}^+} + f^{\mathcal{O}}_{-\mathbf{i}^+}$, we see $\mathscr{H}_n \cong \mathscr{H}_n^{\#} \oplus \varepsilon\mathscr{H}_n^{\#}$, which gives the result by Theorem 3.1.6 since any $x \in \mathscr{H}_n$ may be written as $x = \sum_{\mathbf{i} \in I^n} x f^{\mathcal{O}}_{\mathbf{i}} = \sum_{[\mathbf{i}] \in I^n_\sim} x f^{\mathcal{O}}_{[\mathbf{i}]}$ and so we can write

$$x = \frac{1}{2} \sum_{[\mathbf{i}] \in I^n_\sim} (x + x^{\#}) f^{\mathcal{O}}_{[\mathbf{i}]} + \frac{1}{2} \varepsilon \sum_{[\mathbf{i}] \in I^n_\sim} (x - x^{\#}) f^{\mathcal{O}}_{[\mathbf{i}]}$$

provided $\frac{1}{2} \in \mathcal{O}$. $\square$

**Remark 3.5.25.** In particular, by our choice of rings in this section, we obtain the dimension of the alternating cyclotomic Hecke algebra over the field $F$ by observing $\dim_F \mathscr{H}_n(F)^{\#} = \dim_F \mathscr{H}_n(\mathcal{O})^{\#} \otimes_{\mathcal{O}} F = \mathrm{rk}_{\mathcal{O}} \mathscr{H}_n(\mathcal{O})^{\#}$.

## 3.6. Semisimple representations of alternating cyclotomic Hecke algebras

In this section we construct the semisimple representations of alternating cyclotomic Hecke algebras using the seminormal form from §3.4. Our goal is to construct a full set of pairwise non-isomorphic irreducible modules for $\mathscr{H}_n(\mathcal{K})^{\#}$

for certain fields $\mathcal{K}$. To begin with, we need to place some more restrictive conditions on our seminormal coefficient system $\boldsymbol{\alpha}$.

**Definition 3.6.1.** An *alternating coefficient system* is a $*$-seminormal coefficient system $\boldsymbol{\alpha} = \{\alpha_r(\mathtt{s}) \mid 1 \le r \le n \text{ and } \mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^r)\}$ such that $\alpha_r(\mathtt{s}) = -\alpha_r(\mathtt{s}')$, for all $1 \le r < n$ and $\mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^r)$.

**Remark 3.6.2.** Example 3.4.9 shows that alternating seminormal coefficient systems exist.

For the remainder of this section, fix a field $\mathcal{K}$ with $\xi \in \mathcal{K}^\times$ with quantum characteristic $e > 2$ and such that $P_{\mathscr{H}}$ is nonzero in $\mathcal{K}$, a symmetric multicharge $\boldsymbol{\kappa}$, a seminormal basis $\{f_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ for $\mathscr{H}_n(\mathcal{K})$ and an alternating seminormal coefficient system $\boldsymbol{\alpha}$.

**Remark 3.6.3.** Since we will need these coefficients explicitly later, we now compute $\alpha_r(\mathtt{t})$ for some particularly important $r$ and $\mathtt{t}$. Note that by (3.4.7), if $\mathtt{t}$ is any tableau with 2 in the first row and 3 in the first column,

$$\alpha_2(\mathtt{t})\alpha_2(s_2 \cdot \mathtt{t}) = \frac{[3][-1]}{[2][-2]} = \frac{-t^{-1}[3][1]}{-t^{-2}[2]^2} = \frac{t[3]}{[2]^2}.$$

In order to satisfy the requirement of an alternating seminormal coefficient system, we make the following choice for the sake of definiteness:

$$\alpha_2(\mathtt{t}) = \begin{cases} \frac{\sqrt{-1}\sqrt{t}\sqrt{[3]}}{[2]}, & \text{if 2 is in the first row of } \mathtt{t} \\ -\frac{\sqrt{-1}\sqrt{t}\sqrt{[3]}}{[2]}, & \text{if 2 is in the second row of } \mathtt{t} \end{cases} \tag{3.6.4}$$

This has the effect that $\alpha_2(\mathtt{s}) = \alpha_2(\widetilde{\mathtt{s}})$ for any tableaux $\mathtt{s}, \widetilde{\mathtt{s}}$ with $\mathtt{s}\downarrow_3 = \widetilde{\mathtt{s}}\downarrow_3$.

The following lemma continues some calculations we began in the previous section.

**Lemma 3.6.5.** *Suppose that $\mathtt{s}, \mathtt{u} \in \operatorname{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ are standard tableaux such that $\mathtt{u} = s_r \cdot \mathtt{s}$, where $1 \le r < n$. Then*

$$f_{\mathtt{us}}^{\#} = -\frac{\alpha_r(\mathtt{s}')\gamma_{\mathtt{s}}}{\alpha_r(\mathtt{s})\gamma_{\mathtt{s}'}} f_{\mathtt{u}'\mathtt{s}'}.$$

**Proof.** By Theorem 3.4.11, $f_{\mathtt{us}} = \frac{1}{\alpha_r(\mathtt{s})}\Big(T_r + \frac{1}{[\rho_r(\mathtt{s})]}\Big)f_{\mathtt{ss}}$. Hence, using Definition 3.5.4 and Corollary 3.5.11 for the second equality,

$$\begin{aligned}
f_{\mathtt{us}}^{\#} &= \frac{1}{\alpha_r(\mathtt{s})}\Big(T_r + \frac{1}{[\rho_r(\mathtt{s})]}\Big)^{\#} f_{\mathtt{ss}}^{\#} \\
&= \frac{\gamma_{\mathtt{s}}}{\alpha_r(\mathtt{s})\gamma_{\mathtt{s}'}}\Big( - T_r + t - 1 + \frac{1}{[\rho_r(\mathtt{s})]}\Big) f_{\mathtt{s}'\mathtt{s}'} \\
&= -\frac{\gamma_{\mathtt{s}}}{\alpha_r(\mathtt{s})\gamma_{\mathtt{s}'}}\Big(T_r - \frac{t^{\rho_r(\mathtt{s})}}{[\rho_r(\mathtt{s})]}\Big) f_{\mathtt{s}'\mathtt{s}'} \\
&= -\frac{\gamma_{\mathtt{s}}}{\alpha_r(\mathtt{s})\gamma_{\mathtt{s}'}}\Big(T_r + \frac{1}{[\rho_r(\mathtt{s}')]}\Big) f_{\mathtt{s}'\mathtt{s}'}
\end{aligned}$$

since $[\rho_r(\mathtt{s})] = -t^{\rho_r(\mathtt{s})}[-\rho_r(\mathtt{s})] = -t^{\rho_r(\mathtt{s})}[\rho_r(\mathtt{s}')]$. Observe that $\mathtt{u}' = s_r\mathtt{s}'$. Therefore, the result follows by another application of Theorem 3.4.11. □

By Theorem 3.4.11 any $*$-seminormal basis is uniquely determined by a seminormal coefficient system and a choice of scalars $\{\gamma_{\mathtt{t}^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}\}$. We now determine these scalars for the seminormal basis

$$\{f_{\mathtt{st}}^{\#} \mid \mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}\}.$$

**Proposition 3.6.6.** *The collection $\{f_{\mathtt{st}}^{\#} \mid \mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}\}$ is the seminormal basis of $\mathscr{H}_n(\mathcal{K})$ determined by the seminormal coefficient system*

$$\{-\alpha_r(\mathtt{s}) \mid \mathtt{s} \in \operatorname{Std}(\boldsymbol{\lambda}),\ \boldsymbol{\lambda} \in \mathcal{P}_n^{\ell} \textit{ and } 1 \le r < n\}$$

*together with the $\gamma$-coefficients $\{\gamma_{\mathtt{t}_{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}\}$. That is, if $\mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ and $1 \le r < n$ then*

$$T_r f_{\mathtt{st}}^{\#} = -\alpha_r(\mathtt{s}) f_{\mathtt{ut}}^{\#} - \frac{1}{[\rho_r(\mathtt{s}')]} f_{\mathtt{st}}^{\#},$$

*where $\mathtt{u} = s_r \cdot \mathtt{s}$. Moreover, $f_{\mathtt{st}}^{\#} f_{\mathtt{uv}}^{\#} = \delta_{\mathtt{tu}}\gamma_{\mathtt{t}} f_{\mathtt{sv}}^{\#}$, for $\mathtt{s}, \mathtt{t}, \mathtt{u}, \mathtt{v} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$.*

**Proof.** Using Theorem 3.4.11, if $\mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ we compute

$$\begin{aligned}
T_r f_{\mathtt{st}}^{\#} &= \Big(T_r^{\#} f_{\mathtt{st}}\Big)^{\#} = \Big((-T_r + t - 1) f_{\mathtt{st}}\Big)^{\#} \\
&= \Big( - \alpha_r(\mathtt{s}) f_{\mathtt{ut}} + (t - 1 + \frac{1}{[\rho_r(\mathtt{s})]}) f_{\mathtt{st}}\Big)^{\#}
\end{aligned}$$

$$= \Big( - \alpha_r(\mathtt{s}) f_{\mathtt{ut}} - \frac{1}{[\rho_r(\mathtt{s}')]} f_{\mathtt{st}} \Big)^{\#}$$
$$= -\alpha_r(\mathtt{s}) f_{\mathtt{ut}}^{\#} - \frac{1}{[\rho_r(\mathtt{s}')]} f_{\mathtt{st}}^{\#}.$$

Similarly, $f_{\mathtt{st}}^{\#} f_{\mathtt{uv}}^{\#} = (f_{\mathtt{st}} f_{\mathtt{uv}})^{\#} = \delta_{\mathtt{tu}} \gamma_{\mathtt{t}} f_{\mathtt{sv}}^{\#}$. By Theorem 3.5.9 $f_{\mathtt{st}}^{\#} \in H_{\mathtt{s}'\mathtt{t}'}$, so the $\alpha$-coefficient corresponding to $f_{\mathtt{st}}^{\#}$ is naturally indexed by $\mathtt{s}'$ (and not by $\mathtt{s}$). Similarly, the labelling for the $\gamma$-coefficients involves conjugation because $F_{\mathtt{t}} = \frac{1}{\gamma_{\mathtt{t}'}} f_{\mathtt{t}'\mathtt{t}'}^{\#}$ by Corollary 3.5.11. Hence, the result follows by Theorem 3.4.11. ☐

We now define Specht modules for semisimple cyclotomic Hecke algebras.

**Definition 3.6.7.** Let $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$. The Specht module $\underline{S^{\boldsymbol{\lambda}}}$ for the algebra $\mathscr{H}_n(\mathcal{K})$ is the vector space with basis $\{f_{\mathtt{t}} \mid \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})\}$ and with $\mathscr{H}_n(\mathcal{K})$-action given by

$$L_k f_{\mathtt{t}} = [c_k(\mathtt{t})] f_{\mathtt{t}}$$
$$T_r f_{\mathtt{t}} = \alpha_r(\mathtt{t}) f_{\mathtt{u}} - \frac{1}{[\rho_r(\mathtt{t})]} f_{\mathtt{t}}$$

for $1 \leq k \leq n$ and $1 \leq r < n$, where $\mathtt{u} = s_r \cdot \mathtt{s}$ (and $\alpha_r(\mathtt{t}) = 0$ if $\mathtt{u}$ is not standard).

**Example 3.6.8.** Let us calculate the actions of the generators $T_1$ and $T_2$ on the Specht module $\underline{S^{(21)}}$ for the algebra $\mathscr{H}_{3,1}(\mathbb{C}, 1, 0) \cong \mathbb{C}\mathfrak{S}_3$, noting that $\mathbb{C}$ clearly contains the required alternating seminormal coefficient system computed in Remark 3.6.3. We see that $T_1$ and $T_2$ respectively act as the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}i}{2} \\ -\frac{\sqrt{3}i}{2} & -\frac{1}{2} \end{pmatrix}$$

which the reader can check square to the identity (the first column of each matrix corresponds to the vector $f_{12/3}$ and the second to $f_{13/2}$).

**Remark 3.6.9.** The notation for our Specht modules using an underline is a deliberate foreshadowing of Chapter 7, where we will observe graded lifts of these modules in the sense of Definition 2.2.3 in some cases.

**Theorem 3.6.10.** [41, Lemma 5.12] *Let $\mathcal{K}$ be a field containing a seminormal coefficient system and such $P_{\mathscr{H}}$ is nonzero. Then for each $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, $\underline{S^{\boldsymbol{\lambda}}}$ is an irreducible $\mathscr{H}_n(\mathcal{K})$-module. Moreover, the collection $\{\underline{S^{\boldsymbol{\lambda}}} \mid \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ is a complete list of irreducible modules for the semisimple algebra $\mathscr{H}_n(\mathcal{K})$.*

**Definition 3.6.11.** Let $\sim$ be the equivalence relation on $\mathcal{P}_n^\ell$ generated by $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$ if $\boldsymbol{\mu} = \boldsymbol{\lambda}'$. We write $[\boldsymbol{\lambda}]$ for the equivalence class of $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ under this equivalence relation, and $(\mathcal{P}_n^\ell)_\sim$ for the set of all equivalence classes.

If $A$ is an $\mathcal{O}$-algebra and $B$ an $\mathcal{O}$-subalgebra of $A$, for an $A$-module $M$ we write $M{\downarrow}_B^A$ for the $B$-module obtained by restriction.

**Proposition 3.6.12.** *Let $\mathcal{K}$ be a field containing a seminormal coefficient system and such that $P_{\mathscr{H}}$ is nonzero. Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ be such that $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$. Then*

$$\underline{S^{\boldsymbol{\lambda}}}{\downarrow}^{\mathscr{H}_n(\mathcal{K})}_{\mathscr{H}_n(\mathcal{K})^\#} \cong \underline{S^{\boldsymbol{\lambda}'}}{\downarrow}^{\mathscr{H}_n(\mathcal{K})}_{\mathscr{H}_n(\mathcal{K})^\#}$$

*as $\mathscr{H}_n(\mathcal{K})^\#$-modules.*

**Proof.** Define an map of vector spaces $\tau : S^{\boldsymbol{\lambda}} \to S_{\boldsymbol{\lambda}}$ by $v_{\mathtt{t}} \mapsto v_{\mathtt{t}'}$; this is clearly an isomorphism of vector spaces since $|\mathrm{Std}(\boldsymbol{\lambda})| = |\mathrm{Std}(\boldsymbol{\lambda}')|$ by Lemma 3.2.7. Then by Proposition 3.6.6, $\tau(T_r v_{\mathtt{t}}) = T_r^\# v_{\mathtt{t}'}$ and by Lemma 3.5.9 $L_k v_{\mathtt{t}} = L_k^\# v_{\mathtt{t}'}$ so $\tau$ is an $\mathscr{H}_n(\mathcal{K})^\#$-module isomorphism between the restricted modules. $\square$

**Definition 3.6.13.** If $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ is such that $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$, we write $S^{[\boldsymbol{\lambda}]}$ for the $\mathscr{H}_n(\mathcal{K})^\#$-module from Proposition 3.6.12.

**Corollary 3.6.14.** *Let $\mathcal{K}$ be a field containing a seminormal coefficient system and such that $P_{\mathscr{H}}$ is nonzero. Then for $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ such that $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$, $S^{[\boldsymbol{\lambda}]}$ is an irreducible $\mathscr{H}_n(\mathcal{K})^\#$-module.*

**Proof.** Let $v$ be a nonzero vector in $S^{[\boldsymbol{\lambda}]} = \underline{S^{\boldsymbol{\lambda}}}{\downarrow}^{\mathscr{H}_n(\mathcal{K})}_{\mathscr{H}_n(\mathcal{K})^\#}$. Writing $v = \sum_{\mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})} r_{\mathtt{t}} v_{\mathtt{t}}$ as an arbitrary linear combination, we can surely choose some $\mathtt{t}$ with $r_{\mathtt{t}} \neq 0$. Then $r_{\mathtt{t}} f_{\mathtt{t}} = (F_{\mathtt{t}} + F_{\mathtt{t}'})v \in \mathscr{H}_n^\# v$ and so $f_{\mathtt{t}} \in \mathscr{H}_n^\# v$. We now observe that it is possible to move from any one basis vector to another by applying a sequence of

$\#$-invariant elements; one can check that for $\mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ with $\mathtt{u} = s_r \cdot \mathtt{t}$ standard, $(F_{\mathtt{u}}T_r + F_{\mathtt{u}'}T_r^{\#})f_{\mathtt{t}} = \frac{\alpha_r(\mathtt{t})}{\gamma_{\mathtt{u}}} f_{\mathtt{u}}$. We are done since we can clearly move all the way from the basis vector $f_{\mathtt{t}^{\lambda}}$ to the vector $f_{\mathtt{t}_{\lambda}}$ in such a way; the element we are acting by is $\#$-invariant by Lemma 3.5.10. So $\mathscr{H}_n^{\#}v = S^{[\boldsymbol{\lambda}]}$ and we are done. $\square$

**Definition 3.6.15.** Let $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ be such that $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ and suppose that 2 is invertible in $\mathcal{K}$. Denote by $S_+^{\boldsymbol{\lambda}}$ the vector space with basis $\{\frac{1}{2}(f_{\mathtt{t}} + f_{\mathtt{t}'}) \mid \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})^+\}$ and by $S_-^{\boldsymbol{\lambda}}$ the vector space with basis $\{\frac{1}{2}(f_{\mathtt{t}} - f_{\mathtt{t}'}) \mid \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})^+\}$.

Note that for the next proposition, it is important that we include the additional assumption that $\mathcal{K}$ contains an *alternating* seminormal coefficient system.

**Proposition 3.6.16.** *Let $\mathcal{K}$ be a field of characteristic greater than 2 containing an alternating seminormal coefficient system and such that $P_{\mathscr{H}}$ is nonzero. Then for $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ with $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, $S_+^{\boldsymbol{\lambda}}$ and $S_-^{\boldsymbol{\lambda}}$ are irreducible $\mathscr{H}_n(\mathcal{K})^{\#}$-modules.*

**Proof.** Similarly to the proof of Corollary 3.6.14, suppose we need to move from $\frac{1}{2}(f_{\mathtt{t}} \pm f_{\mathtt{t}'})$ to $\frac{1}{2}(f_{\mathtt{u}} \pm f_{\mathtt{u}'})$, where $\mathtt{u} = s_r \cdot \mathtt{t}$ is standard. Then we compute, using Theorem 3.4.11 and Proposition 3.6.6,

$$(F_{\mathtt{u}}T_r + F_{\mathtt{u}'}T_r^{\#}) \cdot \frac{1}{2}(f_{\mathtt{t}} \pm f_{\mathtt{t}'}) = \frac{\alpha_r(\mathtt{t})}{\gamma_{\mathtt{u}}} \frac{1}{2}(f_{\mathtt{u}} \pm f_{\mathtt{u}'})$$

using the fact that $\boldsymbol{\alpha}$ is an alternating seminormal coefficient system. $\square$

**Example 3.6.17.** Continuing with Example 3.6.8, noting that $\mathbb{C}$ contains an alternating seminormal coefficient system, and that the generator for $\mathscr{H}_3(\mathbb{C})^{\#}$ is $T_1T_2$ by Proposition 3.5.17, using the matrices from Example 3.6.8 we obtain two one-dimensional modules on which $T_1T_2$ act by $\omega$ and $\omega^2$ respectively, where $\omega = \frac{1}{2}(1 + \sqrt{-3}) \in \mathbb{C}$ is a cube root of unity.

Notice that we have the following immediate corollaries of the definitions of the respective modules.

**Corollary 3.6.18.** *Let $\boldsymbol{\lambda} \in \mathcal{P}_n^{\ell}$ and let $\mathcal{K}$ be a field of characteristic greater than 2.*

(i) *If* $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$, $\dim_{\mathcal{K}} S^{[\boldsymbol{\lambda}]} = \dim_{\mathcal{K}} S^{\boldsymbol{\lambda}} = |\mathrm{Std}(\boldsymbol{\lambda})|$.

(ii) *If* $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$, $\dim_{\mathcal{K}} S^{\boldsymbol{\lambda}}_{+} = \dim_{\mathcal{K}} S^{\boldsymbol{\lambda}}_{-} = \frac{1}{2}\dim_{\mathcal{K}} S^{\boldsymbol{\lambda}} = \frac{1}{2}|\mathrm{Std}(\boldsymbol{\lambda})|$.

We can now give a classification of irreducible representations for semisimple alternating cyclotomic Hecke algebras.

**Proposition 3.6.19.** *Let $\mathcal{K}$ be a field of characteristic greater than 2 containing an alternating seminormal coefficient system and such that $P_{\mathscr{H}}$ is nonzero. Then the collection*

$$\{S^{[\boldsymbol{\lambda}]} \mid [\boldsymbol{\lambda}] \in (\mathcal{P}^{\ell}_n)_{\sim} \text{ with } |[\boldsymbol{\lambda}]| = 2\} \cup \{S^{\boldsymbol{\lambda}}_{+}, S^{\boldsymbol{\lambda}}_{-} \mid [\boldsymbol{\lambda}] \in (\mathcal{P}^{\ell}_n)_{\sim} \text{ with } |[\boldsymbol{\lambda}]| = 1\}$$

*is a complete list of pairwise non-isomorphic irreducible modules for the semisimple alternating cyclotomic Hecke algebra $\mathscr{H}_n(\mathcal{K})^{\#}$.*

**Proof.** By Theorem 3.6.10, $\sum_{\boldsymbol{\lambda}\in\mathcal{P}^{\ell}_n} |\mathrm{Std}(\boldsymbol{\lambda})|^2 = \ell^n n!$. Moreover, the modules $S^{[\boldsymbol{\lambda}]}$ for $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$ and $S^{\boldsymbol{\lambda}}_{+}$ and $S^{\boldsymbol{\lambda}}_{-}$ for $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$ are clearly pairwise non-isomorphic by Lemma 3.4.1 and the orthogonality of the idempotents $\{F_{\mathtt{t}} \mid \mathtt{t} \in \mathrm{Std}(\mathcal{P}^{\ell}_n)\}$. Hence since

$$\begin{aligned}\sum_{\substack{[\boldsymbol{\lambda}]\in(\mathcal{P}^{\ell}_n)_{\sim}\\ |[\boldsymbol{\lambda}]|=2}} |\mathrm{Std}(\boldsymbol{\lambda})|^2 + \sum_{\substack{[\boldsymbol{\lambda}]\in(\mathcal{P}^{\ell}_n)_{\sim}\\ |[\boldsymbol{\lambda}]|=1}} 2\Big(\frac{1}{2}|\mathrm{Std}(\boldsymbol{\lambda})|\Big)^2 &= \sum_{\substack{[\boldsymbol{\lambda}]\in(\mathcal{P}^{\ell}_n)_{\sim}\\ |[\boldsymbol{\lambda}]|=2}} |\mathrm{Std}(\boldsymbol{\lambda})|^2 + \frac{1}{2}\sum_{\substack{[\boldsymbol{\lambda}]\in(\mathcal{P}^{\ell}_n)_{\sim}\\ |[\boldsymbol{\lambda}]|=1}} |\mathrm{Std}(\boldsymbol{\lambda})|^2 \\ &= \frac{\ell^n n!}{2},\end{aligned}$$

we must have a full list of irreducible representations by Corollary 3.6.14, Proposition 3.6.16 and Proposition 3.5.24. □

As a result we obtain the folowing corollary which again highlights the importance of our seminormal coefficient system. The reader should compare this with the results in [**79**] and [**94**].

**Corollary 3.6.20.** *Let $\mathcal{K}$ be a field of characteristic different from 2 containing an alternating seminormal coefficient system and such that $P_{\mathscr{H}}$ is nonzero. Then $\mathcal{K}$ is a splitting field for $\mathscr{H}_n(\mathcal{K})^{\#}$.*

**Proof.** The arguments given in this section work over any extension field of $\mathcal{K}$ so the Specht modules constructed in this section are irreducible over any such extension. Hence $\mathcal{K}$ is a splitting field for $\mathscr{H}_n^{\#}$. $\square$

Chapter 4

# Quiver Hecke algebras

In a landmark series of papers [53, 54], Khovanov and Lauda introduced a remarkable new family of algebras, which were also independently discovered by Rouquier [95]. Initially referred to in the literature as Khovanov-Lauda-Rouquier algebras (KLR algebras), these algebras have come to be known as quiver Hecke algebras due to an astonishing connection discovered by Brundan and Kleshchev [17]. In this chapter we discuss the theory of quiver Hecke algebras and their cyclotomic quotients, as well as the isomorphism theorem of Brundan and Kleshchev which establishes the connection with the classical theory of cyclotomic Hecke algebras.

## 4.1. Quiver Hecke algebras

In this section we introduce quiver Hecke algebras. Throughout this thesis we are most interested in their *cyclotomic quotients* (see §4.2), but for several results we will require the greater generality of this section. At this point we need to introduce a considerable amount of notation, which for our purposes amounts to little more than bookkeeping, but which foreshadows the deep and beautiful connections with Lie theory uncovered by Ariki [3], Grojnowski [36] and Lascoux, Leclerc and Thibon [67].

Fix an integer $e > 2$, or let $e = \infty$, and let $\Gamma_e$ be the oriented quiver with vertex set

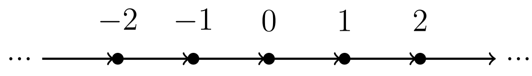
$$I = \begin{cases} \mathbb{Z}/e\mathbb{Z}, & \text{if } e < \infty \\ \mathbb{Z}, & \text{if } e = \infty \end{cases} \tag{4.1.1}$$

and with directed edges

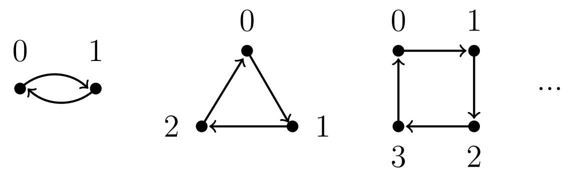
$$\{i \to i+1 \mid i \in I\}.$$

$\Gamma_e$ is the quiver of type $A_\infty$ if $e = \infty$, or the finite quiver of type $A^{(1)}_{e-1}$ if $e$ is finite.

$A_\infty$:




$A^{(1)}_{e-1}$:




**Remark 4.1.2.** We note that it is more common in the literature, for example [17], to use $e = 0$ to refer to the infinite quiver case; we prefer $e = \infty$ since then $e$ is consistently equal to the cardinality of the vertex set. This is consistent with [41] and [62].

The *Cartan matrix* corresponding to $\Gamma_e$ is the $e \times e$ integer matrix $C = (c_{ij})_{i,j\in I}$, given by

$$c_{ij} = \begin{cases} 2, & \text{if } i = j \\ 0, & \text{if } i \not\to j \\ -1, & \text{if } i \to j \text{ or } j \to i \end{cases} \tag{4.1.3}$$

where $i \not\to j$ means $i$ and $j$ are not connected by an edge in $\Gamma_e$.

**Remark 4.1.4.** We have omitted one case which appears in [17] since $i \leftrightarrows j$ can only occur when $e = 2$, and we are specifically avoiding this case.

Additionally from the language of Kac-Moody algebras [52] we require two $\mathbb{Z}$-lattices

$$P_e = \bigoplus_{i\in I} \mathbb{Z}\Lambda_i \quad \text{and} \quad Q_e = \bigoplus_{i\in I} \mathbb{Z}\alpha_i \tag{4.1.5}$$

which are paired under the bilinear form $(\cdot,\cdot) : P_e \times Q_e \to \mathbb{Z}$ defined by

$$(\Lambda_i, \alpha_j) = \delta_{ij}. \tag{4.1.6}$$

We write $\langle \cdot,\cdot\rangle$ for the pairing $Q_e \times Q_e \to \mathbb{Z}$ which satisfies $\langle \alpha_i, \alpha_j\rangle = c_{ij}$. We write $P_e^+$ for the corresponding lattice with non-negative coefficients.

The vectors $\{\Lambda_i \mid i \in I\}$ are the fundamental weights and the vectors $\{\alpha_i \mid i \in I\}$ are the simple roots of the affine Lie algebra $\widehat{\mathfrak{sl}}_e$, which is the Kac-Moody algebra associated to the quiver $\Gamma_e$.

For various technical reasons, it is more convenient and clean to work with the *blocks* of quiver Hecke algebras instead of the full algebras. We will need some additional notation to describe this decomposition into blocks (see (4.1.11)).

The symmetric group $\mathfrak{S}_n$ acts on $I^n$ by place permutation. Notice that we can decompose $I^n$ into a disjoint union of $\mathfrak{S}_n$-orbits using our notation above as

$$I^n = \bigsqcup_{\alpha \in Q_e} I^\alpha, \tag{4.1.7}$$

where $I^\alpha = \{\mathbf{i} \in I^n \mid \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_n} = \alpha\}$.

We are now ready to define quiver Hecke algebras. Our definition below is slightly simpler than the usual definition in the literature [**17, 53**] since we are excluding the case $e = 2$. Recall $\mathcal{Z}$ is a unital integral domain.

**Definition 4.1.8.** Let $n \geq 0$, $e > 2$ and $\alpha \in Q^+$. The *quiver Hecke algebra* of type $\Gamma_e$ is the unital associative $\mathcal{Z}$-algebra $\mathcal{R}_\alpha = \mathcal{R}_\alpha(e, \mathcal{Z})$ with generators

$$\{\psi_1, \psi_2, \ldots, \psi_{n-1}\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha\}$$

subject to the relations

$$e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}} e(\mathbf{i}), \qquad \sum_{e(\mathbf{i}) \in I^\alpha} e(\mathbf{i}) = 1$$

$$y_r e(\mathbf{i}) = e(\mathbf{i}) y_r$$

$$\psi_r e(\mathbf{i}) = e(s_r \cdot \mathbf{i}) \psi_r$$

$$y_r y_s = y_s y_r$$

$$\psi_r y_s = y_s \psi_r \quad \text{if } s \neq r, r+1$$

$$\psi_r \psi_s = \psi_s \psi_r \quad \text{if } |r - s| > 1$$

$$\psi_r y_{r+1} e(\mathbf{i}) = \begin{cases} (y_r\psi_r + 1)e(\mathbf{i}) & \text{if } i_r = i_{r+1} \\ y_r\psi_r e(\mathbf{i}) & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$y_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_r y_r + 1)e(\mathbf{i}) & \text{if } i_r = i_{r+1} \\ \psi_r y_r e(\mathbf{i}) & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$\psi_r^2 e(\mathbf{i}) = \begin{cases} 0 & \text{if } i_r = i_{r+1} \\ e(\mathbf{i}) & \text{if } i_r \not- i_{r+1} \\ (y_{r+1} - y_r)e(\mathbf{i}) & \text{if } i_r \to i_{r+1} \\ (y_r - y_{r+1})e(\mathbf{i}) & \text{if } i_r \leftarrow i_{r+1} \end{cases}$$

$$\psi_r\psi_{r+1}\psi_r e(\mathbf{i}) = \begin{cases} (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}) & \text{if } i_{r+2} = i_r \to i_{r+1} \\ (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(\mathbf{i}) & \text{if } i_{r+2} = i_r \leftarrow i_{r+1} \\ \psi_{r+1}\psi_r\psi_{r+1} e(\mathbf{i}) & \text{otherwise} \end{cases}$$

for $\mathbf{i}, \mathbf{j} \in I^\alpha$ and all admissible $r$ and $s$.

**Remark 4.1.9.**

(i) The relations in Definition 4.1.8 look different to their original form in [**53**, p313], where they are written down using diagrams. We eschew the diagrammatic approach entirely in this thesis as, although it is useful in some proofs, a purely algebraic approach is neater for our purposes.

(ii) Khovanov and Lauda [**53**] and Rouquier [**95**] considered quiver Hecke algebras for arbitrary quivers, and allowed for greater diversity in their quadratic and braid relations. In particular, Rouquier allowed for dependence on a coefficient matrix $Q$ with entries in a polynomial ring $\mathcal{Z}[u, v]$. See also [**60**] for more details. Using their notation, our matrix $Q = (Q_{ij})_{i,j\in I}$ is

$$Q_{ij} = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } i \not- j \\ v - u, & \text{if } i \to j \\ u - v, & \text{if } i \leftarrow j \end{cases}$$

however this is largely inconsequential, as by [95, Proposition 3.12], subject to some mild constraints, the algebra $\mathcal{R}_\alpha$ in the end does not depend on the choice of matrix $Q$.

Using (4.1.7), if we define

$$e_\alpha = \sum_{\mathbf{i}\in I^\alpha} e(\mathbf{i}) \tag{4.1.10}$$

for $\alpha \in Q_e$ then the full quiver Hecke algebra $\mathcal{R}_n := \mathcal{R}_n(e, \mathcal{Z})$ is defined as

$$\mathcal{R}_n = \bigoplus_{\alpha\in Q_e} \mathcal{R}_\alpha. \tag{4.1.11}$$

The algebras above are actually the *blocks* of $\mathcal{R}_n$ [53, Corollary 2.11] (the *blocks* of an associative algebra are its indecomposable two-sided ideals) and the summand $\mathcal{R}_\alpha = \mathcal{R}_\alpha(e, \mathcal{Z}) = e_\alpha \mathcal{R}_n e_\alpha$ is a two-sided ideal of the algebra $\mathcal{R}_n$.

Our next goal is to give a basis theorem for blocks of quiver Hecke algebras due to Khovanov-Lauda and Rouquier. For each permutation $\omega \in \mathfrak{S}_n$, fix a reduced expression $\omega = s_{i_1} s_{i_2} \cdots s_{i_n}$ and define

$$\psi_\omega = \psi_{i_1}\psi_{i_2}\cdots\psi_{i_n}.$$

Importantly, because the braid relations for the elements $\{\psi_1, \psi_2, \ldots, \psi_{n-1}\}$ are more complicated than the symmetric group braid relations, the elements $\psi_\omega$ *do* depend on the choice of reduced expression.

**Example 4.1.12** (Rank-two quiver Hecke algebras)**.** Let $n = 2$ and suppose $2 < e < \infty$. Then, remembering $I = \mathbb{Z}/e\mathbb{Z}$, $\mathcal{R}_2(e, \mathcal{Z})$ is the $\mathcal{Z}$-algebra generated by $\psi = \psi_1, y_1, y_2$ and $\{e(ij) \mid i, j \in I\}$ subject to the relations

$$e(ij)e(kl) = \delta_{ik}\delta_{jl}e(ij), \qquad \sum_{i,j\in I} e(ij) = 1$$

$$y_r e(ij) = e(ij) y_r$$

$$\psi e(ij) = e(ji)\psi$$

$$\psi y_2 e(ij) = (y_1\psi + \delta_{ij})e(ij)$$

$$\psi^2 e(ij) = \begin{cases} (y_1 - y_2)e(ij), & \text{if } i \to j \\ (y_2 - y_1)e(ij), & \text{if } i \leftarrow j \\ 0, & \text{if } i = j \\ e(ij), & \text{otherwise} \end{cases}$$

Using these relations it is easy to see that

$$\{\psi^a y_1^b y_2^c e(ij) \mid a \in \{0,1\},\ b, c \geq 0,\ i, j \in I\}$$

is a spanning set for the algebra $\mathcal{R}_2$. In fact, elements like these always give a basis for the quiver Hecke algebras.

The following theorem appears as [**53**, Theorem 2.5] and [**95**, Theorem 3.7].

**Theorem 4.1.13** (Basis theorem for quiver Hecke algebras)**.** *Let $\alpha \in Q_e$. Then $\mathcal{R}_\alpha(e, \mathcal{Z})$ is a free $\mathcal{Z}$-algebra with homogeneous basis*

$$\{\psi_\omega y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n} e(\mathbf{i}) \mid \omega \in \mathfrak{S}_n,\ a_1, a_2, \ldots, a_n \in \mathbb{N},\ \mathbf{i} \in I^\alpha\}.$$

The above result can be thought of the first reason to be interested in studying quiver Hecke algebras, especially when one considers its tantalising similarity to the basis theorem for cyclotomic Hecke algebras (Theorem 3.1.6). We will return to quiver Hecke algebras in the next chapter, where we prove a similar basis theorem for a certain fixed-point subalgebra.

**Example 4.1.14** (Rank-one algebras)**.** Let $n = 1$ and suppose $e < \infty$. Then, remembering $I = \mathbb{Z}/e\mathbb{Z}$,

$$\begin{aligned} \mathcal{R}_1(e, \mathcal{Z}) &= \left\langle y, e(i) \;\middle|\; i \in I,\ ye(i) = e(i)y,\ \sum_{i \in I} e(i) = 1,\ e(i)e(j) = \delta_{ij} e(i) \right\rangle \\ &\cong \bigoplus_{i \in I} \mathcal{Z}[y]e(i). \end{aligned}$$

where $\deg y = 2$ and $\deg e(i) = 0$. This has the obvious homogeneous basis

$$\{y^k e(i) \mid k \geq 0, i \in I\}.$$

## 4.2. Cyclotomic quiver Hecke algebras

For most of this thesis we are concerned with certain cyclotomic quotients of the quiver Hecke algebras, which were originally defined by Brundan and Kleshchev in their landmark paper [17].

**Definition 4.2.1** (Cyclotomic quiver Hecke algebra)**.** For $\Lambda \in P_e$ the *cyclotomic quiver Hecke algebra* of type $\Gamma_e$, weight $\Lambda$ and corresponding to $\alpha \in Q_e$ is the quotient

$$\mathcal{R}_\alpha^\Lambda = \mathcal{R}_\alpha^\Lambda(e, \mathcal{Z}) = \mathcal{R}_\alpha(e, \mathcal{Z})/\mathcal{I}_\alpha^\Lambda,$$

where $\mathcal{I}_\alpha^\Lambda$ is the ideal $\langle y_1^{(\Lambda, \alpha_{i_1})} e(\mathbf{i}) \mid \mathbf{i} \in I^\alpha \rangle$. The *cyclotomic quiver Hecke algebra* of type $\Gamma_e$ and weight $\Lambda$ is the direct sum

$$\mathcal{R}_n^\Lambda = \bigoplus_{\alpha \in Q_e} \mathcal{R}_\alpha^\Lambda. \tag{4.2.2}$$

The algebras $\mathcal{R}_\alpha^\Lambda$ are the blocks of the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$.

**Definition 4.2.3.** An $\mathcal{R}_n^\Lambda$-module $M$ is said to *belong* to the block $\mathcal{R}_\alpha^\Lambda$ if $M = \mathcal{R}_\alpha^\Lambda M$.

Although it is not immediately obvious, the following result shows that the algebras $\mathcal{R}_\alpha^\Lambda$ (and hence the algebras $\mathcal{R}_n^\Lambda$) are finite-dimensional. We use a convenient abuse of notation and write $\psi_r \in \mathcal{R}_\alpha^\Lambda$ for the image of $\psi_r \in \mathcal{R}_\alpha$ under the canonical projection map for $1 \leq r < n$, and similarly for the image of $y_r$ for $1 \leq r \leq n$ and of $e(\mathbf{i})$ for $\mathbf{i} \in I^\alpha$.

**Proposition 4.2.4.** [17, Lemma 2.1] *The elements $\{y_1, y_2, \ldots, y_n\}$ in $\mathcal{R}_\alpha^\Lambda$ are nilpotent.*

We will need the following result several times below.

**Proposition 4.2.5.** [41, Lemma 4.1] *Let $\mathbf{i} \in I^n$. Then in the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$, $e(\mathbf{i}) \neq 0$ if and only if $\mathbf{i}$ is the residue sequence of a standard tableau.*

**Example 4.2.6.** Let $n = 3$, $e = 3$ and $\Lambda = \Lambda_0$. The cyclotomic relation immediately gives $y_1 = 0$. Using the commutativity relation between $\psi_1$ and the idempotents, Proposition 4.2.5 shows that $\psi_1 = 0$ as well, since $(012)$ and $(021)$ are the only sequences $\mathbf{i} \in I^3$ with $e(\mathbf{i}) \neq 0$. By the quadratic relation,

$$\psi_1^2 e(\mathbf{i}) = 0 = \begin{cases} y_2 e(\mathbf{i}), & \text{if } i_2 = 1 \\ -y_2 e(\mathbf{i}), & \text{if } i_2 = 2 \end{cases}$$

so that $y_2 = 0$ as well. Hence by Proposition 4.2.5, there are four nonzero generators: $\psi_2, y_3, e(012)$ and $e(021)$. Observing the quadratic relation again we see that

$$\psi_2^2 e(\mathbf{i}) = \begin{cases} y_3 e(\mathbf{i}), & \mathbf{i} = (012) \\ -y_3 e(\mathbf{i}), & \mathbf{i} = (021) \end{cases}$$

Finally, as in the proof of Proposition 4.2.4, $y_3^2 e(\mathbf{i}) = \pm y_3 \psi_2 e(\mathbf{i}) = \pm \psi_2 y_2 e(\mathbf{i}) = 0$, whence $y_3^2 = 0$. Therefore we have six elements

$$e(012),\ e(021),\ \psi_2 e(012),\ \psi_2 e(021),\ y_3 e(012),\ y_3 e(021)$$

which span our algebra. We will see shortly that these are a basis for this 6-dimensional algebra (indeed, up to sign, they are precisely a certain distinguished homogeneous graded cellular basis (§7.1)).

**Example 4.2.7** (Young's seminormal form)**.** [**17**, §5] Let $n > 1$, $e = \infty$ and $\Lambda = \Lambda_0$. Then it is easy to describe the representation theory of the cyclotomic quiver Hecke algebra $\mathcal{R}_n^{\Lambda_0}(\infty, F)$ for a field $F$ as in [**17**, §5]. Indeed, for each $\lambda \in \mathcal{P}_n$ there is a unique irreducible graded $\mathcal{R}_n^{\Lambda_0}$ module $S^\lambda$ with basis

$$\{v_{\mathtt{t}} \mid \mathtt{t} \in \operatorname{Std}(\lambda)\},$$

which is homogeneous with respect to the trivial degree function $\deg v_{\mathtt{t}} = 0$ for all $\mathtt{t} \in \operatorname{Std}(\lambda)$ and with left $\mathcal{R}_n^{\Lambda_0}$-action given by

$$e(\mathbf{i}) v_{\mathtt{t}} = \begin{cases} v_{\mathtt{t}}, & \text{if } \mathbf{i} = \mathbf{i}_{\mathtt{t}} \\ 0, & \text{otherwise} \end{cases}$$

$$y_r v_{\mathtt{t}} = 0,$$

$$\psi_r v_{\mathtt{t}} = \begin{cases} v_{s_r \cdot \mathtt{t}}, & \text{if } s_r \cdot \mathtt{t} \in \mathrm{Std}(\lambda) \\ 0, & \text{otherwise} \end{cases}$$

Using the formulas in the next section, one can obtain the usual coefficients from Young's seminormal form [**107**] from this action.

## 4.3. The Brundan-Kleshchev isomorphism

Quiver Hecke algebras were originally presented by Khovanov and Lauda [**53**] and Rouquier [**95**] as a solution to the problem of categorifying $U_q^-(\mathfrak{g})$, the negative half of the quantised universal enveloping algebra of $\mathfrak{g}$, where $\mathfrak{g}$ is the Kac-Moody Lie algebra associated with the appropriate quiver. The first indication of their deep link with the theory of Hecke algebras was given by Brundan and Kleshchev [**17**], who proved a remarkable isomorphism theorem, which was conjectured by Khovanov and Lauda.

Given $\Lambda \in P_e$, we may form a multicharge (see Definition 3.1.1) $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\Lambda)$ by taking $\boldsymbol{\kappa} \in I^\ell$ such that $\Lambda = \Lambda_{\overline{\kappa_1}} + \ldots + \Lambda_{\overline{\kappa_\ell}}$.

**Definition 4.3.1.** For a dominant weight $\Lambda$, we write $\mathscr{H}_n^\Lambda$ for the cyclotomic Hecke algebra $\mathscr{H}_{n,\ell}(\mathcal{Z}, \xi, \boldsymbol{\kappa}(\Lambda))$.

For the remainder of this section, let $F$ be an arbitrary field with $\xi \in F^\times$ such that the quantum characteristic of $\xi$ in $F$ is $e > 2$. Fix a modular system $(\mathcal{K}, \mathcal{O}, F)$ as in Definition 3.5.7.

As we are stating our theorems about quiver Hecke algebras blockwise, we need to classify the blocks of cyclotomic Hecke algebras in order to state Brundan-Kleshchev's isomorphism theorem. Given $\alpha \in Q_e$, using the idempotents from (3.5.18), define

$$f_\alpha^{\mathcal{O}} = \sum_{\mathbf{i} \in I^\alpha} f_{\mathbf{i}}^{\mathcal{O}} \in \mathscr{H}_n^\Lambda(\mathcal{O}),$$

where $f_{\mathbf{i}}^{\mathcal{O}}$ is defined in (3.5.18). By [**16**, Theorem 1] and [**71**], $f_\alpha^{\mathcal{O}}$ is either zero or it is a primitive central idempotent in $\mathscr{H}_n$, so the algebra $f_\alpha^{\mathcal{O}} \mathscr{H}_n^\Lambda f_\alpha^{\mathcal{O}}$ is either zero or a block of $\mathscr{H}_n^\Lambda$. We denote such a block by $\mathscr{H}_\alpha^\Lambda = \mathscr{H}_{\alpha,\ell}(\mathcal{O}, \xi, \boldsymbol{\kappa}(\Lambda))$.

We also obtain blocks $\mathscr{H}_{\alpha,\ell}(F,\xi,\boldsymbol{\kappa}(\Lambda))$ over $F$ by taking $e(\alpha)\mathscr{H}_{\alpha}^{\Lambda}(F)e(\alpha)$, where $e(\alpha) = f_{\alpha}^{\mathcal{O}} \otimes_{\mathcal{O}} 1_F$.

**Theorem 4.3.2** (Brundan-Kleshchev isomorphism)**.** *Let $F$ be a field and suppose $\xi \in F^{\times}$ has quantum characteristic $e$. For $\Lambda \in P_e^+$ and $\alpha \in Q_e$,*

$$\mathcal{R}_{\alpha}^{\Lambda}(e,F) \cong \mathscr{H}_{\alpha,\ell}(F,\xi,\boldsymbol{\kappa}(\Lambda))$$

*as $F$-algebras. In particular, $\mathcal{R}_n^{\Lambda} \cong \mathscr{H}_n^{\Lambda}$ as $F$-algebras and*

$$\dim_F \mathcal{R}_n^{\Lambda} = \ell^n n!. \tag{4.3.3}$$

Rather than using complicated machinery or deep geometric arguments, which have been used in proofs of Theorem 4.3.2 since [**95, 103**], Brundan and Kleshchev construct explicit isomorphisms in both directions and calculate that they compose to the identity. In full detail, $e(\mathbf{i}) \mapsto f_{\mathbf{i}}^{\mathcal{O}}$ and

$$\begin{aligned} y_r e(\mathbf{i}) &\mapsto \xi^{-i_r}(L_r - [i_r]) f_{\mathbf{i}}^{\mathcal{O}} \\ \psi_r e(\mathbf{i}) &\mapsto (T_r + P_r(\mathbf{i}))\frac{1}{Q_r(\mathbf{i})} f_{\mathbf{i}}^{\mathcal{O}}, \end{aligned} \tag{4.3.4}$$

where $P_r(\mathbf{i})$ and $Q_r(\mathbf{i})$ are certain rational functions in $F(y_r, y_{r+1})$ (see [**17**, (3.32), (3.30), (4.27) and (4.36)]). The inverse isomorphism is then given by $f_{\mathbf{i}}^{\mathcal{O}} \mapsto e(\mathbf{i})$ and

$$\begin{aligned} L_r &\mapsto \sum_{\mathbf{i}\in I^n} (\xi^{i_r} y_r + [i_r]) e(\mathbf{i}) \\ T_r &\mapsto \sum_{\mathbf{i}\in I^n} (\psi_r Q_r(\mathbf{i}) - P_r(\mathbf{i})) e(\mathbf{i}). \end{aligned}$$

We will not go into detail of the original proof of this theorem here. The *Brundan-Kleshchev polynomials* $P_r(\mathbf{i})$ and $Q_r(\mathbf{i})$ are interesting as the $Q_r(\mathbf{i})$ are not unique; we will see the shadow of this again later.

It is worthwhile giving an example of how this isomorphism works in practice, as such examples are (as far as the author is aware) currently non-existent in the literature.

**Example 4.3.5.** Suppose $\xi = 1$ and $e = 3$ and let $F = \mathbb{F}_3 = \{0, 1, 2\}$. Then as in Example 4.2.6 there are two idempotents, $e(012)$ and $e(021)$, which can be shown to map to the idempotents

$$2 + 2s_1 \quad \text{and} \quad 2 + s_1$$

respectively (as in Example 3.3.9, after reducing modulo 3). Moreover, $y_1 \mapsto 0$ and $y_2 \mapsto 0$. Now

$$\begin{aligned} y_3 e(012) &= (s_2 + s_1 s_2 s_1 - 2)(2 + 2s_1) \\ &= 2 + 2s_1 + 2s_2 + 2s_1 s_2 + 2s_2 s_1 + 2s_1 s_2 s_1 \end{aligned}$$

and

$$\begin{aligned} y_3 e(021) &= (s_2 + s_1 s_2 s_1 - 1)(2 + s_1) \\ &= 1 + 2s_1 + 2s_2 + s_1 s_2 + s_2 s_1 + 2s_1 s_2 s_1 \end{aligned}$$

and so $y_3 \mapsto s_1 + s_2 + s_1 s_2 s_1$. One can check using the formulas in [**17**, (3.22)] that

$$\begin{aligned} P_2(012) &= 2 + y_3 = 2 + s_1 + s_2 + s_1 s_2 s_1 \\ P_2(021) &= 1 + y_3 = 1 + s_1 + s_2 + s_1 s_2 s_1 \end{aligned}$$

where to compute $P_2(\mathbf{i})$ we use the fact that $y_3^2 = 0$ from Example 4.2.6, as well as several applications of the geometric series formula, finally converting the resulting polynomials in $y_3$ into the group basis. Using [**17**, (3.30)], the $Q_r(\mathbf{i})$ polynomials in this case are simply the scalars

$$Q_2(012) = 2 \quad \text{and} \quad Q_2(021) = 1.$$

Hence

$$\begin{aligned} \psi_2 e(012) &= 2[s_2 + (2 + s_1 + s_2 + s_1 s_2 s_1)](2 + 2s_1) \\ &= 2s_2 + s_1 s_2 + 2s_2 s_1 + s_1 s_2 s_1 \end{aligned}$$

$$\begin{aligned}\psi_2 e(021) &= [s_2 + (1 + s_1 + s_2 + s_1 s_2 s_1)](2 + s_1)\\ &= s_2 + s_1 s_2 + 2 s_2 s_1 + 2 s_1 s_2 s_1\end{aligned}$$

and so

$$\psi_2 = 2 s_1 s_2 + s_2 s_1.$$

It is not too hard to see that $\{e(012), e(021), y_3 e(012), y_3 e(021), \psi_2 e(012), \psi_2 e(021)\}$ gives a basis for the algebra and so the Brundan-Kleshchev isomorphism, as a linear map $\mathcal{R}_3 \xrightarrow{\sim} \mathbb{F}_3 \mathfrak{S}_3$ with respect to the group basis, has matrix

$$\begin{pmatrix} 2 & 2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 & 1 & 2 \end{pmatrix}.$$

**Example 4.3.6.** By Theorem 4.3.2, the algebra $\mathcal{R}_n(\infty, F)$ in Example 4.2.7 is isomorphic to $\mathscr{H}_{n,1}(F, 1, 0) \cong F\mathfrak{S}_n$ when $F$ is a field of characteristic zero.

## Chapter 5

# Alternating quiver Hecke algebras

In this chapter, we define a new family of algebras called alternating quiver Hecke algebras. For quiver Hecke algebras corresponding to a certain class of dominant weights $\Lambda$ which we call *symmetric* dominant weights, it is possible to define a homogeneous involution on the quiver Hecke algebra; studying the fixed-point subalgebra under this involution gives rise to our family of algebras. We discuss these algebras using a version of Clifford theory for associative algebras, and construct a homogeneous basis and a presentation by homogeneous generators and relations which are reminiscent of Khovanov and Lauda [**53**] and Rouquier's [**95**] theorems for quiver Hecke algebras. We then discuss cyclotomic quotients of these algebras which we call alternating cyclotomic quiver Hecke algebras.

## 5.1. Clifford theory for associative algebras

In order to deduce results about our fixed-point subalgebras from the corresponding results for the full algebra, we will use the language of Clifford theory. Clifford theory was initially developed to study the representations of normal subgroups of finite groups [**23**]. Here we adapt it to cover associative algebras with a $C_2$-graded Clifford system; details in the finite group case can be found in [**25**], and a slightly different and more general treatment for associative algebras is given in [**92**]. Below we write the two elements of the cyclic group $C_2$ as $\{+, -\}$, where signs multiply according to the usual rules. We also note that Mitsuhashi referred to these as $\mathbb{Z}_2$-graded Clifford systems in [**84**].

**Definition 5.1.1.** Let $A$ be a $\mathcal{Z}$-algebra. A *$C_2$-graded Clifford system* for $A$ is a family $\{A_s \mid s \in C_2\}$ of two $\mathcal{Z}$-submodules of $A$ such that

(i) $A_sA_t = A_{st}$, for $s, t \in C_2$;

(ii) there exists a distinguished central element $\varepsilon \in A$ such that $\varepsilon^2 = 1$ and $A_+ = \varepsilon A_-$;

(iii) $A = A_+ \oplus A_-$; and

(iv) $1 \in A_+$.

$A_+$ is called the *even* part of the algebra; $A_-$ is called the *odd* part.

Clifford theory allows a neat description of the representation theory of the subalgebra $A_+$ given knowledge of the representation theory of $A$. Specifically, for an $A_+$-module $M$, we may twist the $A_+$-action by $\varepsilon$ to define the module $M^\varepsilon$, which is $M$ as an $\mathcal{O}$-module and where $a \in M$ acts as

$$a \cdot m = \varepsilon a \varepsilon m.$$

Using Definition 5.1.1(ii) to realise $C_2$ as $\{1, \varepsilon\}$, the *inertia group* of $M$ is

$$\mathcal{I}(M) = \{x \in C_2 \mid M^x \cong M\} \trianglelefteq C_2.$$

The size of the inertia group (either 1 or 2) determines the behaviour of an irreducible representation $N$ restricted from $A$ to $A_+$, which we denote by $\mathrm{Res}^A_{A_+} N$, in the following sense. We refer the reader to [**24**, pp344–345] for a proof of the following result, noting that inverting 2 is necessary to prove the direct sum decomposition in (ii).

**Proposition 5.1.2** (Clifford's theorem for $C_2$-graded associative algebras)**.** *Let 2 be invertible in $\mathcal{Z}$ and let $A$ be an associative $\mathcal{Z}$-algebra with a $C_2$-graded Clifford system. Let $N$ be an irreducible $A$-module. Then*

(i) *If $\mathcal{I}(\mathrm{Res}^A_{A_+} N) = C_2$ then $\mathrm{Res}^A_{A_+} N \cong \left(\mathrm{Res}^A_{A_+} N\right)^\varepsilon$ is an irreducible $A_+$-module.*

(ii) *If $\mathcal{I}(\mathrm{Res}^A_{A_+} N) = 1$ then $\mathrm{Res}^A_{A_+} N = M_+ \oplus M_-$ is the direct sum of two irreducible $A_+$-modules, related under the conjugation map (i.e. $M_+ = M_-^\varepsilon$).*

*Moreover,* all *irreducible $A_+$-modules arise in one of these two ways.*

## 5.2. Alternating quiver Hecke algebras

We have seen in §3.5 how to define alternating cyclotomic Hecke algebras by taking the subalgebra of fixed-points under the hash map, which is an involution defined on cyclotomic Hecke algebras with symmetric multicharge. Our goal now is to generalise the Brundan-Kleshchev isomorphism framework to these subalgebras in level 1, and the correct objects on the other side of our isomorphism theorem are what we call alternating quiver Hecke algebras, which we define for arbitrary level. These are defined as subalgebras of fixed-points under a new homogeneous involution.

**For the remainder of this section, $\mathcal{Z}$ is a unital integral domain with $\xi \in \mathcal{Z}^\times$ such that the quantum characteristic $e$ of $\xi$ in $\mathcal{Z}$ is greater than 2 (see Definition 3.1.3).**

Let us work again with the quiver Hecke algebra $\mathcal{R}_n = \mathcal{R}_n(e, \mathcal{Z})$ from §4.1. Recall the definition of $I$ from (4.1.1) and, for $\mathbf{i} \in I^n$, of $-\mathbf{i} \in I^n$ from (3.5.22).

**Definition 5.2.1** (Kleshchev, Mathas and Ram [62])**.** The *graded sign map* $\mathsf{sgn} : \mathcal{R}_n \to \mathcal{R}_n$ is the map defined on generators as

$$e(\mathbf{i}) \mapsto e(-\mathbf{i}), \qquad y_r \mapsto -y_r, \qquad \psi_r \mapsto -\psi_r.$$

**Proposition 5.2.2.** [62] *The graded sign map is a well-defined homogeneous algebra involution $\mathcal{R}_n \to \mathcal{R}_n$.*

**Proof.** That the map is a homogeneous involution is clear from its definition; checking it is a well-defined algebra homomorphism amounts to checking it respects the list of relations in Definition 4.1.8, a straightforward exercise which we leave to the reader. □

We need to determine how the $\mathtt{sgn}$ map restricts to the blocks of the quiver Hecke algebras. For a given $\alpha \in Q_e$ with $\alpha = \alpha_{j_1} + \alpha_{j_2} + \ldots + \alpha_{j_n}$, we can define

$$\alpha' = \alpha_{-j_1} + \alpha_{-j_2} + \ldots + \alpha_{-j_n},$$

recalling that $j_i \in \mathbb{Z}/e\mathbb{Z}$ and so $-j_i \in \mathbb{Z}/e\mathbb{Z}$ for $i = 1, 2, \ldots, n$. The following fact is easy to prove from the definitions.

**Lemma 5.2.3.** *For $\mathbf{i} \in I^n$, $\mathbf{i} \in I^\alpha$ if and only if $-\mathbf{i} \in I^{\alpha'}$.*

The above lemma means that, on the level of blocks, $\mathtt{sgn}$ maps between $\mathcal{R}_\alpha$ and $\mathcal{R}_{\alpha'}$ and so is an involution on the block $\mathcal{R}_\alpha$ if $\alpha = \alpha'$, or on the direct sum $\mathcal{R}_\alpha \oplus \mathcal{R}_{\alpha'}$ of blocks if $\alpha \neq \alpha'$.

**Definition 5.2.4.** We write $[\alpha]$ for the equivalence class of $\alpha \in Q_e$ under the equivalence relation $\sim$ generated by $\alpha \sim \beta$ if $\beta = \alpha'$. We write $Q_e^+$ for the set of all equivalence classes with two elements (i.e. when $\alpha \neq \alpha'$), and $Q_e^\pm$ for the set of all equivalence classes with one element.

We are now ready to define alternating quiver Hecke algebras, which we will study for the remainder of this thesis. The definition of alternating quiver Hecke algebras can be motivated from the fact that the alternating group algebra is the fixed-point subalgebra of the symmetric group algebra under the ungraded sign map [49, §2.1].

**Definition 5.2.5.** The *alternating quiver Hecke algebra* is the fixed-point subalgebra $\mathcal{R}_n^{\mathtt{sgn}}$ of the quiver Hecke algebra under the graded sign map.

**Remark 5.2.6.** We note that the $\alpha$ which gives the pathological sequence $\mathbf{i} = \underbrace{(0, 0, \ldots, 0)}_{n \text{ zeroes}}$ is $\alpha = n\alpha_0$. In some results below we need to exclude this case.

Using the equivalence classes and distinguished elements from p40, we now define an element $\varepsilon \in \mathcal{R}_n$ which will be very important in studying alternating quiver

Hecke algebras and their cyclotomic quotients:

$$\varepsilon = \sum_{\substack{[\mathbf{i}]\in I^n{}_\sim \\ \mathbf{i}^+\in[\mathbf{i}]}} \Big(e(\mathbf{i}^+) - e(-\mathbf{i}^+)\Big). \tag{5.2.7}$$

Recall the idempotents $e_\alpha$ from (4.1.10), which define the blocks of quiver Hecke algebras. If $\alpha \neq \alpha'$, we define new idempotents

$$e_{[\alpha]} = \sum_{\beta\in[\alpha]} e_\beta, \tag{5.2.8}$$

noting that since $e_\alpha^{\mathtt{sgn}} = e_{\alpha'}$, $e_{[\alpha]}$ is $\mathtt{sgn}$-invariant; in particular if $\alpha = \alpha'$, $e_\alpha$ is already a $\mathtt{sgn}$-invariant idempotent. Any given equivalence class $[\mathbf{i}] \in I^n_\sim$ will contain a sequence from $I^\alpha$ and $I^{\alpha'}$; we write $I^{[\alpha]}_\sim$ for the collection of all such equivalence classes. Finally, for an equivalence class $[\mathbf{i}] \in I^n_\sim$ we write

$$e[\mathbf{i}] = \sum_{\mathbf{j}\in[\mathbf{i}]} e(\mathbf{j}). \tag{5.2.9}$$

We now give a basis theorem for alternating quiver Hecke algebras analogous to Theorem 4.1.13 for quiver Hecke algebras. It is worthwhile to first consider the following simple rank-one example which exhibits the basic features of the basis in general.

**Example 5.2.10.** Recall the (infinite-dimensional) algebra $\mathcal{R}_1$ from Example 4.1.14. Suppose that $e = 3$ and that $\mathrm{char}(\mathcal{Z}) \neq 2$; then it is not too hard to see that the following is a homogeneous basis for the (again infinite-dimensional) fixed-point subalgebra $\mathcal{R}_1(3, \mathcal{Z})^{\mathtt{sgn}}$, where we have grouped basis vectors by degree:

$$\begin{aligned}
\deg 0: &\quad 2e(0), \quad e(1) + e(2) \\
\deg 2: &\quad y\Big(e(1) - e(2)\Big) \\
\deg 4: &\quad 2y^2e(0), \quad y^2\Big(e(1) + e(2)\Big) \\
\deg 6: &\quad y^3\Big(e(1) - e(2)\Big)
\end{aligned}$$

$$\vdots$$

where $y = y_1$. For general $e > 2$, we have the homogeneous basis

$$\Big\{y^k\Big(e(i) + e(-i)\Big) \mid k \text{ even}\Big\} \cup \Big\{y^k\Big(e(i) - e(-i)\Big) \mid k \text{ odd},\ i \neq 0\Big\}$$

for $\mathcal{R}_n(e, \mathcal{Z})^{\mathtt{sgn}}$. Note that

$$\dim\Big([\mathcal{R}_n(e, \mathcal{Z})^{\mathtt{sgn}}]_{2k} \oplus [\mathcal{R}_n(e, \mathcal{Z})^{\mathtt{sgn}}]_{2(k+1)}\Big) \\ = \frac{1}{2}\dim\Big([\mathcal{R}_n(e, \mathcal{Z})]_{2k} \oplus [\mathcal{R}_n(e, \mathcal{Z})]_{2(k+1)}\Big)$$

for all $k > 0$. Finally, the reader should check that $\mathcal{R}_1(3, \mathcal{Z})e^+([1])$ has the $C_2$-Clifford decomposition

$$\mathcal{R}_1(3, \mathcal{Z})e(1) \oplus \mathcal{R}_1(3, \mathcal{Z})e(2) \cong \Big[\mathcal{R}_1(3, \mathcal{Z})e(1) \oplus \mathcal{R}_1(3, \mathcal{Z})e(2)\Big]^{\mathtt{sgn}} \\ \oplus \varepsilon\Big[\mathcal{R}_1(3, \mathcal{Z})e(1) \oplus \mathcal{R}_1(3, \mathcal{Z})e(2)\Big]^{\mathtt{sgn}},$$

where $\varepsilon = e(1) - e(2)$, and that the transition matrix between this basis and the basis from Example 4.1.14 requires inverting 2.

Recall that $Q_e^+$ (resp. $Q_e^{\pm}$) is the set of all equivalence classes $[\alpha]$ with two (resp. one) elements (resp. element).

**Definition 5.2.11.** For $[\alpha] \in Q_e^+$, define $\mathcal{R}_{[\alpha]}^{\mathtt{sgn}} = \mathcal{R}_{[\alpha]}(e, \mathcal{Z})^{\mathtt{sgn}} = e_{[\alpha]}\mathcal{R}_n^{\mathtt{sgn}}$ to be the fixed-point subalgebra of the direct sum $\mathcal{R}_\alpha \oplus \mathcal{R}_{\alpha'}$ under the $\mathtt{sgn}$ involution. If $[\alpha] \in Q_e^{\pm}$ and $\alpha \neq n\alpha_0$), write $\mathcal{R}_{[\alpha]}^{\mathtt{sgn}}$ for the fixed-point subalgebra of $\mathcal{R}_\alpha$ under the $\mathtt{sgn}$ involution. Finally, if $\alpha = n\alpha_0$, write $\mathcal{R}_{n\alpha_0}^{\mathtt{sgn}}$ for the fixed-point subalgebra of $\mathcal{R}_{n\alpha_0}$ under the $\mathtt{sgn}$ involution.

**Remark 5.2.12.** By the definition above and the decomposition in (4.2.2), the alternating quiver Hecke algebra is equal to the direct sum

$$\mathcal{R}_n^{\mathtt{sgn}} = \Big(\bigoplus_{[\alpha]\in Q_e^+} \mathcal{R}_{[\alpha]}^{\mathtt{sgn}}\Big) \oplus \Big(\bigoplus_{\substack{[\alpha]\in Q_e^{\pm} \\ \alpha\neq n\alpha_0}} \mathcal{R}_{[\alpha]}^{\mathtt{sgn}}\Big) \oplus \mathcal{R}_{n\alpha_0}^{\mathtt{sgn}}. \tag{5.2.13}$$

It is important to note that in general these are not the *blocks* of the alternating quiver Hecke algebras, as they are not guaranteed to be indecomposable (although most of them are).

For $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$, write $y^{\mathbf{a}} \in \mathcal{R}_n$ for the monomial $y_1^{a_1} y_2^{a_2} \cdots y_n^{a_n}$ and let $|\mathbf{a}| = \sum_{i=1}^n a_i$.

**Theorem 5.2.14** (Basis theorem for alternating quiver Hecke algebras)**.** *Let 2 be invertible in $\mathcal{Z}$.*

(i) *If $[\alpha] \in Q_e^+ \cup (Q_e^\pm \setminus \{n\alpha_0\})$ then the alternating quiver Hecke algebra $\mathcal{R}_{[\alpha]}(e, \mathcal{Z})^{\mathtt{sgn}}$ has homogeneous basis*

$$\left\{ \psi_\omega y^{\mathbf{a}} \varepsilon^b e[\mathbf{i}] \;\middle|\; \begin{array}{c} \omega \in \mathfrak{S}_n, \quad \mathbf{a} \in \mathbb{N}^n, \quad [\mathbf{i}] \in I_{\sim}^{[\alpha]}, \\ b \in \{0,1\}, \quad \ell(\omega) + |\mathbf{a}| + b \equiv 0 \ mod \ 2 \end{array} \right\}$$

(ii) *If $\alpha = n\alpha_0$ then the alternating quiver Hecke algebra $\mathcal{R}_{n\alpha_0}(e, \mathcal{Z})^{\mathtt{sgn}}$ has homogeneous basis*

$$\{\psi_\omega y^{\mathbf{a}} e \mid \omega \in \mathfrak{S}_n, \ \ \mathbf{a} \in \mathbb{N}^n, \ \ \ell(\omega) + |\mathbf{a}| \equiv 0 \ mod \ \ 2\},$$

*where* $e = \underbrace{(0, 0, \ldots, 0)}_{n \ zeroes}$.

**Proof.** For $[\alpha] \in Q_e^+ \cup (Q_e^\pm \setminus \{n\alpha_0\})$, to observe that the specified set spans $\mathcal{R}_{[\alpha]}^{\mathtt{sgn}}$, let us write an arbitrary element of $\mathcal{R}_\alpha \oplus \mathcal{R}_{\alpha'}$ as a finite sum

$$x = \sum_{\omega, \mathbf{a}, \mathbf{i}} \lambda_{\omega, \mathbf{a}, \mathbf{i}} \psi_\omega y^{\mathbf{a}} e(\mathbf{i}),$$

for $\omega \in \mathfrak{S}_n$, $\mathbf{a} \in \mathbb{N}^n$, $\mathbf{i} \in I^\alpha \cup I^{\alpha'}$ and $\lambda_{\omega, \mathbf{a}, \mathbf{i}} \in \mathcal{Z}$ using Theorem 4.1.13. In order that $x$ be an element of $\mathcal{R}_{[\alpha]}^{\mathtt{sgn}}$ we require $x^{\mathtt{sgn}} = x$, i.e. that

$$\sum_{\omega, \mathbf{a}, \mathbf{i}} (-1)^{\ell(\omega) + \sum_i a_i} \lambda_{\omega, \mathbf{a}, -\mathbf{i}} \psi_\omega y^{\mathbf{a}} e(\mathbf{i}) = \sum_{\omega, \mathbf{a}, \mathbf{i}} \lambda_{\omega, \mathbf{a}, \mathbf{i}} \psi_\omega y^{\mathbf{a}} e(\mathbf{i})$$

which implies that the given elements span by applying Theorem 4.1.13 and equating coefficients.

For linear independence, take a linear combination

$$\sum_{[\mathbf{i}]\in I^{\alpha/\alpha'}_{\sim}}\Big(\sum_{\substack{\omega\in\mathfrak{S}_n,\mathbf{a}\in\mathbb{N}^n\\ \ell(\omega)+\sum_i a_i\equiv 0 \bmod 2}}\lambda_{\mathbf{a},\omega,\mathbf{i}}\psi_\omega y^{\mathbf{a}}\sum_{\mathbf{j}\in[\mathbf{i}]}e(\mathbf{j})$$
$$+\sum_{\substack{\omega\in\mathfrak{S}_n,\mathbf{a}\in\mathbb{N}^n\\ \ell(\omega)+\sum_i a_i\equiv 1 \bmod 2}}\lambda_{\mathbf{a},\omega,\mathbf{i}}\psi_\omega y^{\mathbf{a}}\varepsilon\sum_{\mathbf{j}\in[\mathbf{i}]}e(\mathbf{j})\Big)=0$$

for coefficients $\lambda_{\mathbf{a},\omega,\mathbf{i}}\in\mathcal{Z}$ and project the above sum onto the idempotent $e(\mathbf{i})$ for each $\mathbf{i}\in I^\alpha\cup I^{\alpha'}$; then since $\mathbf{i}\neq-\mathbf{i}$ for any idempotents in this case, we obtain sums of basis vectors of the form from Theorem 4.1.13 which are linearly independent.

For $\alpha=n\alpha_0$, the same argument as above shows the set spans the $\mathtt{sgn}$-invariant subalgebra. Since this basis is a subset of the basis from Theorem 4.1.13, linear independence, and hence the result, follows. □

We can obtain a $C_2$-graded Clifford decomposition for direct sums of blocks of quiver Hecke algebras using the element $\varepsilon$ defined in (5.2.7), provided $\alpha$ is not equal to the "pathological" root $n\alpha_0$.

**Proposition 5.2.15.** *Let 2 be invertible in $\mathcal{Z}$ and suppose that $\alpha\neq n\alpha_0$. Then we have the $C_2$-graded Clifford decomposition*

$$\bigoplus_{\beta\in[\alpha]}\mathcal{R}_\beta\cong\mathcal{R}^{\mathtt{sgn}}_{[\alpha]}\oplus\varepsilon\mathcal{R}^{\mathtt{sgn}}_{[\alpha]}.$$

**Proof.** To demonstrate the Clifford decomposition, we check the requirements in Definition 5.1.1. Condition (i) follows easily; since $\varepsilon^{\mathtt{sgn}}=-\varepsilon$, if $x\in\mathcal{R}^{\mathtt{sgn}}_n$ and $y=\varepsilon z\in\varepsilon\mathcal{R}^{\mathtt{sgn}}_n$ then $(xy)^{\mathtt{sgn}}=-xy$ giving the required multiplicative property. Conditions (ii) and (iv) follow by definition, so it remains to demonstrate the direct sum decomposition, which follows from the same argument as in Theorem 3.5.24: since any $x\in\mathcal{R}_\alpha\oplus\mathcal{R}_{\alpha'}$ may be written as $x=\sum_{\mathbf{i}\in I^\alpha\cup I^{\alpha'}}xe(\mathbf{i})=\sum_{\mathbf{i}\in I^\alpha}x\Big[e(\mathbf{i})+e(-\mathbf{i})\Big]$, we can write

$$x=\frac{1}{2}\sum_{\mathbf{i}\in I^\alpha}(x+x^{\mathtt{sgn}})\Big[e(\mathbf{i})+e(-\mathbf{i})\Big]+\frac{1}{2}\varepsilon\sum_{\mathbf{i}\in I^\alpha}(x-x^{\mathtt{sgn}})\Big[e(\mathbf{i})+e(-\mathbf{i})\Big]$$

which gives the required decomposition provided $\frac{1}{2} \in \mathcal{Z}$. $\square$

**Remark 5.2.16.**

(i) We exclude the case $\alpha = n\alpha_0$ in Proposition 5.2.15 because in that case the projection of $\varepsilon$ onto the block $\mathcal{R}_\alpha$ is zero, since if $\alpha = n\alpha_0$, there is a single idempotent $e_{n\alpha_0} = e(0, 0, \ldots, 0)$ which maps to itself under $\mathtt{sgn}$ so $\varepsilon e_{n\alpha_0} = 0$.

(ii) For any $i \in I$, the algebra $\mathcal{R}_{n\alpha_i}$ is called a *nil-Hecke algebra*. Our algebra $\mathcal{R}_{n\alpha_0}^{\mathtt{sgn}}$, which behaves slightly differently to the other summands of the alternating quiver Hecke algebra from (5.2.13), may be thought of as an alternating nil-Hecke algebra.

(iii) The reader may have noticed the dependence of our element $\varepsilon$ on choices of equivalence class representatives for each class $[\mathbf{i}] \in I^n_\sim$. Indeed, there are many choices of Clifford element which give rise to different Clifford decompositions, much in the way that different choices of coset representatives give rise to different Clifford decompositions of representations of finite groups with respect to normal subgroups [**24**].

**Example 5.2.17.** It is worth giving an example of how a Clifford decomposition like that in Proposition 5.2.15 is impossible when $\alpha = n\alpha_0$, as mentioned in Remark 5.2.16(i). Indeed, let $n = 1$ and consider the block $\mathcal{R}_0 = e(0)\mathcal{R}_1$ of the algebra from Example 4.1.14. Then $\mathcal{R}_0 = \mathcal{Z}[y]e(0)$ so $\mathcal{R}_0^{\mathtt{sgn}} = \mathcal{Z}[y^2]e(0)$ since $e(0)^{\mathtt{sgn}} = e(0)$. Since $\varepsilon e(0) = 0$ we do not have a Clifford decomposition in this case. We can also easily compute the basis $\{1, y^2, y^4, \ldots\}$ from Theorem 5.2.14(iii).

## 5.3. Generators and relations for alternating quiver Hecke algebras

Using the $C_2$-graded Clifford decomposition from Theorem 5.2.14, we can give a presentation for the alternating quiver Hecke algebra $\mathcal{R}_n^{\mathtt{sgn}}$ by homogeneous generators and relations, akin to the Khovanov-Lauda presentation for the quiver

Hecke algebras (Definition 4.1.8). The following elements will play the role of the generators from Definition 4.1.8 for alternating quiver Hecke algebras.

**Definition 5.3.1** (Generators for alternating quiver Hecke algebras)**.** For $1 \leq r < n$, define $\Psi_r = \psi_r\varepsilon$, and for $1 \leq r \leq n$, define $\mathcal{Y}_r = y_r\varepsilon$. Finally, for $[\mathbf{i}] \in I^n_\sim$ recall the definition of $e[\mathbf{i}]$ from (5.2.9).

As a corollary to Theorem 5.2.14 we see that the elements defined above generate the alternating quiver Hecke algebras.

**Corollary 5.3.2.** *Let $n > 1$ and $e > 2$ and suppose 2 is invertible in $\mathcal{Z}$.*

(i) *If $\alpha \neq n\alpha_0$, the alternating quiver Hecke algebra $\mathcal{R}^{\mathbf{sgn}}_{[\alpha]}$ is generated by the collection*

$$\{\Psi_1, \Psi_2, \dots, \Psi_{n-1}\} \cup \{\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_n\} \cup \{e[\mathbf{i}] \mid [\mathbf{i}] \in I^{[\alpha]}_\sim\}.$$

(ii) *If $\alpha = n\alpha_0$, the alternating quiver Hecke algebra $\mathcal{R}^{\mathbf{sgn}}_{n\alpha_0}$ is generated by all even products of the generators $\{\psi_1, \psi_2, \dots, \psi_{n-1}\} \cup \{y_1, y_2, \dots, y_n\}$ and $\{e\}$ where $e = e(0, 0, \dots, 0)$.*

**Proof.** Suppose $\alpha \neq n\alpha_0$. We proceed to write each basis vector from Theorem 5.2.14(i) and (ii) in terms of the proposed generators; since the $y_r$ commute with $e(\mathbf{i})$, and since $\varepsilon^2 = 1$,

$$y_1^{a_1} \cdots y_n^{a_n} = \begin{cases} \mathcal{Y}_1^{a_1} \cdots \mathcal{Y}_n^{a_n}, & \text{if } |\mathbf{a}| \equiv 0 \bmod 2 \\ \varepsilon\mathcal{Y}_1^{a_1} \cdots \mathcal{Y}_n^{a_n}, & \text{if } |\mathbf{a}| \equiv 1 \bmod 2 \end{cases}$$

Moreover, since $\psi_\omega e(\mathbf{i}) = e(\omega \cdot \mathbf{i})\psi_\omega$ by the relations in Definition 4.1.8,

$$\psi_\omega = \begin{cases} \Psi_\omega, & \text{if } \ell(\omega) \equiv 0 \bmod 2 \\ \Psi_\omega\varepsilon, & \text{if } \ell(\omega) \equiv 1 \bmod 2 \end{cases}$$

There are four cases to consider; we just give two as illustration of the method of proof. If $|\mathbf{a}| \equiv 0 \bmod 2$ and $\ell(\omega) \equiv 0 \bmod 2$, then the basis vector $\psi_\omega y_1^{a_1} \cdots y_n^{a_n} e[\mathbf{i}]$ is equal to $\Psi_{i_1} \cdots \Psi_{i_d}\mathcal{Y}_1^{a_1} \cdots \mathcal{Y}_n^{a_n} e[\mathbf{i}]$ where $\omega = s_{i_1} \cdots s_{i_d}$. If $|\mathbf{a}| \equiv 1 \bmod 2$ and

$\ell(\omega) \equiv 0 \bmod 2$ then the basis vector

$$\begin{aligned}\psi_\omega y_1^{a_1} \cdots y_n^{a_n}[e(\mathbf{i}) - e(-\mathbf{i})] &= \psi_\omega y_1^{a_1} \cdots y_n^{a_n} \varepsilon e[\mathbf{i}] \\ &= \Psi_{i_1} \cdots \Psi_{i_d} \varepsilon \mathcal{Y}_1^{a_1} \cdots \mathcal{Y}_n^{a_n} \varepsilon e[\mathbf{i}] \\ &= \Psi_{i_1} \cdots \Psi_{i_d} \mathcal{Y}_1^{a_1} \cdots \mathcal{Y}_n^{a_n} e[\mathbf{i}]\end{aligned}$$

since $\varepsilon$ commutes with the $\mathcal{Y}_r$ and squares to 1.

When $\alpha = n\alpha_0$, the argument is essentially the same, and easier than, the case above and we leave this to the reader. □

Our goal now is to obtain a set of *relations* for the generators from Corollary 5.3.2. Let us start by defining a new abstract algebra with a slightly different presentation and the additional structure of a $\mathbb{Z} \times C_2$-grading; as in §5.1, let us realise $C_2$ as the sign group $\{+, -\}$ with usual multiplication of signs.

**Definition 5.3.3.** Let $n \geq 0$ and suppose $2 < e \leq \infty$. The *signed quiver Hecke algebra* of type $\Gamma_e$ and corresponding to $[\alpha] \in Q_e^+ \cup Q_e^\pm$ is the unital associative $\mathcal{Z}$-algebra $\mathcal{R}'_{[\alpha]} = \mathcal{R}'_{[\alpha]}(e, \mathcal{Z})$ with generators

$$\{\psi'_1, \psi'_2, \ldots, \psi'_{n-1}\} \cup \{y'_1, y'_2, \ldots, y'_n\} \cup \left\{\varepsilon_a(\mathbf{i}) \mid \mathbf{i} \in \bigcup_{\beta \in [\alpha]} I^\beta,\ a \in C_2\right\}$$

subject to the relations

$$\begin{aligned}\varepsilon_a(\mathbf{i})\varepsilon_b(\mathbf{j}) &= \delta_{\mathbf{ij}}\varepsilon_{ab}(\mathbf{i}), \qquad \sum_{\mathbf{i} \in I^{[\alpha]}} \varepsilon_+(\mathbf{i}) = 1 \\ \varepsilon_a(\mathbf{i}) &= a\varepsilon_a(-\mathbf{i}) \\ y'_r \varepsilon_a(\mathbf{i}) &= \varepsilon_a(\mathbf{i}) y'_r \\ \psi'_r \varepsilon_a(\mathbf{i}) &= \varepsilon_a(s_r \cdot \mathbf{i}) \psi'_r \\ y'_r y'_s \varepsilon_-(\mathbf{i}) &= y'_s y'_r \varepsilon_-(\mathbf{i}) \\ \psi'_r y'_s \varepsilon_-(\mathbf{i}) &= y'_s \psi'_r \varepsilon_-(\mathbf{i}) \quad \text{if } s \neq r, r+1 \\ \psi'_r \psi'_s \varepsilon_-(\mathbf{i}) &= \psi'_s \psi'_r \varepsilon_-(\mathbf{i}) \quad \text{if } |r - s| > 1\end{aligned}$$

$$\psi'_r y'_{r+1}\varepsilon_a(\mathbf{i}) = \begin{cases} (y'_r\psi'_r + 1)\varepsilon_a(\mathbf{i}), & \text{if } i_r = i_{r+1} \\ y'_r\psi'_r\varepsilon_a(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$y'_{r+1}\psi'_r\varepsilon_a(\mathbf{i}) = \begin{cases} (\psi'_r y'_r + 1)\varepsilon_a(\mathbf{i}), & \text{if } i_r = i_{r+1} \\ \psi'_r y'_r\varepsilon_a(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$(\psi'_r)^2\varepsilon_-(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1} \\ (y'_r - y'_{r+1})\varepsilon_+(\mathbf{i}), & \text{if } i_r \to i_{r+1} \\ (y'_{r+1} - y'_r)\varepsilon_+(\mathbf{i}), & \text{if } i_r \leftarrow i_{r+1} \\ \varepsilon_-(\mathbf{i}), & \text{otherwise} \end{cases}$$

$$\psi'_r\psi'_{r+1}\psi'_r\varepsilon_+(\mathbf{i}) = \begin{cases} \psi'_{r+1}\psi'_r\psi'_{r+1}\varepsilon_+(\mathbf{i}) + \varepsilon_-(\mathbf{i}), & \text{if } i_{r+2} = i_r \to i_{r+1} \\ \psi'_{r+1}\psi'_r\psi'_{r+1}\varepsilon_+(\mathbf{i}) - \varepsilon_-(\mathbf{i}), & \text{if } i_{r+2} = i_r \leftarrow i_{r+1} \\ \psi'_{r+1}\psi'_r\psi'_{r+1}\varepsilon_+(\mathbf{i}), & \text{otherwise} \end{cases}$$

for $\mathbf{i}, \mathbf{j} \in I^\alpha$, $a, b \in C_2$ and all admissible $r$ and $s$.

**Remark 5.3.4.**

(i) As usual, the algebra $\mathcal{R}'_{[\alpha]}$ is a block of the full signed quiver Hecke algebra $\mathcal{R}'_n$, which decomposes as

$$\mathcal{R}'_n = \mathcal{R}'_n(e, \mathcal{Z}) = \bigoplus_{[\alpha]\in Q_e^+ \cup Q_e^\pm} \mathcal{R}'_{[\alpha]}$$

and $\mathcal{R}'_\alpha = \varepsilon^+_\alpha \mathcal{R}'_n$ where $\varepsilon^+_\alpha = \sum_{\mathbf{i}\in I^\alpha} \varepsilon_+(\mathbf{i})$.

(ii) The generators in Definition 5.3.3 are somewhat superfluous: for example we can use the relations to see that $\varepsilon_a(\mathbf{i}) = 0$ whenever $\mathbf{i} = -\mathbf{i}$. These extra generators do however mean we can more easily compare the algebra $\mathcal{R}'_{[\alpha]}$ with the quiver Hecke algebra, which we are about to do.

The proof of the following lemma requires nothing more than an inspection of the relations in Definition 5.3.3 which we leave to the reader (see §2.1 for general remarks on algebras graded by a finite group $G$).

**Lemma 5.3.5.** *The relations in Definition 5.3.3 are homogeneous with respect to the degree function* $\deg_2 : \mathcal{R}'_n \to (\mathbb{Z} \times C_2)$ *given by*

$$\deg_2 \varepsilon_a(\mathbf{i}) = (0, a), \qquad \deg_2 y'_r = (2, -), \qquad \begin{array}{l} \deg_2 \psi'_r \varepsilon_+(\mathbf{i}) = (-c_{i_r i_{r+1}}, -) \\ \deg_2 \psi'_r \varepsilon_-(\mathbf{i}) = (-c_{i_r i_{r+1}}, +) \end{array}$$

*for all* $\mathbf{i} \in I^n$*,* $a \in C_2$ *and* $1 \leq r \leq n$*. In particular,* $\mathcal{R}'_n$ *is a* $(\mathbb{Z} \times C_2)$*-graded* $\mathcal{Z}$*-algebra.*

**Remark 5.3.6.** By forgetting the $C_2$-grading (but *not* the $\mathbb{Z}$-grading) on $\mathcal{R}'_n$ we obtain a $\mathbb{Z}$-graded algebra.

**Proposition 5.3.7.** *Let* $n > 0$ *and suppose* $e > 2$*. If* $2$ *is invertible in* $\mathcal{Z}$*, then for* $\alpha \in Q_e$*,*

$$\mathcal{R}'_{[\alpha]}(e, \mathcal{Z}) \cong \bigoplus_{\beta \in [\alpha]} \mathcal{R}_\beta(e, \mathcal{Z})$$

*as* $\mathbb{Z}$*-graded* $\mathcal{Z}$*-algebras.*

**Proof.** Define a map $\vartheta : \mathcal{R}'_{[\alpha]}(e, \mathcal{Z}) \to \bigoplus_{\beta \in [\alpha]} \mathcal{R}_\beta(e, \mathcal{Z})$ on generators by

$$\vartheta(y'_r) = y_r$$
$$\vartheta(\psi'_s) = \psi_s$$
$$\vartheta(\varepsilon_a(\mathbf{i})) = \begin{cases} e(\mathbf{i}) + e(-\mathbf{i}), & \text{if } a = + \\ e(\mathbf{i}) - e(-\mathbf{i}), & \text{if } a = - \end{cases}$$

for $1 \leq r \leq n$ and $1 \leq s < n$ and $\mathbf{i} \in I^\alpha$. We must check $\vartheta$ is an algebra homomorphism of degree zero; this amounts to the largely tedious and straightforward calculation of checking it preserves the relations. We check two relations so the reader can obtain a taste for how they are done. For example, let $\mathbf{i}, \mathbf{j} \in I^\alpha$. Then we have

$$\begin{aligned} \vartheta(\varepsilon_a(\mathbf{i})\varepsilon_b(\mathbf{j})) &= \vartheta(\delta_{\mathbf{ij}}\varepsilon_{ab}(\mathbf{i})) \\ &= \begin{cases} \delta_{\mathbf{ij}}(e(\mathbf{i}) + e(-\mathbf{i})), & \text{if } a = b \\ \delta_{\mathbf{ij}}(e(\mathbf{i}) - e(-\mathbf{i})), & \text{if } a \neq b \end{cases} \end{aligned}$$

$$= \begin{cases} \delta_{\mathbf{ij}}(e(\mathbf{i}) + e(-\mathbf{i})), & \text{if } a = b \\ \delta_{\mathbf{ij}}(e(\mathbf{i}) - e(-\mathbf{i})), & \text{if } a \neq b \end{cases}$$

$$= \vartheta(\varepsilon_a(\mathbf{i}))\vartheta(\varepsilon_b(\mathbf{j})).$$

Now suppose that $\mathbf{i} \in I^\alpha$ is such that $i_{r+2} = i_r \to i_{r+1}$. Then

$$\begin{aligned} \vartheta(\psi'_r\psi'_{r+1}\psi'_r\varepsilon_+(\mathbf{i})) &= \vartheta(\psi'_{r+1}\psi'_r\psi'_{r+1}\varepsilon_+(\mathbf{i}) + \varepsilon_-(\mathbf{i})) \\ &= \psi_{r+1}\psi_r\psi_{r+1}[e(\mathbf{i}) + e(-\mathbf{i})] + e(\mathbf{i}) - e(-\mathbf{i}) \\ &= (\psi_{r+1}\psi_r\psi_{r+1} + 1)e(\mathbf{i}) + (\psi_{r+1}\psi_r\psi_{r+1} - 1)e(-\mathbf{i}) \\ &= \psi_r\psi_{r+1}\psi_r(e(\mathbf{i}) + e(-\mathbf{i})) \\ &= \vartheta(\psi'_r)\vartheta(\psi'_{r+1})\vartheta(\psi'_r)\vartheta(\varepsilon_+(\mathbf{i})) \end{aligned}$$

as required. We leave the remaining checks to the reader; this amounts to proving $\vartheta$ is surjective. Similarly, define a map $\varsigma : \bigoplus_{\beta\in[\alpha]} \mathcal{R}_\beta \to \mathcal{R}'_{[\alpha]}$ on generators by

$$\begin{aligned} \varsigma(y_r) &= y'_r \\ \varsigma(\psi_s) &= \psi'_s \\ \varsigma(e(\mathbf{i})) &= \frac{1}{2}\left(\varepsilon_+(\mathbf{i}) + \varepsilon_-(\mathbf{i})\right) \end{aligned}$$

for $1 \leq r \leq n$, $1 \leq s < n$ and $\mathbf{i} \in \bigcup_{\beta\in[\alpha]} I^\beta$. We must check that $\varsigma$ extends to an algebra homomorphism, again by checking it preserves all relations: note that since $\varepsilon_a(\mathbf{i})\varepsilon_b(\mathbf{i}) = \varepsilon_{a+b}(\mathbf{i})$ for all $\mathbf{i} \in I^n$ and $a, b \in \mathbb{Z}_2$, we can multiply all the relations in Definition 5.3.3 by $\varepsilon_-(\mathbf{i})$ to obtain a list of additional relations which also hold in $\mathcal{R}'_{[\alpha]}$, obtaining relations like

$$\psi'_r y'_s \varepsilon_+(\mathbf{i}) = y'_s \psi'_r \varepsilon_+(\mathbf{i}).$$

These, together with the original list of relations, allow one to check that $\varsigma$ respects all of the relations. For example, if we compute

$$\varsigma(y_r\psi_s e(\mathbf{i})) = \frac{1}{2}\Big(y'_r\psi'_s\varepsilon_+(\mathbf{i}) + y'_r\psi'_s\varepsilon_-(\mathbf{i})\Big)$$

we can see that both terms on the right-hand side are indeed relations in $\mathcal{R}'_{[\alpha]}$ (with the first term being a relation from Definition 5.3.3 multiplied on the right by $\varepsilon_-(\mathbf{i})$). Since $\varsigma$ is also homogeneous of degree zero and since we clearly have $\vartheta \circ \varsigma = \mathrm{id}$ and $\varsigma \circ \vartheta = \mathrm{id}$, this establishes the required isomorphism of $\mathbb{Z}$-graded algebras. □

Using the $C_2$-grading, we can write the decomposition of $\mathcal{R}'_n$ into odd and even parts afforded by its $C_2$ grading as $\mathcal{R}'_n = (\mathcal{R}'_n)_+ \oplus (\mathcal{R}'_n)_-$. This, combined with Theorem 5.2.14, gives the following corollary regarding the alternating quiver Hecke algebra.

**Corollary 5.3.8.** *Let $n \geq 0$, suppose $e \neq 2$ and let 2 be invertible in $\mathcal{O}$. Then $(\mathcal{R}'_n)_+ \cong \mathcal{R}^{\mathtt{sgn}}_n$.*

**Theorem 5.3.9** (Generators and relations for alternating quiver Hecke algebras)**.** *Let $n > 1$, let $e > 2$ and suppose 2 is invertible in $\mathcal{Z}$. Let $[\alpha] \in Q_e^+ \cup (Q_e^\pm \setminus \{n\alpha_0\})$. Then the alternating quiver Hecke algebra $\mathcal{R}^{\mathtt{sgn}}_{[\alpha]}$ is generated by the elements*

$$\{\Psi_1, \Psi_2, \ldots, \Psi_{n-1}\} \cup \{\mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_n\} \cup \{e[\mathbf{i}] \mid [\mathbf{i}] \in I^{[\alpha]}_{\sim}\}$$

*subject to the relations*

$$e[\mathbf{i}]e[\mathbf{j}] = \delta_{[\mathbf{i}][\mathbf{j}]}e[\mathbf{i}], \qquad \sum_{[\mathbf{i}]\in I^{[\alpha]}_{\sim}} e[\mathbf{i}] = 1$$

$$\mathcal{Y}_r e[\mathbf{i}] = e[\mathbf{i}]\mathcal{Y}_r$$

$$\Psi_r e[\mathbf{i}] = e[s_r \cdot \mathbf{i}]\Psi_r$$

$$\mathcal{Y}_r\mathcal{Y}_s = \mathcal{Y}_s\mathcal{Y}_r$$

$$\Psi_r\mathcal{Y}_s e[\mathbf{i}] = \mathcal{Y}_s\Psi_r e[\mathbf{i}] \quad \text{if } s \neq r, r+1$$

$$\Psi_r\Psi_s e[\mathbf{i}] = \Psi_s\Psi_r e[\mathbf{i}], \quad \text{if } |r-s| > 1$$

$$\Psi_r\mathcal{Y}_{r+1}e[\mathbf{i}] = \begin{cases} (\mathcal{Y}_r\Psi_r + 1)e[\mathbf{i}], & \text{if } i_r = i_{r+1} \\ \mathcal{Y}_r\Psi_r e[\mathbf{i}], & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$\mathcal{Y}_{r+1}\Psi_r e[\mathbf{i}] = \begin{cases} (\Psi_r\mathcal{Y}_r + 1)e[\mathbf{i}], & \text{if } i_r = i_{r+1} \\ \Psi_r\mathcal{Y}_r e[\mathbf{i}], & \text{if } i_r \neq i_{r+1} \end{cases}$$

$$\Psi_r^2 e[\mathbf{i}] = \begin{cases} 0, & \text{if } i_r = i_{r+1} \\ (\mathcal{Y}_r - \mathcal{Y}_{r+1})e[\mathbf{i}], & \text{if } i_r \to i_{r+1} \\ (\mathcal{Y}_{r+1} - \mathcal{Y}_r)e[\mathbf{i}], & \text{if } i_r \leftarrow i_{r+1} \\ e[\mathbf{i}], & \text{otherwise} \end{cases}$$

$$\Psi_r\Psi_{r+1}\Psi_r e[\mathbf{i}] = \begin{cases} (\Psi_{r+1}\Psi_r\Psi_{r+1} - 1)e[\mathbf{i}], & \\ \qquad \text{if } i_r = i_{r+2} \to i_{r+1} & \\ (\Psi_{r+1}\Psi_r\Psi_{r+1} + 1)e[\mathbf{i}], & \\ \qquad \text{if } i_r = i_{r+2} \leftarrow i_{r+1} & \\ \Psi_{r+1}\Psi_r\Psi_{r+1}e[\mathbf{i}], & \\ \qquad \text{otherwise} & \end{cases}$$

*for all* $[\mathbf{i}] \in I_\sim^{[\alpha]}$ *and all admissible* $r$ *and* $s$. *Moreover,* $\mathcal{R}_{[\alpha]}^{\mathbf{sgn}}$ *is* $\mathbb{Z}$*-graded with degree function*

$$\deg \Psi_r e[\mathbf{i}] = -a_{i_r i_{r+1}}, \quad \deg \mathcal{Y}_s = 2, \quad \deg e[\mathbf{i}] = 0.$$

*for all* $1 \leq r < n$, $1 \leq s \leq n$ *and* $[\mathbf{i}] \in I_\sim^{[\alpha]}$.

**Proof.** First we note that all the requirements on the residue sequences $\mathbf{i}$ in fact depend only on the *equivalence class* of the sequence because of the symmetry of the Cartan matrix from (4.1.3). By Corollary 5.3.8, it is enough to prove that the abstract algebra $A_{[\alpha]}$ defined in the statement of the theorem is isomorphic to $(\mathcal{R}'_{[\alpha]})_+$. Define a map $\varrho : A_{[\alpha]} \to (\mathcal{R}'_{[\alpha]})_+$ by

$$e[\mathbf{i}] \mapsto \varepsilon_+(\mathbf{i}), \quad \mathcal{Y}_r \mapsto y'_r\varepsilon_-(\mathbf{i}), \quad \Psi_s \mapsto \psi'_s\varepsilon_-(\mathbf{i}),$$

for all $[\mathbf{i}] \in I_\sim^{[\alpha]}$, $1 \leq r \leq n$ and $1 \leq s < n$. It is a straightforward check that all the relations of $A_{[\alpha]}$ are satisfied in $(\mathcal{R}'_{[\alpha]})_+$, so $\varrho$ determines a well-defined homogeneous algebra homomorphism of degree zero.

By definition, $(\mathcal{R}'_{[\alpha]})_+$ is generated by even words in the generators of $\mathcal{R}'_{[\alpha]}$. However the only even *generators* of $\mathcal{R}'_{[\alpha]}$ are the idempotents $\varepsilon_+(\mathbf{i})$ for $\mathbf{i} \in I^\alpha \cup I^{\alpha'}$, so $(\mathcal{R}'_{[\alpha]})_+$ is generated by these idempotents together with all words of even length in the *odd* generators of $\mathcal{R}'_{[\alpha]}$. It is now easy to see that $(\mathcal{R}'_{[\alpha]})_+$ is generated by the images of the generators of $A_{[\alpha]}$ under $\varrho$ and so $\varrho$ is surjective.

The algebra $\mathcal{R}'_{[\alpha]}$ is defined by generators and relations, so $(\mathcal{R}'_{[\alpha]})_+$ is the subalgebra of $\mathcal{R}'_{[\alpha]}$ generated by the even words in the generators of $\mathcal{R}'_{[\alpha]}$ modulo the even part of the relation ideal defining $\mathcal{R}'_{[\alpha]}$, which is the set of all linear combinations of arbitrary products of even relations multiplied by even products of odd relations. However the only even relations in $\mathcal{R}'_{[\alpha]}$ are given by idempotents and commutation relations. Therefore, the even part of the relation ideal for $\mathcal{R}'_{[\alpha]}$ is generated by the even relations in $\mathcal{R}'_{[\alpha]}$ together with all products of the odd relations in $\mathcal{R}'_{[\alpha]}$. It follows that all the remaining relations are generated by even products of odd relations, together with odd relations multiplied by $\varepsilon$: in this way we obtain the complete set of relations for $(\mathcal{R}'_{[\alpha]})_+$. One checks these are precisely the relations written above for $A_{[\alpha]}$; for example multiplying the relation

$$y'_{r+1}\psi'_r\varepsilon_-(\mathbf{i}) = \begin{cases} (\psi'_r y'_r + 1)\varepsilon_-(\mathbf{i}), & \text{if } i_r = i_{r+1} \\ \psi'_r y'_r \varepsilon_-(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

by $\varepsilon_-(\mathbf{i})$ and using the idempotent relations in $\mathcal{R}'_{[\alpha]}$ gives the relation

$$y'_{r+1}\varepsilon_-(\mathbf{i})\psi'_r\varepsilon_-(\mathbf{i})\varepsilon_+(\mathbf{i}) = \begin{cases} (\psi'_r\varepsilon_-(\mathbf{i})y'_{r+1}\varepsilon_-(\mathbf{i}) + 1)\varepsilon_+(\mathbf{i}), & \text{if } i_r = i_{r+1} \\ \psi'_r\varepsilon_-(\mathbf{i})y'_r\varepsilon_-(\mathbf{i})\varepsilon_+(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \end{cases}$$

which is precisely the image of the relation

$$\mathcal{Y}_{r+1}\Psi_r e[\mathbf{i}] = \begin{cases} (\Psi_r\mathcal{Y}_r + 1)e[\mathbf{i}], & \text{if } i_r = i_{r+1} \\ \Psi_r\mathcal{Y}_r e[\mathbf{i}], & \text{if } i_r \neq i_{r+1} \end{cases}$$

under $\varrho$. Continuing in this way we see $\varrho$ is an isomorphism.

Finally, we note that the given degree function is well-defined; since the Cartan matrix $(c_{ij})_{i,j\in I}$ is symmetric, the entries only depend on equivalence classes of residue sequences under $\sim$.

□

**Remark 5.3.10.** Since we have given an isomorphism between the alternating quiver Hecke algebra and the even part of the abstract algebra $\mathcal{R}'_n$, for which we gave an abstract presentation by generators and relations, we see that the generators from Definition 5.3.1 do not depend on the choice of equivalence class representatives $\mathbf{i}^+$ (see p40).

## 5.4. Alternating cyclotomic quiver Hecke algebras

In this section we define the cyclotomic quotients of alternating quiver Hecke algebras from §5.2 which we will study for the remainder of this thesis. For a given weight $\Lambda = \Lambda(\boldsymbol{\kappa})$ with multicharge $\boldsymbol{\kappa}$, we set

$$\Lambda' = \Lambda'(\boldsymbol{\kappa}') = \Lambda_{\overline{-\kappa_\ell}} + \Lambda_{\overline{-\kappa_{\ell-1}}} + \ldots + \Lambda_{\overline{-\kappa_1}}.$$

**Proposition 5.4.1.** [62] *There is a unique homogeneous algebra isomorphism* $\mathtt{sgn} : \mathcal{R}^\Lambda_\alpha \to \mathcal{R}^{\Lambda'}_{\alpha'}$ *of degree zero satisfying*

$$e(\mathbf{i}) \mapsto e(-\mathbf{i}), \qquad y_r \mapsto -y_r, \qquad \psi_r \mapsto -\psi_r.$$

*Moreover, this map lifts to the graded sign map* $\mathtt{sgn} : \mathcal{R}_n \to \mathcal{R}_n$ *from Definition 5.2.1.*

**Proof.** This follows immediately since $\mathtt{sgn}$ maps the cyclotomic ideal $\mathcal{I}^\Lambda_\alpha$ to the cyclotomic ideal $\mathcal{I}^{\Lambda'}_{\alpha'}$. □

We are interested in the case when this cyclotomic version of $\mathtt{sgn}$ is an *involution* of graded algebras, i.e. when $\Lambda = \Lambda'$. Under the Brundan-Kleshchev isomorphism (Theorem 4.3.2), cyclotomic quiver Hecke algebras of weight $\Lambda$ such that $\Lambda = \Lambda'$ correspond to cyclotomic Hecke algebras with symmetric multicharges (Definition 3.5.12); this follows immediately from translating between our notation $\boldsymbol{\kappa}$ for multicharges and $\Lambda$ for dominant weights. This, together with the analogy between the # and $\mathtt{sgn}$ involutions, motivates the following definition.

**Definition 5.4.2.** Let $\Lambda = \Lambda'$. The *alternating cyclotomic quiver Hecke algebra* is the fixed-point subalgebra $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$ of the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$ under the graded sign involution.

**Example 5.4.3.** Let $n = 3$, $\Lambda = \Lambda_0$, and suppose that $F$ is a field of characteristic 3. We have seen in Example 4.2.6 that the collection

$$e(012),\ e(021),\ y_3e(012),\ y_3e(021),\ \psi_2e(012) \quad \text{and} \quad \psi_2e(021)$$

gives a basis for the cyclotomic quiver Hecke algebra $\mathcal{R}_3^{\Lambda_0}$. One can check that

$$1 = e(012) + e(021), \quad \Psi = \psi_2(e(012) - e(021)) \quad \text{and} \quad Y = y_3(e(012) - e(021))$$

span the fixed-point subalgebra $(\mathcal{R}_3^{\Lambda_0})^{\mathtt{sgn}}$, since $\Psi^2 = -Y$ and $Y^2 = 0$. Using the isomorphism in Example 4.3.5, we obtain

$$\begin{aligned} e(012) + e(021) &= 1 \\ y_3(e(012) - e(021)) &= 1 + s_1s_2 + s_2s_1 \\ \psi_2(e(012) - e(021)) &= 2s_1s_2s_1 + s_2, \end{aligned}$$

so the isomorphism from that example does *not* restrict to give an isomorphism $(\mathcal{R}_3^{\Lambda_0})^{\mathtt{sgn}} \xrightarrow{\sim} k\mathfrak{A}_3$ (since $\psi_2^{\mathtt{sgn}} = -\psi_2 = s_1s_2 + 2s_2s_1 \neq 2s_1s_2 + s_2s_1 = \psi_2^{\#}$). It is easy to see that $(\mathcal{R}_3^{\Lambda_0})^{\mathtt{sgn}} \cong F[x]/(x^3)$: an isomorphism is given by $\Psi \mapsto x$. If we put $x \in F[x]/(x^3)$ in degree 1 as in Example 2.3.2(i), this is even an isomorphism of graded algebras since $\deg \Psi = 1$. In turn, $F[x]/(x^3)$ is isomorphic to $F\mathfrak{A}_3$ via $x \mapsto 1 - s_1s_2$. We will soon see that a slightly different choice of Brundan-Kleshchev style isomorphism allows us to see this correspondence more clearly.

On the level of blocks, which is where we need to state most of our results for technical reasons, we need to take a slightly more subtle approach. The following lemma is immediate from Proposition 5.4.1.

**Lemma 5.4.4.** *Let $\Lambda \in P_e$ be such that $\Lambda = \Lambda'$ and let $\alpha \in Q_e$.*

(i) *If $\alpha = \alpha'$ then $\mathtt{sgn} : \mathcal{R}_\alpha^\Lambda \to \mathcal{R}_\alpha^\Lambda$ is an involution.*

(ii) *If* $\alpha \neq \alpha'$ *then* $\mathtt{sgn} : \mathcal{R}^\Lambda_\alpha \oplus \mathcal{R}^\Lambda_{\alpha'} \to \mathcal{R}^\Lambda_\alpha \oplus \mathcal{R}^\Lambda_{\alpha'}$ *is an involution.*

We now want to show that the alternating cyclotomic quiver Hecke algebra is isomorphic to the subalgebra of the cyclotomic quiver Hecke algebra of points fixed under $\mathtt{sgn}$.

Following Definition 4.2.1, for $[\alpha] \in Q_e^+ \cup (Q_e^\pm \setminus \{n\alpha_0\}$, note that $e[\mathbf{i}] \in \oplus_{\beta\in[\alpha]}\mathcal{R}_\beta$ for all $[\mathbf{i}] \in I^{[\alpha]}_\sim$. Hence define the ideal

$$\overline{\mathcal{I}^\Lambda_{[\alpha]}} = \left\langle \mathcal{Y}_1^{\langle\Lambda,\alpha_{i_1}\rangle} e[\mathbf{i}] \;\middle|\; [\mathbf{i}] \in I^{[\alpha]}_\sim \right\rangle$$

of $\bigoplus_{\beta\in[\alpha]} \mathcal{R}_\beta$. Importantly, $\mathtt{sgn}$ fixes $\overline{\mathcal{I}^\Lambda_{[\alpha]}}$ as a set and so $\overline{\mathcal{I}^\Lambda_{[\alpha]}}$ is actually an ideal of $\mathcal{R}^{\mathtt{sgn}}_{[\alpha]}$.

**Lemma 5.4.5.** *Let* $2$ *be invertible in* $\mathcal{Z}$. *If* $\sigma$ *is an involution on* $\mathcal{Z}$*-algebras* $X$ *and* $Y$ *with* $Y \lhd X$ *and* $Y^\sigma = Y$ *as a set then* $(X/Y)^\sigma \cong X^\sigma/Y^\sigma$.

**Proof.** Define a map $\rho : (X/Y)^\sigma \to X^\sigma/Y^\sigma$ by

$$x + Y \mapsto \frac{1}{2}(x + x^\sigma) + Y^\sigma.$$

This map is clearly well-defined since 2 is invertible in $\mathcal{Z}$; moreover it is injective since if $\frac{1}{2}(x + x^\sigma - x' - (x')^\sigma) \in Y$ for $x, x' \in X$ then $\frac{1}{2}(x + x^\sigma - x' - (x')^\sigma) = \frac{1}{2}(x^\sigma + x - (x')^\sigma - x')$, since $Y$ is fixed by $\sigma$, and so $(x - x') = (x - x')^\sigma$ and $x + Y = x' + Y$. Surjectivity follows since if $z + Y^\sigma \in X^\sigma + Y^\sigma$ then clearly $z + Y^\sigma = \rho(z + Y)$, since $z$ is fixed by $\sigma$. □

**Proposition 5.4.6.** *Let* $\Lambda \in P_e$ *be such that* $\Lambda = \Lambda'$. *Then for* $[\alpha] \in Q_e^+ \cup (Q_e^\pm \setminus \{n\alpha_0\})$, *we have the following isomorphism of* $\mathcal{Z}$*-algebras:*

$$\bigoplus_{\beta\in[\alpha]} (\mathcal{R}^\Lambda_\beta)^{\mathtt{sgn}} \cong \left(\bigoplus_{\beta\in[\alpha]} \mathcal{R}_\beta\right)^{\mathtt{sgn}} \Big/ \overline{\mathcal{I}^\Lambda_{[\alpha]}}$$

**Proof.** Note that the algebra on the left-hand-side of the isomorphism is equal to $\left[\left(\bigoplus_{\beta\in[\alpha]} \mathcal{R}_\beta\right)/\left(\bigoplus_{\beta\in[\alpha]} \mathcal{I}^\Lambda_\beta\right)\right]^{\mathtt{sgn}}$ by orthogonality of idempotents in unequal blocks (note the difference between the ideal $\mathcal{I}^\Lambda_\alpha$ from Definition 4.2.1 and the ideal $\overline{\mathcal{I}^{\Lambda}_{[\alpha]}}$ as defined above). Hence by Lemma 5.4.5 it remains to show that $\overline{\mathcal{I}^{\Lambda}_{[\alpha]}}$ is precisely the set of fixed points in $\bigoplus_{\beta\in[\alpha]}\mathcal{I}^{\Lambda}_{\beta}$ under $\mathtt{sgn}$. Since

$$\mathcal{Y}_1^{(\Lambda,\alpha_{i_1})}e[\mathbf{i}] = \begin{cases} \varepsilon y_1^{(\Lambda,\alpha_{i_1})}e[\mathbf{i}], & \text{if } (\Lambda,\alpha_{i_1}) \text{ is odd} \\ y_1^{(\Lambda,\alpha_{i_1})}e[\mathbf{i}], & \text{if } (\Lambda,\alpha_{i_1}) \text{ is even,} \end{cases} \tag{5.4.7}$$

all elements of $\overline{\mathcal{I}^{\Lambda}_{[\alpha]}}$ are $\mathtt{sgn}$-invariant and belong to $\left(\bigoplus_{\beta\in[\alpha]}\mathcal{I}^{\Lambda}_{\beta}\right)^{\mathtt{sgn}}$. To demonstrate the other inclusion, let $x \in \left(\bigoplus_{\beta\in[\alpha]}\mathcal{I}^{\Lambda}_{\beta}\right)^{\mathtt{sgn}}$. Write $x$ as

$$x = \sum_{\beta\in[\alpha]} x_\beta \sum_{\mathbf{i}\in I^\beta} y_1^{(\Lambda,\beta i_1)}e(\mathbf{i})$$

for $x_\beta \in \mathcal{R}_\beta$. Then since $x = x^{\mathtt{sgn}}$,

$$\begin{aligned} x = x^{\mathtt{sgn}} &= \sum_{\beta\in[\alpha]} x_\beta^{\mathtt{sgn}} \sum_{\mathbf{i}\in I^\beta} (-1)^{(\Lambda,\beta_{i_1})} y_1^{(\Lambda,\beta i_1)} e(-\mathbf{i}) \\ &= \sum_{\beta\in[\alpha]} x_\beta^{\mathtt{sgn}} \prod_{\mathbf{i}\in I^{\beta'}} (-1)^{(\Lambda,\beta_{i_1})} y_1^{(\Lambda,\beta_{i_1})} e(\mathbf{i}) \end{aligned}$$

by Lemma 5.2.3 and since $\Lambda$ is a symmetric dominant weight. Projecting onto idempotents gives $x_\beta e(\mathbf{i}) = (-1)^{(\Lambda,\beta_{i_1})} x_{\beta'}^{\mathtt{sgn}} e(\mathbf{i})$ for all $\beta \in [\alpha]$ and $\mathbf{i} \in I^\beta$ and this allows us to write $x$ in the required form:

$$\begin{aligned} x &= \sum_{\mathbf{i}\in I^{\beta^+}} x_{\beta^+} y_1^{(\Lambda,\beta^+_{i_1})} e(\mathbf{i}) + \sum_{\mathbf{i}\in I^{\beta^{+\prime}}} (-1)^{(\Lambda,\beta^+_{i_1})} x_{\beta^+} y_1^{(\Lambda,\beta^+_{i_1})} e(\mathbf{i}) \\ &= x_{\beta^+} \sum_{\mathbf{i}\in I^{\beta^+}} \left[ y_1^{(\Lambda,\beta^+_{i_1})} \left( e(\mathbf{i}) + (-1)^{(\Lambda,\beta^+_{i_1})} e(-\mathbf{i}) \right) \right] \end{aligned}$$

for some distinguished $\beta^+ \in [\alpha]$, which belongs to $\overline{\mathcal{I}^{\Lambda}_{[\alpha]}}$. □

Finally, we deal with the cyclotomic relation when $\alpha = n\alpha_0$. This is the least important case, since in level 1 it does not occur.

**Proposition 5.4.8.** *Let $n > 1$ and $e > 2$.*

(i) $(\mathcal{R}^{\Lambda}_{n\alpha_0})^{\mathtt{sgn}} \cong \mathcal{R}^{\mathtt{sgn}}_{n\alpha_0}/\langle y_1^{(\Lambda,\alpha_0)} e\rangle$, *where* $e = \underbrace{e(0,0,\dots,0)}_{n \text{ zeroes}}$.

(ii) *If* $(\Lambda,\alpha_0) = 1$, $\mathcal{R}^{\Lambda}_{n\alpha_0} = 0$.

**Proof.** The first part follows from Lemma 5.4.5 since the ideal is fixed setwise by $\mathtt{sgn}$. For the second part, we deduce from the cyclotomic relation in the algebra that $y_1 = 0$; then, since $\psi_1 y_2 e = y_1\psi_1 e + e$ we have $\psi_1 y_2 e = e$. Multiplying by $\psi_1$ and using $\psi_1^2 = 0$ gives $\psi_1 e = 0$ which in turn gives $e = 0$, showing the algebra must be zero since $e$ is the only idempotent. □

We now obtain the following presentation for alternating cyclotomic quiver Hecke algebras as an immediate corollary to Proposition 5.4.6 and Theorem 5.3.9. If $(\Lambda, \alpha_0) \equiv 1 \bmod 2$ we do not know how to characterise the cyclotomic ideal.

**Theorem 5.4.9.** *Let $n > 1$ and $e > 2$. Suppose $\Lambda \in P_e$ is such that $\Lambda = \Lambda'$. Let $[\alpha] \in Q_e^+ \cup (Q_e^\pm \setminus \{n\alpha_0\})$. The alternating cyclotomic quiver Hecke algebra $(\mathcal{R}^\Lambda_{[\alpha]})^{\mathtt{sgn}}$ is generated by*

$$\{\Psi_1, \Psi_2, \dots, \Psi_{n-1}\} \cup \{\mathcal{Y}_1, \mathcal{Y}_2, \dots, \mathcal{Y}_n\} \cup \{e[\mathbf{i}] \mid \mathbf{i} \in I^{[\alpha]}_\sim\}$$

*subject to all of the relations in Theorem 5.3.9(i) together with the cyclotomic relation*

$$\mathcal{Y}_1^{\langle \Lambda, \alpha_{i_1}\rangle} e[\mathbf{i}] = 0$$

*for $[\mathbf{i}] \in I^{[\alpha]}_\sim$.*

Finally, we compute the rank of the alternating cyclotomic quiver Hecke algebras.

**Proposition 5.4.10.** *Let 2 be invertible in $\mathcal{Z}$, let $e > 2$ and suppose $\Lambda$ is a dominant weight such that $\Lambda = \Lambda'$ and $(\Lambda, \alpha_0) < n$. Then*

$$\mathrm{rk}_{\mathcal{Z}} (\mathcal{R}^\Lambda_n)^{\mathtt{sgn}} = \frac{\ell^n n!}{2} = \frac{1}{2}\mathrm{rk}_{\mathcal{Z}} \mathcal{R}^\Lambda_n.$$

**Proof.** Since the condition on $\Lambda$ means the block $\mathcal{R}^\Lambda_{n\alpha_0}$ is zero, $\varepsilon$ is a Clifford element which splits $\mathcal{R}^\Lambda_n$ into a direct sum of two $\mathcal{Z}$-modules $(\mathcal{R}^\Lambda_n)^{\mathtt{sgn}}$ and $\varepsilon(\mathcal{R}^\Lambda_n)^{\mathtt{sgn}}$ of the same rank by the same argument as in the affine case (Proposition 5.2.15). □

CHAPTER 6

# A graded isomorphism theorem for alternating quiver Hecke algebras

In this chapter, using a similar approach to Hu and Mathas [**42**] to define certain deformed integral versions of alternating cyclotomic quiver Hecke algebras in level 1, we show that, over a large enough field, these are isomorphic to the alternating cyclotomic Hecke algebras from Chapter 3. In particular, this alternating analogue of Brundan and Kleshchev's isomorphism theorem implies the existence of a $\mathbb{Z}$-grading on the modular group algebras of alternating groups, and on Mitsuhashi's alternating Hecke algebras [**83**].

## 6.1. Integral cyclotomic quiver Hecke algebras

In this section we give a slightly modified version of Hu-Mathas' deformation [**43**] of the Brundan-Kleshchev isomorphism theorem using seminormal forms.

Let us fix a field $\mathcal{K}$ and an $e$-idempotent subring $\mathcal{O}$ (Definition 3.5.7); we will make a specific choice of $\mathcal{O}$ in the next section. We will be working with $\mathscr{H}_n^\Lambda(\mathcal{O})$ and $\mathscr{H}_n^\Lambda(\mathcal{K}) = \mathscr{H}_n^\Lambda(\mathcal{O}) \otimes_\mathcal{O} \mathcal{K}$. $\mathscr{H}_n^\Lambda(\mathcal{K})$ is semisimple and therefore has a seminormal basis $\{f_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$ together with its associated seminormal coefficient system $\boldsymbol{\alpha}$. From this seminormal basis we may obtain idempotents $f_{\mathbf{i}}^{\mathcal{O}}$ as in (3.5.18), which belong to $\mathscr{H}_n^\Lambda(\mathcal{O})$ by Lemma 3.5.19.

Let $M_r = 1 - L_r + tL_{r+1}$ for $1 \le r \le n$. Observe that by Theorem 3.4.11 if $(\mathtt{s}, \mathtt{t}) \in \mathrm{Std}^2(\mathcal{P}_n)$ then

$$M_r f_{\mathtt{st}} = t^{c_r(\mathtt{s})}[1 - \rho_r(\mathtt{s})] f_{\mathtt{st}}, \quad \text{and} \quad (L_{r+1} - L_r) f_{\mathtt{st}} = -t^{c_{r+1}(\mathtt{s})}[\rho_r(\mathtt{s})]. \quad (6.1.1)$$

It is shown in [**43**] that these elements are locally invertible in the following sense.

**Corollary 6.1.2** ([**43**, Corollary 4.8]). *Suppose that $1 \le r < n$ and $\mathbf{i} \in I^n$.*

a) *If $i_r \neq i_{r+1} + 1$ then* $\displaystyle\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_r(\mathfrak{t})}}{[1-\rho_r(\mathfrak{s})]} \in \mathscr{H}_n^\Lambda(\mathcal{O})$ *and*

$$\Big(\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_r(\mathfrak{t})}}{[1-\rho_r(\mathfrak{s})]}\Big) M_r f_{\mathbf{i}}^{\mathcal{O}} = M_r\Big(\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_r(\mathfrak{t})}}{[1-\rho_r(\mathfrak{s})]}\Big) f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}}.$$

b) *If $i_r \neq i_{r+1}$ then* $\displaystyle\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{-t^{-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{s})]} F_{\mathfrak{s}} \in \mathscr{H}_n^\Lambda(\mathcal{O})$ *and*

$$\begin{aligned}\Big(\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{-t^{-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{s})]} F_{\mathfrak{s}}\Big)(L_r - L_{r+1}) f_{\mathbf{i}}^{\mathcal{O}}& \\ = (L_r - L_{r+1})\Big(\sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{-t^{-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{s})]} F_{\mathfrak{s}}\Big) f_{\mathbf{i}}^{\mathcal{O}} &= f_{\mathbf{i}}^{\mathcal{O}}.\end{aligned}$$

**Remark 6.1.3.** The upshot of Corollary 6.1.2 is that we are justified in the abuses of notation

$$\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{t^{-c_r(\mathfrak{t})}}{[1-\rho_r(\mathfrak{s})]} F_{\mathfrak{s}}$$

when $i_r \neq i_{r+1} + 1$ and

$$\frac{1}{L_r - L_{r+1}} f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathfrak{s}\in\operatorname{Std}(\mathbf{i})} \frac{-t^{-c_{r+1}(\mathfrak{t})}}{[\rho_r(\mathfrak{s})]} F_{\mathfrak{s}}$$

when $i_r \neq i_{r+1}$. When we use these in the next section we are implicitly appealing to Corollary 6.1.2.

Following [**43**] we now define a deformation of $\mathcal{R}_n^\Lambda(e,F) \cong \mathscr{H}_n^\Lambda(F)$ over $\mathcal{O}$. We will prove our main theorem in the next section by adapting these ideas. Our generators in this section are adorned with superscripts which do not appear in [**43**]; the reason for this will become clear in the next section.

The results of [**43**] depend upon choosing an arbitrary section of the natural quotient map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/e\mathbb{Z}$ when $e < \infty$. If $i \in I$ let $\hat{\imath} \ge 0$ be the smallest non-negative integer such that $i = \hat{\imath} + e\mathbb{Z}$. This defines an embedding $I \hookrightarrow \mathbb{Z}; i \mapsto \hat{\imath}$.

**Definition 6.1.4.** [43, Definition 4.14] Suppose that $1 \le r < n$. Define $\psi_r^+ = \sum_{\mathbf{i}\in I^n} \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}}$ by setting

$$\psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (1+T_r)\frac{t^{\hat{\imath}_r}}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+1}, \\ (T_rL_r - L_rT_r)t^{-\hat{\imath}_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \to i_{r+1} \\ (T_rL_r - L_rT_r)\frac{1}{M_r} f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

for $\mathbf{i} \in I^n$. Moreover, for $1 \le r \le n$ define $y_r^+ = \sum_{\mathbf{i}\in I^n} t^{-\hat{\imath}_r}(L_r - [\hat{\imath}_r]) f_{\mathbf{i}}^{\mathcal{O}}$.

**Theorem 6.1.5** (Hu-Mathas [43, Theorem A])**.** *Let $\mathcal{K}$ be a field and $(\mathcal{O}, t)$ an $e$-idempotent subring of $\mathcal{K}$, where $e < \infty$. For $\alpha \in Q_e$, the block $\mathscr{H}_\alpha^\Lambda(\mathcal{O})$ of the cyclotomic Hecke algebra $\mathscr{H}_n^\Lambda(\mathcal{O})$ is generated as an $\mathcal{O}$-algebra by the elements*

$$\{\, f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^\alpha \,\} \cup \{\, \psi_r^+ \mid 1 \le r < n \,\} \cup \{\, y_s^+ \mid 1 \le s \le n \,\}$$

*subject to the relations*

$$\prod_{\substack{1 \le \ell < r \\ \kappa_i \equiv \hat{\imath}_1 \bmod e}} (y_1^+ - [\kappa_\ell - \hat{\imath}_1]) f_{\mathbf{i}}^{\mathcal{O}} = 0$$

$$f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} = \delta_{\mathbf{ij}} f_{\mathbf{i}}^{\mathcal{O}}, \qquad \sum_{\mathbf{i}\in I^\alpha} f_{\mathbf{i}}^{\mathcal{O}} = 1$$

$$y_r^+ f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} y_r^+, \qquad \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} = f_{s_r\cdot\mathbf{i}}^{\mathcal{O}} \psi_r^+, \qquad y_r^+ y_s^+ = y_s^+ y_r^+$$

$$\psi_r^+ y_{r+1}^+ f_{\mathbf{i}}^{\mathcal{O}} = (y_r^+ \psi_r^+ + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, \qquad y_{r+1}^+ \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} = (\psi_r^+ y_r^+ + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}$$

$$\psi_r^+ y_s^+ = y_s^+ \psi_r^+, \qquad \textit{if } s \ne r, r+1,$$

$$\psi_r^+ \psi_s^+ = \psi_s^+ \psi_r^+, \qquad \textit{if } |r-s| > 1,$$

$$(\psi_r^+)^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^+) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \to i_{r+1}, \\ (y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^+) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \leftarrow i_{r+1}, \\ 0, & \textit{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^{\mathcal{O}}, & \textit{otherwise,} \end{cases}$$

$$\psi_r^+\psi_{r+1}^+\psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (\psi_{r+1}^+\psi_r^+\psi_{r+1}^+ - t^{1+\rho_r(\mathbf{i})}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+2} \to i_{r+1}, \\ (\psi_{r+1}^+\psi_r^+\psi_{r+1}^+ + 1) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ \psi_{r+1}^+\psi_r^+\psi_{r+1}^+ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

*where* $y_r^{\langle d\rangle} f_{\mathbf{i}}^{\mathcal{O}} = (t^d y_r^+ - [d]) f_{\mathbf{i}}^{\mathcal{O}}$ *for* $d \in \mathbb{Z}$*, and* $\rho_r(\mathbf{i}) = \hat{\imath}_r - \hat{\imath}_{r+1}$*, for* $\mathbf{i}, \mathbf{j} \in I^\alpha$ *and all admissible* $r, s$*.*

**Remark 6.1.6.**

(i) Note that if $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}$ then (the image of) $t$ has quantum characteristic $e$ in the residue field $F = \mathcal{O}/\mathfrak{m}$, and so the relations above reduce to the familiar relations for cyclotomic quiver Hecke algebras in Definition 4.1.8 when tensoring with $F$ over $\mathcal{O}$: indeed, it is easily checked that all powers of $t$ that appear are $t^0$ or $t^{me}$ for some $m \in \mathbb{Z}$; since $\xi^e = 1$ if $\xi \neq 1$, this gives the familar relations in all cases above.

(ii) Although we have to assume $e < \infty$ to invoke Theorem 6.1.5, this is not a problem since by [**42**, Corollary 2.15], the case $e = \infty$ is equivalent to the case $e \gg 0$.

## 6.2. Integral alternating cyclotomic quiver Hecke algebras

In this section we restrict to level 1 (notice that $\Lambda = \Lambda_0$ is the only symmetric dominant weight in level 1) and give our main result, proving that the fixed-point subalgebra we have defined as the alternating cyclotomic Hecke algebra (Definition 3.5.15), and the fixed-point subalgebra we have defined as the alternating cyclotomic quiver Hecke algebra (Definition 5.4.2), coincide, provided we work over a large enough field. This gives an alternating analogue of Brundan-Kleshchev's isomorphism theorem in level 1. As a first guess, one might expect Brundan and Kleshchev's map to simply intertwine the # and `sgn` maps; unfortunately such an easy proof of our main result is not possible, as we saw in Example 5.4.3.

At this point, we need to make a careful choice of modular system for our algebras in order to ensure our rings have enough square roots for our computations.

**Definition 6.2.1.** Let $\mathbb{F}$ be a field with $\xi \in \mathbb{F}^\times$ and let $x$ be an indeterminate over $\mathbb{F}$. Let $e$ be the quantum characteristic of $\xi$ in $\mathbb{F}$. Let $t = x + \xi$ and define the local ring

$$\mathcal{O} = \mathbb{F}[\sqrt{t}, \sqrt{-1}, \{\sqrt{[h]_t} \mid 1 < h \leq n\}]_{(\sqrt{t})}.$$

For $-n \leq t < -1$ define $\sqrt{[h]_t} = \sqrt{-1}\sqrt{t}^h\sqrt{[-h]}$. Note that the compatibility requirement makes sense because $[h]_t = -t^h[-h]_t$ for $h < 0$. Finally, let $\mathcal{K}$ be the field of fractions of $\mathcal{O}$.

**Lemma 6.2.2.** *Let $\mathcal{O}$ be the local ring defined in Definition 6.2.1. Then $(\mathcal{O}, t)$ is an $e$-idempotent subring. Moreover, if $\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}$, then in the residue field $F = \mathcal{O}/\mathfrak{m}$, the image of $\xi$ has quantum characteristic $e$.*

**Proof.** Note that $\mathcal{O} \cong \widetilde{\mathcal{O}}[\sqrt{-1}, \{\sqrt{[h]_t} \mid 1 < h \leq n\}]$ where $\widetilde{\mathcal{O}} = \mathbb{F}[\sqrt{t}]_{(\sqrt{t})}$. The ring $\widetilde{\mathcal{O}}$ is an $e$-idempotent subring of $\mathcal{K}$ by [**41**, Example 4.2(b)] and it is easy to show that algebraic extensions of $e$-idempotent subrings are again $e$-idempotent subrings. □

**Let us establish the fixed notation from Definition 6.2.1, including the modular system $\mathcal{K} \supset \mathcal{O} \twoheadrightarrow F$ for the algebra $\mathscr{H}_n^{\Lambda_0}(\mathcal{K})$, where $\mathcal{O}$, $F$ and $\mathcal{K}$ are defined above.** We write

- $\mathscr{H}_n(F)$ for $\mathscr{H}_{n,1}(F, \xi, 0)$
- $\mathscr{H}_n(\mathcal{O})$ for $\mathscr{H}_{n,1}(\mathcal{O}, t, 0)$
- $\mathscr{H}_n(\mathcal{K})$ for $\mathscr{H}_{n,1}(\mathcal{K}, t, 0)$.

We also assume that $e < \infty$ (see Remark 6.1.6(ii)).

**Remark 6.2.3.**

(i) Note that $\mathscr{H}_n(\mathcal{K}) \cong \mathscr{H}_n(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{K}$ and $\mathscr{H}_n(F) \cong \mathscr{H}_n(\mathcal{O}) \otimes_{\mathcal{O}} F$.

(ii) Although if $\xi = 1$ it is guaranteed that $F$ is a field of characteristic $e$ by Lemma 6.2.2(i), there is no guarantee it will be a small field ($\mathbb{F}_e$ for example). Indeed, the modular system is required precisely to ensure that the resulting residue field $F$ is large enough.

(iii) By Proposition 5.4.8(ii), if $n > 1$, since $(\Lambda_0, \alpha_0) = 1$, the block $\mathscr{H}^{\Lambda_0}_{n\alpha_0}$ is zero and so we do not have to consider this case in this section.

Recall the definition of our generators $\psi_r^+$ and $y_r^+$ for $\mathscr{H}_n(\mathcal{O})$ from the previous section. We want to define new generators which are their images under the hash involution.

**Definition 6.2.4.** Let $\psi_r^- = (\psi_r^+)^\#$ and $y_s^- = (y_s^+)^\#$, for $1 \le r < n$ and $1 \le s \le n$.

Since $\#$ is an automorphism, the collection of elements

$$\{\psi_1^-, \psi_2^-, \dots, \psi_{n-1}^-\} \cup \{y_1^-, y_2^-, \dots, y_n^-\} \cup \{f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n\}$$

also generate $\mathscr{H}_n(\mathcal{O})$ subject to relations very similar to those given in Theorem 6.1.5. As we need this result below we state it in full for easy reference. In effect, Lemma 3.5.23 means that the orientation of the quiver $\Gamma_e$ in the relations below is reversed as compared to the relations in Theorem 6.1.5.

**Proposition 6.2.5.** *Let $n > 1$ and $\infty > e > 2$. Then for $\alpha \in Q_e$, the block $\mathscr{H}_\alpha(\mathcal{O})$ of the cyclotomic Hecke algebra $\mathscr{H}_n(\mathcal{O})$ is generated as an $\mathcal{O}$-algebra by the elements*

$$\{\psi_r^- \mid 1 \le r < n\} \cup \{y_s^- \mid 1 \le s \le n\} \cup \{f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^\alpha\}$$

*subject to the relations*

$$\prod_{\mathbf{i}\in I^\alpha} (y_1^-)^{(\Lambda_0,\alpha_{i_1})} e(\mathbf{i}) = 0, \qquad f_{\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} = \delta_{\mathbf{i}\mathbf{j}} f_{\mathbf{i}}^{\mathcal{O}}, \qquad \sum_{\mathbf{i}\in I^\alpha} f_{\mathbf{i}}^{\mathcal{O}} = 1$$

$$y_r^- f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}} y_r^-, \qquad \psi_r^- f_{\mathbf{i}}^{\mathcal{O}} = f_{s_r\cdot\mathbf{i}}^{\mathcal{O}} \psi_r^-, \qquad y_r^- y_s^- = y_s^- y_r^-$$

$$\psi_r^- y_{r+1}^- f_{\mathbf{i}}^{\mathcal{O}} = (y_r^- \psi_r^- + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}, \qquad y_{r+1}^- \psi_r^- f_{\mathbf{i}}^{\mathcal{O}} = (\psi_r^- y_r^- + \delta_{i_r i_{r+1}}) f_{\mathbf{i}}^{\mathcal{O}}$$

$$\psi_r^- y_s^- = y_s^- \psi_r^-, \qquad \textit{if } s \ne r, r+1,$$

$$\psi_r^- \psi_s^- = \psi_s^- \psi_r^-, \qquad \textit{if } |r - s| > 1,$$

$$(\psi_r^-)^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (y_r^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_{r+1}^-) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \leftarrow i_{r+1}, \\ (y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_r^-) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r \rightarrow i_{r+1}, \\ 0, & \text{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

$$\psi_r^- \psi_{r+1}^- \psi_r^- f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (\psi_{r+1}^- \psi_r^- \psi_{r+1}^- - t^{1-\rho_r(\mathbf{i})}) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+2} \leftarrow i_{r+1}, \\ (\psi_{r+1}^- \psi_r^- \psi_{r+1}^- + 1) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r = i_{r+2} \rightarrow i_{r+1}, \\ \psi_{r+1}^- \psi_r^- \psi_{r+1}^- f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

*where* $y_r^{\langle d\rangle} f_{\mathbf{i}}^{\mathcal{O}} = (t^d y_r^- - [d]) f_{\mathbf{i}}^{\mathcal{O}}$*, for all* $d \in \mathbb{Z}$*, for* $\mathbf{i}, \mathbf{j} \in I^\alpha$ *and all admissible* $r$ *and* $s$*.*

**Proof.** This result is immediate from Theorem 6.1.5 since $\#$ is an algebra automorphism. □

**Remark 6.2.6.** To avoid unsightly double superscripts, we are using the same notation $y_r^{\langle d\rangle}$ for $(y_r^+)^{\langle d\rangle}$ from Theorem 6.1.5 and $(y_r^-)^{\langle d\rangle}$ above. This mild abuse should not cause any confusion, even when we use the same notation again for a third element later in this section. We are deliberately careful with the cyclotomic relation and do use double superscripts there.

To prove our main result we need an isomorphism from $\mathcal{R}_n^{\Lambda_0}(F)$ to $\mathscr{H}_n(F)$ which intertwines the $\mathtt{sgn}$ and $\#$ involutions on the two algebras. The following definitions will guarantee that we have this intertwining property once we show that this choice of elements indeed defines an isomorphism from $\mathcal{R}_n^{\Lambda_0}(F)$ to $\mathscr{H}_n(F)$.

When $\Lambda = \Lambda_0$ we are able to make a much more specific choice of representative $\mathbf{i}^+ \in [\mathbf{i}]$ from each equivalence class $[\mathbf{i}] \in I^n_\sim$ (see p40). Precisely, we define sets $I^n_+$ and $I^n_-$ by

$$I^n_+ = \{\, \mathbf{i} \in I_n \mid i_2 = 1 \,\} \qquad \text{and} \qquad I^n_- = \{\, \mathbf{i} \in I_n \mid i_2 = e-1 \,\}$$

and note that these sets are disjoint by our assumption that $e \neq 2$.

**Lemma 6.2.7.** *Suppose that* $\mathbf{i} \in I^n$ *and* $f_{\mathbf{i}}^{\mathcal{O}} \neq 0$*. Then* $\mathbf{i} \in I^n_+$ *or* $\mathbf{i} \in I^n_-$*.*

**Proof.** By Proposition 4.2.5, $f_{\mathbf{i}}^{\mathcal{O}} \neq 0$ if and only if $\operatorname{Std}(\mathbf{i}) \neq \emptyset$ or, equivalently, $\mathbf{i} = \operatorname{res}(\mathtt{t})$ for some standard tableau $\mathtt{t} \in \operatorname{Std}(\mathcal{P}_n)$. Therefore, $i_2 = \operatorname{res}_2(\mathtt{t}) = \pm 1$ so that $\mathbf{i} \in I^n_{\pm}$. □

In particular, we have that if $h \in \mathscr{H}_n(\mathcal{O})$ then $h = \sum_{\mathbf{i} \in I^n_+}(hf_{\mathbf{i}}^{\mathcal{O}} + hf_{-\mathbf{i}}^{\mathcal{O}})$. Furthermore, observe that $x = y$ in $\mathscr{H}_n(\mathcal{O})$ if and only if $xf_{\mathbf{i}}^{\mathcal{O}} = yf_{\mathbf{i}}^{\mathcal{O}}$, for all $\mathbf{i} \in I^n$. Further, since $f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\mathtt{t} \in \operatorname{Std}(\mathbf{i})} \frac{1}{\gamma} f_{\mathtt{tt}}$ by (3.5.18), we have that $x = y$ if and only if $xf_{\mathtt{tt}} = yf_{\mathtt{tt}}$, for all $\mathtt{t} \in \operatorname{Std}(\mathcal{P}_n)$. We apply Lemma 6.2.7, and these observations, without mention below.

**Lemma 6.2.8.** *Let $n > 1$ and $e > 2$. Then the ring $\mathcal{O}$ defined in Lemma 6.2.2 contains an alternating seminormal coefficient system.*

**Proof.** It is easy to check that the seminormal coefficient system $\boldsymbol{\alpha}$ in Example 3.4.9 satisfies the additional constraint from Definition 3.6.1 in order to be an alternating seminormal coefficient system. We have seen in Example 3.4.9 that our ring $\mathcal{O}$ must be extended by exactly the elements that we include in Definition 6.2.1 in order to define this system. □

**Definition 6.2.9.** Suppose that $1 \leq r < n$, that $r \neq 2$ and $1 \leq s \leq n$. Set

$$\psi_r^{\mathcal{O}} = \sum_{\mathbf{i} \in I^n_+} \psi_r^+ f_{\mathbf{i}}^{\mathcal{O}} - \sum_{\mathbf{i} \in I^n_-} \psi_r^- f_{\mathbf{i}}^{\mathcal{O}} \quad \text{and} \quad y_s^{\mathcal{O}} = \sum_{\mathbf{i} \in I^n_+} y_s^+ f_{\mathbf{i}}^{\mathcal{O}} - \sum_{\mathbf{i} \in I^n_-} y_s^- f_{\mathbf{i}}^{\mathcal{O}},$$

For $r = 2$, separately define

$$\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} \sqrt{t}(T_2L_2 - L_2T_2)f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_+ \\ \sqrt{t}(T_2L_2^{\#} - L_2^{\#}T_2)f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_- \end{cases} \tag{6.2.10}$$

if $e = 3$, and by defining

$$\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (T_2 - t)f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_+ \text{ and } i_3 = 2 \\ \frac{\sqrt{t}}{\sqrt{[3]}}(T_2L_2 - L_2T_2)f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_+ \text{ and } i_3 = e-1 \\ \frac{\sqrt{t}}{\sqrt{[3]}}(L_2^{\#}T_2 - T_2L_2^{\#})f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_- \text{ and } i_3 = 1 \\ (T_2 + 1)f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_- \text{ and } i_3 = e-2 \end{cases} \tag{6.2.11}$$

if $e > 3$.

By definition, and by our choice of $\mathcal{O}$, $\{\psi_r^{\mathcal{O}} \mid 1 \leq r < n\}$ and $\{y_s^{\mathcal{O}} \mid 1 \leq s \leq n\}$ are elements of $\mathscr{H}_n(\mathcal{O})$. The aim is now to show that the elements

$$\{\psi_1^{\mathcal{O}}, \psi_2^{\mathcal{O}}, \ldots, \psi_{n-1}^{\mathcal{O}}\} \cup \{y_1^{\mathcal{O}}, y_2^{\mathcal{O}}, \ldots, y_n^{\mathcal{O}}\} \cup \{f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^n\}$$

generate $\mathscr{H}_n(\mathcal{O})$ subject to relations like those in Theorem 6.1.5. This will imply that these elements induce an isomorphism $\mathcal{R}_n^{\Lambda_0} \xrightarrow{\sim} \mathscr{H}_n(F)$.

The following easy lemma involving the hash involution is used in [**43**] in the course of the proof of Theorem 3.4.11; we will use it to compute coefficients for the action of generators from [**17**] on the seminormal basis.

**Lemma 6.2.12.** *For $1 \leq r < n$ we have*

$$(T_rL_r - L_rT_r) = T_r(L_r - L_{r+1}) + 1 + (t-1)L_{r+1}$$
$$-(T_rL_r - L_rT_r)^{\#} = (T_rL_r^{\#} - L_r^{\#}T_r) = T_r(L_r^{\#} - L_{r+1}^{\#}) - 1 - (t-1)L_r^{\#}.$$

**Proof.** This follows immediately from manipulating the final relation in Definition 3.1.2. $\square$

Once our relations are established, the next result will imply that the isomorphism induced by our elements intertwines the `sgn` and # involutions.

**Corollary 6.2.13.** *Suppose that $1 \leq r < n$ and $1 \leq s \leq n$. Then $(\psi_r^{\mathcal{O}})^{\#} = -\psi_r^{\mathcal{O}}$. and $(y_s^{\mathcal{O}})^{\#} = -y_s^{\mathcal{O}}$.*

**Proof.** Note that $(T_2 + 1)^{\#} = (-T_2 + (t-1) + 1) = -(T_2 - t)$. As # is an involution the result now follows directly from Definition 6.2.9, Lemma 6.2.12, the explicit equations (6.2.10) and (6.2.11) defining $\psi_2^{\mathcal{O}}$, and Lemma 3.5.23. $\square$

**Proposition 6.2.14.** *For $\alpha \in Q_e$, the block $\mathscr{H}_\alpha(\mathcal{O})$ of the cyclotomic Hecke algebra $\mathscr{H}_n(\mathcal{O})$ is generated as an $\mathcal{O}$-algebra by the elements*

$$\{\psi_1^{\mathcal{O}}, \psi_2^{\mathcal{O}}, \ldots, \psi_{n-1}^{\mathcal{O}}\} \cup \{y_1^{\mathcal{O}}, y_2^{\mathcal{O}}, \ldots, y_n^{\mathcal{O}}\} \cup \{f_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in I^\alpha\}$$

**Proof.** Let $H$ be the $\mathcal{O}$-subalgebra of $\mathscr{H}_\alpha(\mathcal{O})$ generated by the elements in the statement of the proposition. Since the elements $\{T_1, T_2, \ldots, T_{n-1}\}$ generate

$\mathscr{H}_\alpha(\mathcal{O})$, and since the elements $\{f^{\mathcal{O}}_{\mathbf{i}} \mid \mathbf{i} \in I^\alpha\}$ give a complete set of idempotents, to prove the proposition it is enough to show that $T_r f^{\mathcal{O}}_{\mathbf{i}} \in H$, whenever $1 \le r < n$ and $\mathbf{i} \in (I^n_+ \cup I^n_-) \cap I^\alpha$. For $r > 3$, if $\mathbf{i} \in I^n_+$ then $T_r f^{\mathcal{O}}_{\mathbf{i}} \in H$ by Theorem 6.1.5 whereas if $\mathbf{i} \in I^n_-$ then $T_r f^{\mathcal{O}}_{\mathbf{i}} \in H$ by Proposition 6.2.5. On the other hand, when $r = 2$ we can rewrite equations (6.2.10) and (6.2.11) using Lemma 6.2.12:

$$T_2 f^{\mathcal{O}}_{\mathbf{i}} = \begin{cases} (t^{-\frac{1}{2}}\psi^{\mathcal{O}}_2 - 1 - (t-1)L_3)\frac{1}{L_2 - L_3} f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } e = 3 \text{ and } \mathbf{i} \in I^n_+ \\ (t^{-\frac{1}{2}}\psi^{\mathcal{O}}_2 + 1 + (t-1)L^{\#}_3)\frac{1}{L^{\#}_2 - L^{\#}_3} f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } e = 3 \text{ and } \mathbf{i} \in I^n_- \\ (\psi^{\mathcal{O}}_2 + t) f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } e > 3 \text{ and } i_3 = 2 \\ \left(\frac{\sqrt{[3]}}{\sqrt{t}}\psi^{\mathcal{O}}_2 - 1 - (t-1)L_3\right)\frac{1}{L_2 - L_3} f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } e > 3 \text{ and } i_3 = e-1 \\ \left(\frac{\sqrt{[3]}}{\sqrt{t}}\psi^{\mathcal{O}}_2 + 1 + (t-1)L^{\#}_3\right)\frac{1}{L^{\#}_2 - L^{\#}_3} f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } e > 3 \text{ and } i_3 = 1 \\ (\psi^{\mathcal{O}}_2 - 1) f^{\mathcal{O}}_{\mathbf{i}}, & \text{if } e > 3 \text{ and } i_3 = e-2 \end{cases}$$

noting that $\frac{1}{L_2 - L_3} f^{\mathcal{O}}_{\mathbf{i}}$ and its hashed version make sense by Corollary 6.1.2(ii). So $T_2 f^{\mathcal{O}}_{\mathbf{i}} \in H$ in all cases, since $\mathscr{L} = \langle y^{\mathcal{O}}_1, \dots, y^{\mathcal{O}}_r \rangle$ by the definition of $y^{\mathcal{O}}_r$. □

We now determine the relations satisfied by the generators of $\mathscr{H}_n(\mathcal{O})$ given in Proposition 6.2.14. Fortunately, much of the work has been done already because Theorem 6.1.5 and Proposition 6.2.5 give us a large number of these relations for free. More precisely we have the following list of relations which do **not** involve a $\psi^{\mathcal{O}}_2$.

**Lemma 6.2.15.** *For $\alpha \in Q_e$, the following identities hold in the block $\mathscr{H}_\alpha(\mathcal{O})$ of the cyclotomic Hecke algebra $\mathscr{H}_n(\mathcal{O})$:*

$$\prod_{\mathbf{i} \in I^\alpha} (y^{\mathcal{O}}_1)^{(\Lambda_0, \alpha_{i_1})} e(\mathbf{i}) = 0, \qquad f^{\mathcal{O}}_{\mathbf{i}} f^{\mathcal{O}}_{\mathbf{j}} = \delta_{\mathbf{ij}} f^{\mathcal{O}}_{\mathbf{i}}, \qquad \textstyle\sum_{\mathbf{i} \in I^\alpha} f^{\mathcal{O}}_{\mathbf{i}} = 1$$

$$y^{\mathcal{O}}_t f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{\mathbf{i}} y^{\mathcal{O}}_t, \qquad \psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = f^{\mathcal{O}}_{s_r \cdot \mathbf{i}} \psi^{\mathcal{O}}_r, \qquad y^{\mathcal{O}}_r y^{\mathcal{O}}_t = y^{\mathcal{O}}_t y^{\mathcal{O}}_r$$

$$\psi^{\mathcal{O}}_r y^{\mathcal{O}}_{r+1} f^{\mathcal{O}}_{\mathbf{i}} = (y^{\mathcal{O}}_r \psi^{\mathcal{O}}_r + \delta_{i_r i_{r+1}}) f^{\mathcal{O}}_{\mathbf{i}}, \qquad y^{\mathcal{O}}_{r+1} \psi^{\mathcal{O}}_r f^{\mathcal{O}}_{\mathbf{i}} = (\psi^{\mathcal{O}}_r y^{\mathcal{O}}_r + \delta_{i_r i_{r+1}}) f^{\mathcal{O}}_{\mathbf{i}}$$

$$\psi^{\mathcal{O}}_r y^{\mathcal{O}}_t = y^{\mathcal{O}}_t \psi^{\mathcal{O}}_r, \qquad \textit{if } t \ne r, r+1,$$

$$\psi^{\mathcal{O}}_r \psi^{\mathcal{O}}_s = \psi^{\mathcal{O}}_s \psi^{\mathcal{O}}_r, \qquad \textit{if } |r-s| > 1,$$

$$(\psi_r^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle} - y_{r+1}^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \to i_{r+1} \textit{ and } \mathbf{i} \in I_+^n, \\ (y_r^{\mathcal{O}} - y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \to i_{r+1} \textit{ and } \mathbf{i} \in I_-^n, \\ (y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle} - y_r^{\mathcal{O}}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \leftarrow i_{r+1} \textit{ and } \mathbf{i} \in I_+^n \\ (y_{r+1}^{\mathcal{O}} - y_r^{\langle 1-\rho_r(\mathbf{i})\rangle}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r \leftarrow i_{r+1} \textit{ and } \mathbf{i} \in I_-^n \\ 0, & \textit{if } i_r = i_{r+1}, \\ f_{\mathbf{i}}^{\mathcal{O}}, & \textit{otherwise}, \end{cases}$$

$$\psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (\psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} - t^{1+\rho_r(\mathbf{i})}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r = i_{r+2} \to i_{r+1}, \textit{ and } \mathbf{i} \in I_+^n \\ (\psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} - 1) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r = i_{r+2} \to i_{r+1}, \textit{ and } \mathbf{i} \in I_-^n \\ (\psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} + 1) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r = i_{r+2} \leftarrow i_{r+1} \textit{ and } \mathbf{i} \in I_+^n, \\ (\psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} + t^{1-\rho_r(\mathbf{i})}) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } i_r = i_{r+2} \leftarrow i_{r+1} \textit{ and } \mathbf{i} \in I_-^n, \\ \psi_{r+1}^{\mathcal{O}} \psi_r^{\mathcal{O}} \psi_{r+1}^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{otherwise}, \end{cases}$$

*where* $y_r^{\langle d\rangle} f_{\mathbf{i}}^{\mathcal{O}} = (t^d y_r^{\mathcal{O}} - [d]) f_{\mathbf{i}}^{\mathcal{O}}$, *for all admissible* $\mathbf{i}, \mathbf{j} \in (I_+^n \cup I_-^n) \cap I^\alpha$ *such that* $r, s, t$ *satisfy* $2 < r, s < n$ *and* $1 \le t \le n$.

**Proof.** The first three identities follow directly from Theorem 6.1.5 and Proposition 6.2.5. For the remaining formulas, observe that if $2 < r < n$ then $\mathbf{i} \in I_+^n$ if and only if $s_r \cdot \mathbf{i} \in I_+^n$ and, similarly, $\mathbf{i} \in I_-^n$ if and only if $s_r \cdot \mathbf{i} \in I_-^n$. Therefore, if $\mathbf{i} \in I_+^n$ the relations hold by Theorem 6.1.5 and if $\mathbf{i} \in I_-^n$ then they hold by Proposition 6.2.5. Note that if $\mathbf{i} \in I_-^n$ then there is a sign change in the last two relations, in comparison with Proposition 6.2.5, because $\psi_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = -\psi_r^- f_{\mathbf{i}}^{\mathcal{O}}$ for $1 \le r < n$ and $y_s^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = -y_s^- f_{\mathbf{i}}^{\mathcal{O}}$ for $1 \le s \le n$. □

**Remark 6.2.16.**

(i) Since $\Lambda = \Lambda_0$, $i_1 = 0$ for all $\mathbf{i} \in I^n$ and so $\psi_1^{\mathcal{O}} = 0$ by the relations in Theorem 6.1.5 and Proposition 6.2.5 hence, once we have established the remaining relations involving $\psi_2^{\mathcal{O}}$, there is no issue in not including relations involving $\psi_1^{\mathcal{O}}$ in the list in Lemma 6.2.15.

(ii) Observe that, in all of the cases in Lemma 6.2.15, whenever an exponent $1 + \rho_r(\mathbf{i})$ appears in a relation it is always a multiple of $e$, and likewise whenever $1-\rho_r(\mathbf{i})$ appears in a relation it is always a multiple of $e$. Hence, modulo reduction to $F$, all of the relations in Lemma 6.2.15 are compatible with Definition 4.1.8. It remains to determine the relations which involve the generator $\psi_2^{\mathcal{O}}$, since we have explicitly excluded $\psi_2$ from the list of relations in Lemma 6.2.15. We will consider the cases $e=3$ and $e>3$ separately.

To describe how $\psi_r^{\mathcal{O}}$ acts on the seminormal basis $\{f_{\mathtt{st}} \mid \mathtt{s},\mathtt{t}\in \operatorname{Std}(\lambda),\ \lambda\in\mathcal{P}_n\}$ we need generalisations of the $\beta$-coefficients defined in [**43**]. For $\mathbf{i}\in I^n$, suppose that $\mathtt{s}\in\operatorname{Std}(\mathbf{i})$ and $\mathtt{u}=s_r\cdot\mathtt{s}$, where $1\le r<n$. If $\mathtt{u}\notin\operatorname{Std}(\lambda)$ for any $\lambda\in\mathcal{P}_n$, set $\beta_r(\mathtt{s})=0$. On the other hand, if $\mathtt{u}\in\operatorname{Std}(\lambda)$ for some $\lambda\in\mathcal{P}_n$ then define

$$\beta_r(\mathtt{s})=\begin{cases}\beta_r(\mathtt{s}'), & \text{if } \mathbf{i}\in I^n_-,\\ \dfrac{t^{i_r-c_r(\mathtt{s})}\alpha_r(\mathtt{s})}{[1-\rho_r(\mathtt{s})]}, & \text{if } i_r=i_{r+1},\\ t^{c_{r+1}(\mathtt{s})-i_r}\alpha_r(\mathtt{s})[\rho_r(\mathtt{s})], & \text{if } i_r=i_{r+1}+1,\\ \dfrac{t^{-\rho_r(\mathtt{s})}\alpha_r(\mathtt{s})[\rho_r(\mathtt{s})]}{[1-\rho_r(\mathtt{s})]}, & \text{otherwise.}\end{cases} \tag{6.2.17}$$

Observe that the four cases in (6.2.17) are mutually exclusive because $\operatorname{res}(\mathtt{s}')=-\operatorname{res}(\mathtt{s})$. Therefore, $\operatorname{res}(\mathtt{s})\in I^n_\pm$ if and only if $\operatorname{res}(\mathtt{s}')\in I^n_\mp$. We need to be a little careful, however, because if $\mathbf{i}=\operatorname{res}(\mathtt{s})$ and $\mathbf{j}=\operatorname{res}(\mathtt{s}')$ then even though $\mathbf{j}=-\mathbf{i}$ it is not usually true that $\hat{\jmath}_r=-\hat{\imath}_r$, for $1\le r\le n$. Nonetheless, we do always have $i_r+j_r\equiv 0 \pmod e$, for $1\le r\le n$.

**Proposition 6.2.18.** *Suppose that* $\mathtt{s},\mathtt{t}\in\operatorname{Std}(\boldsymbol{\lambda})$ *for* $\boldsymbol{\lambda}\in\mathcal{P}_n$ *and let* $\mathbf{i}=\operatorname{res}(\mathtt{s})$, $\mathbf{j}=\operatorname{res}(\mathtt{s}')$ *and* $\mathtt{u}=s_r\cdot\mathtt{s}$. *Then* $\psi_1^{\mathcal{O}}f_{\mathtt{st}}=0$ *and for* $3\le r<n$,

$$\psi_r^{\mathcal{O}}f_{\mathtt{st}}=\begin{cases}\beta_r(\mathtt{s})f_{\mathtt{ut}}-\delta_{i_ri_{r+1}}\dfrac{t^{i_{r+1}-c_{r+1}(\mathtt{s})}}{[\rho_r(\mathtt{s})]}f_{\mathtt{ss}}, & \textit{if } \mathbf{i}\in I^n_+,\\ \beta_r(\mathtt{s})f_{\mathtt{ut}}-\delta_{i_ri_{r+1}}\dfrac{t^{\hat{\jmath}_{r+1}-c_r(\mathtt{s})}}{[\rho_r(\mathtt{s})]}f_{\mathtt{ss}}, & \textit{if } \mathbf{i}\in I^n_-.\end{cases}$$

*Moreover, if $1 \le r \le n$ then*

$$y_r^{\mathcal{O}} f_{\mathtt{st}} = \begin{cases} [c_r(\mathtt{s}) - i_r] f_{\mathtt{st}}, & \text{if } \mathbf{i} \in I_+^n, \\ -[c_r(\mathtt{s}') - \hat{\jmath}_r] f_{\mathtt{st}}, & \text{if } \mathbf{i} \in I_-^n. \end{cases}$$

**Proof.** We observed in Remark 6.2.16(i) that $\psi_1^{\mathcal{O}} = 0$. Observe that, without loss of generality, we can assume that $\mathtt{t} = \mathtt{s}$ by Theorem 3.4.11. If $\mathbf{i} \in I_+^n$ then $\psi_r^{\mathcal{O}} f_{\mathtt{ss}} = \psi_r^+ f_{\mathtt{ss}}$ and the lemma is a restatement of [**43**, Lemma 4.23]. Now suppose that $\mathbf{i} \in I_-^n$, so that $\mathbf{j} \in I_+^n$ and $\psi_r^{\mathcal{O}} f_{\mathtt{s}'\mathtt{s}'}$ is again given by [**43**]. As $\#$ is an involution, using Corollary 3.5.11 and Lemma 3.6.5,

$$\begin{aligned} \psi_r^{\mathcal{O}} f_{\mathtt{ss}} &= -(\psi_r^+)^\# f_{\mathtt{ss}} = -\big(\psi_r^+ f_{\mathtt{ss}}^\#\big)^\# = -\frac{\gamma_{\mathtt{s}}}{\gamma_{\mathtt{s}'}} \big(\psi_r^+ f_{\mathtt{s}'\mathtt{s}'}\big)^\#, \\ &= -\frac{\gamma_{\mathtt{s}}}{\gamma_{\mathtt{s}'}} \Big(\beta_r(\mathtt{s}') f_{\mathtt{u}'\mathtt{s}'} - \delta_{j_r j_{r+1}} \frac{t^{\hat{\jmath}_{r+1} - c_{r+1}(\mathtt{s}')}}{[\rho_r(\mathtt{s}')]} f_{\mathtt{s}'\mathtt{s}'}\Big)^\# \\ &= \frac{\alpha_r(\mathtt{s})\beta_r(\mathtt{s}')}{\alpha_r(\mathtt{s}')} f_{\mathtt{us}} - \delta_{i_r i_{r+1}} \frac{t^{\hat{\jmath}_{r+1} - c_r(\mathtt{s})}}{[\rho_r(\mathtt{s})]} f_{\mathtt{ss}}, \end{aligned}$$

since $[\rho_r(\mathtt{s}')] = [-\rho_r(\mathtt{s})] = -t^{-\rho_r(\mathtt{s})}[\rho_r(\mathtt{s})]$. By Definition 3.6.1, $\alpha_r(\mathtt{s}') = -\alpha_r(\mathtt{s})$, so the coefficient of $f_{\mathtt{us}}$ in $\psi_r^{\mathcal{O}} f_{\mathtt{ss}}$ is $\beta_r(\mathtt{s}) = \beta_r(\mathtt{s}')$ by (6.2.17).

For the action of $y_r^{\mathcal{O}}$, if $\mathbf{i} \in I_+^n$ then $y_r^{\mathcal{O}} f_{\mathtt{ss}} = y_r^+ f_{\mathtt{ss}} = [c_r(\mathtt{s}) - i_r] f_{\mathtt{ss}}$ by [**43**, Lemma 4.23]. On the other hand, if $\mathbf{i} \in I_-^n$ then, using Lemma 3.5.23 twice,

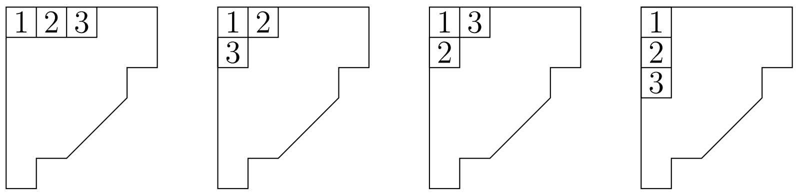
$$y_r^{\mathcal{O}} f_{\mathtt{ss}} = -\frac{\gamma_{\mathtt{s}}}{\gamma_{\mathtt{s}'}} (y_r^+ f_{\mathtt{s}'\mathtt{s}'})^\# = -\frac{\gamma_{\mathtt{s}}}{\gamma_{\mathtt{s}'}} \big([c_r(\mathtt{s}') - \hat{\jmath}_r] f_{\mathtt{s}'\mathtt{s}'}\big)^\# = -[c_r(\mathtt{s}') - \hat{\jmath}_r] f_{\mathtt{ss}}.$$

as required. □

There are four equivalency classes of standard tableaux according to the positions of the numbers $1, 2, 3$. More precisely, let

$$\begin{aligned} \operatorname{Std}_{\rightarrow}(\mathcal{P}_n) &= \{\mathtt{t} \in \mathcal{P}_n \mid c_{\mathtt{t}}(1) = 0, c_{\mathtt{t}}(2) = 1 \text{ and } c_{\mathtt{t}}(3) = 2\} \\ \operatorname{Std}_{\swarrow}(\mathcal{P}_n) &= \{\mathtt{t} \in \mathcal{P}_n \mid c_{\mathtt{t}}(1) = 0, c_{\mathtt{t}}(2) = 1 \text{ and } c_{\mathtt{t}}(3) = -1\} \\ \operatorname{Std}_{\nearrow}(\mathcal{P}_n) &= \{\mathtt{t} \in \mathcal{P}_n \mid c_{\mathtt{t}}(1) = 0, c_{\mathtt{t}}(2) = -1 \text{ and } c_{\mathtt{t}}(3) = 1\} \\ \operatorname{Std}_{\downarrow}(\mathcal{P}_n) &= \{\mathtt{t} \in \mathcal{P}_n \mid c_{\mathtt{t}}(1) = 0, c_{\mathtt{t}}(2) = -1 \text{ and } c_{\mathtt{t}}(3) = -2\} \end{aligned} \tag{6.2.19}$$

The following mnemonic illustrates the definition above, remembering we allow tableaux to extend to the east and south.




$\mathrm{Std}_{\rightarrow}(\mathcal{P}_n)$ $(012\cdots)$ $\quad$ $\mathrm{Std}_{\swarrow}(\mathcal{P}_n)$ $(01-1\cdots)$ $\quad$ $\mathrm{Std}_{\nearrow}(\mathcal{P}_n)$ $(0(e-1)1\cdots)$ $\quad$ $\mathrm{Std}_{\downarrow}(\mathcal{P}_n)$ $(0(e-1)(e-2)\cdots)$

We will also need the corresponding notation

$$\begin{aligned} I^n_{\rightarrow} &= \{\mathbf{i} \in I^n \mid i_3 = 2\} \\ I^n_{\swarrow} &= \{\mathbf{i} \in I^n \mid i_3 = e-1\} \\ I^n_{\nearrow} &= \{\mathbf{i} \in I^n \mid i_3 = 1\} \\ I^n_{\downarrow} &= \{\mathbf{i} \in I^n \mid i_3 = e-2\}. \end{aligned} \tag{6.2.20}$$

Notice that, if $e > 3$, we have the following **disjoint** unions:

$$I^n_+ = I^n_{\rightarrow} \sqcup I^n_{\swarrow} \quad \text{and} \quad I^n_- = I^n_{\nearrow} \sqcup I^n_{\downarrow}.$$

**Relations involving $\psi_2^{\mathcal{O}}$ when $e = 3$**

We first determine the scalars by which $\psi_2^{\mathcal{O}}$ acts on the seminormal basis in the case when $e = 3$; this computation will allow us to check the relations by comparing coefficients as in our remarks above. Notice that 3-residue sequences alone can **not** distinguish between the four equivalence classes of tableaux in (6.2.19).

**Lemma 6.2.21.** *Let $e = 3$ and let $\mathsf{s} \in \mathrm{Std}(\mathbf{i})$ for some $\mathbf{i} \in I^n_+ \cup I^n_-$. Then*

$$\psi_2^{\mathcal{O}} f_{\mathsf{ss}} = \begin{cases} 0, & \text{if } \mathsf{s} \in \mathrm{Std}_{\rightarrow}(\mathcal{P}_n) \cup \mathrm{Std}_{\downarrow}(\mathcal{P}_n) \\ \sqrt{-1}\sqrt{[3]} f_{s_2\cdot\mathsf{s},\mathsf{s}}, & \text{if } \mathsf{s} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n) \\ -\sqrt{-1}\sqrt{[3]} f_{s_2\cdot\mathsf{s},\mathsf{s}}, & \text{if } \mathsf{s} \in \mathrm{Std}_{\nearrow}(\mathcal{P}_n) \end{cases}$$

**Proof.** If $\mathbf{i} \in I^n_+$, by (6.2.10), for $\mathtt{s} \in \mathrm{Std}(\mathbf{i})$,

$$\psi_2^{\mathcal{O}} f_{\mathtt{ss}} = \sqrt{\mathtt{t}}\alpha_2(\mathtt{s})\Big([c_2(\mathtt{s})] - [c_2(s_2 \cdot \mathtt{s})]\Big) f_{s_2\cdot\mathtt{s},\mathtt{s}}.$$

If $\mathtt{s} \in \mathrm{Std}_{\rightarrow}(\mathcal{P}_n)$, then $s_2 \cdot \mathtt{s} = 0$ and so $\psi_2^{\mathcal{O}} f_{\mathtt{ss}} = 0$. On the other hand, if $\mathtt{s} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n)$, then $[c_2(\mathtt{s})] - [c_2(s_2 \cdot \mathtt{s})] = [1] - [-1] = t^{-1}[2]$, and so

$$\begin{aligned}\psi_2^{\mathcal{O}} f_{\mathtt{ss}} &= \sqrt{t}t^{-1}\alpha_2(\mathtt{s})[2] f_{s_2\cdot\mathtt{s},\mathtt{s}} \\ &= \sqrt{-1}\sqrt{[3]} f_{s_2\cdot\mathtt{s},\mathtt{s}}\end{aligned}$$

by (3.6.4). The computation for $\mathbf{i} \in \mathrm{Std}_{\nearrow}(\mathcal{P}_n)$ is similar, upon noting that $\mathrm{Std}_{\nearrow}(\mathcal{P}_n) = \{s_2 \cdot \mathtt{t} \mid \mathtt{t} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n)\}$. $\square$

**Lemma 6.2.22.** *Let $e = 3$, let $\mathbf{i} \in I^n_+ \cup I^n_-$ and let $\mathbf{j} = s_2 \cdot \mathbf{i}$. Then $\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = f_{\mathbf{j}}^{\mathcal{O}} \psi_2^{\mathcal{O}}$.*

**Proof.** Suppose $e = 3$ and suppose $\mathbf{i} \in I^n_+$. Then

$$\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \sum_{\substack{\mathtt{s} \in \mathrm{Std}_{\rightarrow}(\mathcal{P}_n) \\ \mathrm{res}(\mathtt{s}) = \mathbf{i}}} \psi_2^{\mathcal{O}} f_{\mathtt{ss}} + \sum_{\substack{\mathtt{t} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n) \\ \mathrm{res}(\mathtt{t}) = \mathbf{i}}} \psi_2^{\mathcal{O}} f_{\mathtt{tt}} = \sqrt{-1}\sqrt{[3]} \sum_{\substack{\mathtt{t} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n) \\ \mathrm{res}(\mathtt{t}) = \mathbf{i}}} f_{s_2\cdot\mathtt{t},\mathtt{t}}.$$

On the other hand, since $s_2 \cdot \mathbf{i} \in I^n_-$,

$$\begin{aligned} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} \psi_2^{\mathcal{O}} &= \sum_{\mathbf{j} \in I^n} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} \psi_2^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} \\ &= \sum_{\mathbf{j} \in I^n} \Big( \sqrt{-1}\sqrt{[3]} \sum_{\substack{\mathtt{t} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n) \\ \mathrm{res}(\mathtt{t}) = \mathbf{j}}} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} f_{s_2\cdot\mathtt{t},\mathtt{t}} - \sqrt{-1}\sqrt{[3]} \sum_{\substack{\mathtt{u} \in \mathrm{Std}_{\nearrow}(\mathcal{P}_n) \\ \mathrm{res}(\mathtt{u}) = \mathbf{j}}} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} f_{s_2\cdot\mathtt{u},\mathtt{u}} \Big). \end{aligned}$$

But if $\mathtt{u} \in \mathrm{Std}_{\nearrow}(\mathcal{P}_n)$, $s_2 \cdot \mathtt{u} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n)$, so $f_{s_2\cdot\mathtt{u},\mathtt{u}} \in f_{\mathbf{j}}^{\mathcal{O}} \mathscr{H}_n(\mathcal{O})$ for some $\mathbf{j} \in I^n_+$. So $f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} f_{s_2\cdot\mathtt{u},\mathtt{u}} = 0$ for all $\mathtt{u} \in \mathrm{Std}_{\nearrow}(\mathcal{P}_n)$ and $f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} f_{s_2\cdot\mathtt{t},\mathtt{t}} = \delta_{\mathrm{res}(s_2\cdot\mathtt{t}),s_2\cdot\mathbf{i}} f_{s_2\cdot\mathtt{t},\mathtt{t}}$, so since $\mathrm{res}(s_2 \cdot \mathtt{t}) = s_2 \cdot \mathbf{i}$ if and only if $\mathrm{res}(\mathtt{t}) = \mathbf{i}$,

$$f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} \psi_2^{\mathcal{O}} = \sqrt{-1}\sqrt{[3]} \sum_{\substack{\mathtt{t} \in \mathrm{Std}_{\swarrow}(\mathcal{P}_n) \\ \mathrm{res}(\mathtt{t}) = \mathbf{i}}} f_{s_2\cdot\mathtt{t},\mathtt{t}} = \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}.$$

The computation for $\mathbf{i} \in I^n_-$ is nearly identical. $\square$

It remains to determine the relations between $\psi_2^{\mathcal{O}}$ and the elements

$$\{y_r^{\mathcal{O}} \mid 3 \le r \le n\} \cup \{\psi_r^{\mathcal{O}} \mid 3 \le r < n\}.$$

To do this we follow the strategy in [**43**] and determine these relations by computing actions on the seminormal basis and comparing coefficients. Unlike the relations in Lemma 6.2.15, the remaining relations do move between weight spaces indexed by sequences in $I_+^n$ and those indexed by sequences in $I_-^n$.

**Lemma 6.2.23.** *Suppose that $e = 3$ and $\mathbf{i} \in I^n$. Then*

$$\psi_2^{\mathcal{O}} y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-3} y_2^{\langle 3\rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_+^n \\ (y_2^{\mathcal{O}} + t^{-3}[3])\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_-^n \end{cases}$$

$$\psi_2^{\mathcal{O}} y_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-3} y_3^{\langle 3\rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_+^n \\ (y_3^{\mathcal{O}} + t^{-3}[3])\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_-^n \end{cases}$$

**Proof.** Using the formulas in Proposition 6.2.18 and Lemma 6.2.21, we observe that if $\mathbf{i} \in I_+^n$,

$$\begin{aligned} \psi_2^{\mathcal{O}} y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= \sum_{\mathsf{s} \in \mathrm{Std}(\mathbf{i}) \cap \mathrm{Std}_{\rightarrow}(\mathcal{P}_n)} \psi_2^{\mathcal{O}} y_3^{\mathcal{O}} f_{\mathsf{ss}} + \sum_{\mathsf{s} \in \mathrm{Std}(\mathbf{i}) \cap \mathrm{Std}_{\swarrow}(\mathcal{P}_n)} \psi_2^{\mathcal{O}} y_3^{\mathcal{O}} f_{\mathsf{ss}} \\ &= \sum_{\mathsf{s} \in \mathrm{Std}(\mathbf{i}) \cap \mathrm{Std}_{\swarrow}(\mathcal{P}_n)} \sqrt{-1}[-3]\sqrt{[3]} f_{s_2 \cdot \mathsf{s}, \mathsf{s}} \\ &= (y_2^{\mathcal{O}} + [-3])\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} \\ &= t^{-3} y_2^{\langle 3\rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} \end{aligned}$$

using the definition of $y_r^{\langle d\rangle}$. Similarly, if $\mathbf{i} \in I_-^n$,

$$\begin{aligned} \psi_2^{\mathcal{O}} y_3^{\mathcal{O}} &= \sum_{\mathsf{s} \in \mathrm{Std}(\mathbf{i}) \cap \mathrm{Std}_{\nearrow}(\mathcal{P}_n)} -[-3]\psi_2^{\mathcal{O}} f_{\mathsf{ss}} \\ &= \sum_{\mathsf{s} \in \mathrm{Std}(\mathbf{i}) \cap \mathrm{Std}_{\nearrow}(\mathcal{P}_n)} \sqrt{-1}[-3]\sqrt{[3]} f_{s_2 \cdot \mathsf{s}, \mathsf{s}} \\ &= (y_2^{\mathcal{O}} - [-3])\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}. \end{aligned}$$

The first formula now follows by the definition of $y_r^{\langle d\rangle}$. The proof of the other formula is similar. $\square$

**Lemma 6.2.24.** *Suppose that $e = 3$. Then for $r > 3$ we have*

$$\psi_2^{\mathcal{O}} y_r^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = y_r^{\mathcal{O}} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}.$$

*for all $\mathbf{i} \in I^n$.*

**Proof.** For $\mathbf{i} \in I^n$, let $\mathsf{s} \in \operatorname{Std}(\mathbf{i})$ and note that by the formulas in Proposition 6.2.18,

$$y_r^{\mathcal{O}} f_{\mathsf{ss}} = \begin{cases} [c_r(\mathsf{s}) - i_r] f_{\mathsf{ss}}, & \text{if } \mathbf{i} \in I_+^n \\ -[c_r(\mathsf{s}') - \hat{\jmath}_r], & \text{if } \mathbf{i} \in I_-^n \end{cases}$$

whereas

$$y_r^{\mathcal{O}} f_{s_2 \cdot \mathsf{s},\mathsf{s}} = \begin{cases} [c_r(s_2 \cdot \mathsf{s}) - (s_2 \cdot i)_r] f_{s_2 \cdot \mathsf{s},\mathsf{s}}, & \text{if } s_2 \cdot \mathbf{i} \in I_+^n \\ -[c_r(s_2 \cdot \mathsf{s}') - (s_2 \cdot \mathbf{j})_r] f_{s_2 \cdot \mathsf{s},\mathsf{s}}, & \text{if } s_2 \cdot \mathbf{i} \in I_-^n \end{cases}$$

For $r > 3$, $c_r(s_2 \cdot \mathsf{s}) = c_r(\mathsf{s})$ and $i_r = (s_2 \cdot i)_r$ for all $\mathsf{s}$, since $s_2$ only changes the second and third entries of the content sequence and residue sequence, which are not involved in these quantities. □

**Lemma 6.2.25.** *Suppose $e = 3$ and that $3 < r \leq n$. Then $\psi_2^{\mathcal{O}} \psi_r^{\mathcal{O}} = \psi_r^{\mathcal{O}} \psi_2^{\mathcal{O}}$.*

**Proof.** Let $\mathbf{i} \in I_+^n \cup I_-^n$ and suppose $i_r \neq i_{r+1}$; then $(s_2 \cdot \mathbf{i})_r \neq (s_2 \cdot \mathbf{i})_{r+1}$ and so it is easy to check that, for $\mathsf{s} \in \operatorname{Std}(\mathbf{i})$,

$$\psi_2^{\mathcal{O}} \psi_r^{\mathcal{O}} f_{\mathsf{ss}} = \begin{cases} 0, & \text{if } s_r \cdot \mathsf{s} \in \operatorname{Std}_{\rightarrow}(\mathcal{P}_n) \cup \operatorname{Std}_{\downarrow}(\mathcal{P}_n) \\ \sqrt{-1}\sqrt{[3]}\beta_r(\mathsf{s}) f_{s_2 s_r \cdot \mathsf{s},\mathsf{s}}, & \text{if } s_r \cdot \mathsf{s} \in \operatorname{Std}_{\swarrow}(\mathcal{P}_n) \\ -\sqrt{-1}\sqrt{[3]}\beta_r(\mathsf{s}) f_{s_2 s_r \cdot \mathsf{s},\mathsf{s}}, & \text{if } s_r \cdot \mathsf{s} \in \operatorname{Std}_{\nearrow}(\mathcal{P}_n) \end{cases}$$

whereas

$$\psi_r^{\mathcal{O}} \psi_2^{\mathcal{O}} f_{\mathsf{ss}} = \begin{cases} 0, & \text{if } \mathsf{s} \in \operatorname{Std}_{\rightarrow}(\mathcal{P}_n) \cup \operatorname{Std}_{\downarrow}(\mathcal{P}_n) \\ \sqrt{-1}\sqrt{[3]}\beta_r(s_2 \cdot \mathsf{s}) f_{s_r s_2 \cdot \mathsf{s},\mathsf{s}}, & \text{if } \mathsf{s} \in \operatorname{Std}_{\swarrow}(\mathcal{P}_n) \\ -\sqrt{-1}\sqrt{[3]}\beta_r(s_2 \cdot \mathsf{s}) f_{s_r s_2 \cdot \mathsf{s},\mathsf{s}}, & \text{if } \mathsf{s} \in \operatorname{Std}_{\nearrow}(\mathcal{P}_n). \end{cases}$$

Since $|r - 2| > 1$, $s_r s_2 = s_2 s_r$ by the symmetric group relations, and an inspection of the formulas for $\beta_r(\mathsf{s})$ in (6.2.17) gives that this quantity is invariant under changing $\mathsf{s}$ to $s_2 \cdot \mathsf{s}$ provided $r > 3$, so the lemma follows. □

**Lemma 6.2.26.** *Suppose that $e = 3$ and that $\mathbf{i} \in I^n_+ \cup I^n_-$. Then*

$$(\psi_2^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} -t^3 y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I^n_+, \\ t^3 y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I^n_-. \end{cases}$$

**Proof.** Observe that by Proposition 6.2.18, for $\mathsf{s} \in \operatorname{Std}(\mathbf{i})$ for some $\mathbf{i} \in I^n_+ \cup I^n_-$,

$$y_3^{\mathcal{O}} f_{\mathsf{ss}} = \begin{cases} 0, & \text{if } \mathsf{s} \in \operatorname{Std}_{\rightarrow}(\mathcal{P}_n) \cup \operatorname{Std}_{\downarrow}(\mathcal{P}_n) \\ [-3] f_{\mathsf{ss}}, & \text{if } \mathsf{s} \in \operatorname{Std}_{\swarrow}(\mathcal{P}_n) \\ -[-3] f_{\mathsf{ss}}, & \text{if } \mathsf{s} \in \operatorname{Std}_{\nearrow}(\mathcal{P}_n) \end{cases}$$

and that by Lemma 6.2.21,

$$(\psi_2^{\mathcal{O}})^2 f_{\mathsf{ss}} = \begin{cases} 0, & \text{if } \mathsf{s} \in \operatorname{Std}_{\rightarrow}(\mathcal{P}_n) \cup \operatorname{Std}_{\downarrow}(\mathcal{P}_n) \\ [3] f_{\mathsf{ss}}, & \text{if } \mathsf{s} \in \operatorname{Std}_{\swarrow}(\mathcal{P}_n) \cup \operatorname{Std}_{\nearrow}(\mathcal{P}_n) \end{cases}$$

The result now follows since $[-3] = -t^{-3}[3]$. ∎

**Lemma 6.2.27.** *Suppose that $e = 3$. Then for $\mathbf{i} \in I^n$,*

$$\psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (\psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} + t^3) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} = (0212\cdots) \\ (\psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} - 1) f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} = (0121\cdots) \\ \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{otherwise} \end{cases}$$

**Proof.** Observe the following exhaustive list of the first four entries of standard tableaux (tableaux have $n$ entries and may extend to the east and south); below each we have written the first four entries of their 3-residue sequences:

$$\begin{array}{ccccc}
\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau} & \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} & \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau} & \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau} & \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau} \\
0120 & 0122 & 0122 & 0212 & 0120 \\
\begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau} & \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau} & \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau} & \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau} & \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau} \\
0210 & 0121 & 0211 & 0211 & 0210
\end{array} \tag{6.2.28}$$

Note that, since $\psi_2^{\mathcal{O}}\psi_3^{\mathcal{O}}\psi_2^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = f_{s_2s_3s_2\mathbf{i}}^{\mathcal{O}}\psi_2^{\mathcal{O}}\psi_3^{\mathcal{O}}\psi_2^{\mathcal{O}}$ and

$$\psi_3^{\mathcal{O}}\psi_2^{\mathcal{O}}\psi_3^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = f_{s_3s_2s_3}^{\mathcal{O}}\psi_3^{\mathcal{O}}\psi_2^{\mathcal{O}}\psi_3^{\mathcal{O}} = f_{s_2s_3s_2\mathbf{i}}^{\mathcal{O}}\psi_3^{\mathcal{O}}\psi_2^{\mathcal{O}}\psi_3^{\mathcal{O}}$$

by the braid relation in the symmetric group, the formula given in the statement of the lemma holds for all $\mathbf{i} \notin \{(0212\cdots), (0121\cdots)\}$ by virtue of both sides being zero.

If $\mathbf{i} = (0212\cdots)$ then,

$$\begin{aligned}
\psi_2^{\mathcal{O}}\psi_3^{\mathcal{O}}\psi_2^{\mathcal{O}}f_{134/2,134/2} &= -\sqrt{-1}\sqrt{[3]}\psi_2^{\mathcal{O}}\Big(\beta_3(124/3)f_{123/4,134/2} - \frac{1}{[\rho_3(124/3)]}f_{124/3,134/2}\Big)\\
&= \frac{-[3]}{[\rho_3(124/3)]}f_{134/2,134/2}\\
&= t^3 f_{134/2,134/2}
\end{aligned}$$

since $\rho_3(124/3) = -1-2 = -3$. Since $\psi_3^{\mathcal{O}}f_{134/2,134/2} = 0$ by Proposition 6.2.18, because if $\mathbf{j} = \mathrm{res}(134/2) = (0212)$ then $j_3 \neq j_4$ and $\alpha_3(134/2) = 0$, this gives the first line of the formula. On the other hand, if $\mathbf{i} = (0121\cdots)$ then a similar calculation shows

$$\begin{aligned}
\psi_2^{\mathcal{O}}\psi_3^{\mathcal{O}}\psi_2^{\mathcal{O}}f_{12/3/4,12/3/4} &= \frac{-[3]}{[\rho_3(13/2/4)]}f_{12/3/4,12/3/4}\\
&= -f_{12/3/4,12/3/4}
\end{aligned}$$

which gives the second line of the formula since $\psi_3^{\mathcal{O}}f_{12/3/4} = 0$ by Proposition 6.2.18. □

We summarise the relations from this subsection in the following proposition:

**Proposition 6.2.29.** *Suppose $e = 3$. Then the following relations hold in the algebra $\mathscr{H}_n(\mathcal{O})$.*

$$\psi_2^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = f_{s_2\mathbf{i}}^{\mathcal{O}}\psi_2^{\mathcal{O}}$$

$$\psi_2^{\mathcal{O}}y_3^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-3}y_2^{\langle 3\rangle}\psi_2^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I_+^n\\ (y_2^{\mathcal{O}} + t^{-3}[3])\psi_2^{\mathcal{O}}f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I_-^n\end{cases}$$

$$\psi_2^{\mathcal{O}} y_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-3} y_3^{\langle 3\rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_+^n \\ (y_3^{\mathcal{O}} + t^{-3}[3]) \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_-^n \end{cases}$$

$$\psi_2^{\mathcal{O}} y_r^{\mathcal{O}} = y_r^{\mathcal{O}} \psi_2^{\mathcal{O}} \qquad \text{for } n \geq r > 3$$

$$\psi_2^{\mathcal{O}} \psi_r^{\mathcal{O}} = \psi_r^{\mathcal{O}} \psi_2^{\mathcal{O}} \qquad \text{for } n > r > 3$$

$$(\psi_2^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} -t^3 y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_+^n, \\ t^3 y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_-^n. \end{cases}$$

$$\psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (\psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} + t^3) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} = (0212\cdots) \\ (\psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} - 1) f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} = (0121\cdots) \\ \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise} \end{cases}$$

*for all* $\mathbf{i} \in I_+^n \cup I_-^n$.

**Remark 6.2.30.** Note that all of the relations above reduce to the relations from Definition 4.1.8 upon tensoring with $F$ (recalling our assumptions on the field $F$).

## Relations involving $\psi_2^{\mathcal{O}}$ when $e > 3$

We now determine the scalars by which $\psi_2^{\mathcal{O}}$ acts on the seminormal basis in the case when $e > 3$. Recall the definition of $\psi_2^{\mathcal{O}}$ from (6.2.11).

**Lemma 6.2.31.** *For $e > 3$ and $\mathbf{i} \in I^n$, we have*

$$\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} 0, & \text{if } \mathbf{i} \in I_{\rightarrow}^n \cup I_{\downarrow}^n \\ \sqrt{-1} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_{\swarrow}^n \\ -\sqrt{-1} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_{\nearrow}^n \end{cases}$$

**Proof.** The important idea here is that all quantities act on $f_{\mathbf{i}}^{\mathcal{O}}$ as the scalars they act by on the seminormal basis vectors because when $r = 2$ and $e > 3$, residue sequence classes uniquely determine tableau classes. Since

$$T_2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} -f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } f_{\mathbf{i}}^{\mathcal{O}} \in I_{\rightarrow}^n \\ t f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } f_{\mathbf{i}}^{\mathcal{O}} \in I_{\downarrow}^n \end{cases}$$

this gives the first line of the formula. If $\mathbf{i} \in I^n_{\swarrow}$, so $\mathbf{i} = (0, 1, e-1, \ldots)$ and $c(\mathbf{i}) = (0, 1, -1, \ldots)$,

$$\begin{aligned}
\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} &= \Big[\Big(T_2(L_2 - L_3) + 1 + (t-1)L_3\Big)\Big] f_{\mathbf{i}}^{\mathcal{O}} \\
&= \frac{\sqrt{t}}{\sqrt{[3]}}\Big(([1]-[-1])T_2 f_{\mathbf{i}}^{\mathcal{O}} + t^{-1} f_{\mathbf{i}}^{\mathcal{O}}\Big) \\
&= \frac{\sqrt{t}}{\sqrt{[3]}}\Big[t^{-1}[2]\Big(\alpha_2(\mathbf{i}) f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} - \frac{1}{[\rho_2(\mathbf{i})]} f_{\mathbf{i}}^{\mathcal{O}}\Big) + t^{-1} f_{\mathbf{i}}^{\mathcal{O}}\Big] \\
&= \frac{[2]\alpha_2(\mathbf{i})}{\sqrt{[3]}\sqrt{t}} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} \\
&= \sqrt{-1} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}}
\end{aligned}$$

since $\alpha_2(\mathbf{i}) = \frac{\sqrt{-1}\sqrt{[3]}\sqrt{t}}{[2]}$ by (3.6.4). The calculation for $\mathbf{i} \in I^n_{\nearrow}$ is similar, noting that

$$\frac{\sqrt{t}}{\sqrt{[3]}}(L_2^{\#}T_2 - T_2L_2^{\#}) f_{\mathbf{i}}^{\mathcal{O}} = \Big[-\frac{\sqrt{t}}{\sqrt{[3]}}(L_2T_2 - T_2L_2) f_{-\mathbf{i}}^{\mathcal{O}}\Big]^{\#}$$

in this case. □

**Lemma 6.2.32.** *For $e > 3$ and $\mathbf{i} \in I^n$, we have $\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} \psi_2^{\mathcal{O}}$.*

**Proof.** For $\mathbf{i} \in I^n_{\rightarrow} \cup I^n_{\downarrow}$ the statement is trivial since both sides are zero. For $\mathbf{i} \in I^n_{\swarrow}$, $\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \sqrt{-1} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}}$ whereas

$$f_{s_2\cdot\mathbf{i}}^{\mathcal{O}}\psi_2^{\mathcal{O}} = \sum_{\mathbf{j}\in I^n_{\swarrow}} \sqrt{-1} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} + \sum_{\mathbf{j}\in I^n_{\nearrow}} -\sqrt{-1} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} f_{\mathbf{j}}^{\mathcal{O}} = \sqrt{-1} f_{s_2\cdot\mathbf{i}}^{\mathcal{O}}.$$

The case when $\mathbf{i} \in I^n_{\nearrow}$ is similar. □

The proof of the following lemma follows from a computation entirely analogous to Lemma 6.2.23 which we leave to the reader.

**Lemma 6.2.33.** *For $e > 3$ and $\mathbf{i} \in I^n$, we have*

$$\psi_2^{\mathcal{O}} y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-e} y_2^{\langle e\rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_+ \\ (y_2^{\mathcal{O}} + t^{-e}[e])\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I^n_- \end{cases}$$

$$\psi_2^{\mathcal{O}} y_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-e} y_3^{\langle e \rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I_+^n \\ (y_3^{\mathcal{O}} + t^{-e}[e]) \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I_-^n \end{cases}$$

The proof of the following lemma proceeds exactly as in Lemma 6.2.24 and Lemma 6.2.25; we also leave this for the reader to check.

**Lemma 6.2.34.** *For $e > 3$, we have $\psi_2^{\mathcal{O}} y_r^{\mathcal{O}} = y_r^{\mathcal{O}} \psi_2^{\mathcal{O}}$ and $\psi_2^{\mathcal{O}} \psi_r^{\mathcal{O}} = \psi_r^{\mathcal{O}} \psi_2^{\mathcal{O}}$ for $r > 3$.*

**Lemma 6.2.35.** *For $e > 3$ and $\mathbf{i} \in I^n$, we have*

$$(\psi_2^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} -y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I_{\rightarrow}^n \\ y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I_{\downarrow}^n \\ f_{\mathbf{i}}^{\mathcal{O}}, & \textit{if } \mathbf{i} \in I_{\swarrow}^n \cup I_{\nearrow}^n \end{cases}$$

**Proof.** If $\mathbf{i} \in I_{\rightarrow}^n$ then $(\psi_2^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = 0 = -y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}$; similarly if $\mathbf{i} \in I_{\downarrow}^n$. If $\mathbf{i} \in I_{\swarrow}^n$ then

$$(\psi_2^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \sqrt{-1} \psi_2^{\mathcal{O}} f_{s_2 \cdot \mathbf{i}}^{\mathcal{O}} = f_{\mathbf{i}}^{\mathcal{O}}$$

and similarly for $\mathbf{i} \in I_{\nearrow}^n$. □

**Remark 6.2.36.** The reader may note that all of the other relations work with $\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \chi(\mathbf{i}) f_{s_2 \cdot \mathbf{i}}^{\mathcal{O}}$ where $\{\chi(\mathbf{i}) \mid \mathbf{i} \in I^n\}$ is any collection of scalars satisfying the property $\chi(s_2 \cdot \mathbf{i}) = -\chi(\mathbf{i})$; it is only in the proof of Lemma 6.2.35 where we genuinely require the particular choice of $\chi(\mathbf{i})$ that we have made.

**Lemma 6.2.37.** *When $e > 3$ and $r = 2$, the cases $i_r = i_{r+2} \leftarrow i_{r+1}$ and $i_r = i_{r+2} \rightarrow i_{r+1}$ can never occur for $\mathbf{i} \in I_+^n \cup I_-^n$. Moreover,*

$$\psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}$$

*for all $\mathbf{i} \in I_+^n \cup I_-^n$.*

**Proof.** An inspection of the list of possible residue sequence classes in (6.2.28) easily gives the first half of the lemma. For the second half we note that for all tableaux in the list, $s_2 s_3 s_2 \cdot \mathbf{s} = (2, 4) \cdot \mathbf{s}$ is non-standard; since by the first half we are never in a case where we need to consider error terms, the statement now follows by virtue of both sides being zero in all cases. □

Again, we collect the relations from this subsection into a single proposition for easy reference:

**Proposition 6.2.38.** *Suppose $e > 3$. Then the following relations hold in the algebra $\mathscr{H}_n(\mathcal{O})$:*

$$\psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = f_{s_2\cdot\mathbf{i}}^{\mathcal{O}} \psi_2^{\mathcal{O}}$$

$$\psi_2^{\mathcal{O}} y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-e} y_2^{\langle e\rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_+^n \\ (y_2^{\mathcal{O}} + t^{-e}[e]) \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_-^n \end{cases}$$

$$\psi_2^{\mathcal{O}} y_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-e} y_3^{\langle e\rangle} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_+^n \\ (y_3^{\mathcal{O}} + t^{-e}[e]) \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_-^n \end{cases}$$

$$\psi_2^{\mathcal{O}} y_r^{\mathcal{O}} = y_r^{\mathcal{O}} \psi_2^{\mathcal{O}}, \qquad \text{for } r > 3$$

$$\psi_2^{\mathcal{O}} \psi_r^{\mathcal{O}} = \psi_r^{\mathcal{O}} \psi_2^{\mathcal{O}}, \qquad \text{for } r > 3$$

$$(\psi_2^{\mathcal{O}})^2 f_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} -y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_{\rightarrow}^n \\ y_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_{\downarrow}^n \\ f_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i} \in I_{\swarrow}^n \cup I_{\nearrow}^n \end{cases}$$

$$\psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}} = \psi_3^{\mathcal{O}} \psi_2^{\mathcal{O}} \psi_3^{\mathcal{O}} f_{\mathbf{i}}^{\mathcal{O}}$$

*for all $\mathbf{i} \in I^n$.*

**Definition 6.2.39.** For $\alpha \in Q_e$, let $R_\alpha(\mathcal{O})$ be the abstract algebra with generators

$$\{\widetilde{\psi}_1^{\mathcal{O}}, \widetilde{\psi}_2^{\mathcal{O}}, \dots, \widetilde{\psi}_{n-1}^{\mathcal{O}}\} \cup \{\widetilde{y}_1^{\mathcal{O}}, \widetilde{y}_2^{\mathcal{O}}, \dots, \widetilde{y}_n^{\mathcal{O}}\} \cup \{\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} \mid \mathbf{i} \in (I_+^n \cup I_-^n) \cap I^\alpha\}$$

subject to the relations

$$\prod_{\mathbf{i}\in I^\alpha} (\widetilde{y}_1^{\mathcal{O}})^{(\Lambda_0,\alpha_{i_1})} e(\mathbf{i}) = 0, \qquad \widetilde{f}_{\mathbf{i}}^{\mathcal{O}} \widetilde{f}_{\mathbf{j}}^{\mathcal{O}} = \delta_{\mathbf{ij}} \widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, \qquad \textstyle\sum_{\mathbf{i}\in I^\alpha} \widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = 1$$

$$\widetilde{y}_t^{\mathcal{O}} \widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = \widetilde{f}_{\mathbf{i}}^{\mathcal{O}} \widetilde{y}_t^{\mathcal{O}}, \qquad \widetilde{\psi}_r^{\mathcal{O}} \widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = \widetilde{f}_{s_r\cdot\mathbf{i}}^{\mathcal{O}} \widetilde{\psi}_r^{\mathcal{O}}, \qquad \widetilde{y}_r^{\mathcal{O}} \widetilde{y}_t^{\mathcal{O}} = \widetilde{y}_t^{\mathcal{O}} \widetilde{y}_r^{\mathcal{O}}$$

$$\widetilde{\psi}_2^{\mathcal{O}}\widetilde{y}_3^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-e}\widetilde{y}_2^{\langle e\rangle}\widetilde{\psi}_2^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i}\in I_+^n \\ (\widetilde{y}_2^{\mathcal{O}}+t^{-e}[e])\widetilde{\psi}_2^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i}\in I_-^n \end{cases}$$

$$\widetilde{\psi}_2^{\mathcal{O}}\widetilde{y}_2^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} t^{-e}\widetilde{y}_3^{\langle e\rangle}\widetilde{\psi}_2^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i}\in I_+^n \\ (\widetilde{y}_3^{\mathcal{O}}+t^{-e}[e])\widetilde{\psi}_2^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } \mathbf{i}\in I_-^n \end{cases}$$

$$\widetilde{\psi}_r^{\mathcal{O}}\widetilde{y}_{r+1}^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = (\widetilde{y}_r^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}+\delta_{i_ri_{r+1}})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, \qquad r>2$$

$$\widetilde{y}_{r+1}^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = (\widetilde{\psi}_r^{\mathcal{O}}\widetilde{y}_r^{\mathcal{O}}+\delta_{i_ri_{r+1}})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, \qquad r>2$$

$$\widetilde{\psi}_r^{\mathcal{O}}\widetilde{y}_t^{\mathcal{O}} = \widetilde{y}_t^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}, \quad \text{if } t\neq r,r+1,$$

$$\widetilde{\psi}_r^{\mathcal{O}}\widetilde{\psi}_s^{\mathcal{O}} = \widetilde{\psi}_s^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}, \quad \text{if } |r-s|>1,$$

$$(\widetilde{\psi}_r^{\mathcal{O}})^2\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (y_r^{\langle 1+\rho_r(\mathbf{i})\rangle}-\widetilde{y}_{r+1}^{\mathcal{O}})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r\to i_{r+1} \text{ and } \mathbf{i}\in I_+^n \\ (\widetilde{y}_r^{\mathcal{O}}-y_{r+1}^{\langle 1+\rho_r(\mathbf{i})\rangle})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r\to i_{r+1} \text{ and } \mathbf{i}\in I_-^n \\ (y_{r+1}^{\langle 1-\rho_r(\mathbf{i})\rangle}-\widetilde{y}_r^{\mathcal{O}})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r\leftarrow i_{r+1} \text{ and } \mathbf{i}\in I_+^n \\ (\widetilde{y}_{r+1}^{\mathcal{O}}-y_r^{\langle 1-\rho_r(\mathbf{i})\rangle})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r\leftarrow i_{r+1} \text{ and } \mathbf{i}\in I_-^n \\ 0, & \text{if } i_r=i_{r+1}, \\ \widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

$$\widetilde{\psi}_r^{\mathcal{O}}\widetilde{\psi}_{r+1}^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = \begin{cases} (\widetilde{\psi}_{r+1}^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}\widetilde{\psi}_{r+1}^{\mathcal{O}}-t^{1+\rho_r(\mathbf{i})})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+2}\to i_{r+1} \text{ and } \mathbf{i}\in I_+^n, \\ (\widetilde{\psi}_{r+1}^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}\widetilde{\psi}_{r+1}^{\mathcal{O}}-1)\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+2}\to i_{r+1} \text{ and } \mathbf{i}\in I_-^n, \\ (\widetilde{\psi}_{r+1}^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}\widetilde{\psi}_{r+1}^{\mathcal{O}}+1)\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+2}\leftarrow i_{r+1} \text{ and } \mathbf{i}\in I_+^n, \\ (\widetilde{\psi}_{r+1}^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}\widetilde{\psi}_{r+1}^{\mathcal{O}}+t^{1-\rho_r(\mathbf{i})})\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{if } i_r=i_{r+2}\leftarrow i_{r+1} \text{ and } \mathbf{i}\in I_-^n, \\ \widetilde{\psi}_{r+1}^{\mathcal{O}}\widetilde{\psi}_r^{\mathcal{O}}\widetilde{\psi}_{r+1}^{\mathcal{O}}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}, & \text{otherwise,} \end{cases}$$

where $\widetilde{y}_r^{\langle d\rangle}\widetilde{f}_{\mathbf{i}}^{\mathcal{O}} = (t^d\widetilde{y}_r^{\mathcal{O}}-[d])\widetilde{f}_{\mathbf{i}}^{\mathcal{O}}$ for $d\in\mathbb{Z}$, and $\rho_r(\mathbf{i})=i_r-i_{r+1}$, for all admissible $\mathbf{i},\mathbf{j}\in I_+^n\cup I_-^n$ and all admissible $r,s$ and $t$.

For technical reasons we need the following lemma, which is essentially a deformation of the nilpotency result Proposition 4.2.4 to our $\mathcal{O}$-algebras.

**Lemma 6.2.40.** *Suppose that $2 \leq r \leq n$ and $\mathbf{i} \in I^n$. Then there exists a multiset $X_r(\mathbf{i}) \subseteq e\mathbb{Z}$ such that*

$$\prod_{c \in X_r(\mathbf{i})} (\widetilde{y}_r^{\mathcal{O}} - [c]) f_{\mathbf{i}}^{\mathcal{O}} = 0$$

*in $R_\alpha(\mathcal{O})$.*

**Proof.** Since our $\widetilde{y}_r^{\mathcal{O}}$'s are either precisely the $y_r^{\mathcal{O}}$'s appearing in [**41**, Lemma 4.31], or their images under the hash involution, we obtain the result immediately from [**41**, Lemma 4.31]. □

Finally, we are able to prove the enhanced version of Theorem 6.1.5 that we will use to prove our main result.

**Theorem 6.2.41.** *For $\alpha \in Q_e$, the block $\mathscr{H}_\alpha(\mathcal{O})$ of the cyclotomic Hecke algebra $\mathscr{H}_n(\mathcal{O})$ is isomorphic to $R_\alpha(\mathcal{O})$.*

**Proof.** By all of the results in this section, the elements given in Definition 6.2.9 satisfy all of the relations in $R_\alpha(\mathcal{O})$ and so there is a surjective $\mathcal{O}$-algebra homomorphism

$$\vartheta : R_\alpha(\mathcal{O}) \to \mathscr{H}_\alpha(\mathcal{O}),$$

which maps the generators of $R_\alpha(\mathcal{O})$ to the corresponding elements of $\mathscr{H}_\alpha(\mathcal{O})$ unadorned with tildes. It is easy to check as in the proof of [**41**, Theorem 4.32] that the relations above together with Lemma 6.2.40 guarantee that $R_\alpha(\mathcal{O})$ is finitely generated as an $\mathcal{O}$-module.

Using our modular system from Lemma 6.2.2, we see that $y_r^{\langle de \rangle} \otimes_{\mathcal{O}} 1_F = y_r^{\mathcal{O}} \otimes_{\mathcal{O}} 1_F$ for all $d \in \mathbb{Z}$. Since also $t^{-e}[e] = 0$ in $F$, and since all of the shifts $1 \pm \rho_r(\mathbf{i})$ appearing in the statement of the theorem are equal to 0 or $e$, and since $y_2^{\mathcal{O}} = 0$, upon base change to $F$ the relations of $R_\alpha(\mathcal{O}) \otimes_{\mathcal{O}} F$ are precisely the relations of the cyclotomic quiver Hecke algebra $\mathcal{R}_\alpha^\Lambda(F)$ from Definition 4.1.8. Hence $R_\alpha(\mathcal{O}) \otimes_{\mathcal{O}} F \cong \mathcal{R}_\alpha^\Lambda(F)$; in particular $\dim R_\alpha(\mathcal{O}) \otimes_{\mathcal{O}} F = \dim \mathscr{H}_\alpha^\Lambda(F)$ by [**18**, Theorem 4.20]. Since this argument works for any maximal ideal $\mathfrak{m}$ of $\mathcal{O}$, an argument using Nakayama's Lemma as in the proof of [**43**, Theorem 4.32] shows that $\theta$ is in fact an isomorphism of $\mathcal{O}$-algebras as required. □

**Theorem 6.2.42.** *Let $n > 1$ and suppose $\xi \in F^\times$ is such that $e > 2$. Then, under the conditions specified in Lemma 6.2.2, the alternating cyclotomic Hecke algebra $\mathscr{H}_n(F)^\#$ is isomorphic to the alternating cyclotomic quiver Hecke algebra $(\mathcal{R}_n(F))^{\mathtt{sgn}}$.*

**Proof.** By definition, $\mathcal{R}_n(F) \cong \mathcal{R}_n(\mathcal{O}) \otimes_{\mathcal{O}} F$. Since all the exponents of the generators $y_1, y_2, \dots, y_n$ in the relations above are multiples of $e$, we can define a map $\widetilde{\mathtt{sgn}}$ on generators $\psi_r^{\mathcal{O}}, y_r^{\mathcal{O}}, f_{\mathbf{i}}^{\mathcal{O}}$ in $\mathcal{R}_n(\mathcal{O})$ which becomes the usual $\mathtt{sgn}$ map in $\mathcal{R}_n(F)$. We observed that this agrees with the $\#$ map on $\mathscr{H}_n(\mathcal{O})$ in Corollary 6.2.13 and so they will still agree on reduction to $F$; hence the fixed-point subalgebras are also isomorphic as required. □

The main upshot of Theorem 6.2.42 is that there is now a $\mathbb{Z}$-grading on the group algebras of the alternating groups, and on Mitsuhashi's alternating Hecke algebras, provided the field $F$ is large enough.

**Corollary 6.2.43.** *Let $n > 1$ and suppose that $\xi \in F$ is such that $e > 2$. Then, provided that $F$ comes from a modular system as in Lemma 6.2.2, the alternating cyclotomic Hecke algebra $\mathscr{H}_n(F)^\# \cong \mathcal{R}_n(F)^{\mathtt{sgn}}$ is $\mathbb{Z}$-graded.*

**Remark 6.2.44.** In order to define the isomorphism in Theorem 6.2.42, we required the existence in $\mathcal{O}$ of all the square roots from Lemma 6.2.2 to ensure the existence of an alternating seminormal coefficient system $\boldsymbol{\alpha}$; for example when $\xi = 1$ and $e = 3$, we require $\sqrt{3}$, $\sqrt{2}$ and $\sqrt{-1}$ to belong to $\mathcal{O}$.

**Example 6.2.45.** In this example we illustrate how our different choice of elements gives rise to a different Brundan-Kleshchev isomorphism to Example 4.3.5 which *does* intertwine the $\#$ and $\mathtt{sgn}$ maps. This amounts to making a choice of Brundan-Kleshchev polynomials (see (4.3.4)) on one "half" of the algebra, corresponding to idempotents with $\mathbf{i} \in I_+^n$, and the "hash" of this isomorphism on the other half (idempotents with $\mathbf{i} \in I_-^n$). Working in the cyclotomic Hecke algebra $\mathscr{H}_{3,1}(1, \mathbb{F}, 0)$, where $\mathbb{F}$ is an arbitrary field of characteristic three, we use the formulas in this section to obtain

$$\begin{aligned} y_3^{\mathcal{O}} &= y_3^{\mathcal{O}}(f_{\mathtt{ss}} + f_{\mathtt{tt}}) + y_3^{\mathcal{O}}(f_{\mathtt{uu}} + f_{\mathtt{vv}}) \\ &= -3f_{\mathtt{tt}} + 3f_{\mathtt{uu}} \\ \psi_2^{\mathcal{O}} &= \psi_2^{\mathcal{O}}(f_{\mathtt{ss}} + f_{\mathtt{tt}}) + \psi_2^{\mathcal{O}}(f_{\mathtt{uu}} + f_{\mathtt{vv}}) \\ &= \sqrt{3}\sqrt{-1}f_{\mathtt{ut}} - \sqrt{3}\sqrt{-1}f_{\mathtt{tu}}. \end{aligned}$$

and so $y_3^{\mathcal{O}} \otimes_{\mathcal{O}} 1_{\mathbb{F}} = s_1 + s_2 + s_1 s_2 s_1$, the same as in Example 4.3.5. By definition,

$$f_{\mathtt{ut}} = \frac{1}{\alpha_2(\mathtt{t})}\Big(s_2 f_{\mathtt{tt}} + \frac{1}{2} f_{\mathtt{tt}}\Big) \quad \text{and} \quad f_{\mathtt{tu}} = \frac{1}{\alpha_2(\mathtt{u})}\Big(s_2 f_{\mathtt{uu}} - \frac{1}{2} f_{\mathtt{uu}}\Big)$$

and so we can compute that

$$\psi_2^{\mathcal{O}} \otimes_{\mathcal{O}} 1_{\mathbb{F}} = s_2 + 2s_1 s_2 s_1$$

using Example 3.3.9 and Remark 3.6.3. This element has the property that $(\psi_2^{\mathcal{O}} \otimes_{\mathcal{O}} 1_{\mathbb{F}})^{\#} = -\psi_2^{\mathcal{O}} \otimes_{\mathcal{O}} 1_{\mathbb{F}}$ as required. Since $\varepsilon = s_1$, this gives the basis

$$e^+[012] = 1, \quad \mathcal{Y}_3 = 1 + s_1 s_2 + s_2 s_1, \quad \Psi_2 = s_1 s_2 + 2 s_2 s_1$$

for the alternating group algebra $\mathbb{F}\mathfrak{A}_3$, which has transition matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

with the group basis. The observant reader will notice that $\Psi_2^2 = \mathcal{Y}$ in this example whereas in Example 5.4.3 we had $\Psi_2^2 = -\mathcal{Y}$; this is due to the slightly different choice of relations in this example (which is consistent with [**43**]) to those in [**17**].

**Remark 6.2.46.** In this thesis we have explicitly avoided discussing alternating cyclotomic Hecke algebras in the case when the quantum characteristic $e$ is 2 as then we can neither apply Clifford theory, nor are residue sequences guaranteed not to be equal to their images under the $\mathtt{sgn}$ map – at many points in this chapter we explicitly and implicitly used both of these assumptions. It would be interesting to see which of our results still hold in the even quantum characteristic case, and to obtain new answers for those which don't.

In this section we have restricted to the case when $\Lambda = \Lambda_0$. We conjecture that the result is true for general symmetric $\Lambda$, but the proof will require different techniques.

**Conjecture 6.2.47.** *Suppose the quantum characteristic of $\xi \in F$ is not equal to 2. Then the alternating Hecke algebra $\mathscr{H}_n^\Lambda(F)^\#$ is isomorphic to the alternating cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda(F)^{\mathtt{sgn}}$ for any dominant weight $\Lambda$ such that $\Lambda = \Lambda'$, provided $F$ is large enough.*

CHAPTER 7

# Alternating graded Specht modules

In this chapter, we discuss the representation theoretic consequences of our graded isomorphism theorem for alternating cyclotomic quiver Hecke algebras from Chapter 6. We start by reviewing the graded cellular bases for cyclotomic Hecke algebras of Hu and Mathas [41], and use these to give a homogeneous basis and graded dimension formula for alternating cyclotomic Hecke algebras. We then review the graded Specht module theory of Brundan, Kleshchev and Wang [20], Hu and Mathas [41] and Kleshchev, Mathas and Ram [62], before defining graded modules for our alternating subalgebras in certain cases, which we call alternating graded Specht modules. Using the Mullineux map [5, 30, 85] and Clifford theory (§5.1), we classify the graded simple modules for these algebras. We also define graded decomposition numbers for alternating cyclotomic quiver Hecke algebras in certain cases and give addition formulas for these graded decomposition numbers which lead to a partial algorithm for computing decomposition matrices for alternating cyclotomic Hecke algebras when the corresponding matrices for cyclotomic Hecke algebras are known (for example, by the LLT algorithm [67]), and a generalisation of our main theorem from Chapter 6 to arbitrary level in the semisimple case.

## 7.1. Graded cellular bases for cyclotomic quiver Hecke algebras

In this section we introduce the two graded cellular bases for quiver Hecke algebras which were originally defined by Hu and Mathas [41]. We require some new combinatorial quantities in order to construct these bases and define a degree function. Since both of our graded cellular bases will be indexed by pairs of standard tableaux of the same shape, we would like a combinatorial degree function on tableaux. This was originally defined by Brundan, Kleshchev and Wang [20].

For $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and two nodes $A = (r, c, l)$ and $B = (r', c', l')$ of $\boldsymbol{\lambda}$ we say $A$ is strictly *above* $B$, or $B$ is strictly *below* $A$, if $l' > l$, or $l' = l$ and $r' > r$.

**Definition 7.1.1.** Let $A$ be a node of the multipartition $\boldsymbol{\lambda}$ with residue $i$. Define integers

$$d_A(\boldsymbol{\lambda}) = \#\left\{\begin{array}{c}\text{addable } i\text{-nodes of } \boldsymbol{\lambda}\\ \text{strictly below A}\end{array}\right\} - \#\left\{\begin{array}{c}\text{removeable } i\text{-nodes of } \boldsymbol{\lambda}\\ \text{strictly below } A\end{array}\right\},$$

$$d^A(\boldsymbol{\lambda}) = \#\left\{\begin{array}{c}\text{addable } i\text{-nodes of } \boldsymbol{\lambda}\\ \text{strictly above A}\end{array}\right\} - \#\left\{\begin{array}{c}\text{removeable } i\text{-nodes of } \boldsymbol{\lambda}\\ \text{strictly above } A\end{array}\right\}.$$

Note that the integers $d_A(\boldsymbol{\lambda})$ and $d^A(\boldsymbol{\lambda})$ depend on both $e$ and the choice of multicharge $\boldsymbol{\kappa}$ (because the residue of any given node depends on both of these quantities – see Definition 3.2.8).

Using these integers, we define the notion of degree and codegree for a standard $\boldsymbol{\lambda}$-tableau $\mathtt{t}$ with $n$ boxes inductively as follows. Set $\deg \mathtt{t} = \operatorname{codeg} \mathtt{t} = 0$ for tableaux $\mathtt{t}$ with $n = 0$ boxes and for a general tableau $\mathtt{t}$ let

$$\deg \mathtt{t} = \deg \mathtt{t}_{n-1} + d_A(\boldsymbol{\lambda})$$
$$\operatorname{codeg} \mathtt{t} = \operatorname{codeg} \mathtt{t}_{n-1} + d^A(\boldsymbol{\lambda}),$$

where $A$ is the node in $\mathtt{t}$ containing $n$ and $\mathtt{t}_{n-1}$ is the tableau with $n-1$ boxes obtained by deleting A from $\mathtt{t}$. Now inductively define sequences of integers $d_1^{\boldsymbol{\lambda}}, \ldots, d_n^{\boldsymbol{\lambda}}$ and $d_{\boldsymbol{\lambda}}^1, \ldots, d_{\boldsymbol{\lambda}}^n$ by the requirement that

$$d_1^{\boldsymbol{\lambda}} + \ldots + d_k^{\boldsymbol{\lambda}} = \deg \mathtt{t}_k^{\boldsymbol{\lambda}}$$
$$d_{\boldsymbol{\lambda}}^1 + \ldots + d_{\boldsymbol{\lambda}}^k = \operatorname{codeg} (\mathtt{t}_{\boldsymbol{\lambda}})_k$$

**Example 7.1.2.** Suppose $\lambda = (4, 2, 1^2)$, $e = 3$, and $\mathtt{t}$ is the standard tableau

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | | |
| 7 | | | |
| 8 | | | |

Then we record the above information in the following table (we have drawn the "residue diagram" of $\lambda$ to the right of the table, with addable nodes coloured light grey and removeable nodes dark grey).

| $k$ | $d_k^\lambda$ | $d_\lambda^k$ |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0 | 0 |
| 3 | 1 | 0 |
| 4 | 0 | 0 |
| 5 | 0 | 0 |
| 6 | 0 | $-1$ |
| 7 | 0 | 2 |
| 8 | 0 | $-2$ |

| | | | | |
|---|---|---|---|---|
| 0 | 1 | 2 | 0 | 1 |
| 2 | 0 | 1 | | |
| 1 | 2 | | | |
| 0 | | | | |
| 2 | | | | |

Summing the columns respectively gives $\deg \mathtt{t} = 1$ and $\operatorname{codeg} \mathtt{t} = -1$.

The degree and codegree of a tableau are closely related.

**Definition 7.1.3.** Let $\beta \in Q_e$. Then the *defect* of $\beta$ is the quantity

$$\operatorname{def}\beta = (\Lambda, \beta) - \frac{1}{2}(\beta, \beta),$$

where $(\cdot, \cdot)$ is the pairing defined in (4.1.6).

**Remark 7.1.4.** Like the degree and codegree, the defect depends on $\Lambda_0$, the multicharge $\boldsymbol{\kappa}$ *and* the quantum characteristic $e$.

**Lemma 7.1.5.** [20, Lemmata 3.11, 3.12] *For* $\mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^\ell)$ *in block* $\beta$,

$$\deg \mathtt{s} + \operatorname{codeg} \mathtt{s} = \operatorname{def}\beta.$$

The degree function defined on quiver Hecke algebras whose weights allow us to define alternating quiver Hecke subalgebras has the following simple combinatorial property which is very important.

**Lemma 7.1.6.** *Let $\mathcal{R}_n^\Lambda$ be a quiver Hecke algebra with weight $\Lambda$ such that $\Lambda = \Lambda'$. Then* $\deg \mathtt{s} = \operatorname{codeg} \mathtt{s}'$.

**Proof.** Since $\boldsymbol{\kappa}(\Lambda) = \boldsymbol{\kappa}(\Lambda') = (\boldsymbol{\kappa}(\Lambda))'$, for $i \in I$ there is a bijection between the addable (resp. removeable) $i$-nodes above $(r, c, l)$ and the addable (resp. removeable) $-i$-nodes below $(c, r, \ell - l + 1)$. This implies the result; compare with [42, Lemma 3.25]. □

For $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, define elements

$$y_{\boldsymbol{\lambda}} = y_1^{d_1^{\boldsymbol{\lambda}}} y_2^{d_2^{\boldsymbol{\lambda}}} \cdots y_n^{d_n^{\boldsymbol{\lambda}}}$$
$$y'_{\boldsymbol{\lambda}} = y_1^{d^1_{\boldsymbol{\lambda}}} y_2^{d^2_{\boldsymbol{\lambda}}} \cdots y_n^{d^n_{\boldsymbol{\lambda}}}$$

of $\mathcal{R}_n^\Lambda$.

**Lemma 7.1.7.** *For $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ we have*

$$y_{\boldsymbol{\lambda}} = y'_{\boldsymbol{\lambda}'} \quad \text{and} \quad (y_{\boldsymbol{\lambda}})^{\mathtt{sgn}} = (-1)^{\deg \mathtt{t}^{\boldsymbol{\lambda}}} y_{\boldsymbol{\lambda}}.$$

**Proof.** Both equations follow from the same comparison of the integers $\{d_i^{\boldsymbol{\lambda}}\}$ and $\{d^i_{\boldsymbol{\lambda}}\}$ that gives Lemma 7.1.6, and the definition of the graded sign map. □

We can now define Hu and Mathas' graded cellular bases for cyclotomic quiver Hecke algebras. Recall that for a tableau $\mathtt{t}$, $\mathbf{i}_\mathtt{t}$ is the residue sequence of $\mathtt{t}$; furthermore recall the definition of the permutations $d(\mathtt{s})$ and $d'(\mathtt{s})$ for a standard tableau $\mathtt{s}$ from (3.3.2).

**Definition 7.1.8** (Graded cellular bases)**.** For any pair $(\mathtt{s}, \mathtt{t})$ of standard tableaux of the same shape $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, define elements

$$\psi_{\mathtt{st}} = \psi_{d(\mathtt{s})} y_{\boldsymbol{\lambda}} e(\mathbf{i}_{\mathtt{t}^{\boldsymbol{\lambda}}}) \psi^*_{d(\mathtt{t})}$$

and

$$\psi'_{\mathtt{st}} = \psi_{d'(\mathtt{s})} y'_{\boldsymbol{\lambda}} e(\mathbf{i}_{\mathtt{t}_{\boldsymbol{\lambda}}}) \psi^*_{d'(\mathtt{t})}.$$

of $\mathcal{R}_n^\Lambda$.

**Remark 7.1.9.** The reader should note that we follow the convention of [42] and [62] here rather than the original paper [41]; our element $\psi'_{\mathtt{st}}$ is equal to $\psi'_{\mathtt{s}'\mathtt{t}'}$ in the original notation (but is equal to $\psi'_{\mathtt{st}}$ in [42]). This change makes a number of formulas more aesthetically pleasing. It also eliminates some ambiguities and errors from the original paper; however since these errors only appear in the case when the multicharge not symmetric, there is no need to discuss this further in this thesis.

**Theorem 7.1.10** (Graded cellular basis theorems [41])**.** *Let $\Lambda \in P_e$ for $e > 2$ and let $n \geq 1$. Then the collections*

$$\{\psi_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \textit{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$$

*and*

$$\{\psi'_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \textit{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^\ell\}$$

*are graded cellular bases for the cyclotomic quiver Hecke algebra $\mathcal{R}_n^\Lambda$, with degree functions*

$$\deg \psi_{\mathtt{st}} = \deg \mathtt{s} + \deg \mathtt{t}$$

$$\deg \psi'_{\mathtt{st}} = \operatorname{codeg} \mathtt{s}' + \operatorname{codeg} \mathtt{t}'$$

*and weight posets $(\mathcal{P}_n^\ell, \trianglerighteq)$ and $(\mathcal{P}_n^\ell, \trianglelefteq)$, respectively.*

**Remark 7.1.11.** It is important to note that the bases defined above *depend on the choices of reduced expression* for $\omega \in \mathfrak{S}_n$. [41, Example 5.6] gives an example of how different choices of reduced expression can give rise to genuinely different basis vectors. This is an unfortunate consequence of the complicated nature of the deformed braid relations in quiver Hecke algebras. Although this is a slightly unsavoury detail, [41, Lemma 5.7] ensures it is not a particularly important one, as different choices of reduced expression only affect the resulting basis in a minor way. Hu and Mathas have since defined a new graded cellular basis which is independent of the choice of reduced expression [43], but which currently only has an inductive definition which is exceedingly difficult to work with. The survey paper [79] by Mathas gives many more remarks in this direction (see in particular [79, §4.3]).

**Example 7.1.12.** Let $n = e = 3$. One can check that $y_{(3)} = y_3$ and $y_{(2,1)} = y_{(1^3)} = 1$. Hence we can easily write down the six graded cellular basis vectors for the algebra $\mathcal{R}_3^{\Lambda_0}(3, \mathbb{F}_3)$ from Example 4.2.6, where $\mathtt{s}, \mathtt{t}, \mathtt{u}$ and $\mathtt{v}$ respectively are the standard tableaux

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$$

$$\begin{aligned}
\psi_{\mathtt{ss}} &= y_3 e(012)\\
\psi_{\mathtt{tt}} &= e(012)\\
\psi_{\mathtt{tu}} &= e(012)\psi_2 = \psi_2 e(021)\\
\psi_{\mathtt{ut}} &= \psi_2 e(012)\\
\psi_{\mathtt{uu}} &= \psi_2 e(012)\psi_2 = \psi_2^2 e(021) = -y_3 e(021)\\
\psi_{\mathtt{vv}} &= e(021).
\end{aligned}$$

Notice that this agrees, up to sign, with Example 4.2.6. We leave it to the reader to compute the dual basis vectors $\psi'_{\mathtt{st}}$; he or she may wish to interpret this computation in light of Conjecture 7.1.15 below.

Our goal is to use the two graded cellular bases for cyclotomic quiver Hecke algebras from Theorem 7.1.10 to produce a related basis for the alternating cyclotomic quiver Hecke algebras. The next key result links the two graded cellular bases and the sign automorphism.

**Proposition 7.1.13.** [42, Proposition 3.26] *Let $\Lambda \in P_e$ be a dominant weight such that $\Lambda = \Lambda'$. Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$. Then*

$$\psi_{\mathtt{st}}^{\mathtt{sgn}} = (-1)^{\ell(d(\mathtt{s}))+\ell(d(\mathtt{t}))+\deg \mathtt{t}^{\boldsymbol{\lambda}}} \psi'_{\mathtt{s}'\mathtt{t}'}$$

*in the algebra $\mathcal{R}_n^\Lambda$.*

**Proof.** Note that by the definition of the $\mathtt{sgn}$ involution, for any standard tableau $\mathtt{v}$, we have $\psi_{d(\mathtt{v})}^{\mathtt{sgn}} = (-1)^{\ell(d(\mathtt{v}))} \psi_{d(\mathtt{v})}$. The result now follows since $e(-\mathbf{i}_{\mathtt{t}^{\boldsymbol{\lambda}}}) = e(\mathbf{i}_{\mathtt{t}^{\boldsymbol{\lambda}'}})$ and using Lemma 7.1.7, noting that for the $\psi'_{\mathtt{st}}$ basis, we use $d'(\mathtt{s})$, which moves from the *final* tableau *up* to $\mathtt{s}$, which is why the conjugates appear. □

**Remark 7.1.14.** The reader may have noticed in Example 7.1.12 that $\psi_{\mathtt{st}}^{\mathtt{sgn}} = \pm\psi_{\mathtt{uv}}$ for some tableaux $\mathtt{u}, \mathtt{v}$ in every case. Several computations have been carried out on a computer in GAP to establish a conjecture, which can be verified for $n \leq 5$.

**Conjecture 7.1.15.** *Let $n > 1$ and for each $\omega \in \mathfrak{S}_n$ fix a reduced expression for $\omega$. Then for these particular choices of reduced expression, for all $\mathtt{s}, \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})$ with $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$, $\psi_{\mathtt{st}}^{\mathtt{sgn}} = \pm\psi_{\mathtt{s}^!\mathtt{t}^!}$, where $! : \operatorname{Std}(\mathcal{P}_n^\ell) \to \operatorname{Std}(\mathcal{P}_n^\ell)$ is some combinatorially described involution on tableaux.*

We finish this section by combining the two cellular bases we have seen into a basis for the alternating cyclotomic quiver Hecke algebras of Chapter 5. This will have as an immediate corollary a formula for the graded dimension of these algebras. Define an equivalence relation $\sim$ on $\operatorname{Std}(\mathcal{P}_n^\ell)$ to be generated by $\mathtt{s} \sim \mathtt{t}$ if $\mathtt{t} = \mathtt{s}'$. Let $\operatorname{Std}(\mathcal{P}_n^\ell)_\sim$ denote the set of equivalence classes under this relation and in each equivalence class $[\mathtt{s}]$, choose a representative $\mathtt{s}^+$. Define

$$\operatorname{Std}(\mathcal{P}_n^\ell)^+ = \{\mathtt{s}^+ \mid [\mathtt{s}] \in \operatorname{Std}(\mathcal{P}_n^\ell)_\sim\}$$

$$\operatorname{Std}(\mathcal{P}_n^\ell)^- = \{\mathtt{s}' \mid \mathtt{s} \in \operatorname{Std}(\mathcal{P}_n^\ell)_+\}.$$

Since $e > 2$, $\mathbf{i}_{\mathtt{s}}$ is always distinct from $\mathbf{i}_{\mathtt{s}'}$ and so $\mathrm{Std}(\mathcal{P}_n^\ell)^+ \sqcup \mathrm{Std}(\mathcal{P}_n^\ell)^- = \mathrm{Std}(\mathcal{P}_n^\ell)$ – both sets have cardinality $\frac{1}{2}\left|\mathrm{Std}(\mathcal{P}_n^\ell)\right|$, and all equivalence classes $[\mathtt{s}] \in \mathrm{Std}(\mathcal{P}_n^\ell)_\sim$ contain two elements.

**Definition 7.1.16.** Let $e > 2$, let $\Lambda \in P_e$ be such that $\Lambda = \Lambda'$ and let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$. For $\mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda})$, define an element

$$\Psi_{\mathtt{st}} = \psi_{\mathtt{st}} + \psi_{\mathtt{st}}^{\mathtt{sgn}} \in (\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}.$$

**Example 7.1.17.** Continuing with Example 7.1.12 and using the notation from §5.1 for generators of the alternating cyclotomic Hecke algebra, observe that

$$\Psi_{\mathtt{ss}} = \mathcal{Y}_3 = -\Psi_{\mathtt{uu}}$$
$$\Psi_{\mathtt{tt}} = 1 = \Psi_{\mathtt{vv}}$$
$$\Psi_{\mathtt{tu}} = \Psi_2 = -\Psi_{\mathtt{tu}}.$$

Hence taking the collection $\{\Psi_{\mathtt{st}} \mid \mathtt{s} \in \mathrm{Std}(\mathcal{P}_n)^+\}$ precisely gives the basis we saw in Example 5.4.3. Moreover, since $\deg \mathtt{s} = 1$, $\deg \mathtt{u} = 1$ and $\deg \mathtt{t} = 0$, we see that this algebra has the same graded dimension as $\mathbb{F}[x]/(x^3)$ as we saw in Example 2.4.12:

$$\mathrm{qdim}_{\mathbb{F}_3} \mathcal{R}_3^{\Lambda_0}(3, \mathbb{F}_3)^{\mathtt{sgn}} = 1 + q + q^2.$$

**Theorem 7.1.18.** *Let $e > 2$ and let $\Lambda \in P_e^+$ be such that $\Lambda = \Lambda'$. Then the collection*

$$\{\Psi_{\mathtt{st}} \mid \mathtt{s} \in \mathrm{Std}(\mathcal{P}_n^\ell)^+, \mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell) \text{ with } \mathrm{sh}(\mathtt{s}) = \mathrm{sh}(\mathtt{t})\}$$

*is a homogeneous basis for the alternating cyclotomic quiver Hecke algebra $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$.*

**Proof.** Since $\dim_{\mathcal{O}}(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}} = \frac{1}{2}\dim_{\mathcal{O}} \mathcal{R}_n^\Lambda$ by Proposition 5.4.10, and there are precisely this number of vectors in the collection above, it suffices to prove that the given collection is $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$-linearly independent. Let $x = \displaystyle\sum_{\substack{\mathtt{s} \in \mathrm{Std}(\mathcal{P}_n)^+ \\ \mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)}} a_{\mathtt{st}} \Psi_{\mathtt{st}}$ for some coefficients $a_{\mathtt{st}} \in \mathcal{O}$. Then if $\mathtt{s} \in \mathrm{Std}(\mathcal{P}_n^\ell)^+$, since $e > 2$,

$$xe(\mathbf{i}) = \begin{cases} \sum_{\mathrm{res}\,\mathtt{s}=\mathbf{i}_\mathtt{s}} a_{\mathtt{st}}\psi_{\mathtt{st}}, & \text{if } \mathbf{i} = \mathbf{i}_\mathtt{s} \\ \sum_{\mathrm{res}\,\mathtt{s}=\mathbf{i}_{\mathtt{s}'}} a_{\mathtt{st}}\psi_{\mathtt{st}}^{\mathtt{sgn}}, & \text{if } \mathbf{i} = \mathbf{i}_{\mathtt{s}'} \end{cases}$$

for any $\mathbf{i} \in I^n$, which gives linear independence by Theorem 7.1.10 and Proposition 7.1.13. □

By taking degrees of each homogeneous basis vector we immediately obtain the following corollary.

**Corollary 7.1.19.** *Let $e > 2$ and let $\Lambda \in P_e^+$ be such that $\Lambda = \Lambda'$. Then the graded dimension of the alternating cyclotomic quiver Hecke algebra $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$ is*

$$\mathrm{qdim}(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}} = \sum_{\substack{\mathtt{s}\in\mathrm{Std}(\mathcal{P}_n^\ell)^+ \\ \mathtt{t}\in\mathrm{Std}(\mathcal{P}_n^\ell) \\ \mathrm{sh}(\mathtt{s}) = \mathrm{sh}(\mathtt{t})}} q^{\deg \mathtt{s} + \deg \mathtt{t}}.$$

**Remark 7.1.20.**

(i) As in Example 7.1.17, the set $\{\Psi_{\mathtt{st}} \mid \mathtt{s} \in \mathrm{Std}(\mathcal{P}_n^\ell)^-,\ \mathtt{t} \in \mathrm{Std}(\mathcal{P}_n^\ell)\}$ can also be shown to be a basis for the alternating cyclotomic quiver Hecke algebra by an identical argument; according to Conjecture 7.1.15, this basis is the same up to sign as the basis from Theorem 7.1.18. Moreover, by Lemma 7.1.6, this basis gives rise to the same graded dimension formula as Corollary 7.1.19.

(ii) Since precisely half of the standard tableaux with $n$ boxes belong to $\mathrm{Std}(\mathcal{P}_n^\ell)^+$ (and the other half to $\mathrm{Std}(\mathcal{P}_n^\ell)^-$), the graded dimension formula in Corollary 7.1.19 reduces to

$$\mathrm{qdim}_\mathcal{O}(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}(1) = \frac{\ell^n n!}{2},$$

which agrees with Proposition 5.4.10.

(iii) It is unclear, and would be interesting to determine, whether or not the alternating cyclotomic Hecke algebras are cellular, or graded cellular, algebras.

## 7.2. Graded Specht modules

Using the general machinery of graded cellular theory from §2.3 and §2.4, we may obtain from the two graded cellular bases defined in the previous section two collections of graded cell modules. We call these *graded Specht modules*. We work with cyclotomic Hecke algebras in this section because this is the notation more commonly used in the literature; of course over a field this is equivalent to working with cyclotomic quiver Hecke algebras by the Brundan-Kleshchev Isomorphism Theorem 4.3.2. In particular, the cyclotomic Hecke algebras are graded cellular algebras with the two graded cellular bases from Theorem 7.1.10.

**Definition 7.2.1** (Graded Specht modules)**.** For $\boldsymbol{\lambda} \in \mathcal{P}_n^r$ let $S^{\boldsymbol{\lambda}}$ denote the graded cell module obtained from the graded cellular basis

$$\{\psi_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^r\}$$

and let $S_{\boldsymbol{\lambda}}$ denote the graded cell module obtained from the basis

$$\{\psi'_{\mathtt{st}} \mid \mathtt{s}, \mathtt{t} \in \mathrm{Std}(\boldsymbol{\lambda}) \text{ for } \boldsymbol{\lambda} \in \mathcal{P}_n^r\},$$

using the process of cell theory outlined in §2.4 (specifically Definition 2.4.1). These are called the *row* and *column* graded Specht modules for $\mathcal{R}_n^\Lambda$, respectively.

Although we automatically obtain bases for our Specht modules by the above cell theoretic construction, there is another basis that is more useful for our purposes.

**Definition 7.2.2.** For any $\boldsymbol{\lambda} \in \mathcal{P}_n^r$ and $\boldsymbol{\lambda}$-tableau $\mathtt{t}$, define an element

$$z_{\boldsymbol{\lambda}} = \psi_{\mathtt{t}^{\boldsymbol{\lambda}}\mathtt{t}^{\boldsymbol{\lambda}}} + (\mathscr{H}_n^\Lambda)^{\rhd\boldsymbol{\lambda}} \in \mathscr{H}_n^\Lambda/(\mathscr{H}_n^\Lambda)^{\rhd\boldsymbol{\lambda}}$$

and vectors

$$v_{\mathtt{t}} = \psi_{d(\mathtt{t})} z_{\boldsymbol{\lambda}}.$$

Define the degree of $v_{\mathtt{t}}$ to be $\deg v_{\mathtt{t}} = \deg \mathtt{t}$. For any $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ and $\boldsymbol{\lambda}$-tableau $\mathtt{t}$, define the generator

$$z'_{\boldsymbol{\lambda}} = \psi'_{\mathtt{t}_{\boldsymbol{\lambda}}\mathtt{t}_{\boldsymbol{\lambda}}} + (\mathscr{H}_n^\Lambda)^{\lhd\boldsymbol{\lambda}} \in \mathscr{H}_n^\Lambda/(\mathscr{H}_n^\Lambda)^{\lhd\boldsymbol{\lambda}}$$

and vectors

$$v'_{\mathtt{t}} = \psi_{d'(\mathtt{t})} z'_{\lambda}$$

Define the degree of $v'_{\mathtt{t}}$ to be $\deg v'_{\mathtt{t}} = \operatorname{codeg} \mathtt{t}$.

**Lemma 7.2.3.** [62, Theorem 8.5] *For $\boldsymbol{\lambda} \in \mathcal{P}^{\ell}_n$, $z^{\mathtt{sgn}}_{\boldsymbol{\lambda}} = z'_{\boldsymbol{\lambda}'}$.*

**Proposition 7.2.4** (Brundan, Kleshchev and Wang, Hu-Mathas [20, 41])**.** *Let $\boldsymbol{\lambda} \in \mathcal{P}^{\ell}_n$. Then the graded $\mathscr{H}^{\Lambda}_n$-module $S^{\boldsymbol{\lambda}}$ has basis $\{v_{\mathtt{t}} \mid \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})\}$ and the graded $\mathscr{H}^{\Lambda}_n$-module $S_{\boldsymbol{\lambda}}$ has basis $\{v'_{\mathtt{t}} \mid \mathtt{t} \in \operatorname{Std}(\boldsymbol{\lambda})\}$.*

For a $\mathcal{R}^{\Lambda}_n$-module $M$, we denote by $M^{\mathtt{sgn}}$ the $\mathcal{R}^{\Lambda}_n$-module which is $M$ as a graded vector space and whose $\mathcal{R}^{\Lambda}_n$-action is given by

$$a \cdot m = a^{\mathtt{sgn}} m$$

for all $a \in \mathcal{R}^{\Lambda}_n$ and $m \in M$.

**Proposition 7.2.5.** [62, Theorem 8.5] *Let $\mathscr{H}^{\Lambda}_n$ be a cyclotomic Hecke algebra with symmetric parameters. Then for all $\boldsymbol{\lambda} \in \mathcal{P}^{\ell}_n$,*

$$S^{\boldsymbol{\lambda}} \cong S^{\mathtt{sgn}}_{\boldsymbol{\lambda}'}$$

*as graded $\mathscr{H}^{\Lambda}_n$-modules.*

Finally, we have the following important relationship between a Specht module and its contragredient graded dual (as in §2.3). Notice the important distinction in terminology between "dual Specht module" and "contragredient dual of a Specht module" which arises from this result, and the difference in notation compared with [41] (see Remark 7.1.11).

**Proposition 7.2.6.** [41, Proposition 6.19] *Let $\boldsymbol{\lambda} \in \mathcal{P}^{\ell}_n$. Then $S^{\boldsymbol{\lambda}} \cong S^{\circledast}_{\boldsymbol{\lambda}}\langle \operatorname{def} \lambda \rangle$ as graded $\mathscr{H}^{\Lambda}_n$-modules.*

**Example 7.2.7.** We calculate the graded Specht modules for our running example, continuing with the notation from Example 7.1.12. Let $n = e = 3$ and $\Lambda = \Lambda_0$. The graded Specht module $S^{(3)}$ for the cyclotomic quiver Hecke algebra $\mathcal{R}_3^{\Lambda_0}(3,\mathcal{O})$ has basis $\{v_{\mathtt{s}}\}$, with $\deg v_{\mathtt{s}} = 1$, on which the generators $\psi_1,\psi_2,y_1,y_2,y_3$ and $e(021)$ all act as zero and the generator $e(012)$ acts as the identity. Similarly, the graded Specht module $S^{(1^3)}$ has basis $\{v_{\mathtt{v}}\}$, with $\deg v_{\mathtt{v}} = 0$, on which the generators $\psi_1,\psi_2,y_1,y_2,y_3$ and $e(012)$ all act as zero and the generator $e(021)$ acts as the identity.

Finally, the Specht module $S^{(2,1)}$ has basis $\{v_{\mathtt{t}},v_{\mathtt{u}}\}$ with $\deg v_{\mathtt{t}} = 0$ and $\deg v_{\mathtt{u}} = 1$ and on which the generators $\psi_1,y_1,y_2$ and $y_3$ act as zero and the generators $\psi_2, e(012)$ and $e(021)$ act via the matrices

$$\left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right),\quad \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)\quad\text{and}\quad \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$$

respectively (one can obtain these formulas using [20]). We leave it to the reader to compute the dual graded Specht modules $S_\lambda$ for this algebra.

### 7.3. Semisimple alternating graded Specht modules

In this section we demonstrate how to obtain analogues of graded Specht modules for semisimple alternating cyclotomic quiver Hecke algebras.

**For the remainder of this section, as in Chapter 6, we specialise to the case $\Lambda = \Lambda_0$ and work with the same modular system of rings defined on page 87.**

Since $\mathcal{K}$ is a field, by the Brundan-Kleshchev isomorphism theorem 4.3.2, the algebra $\mathscr{H}_n(\mathcal{K})$ is isomorphic to the cyclotomic quiver Hecke algebra $\mathcal{R}_n(\mathcal{K})$. For the same reason, $\mathscr{H}_n(F) \cong \mathcal{R}_n(F)$. We also have, by Theorem 6.2.41, that $\mathcal{H}_n(\mathcal{O})$ is isomorphic to the integral cyclotomic quiver Hecke algebra $\mathcal{R}_n(\mathcal{O})$.

In particular, the framework of the previous section gives us a collection $\{S^\lambda(\mathcal{O}) \mid \lambda\in\mathcal{P}_n\}$ of Specht modules for the algebra $\mathscr{H}_n(\mathcal{O})$. Our goal is to obtain analogues of these $\mathcal{O}$-modules for $\mathscr{H}_n(\mathcal{O})^{\mathtt{sgn}}$ which can be reduced to modules for $\mathscr{H}_n(F)^{\mathtt{sgn}}$ that may be thought of as alternating graded Specht modules, and about which we can ask questions of graded decomposition numbers. First we need to further develop the semisimple theory.

**Definition 7.3.1.** Let $\lambda \in \mathcal{P}_n$. The Specht module $S^\lambda(\mathcal{K})$ for $\mathscr{H}_n(K) \cong \mathcal{R}_n(K)$ is the vector space with basis $\{f_{\mathtt{t}} \mid \mathtt{t} \in \mathrm{Std}(\lambda)\}$ and action given by

$$f^{\mathcal{O}}_{\mathbf{i}} f_{\mathtt{t}} = \delta_{\mathbf{i}_{\mathtt{t}}\mathbf{i}} f_{\mathtt{t}}$$

$$\psi^{\mathcal{O}}_r f_{\mathtt{t}} = \begin{cases} \beta_r(\mathtt{t}) f_{\mathtt{u}} - \delta_{i_r i_{r+1}} \dfrac{t^{i_{r+1}-c_{r+1}(\mathtt{t})}}{[\rho_r(\mathtt{t})]} f_{\mathtt{t}}, & \text{if } \mathbf{i} \in I^n_+ \\ \beta_r(\mathtt{t}) f_{\mathtt{u}} - \delta_{i_r i_{r+1}} \dfrac{t^{\hat{\jmath}_{r+1}-c_r(\mathtt{t})}}{[\rho_r(\mathtt{t})]} f_{\mathtt{t}}, & \text{if } \mathbf{i} \in I^n_- \end{cases}$$

$$y^{\mathcal{O}}_r f_{\mathtt{t}} = \begin{cases} [c_r(\mathtt{t}) - i_r] f_{\mathtt{t}}, & \text{if } \mathbf{i} \in I^n_+ \\ -[c_r(\mathtt{t}') - \hat{\jmath}_r] f_{\mathtt{t}}, & \text{if } \mathbf{i} \in I^n_- \end{cases}$$

where we note that, although these generators live inside $\mathscr{H}_n(\mathcal{O}) \hookrightarrow \mathscr{H}_n(\mathcal{K})$, the *coefficients* used to define this action in general are in $\mathcal{K}$.

**Example 7.3.2.** Let us continue our running example when $n = e = 3$ and $\Lambda = \Lambda_0$. Since they are trivial, we leave the computation of the modules $S^{(3)}(\mathcal{K})$ and $S^{(1^3)}(\mathcal{K})$ to the reader and proceed to compute the actions of $y^{\mathcal{O}}_3$ and $\psi^{\mathcal{O}}_2$ on $S^{(21)}(\mathcal{K})$, since these are the only generators which give nonzero matrices:

$$y^{\mathcal{O}}_3 \mapsto \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}, \qquad \psi^{\mathcal{O}}_2 \mapsto \begin{pmatrix} 0 & -\sqrt{3}i \\ \sqrt{3}i & 0 \end{pmatrix}.$$

**Example 7.3.3.** Suppose $e = \infty$ and $\xi \neq 1$, so that $\mathcal{K} = \mathbb{C}$ and we are in the same territory as Example 4.2.7. Then since $c_r(\mathtt{t}) = (\mathbf{i}_t)_r$ for all $\mathtt{t}$, $y^{\mathcal{O}}_r f_{\mathtt{t}} = 0$ for all $\mathtt{t}$. Moreover, $i_r \neq i_{r+1}$ for any $\mathbf{i}$, so $\psi^{\mathcal{O}}_r f_{\mathtt{t}} = \beta_r(\mathtt{t}) f_{s_r \cdot \mathtt{t}}$ if $s_r \cdot \mathtt{t} \in \mathrm{Std}(\lambda)$. Hence these Specht modules are clearly isomorphic to those given in Example 4.2.7. One checks these are compatible with the $\mathcal{O}$-modules from Example 7.2.7 by noting that $f_{\mathtt{t}} = v_{\mathtt{t}}$ and $v_{\mathtt{u}} = \psi_2 f_{\mathtt{t}} = \sqrt{3}i f_{\mathtt{u}}$. Hence $\psi_2 v_{\mathtt{u}} = \sqrt{3}i \psi_2 f_{\mathtt{u}} = 3 f_{\mathtt{t}} \equiv 0$ in $\mathcal{O}$.

Since the alternating cyclotomic quiver Hecke algebra is precisely the fixed-point subalgebra under this involution, the restricted actions of $(\mathcal{R}^\Lambda_n)^{\mathtt{sgn}}$ on $M$ and

$M^{\mathtt{sgn}}$ coincide and we obtain the following result as a corollary to Proposition 7.2.5.

**Lemma 7.3.4.** [41, §5.2] *The modules* $\{S^\lambda(\mathcal{K}) \mid \lambda \in \mathcal{P}_n\}$ *give a complete collection of irreducible modules for the algebra* $\mathscr{H}_n(\mathcal{K})$*. Moreover, for* $\lambda \in \mathcal{P}_n$ *we have* $S^\lambda(\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{K} \cong S^\lambda(\mathcal{K})$*.*

**Corollary 7.3.5.** *Let* $\lambda \in \mathcal{P}_n$*. Then*

$$S^\lambda(\mathcal{O}) \downarrow^{\mathcal{R}_n(\mathcal{O})}_{(\mathcal{R}_n(\mathcal{O}))^{\mathtt{sgn}}} \cong S_{\lambda'}(\mathcal{O}) \downarrow^{\mathcal{R}_n(\mathcal{O})}_{(\mathcal{R}_n(\mathcal{O}))^{\mathtt{sgn}}}$$

*as graded* $(\mathcal{R}_n(\mathcal{O}))^{\mathtt{sgn}}$*-modules, where the homogeneous isomorphism of degree zero is explicitly given by*

$$v_{\mathtt{t}} \mapsto (-1)^{\ell(d(\mathtt{t}))} v'_{\mathtt{t}'}.$$

**Proof.** The first part is clear; for the second part we simply compute that, by definition and Lemma 7.2.3,

$$(v_{\mathtt{t}})^{\mathtt{sgn}} = (\psi_{d(\mathtt{t})} z_\lambda)^{\mathtt{sgn}} = (-1)^{\ell(d(\mathtt{t}))} \psi_{d(\mathtt{t})} z'_{\lambda'} = (-1)^{\ell(d(\mathtt{t}))} \psi_{d'(\mathtt{t}')} z'_{\lambda'} = (-1)^{\ell(d(\mathtt{t}))} v'_{\mathtt{t}'}$$

which gives the desired isomorphism. $\square$

The next result follows immediately from Lemma 7.3.4 and will allow us to use Clifford theory to classify the irreducible $\mathcal{R}_n^{\Lambda_0}(\mathcal{K})^{\mathtt{sgn}}$-modules.

**Corollary 7.3.6.** *For* $\lambda \in \mathcal{P}_n$ *we have* $S^\lambda(\mathcal{K}) \downarrow^{\mathcal{R}_n(\mathcal{K})}_{(\mathcal{R}_n(\mathcal{K}))^{\mathtt{sgn}}} \cong S_{\lambda'}(\mathcal{K}) \downarrow^{\mathcal{R}_n(\mathcal{K})}_{(\mathcal{R}_n(\mathcal{K}))^{\mathtt{sgn}}}$ *as graded* $(\mathcal{R}_n(\mathcal{K}))$*-modules.*

**Definition 7.3.7.** If $\lambda \neq \lambda'$, we write $S^{[\lambda]}(\mathcal{K})$ for the restricted module

$$S^\lambda(\mathcal{K}) \downarrow^{\mathcal{R}_n^{\Lambda_0}(\mathcal{K})}_{(\mathcal{R}_n^{\Lambda_0}(\mathcal{K}))^{\mathtt{sgn}}} \cong S_{\lambda'}(\mathcal{K}) \downarrow^{\mathcal{R}_n^{\Lambda_0}(\mathcal{K})}_{(\mathcal{R}_n^{\Lambda_0}(\mathcal{K}))^{\mathtt{sgn}}}$$

from Corollary 7.3.6. If $\lambda = \lambda'$, $S^\lambda(\mathcal{K}) \downarrow^{\mathcal{R}_n^{\Lambda_0}(\mathcal{K})}_{(\mathcal{R}_n^{\Lambda_0}(\mathcal{K}))^{\mathtt{sgn}}}$ is the direct sum of two irreducible $(\mathcal{R}_n^{\Lambda_0}(\mathcal{K}))^{\mathtt{sgn}}$-modules by Proposition 5.1.2 which we write as $S^\lambda_+(\mathcal{K})$ and $S^\lambda_-(\mathcal{K})$.

Since $\mathcal{R}_n^{\Lambda_0}(\mathcal{K})$ is a semisimple algebra and since $(\Lambda_0, \alpha_0) = 1$, by Proposition 5.1.2 and Theorem 2.2.12 we obtain the following collection of irreducible modules for the alternating cyclotomic quiver Hecke algebras $\mathcal{R}_n^{\Lambda_0}(\mathcal{K})^{\mathtt{sgn}}$.

**Theorem 7.3.8.** *If $n > 1$ and $e > 2$, then the collection*

$$\{S^{[\lambda]}(\mathcal{K})\langle k\rangle \mid [\lambda] \in (\mathcal{P}_n)_\sim \textit{ with } |[\lambda]| = 2,\ k \in \mathbb{Z}\}$$
$$\cup \{S_+^\lambda(\mathcal{K})\langle k\rangle, S_-^\lambda(\mathcal{K})\langle k\rangle \mid [\lambda] \in (\mathcal{P}_n)_\sim \textit{ with } |[\lambda]| = 1,\ k \in \mathbb{Z}\}$$

*is a complete list of irreducible graded $\mathcal{R}_n^{\Lambda_0}(\mathcal{K})^{\mathtt{sgn}}$-modules.*

We call the modules from Theorem 7.3.8 semisimple alternating graded Specht modules.

## 7.4. $\mathcal{O}$-forms for alternating graded Specht modules

Our goal now is to obtain $\mathcal{O}$-module analogues of the alternating graded Specht modules from §7.3 for $\mathcal{R}_n(\mathcal{O})^{\mathtt{sgn}}$ which can be reduced modulo the maximal ideal $\mathfrak{m}$ of $\mathcal{O}$ to yield analogues of graded Specht modules for the algebras $\mathscr{H}_n(F)^{\mathtt{sgn}}$ that we are ultimately interested in. In the case when $\lambda \neq \lambda'$ this is easy.

**Definition 7.4.1.** Let $\lambda \in \mathcal{P}_n^\alpha$ be such that $\lambda \neq \lambda'$. Then the *alternating graded Specht module* is the graded $(\mathcal{R}_n^{\Lambda_0}(\mathcal{O}))^{\mathtt{sgn}}$-module

$$S^{[\lambda]}(\mathcal{O}) = S^\lambda(\mathcal{O})\langle -\tfrac{\operatorname{def}\alpha}{2}\rangle \downarrow^{\mathcal{R}_n^{\Lambda_0}}_{(\mathcal{R}_n^{\Lambda_0})^{\mathtt{sgn}}}$$

We have chosen the half-integer degree shift so that the following duality property holds; its proof follows immediately from Proposition 7.2.6 and Definition 7.4.1.

**Proposition 7.4.2.** *Let $\lambda \in \mathcal{P}_n^\alpha$ be such that $\lambda \neq \lambda'$. Then*

$$S^{[\lambda]}(\mathcal{O}) \cong (S^{[\lambda]}(\mathcal{O}))^{\circledast}.$$

**Remark 7.4.3.**

(i) The reader may wonder why we do not need to define two types of alternating graded Specht module, corresponding to the row and column Specht modules we have for the quiver Hecke algebras. By Proposition 7.2.5 however, modules with a subscript would be equivalent to those we have defined (up to some shift).

(ii) We will abuse notation and write $v_{\mathtt{t}} \in S^{[\lambda]}(\mathcal{O})$ for the image of $v_{\mathtt{t}}$ in the restricted module, remembering that this vector has a different degree in $S^{[\lambda]}(\mathcal{O})$ (where its degree is $\deg \mathtt{t} - \frac{\operatorname{def} \alpha}{2}$) than it does in $S^{\lambda}(\mathcal{O})$ (where its degree is $\deg \mathtt{t}$).

Constructing graded Specht modules for alternating cyclotomic quiver Hecke algebras for self-conjugate partitions, where we expect (but cannot be guaranteed) some sort of splitting behaviour, is far more difficult. The following example demonstrates the kind of calculation we expect to be necessary in general.

**Example 7.4.4.** We saw in Example 7.3.2 the matrix for $\psi_2^{\mathcal{O}}$; from this we can obtain the matrix for the generator $\Psi_2$ of the alternating cyclotomic quiver Hecke algebra as

$$\begin{pmatrix} 0 & \sqrt{3}i \\ \sqrt{3}i & 0 \end{pmatrix}.$$

Since there is only one idempotent, the identity, for this algebra, this module decomposes as a direct sum of $\Psi_2$-eigenspaces, the basis vectors for which are $f_{\mathtt{t}} + f_{\mathtt{u}}$ and $-f_{\mathtt{t}} + f_{\mathtt{u}}$. These vectors do not live in the $\mathcal{O}$-form of the module, since $\pm f_{\mathtt{t}} + f_{\mathtt{u}} = \pm v_{\mathtt{t}} + \frac{1}{\sqrt{3}i} f_{\mathtt{u}}$. By clearing denominators however, we can define vectors $w_+ = \sqrt{3}i(f_{\mathtt{t}} + f_{\mathtt{u}})$ and $w_- = \sqrt{3}i(-f_{\mathtt{t}} + f_{\mathtt{u}})$ such that $w_+, w_-$ do belong to the $\mathcal{O}$-form (recalling we have defined $\mathcal{O}$ to contain these square roots).

## 7.5. Graded simple modules and the Mullineux map

In this section we return to discussing general dominant weights $\Lambda$ such that $\Lambda = \Lambda'$. Using graded cellular theory from Chapter 2, we can obtain a full set of graded simple modules $\{D^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in (\mathcal{P}_n^r)^0\}$. We can then use the Clifford theory from §5.1 to obtain simple modules for the alternating cyclotomic quiver Hecke algebras. In order to do this, we need to understand the effect of twisting a simple module by the sign involution. This question is much more difficult than for Specht modules, and was only settled in the 1990s.

The first question however is for which multipartitions $\boldsymbol{\lambda}$ is $D_{\boldsymbol{\lambda}} \neq 0$; by Definition 2.4.8 we refer to this set as $(\mathcal{P}_n^\ell)^0$. These are known as *Kleshchev multipartitions*; they generalise $p$-restricted partitions.

**Definition 7.5.1.** Let $\boldsymbol{\lambda} \in \mathcal{P}_n^\ell$ be a multipartition where $\mathcal{P}_n^\ell$ corresponds to the choice of multicharge $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\Lambda)$. Then $\boldsymbol{\lambda}$ is $\Lambda$-*Kleshchev* if the simple module $D_{\boldsymbol{\lambda}} \neq 0$. We refer to $\Lambda_0$-Kleshchev multipartitions simply as Kleshchev partitions.

**Remark 7.5.2.**

(i) In [**41**], Kleshchev partitions are defined as those for which the *ungraded* simple module $\dot{D}_{\boldsymbol{\lambda}}$ (we have not used this notation here) is nonzero; it is then a theorem [**41**, Corollary 5.11] that these are the same partitions as we have defined above.

(ii) There is an explicit combinatorial definition of $\Lambda$-Kleshchev multipartitions, which are often referred to in the literature as restricted multipartitions; we will see this below and a full proof can be found in [**18**, (3.27)].

In contrast to the theory of graded Specht modules, where it is not at all clear how to define alternating analogues of the graded Specht modules (see §7.4), we *do* get "God-given" simple modules for our algebras by Clifford theory.

Using the Clifford theory for $C_2$-graded algebras which we developed in §5.1, there will be two cases, depending on the cardinality of the inertia group $\mathcal{I}(\mathrm{Res}^{\mathcal{R}_n^\Lambda}_{(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}} D_{\boldsymbol{\lambda}})$ for a simple $\mathcal{R}_n^\Lambda$-module $D_{\boldsymbol{\lambda}}$. Since the twisted module $D_{\boldsymbol{\lambda}}^{\mathtt{sgn}}$ will again be a simple $\mathcal{R}_n^\Lambda$-module, it must be equal to $D_{m(\boldsymbol{\lambda})}$ for some Kleshchev partition $m(\boldsymbol{\lambda})$ and so the size of the inertia group is determined by this unknown involution.

In 1979, Mullineux [85] gave a conjectural formula for this involution in the case $\ell = 1$ and fifteen years later Ford and Kleshchev [30] proved what had become known as the Mullineux conjecture. There have since been many alternative formulations of Mullineux's involution, known as the *Mullineux map*, as well as generalisations to higher levels. In order to define this involution, we need to introduce some additional combinatorics.

**Definition 7.5.3.** Given $0 \le i < p$, define the *$i$-node sequence* of a multipartition as follows: read along the rows of successive components from left-to-right starting with the top row and finishing with the bottom, recording in order the removable $i$-nodes with an $R$ and the addable $i$-nodes with an $A$. Given an $i$-node sequence, we refine it to a *good $i$-node sequence* by deleting subwords $RA$ from the $i$-node sequence until this is no longer possible (note these may be nested) – the $i$-nodes corresponding to the remaining $R$s are called *normal $i$-nodes*. The $i$-node corresponding to the left-most $R$ remaining in the good $i$-node sequence, if it exists, is called the *good $i$-node*. A node in $\boldsymbol{\lambda}$ is *good* (more precisely, $p$-good) if it is a good $i$-node for some $0 \le i < p$.

**Proposition 7.5.4.** [18, (3.27)] *The Kleshchev multipartitions are precisely those obtained by adding sequences of good nodes to the empty partition $\emptyset$.*

Notice that, by our inductive definition and Proposition 7.5.4, for any $\Lambda$-Kleshchev partition $\boldsymbol{\lambda} \vdash n$ there is a unique sequence

$$\emptyset = \boldsymbol{\lambda}^0 \to \boldsymbol{\lambda}^1 \to \boldsymbol{\lambda}^2 \to \cdots \to \boldsymbol{\lambda}^n = \boldsymbol{\lambda}$$

of partitions $\{\boldsymbol{\lambda}^k \vdash_\ell k\}$ such that $\boldsymbol{\lambda}^{k+1}$ is obtained from $\boldsymbol{\lambda}^k$ by adding a good node of residue $i_k$.

**Theorem 7.5.5.** [5][18, Theorem 4.12] *If $\boldsymbol{\lambda} \in (\mathcal{P}^\ell_n)_0$, there exists a unique sequence of multipartitions $m(\boldsymbol{\lambda})_0, m(\boldsymbol{\lambda})_1, \dots, m(\boldsymbol{\lambda})_n = m(\boldsymbol{\lambda})$ such that $m(\boldsymbol{\lambda})_{k+1}$ is obtained from $m(\boldsymbol{\lambda})_k$ by adding a good $-i_k$-node. Moreover, $m(\boldsymbol{\lambda})_k \in (\mathcal{P}^\ell_k)_0$ for $1 \le k \le n$.*

**Definition 7.5.6** (Mullineux map)**.** The partition obtained is called the *Mullineux conjugate* of $\boldsymbol{\lambda}$ and is denoted $m(\boldsymbol{\lambda})$.

**Example 7.5.7.** Suppose that $e = 3$; then it is not to hard to show that one can build the partition $\lambda = (3,2)$ using the following sequence of good nodes:

$$\emptyset \xrightarrow[0]{(1,1)} (1) \xrightarrow[2]{(2,1)} (1^2) \xrightarrow[1]{(1,2)} (2,1) \xrightarrow[0]{(2,2)} (2^2) \xrightarrow[2]{(1,3)} (3,2)$$

Adding a good node of the opposite residue at every stage gives the following sequence resulting in the Mullineux conjugate partition $m(\lambda) = (5)$:

$$\emptyset \xrightarrow[0]{(1,1)} (1) \xrightarrow[1]{(1,2)} (2) \xrightarrow[2]{(1,3)} (3) \xrightarrow[0]{(1,4)} (4) \xrightarrow[1]{(1,5)} (5)$$

We can now apply our knowledge of Clifford theory to alternating quiver Hecke algebras and obtain a classification of graded simple modules.

**Definition 7.5.8.** Let $\sim$ be the equivalence relation defined in $(\mathcal{P}_n^r)^0$ by $\boldsymbol{\lambda} \sim \boldsymbol{\mu}$ if $\boldsymbol{\mu} = m(\boldsymbol{\lambda})$. We write $[\boldsymbol{\lambda}]$ for the equivalence class of $\boldsymbol{\lambda} \in (\mathcal{P}_n^r)^0$ under this equivalence relation, and $(\mathcal{P}_n^r)^0_\sim$ for the set of all equivalence classes.

**Definition 7.5.9.** Let $\Lambda \in P_e$ be a dominant weight such that $\Lambda = \Lambda'$. Let $\boldsymbol{\lambda} \in (\mathcal{P}_n^r)^0$. If $\boldsymbol{\lambda} \neq m(\boldsymbol{\lambda})$, $D_{\boldsymbol{\lambda}}\downarrow^{\mathcal{R}_n^\Lambda}_{(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}} \cong D_{m(\boldsymbol{\lambda})}\downarrow^{\mathcal{R}_n^\Lambda}_{(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}}$ and so we write $D_{[\boldsymbol{\lambda}]}$ for this restricted module. If $\boldsymbol{\lambda} = m(\boldsymbol{\lambda})$, $D_{\boldsymbol{\lambda}}\downarrow^{\mathcal{R}_n^\Lambda}_{(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}}$ is a direct sum of two irreducible $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$-modules by Proposition 5.1.2 which we write as $D_{\boldsymbol{\lambda}}^+$ and $D_{\boldsymbol{\lambda}}^-$

We can now use Clifford theory to give a classification of irreducible graded modules for alternating cyclotomic quiver Hecke algebras.

**Theorem 7.5.10.** *Let $n > 1$, $e > 2$ and suppose $\Lambda \in P_e$ is such that $\Lambda = \Lambda'$, and such that $(\Lambda, \alpha_0) < n$. Then the collection*

$$\{D_{[\boldsymbol{\lambda}]}\langle k\rangle \mid [\boldsymbol{\lambda}] \in (\mathcal{P}_n^r)^0_\sim \;\; \textit{with} \;\; |[\boldsymbol{\lambda}]| = 2,\; k \in \mathbb{Z}\}$$
$$\cup \{D_{\boldsymbol{\lambda}}^+\langle k\rangle, D_{\boldsymbol{\lambda}}^-\langle k\rangle \mid [\boldsymbol{\lambda}] \in (\mathcal{P}_n^r)^0_\sim \;\; \textit{with} \;\; |[\boldsymbol{\lambda}]| = 1,\; k \in \mathbb{Z}\}.$$

*is a complete list of irreducible graded $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$-modules.*

**Proof.** The classification follows immediately from Proposition 5.1.2 and our results on graded representation theory from Chapter 2, provided we use the assumption that $(\Lambda, \alpha_0) < n$ to avoid being unable to use Clifford theory. □

**Example 7.5.11.** Let us continue with the running example that we last encountered in Example 7.1.17. Since $\mathcal{P}_3^0 = \{(3), (21)\}$, and $m(3) = (21)$, there will be one irreducible graded $(\mathcal{R}_3^{\Lambda_0})^{\mathtt{sgn}}$-module up to shift: $D_{[(3)]}$. This agrees with Example 5.4.3, since there is a unique irreducible representation of the cyclic group $C_3$ over a field of characteristic three [**1**, Corollary 3.3].

**Corollary 7.5.12.** *Let $\mathcal{K}$ be a field of characteristic greater than 2 which contains an alternating seminormal coefficient system. Let $\Lambda \in P_e$ be such that $\Lambda = \Lambda'$. If $\mathcal{R}_n^\Lambda(\mathcal{K})$ is semisimple then*

$$\mathscr{H}_n^\Lambda(\mathcal{K})^\# \cong \mathcal{R}_n^\Lambda(\mathcal{K})^{\mathtt{sgn}}.$$

**Proof.** If $\mathcal{R}_n^\Lambda(\mathcal{K})$ is semisimple, $(\mathcal{P}_n^\ell)_0 = \mathcal{P}_n^\ell$ and so the irreducible modules from Theorem 7.5.10 have the same dimensions and are of the same number as the Specht modules for $\mathscr{H}_n^\Lambda(\mathcal{K})$ from Theorem 3.6.19. The result now follows from the Wedderburn decomposition [**75**, Theorem A20]. □

## 7.6. Graded decomposition numbers

In this section we work in the graded Grothendieck group $K_0((\mathcal{R}_n^{\Lambda_0})^{\mathtt{sgn}})$ of the alternating cyclotomic quiver Hecke algebra with dominant weight $\Lambda = \Lambda_0$. To avoid confusion, we write $[M : D]_q^{\mathfrak{A}}$ for the graded composition multiplicity of the graded simple $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$-module $D$ in the graded simple $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$-module $M$ and reserve the notation $[M : D]_q$ for graded decomposition numbers of cyclotomic quiver Hecke algebras. Using the results of the last two sections, we obtain equalities of graded decomposition numbers for alternating cyclotomic quiver Hecke algebras, relating them to the corresponding graded decomposition numbers for cyclotomic quiver Hecke algebras, in the case where the partition $\lambda$ whose Specht module we are interested in computing multiplicities in is not self-conjugate.

**Proposition 7.6.1.** *Suppose $\lambda \in \mathcal{P}_n^\alpha$ is such that $\lambda \neq \lambda'$. Then*

(i) *if $\mu \in \mathcal{P}_n^0$ is such that $\mu \neq m(\mu)$,*

$$[S^{[\lambda]} : D_{[\mu]}]_q^{\mathfrak{A}} = q^{-\frac{\operatorname{def}\alpha}{2}}\Big([S^\lambda : D_\mu]_q + [S^\lambda : D_{m(\mu)}]_q\Big).$$

(ii) *if $\mu \in \mathcal{P}_n^0$ is such that $\mu = m(\mu)$,*

$$[S^{[\lambda]} : D_\mu^+]_q^{\mathfrak{A}} = [S^{[\lambda]} : D_\mu^-]_q^{\mathfrak{A}} = q^{-\frac{\operatorname{def}\alpha}{2}}[S^\lambda : D_\mu]_q.$$

**Proof.** By definition,

$$[S^\lambda] = \sum_{\mu\in\mathcal{P}_n}[S^\lambda : D_\mu]_q[D_\mu]. \tag{7.6.2}$$

Restricting this formula to $(\mathcal{R}_n^{\Lambda_0})^{\mathtt{sgn}}$ gives

$$\begin{aligned} q^{\frac{\operatorname{def}\alpha}{2}}[S^{[\lambda]}]^{\mathfrak{A}} = [S^\lambda\downarrow^{\mathcal{R}_n^{\Lambda_0}}_{(\mathcal{R}_n^{\Lambda_0})^{\mathtt{sgn}}}]^{\mathfrak{A}} &= \sum_{\mu\in\mathcal{P}_n}[S^\lambda : D_\mu]_q[D_\mu\downarrow^{\mathcal{R}_n^{\Lambda_0}}_{(\mathcal{R}_n^{\Lambda_0})^{\mathtt{sgn}}}]^{\mathfrak{A}} \\ &= \sum_{\substack{\mu\in\mathcal{P}_n\\ \mu>m(\mu)}}\Big([S^\lambda : D_\mu] + [S^\lambda : D_{m(\mu)}]\Big)[D_{[\mu]}] \qquad (7.6.3) \\ &\qquad + \sum_{\substack{\mu\in\mathcal{P}_n\\ \mu=m(\mu)}}[S^\lambda : D_\mu]_q\Big([D_\mu^+]^{\mathfrak{A}} + [D_\mu^-]^{\mathfrak{A}}\Big) \end{aligned}$$

which gives the result in both cases. □

**Example 7.6.4** (Decomposition matrix for $(\mathcal{R}_{3,3,\mathbb{C}}^{\Lambda_0})^{\mathtt{sgn}}$)**.** Let $(\mathcal{R}_{3,3,\mathbb{C}}^{\Lambda_0})^{\mathtt{sgn}}$ be the alternating Hecke algebra over $\mathbb{C}$ at a third root of unity. Using the tables of decomposition numbers computed using the LLT algorithm [**67**] (which give the *graded* decomposition numbers by results of Brundan and Kleshchev [**18**]), we can compute the graded decomposition matrix for the alternating Hecke algebra using the formulas above. As we saw in Example 7.5.11, there is only one simple module, $D_{[(3)]}$, and we compute its graded multiplicity in each of the three alternating graded Specht modules as follows.

$$[S^{[(3)]} : D_{[(3)]}]_{\mathfrak{A}} = q^{-\frac{1}{2}}\Big([S^3 : D_{(3)}] + [S^{(3)} : D_{(21)}]\Big) = q^{-\frac{1}{2}}$$

by Proposition 7.6.1. Moreover, by Example 7.4.4, assuming we have two modules $S^{(21)}_{+}$ and $S^{(21)}_{-}$ which are direct summands of the module $S^{(21)}\downarrow^{\mathcal{R}_3^{\Lambda_0}}_{(\mathcal{R}_3^{\Lambda_0})^{\mathtt{sgn}}}$,

$$q^{-1}[S^{(21)}_{+} : D_{(3)/(21)}]_{\mathfrak{A}} + q[S^{(21)}_{-} : D_{(3)/(21)}]_{\mathfrak{A}} = [S^{(21)} : D_{(3)}] + [S^{(21)} : D_{(21)}] = q + 1.$$

Since exponents of graded dimensions are bounded above and below by $\operatorname{def}\alpha$ and $-\operatorname{def}\alpha$ respectively [**41**], we must have

$$[S^{(21)}_{+} : D_{(3)/(21)}]_{\mathfrak{A}} = q, \quad \text{and} \quad [S^{(21)}_{-} : D_{(3)/(21)}]_{\mathfrak{A}} = q^{-1}$$

which gives the following graded decomposition matrix:

| | [(3)] |
|---|---|
| [(3)] | $q^{-\frac{1}{2}}$ |
| $(21)^{+}$ | $q$ |
| $(21)^{-}$ | $q^{-1}$ |

**Remark 7.6.5.** Since there is no reason for our algebras to have filtrations by our graded Specht modules, this decomposition matrix is not in contradiction with the $\mathbb{F}_3[x]/(x^3)$ decomposition matrix we computed in Chapter 2.

# Index of notation

| Notation | Meaning |
|---|---|
| $A$ | a graded algebra |
| $A$-**Mod** | the category of graded $A$-modules |
| $\underline{A}$ | the ungraded algebra obtained by forgetting the grading on $A$ |
| $\underline{A}$-**Mod** | the category of $A$-modules |
| $\mathfrak{A}_n$ | the alternating group on $n$ letters |
| $\mathscr{A}_{\mathtt{t}}(k)$ | $\{$addable nodes $c$ of $\mathrm{sh}(\mathtt{t}_k) \mid c$ is below $\mathtt{t}^{-1}(k)\}$ |
| $\mathscr{A}_{\mathtt{t}}^{\Lambda}(k)$ | $\{c \in \mathscr{A}_{\mathtt{t}}(k) \mid \mathrm{res}(c) = \mathrm{res}_{\mathtt{t}}(k)\}$ |
| $\mathscr{A}_{\mathtt{t}}(k)'$ | $\{$addable nodes $c$ of $\mathrm{sh}(\mathtt{t}_k) \mid c$ is above $\mathtt{t}^{-1}(k)\}$ |
| $\alpha$ | an element of the root lattice $Q_e$ |
| $\alpha_r(\mathtt{t})$ | element of seminormal coefficient system |
| $[\alpha]$ | equivalence class of $\alpha$ under $\sim$ |
| $\mathscr{A}_{\mathtt{t}}^{\Lambda}(k)'$ | $\{c \in \mathscr{A}_{\mathtt{t}}(k)' \mid \mathrm{res}(c) = \mathrm{res}_{\mathtt{t}}(k)\}$ |
| $\boldsymbol{\alpha}$ | $*$-seminormal coefficient system of coefficients $\alpha_r(\mathtt{t})$ |
| $\beta_r(\mathtt{t})$ | coefficient for action of $\psi_r$ on seminormal basis |
| $c_{\mathtt{t}}(k)$ | the content of $k$ in $\mathtt{t}$ |
| $\delta$ | the Kronecker delta |
| $D^{\boldsymbol{\mu}}$ | simple module for cyclotomic quiver Hecke algebra |
| $D^{[\boldsymbol{\mu}]}$ | simple module for alternating cyclotomic quiver Hecke algebra when $\boldsymbol{\mu} \neq m(\boldsymbol{\mu})$ |
| $D^{\boldsymbol{\mu}}_{+}, D^{\boldsymbol{\mu}}_{-}$ | simple modules for alternating cyclotomic quiver Hecke algebra when $\boldsymbol{\mu} = m(\boldsymbol{\mu})$ |
| deg | the degree function on a graded algebra |
| $d_{\lambda\mu}(q)$ | the graded composition multiplicity of $D_\mu$ in $S_\lambda$ |
| $d(\mathtt{s})$ | the permutation taking a tableau $\mathtt{s}$ to the initial tableau |

| | |
|---|---|
| $d'(\mathtt{s})$ | the permutation taking a tableau $\mathtt{s}$ to the final tableau |
| $e(\mathbf{i})$ | quiver Hecke algebra idempotent corresponding to the residue sequence $\mathbf{i}$ |
| $F$ | a field |
| $F_{\mathtt{t}}$ | idempotent in $\mathscr{H}$ corresponding to standard tableau $\mathtt{t}$ |
| $f_{\mathtt{st}}$ | seminormal basis vector |
| $\Gamma_e$ | the oriented quiver of type $A^{\infty}_{e-1}$ or $A_{\infty}$ |
| $\mathscr{H}_{n,\ell}(\mathcal{O},\xi,\mathbf{Q})$ | the cyclotomic Hecke algebra of type $(\ell,n)$ with parameters $\xi$ and $\mathbf{Q}$ |
| $\mathrm{HOM}_A(M,N)$ | graded $A$-module homomorphisms from $M$ to $N$ |
| $\mathrm{Irr}(\underline{A})$ | the irreducible objects in $A$-**Mod** |
| $\mathrm{Irr}(A)$ | the irreducible objetcs in $\underline{A}$-**Mod** |
| $\mathbf{i},\mathbf{j}$ | residue sequences |
| $[\mathbf{i}]$ | equivalence classes of residue sequences under $\sim$ |
| $\mathbf{i}_{\mathtt{t}}$ | residue sequence of the standard tableau $\mathtt{t}$ |
| $I$ | $\mathbb{Z}/e\mathbb{Z}$ |
| $I^{\alpha}$ | $\{\mathbf{i}\in I^n \mid \alpha_{i_1}+\alpha_{i_2}+\ldots+\alpha_{i_n}=\alpha\}$ |
| $I^n_{\sim}$ | $I^n/\sim$ |
| $I^n_{\bullet}$ | where $\bullet=\rightarrow,\swarrow,\nearrow,\downarrow$, equivalence classes of residue sequences depending on first three entries |
| $I^{[\alpha]}_{\sim}$ | equivalence classes of sequences in $I^{\alpha}$ and $I^{\alpha'}$ for $\alpha\neq\alpha'$ |
| $I^{\alpha}_{\sim}$ | equivalence classes of sequences in $I^{\alpha}$ for $\alpha=\alpha'$ |
| $K_0(A\text{-}\mathbf{Mod})$ | the graded Groethendieck group of $A$ |
| $[k]_{\xi}$ | the quantum integer corresponding to $k$ with paramter $\xi$ |
| $\mathcal{K}$ | a field |
| $\kappa$ | multicharge corresponding to a cyclotomic Hecke algebra |
| $\boldsymbol{\lambda},\boldsymbol{\mu}$ | multipartitions |
| $L_i$ | Jucys-Murphy elements |
| $\Lambda$ | dominant weight for Kac-Moody algebra $\widehat{\mathfrak{sl}}_e$ |
| $M,N$ | modules over a graded algebra |
| $M\langle k\rangle$ | $M$ with grading shifted by $k$ |

| | |
|---|---|
| $\underline{M}$ | the ungraded module obtained by forgetting the grading on $M$ |
| $[M]$ | the isomorphism class of $M$ |
| $M^{\circledast}$ | the contragredient graded dual of $M$ |
| $[M : D\langle k\rangle]$ | the multiplicity of the simple module $D\langle k\rangle$ as a graded composition factor of $M$ |
| $m_{\mathtt{st}}$ | Murphy basis vector $T_{d(\mathtt{s})}^{-1}u_\lambda x_\lambda T_{d(\mathtt{t})}$ |
| $n_{\mathtt{st}}$ | dual Murphy basis vector $(-\xi)^{-\ell(d(\mathtt{s}))-\ell(d(\mathtt{t}))}T_{d'(\mathtt{s})^{-1}}u'_\lambda y_\lambda T_{d'(\mathtt{t})}$ |
| $\mathcal{O}$ | an $e$-idempotent subring |
| $\Omega$ | the fixed data of a reduced expression for each permutation in $\mathfrak{S}_n$ |
| $\psi_r$ | generator of quiver Hecke algebra |
| $\psi_\omega$ | element $\psi_{i_1}\psi_{i_2}\cdots\psi_{i_r}$ corresponding to $\omega = s_{i_1}s_{i_2}\cdots s_{i_r}$ |
| $\Psi_r$ | generator of alternating quiver Hecke algebra |
| $\psi_r^+$ | generator of integral cyclotomic quiver Hecke algebra |
| $\psi_r^-$ | hashed generator of integral cyclotomic quiver Hecke algebra |
| $\psi_{\mathtt{st}}$ | graded cellular basis vector |
| $\psi'_{\mathtt{st}}$ | dual graded cellular basis vector |
| $\mathcal{P}_n^\ell$ | the set of $\ell$-multipartitions of $n$ ($\boldsymbol{\lambda} \vdash_\ell n$) |
| $P_e$ | dominant weight lattice |
| $Q_e$ | root lattice |
| $\operatorname{qdim} M$ | the graded dimension of $M$ |
| $\mathcal{R}_n$ | quiver Hecke algebra |
| $\mathcal{R}_n^{\mathtt{sgn}}$ | alternating quiver Hecke algebra |
| $\mathcal{R}_n^\Lambda$ | cyclotomic quiver Hecke algebra |
| $(\mathcal{R}_n^\Lambda)^{\mathtt{sgn}}$ | alternating cyclotomic quiver Hecke algebra |
| $\operatorname{res}_{\mathtt{t}}(k)$ | the residue of $k$ in $\mathtt{t}$ modulo $e$ |
| $\rho_r(\mathtt{t})$ | axial distance from $r+1$ to $r$ in $\mathtt{t}$ |
| $S^{\boldsymbol{\lambda}}$ | graded row Specht module (obtained from graded cellular basis) |
| $S_{\boldsymbol{\lambda}}$ | graded column /dual Specht module (obtained from dual graded cellular basis) |
| $S^{[\lambda]}$ | alternating graded Specht module for $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$ |

| | |
|---|---|
| $\mathtt{sgn}$ | the homogeneous sign involution |
| $M^{\mathtt{sgn}}$ | the graded $\mathcal{R}_n^\Lambda$-module $M$ twisted by $\mathtt{sgn}$ |
| $\mathfrak{S}_n$ | the symmetric group on $n$ letters |
| $\sim$ | used for equivalence relations on $I^n$ and $Q_e$ |
| $\mathtt{s},\mathtt{t},\mathtt{u},\mathtt{v}$ | standard tableaux |
| $\mathrm{Std}(\boldsymbol{\lambda})$ | the set of standard multitableaux of shape $\boldsymbol{\lambda}$ |
| $\mathrm{Std}(\mathcal{P}_n^\ell)$ | the set of all $\ell$-multitableaux with $n$ boxes |
| $\mathrm{Std}_\bullet(\mathcal{P}_n)$ | where $\bullet = \rightarrow, \swarrow, \nearrow, \downarrow$, equivalence classes of standard tableaux of entries $1,2,3$ |
| $\mathtt{t}^{\boldsymbol{\lambda}}, \mathtt{t}_{\boldsymbol{\lambda}}$ | the initial and final standard $\boldsymbol{\lambda}$-tableaux |
| $T_i$ | generators of cyclotomic Hecke algebras |
| $v_{\mathtt{t}}$ | homogeneous basis vector for graded Specht module |
| $v'_{\mathtt{t}}$ | homogeneous basis vector for dual graded Specht module |
| $y_r$ | generator of quiver Hecke algebra |
| $\mathcal{Y}_r$ | generator of alternating quiver Hecke algebra |
| $y_\lambda$ | $\prod_{k=1}^n y_k^{\lvert \mathscr{A}^\Lambda_{\mathtt{t}^\lambda}(k)\rvert}$ |
| $y'_\lambda$ | $\prod_{k=1}^n y_k^{\lvert \mathscr{A}^\Lambda_{\mathtt{t}^\lambda}(k)'\rvert}$ |
| $z_\lambda$ | homogeneous generator of the graded Specht module |
| $z'_\lambda$ | homogeneous generator of the dual graded Specht module |
| $\mathcal{Z}$ | a unital integral domain |
| $\trianglerighteq$ | the dominance order on partitions |
| $\#$ | hash involution which defines the alternating cyclotomic Hecke algebra |
| $\sqcup$ | disjoint union |
| $*$ | the antiautomorphism for a cellular algebra |

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