



COPYRIGHT AND USE OF THIS THESIS

This thesis must be used in accordance with the provisions of the Copyright Act 1968.

Reproduction of material protected by copyright may be an infringement of copyright and copyright owners may be entitled to take legal action against persons who infringe their copyright.

Section 51 (2) of the Copyright Act permits an authorized officer of a university library or archives to provide a copy (by communication or otherwise) of an unpublished thesis kept in the library or archives, to a person who satisfies the authorized officer that he or she requires the reproduction for the purposes of research or study.

The Copyright Act grants the creator of a work a number of moral rights, specifically the right of attribution, the right against false attribution and the right of integrity.

You may infringe the author's moral rights if you:

- fail to acknowledge the author of this thesis if you quote sections from the work
- attribute this thesis to another author
- subject this thesis to derogatory treatment which may prejudice the author's reputation

For further information contact the University's Director of Copyright Services

sydney.edu.au/copyright

Managing ecological systems
with unknown threshold locations

David John Rogers

B. ResEc (Hons.) University of Sydney

Faculty of Agriculture and Environment
Discipline of Agricultural and Resource Economics
University of Sydney

Submitted in fulfilment of the requirements of the degree of
Doctor of Philosophy

December 2013

Abstract

The optimal management of ecological systems is challenging because the locations of thresholds between desirable and undesirable regimes are generally unknown to the decision-maker. However, it is possible to learn about the resilience of an ecological system by intelligently perturbing the system using adaptive management (Arrow *et al.* 1995). Previous research has modelled optimal decisions in systems with hysteretic thresholds (Mäler *et al.* 2003), derived necessary conditions for optimal control when the locations of thresholds are unknown (Nævdal 2006; Nævdal and Oppenheimer 2007), and used stochastic dynamic programming to examine the effect of this form of uncertainty on risk averse behaviour (Brozovic and Schlenker 2011). This thesis extends previous research to model the effect on optimal decisions of learning about the locations of thresholds via a process of adaptive management. A dynamic programming framework is developed and applied to various ecological contexts, including numerical simulations of a shallow lake ecosystem, and used to demonstrate the role of learning.

This thesis demonstrates that learning can be modelled by updating the prior probability distribution for a threshold's location and by adjusting the boundary between the regions of a system's state-space that could and could not contain the threshold. The model captures the trade-off faced by the decision-maker between the costs of crossing a threshold and shifting to an undesirable alternative regime, and the benefits of learning about the threshold location. Explicit consideration of the value of information means the decision-maker will generally make decisions that incur a greater risk of crossing the threshold in order to learn about its location. This finding is independent of the initial prior probability distribution used to model threshold location and the type of ecosystem dynamics considered. By explicitly modelling the value of information, this thesis better demonstrates the nature of optimal decision-making in the adaptive management of ecological systems.

Table of Contents

ABSTRACT	II
List of Tables.....	v
List of Figures	vi
Acknowledgements	vii
Statement of originality	viii
CHAPTER 1. INTRODUCTION	1
1.1 Ecological systems	3
1.2 Modelling and managing ecological systems.....	5
1.3 The research problem.....	7
1.3.1 Research aims	8
1.3.2 An overview of the methodology	10
1.3.3 Research questions and propositions.....	13
1.4 An outline of the thesis.....	16
CHAPTER 2. LITERATURE REVIEW	18
2.1 Managing ecological systems with unknown threshold locations.....	18
2.2 Modelling the management of ecological systems.....	23
2.2.1 Marginal resilience pricing.....	24
2.2.2 Dynamic optimisation with known or unknown threshold locations.....	25
2.2.3 Alternatives to standard optimisation techniques	27
2.3 Extensions to previous approaches.....	30
2.3.1 Consideration of the impacts of active learning	30
2.3.2 Provision of specific recommendations for control variables.....	33
2.3.3 Conceptual inconsistencies.....	35
2.4 Requirements of a new approach	37
CHAPTER 3. MODELLING FRAMEWORK	44
3.1 Introduction to the model	45
3.1.1 Notation	45
3.1.2 The decision-maker's objective.....	46
3.1.3 Model characteristics.....	47
3.1.4 A system with hysteretic dynamics	49
3.1.5 A system with an irreversibility.....	53
3.2 Updating prior beliefs about threshold locations.....	55
3.2.1 Modelling of probability density functions for unknown threshold locations	55
3.2.2 Deterministic evolution of the underlying slow ecosystem variable	60
3.2.3 General updating rules for the four state variables	64
3.2.4 Conversion of updating rules for use in simulation model	67
3.3 Equation of motion (EOM) for wealth	74
3.4 The value function: the NPV of expected future utility	76
3.4.1 Sources of utility.....	76
3.4.2 The value function.....	79
CHAPTER 4. MODELLING COMMERCIAL PRODUCTION WITH UNCERTAIN ECOLOGICAL THRESHOLDS	82
4.1 Shallow lake and adjoining agricultural land	83
4.1.1 Shallow lake: State variables	84
4.1.2 Shallow lake: Control variables.....	84
4.1.3 Shallow lake: Trade-offs	85
4.1.4 Shallow lake: EOM for stock of phosphorus in the lake (X)	87
4.1.5 Shallow lake: Critical ecological thresholds.....	88
4.1.6 Shallow lake: Sources of utility.....	89
4.2 Savannah-forest system.....	92
4.2.1 Savannah: State variables	93

4.2.2	Savannah: Control variables	93
4.2.3	Savannah: Trade-offs	94
4.2.4	Savannah: Critical ecological thresholds	98
4.2.5	Savannah: Population growth functions	99
4.2.6	Savannah: Sources of utility	102
CHAPTER 5. MODELLING ENVIRONMENTAL QUALITY WITH UNCERTAIN ECOLOGICAL THRESHOLDS		105
5.1	Reef fishery (reef-mangrove interaction)	105
5.1.1	Reef fishery: State variables	106
5.1.2	Reef fishery: Control variables	107
5.1.3	Reef fishery: Trade-offs	107
5.1.4	Reef fishery: EOM for physical distance between reef and mangrove systems (X)	110
5.1.5	Reef fishery: Critical ecological thresholds	112
5.1.6	Reef fishery: Population growth functions	113
5.1.7	Reef fishery: Sources of utility	115
5.2	Koala preservation and housing development	117
5.2.1	Koalas: State variables	118
5.2.2	Koalas: Control variables	118
5.2.3	Koalas: Trade-offs	118
5.2.4	Koalas: EOM for habitat quality (X)	122
5.2.5	Koalas: Critical ecologic thresholds	122
5.2.6	Koalas: Population growth functions	123
5.2.7	Koalas: Sources of utility	125
CHAPTER 6. SHALLOW LAKE SIMULATION MODEL AND RESULTS		127
6.1	Simulation model (abridged 'shallow lake' model)	128
6.1.1	The 'Learning' model	128
6.1.2	The 'No learning' model	132
6.2	Simulation methodology	133
6.3	Simulation results	136
6.3.1	Expected net present value calculations and the value of information	137
6.3.2	Extent of risk averse activity and willingness to incur risk	143
6.3.3	Subjective choice of prior distribution for unknown threshold location	148
6.3.4	Relative contributions of direct and consumption utility	156
CHAPTER 7. CONCLUSIONS AND IMPLICATIONS		160
7.1	Impact of active learning on optimal decisions	160
7.2	Sensitivity of optimal decisions to particular ecological dynamics	162
7.3	Sensitivity of optimal decisions to the subjective choice of prior distribution	164
7.4	Avenues for future research	166
	<i>Conclusion</i>	168
	References	169
	Appendix A: Reverse framing of updating rules for prior distribution	179
	Appendix B: MATLAB code for 'Learning' model	180

List of Tables

Table 1-1 Key features of the four representative case studies	16
Table 3-1 Variables used in the mathematical modelling framework	45
Table 4-1 Key features of the shallow lake and savannah case studies.....	82
Table 4-2 State variables of the shallow lake case study.....	84
Table 4-3 Control variables of the shallow lake case study	84
Table 4-4 State variables of the savannah case study.....	93
Table 4-5 Control variables of the savannah case study.....	93
Table 5-1 Key features of the reef fishery and koala case studies.....	105
Table 5-2 State variables of the reef fishery case study	106
Table 5-3 Control variables of the reef fishery case study	107
Table 5-4 State variables of the koala case study	118
Table 5-5 Control variables of the koala case study.....	118

List of Figures

Figure 2-1 Ecosystem dynamics and threshold effects.....	22
Figure 3-1 A system with two alternative regimes and hysteretic dynamics	50
Figure 3-2 Decision tree for the optimisation of a system with hysteretic dynamics	51
Figure 3-3 Decision tree for the optimisation of a system with an irreversible threshold	54
Figure 4-1 Transfer coefficient for agricultural runoff to the lake	85
Figure 4-2 Crop response function	86
Figure 4-3 Alternative system regimes for the cycling of phosphorus in the shallow lake	89
Figure 4-4 Direct utility from recreational use of the shallow lake	91
Figure 4-5 Natural growth of grass, net of losses from fire and grazing	95
Figure 4-6 Natural growth of trees, net of losses from fire and grazing.....	96
Figure 4-7 Proportional losses of vegetation due to fire.....	97
Figure 4-8 Effect of livestock grazing on grass cover	97
Figure 4-9 Effect of deliberate clearing on tree cover	98
Figure 4-10 Alternative system regimes for prevalence of grass cover.....	99
Figure 5-1 Fishing harvest.....	108
Figure 5-2 Amount of pollution released after filtering	109
Figure 5-3 Amount of pollution deliberately removed from the system (excl. assimilation).....	110
Figure 5-4 Per-period natural growth of reef for various levels of harvestable fish stock, and reef- mangrove distance.....	111
Figure 5-5 Per-period damage caused by accumulated pollution	112
Figure 5-6 Alternative system regimes for harvestable fish population	113
Figure 5-7 Habitat size vs. Housing development.....	119
Figure 5-8 Habitat connectivity vs. Housing development	120
Figure 5-9 Habitat quality vs. Housing development.....	121
Figure 5-10 Alternative system regimes for koala population	123
Figure 5-11 An illustrative path of the koala population when habitat size remains constant	124
Figure 5-12 An illustrative path of the koala population when habitat size is occasionally reduced ...	125
Figure 6-1 Alternative system regimes in abridged 'shallow lake' model	132
Figure 6-2 Expected NPV of utility when the lower bound is varied.....	138
Figure 6-3 Expected NPV of utility when the upper bound is varied.....	138
Figure 6-4 Value of information when the lower bound is varied.....	141
Figure 6-5 Value of information when the upper bound is varied.....	142
Figure 6-6 Optimal first-period level of phosphorus when in the preferred regime	145
Figure 6-7 Optimal first-period level of phosphorus when in the non-preferred regime.....	146
Figure 6-8 Optimal first-period level of phosphorus with an irreversible threshold	148
Figure 6-9 Sensitivity of control variable to choice of prior bounds - 'Learning' model.....	149
Figure 6-10 Sensitivity of control variable to choice of prior bounds - 'No learning' model (Preferred regime)	150
Figure 6-11 Sensitivity of control variable to choice of prior bound - 'Learning' model	152
Figure 6-12 Sensitivity of control variable to choice of prior bounds - 'No learning' model	153
Figure 6-13 Optimal first-period level of phosphorus when lower bound is varied.....	154
Figure 6-14 Optimal first-period level of phosphorus when the upper bound is varied (Irreversible threshold)	155
Figure 6-15 Optimal first-period level of phosphorus when the relative importance of consumption utility and the lower bound of the prior distribution are varied	157
Figure 6-16 Optimal first-period level of phosphorus when the relative importance of consumption utility and the upper bound of the prior distribution is varied.....	158

Acknowledgements

It would not have been possible to write this PhD thesis without the help and support of my family, friends and supervisors. Above all, I would like to thank my beautiful wife, Sally, for her love and support during my many ups and downs throughout this process. I look forward to spending many happy years with you. My parents and brothers have provided unwavering support during what has been a long and, at times, frustrating process. I cannot put into words how appreciative I am of this support.

I am most thankful for the assistance, support and patience of my primary supervisor, Dr. Greg Hertzler, and associate supervisor, Dr. Tim Capon. Greg's technical skills, vast experience and dry sense of humour have been invaluable. Tim's most valuable attributes are his relentless enthusiasm, creativity and meticulous nature.

I would like to thank Tristan Lee for his ongoing friendship and for showing me how to maintain a healthy separation between striving to complete a PhD and enjoying all other aspects of one's life. I would also like to thank Dr. Todd Sanderson for his friendship and support, especially during the difficult early years of the PhD process. In the later years, I have developed a strong friendship with Evarn Ooi, with whom I credit many interesting intellectual and philosophical discussions. I'm also very thankful for his contribution to the editing process.

Among the other postgraduate students in the Department of Agricultural and Resource Economics, Elizabeth Bernold deserves special mention for promoting and stimulating a welcoming academic and social environment that will serve as an example to those who succeed her. Last, but by no means least, I thank my many friends for their assistance, encouragement or casual indifference. Some of these friends have already been named, but the others probably do not realise the contribution they have made.

For any errors or inadequacies that remain in this work, of course, the responsibility is entirely my own.

Statement of originality

This work has not been previously submitted for a degree or diploma in any University. To the best of my knowledge and belief, the dissertation contains no material previously published or written by another person except where due reference is made in the dissertation itself.

David John Rogers

Chapter 1. Introduction

The aims of this thesis are (1) to examine how active learning about unknown ecological threshold locations impacts optimal management decisions, (2) to examine the sensitivity of perceived optimal management decisions to the particular dynamics of an ecosystem, that is, the responsiveness of system ‘output’ to changes in system ‘input’, and (3) to examine the sensitivity of perceived optimal management decisions to the subjective choice of prior distribution for unknown threshold locations. Arrow *et al.* (1995) argue in favour of the use of adaptive management [active learning] to learn about the resilience of an ecological system. This thesis extends the existing literature by including the benefits and costs of active learning within a mathematical decision framework.

...ultimately, the resilience of systems may only be tested by intelligently perturbing them and observing the response using what has been called ‘adaptive management’.

Arrow *et al.* (1995, p. 93)

Ecological resilience is the ability of a system to experience disturbances or shocks, yet maintain the same basic structure, function and productivity. Loss of resilience, which is a movement of the system that takes it closer to a critical threshold, is potentially important for at least three reasons. First, crossing a threshold results in a sudden and non-marginal loss of biological productivity. Second, crossing a threshold may imply an irreversible change in the set of options available to both present and future generations. Third, crossing a threshold and experiencing a regime shift from a familiar to an unfamiliar regime, for which the dynamics are less well understood, increases the uncertainties associated with the environmental effects of economic activities (Arrow *et al.* 1995).

An uncertain and irreversible decision causes a reduction in the range of options available to a decision-maker and should result in an adjustment to the expected benefits of this decision (Hart 1942; Arrow and Fisher 1974). For risk averse decision-makers, this adjustment is downward (Cicchetti and Freeman 1971).

Therefore, the idea of an ‘option value’ is central to any mathematical framework that seeks to model uncertain or irreversible decisions over time. Ecosystems provide flows of goods and services that are valued by humans, but may be subject to reversible or irreversible regime shifts. Actions that increase the likelihood of crossing a critical threshold, and experiencing an undesirable regime shift, negatively affect the net present value of expected returns from that system.

In ecosystems, regime shifts may be reversible or irreversible and the exact locations of ecological thresholds may be known or unknown (Nævdal 2003; Nævdal 2006). Previous research has modelled optimal decisions in systems with hysteretic thresholds (Mäler *et al.* 2003), derived necessary conditions for optimal control when the locations of thresholds are unknown (Nævdal 2006; Nævdal and Oppenheimer 2007), and used stochastic dynamic programming to examine the effect of this form of uncertainty on risk averse behaviour (Brozovic and Schlenker 2011). However, these approaches do not incorporate a mechanism for modelling the iterative process of active learning about threshold locations or the benefits of engaging in active learning. This thesis demonstrates that learning about unknown threshold locations can be modelled by first partitioning the system’s state-space into regions that could and could not contain the threshold. Using this approach to model learning extends the ‘risk switching point’ concept introduced by Nævdal (2006).

The modelling framework developed in this thesis extends the line of research described above to consider the role of active learning about an unknown threshold location, and explicitly factor into current decisions the possibility of active learning. An important aspect of the decision problem is neglected if the impact of active learning is not considered. Explicitly factoring active learning into the decision problem acknowledges that the decision-maker is able to exert some control over the speed with which new information is acquired, as well as put this new information to use. The optimal degree of risk averse behaviour¹ is also likely to differ when active learning is considered, compared to the case when learning is not considered.

¹ Risk averse behaviour relates to management actions that move the system away from a critical ecological threshold.

1.1 Ecological systems

For many natural systems, it is possible for the system to exist in one of two or more alternative regimes. Walker and Salt (2006) define a system regime as a set of states, different combinations of underlying slow-moving variables, within which the system will exhibit the same basic structure and function. Examples of regime shifts include (Walker and Salt 2006):

- The large-scale conversion of vegetation in a wetland ecosystem from sawgrass- to cattail-dominated marsh as a result of increased nutrient inflow from agriculture.
- The shift of soil on agricultural land from non-saline to saline as a result of excessive irrigation bringing a saline groundwater table to the soil surface.
- The large-scale conversion of a savannah ecosystem from grass- to shrub-dominated because of increased grazing pressure in combination with low rainfall.
- The large-scale conversion of a reef ecosystem from hard coral- to fleshy seaweed-dominated as a result of overfishing and increased nutrient and sediment runoff from adjacent land.

For systems that can exist in two or more alternative regimes, these regimes are separated by thresholds that usually occur as functions of underlying slow-moving ecosystem variables (Scheffer *et al.* 2001; Walker and Meyers 2004). While the output of the system (sometimes also referred to as flows of goods and services from capital stocks) may display no noticeable change, an underlying slow ecosystem variable that ultimately determines the system's regime may be moving closer to a critical ecological threshold. When the underlying slow variable crosses a critical threshold, a regime shift results and the output of the system will experience an abrupt, non-marginal change. Threshold effects may come in one of three different forms: reversible along the same path, reversible but with a hysteretic return path, and irreversible (Walker and Salt 2006).

Hysteresis is characterised by path dependence. If the system's underlying slow variable crosses a critical threshold in one direction and causes a shift to an alternative regime, the system can only return to the original regime along a different

path of the underlying slow variable. This path will take the slow variable in the opposite direction and beyond the point where the first threshold was crossed. The general design of the model developed in this thesis means that it may be applied to systems with hysteretic dynamics, irreversibilities or threshold effects reversible along the same path of the underlying slow variable.

The ecological literature does not strictly use the metaphor of an ‘ecosystem as a production function’. However, it abounds with cases that fit this description (e.g. Sutherland 1974; May 1977; Carpenter *et al.* 1985; Friedel 1991; Noymeir 1995; Carpenter *et al.* 1999; Scheffer *et al.* 2001; Mäler *et al.* 2003; Collie *et al.* 2004; Petraitis and Dudgeon 2004; Casini *et al.* 2009; Suding and Hobbs 2009; Walker *et al.* 2010). More recently, Baskett and Salomon (2010) examine sea urchin-algae interactions in temperate rocky streams, and Gil-Romera *et al.* (2010) examine bush encroachment in savannah environments. In both cases, ecosystem dynamics are represented using diagrams very similar to Figure 2-1 below (Baskett and Salomon 2010, p. 1768; Gil-Romera *et al.* 2010, p. 624). The level of one ecosystem variable, the system ‘output’, is expressed as a function of another ecosystem variable, the system ‘input’ or underlying slow-moving variable.

In this thesis, a model is developed for a system with two regimes and two ecological thresholds, which represents the simplest general case for an ecosystem with multiple alternative regimes. Such an example is presented in Baskett and Salomon (2010), where algae population is modelled as a function of herbivore mortality. For the model developed in this thesis, the two alternative regimes are ‘high functioning’ and ‘low functioning’. In purely economic terms, the high functioning regime produces greater value. The location of each threshold is assumed unknown; however, a prior probability distribution for its location can be postulated based on historical data, experimental data and/or expert knowledge. A process of active learning occurs every time the system is perturbed, and the aforementioned probability distribution is continually updated as new information becomes available. This information is simply whether or not the relevant critical ecological threshold has been crossed during a particular time period. If a threshold is crossed, the system will shift regimes. The new regime will have different dynamics and a different threshold, compared to the previous regime.

1.2 Modelling and managing ecological systems

Complex ecological systems are characterised by adaptability, unpredictability, feedback effects and thresholds. In contrast, the complicated ecological systems modelled in this thesis are mostly deterministic, yet are susceptible to threshold effects. These natural systems can be represented in the form of a production function² and can be extended to examples where the system output is not easily quantifiable, is a function of multiple inputs and/or is susceptible to threshold effects. Examples include production function models of shallow lake systems (Mäler *et al.* 2003; Brozovic and Schlenker 2011) and models of both genetic and species diversity (Eldridge 2010; Lee *et al.* 2010; Clark *et al.* 2011). Models of ecological systems represent simplifications of the true production functions, the actual ecological dynamics, at play. However, such simplification is necessary to maintain parsimony and tractability when modelling complex or complicated systems.

Many ecological systems are susceptible to threshold effects (Mäler *et al.* 2003; Walker and Salt 2006; Baskett and Salomon 2010; Gil-Romera *et al.* 2010). A threshold is a level of an underlying slow-moving variable where the feedbacks to the rest of the system change; the system shifts to an alternative regime. For example, if the concentration of phosphorus (slow variable) within a shallow lake system becomes too high (crosses a threshold), biological processes within the lake system will be altered, and the lake will change (regime shift) from a clear (oligotrophic) regime to a dirty, murky (eutrophic) regime (Mäler *et al.* 2003). Threshold effects can be partially or fully reversible, or completely irreversible. They represent natural physical limits to substitution between different system inputs. Further, much of the foundational theory and numerous examples of dynamic modelling and optimal control of natural resources is discussed, at length, in Clark (1976).

The nature of the decision-maker's problem, and the character of the resulting optimisation problem, is determined by whether the locations of critical ecological thresholds are known or unknown. In the case where the threshold location is known,

² This form of model specification represents a reductive model of ecosystem function. It by no means captures the true complexity of a complex ecological system, but is necessary to develop a tractable framework for modelling the process of active learning about unknown threshold locations.

the historically standard approach to modelling thresholds was to define the threshold as a state constraint, where the level of the relevant state variable cannot cross the pre-defined threshold level (Seierstad and Sydsaeter 1987; Perrings and Pearce 1994). This approach is problematic for two reasons. First, it is unrealistic to assume that the initial state of the ecological system will always be on the ecologically desirable side of the threshold (i.e. the state constraint has not already been violated). Second, even if the initial state of the system is on the ecologically desirable side of the threshold, in some cases, it may be economically optimal to cross to the other side of the threshold, either briefly or permanently (Nævdal 2001).

In many cases, the exact locations of critical ecological thresholds are unknown to the decision-maker. However, a prior probability distribution can be postulated for an unknown threshold location, based on historical data, experimental data and/or expert knowledge. For a system with a threshold of unknown location, deliberately perturbing the system in the direction of the threshold increases the probability of learning about its location. This process will result in efficiency gains if it is learnt that lower amounts of active control are required to maintain the system in the preferred regime. However, perturbing the system toward the threshold also increases the probability of experiencing an undesirable regime shift and being required to recover the system to the preferred regime.

Prior to this thesis, Nævdal (2006) and Nævdal and Oppenheimer (2007) came closest to including learning about unknown threshold locations within an optimal control framework. Nævdal (2006, p. 1134) suggested the concept of a “risk switching point” that allows the system’s state space to be partitioned into two regions. On one side of the boundary, the decision-maker has information sourced from observations of the system for certain combinations of the state variables (the system state). The ecological threshold is known to have not been crossed for these combinations of the state variables. Therefore, the ecological threshold must be located on the other side of the boundary.

The mathematical framework developed in this thesis extends the concept of the ‘risk switching point’ by using it as a starting point for modelling the benefits of

engaging in active learning in order to reduce or resolve uncertainty³ about unknown threshold locations. Failure to incorporate the benefits of active learning represents an oversight of existing models and is likely to result in perceived optimal outcomes actually being sub-optimal.

1.3 The research problem

When confronted with an uncertain outcome, one may postulate a prior probability distribution that captures the likelihood of a particular event occurring. This means the outcomes are characterised in terms of Knightian risk (Knight 1921). Likewise, when confronted with a threshold of unknown location, the decision-maker may postulate a prior probability distribution that captures the likelihood of a particular location being that of the ecological threshold. Over time, more information is acquired and the extent of uncertainty diminishes. This new information will be factored into management decisions made in the future.

When considering any future management decision, it is necessary to acknowledge that the information set available when making such a decision in the future may, and is highly likely to, differ from the current information set. This section of the introduction summarises the main research problems of the thesis and provides an overview of the methodology, research questions and propositions. Specifically, these research questions concern: the impact of active learning on management decisions, the impact of particular system dynamics on management decisions, and the sensitivity of perceived optimal management decisions to the subjective choice of prior for an unknown threshold location.

Section 1.3.1 outlines the research aims. Section 1.3.2 provides an overview of the methodology used to develop the mathematical modelling framework. Section 1.3.3 describes the research questions and propositions of the thesis.

³ Measured by assuming Knightian risk, rather than Knightian uncertainty.

1.3.1 Research aims

The focus of this thesis is the development of a mathematical framework for the management of ecological systems that are susceptible to threshold effects, where the exact locations of thresholds are unknown to the decision-maker. Despite a lack of perfect knowledge on the part of the decision-maker, it is still possible to postulate a prior probability distribution for each unknown threshold location. The decision-maker is able to learn about the location of a threshold by engaging in active learning. This involves deliberately perturbing the system in the direction of the threshold in order to learn about its location. The benefits of engaging in active learning must be included in the decision-maker's objective function, otherwise they will pursue an incorrectly specified objective. After including the benefits of active learning, the objective of the decision-maker is to make management decisions 'today' (i.e. $t = 1$) that maximise the expected net present value of the stream of net benefits flowing from the ecosystem, while explicitly acknowledging that the information set available in future will be conditional on the management actions undertaken 'today'.

This thesis extends previous research to model the effect on optimal decisions of learning about the locations of critical ecological thresholds. Learning is modelled using a process of adaptive management. A dynamic programming framework is developed and applied to various ecological contexts and used to demonstrate the role of learning. In modelling terms, this means that after each time period (this may be days, weeks, years, etc.), a (hypothetical) signal is provided by the ecosystem, in the form of whether or not a critical ecological threshold was crossed during that time period. This new information can then be used by the decision-maker to update prior beliefs, summarised by the prior probability distribution for the unknown threshold location, to form a posterior distribution. This posterior distribution will form the basis of management decisions in the following time period, and the process is repeated.

Previous research has modelled optimal decisions in ecological systems with hysteretic thresholds (Mäler *et al.* 2003), derived necessary conditions for optimal control when the locations of thresholds are unknown (Nævdal 2006; Nævdal and

Oppenheimer 2007), and used stochastic dynamic programming to examine the effect of this form of uncertainty on risk averse behaviour (Brozovic and Schlenker 2011). However, existing modelling frameworks do not adequately capture the benefits of active learning about unknown threshold locations. In fact, Brozovic and Schlenker (2011) explicitly state that allowing the decision-maker to learn about the uncertain threshold location through time would be a valuable extension to their analysis. This thesis thereby examines the role of active learning in making optimal decisions for the management of ecological systems. This research problem has three components, which are (1) to examine how active learning about unknown ecological threshold locations impacts optimal management decisions, (2) to examine the sensitivity of perceived optimal management decisions to the particular dynamics of an ecosystem, that is, the responsiveness of system ‘output’ to changes in system ‘input’, and (3) to examine the sensitivity of perceived optimal management decisions to the subjective choice of prior distribution for unknown threshold locations.

This new modelling framework captures the trade-off faced by the decision-maker between the costs of crossing a threshold and shifting to an undesirable alternative regime, and the benefits of learning about the threshold location. Explicitly considering the value of information, or the benefits of active learning, means the decision-maker will generally make decisions that incur a greater risk of crossing the threshold in order to learn about its location. This finding is independent of the initial prior probability distribution used to model the unknown threshold location and the type of ecosystem dynamics considered. Generally, the omission of active learning from the problem structure results in sub-optimal management decisions that incur less risk than is deemed optimal by using a problem structure that explicitly includes the benefits of active learning.

For a system with a reversible threshold, the decision-maker will engage in riskier behaviour for two reasons. First, if more is learnt about the location of a threshold without actually crossing it, the decision-maker will know that they can operate in a what is likely to be a more profitable region of the system’s state-space in future time periods without any risk of crossing the threshold. Second, if the threshold is crossed, the information acquired while crossing this threshold can be put to use in future time periods. However, this will only be possible after the system has

recovered to the preferred regime. For a system with an irreversible threshold, only the first reason for riskier behaviour is applicable because there is no possibility of the system recovering from the less preferred to the preferred regime. However, for both threshold types, the benefits of active learning are omitted from a standard dynamic optimisation model that overlooks the role of learning.

1.3.2 An overview of the methodology

This thesis develops a mathematical modelling framework that demonstrates the role of active learning in making optimal decisions concerning the management of ecological ecosystems. The use of a mathematical model removes the need for reliance on intuition when making management decisions. Instead, it allows for results that might seem counter-intuitive, but are actually optimal for a rational decision-maker. For example, deliberately perturbing a system in the direction of a threshold for the purpose of learning about its location.

The model is applied to various ecological contexts, including numerical simulations of a shallow lake ecosystem. Active learning is modelled by updating the prior probability distribution for a threshold's location and by adjusting the boundary between the regions of the system's state-space that could and could not contain the threshold. The model captures the trade-off faced by the decision-maker between the costs of crossing a threshold and shifting to an undesirable alternative regime, and the benefits of learning about the threshold's location.

The important components of the model are as follows:

1. *An ecosystem susceptible to threshold effects.* Crossing a critical ecological threshold causes abrupt, non-marginal changes to the output of the system. Most ecosystems can exist in more than one regime (Walker and Salt 2006). The most general example is a system with hysteretic dynamics. This means that a regime shift is reversible but along a different path of the underlying slow ecosystem variable than the path that caused the original regime shift. The simplest case of hysteretic dynamics is an ecosystem with two alternative system regimes and two critical

ecological thresholds. It is quite easy to show that this general model design can also be applied to systems with irreversibilities or threshold effects reversible along the same path of the underlying slow variable (see Section 2.1).

2. *Fixed but unknown threshold locations.* The locations of the two critical ecological thresholds are assumed to be at fixed positions within the X -space (where X denotes the underlying slow variable). However, these positions are not known to the decision-maker at the beginning of the problem. Instead, a prior probability distribution is postulated for each threshold's position, based on historical or experimental data, or expert knowledge. Over time, more is learnt about the system and the bounds of these prior probability distributions are gradually tightened as more information is (hypothetically) acquired about the locations of thresholds.

3. *Regime-specific production functions.* The two alternative system regimes are modelled as two distinct production functions. System output for Regime i (Y^i) is modelled as a function of system input (X), which is an underlying slow ecosystem variable. These production functions are deterministic in the sense that if the system input (X) is known with certainty, the system output (Y) is also known with certainty. However, if the system input is a random variable, the system output (as a direct function of X) will also be a random variable. A few simple examples of these production functions include fish population as a function of water quality, koala population as a function of habitat size or connectivity, and crop yield as a function of the depth of a saline groundwater table.

4. *The objective of the decision-maker.* The decision-maker's objective is to maximise the net present value (NPV) of expected utility, generated directly and/or indirectly from the system, over a finite time horizon. This is achieved by managing the control variables relating to controlling the underlying slow variable (C) and the amount of harvest effort undertaken for either system output or input (E). The level of consumption expenditure (q) must also be chosen for each time period.

Utility can be obtained from two main sources: (i) consumption utility, which results from consumption expenditure paid for by wealth acquired from any profits

generated by the system; and (ii) direct utility, which is a direct function of the current output of the system but does not require that the output be harvested. For example, direct utility may be obtained from the aesthetic value of a system, which is a function of the population size of a particular bird species. Consumption utility is temporally transferable because acquired wealth can be stored and used for consumption in a future time period. Direct utility, on the other hand, cannot be stored and can only be enjoyed in the current time period.

The two other sources of utility are only applicable in the terminal time period, T . These sources are (iii) utility from terminal wealth, which captures the bequest value (either to a representative agent or individual) of any wealth remaining at the end of the terminal time period; and (iv) the scrap value of the system, which captures utility that may be sourced based on the system's future productive potential and is conditional on the state of the system at the end of the terminal time period. Implicit in (iii) and (iv) is an assumption that both the system and society will continue to exist beyond the finite time horizon modelled.

5. *Updating of prior beliefs (Learning)*. This model assumes Knightian risk (Knight 1921) rather than Knightian uncertainty. The risk context considered is one where the system is deterministic, but the decision-maker possesses incomplete knowledge of it. Exact threshold locations are unknown, but prior probability distributions can be postulated based on historical data, experimental data and/or expert knowledge. A process of learning occurs (i.e. a signal is provided) every time the system is (hypothetically) perturbed. The probability distributions for threshold locations are continually updated as new (hypothetical) information becomes available and the bounds of these prior probability distributions are progressively tightened over time.

Use is made of (hypothetical) active learning to capture how individuals and decision-makers actually make management decisions. That is, to choose what appears to be the best course of action (e.g. level of investment) at the current point in time, conditional on current levels of information and risk. Then, as new information is acquired, and levels of information and risk change, the best course of action is revised to reflect the most recent information set.

6. *Solution method is dynamic programming.* The economic problem analysed in this thesis takes the form of a dynamic optimisation, where the objective is to maximise the expected NPV of utility, generated directly and/or indirectly from the system, over a finite time horizon. The problem is a dynamic optimisation because of the presence of several state variables. Generally, these state variables are: system output (Y), system input (X), acquired wealth (W), and the parameters that determine the shape and/or bounds of the prior probability distributions for unknown threshold locations. The control variables relate to actions (C) that impact on the level of the underlying slow variable, the amount of harvest effort (E) undertaken for either system output or input, and the amount of consumption expenditure (q) for each time period.

The dynamic nature of the problem means that the optimisation problem must be solved recursively. The problem must be first solved for the terminal time period, T , for all feasible levels of the abovementioned state variables. This process will generate a value function that captures the maximum expected utility generated from the system during the terminal time period, T , as a function of the levels of the state variables at the beginning of the same time period. The value function for time period T then forms part of the value function for time period $T - 1$, which captures the maximum expected utility generated from the system during time periods $T - 1$ and T , as a function of the levels of the state variables at the beginning of the time period $T - 1$. This iterative process is repeated until a value function is generated for the first time period of the problem, which captures the maximum expected NPV of utility generated from $t = [1, T]$. The optimal levels of the control variables can then be determined as functions of the initial levels of the state variables. The model simulations are solved using numerical techniques.

1.3.3 Research questions and propositions

To examine the impact on management decisions of active learning about unknown ecological threshold locations, different types of ecosystem dynamics, and the subjective choice of prior distribution for unknown threshold locations, the following research questions and propositions were formulated, based on a review of the relevant literature.

Question 1: What is the impact on expected returns of active learning about unknown ecological threshold locations?

P1.1: The maximum expected NPV of utility generated by an ecosystem will be *higher* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

P1.2: For the first time period of the problem, the amount of effort invested into controlling the ecosystem away from an undesirable threshold (i.e. investment in risk averse actions) will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

P1.3: The value of information (i.e. the difference between the expected NPV of utility when active learning is included compared to when it is *not* included) will have a non-monotonic relationship with the initial level of uncertainty about the threshold location (as measured by standard deviation of prior probability distribution).

Question 2: How are management decisions affected by different types of ecosystem dynamics?

P2.1: For a system with hysteretic dynamics, the optimal level of first-period investment in effort to control the system away from an *undesirable* threshold will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

P2.2: For a system with hysteretic dynamics, the optimal level of first-period investment in effort to control the system in the direction of a *desirable* threshold will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

P2.3: For a system with an irreversibility, the optimal level of first-period investment in effort to control the system away from an undesirable threshold will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

Question 3: Are perceived optimal management decisions sensitive to the subjective choice of prior distribution for unknown threshold locations?

P3.1: For a system with hysteretic dynamics that begins the problem in the *economically-preferred* regime, the optimal level of first-period investment in effort to control the system away from the *undesirable* threshold will be *more* sensitive to changes in the *closer* bound (i.e. crossing this bound means the probability of a regime shift is positive) than changes in the *further* bound (i.e. crossing this bound means the threshold is crossed with certainty) of the prior probability distribution for the unknown threshold location.

P3.2: For a system with hysteretic dynamics that begins the problem in the *economically-non-preferred* regime, the optimal level of first-period investment in effort to control the system in the direction of the *desirable* threshold will be *more* sensitive to changes in the *closer* bound (i.e. crossing this bound means the probability of a regime shift is positive) than changes in the *further* bound (i.e. crossing this bound means the threshold is crossed with certainty) of the prior probability distribution for the unknown threshold location.

P3.3: For a system with an irreversibility that begins the problem in the *economically-preferred* regime, higher uncertainty⁴ about the location of the (sole) *undesirable* threshold will result in a lower optimal level of first-period investment in effort to control the system away from the *undesirable* threshold.

Question 4: How are management decisions affected by the relative contributions of direct and consumption utility to total utility?

P4.1: For a system where utility is sourced from both consumption and direct utility, the optimal level of first-period investment in effort to control the system away from the *undesirable* threshold will be higher *ceteris paribus* when more weight is placed on direct utility, relative to consumption utility.

⁴ Measured as the standard deviation of the prior probability distribution.

1.4 An outline of the thesis

This thesis continues with a literature review in Chapter 2. Chapter 3 presents the modelling framework developed to include the role of active learning within the decision-maker's problem. Chapters 4 and 5 outline several representative case studies of ecosystems that are suitable for simulation exercises. Chapter 6 summarises and discusses the results for each research question using data from model simulations conducted for a shallow lake ecosystem. Chapter 7 discusses the broader implications of the research findings, presents ideas for future research and concludes the thesis.

Table 1-1 below summarises the key features of the four representative case studies described in Chapters 4 and 5.

Table 1-1 Key features of the four representative case studies

Characteristic	Shallow lake	Savannah	Reef fishery	Koalas
Threshold effect	Hysteretic dynamics	Completely reversible	Completely reversible	Hysteretic dynamics
Underlying slow variable	Phosphorus concentration	Grass cover – fuel load	Reef-mangrove distance	Habitat quality
Trade-offs	Better water quality in lake system vs. Harvesting crops for profit	Non-use values from biodiversity in the rangeland system vs. Grazing livestock for profit	Non-use values from the fish population vs. Harvesting fish for profit	Non-use values from koala population vs. Utility from home ownership
Ecological dynamics	Nutrient cycling where phosphorus is added via natural and anthropogenic processes and partially assimilated by the system	Grass and tree prevalence modelled using a space implicit model, where trees are the superior competitor	Fish population modelled using a Lotka-Volterra predator-prey model	Koala population modelled using a logistic growth function
Sources of utility	<i>Direct utility:</i> Recreational use of lake – function of water quality <i>Consumption utility:</i> Paid for using profits from cropping activities	<i>Direct utility:</i> Existence and aesthetic value – function of biodiversity <i>Consumption utility:</i> Paid for using profits from grazing activities	<i>Direct utility:</i> Existence and aesthetic value – function of the fish population <i>Consumption utility:</i> Paid for using profits from fishing activities	<i>Direct utility:</i> Existence and aesthetic value – function of the koala population <i>Direct utility:</i> Utility from home ownership

Summary of introduction

This thesis extends previous research by presenting a mathematical framework to be used for managing ecological systems with thresholds of unknown location. Models that fail to take account of the benefits of reducing or resolving uncertainty⁵ about the locations of thresholds omit an important component of the problem structure. This omission is likely to result in perceived optimal management decisions actually being sub-optimal.

The mathematical framework is used to model the effect on optimal decisions of learning about the locations of thresholds. Learning occurs via a process of adaptive management. The process of learning is modelled by updating the prior probability distribution for a threshold's location and by adjusting the boundary between the regions of a system's state-space that could and could not contain the threshold, as new information is acquired. The model captures the trade-off faced by the decision-maker between the costs of crossing a threshold and shifting to an undesirable alternative regime, and the benefits of learning about the threshold location. Consideration of both the benefits and costs of active learning means the decision-maker will generally make decisions that incur a greater risk of crossing the threshold in order to learn about its location.

⁵ Measured by assuming Knightian risk, rather than Knightian uncertainty.

Chapter 2. Literature review

An ecological system can be represented in the form of a production function, by way of a reductive model. In this way, the system output (Y) can be represented as a function of one or more system inputs (X), where X may be a scalar or vector. Typical for the management of productive systems is the use of optimisation techniques, or the application of benefit-cost analysis (BCA) or similar valuation tools (e.g. Nævdal 2006; Hanley and Barbier 2009; Balıkcıoğlu *et al.* 2011). However, the management of ecological systems is made challenging by the presence of critical ecological thresholds (e.g. Sutherland 1974; May 1977; Carpenter *et al.* 1985; Friedel 1991; Noymeir 1995; Carpenter *et al.* 1999; Scheffer *et al.* 2001; Mäler *et al.* 2003; Collie *et al.* 2004; Petraitis and Dudgeon 2004; Casini *et al.* 2009; Suding and Hobbs 2009; Walker *et al.* 2010). The management of these systems is complicated further if the precise locations of thresholds are unknown, which represents a key challenge to the production function approach to managing ecological systems.

This chapter reviews the methodologies that have been developed to incorporate the components of ecosystems that are not present in simple productive systems.⁶ These include methods for capturing uncertainties within the ecological system and appropriate methods for managing or controlling a system when confronted with these uncertainties. The most notable of these uncertainties are thresholds of unknown location. This chapter describes the development of these methods and evaluates their relative merits for modelling and managing ecological systems.

2.1 Managing ecological systems with unknown threshold locations

Complex ecological systems are characterised by adaptability, unpredictability, feedback effects and thresholds. They are typically self-organising,

⁶ Please note that the author acknowledges the conceptual limitations of the conceptual framework developed in this thesis. The systems modelled here can be classified as complicated, rather than complex. This is because the conceptual framework does not capture the properties of emergence and self-organisation that characterise a complex system. Nonetheless, a discussion of the broader literature is pertinent at this point.

which means a change to one component will cause other components to respond and adapt (Levin 1998; Walker *et al.* 2004). Many natural systems are able to exist in one of two or more alternative regimes, where ecosystem function, dynamics and ‘output’ are different in each regime. For such systems, these regimes are separated by thresholds that usually occur as functions of underlying slow-moving ecosystem variables (Scheffer *et al.* 2001; Walker and Meyers 2004). Although the underlying slow variable may be in a continual state of flux, the output of the system will noticeably change only if a critical ecological threshold is crossed. These threshold effects may be completely or partially reversible, or completely irreversible (Walker and Salt 2006). All of the previously described characteristics, but particularly the presence of thresholds, create challenges for modelling ecological systems and using standard optimisation techniques. However, these challenges are not insurmountable.

An ecological system can be represented in the form of a production function, where the system output (Y) is modelled as a function of one or more system inputs (X). Some examples include (i) crop yield expressed as a function of water and fertiliser application rates (e.g. Anselin *et al.* 2004; Malzer *et al.* 2004; Hurley *et al.* 2005; Tumusiime *et al.* 2011), (ii) manufacturing output expressed as a function of labour, capital and other inputs (e.g. Dobbelaere and Mairesse 2008; Li 2010; Lizal and Galuscak 2012), and (iii) the population of a fish species at a particular point in time expressed as a function of the previous population level and the amount of harvest effort exerted (e.g. Hannesson 2002; Ding and Lenhart 2009; Sarkar 2009; Ewald and Wang 2010).

These examples describe system outputs that are easily observable and quantifiable, and generated by both natural and artificial systems. In the case of natural systems, the metaphor of an ‘ecosystem as a production function’ can also be extended to examples where the system output is not easily quantifiable, is a function of multiple inputs and/or is susceptible to threshold effects (Sutherland 1974; May 1977; Carpenter *et al.* 1985; Friedel 1991; Noymeir 1995; Carpenter *et al.* 1999; Scheffer *et al.* 2001; Mäler *et al.* 2003; Collie *et al.* 2004; Petraitis and Dudgeon 2004; Casini *et al.* 2009; Suding and Hobbs 2009; Walker *et al.* 2010). Examples include production function models of shallow lake systems (Mäler *et al.* 2003; Brozovic and Schlenker 2011) and models of both genetic and species diversity

(Eldridge 2010; Lee *et al.* 2010; Clark *et al.* 2011). The ecosystem services provided by a shallow lake system (output) can be modelled as outputs that are conditional on the concentration of phosphorous within the lake (input). Biodiversity within a geographic area (output) can be modelled as a function of habitat size, shape and connectivity, and within-system feedbacks, plus a multitude of exogenous drivers, such as temperature and precipitation (multiple inputs). Biodiversity and ecosystem services can provide non-use values such as option, bequest and existence values (Hanley *et al.* 2007). Further, at least some of the inputs, or ecological drivers, that determine the levels of outputs, such as biodiversity and ecosystem services, can be controlled through deliberate management actions. Examples of these actions include water filtration to remove undesirable chemicals or nutrients, and the planting of wildlife corridors to improve habitat connectivity. For an optimisation problem, these actions are referred to as control or choice variables.

Models of ecological systems represent simplifications of the true production functions, the actual ecological dynamics, at play. However, such simplifications make the models more tractable and allow them to be solved using a standard optimisation framework, which involves using standard differential calculus for either a constrained or unconstrained optimisation. Key ecological variables are considered to be inputs that can be controlled, or managed, in order to maximise net benefits⁷ accrued from the system's output.

Ecological systems are usually not adequately represented by a single continuous production function. They are 'complex' systems because the many linkages and feedbacks between their components mean that it is almost impossible to predict with certainty what the response will be to any intervention in the system (Walker and Salt 2006). Ecosystems are also adaptive. A complex adaptive system consists of many components that are only loosely connected and typified by variation and unpredictability. This type of system is self-organising, which means that a change to one component will cause other components to respond and adapt. Therefore, complex adaptive systems are characterised by (i) both independent and interacting components, (ii) a selection process that works on these components, and

⁷ Net benefits are expressed in utility or monetary units.

(iii) variation and novelty over time, resulting from the addition and/or removal of components.⁸

Several different analogies have been proposed to describe the manner in which ecological thresholds operate, and the concept of ecological resilience. The first of these analogies is the ‘Ball in a Basin’ metaphor. Here, the ‘state of the system’ is defined by the levels of n state variables, which represent the ‘ball’. Each ‘basin’ represents a different system regime, where the ‘ball’ may move between ‘basins’ and is inclined to move to the bottom of the ‘basin’, toward a stable equilibrium, unless subjected to external forces (Walker and Salt 2006).

A simpler, two-dimensional representation of an ecosystem is proposed by Walker *et al.* (2010). The dependent variable is the state of the capital stock, which can also be described as the ‘fast’ variable or the ‘system output’. The independent variable is the underlying variable, which can also be described as the ‘slow’ variable or ‘system input’. The system output can be modelled as a function of the system input. However, the system may be susceptible to one of several different types of threshold effect. These threshold effects may be (b) completely reversible, (c) reversible, but along a hysteretic return path, or (d) irreversible (see Figure 2-1). Such simplifications of complex ecological systems provide parsimony when demonstrating theoretical concepts and allow applied dynamic optimisation problems to be solved. Dynamic optimisation problems grow exponentially with the number of variables that describe the state of the system. This concept is otherwise known as the ‘curse of dimensionality’ (Bellman 1957).

The model formulation used in this thesis closely follows that proposed by Walker *et al.* (2010); more specifically, case (c) of Figure 2-1 below. Case (c) represents a system with two alternative regimes and hysteretic dynamics, which means that one threshold must be crossed for the system to shift from the first to the second regime, but a different threshold must be crossed for the system to shift from the second to the first regime. Intuitively, it can be seen that cases (b) and (d) are

⁸ The ecological systems modelled in this thesis are complicated systems, rather than complex systems. However, the management of complex systems represents a possible extension to this model.

merely variations of case (c), where these two thresholds either converge, as for case (b), or infinitely diverge, as for case (d).

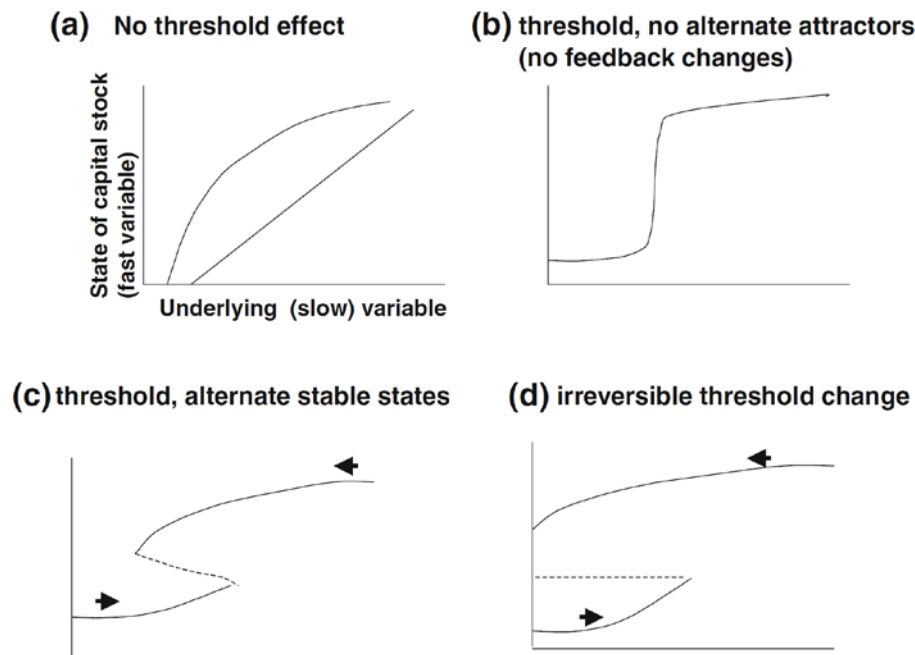


Figure 2-1 Ecosystem dynamics and threshold effects

Source: Walker *et al.* (2010)

The presence of ecological thresholds and the inherent unpredictability of complex ecological systems have led some to advocate a resilience approach to management and dismiss an optimisation approach, as is typically used by economists in a wide variety of settings. For example, Walker and Salt (2006) argue that, for some period of time, it will be possible to manage or control an ecological system within a narrow range of different system states. This can be achieved by breaking a system into its individual components and understanding how each functions in isolation. However, if insufficient consideration is given to feedback effects and the possibility of a threshold being crossed, the system is probably being managed inappropriately. Nevertheless, we will see later in this chapter that an optimisation approach can adequately address these concerns.

It is important to acknowledge that there are biological and physical limits to substitution between different types of capital and between different sources of natural capital (Dasgupta and Mäler 2001; Dietz and Neumayer 2007). The terms

‘substitution’ and ‘substitutability’ relate to the possibility of maintaining the same level of a chosen output (e.g. automobiles or biomass) by substituting between different inputs (Pindyck and Rubinfeld 2005). For example, first, the same number of automobiles can be produced if the number of workers (labour) is reduced, but this reduction is compensated with an increase in the use of machinery and mechanisation (physical capital). Second, it may be possible for an ecosystem to sustain the same amount of biomass following a reduction in its physical size, if the connectivity of the ecosystem is improved. In this sense, an ecological threshold can be considered a natural physical limit to substitution between different system inputs; an unavoidable limit to substitutability. In the case of crossing an ecological threshold, the marginal rate of substitution⁹ between different system inputs is effectively zero because no amount of substitution between different system inputs can maintain the previous level of system output. For example, in the context of fish breeding within a population, the last fertile male fish cannot be substituted with another female fish.

The characteristics of ecological systems discussed above create complications for modelling and using standard optimisation techniques. These complications are not insurmountable, but instead add layers of complexity to the modelling problem. Critical ecological thresholds represents limits to substitutability between system ‘inputs’ and crossing a threshold results in an alteration of the system’s dynamics. In this thesis, ecosystem dynamics are modelled in the form of a production function, with separate functions for each alternative system regime.

2.2 Modelling the management of ecological systems

Several alternative methods have been proposed for modelling the management of ecological systems. These range from mathematical techniques, such as marginal resilience pricing and optimal control, to decision heuristics, such as safe minimum standards, the precautionary principle, minimax and minimax-regret. Option theory and the burden of proof are also discussed in the context of the precautionary principle.

⁹ The marginal rate of substitution is the ratio for which one input can be substituted for another, while maintaining the same level of output.

2.2.1 Marginal resilience pricing

Many ecological systems are complex adaptive systems. It is the ability of these systems to adapt and maintain the same basic function, even after the system state has changed, that is a source of economic value in addition to the standard input-output relationship. In this sense, a system's stock of resilience can be thought of as a form of insurance. New information, in the form of learning about an unknown threshold location, is akin to reducing the risk premium that makes such an insurance contract actuarially fair. A system's ability to recover, or 'bounce back', from a perturbation (i.e. its resilience) can make new information about threshold locations especially valuable. However, the value of new information will be lower in the case of a system with an irreversible threshold. The reason for this distinction between reversible and irreversible thresholds relates to the concept of a quasi-option value, which results from the value of information gained by delaying an irreversible decision (Arrow and Fisher 1974; Freeman 2003). For example, new information gained after experiencing an irreversible regime shift is of little, or no, value because this new information cannot be put to use.

If ecological resilience is defined as the physical distance between the underlying slow variable and the critical ecological threshold, the economic value of an environmental asset can be modelled using an approach known as marginal resilience pricing. Here, a system's 'stock' of resilience is viewed as a capital asset that can be valued (Mäler *et al.* 2007; Walker *et al.* 2010). A more resilient system can absorb a larger shock (disturbance) without shifting into an alternative system regime (Holling 1973; Walker *et al.* 2004) and *ceteris paribus* will have a higher economic value than a less resilient system.

Walker *et al.* (2010) use the example of agricultural land above a saline water table. If the saline water table encroaches within two metres of the soil surface, capillary forces will draw saline water to the soil surface and cause the land to shift from a non-saline to a less productive saline regime. The stock of resilience is defined as the physical distance of the water table from the two-metre critical threshold, such that a water table depth of seven metres would correspond to five metres of resilience

stock. The subsequent valuation of resilience is calculated as a function of (i) the physical distance of the system from the critical ecological threshold, (ii) the associated conditional probability¹⁰ of a regime shift occurring in the future, and (iii) the difference in expected income or utility streams between the current economically-preferred regime and the non-preferred alternative regime.

2.2.2 Dynamic optimisation with known or unknown threshold locations

Dynamic optimisation involves determining optimal levels of control variables, conditional on known or estimated model parameter values. For example, a decision problem may involve determining optimal levels for: fertiliser application rates, the number of trees planted, rates of mechanical pumping of saline groundwater, the amount of fishing effort undertaken, etc. The exact locations of critical ecological thresholds may be known or unknown. Therefore, threshold location is an important model parameter. If the threshold location is known and deterministic, the model parameter is a scalar, whereas, if the threshold location is unknown, the model parameter can be considered as a random variable with an associated probability density function.

Literature concerning optimal control in the presence of deterministic thresholds of known location (e.g. Farzin 1996; Nævdal 2001; Nævdal 2003; Keller *et al.* 2004) has developed in parallel with the literature concerning the optimal control of systems with unknown threshold locations or stochastic thresholds. This literature has developed from the fields of resource economics (Cropper 1976; Kemp 1976; Tsur and Zemel 1995; Tsur and Zemel 1996; Nævdal 2003; Weitzman 2003), and research and development (R&D) (Lucas 1971; Kamien and Schwartz 1991; Zemel *et al.* 2001).

In resource economics, this type of model is most commonly applied to the extraction of non-renewable resources when the stock size is unknown. However, this problem is formally equivalent to the management of a system with an unknown

¹⁰ A conditional cumulative probability distribution is used to estimate the probability of a regime shift having occurred at any time up to pre-determined future point in time. This probability is conditional on the current 'stock' of resilience.

threshold location, given certain conditions (Nævdal 2006). More recent applications are decisions involving potentially catastrophic outcomes and catastrophic risk, such as human-induced climate change (Tsur and Zemel 1996; Nævdal 2003; Nævdal 2006; Nævdal and Oppenheimer 2007), and the management of nutrient loads in shallow lake systems with thresholds of unknown location (Mäler *et al.* 2003; Brozovic and Schlenker 2011).

In the optimal control theory literature that relates to ecosystems, underlying slow ecosystem variables are managed across a finite or infinite time horizon with the aim of maximising the net present value of utility sourced from the ecosystem (e.g. Mäler *et al.* 2003; Nævdal 2003; Nævdal 2006; Brozovic and Schlenker 2011). Essential to this framework is the imposition of Knightian risk (Knight 1921). This means that, although exact locations may be unknown, probability distributions can be postulated for the unknown threshold locations.

For a system with a threshold of unknown location, deliberately perturbing the system in the direction of the threshold increases the probability of learning about its location and may result in future cost savings, or efficiency gains, in terms of the amount of active control required. Perturbing the system toward the threshold also increases the probability of experiencing an undesirable regime shift and being required to recover the system to the preferred regime. Peterson *et al.* (2003) suggest that continual cycles of collapse and recovery may be optimal; however, Brozovic and Schlenker (2011) show that this result is an artefact of an assumption that the decision-maker is *not* allowed to condition their behaviour on the observed state of the system. If allowed to condition their behaviour on the observed state, repeated collapses and recoveries will not occur (Brozovic and Schlenker 2011).

Nævdal (2006) and Nævdal and Oppenheimer (2007) have come closest to including learning about unknown threshold locations within an optimal control framework. Nævdal (2006, p. 1134) identifies what he coins a “risk switching point”. Effectively, the risk switching point represents the boundary of the current data set. On one side of the boundary, the decision-maker has information sourced from observations of the system for certain combinations of the state variables (the system state). One important piece of information is that the threshold is not located at any of

the aforementioned system states. Instead, the threshold must be located on the other side of the boundary. Therefore, crossing the risk switching point will instantly result in a positive probability of crossing the ecological threshold. Partitioning the state space in this manner is one of the important steps of modelling active learning about thresholds of unknown location.

Although not the focus of this thesis, robust control is a powerful alternative toolbox for managing systems with uncertain parameters or dynamics, such as ecosystems with unknown threshold locations. Robust control theory allows the decision-maker to describe physical systems using models that include bounded uncertainty. Robust control methods are designed to maintain a properly functioning system, as long as uncertain parameters or disturbances are within these, typically tight, bounds. Robust control techniques have been applied to the fields of biology, medicine, engineering, neuroscience, computing and economics (Smith and Doyle 1992; Zhou *et al.* 1996; Doyle and Csete 2011). The management of ecological systems represents one such application.

2.2.3 Alternatives to standard optimisation techniques

Several decision rules and valuation tools have been proposed as alternatives to standard optimisation techniques, where each focus on specific aspects of managing ecological systems. These frameworks include, but are not limited to, safe minimum standards, the precautionary principle, real options, minimax, and minimax-regret. Excluding real options, these decision rules are typically advocated for systems where uncertainties, such as threshold locations, are difficult to conceptualise. These uncertainties can be considered as random variables for which probability distributions are difficult to postulate. Given the theme of this thesis, the following review of the literature will focus primarily on uncertainty, or risk, regarding ecological threshold locations. However, other sources of uncertainty will also be discussed.

A safe minimum standard (SMS) is typically advocated in situations where the decision-maker concedes that they don't know enough about an ecological system, or the system is too complex, to adequately characterise it using a mathematical model.

As such, support for a SMS will typically begin with the assertion that a utilitarian optimisation framework is inapplicable (Margolis and Nævdal 2008). Therefore, it is logically impossible to assess the optimality of non-utilitarian rules from within a utilitarian framework. However, Margolis and Nævdal (2008) identify some problem specifications that generate SMS-type results from within a utilitarian framework, but only by building in firm thresholds that force this result. The authors are able to establish a relationship between two parameters that suggest a SMS can be motivated by a conventional utilitarian criterion. These parameters capture (i) the seriousness of the potential catastrophe, and (ii) the magnitude of risk (of catastrophe) associated with the state variable's current position in the state-space.

SMS rules can also be used to trigger shifts from using technical decision criteria, for example, efficiency, to other rules, such as stake-holder dialogue or judicial review. This trigger point signifies the maximum tolerable level of discomfort associated with using technical procedures for decision making (Farmer 2001). The SMS can also be advocated as protection against the piecemeal approach of benefit-cost analysis, where projects are considered on an individual basis and do not always consider flow-on or feedbacks effects on other systems. The SMS guards against the possibility that quasi-option and other non-use values have been excluded from, or inaccurately valued in, a benefit-cost analysis (Randall 1991). Quasi-option values reflect the irreversible nature of many decisions, in the presence of incomplete knowledge. Other non-use values include option value, which is equivalent to an insurance premium for conserving the environment in case it provides a use value in future, and existence value, which captures satisfaction or utility gained merely from knowing that a certain species or system exists (Rolfe 1995).

The minimax and minimax-regret decision rules have been developed as alternatives to a safe minimum standard, with the aim of addressing some of its conceptual limitations. Assume that several experts have postulated different prior probability distributions for the unknown threshold location. For the minimax decision rule, the objective is to minimise the possible loss from a worst-case scenario (i.e. maximum loss). This implies minimisation of the worst-case scenario over the range of different probability distributions postulated (Stoye 2007). Alternatively, this decision rule is used to maximise the minimum gain. For the minimax-regret decision

rule, the objective is to minimise the maximum regret resulting from a decision. Regret is defined as the difference between the expected ex-ante¹¹ outcome and the maximum ex-post¹² outcome. In other words, the difference between the *expected* outcome when the state of nature is not known and the outcome that could have been achieved if the state of nature was known (Stoye 2009a). The minimax-regret criterion explicitly incorporates the opportunity cost of making a ‘wrong’ choice (Palmini 1999).

The precautionary principle, option theory and the burden of proof have developed as distinct methodologies; however, there are enough conceptual similarities and overlaps to allow them to be discussed concurrently. In the context of ecological systems, these methodologies are typically applied to systems susceptible to irreversible threshold effects, where ecosystem dynamics are very poorly understood and/or where damages are assumed to be catastrophic. The precautionary principle can be viewed as a double-negative way of defining a real option (Hertzler 2007). That is, actions are taken today to avoid the possibility of an undesirable regime shift occurring and reducing the number of environmental management options, and consumption possibilities, available in the future. A second interpretation of the precautionary principle is a reversal of the burden of proof, from ‘innocent until proven guilty’ to ‘guilty until proven innocent’ (van den Belt 2003).

Satisfactory arguments in terms of the precautionary principle can be made in two opposite directions. First, that costly preventative actions should not be undertaken until the existence of a high (potentially catastrophic) risk is scientifically proven. Second, that taking preventative action now will lead to greater flexibility and maintain more options in the future, thus resulting in a positive option value when taking preventative action. The central question then becomes one of assigning a burden of proof. This can take the form of either (i) external parties who suffer harm from a management decision being required to prove this harm, or (ii) a decision-maker being required to prove that their management decision does not cause harm to external parties (Ansink and Wesseler 2009).

¹¹ Ex-ante implies that management decisions must be made before the state of nature is known.

¹² Ex-post implies that management decisions can be made after the state of nature is known.

The decision rules described above have been proposed as alternatives to traditional optimisation techniques to counteract the complications associated with modelling ecological systems. However, these rules are largely prescriptive. They are useful for providing ‘rules of thumb’ or decision heuristics, but cannot be used for optimisation in the traditional sense. Nevertheless, each serves as a useful conceptual starting point for more mathematically rigorous analysis and optimisation, where the decision-maker possesses a sufficient amount of information.

This section described several alternative techniques and decision rules that have been proposed for modelling the management of ecological systems. These range from mathematical techniques, such as marginal resilience pricing and optimal control, to decision rules, such as safe minimum standards, the precautionary principle, minimax and minimax-regret. The main limitations of these approaches will be discussed in the following section.

2.3 Extensions to previous approaches

This section details possible extensions to previous approaches proposed for the management and modelling of ecological systems. These extensions relate to the observations that (i) the benefits of active learning are not considered within the decision framework, (ii) the decision framework is unable to provide specific recommendations in terms of levels of control variables, and (iii) there are conceptual inconsistencies with standard economic theory.

2.3.1 Consideration of the impacts of active learning

Most past approaches developed for managing and modelling ecological system have either not considered the benefits of learning or have been unable to include them due to methodological limitations. Conversely, imposing a safe minimum standard strictly precludes any possibility of learning through intelligently perturbing the system.

Marginal resilience pricing can be used to value a system's 'stock' of resilience as a capital asset (Mäler *et al.* 2007; Walker *et al.* 2010). The inclusion of a resilience stock in a capital-theoretic framework, such as the inclusive wealth framework (Arrow *et al.* 2003), can be considered as 'middle ground' between advocates of 'strong' and 'weak' sustainability. This is because it acknowledges that there may be some degree of substitutability between capital stocks, but there are also limits to substitutability, which are defined by critical ecological thresholds (Dasgupta and Mäler 2001; Dietz and Neumayer 2007). However, a major limitation of this approach is that the cumulative probability distribution used to estimate the likelihood of an undesirable regime shift occurring, and therefore place a value on resilience, is independent of the path of the underlying variable over time.

The value of resilience is calculated as a function of (i) the physical distance of the system from the critical ecological threshold, (ii) the associated conditional probability of a regime shift occurring in the future, and (iii) the difference in expected income or utility streams between the current economically preferred regime and the non-preferred alternative regime (Walker *et al.* 2010). The second component described above is of concern because of the possibility of learning is not considered. Instead, the probability $F(X_0, t)$ of experiencing a regime shift at or before any future time period t is modelled as a function of the current stock of resilience, X_0 , and the number of future time periods considered, t . The fact that the specification of F is time-invariant signifies that the functional form cannot be updated as time progresses and new information is gathered.

In the context discussed above, the use of marginal resilience pricing as a component of a capital-theoretic framework appears to be no more than an accounting exercise; a benefit-cost analysis. However, Walker *et al.* (2010) acknowledge the potential for modifying the resilience pricing metric for use in an optimal control framework. Further to this assertion, this thesis develops a dynamic programming model that explicitly solves for the optimal levels of control variables. The model also includes an iterative process of learning about the unknown locations of thresholds, which represents a natural extension of the conceptual framework described above.

In the case of a known and deterministic threshold, there is no need for an optimal control framework to consider the benefits of learning about unknown threshold locations, since there are no opportunities to engage in such learning. However, in many cases, the exact locations of critical ecological thresholds are unknown to the decision-maker, and there are benefits that flow from the possibility of learning. Nævdal (2006) and Nævdal and Oppenheimer (2007) came closest to including learning about unknown threshold locations within an optimal control framework. Nævdal (2006, p. 1134) identifies what he coins a “risk switching point”, which effectively represents the boundary of the current data set. The ‘risk switching point’ allows the state-space (for the underlying slow variable) to be partitioned into regions that can and cannot contain the threshold. This distinction is necessary for the process of modelling learning.

Frequently, the aim of an optimal control analysis is to examine trajectories to an optimal steady state of the system (e.g. Mäler *et al.* 2003; Grune *et al.* 2005; Nævdal 2006; Nævdal and Oppenheimer 2007). In other words, the aim is to determine levels for each state variable that can be maintained across time through active management of these variables. The reasoning is that if the levels of the underlying slow variables of the ecosystem do not change over time, the ecosystem will not experience an undesirable regime shift. However, this approach negates the potential to learn about a threshold’s location and narrow the bounds of the prior distribution that characterises its possible location. This is because the optimal outcome will be one that achieves a steady state of the system and does not necessarily involve perturbing the system toward the threshold for the purpose of learning more about its location.

When the exact location of an ecological threshold is unknown, its location can be modelled using a prior probability distribution. The concept of a ‘risk switching point’ elucidates strong parallels to the rationale for safe minimum standards and risk thresholds (e.g. Margolis and Nævdal 2008) in terms of partitioning the state-space into safe and unsafe regions. However, a risk switching point differs because it relies entirely on data from an observed path of the state variable, whereas a safe minimum standard (SMS) is usually set based on experimental data, expert knowledge or social preferences.

Proponents of the SMS rule argue that it guards against the myopic and irrational nature of human behaviour that fails to account for longer term considerations and the welfare of future generations. If justified on the grounds of short-sightedness, the SMS rule needs to be very inflexible to prevent rent seeking behaviour from parties disadvantaged by the implementation of the SMS (Rolfe 1995). The main drawback of a binding SMS is that it precludes learning about unknown threshold locations. Also, in cases where threshold crossings are thought to be irreversible, it is highly likely that a SMS, option theory or other optimisation techniques would generate similar results that indicate a significant degree of risk averse behaviour is optimal. However, for systems with reversible thresholds, a binding SMS does not allow uncertainty about unknown threshold locations to be reduced because the system will not be perturbed into the region of the state-space that contains the threshold. Reduced uncertainty¹³ about threshold locations may result in significant future efficiency gains. These gains can take the form of cost savings owing to less future active management being required to prevent the system crossing a threshold. Alternatively, these gains can result from being able to operate in a more productive, or more profitable, system state without any risk of crossing a threshold. These potential gains are not considered if a binding SMS is in place.

2.3.2 Provision of specific recommendations for control variables

Some of the decision rules proposed as alternatives to traditional optimisation techniques have practical limitations because they are unable to provide specific recommendations about optimal management decisions. Instead, these alternatives are largely prescriptive, providing decision heuristics or ‘rules of thumb’. Conversely, mathematical optimisation frameworks are used to provide specific recommendations for management decisions.

At its core, the precautionary principle is merely a concept or a guideline that provides no specific notion of how it should be applied in practice. According to Randall (2011), “The precautionary principle is framed as a principle and, as such, cannot be expected to be ready for implementation in particular management and

¹³ Measured by assuming Knightian risk, rather than Knightian uncertainty.

policy situations.” Rather, the precautionary principle is a statement of a normative moral position. It can provide a policy framework that foresees and accounts for competing and conflicting principles. Further, the practical problems with many precautionary principle formulations are the result of, or exacerbated by, unconvincing connections between the possible consequences of uncertainties about a real-world system, the degree of our lack of knowledge and the corrective course of action required (Randall 2011).

While assessing the theoretical validity of the precautionary principle, Gollier and Treich (2003) identify linkages between irreversibility, information, risk and the notion of option value. Although there are various economic interpretations of the precautionary principle, most definitions revolve around the question of how to manage risk in the presence of scientific uncertainty or, in other words, “under conditions of imperfect scientific knowledge” (Gollier and Treich 2003, p. 77). The precautionary principle of scientific management advocates that scientific uncertainty is no reason for failing to act. This is actually a double-negative way of defining a real option. In positive terms, undertaking precautionary actions today will preserve more options for the future. If, later, we decide that we don’t require these options, we can undertake riskier actions (Hertzler 2007).

For irreversible decisions with uncertain benefits, a quasi-option value raises the opportunity cost of a management action due to the prospect of forthcoming information about the currently uncertain benefits (Arrow and Fisher 1974). It is this sequential nature of the decision process that is central to the problem of managing risk in the presence of scientific uncertainty (Gollier and Treich 2003). However, application of the precautionary principle stifles the sequential nature of the decision process by advocating precautionary actions that reduce, or remove, the possibility of learning about an unknown threshold location. Furthermore, for cases with a high degree of uncertainty about threshold locations and where it is expected that any damages will be catastrophic, a mathematical optimisation model is also likely to support precautionary actions, in addition to specifying an optimal amount of precaution.

A similar criticism can be made of the safe minimum standard (SMS). Margolis and Nævdal (2008) use an economic framework to assess the optimality of choosing a SMS. The authors examine the rationale for a SMS policy within both a static and dynamic framework and derive necessary and sufficient conditions for when a SMS can be dismissed or defended as an optimal policy. The authors argue the SMS differs from a simple quantity constraint because, at least up to some level, it is independent of the social benefits available from relaxing the standard. For this reason, the SMS is commonly considered an unappealing policy from the viewpoint of an expected social welfare maximiser. “Indeed, for many authors it is precisely this – the abandonment of the efficiency criterion, rather than the bounding of a physical variable – which defines the SMS” (Margolis and Nævdal 2008, p. 402). Rather than being used to determine an optimal path or optimal steady state of the state variables of the system, the SMS simply imposes a quantity constraint, which may or may not have been set arbitrarily. Therefore, it provides no specific notion of how the system should be optimally managed over time, apart from not crossing the SMS-threshold.

2.3.3 Conceptual inconsistencies

The minimax and minimax-regret decision rules have been developed as alternatives to a safe minimum standard, with the aim of addressing some of its conceptual limitations. Although these rules have experienced a resurgence in recent times (e.g. Manski 2004; Manski 2005; Brock 2006; Eozenou *et al.* 2006; Manski 2006; Schlag 2006; Stoye 2007; Manski 2007a; Manski 2007b; Hirano and Porter 2008; Manski 2008; Stoye 2009a; Stoye 2009b), they have conceptual limitations of their own. For the discussion that follows, assume that several experts postulate different prior probability distributions for the unknown threshold location, but the decision-maker doesn’t know which of these is the correct distribution. One then encounters a decision problem under ambiguity, rather than a decision problem under risk (Stoye 2009a).

Ready and Bishop (1991) use the insurance and lottery games to illustrate that the framing of the decision problem can significantly alter the optimal strategy. In the insurance game, the uncertain future states of the world are independent of the

management options available to the decision-maker. On the other hand, implicit in the lottery game is an interaction between the management option chosen and the possible future states of nature. This means that choosing a particular management option may preclude certain states of nature from occurring. The differing optimal management actions for different framings of the problem are explained by the level of dependence between possible future states of nature (i.e. alternative future events) and the management action chosen (Palmini 1999). Further, the minimax decision rule violates Milnor's bonus invariance axiom for game theoretic decision rules (Milnor 1964; Palmini 1999). This axiom states that if the decision-maker receives a bonus (or penalty) from a source exogenous to the choice set, the decision-maker's optimal strategy should not change.

Palmini (1999) proposes the use of the minimax-regret rule because it results in a consistent optimal strategy, regardless of the framing of the decision problem. The minimax-regret rule minimises the maximum regret resulting from a decision, where regret is defined as the difference between the expected ex-ante¹⁴ outcome and the maximum ex-post¹⁵ outcome. In other words, the difference between the *expected* outcome when the state of nature is not known and the outcome that could have been achieved if the state of nature was known (Stoye 2009a). Palmini (1999) also argues that the minimax rule ignores the standard economic principle that opportunity cost guides the choices people make, even when the person is risk averse.

Further, both the minimax and minimax-regret decision rules have theoretical drawbacks and inconsistencies with standard economic theory. First, both decision rules avoid the explicit use of priors; however, the minimax-regret rule implicitly selects a prior and can be viewed as a prior selection device motivated by a specific notion of uniform quality of decisions.¹⁶ Second, the minimax-regret rule is menu-dependent, which means that adding a new decision rule to the available set of decision rules may alter the benchmark against which regret is measured and can affect their relative rankings (Stoye 2009a). Third, for some cases, both the minimax and minimax-regret decision rules can generate trivial results and no-data rules in

¹⁴ Ex-ante implies that management decisions must be made before the state of nature is known.

¹⁵ Ex-post implies that management decisions can be made after the state of nature is known.

¹⁶ No single decision based on a specific prior is better or more reliable than any other decision based on a different prior.

treatment choice problems (Schlag 2003; Manski 2004; Hirano and Porter 2008; Stoye 2009a).

This section discussed some of the most important limitations of approaches previously advocated for the management and modelling of ecological systems. These range from being unable to include the economic benefits of learning within the decision-making framework, being unable to provide recommendations for optimal levels of control variables, and conceptual inconsistencies with standard economic theory. An optimal control framework that explicitly considers the benefits of active learning addresses these limitations.

2.4 Requirements of a new approach

The previous sections have detailed the management of ecological systems with unknown threshold locations, how the management of ecological systems is modelled, and the limitations of past approaches to management and modelling. This section details the elements that are required for a new approach to modelling and management that focuses particularly on the ability of a decision-maker to engage in active learning to control the speed with which new information is gathered and uncertainty¹⁷ about unknown threshold locations is reduced. This new modelling approach captures the trade-off faced by the decision-maker between the costs of crossing the threshold and shifting to an undesirable alternative regime, and the benefits of active learning. The modelling framework backs away from some of the complexity of ecological modelling to instead focus more productively on the value of active learning through intelligently perturbing the system.

The model developed in this thesis extends previous models by allowing the decision-maker to learn about unknown threshold locations and factoring this potential for learning into the decision-maker's current decision. Learning has been included in some previous models (e.g. Walters 1986; Kelly and Kolstad 1999; Karp and Zhang 2006; Shenton *et al.* 2010; Johnson and Mengersen 2012). In fact, Walters (1986) provided numerous examples of mathematical models similar to that

¹⁷ Measured by assuming Knightian risk, rather than Knightian uncertainty.

developed in this thesis and was a pioneer in the field of adaptive management. One of the intellectual contributions of this thesis is allowing hypothetical future information to be explicitly included in the decision-maker's current decisions, as well as allowing the decision-maker to determine the speed with which new information is acquired. For the specific class of management problems with unknown threshold locations, this has not been done before.

Learning can take several forms and be modelled several different ways.¹⁸ For example, Brozovic and Schlenker (2011) explicitly advocate allowing the decision-maker to learn about an unknown threshold location through time using Bayesian updating. In their seminal paper, Kelly and Kolstad (1999) consider Bayesian learning in the context of greenhouse gas pollution and climate change. Here, there is a clear distinction between passive learning, where the process by which information is acquired is exogenous to the system, and active learning, where the decision-maker has some control over the rate at which information is acquired. The case considered by Kelly and Kolstad (1999), and the case modelled in this thesis, is that of endogenous learning, otherwise known as active learning.

Bayesian updating (see Gelman *et al.* 2003) is the process of using Bayes' theorem to update and refine a prior belief as new information becomes available. For example, a player has a prior expectation that a two-sided coin is 'fair', so the probability of tossing 'heads' or 'tails' is equally likely and equal to $\frac{1}{2}$. However, after several tosses of the coin, the player observes that the coin has landed on 'heads' on each occasion. The player can use Bayes' rule to update their prior belief, using this new information, to form a posterior belief that the probability of tossing 'heads' is greater than $\frac{1}{2}$ and 'tails' is less than $\frac{1}{2}$. As new information is progressively acquired, less weight is placed on the initial prior belief and more weight is placed on the observed data.

¹⁸ Epistemologically, the notion of 'modelling learning' is problematic. Once the rule for updating prior distributions of threshold locations is specified, the model reduces to a discrete time dynamical system. The initial conditions of the problem, as specified by the modeller, will then determine the results. Technically, no learning takes place in the usual sense of the word. However, the process of learning, as modelled in this thesis and elsewhere, nonetheless, captures the possibility of learning occurring and the alertness of the decision-maker to this process. It also adequately captures the ability of the decision-maker to determine the regularity with which new information is acquired; active learning.

The use of a Bayesian updating rule for modelling learning about unknown threshold locations would be most effective in cases where an ecological threshold is thought to be stochastic. In other words, when an ecological threshold is known to exist, but its exact location is continually changing. In such a case, the unknown threshold location can be considered a random variable with an associated probability distribution. As new information is acquired, namely whether an ecological threshold has been crossed at a particular point in time and for particular levels of underlying slow variables, the prior probability distribution is updated.

The rule used in this thesis for updating prior beliefs differs from Bayesian updating because of the assumption of a fixed threshold location. This means that the threshold location does not change over time and is at a fixed point within the state-space. The key point of difference between a fixed and stochastic threshold is that the location of a stochastic threshold will never be known with certainty. Conversely, once discovered, the location of a fixed threshold will always be known. This means it is possible to partition the ecosystem's state-space into a region that is known to *not* contain the threshold, because this region has been traversed previously without crossing the threshold, and a second region that *must* contain the threshold. This is conceptually equivalent to the risk-switching point discussed previously (Nævdal 2006). In terms of an updating rule, this means that the appropriate approach is to truncate and re-scale the initial prior distribution¹⁹ as new information is acquired about where the threshold *could* and *could not* be located. This approach aligns with Kolmogorov's second axiom, which states that there is a probability of one that some elementary event in the entire sample space will occur (Kolmogorov 1956). In this case, the two possible events are the threshold being crossed or not being crossed.

From a decision-maker's perspective, there are several different management options available for ecosystem management. The most notable of these are passive adaptive management (passive learning) and active adaptive management (active learning) (e.g. Kelly and Kolstad 1999; Sabine *et al.* 2004; Prato 2005). The key distinction between passive and active learning is one of intent. If a decision-maker engages in passive learning, they do so simply by observing a system as it changes

¹⁹ The prior probability distribution is truncated and re-scaled such that the posterior distribution has a probability mass of one.

naturally over time. In contrast, active learning involves the deliberate perturbation of a system, with the explicit aim of learning about the system. Arrow *et al.* (1995) argue that it is only possible to learn about the resilience of a complex system by intelligently perturbing the system using adaptive management. By perturbing the system in the direction of a threshold, the decision-maker has actively increased the probability of a threshold being crossed and a regime shift occurring, but has also increased the likelihood of learning about the threshold's location.

If we drop the simplifying assumption that a decision-maker or manager knows the current state and dynamics of the system with certainty, we can assert that adaptive management is well suited to managing these ecosystems. Adaptive management increases the rate at which decision-makers acquire knowledge about ecological relationships and aids decision-making through the use of iterative hypothesis testing. It also enhances information flows among policy makers (Prato 2005). Further, recent research has suggested some generic indicators of close proximity to critical ecological thresholds, such that critical transitions may be anticipated. These indicators include (i) critical slowing down, where the system's ability to recover from small perturbations becomes very slow when near a threshold, and (ii) increased variability in stochastic systems, where larger variance suggests a contrasting regime to which the system may shift if conditions change (Scheffer *et al.* 2012). This field of research is in its infancy, and offers many avenues for cross-disciplinary collaboration.

The main source of economic value in terms of learning, and resulting new information, is the ability of a system to recover or 'bounce back'. For a system with a reversible threshold (either completely reversible or hysteretic), the value of information, acquired through learning, is derived from two separate sources. First, if new information is acquired about the location of an unknown threshold without actually crossing the threshold, future management of the system will be less costly, or more profitable. Learning about the unknown threshold location without actually crossing it involves perturbing the system into the 'risky' sub-space, which is defined as the confidence bounds, or region, within which the unknown threshold location could lie (Nævdal 2006). This process tightens the confidence bounds around the unknown threshold location and will generate a cost saving resulting from requiring a

lower amount of active control of the system in future to certainly avoid crossing the undesirable threshold. Alternatively, the decision-maker can operate in what is likely to be a more profitable region of the state-space without incurring any risk of crossing the undesirable threshold. Second, for a system with a reversible threshold, the damage resulting from an undesirable regime shift may be short-lived and incur a small cost relative to the cost savings that may be made by lowering the level of active control required in future time periods.

The value of information, as defined in this thesis, is conceptual different to the ‘expected value of perfect information’, as defined by Walters (1986). The expected value of perfect information (EVPI) (Walters 1986) requires multiple policy models, with associated probabilities of being the correct model. If no learning is possible, the ex-ante best policy (action) will be a function of the probability-weighted returns from each model. If, however, a magical study is able to resolve all uncertainty (perfect knowledge) and determine the correct model, the optimal policy can be chosen for each model. The set of ‘perfect knowledge’ estimates of returns are then compared with the returns expected if the best ‘no learning’ policy were used, to determine the EVPI. By contrast, the ‘value of information’, as defined in this thesis, is calculated by comparing expected returns when the decision-maker acknowledges that their current actions will have bearing upon the speed with which learning takes place, and expected returns when the decision-maker makes no such acknowledgement. Walters (1986) asserts that, surprisingly often, learning will not be as valuable as intuitively expected. However, in the context of ecological systems with unknown threshold locations, the value of information will depend on the degree of uncertainty around threshold locations, and the magnitude of differences in expected returns across alternative system regimes. The value of information is problem-specific, and may be large or small.

New information, acquired after a threshold has been crossed, can be put to use for future management decisions if the system has a *reversible* threshold, but not an *irreversible* threshold. The model developed in this thesis captures the trade-off faced by the decision-maker between (i) the potential benefits from learning about the threshold location and (ii) the potential costs incurred as a result of an undesirable regime shift. These benefits take the form of a cost saving from requiring less active

control in the direction away from an undesirable threshold, because learning narrows the prior probability bounds for the unknown threshold location. The costs, in the case of a *reversible* threshold, relate to the cost of recovering the system following an undesirable regime shift, which includes both control costs and an opportunity cost in terms of forgone utility or output when in the undesirable regime. The costs, in the case of an *irreversible* threshold, relate to the forgone utility or output from being permanently in the less-preferred regime following an undesirable regime shift. The model then calculates the optimal amount of risk to incur (which could be zero) via some amount (which could be zero) of deliberate perturbation of the system. In this context, risk is defined as the probability of a regime shift occurring, based on the conditional prior probability distribution for the unknown threshold location. It is the ability of the system to recover that, in some cases, may make engaging in risky management behaviour optimal.

A properly specified optimisation framework should acknowledge and include the iterative impact of learning on current and future management decisions (e.g. Walters 1986; Karp and Zhang 2006). Failure to do so omits the value of learning from the optimisation framework. The model developed in this thesis explicitly accounts for the iterative process of learning about unknown threshold locations and the impact of learning on future management decisions. A dynamic programming approach is used to solve recursively for all possible combinations of system state variables and all possible paths of the state variables. The particular paths of state variables will ultimately determine the timing and degree of learning about unknown threshold locations. This process allows a decision-maker to determine the optimal levels of control variables (e.g. fertiliser application rates) in the current time period, while implicitly considering the impact that these choices will have on future management decisions and the expected NPV of the future stream of net benefits from the system.

Summary of literature review

An ecological system can be represented in the form of a production function, by way of a reductive model. In the case of an ecosystem with a critical threshold, the system output (Y) can be represented as a function of one or more system inputs (X),

where at least one threshold exists in terms of the X variable or variables. Many ecological systems are susceptible to threshold effects (e.g. Mäler *et al.* 2003; Walker and Salt 2006; Baskett and Salomon 2010; Gil-Romera *et al.* 2010). A threshold is a level of an underlying slow-moving variable where the dynamics and the ‘output’ of the system change.

Numerous methodologies have been proposed for modelling and managing ecological systems. These include mathematical techniques, such as marginal resilience pricing and optimal control, and decision heuristics, such as safe minimum standards, the precautionary principle, minimax and minimax-regret. Each methodology focuses on specific aspects of managing complicated systems that differ from those associated with the management of simple systems. However, each of these methodologies suffer from conceptual limitations, such as being unable to include the benefits of active learning within the decision framework, being unable to provide specific recommendations in terms of levels of control variables, and conceptual inconsistencies with standard economic theory.

A new approach to modelling and managing ecological systems is required to extend the literature beyond the limitations outlined above. It must focus on the ability of a decision-maker to engage in active learning about the unknown locations of critical ecological thresholds. This new modelling approach captures the trade-off faced by the decision-maker between the costs of crossing the threshold and shifting to an undesirable alternative regime, and the benefits of active learning. It also allows the decision-maker to determine the speed with which new information is acquired, since the potential for learning is conditional on the management actions undertaken.

Chapter 3. Modelling framework

This chapter presents a new modelling framework for the management of ecological systems. The mathematical model demonstrates the role of learning in making optimal decisions by explicitly modelling active learning about unknown threshold locations in complicated ecological systems. Active learning involves occasionally deliberately perturbing the system in the direction of a critical threshold for the purpose of learning about its location. Such deliberate management actions are associated with both benefits and costs. However, previous modelling frameworks have not adequately considered the value of learning about threshold locations.

Learning is modelled by first partitioning the system's state space into two regions. Based on the current information set, one region is known to not contain the ecological threshold, which means that the other region must contain the threshold. A prior probability distribution can then be postulated for the unknown threshold location, based on historical or experimental data, or expert knowledge. The prior distribution is updated over time as more information is hypothetically gathered, where the potential for learning is conditional on the management actions undertaken by the decision-maker.

The objective of the decision-maker is to maximise the expected net present value of utility returns from the system. To this end, each alternative system regime is modelled as a distinct production function. Crossing an ecological threshold results in the system shifting from the original regime to an alternative regime and experiencing an abrupt change in productivity. Utility is modelled as a function of several management actions. These actions determine the instantaneous level of utility, the expected future returns from the system and the likelihood of learning about the unknown threshold location. In other words, the decision-maker must choose management actions 'today', while acknowledging that these actions will ultimately determine the amount of learning that occurs, and the resulting benefits and costs of such learning.

3.1 Introduction to the model

This section provides an introduction of the new mathematical modelling framework developed in this thesis, including the model notation, decision-maker's objective, and the important characteristics of the modelling framework. Section 3.1.1 introduces the notation used in the modelling framework. Section 3.1.2 outlines the decision-maker's objective. Section 3.1.3 describes the important characteristics of the modelling framework. Sections 3.1.4 and 3.1.5 provide graphical representations of the decision process when the system has reversible and irreversible threshold effects, respectively.

3.1.1 Notation

Table 3-1 below lists and describes all of the variables used in the mathematical modelling framework.

Table 3-1 Variables used in the mathematical modelling framework

Variable	Description
t	Discrete time period
i_t	Current system regime – either A or B
j_t	Captures all information captured up to, and including, time period t i.e. a record of when and which thresholds have been crossed
V_{j_t}	Value function at time period t , having arrived at time period t by traversing branches j_t
ρ	Rate of pure time preference
X_{j_t}	Level of the underlying slow ecosystem variable at the end of time t
C_{j_t}	Effort for positive anthropogenic control of the system via investment in measures that determine the level of $X_{j_t} \equiv X'_{j_{t-1}}$
k_t	Marginal cost of effort for positive anthropogenic control of the system
q_{j_t}	Consumption expenditure paid for by acquired wealth
W_{j_t}	Stock of acquired wealth at the end of time t
r_t	Real interest rate on the stock of acquired wealth
a and b	Prior probability density functions (PDFs) for the fixed but unknown locations of Threshold A and Threshold B, respectively

a_{jt}^{min}	Lower bound of the a PDF i.e. Threshold A
a_{jt}^{max}	Upper bound of the a PDF i.e. Threshold A
b_{jt}^{min}	Lower bound of the b PDF i.e. Threshold B
b_{jt}^{max}	Upper bound of the b PDF i.e. Threshold B
Y^i	Output of the system when in Regime i
p_t	Revenue received per unit of output
H_{jt}	Amount of output that is harvested in time period t
E_{jt}	Amount of harvest effort in time period t
h_t	Marginal cost of harvest effort
L_{jt}	Within-system feedbacks, which alter the level of X in the current period t and are a function of the level of X in the previous period $t - 1$
S_t	Vector of exogenous inputs that affect X e.g. temperature, precipitation
D_{jt}	Anthropogenic damages to the underlying slow variable as a result of management practice in the previous period $t - 1$
G_{jt}	Funding from an external source e.g. government or private funding

3.1.2 The decision-maker's objective

In this model, the decision-maker's objective is to maximise the NPV of expected utility from the system over a finite time horizon, $[1, T]$. Excluding the terminal time period T , utility can be obtained from two sources; (i) consumption utility, which results from consumption expenditure paid for by wealth acquired from the profits generated by the system, and (ii) direct utility, which is a direct function of the current 'output' of the system. For example, direct utility may be obtained from the aesthetic value of a system, which may be a function of the population size of a particular bird species. Consumption utility is temporally transferable because acquired wealth can be stored and used for consumption in a future time period. This means that consumption utility can be used to smooth total utility in the event of an undesirable regime shift and significant reduction in the level of direct utility. On the other hand, direct utility cannot be stored and can thus only be enjoyed in the current time period. The two other sources of utility are only applicable in the terminal time period T . These sources are (iii) utility from terminal wealth, which captures the bequest value (either to a representative agent or individual) of any wealth remaining at the end of the terminal time period; and (iv) the scrap value of the system, which

captures utility that may be sourced based on the system's future productive potential, and is conditional on the state of the system at the end of the terminal time period. Implicit in (iii) and (iv) is an assumption that both the ecological system and society will continue to exist beyond the finite time horizon of the problem.

The time-invariant instantaneous utility functions are:

- $U_q(q_{j_t})$, which captures consumption utility
- $U_Y(Y_{j_t})$, which captures direct utility
- $U_{W_T}(W_{j_T})$, which captures utility from terminal wealth
- $U_{X_T}(X_{j_T}, R_{j_T})$, which captures the scrap value of the system, conditional on its state at the end of the terminal time period

3.1.3 Model characteristics

The model is characterised by:

- (i) Two alternative system regimes

A system regime is a set of states in which a system can exist and retain the same basic structure and function. Most ecosystems can exist in more than one regime (Walker and Salt 2006). For example, an agricultural system may exist in a saline or non-saline regime. Also, a lake ecosystem may exist in a variety of regimes, such as oligotrophic and eutrophic.

- (ii) Hysteretic dynamics

This means that a regime change from one regime to another is reversible, but along a different return path. This is because alternative regimes have different dynamics and different critical ecological thresholds. If a system crosses a threshold from the original regime to an alternative regime, it will be required to cross a different threshold, and via different dynamics, to return to the original regime. This

general form of the modelling framework means that it can also be applied to the management of systems with irreversible or fully reversible thresholds, which are merely extreme cases of a system with hysteretic dynamics.

(iii) Fixed but unknown threshold location

The locations of critical ecological thresholds are assumed to be at fixed positions within the X -space. However, these positions are not known to the decision-maker at the beginning of the problem. Instead, prior probability distributions are postulated for the thresholds' position. Over time, more is learnt about the system and the bounds of these prior probability distributions are gradually tightened as more information is (hypothetically) acquired.

(iv) Regime-specific production functions

The two alternative system regimes are modelled as two different production functions. These production functions are deterministic in the sense that if the system 'input' is known with certainty, the system 'output' is also known with certainty. However, if the system 'input' is instead a random variable, the system 'output' will also be a random variable.

'Ecosystem dynamics' and 'ecosystem production function' are two terms that will be used interchangeably unless otherwise stated. A separate production function is assumed for each regime in order to capture the different dynamics across each system regime. Output, Y , is modelled as a function of a single input, X . A few simple examples of these production functions include fish population as a function of water quality, koala population as a function of habitat size or connectivity, and crop yield as a function of the depth of a saline water table.

(v) Learning

This model assumes Knightian risk rather than Knightian uncertainty (Knight 1921). Exact threshold locations are unknown, but prior probability distributions can be postulated based on historical data, experimental data, and/or expert knowledge. A

process of learning occurs (i.e. a signal is provided) every time the system is perturbed. The probability distributions for threshold locations are continually updated as new information becomes available, and the bounds of these prior probability distributions are progressively tightened over time.

3.1.4 A system with hysteretic dynamics

Figure 3-1 provides a visual representation of the ecosystem dynamics assumed in this model. The ecosystem exhibits hysteretic dynamics, which means there is more than one system regime, but regime shifts are reversible along a different path and via different system dynamics. The two alternative regimes are represented by the red and blue curves. Each regime is typified by a different production function, where the production function is stochastic. The near-horizontal *solid* red and blue curves represent the expected value of each production function. The near-horizontal *dotted* red and blue curves represent confidence bounds for each respective production function. The solid red and blue probability density functions (PDFs) at the bottom of the graph represent the prior PDFs for the locations of Threshold A and B, respectively, and are denoted $a_t(X)$ and $b_t(X)$, respectively. The initial PDFs can be supported on a bounded or unbounded interval. They are only required to be bounded in the presence of a natural physical boundary. For example, if the slow ecosystem variable is groundwater table depth, it is bounded from below at $X = 0$ i.e. where the groundwater table reaches the surface. Similarly, if X represents the total area of a species habitat, it cannot take a value below zero. Should the input variable (X) cross a critical ecological threshold, the ecosystem will immediately shift from the current regime to the alternative regime; either red to blue or blue to red. The vertical dotted black arrows represent the means of $a_t(X)$ and $b_t(X)$.

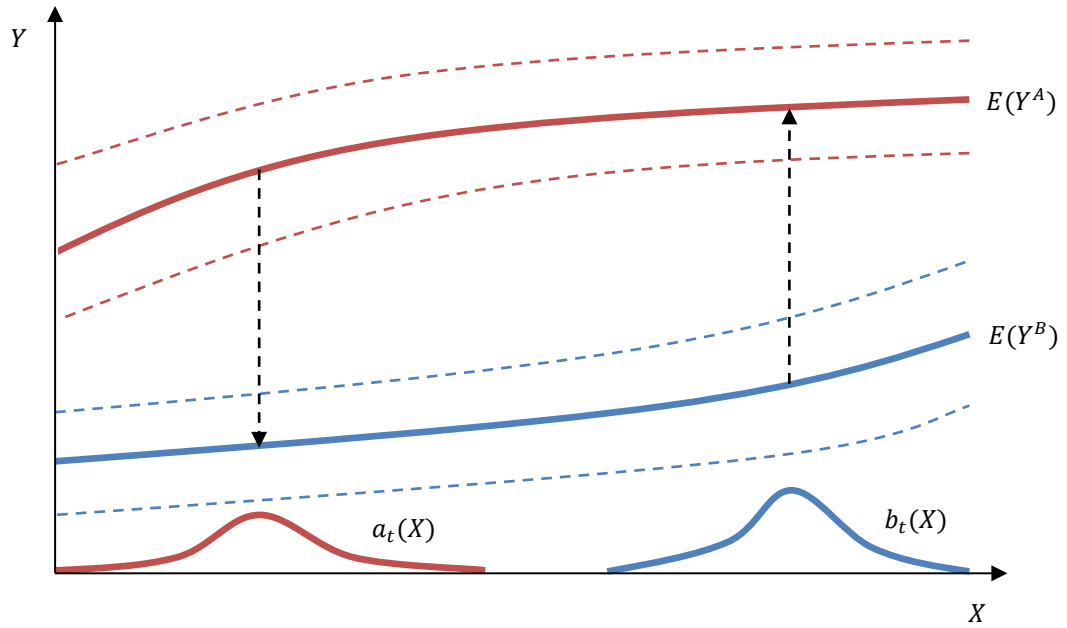


Figure 3-1 A system with two alternative regimes and hysteresis dynamics

Figure 3-2 sets out the period-by-period problem faced by the decision-maker. In this example, the system is known to be in Regime A (R_A) in period 0. Each branch represents the two alternative contingencies, which are ending that period in R_A or R_B (i.e. Regime B). The number alongside each branch (expressed algebraically) represents the probability of following that branch, conditional on the action/s of the decision-maker at the beginning of the period (i.e. the choice of C_{j_t} , which determines $X_{j_t} \equiv X'_{j_{t-1}}$), and the system's regime at the beginning of the period. $A_{j_t}(X_{j_t})$ is a cumulative distribution function (CDF) that gives the probability of remaining in Regime A at the end of period t if the system was in Regime A at the beginning of period t (i.e. the probability of remaining in Regime A). Likewise, $B_{j_t}(X_{j_t})$ is a CDF that gives the probability of remaining in Regime B. $1 - A_{j_t}(X_{j_t})$ and $1 - B_{j_t}(X_{j_t})$ are survival functions, which give the conditional probability of crossing the relevant ecological threshold from Regime A to B, and from Regime B to A respectively.

The subscript j_t is used to denote the path (i.e. branches) the system has traversed, up to and including time period t . For example, X_{AAB} represents the level of the state variable X when the system has (hypothetically) been in Regime A at the end of $t = 1$, Regime A at the end of $t = 2$ and Regime B at the end of $t = 3$. Note that the system is known to be in Regime A at $t = 0$, therefore, there is no need to include

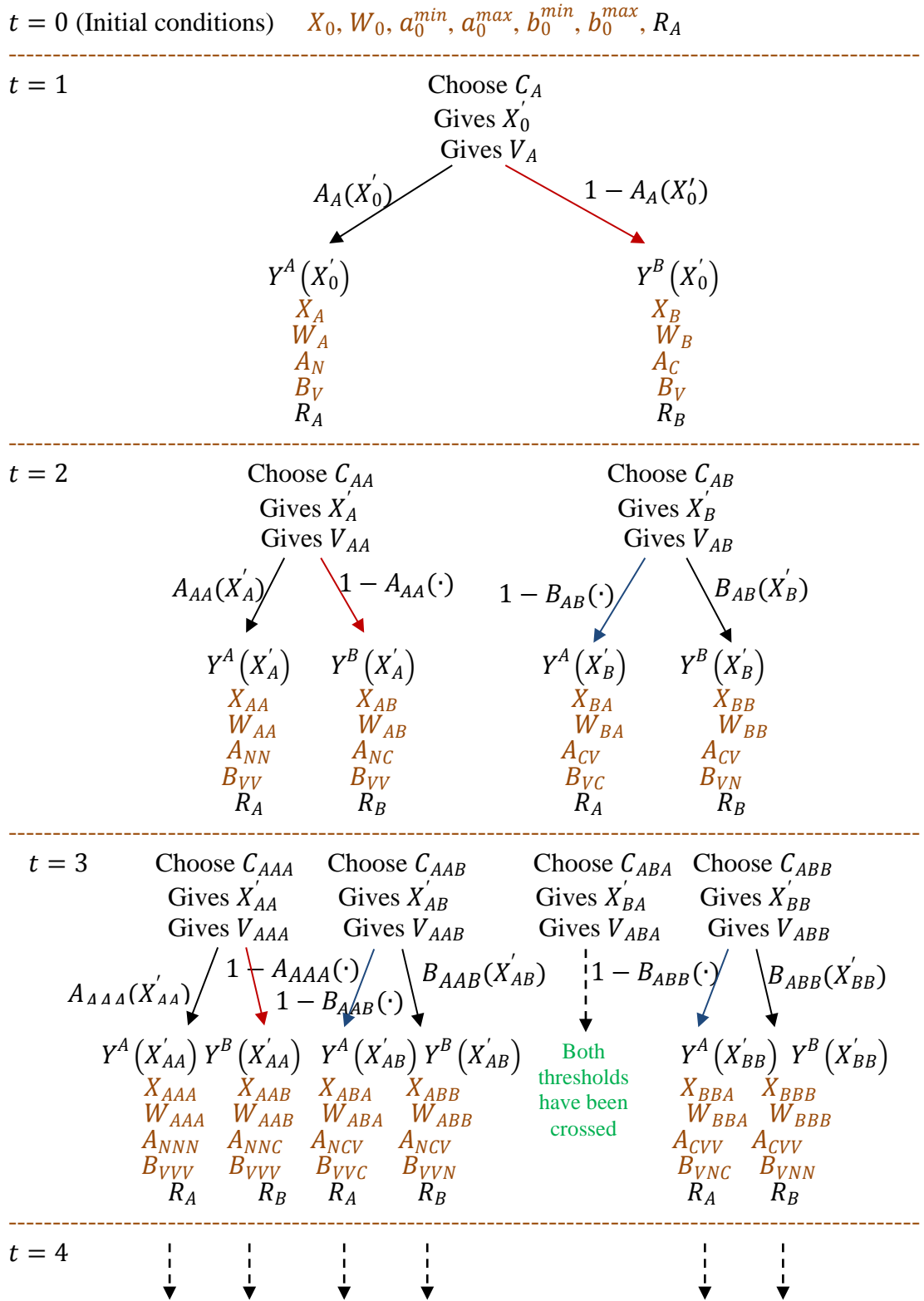


Figure 3-2 Decision tree for the optimisation of a system with hysteretic dynamics

this information in the j_t subscript of a variable. The subscript j_t is also used to capture any information acquired about a variable, up to and including period t . The same notation is used for the parameters of a and b , which are, technically, state variables in the problem. The subscripts on these state variables indicate that, during a particular time period, the relevant threshold was crossed (C), not crossed (N), or the concept is void (V) because the system was in the alternative regime. The updating rules for each of these state variables are discussed in greater detail in Section 3.2.3. Red and blue arrows denote the crossing of Threshold A and Threshold B, respectively.

The brown variables in Figure 3-2 represent the state variables of the decision-maker's problem, measured at the end of a given time period. These variables are:

- X , the underlying slow ecosystem variable.
- W , the decision-maker's level of wealth.
- A , which is a shorthand expression for the lower and upper bounds ($a_{j_{t-1}}^{min}$ and $a_{j_{t-1}}^{max}$, respectively) of the CDF that gives the probability of remaining in Regime A at the end of a time period, if the system was in Regime A at the beginning of that time period.
- B , which is a shorthand expression for the lower and upper bounds ($b_{j_{t-1}}^{min}$ and $b_{j_{t-1}}^{max}$, respectively) of the CDF that gives the probability of remaining in Regime B at the end of a time period, if the system was in Regime B at the beginning of that time period.

The final elements of the decision tree diagram in Figure 3-2 are the value functions, V_{j_t} . Each value function is a sum of (i) a weighted average of the two value functions that succeed it, and (ii) the utility gained in the current period from direct and consumption utility. For this model, any regime shifts occur at the end of the time period. This means that the current regime (A or B) is known for the duration of the current time period, and system output at the end of the time period will be conditional on the system having been in this same regime. The assigned probability weights for the value function are based on the conditional probability of being in Regime A or Regime B at the end of the current period and, therefore, at the beginning of the next period. For example, the equation for V_A is:

$$V_A = \max_{C_A, q_A, E_A} \left\{ U(Y^A(X_A), q_A) \right. \\ \left. + (1 + \rho)^{-1} [A_A(X_A) \cdot V_{AA}(X_A) + (1 - A_A(X_A)) \cdot V_{AB}(X_A)] \right\} \quad (1)$$

The operation $\arg \max \{V_{j_t}\}$ gives the optimal levels for the control variables C_{j_t} , q_{j_t} and E_{j_t} . The optimal levels of these variables can be expressed as functions of the ‘inherited’²⁰ levels of the state variables $X_{j_{t-1}}$, $W_{j_{t-1}}$, $a_{j_{t-1}}^{min}$, $a_{j_{t-1}}^{max}$, $b_{j_{t-1}}^{min}$, $b_{j_{t-1}}^{max}$ as well as the parameter values of several other variables. The value function is discussed in greater detail in Section 3.4.2.

3.1.5 A system with an irreversibility

Figure 3-3 shows a modified version of Figure 3-2. Figure 3-3 shows a system where any regime shift from Regime A to Regime B is irreversible. This has the effect of greatly simplifying the decision tree when compared to a system with hysteretic dynamics, since many of the alternative contingencies (i.e. branches of the tree) are no longer possible. More specifically, it is possible for the system to shift from Regime A to Regime B; however, there is no path of the underlying slow variable (X) that will allow the system to shift back from Regime B to Regime A. Mathematically, this case is defined via the B -CDF. For a system where a regime shift is irreversible, and assuming the preferred and initial regime is Regime A, the B -CDF will be a degenerate distribution where $B = 1$. Therefore, $1 - B = 0$. This means there is no possibility of crossing Threshold B and shifting back to Regime A. For any branches where Threshold A has been crossed, $Y^B(X_{j_t})$ will be received with certainty for all remaining time periods. For any branches where Threshold A has not yet been crossed, the optimisation remains as per that described in the Section 3.1.4. That is, where weights are assigned based on the conditional probabilities of being in either Regime A or B in different time periods.

²⁰ In this context, the word ‘inherited’ is used to describe the levels of the state variables at the end of the previous time period. These levels are inherited in the sense that they will also be the initial levels of the state variables in the current time period.

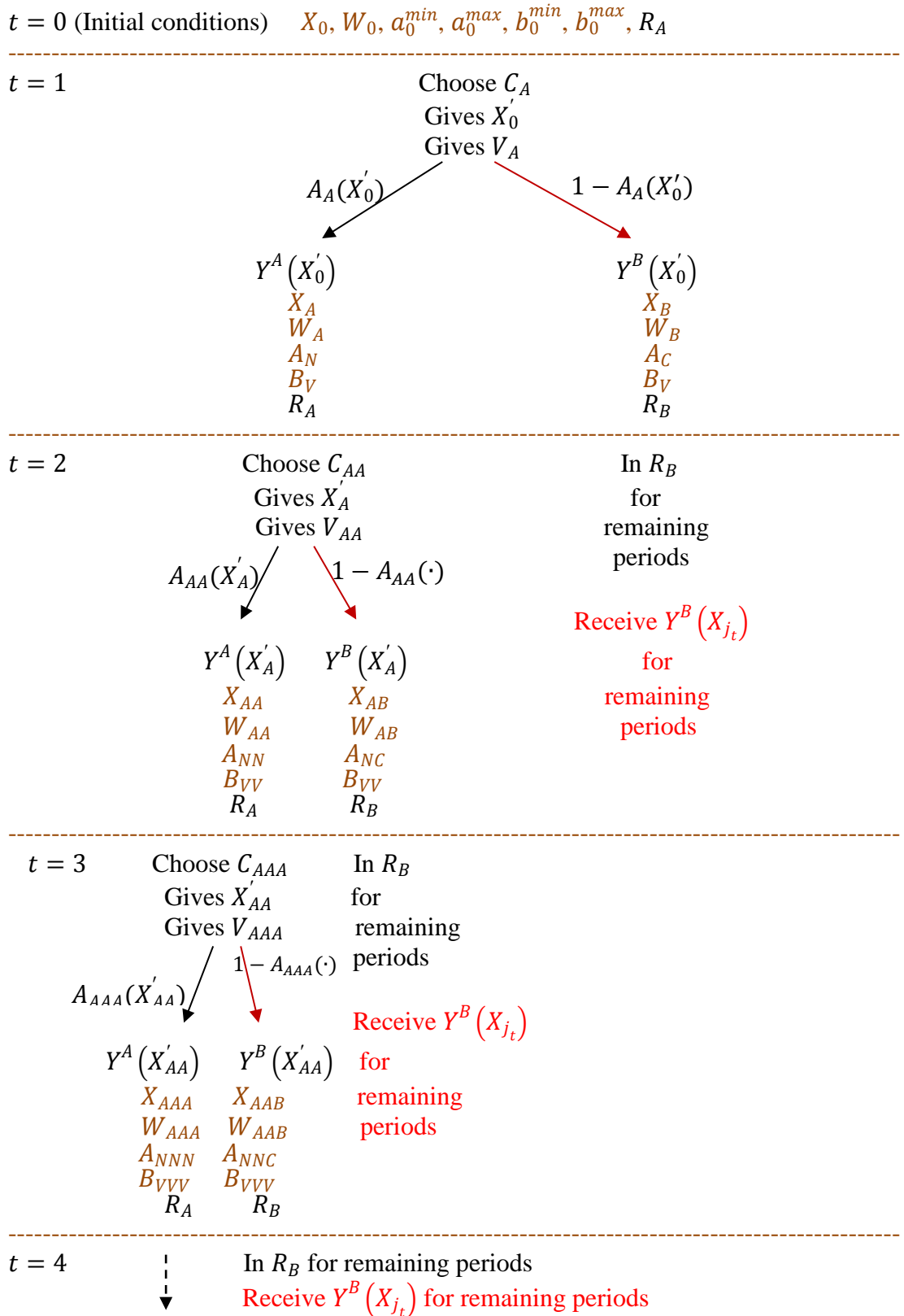


Figure 3-3 Decision tree for the optimisation of a system with an irreversible threshold

3.2 Updating prior beliefs about threshold locations

This section details how active learning is modelled in the conceptual framework. This process involves, first, postulating prior probability distributions for unknown threshold locations. Crossing an ecological threshold results in the system shifting to an alternative system regime, where the productivity of the system experiences an abrupt, non-marginal change. These prior distributions are then progressively updated as new information is gathered, based on (i) the observed path of an underlying slow-moving system variable and (ii) an observation of whether an ecological threshold was or was not crossed after a particular perturbation of the underlying variable.

Section 3.2.1 outlines the process of postulating and modelling probability density functions for unknown threshold locations. Section 3.2.2 explains the process used to model the path of the underlying slow variable. Section 3.2.3 describes how the processes outlined in the two previous sections can be combined to form general updating rules for the prior probability distributions that characterise the unknown threshold locations. Section 3.2.4 explains how these general updating rules can be converted to a form suitable for use in a dynamic programming simulation exercise.

3.2.1 Modelling of probability density functions for unknown threshold locations

Unknown threshold locations. It is assumed that the ecosystem has two separate system regimes (R_A and R_B), each with its own dynamics and critical ecological threshold. These critical ecological thresholds are assumed to be at fixed locations within the X -space. However, the exact locations are unknown to the decision-maker at the beginning of the problem. Using historical or experimental data, or expert knowledge, it is possible to postulate a probability density function (PDF) for each of the two fixed but, as yet, unknown threshold locations. For the initial discrete time period of the decision-maker's problem, the PDFs that characterise two random variables are defined as T_0^A and T_0^B , for Threshold A and Threshold B, respectively. For every other discrete time period, the PDFs that characterise two random variables

are defined as $T_{j_t}^A$ and $T_{j_t}^B$, for Threshold A and Threshold B, respectively. The time subscripts, j_t , allow for updating of the upper and lower bounds as more information is gathered about the probable locations of Thresholds A and B.

Only one restriction applies to the initial prior probability distributions of T_0^A and T_0^B , and necessarily applies to only one of T_0^A and T_0^B . This restriction relates to the presence of data gathered after previous perturbations of the system. For example, if the system has previously been perturbed in the direction of Threshold A [B], without crossing Threshold A [B], the extreme point of this path will represent the *finite* upper or lower bound of the T_0^A [T_0^B] PDF²¹. This point represents the boundary of the sub-space within which Threshold A [B] is known not to exist; the ‘safe’ region. When this case applies for either Regime A or Regime B, the prior PDF for the corresponding unknown threshold location (i.e. Threshold A or B) must be bounded from at least one side. Conversely, if a regime shift from Regime A [B] to Regime B [A] has never been observed, no such restriction applies when the decision-maker postulates the upper and lower bounds of the T_0^B [T_0^A] PDF. Instead, when a particular system regime has not previously been experienced, and data regarding the location of the corresponding threshold is completely lacking, the decision-maker may wish to postulate an unbounded PDF to characterise the unknown location of this threshold.

The upper and lower bounds of the prior PDFs for Threshold A and Threshold B are $a_{j_t}^{max}$ and $a_{j_t}^{min}$, and $b_{j_t}^{max}$ and $b_{j_t}^{min}$, respectively. Each of these parameters²² may be finite or infinite, conditional on the availability of data, as described above. Therefore, the prior PDFs for the unknown threshold locations can be specified as follows:

$$T_{j_t}^A \sim Pr(T^A = X | X_{j_t} = X) = a(X) = a(X, a_0^{max}, a_0^{min}, a_{j_t}^{max}, a_{j_t}^{min}) \quad (2)$$

²¹ The PDF used by the decision-maker to characterise the unknown location of Threshold A, given their level of knowledge at the beginning of the optimisation problem (i.e. at $t = 0$).

²² Technically, $a_{j_t}^{max}$ and $a_{j_t}^{min}$, and $b_{j_t}^{max}$ and $b_{j_t}^{min}$ are state variables in the optimisation problem, and are continually updated as more information is hypothetically gathered about the unknown threshold locations. However, each of these state variables serve as model parameters when specifying the prior PDFs for each discrete time period.

$$T_{j_t}^B \sim Pr(T^B = X | X_{j_t} = X) = b(X) = b(X, b_0^{max}, b_0^{min}, b_{j_t}^{max}, b_{j_t}^{min}) \quad (3)$$

Note: the subscript j_t is used to denote the path followed along the decision tree up to time period t . For example, if the system was in Regime A at the end of time periods $t = 1$ and $t = 2$, and in Regime B at the end of time period $t = 3$, then $j_{t=3} = AAB$.

It is the parameter values for the distributions of these random variables, $T_{j_t}^A$ and $T_{j_t}^B$, that will be updated for each successive node of the decision tree (see Section 3.1.4); conditional on whether or not the relevant ecological threshold has been (hypothetically) crossed during the time period. Only the parameters of $T_{j_t}^A$ will be updated if the previous node was an R_A -node, and only the parameters of $T_{j_t}^B$ will be updated if the previous node was an R_B -node. In other words, for a particular time period, the ecosystem can only cross the threshold of the current regime. Therefore, the decision-maker is only able to learn about the threshold location of the current regime. The method of updating prior beliefs for threshold locations (i.e. $T_{j_t}^A$ and $T_{j_t}^B$) will be discussed in detail in Section 3.2.3.

Underlying slow ecosystem variable. For the purpose of calculating the value function²³ for each node of the decision tree, it is necessary to keep track of the underlying slow ecosystem variable X_{j_t} (a random variable) over time. The exact value of X_{j_t} is only ever known at $t = 0$. The initial value of X_{j_t} is:

$$X_{j_{t=0}} = X_0 = \bar{X}_0 \quad (4)$$

For all other time intervals $t = [1, T]$, the value of the underlying slow ecosystem variable is a function of within-system feedbacks, exogenous inputs, positive anthropogenic control of the system, anthropogenic damages to the system, and whether or not the critical ecological threshold was crossed in the current time period:

²³ The sum of discounted utility in the current time period and discounted expected utility in all future time periods, up to and including the terminal time period T .

i. Within-system feedbacks (L_{j_t}):

For example, X measures phosphorus concentration in a shallow lake or greenhouse gas concentration in the atmosphere. In both cases, the ecosystem will possess some ability to assimilate or break-down these nutrients, which is likely to be a function of the current level of these nutrients.

$$L_{j_t} \sim \mathcal{L}\left(l(E_0[X_{j_{t-1}}]), \sigma_L^2(l(E_0[X_{j_{t-1}}]))\right) = \mathcal{L}_{j_t}(X) \quad (5)$$

ii. Exogenous inputs (S_t):

For example, the average temperature for a region is expected to follow a trend (deterministic), while experiencing some variation around this trend (stochastic).

$$S_t \sim \zeta(E_0[S_t], \sigma_S^2(E_0[S_t])) = \zeta(X) \quad (6)$$

iii. Positive anthropogenic control of the underlying slow ecosystem variable (C_{j_t}):

For example, the removal of saline groundwater via mechanical pumping will cause an expected reduction in the groundwater table (deterministic), but with some uncertainty about the exact reduction (stochastic).

$$C_{j_t} \sim \gamma(E_0[C_{j_t}], \sigma_C^2(E_0[C_{j_t}])) = \gamma_{j_t}(X|E_0[C_{j_t}]) \quad (7)$$

iv. Anthropogenic damages to the underlying slow ecosystem variable (D_{j_t}):

Damages in the current time period t may be a function of the level in the previous time period $t - 1$ of any, or all, of the following variables: system ‘output’ (Y), consumption (q), harvest amount (H), harvest effort (E). For example, assume Y is the fish stock in a fishery and X is the water quality in the fishery. Water quality in the current time period is likely to have been affected by the amount of fishing effort in the previous time period, which generates pollution through fuel leakage, etc.

$$D_{j_t} \sim \delta \left(d \left(E_0 \left[Y_{j_{t-1}}, q_{j_{t-1}}, H_{j_{t-1}}, E_{j_{t-1}} \right] \right), \sigma_D^2 \left(d \left(E_0 \left[Y_{j_{t-1}}, q_{j_{t-1}}, H_{j_{t-1}}, E_{j_{t-1}} \right] \right) \right) \right) \quad (8)$$

$$= \delta_{j_t} \left(X \mid Y_{j_{t-1}}, q_{j_{t-1}}, H_{j_{t-1}}, E_{j_{t-1}} \right)$$

- v. Whether or not the critical ecological threshold was crossed during the current time period (i.e. whether $F_{j_t} = 1$ or $F_{j_t} = 0$) provides information regarding the probability with which X traversed a ‘risky’ path or ‘safe’ path during the current time period. If the threshold was crossed (i.e. $F_{j_t} = 1$), X must have traversed a risky path; one which took it in the direction of the threshold. The critical ecological threshold could not have been crossed if the underlying slow ecosystem variable X remained stationary or moved further away from the threshold. On the other hand, X may have traversed either a risky or safe path if the threshold was not crossed (i.e. $F_{j_t} = 0$). This is so because the underlying slow variable could have moved in the direction of the threshold (risky path) without actually crossing it.

Equation of motion for X_{j_t} . In the absence of any threshold effects or natural physical boundaries in terms of the underlying slow ecosystem variable:

- X_{j_t} evolves according to:

$$X_{j_t} = X_{j_{t-1}} + L_{j_t} + S_t + C_{j_t} + D_{j_t} \quad (9)$$

- The change in X_{j_t} is given by:

$$\begin{aligned} \Delta X_{j_t} = X_{j_t} - X_{j_{t-1}} &= [L_{j_t} + S_t + C_{j_t} + D_{j_t}] \sim [L_{j_t}(X) + \zeta(X) + \gamma_{j_t}(X) + \delta_{j_t}(X)] \\ &= u_{j_t}(X) \end{aligned} \quad (10)$$

However, given the existence of thresholds of unknown location, it is necessary to use two different updating rules for X_{j_t} , conditional on whether $F_{j_t} = 1$ or $F_{j_t} = 0$. However, in the special case where L_{j_t} , S_t , C_{j_t} and D_{j_t} are deterministic, so too is ΔX_{j_t} . In this case, the updating rules for when $F_{j_t} = 1$ and $F_{j_t} = 0$ will converge

to a single updating rule, and ΔX_{j_t} will be a degenerate random variable. However, in all other cases, the updating rules for $F_{j_t} = 1$ and $F_{j_t} = 0$ will remain distinct.

For the explanation that follows, it is assumed that the ecosystem is currently in Regime A. It is also assumed that an increase in the level of X means the system has moved further away from the unknown location of Threshold A, whereas a decrease in the level of X means the system has moved closer to the unknown location of Threshold A. If the reverse is true for a system regime, the equations described below must be reversed. The reverse framing is shown in full in Appendix A.

3.2.2 Deterministic evolution of the underlying slow ecosystem variable

For the deterministic case, ΔX_{j_t} is a degenerate random variable. This scenario represents the extreme case for ΔX_{j_t} , where the exact value of $X_{j_t}(C_{j_t}, X_{j_{t-1}}, q_{j_{t-1}}, H_{j_{t-1}}, E_{j_{t-1}}, S_t)$ will be known for every time period within the time horizon, $[0, T]$. This means that the path traversed by X from $X_{j_{t-1}}$ to X_{j_t} , for which the relevant ecological threshold was either hypothetically crossed or not, is also known with certainty. For this reason, the process of active learning will lead to tighter bounds on the updated priors for the locations of Thresholds A and B than would be the case if ΔX_{j_t} were a random variable. It is assumed that if the threshold is crossed, this crossing occurs at the end of the time period, and the resulting regime shift's effect on system output only manifests itself in the following time period. The process by which the prior distribution for a threshold's location is updated is explained below, by way of example.

The initial value of X is known to be $X_0 = \bar{X}_0$. In addition, initial prior PDFs are postulated for the fixed but unknown threshold locations:

- (i) Threshold A (T^A):

$$\begin{aligned}
Pr(T^A = X | X'_0 = X) &= a_{j_{t=1}}(X) \\
&= a(X, a_0^{max}, a_0^{min}, a_{j_{t=1}}^{max} \equiv a_0^{max}, a_{j_{t=1}}^{min} \equiv a_0^{min})
\end{aligned} \tag{11}$$

(ii) Threshold B (T^B):

$$\begin{aligned}
Pr(T^B = X | X'_0 = X) &= b_{j_{t=1}}(X) \\
&= b(X, b_0^{max}, b_0^{min}, b_{j_{t=1}}^{max} \equiv b_0^{max}, b_{j_{t=1}}^{min} \equiv b_0^{min})
\end{aligned} \tag{12}$$

Where

$a_0^{max}, a_0^{min}, b_0^{max}$ and b_0^{min} are parameters of the model

$a_{j_t}^{max}, a_{j_t}^{min}, b_{j_t}^{max}$ and $b_{j_t}^{min}$ are state variables $\forall t = 1, T$

Since ΔX_{j_t} is purely deterministic, the PDF characterising the probabilistic level of X at the end of $t = 1$ will be identical (in fact, a degenerate random variable) regardless of whether or not the threshold was crossed during that time period:

$$X'_0 = X_A = X_B = \bar{X}_0 + L_0^{i=A}(\bar{X}_0) + D_0^{i=A}(\bar{X}_0) + \bar{S}_0 + C_0'^{i=A} \tag{13}$$

Where

L_0^i is the impact of within-system feedbacks

D_0^i is the impact of anthropogenic damages

S_0 is the impact of exogenous inputs

$C_t'^i$ is the impact of positive anthropogenic control of the system

i is the system regime at the beginning of the time period; in this case, Regime A

At the end of time period $t = 1$, there are two possible (hypothetical) outcomes:

- a. First, $F_A = 0$; which means that after starting the time period in R_A , the ecological threshold has *not* been crossed by the end of the time period.
- b. Second, $F_A = 1$; which means that after starting the time period in R_A , the ecological threshold has been crossed by the end of the time period.

a. If $F_A = 0$

The system will remain in R_A and only the upper bound (i.e. $a_{j_t}^{max}$) of the a -distribution can potentially be updated. Driving the updating process for $a_{j_t}^{max}$ is whether X_A traversed a path which took it into the risky sub-space, where $X_A < a_A^{max}$ (Case 2), or remained in the safe sub-space, where $X_A \geq a_A^{max}$ (Case 1).

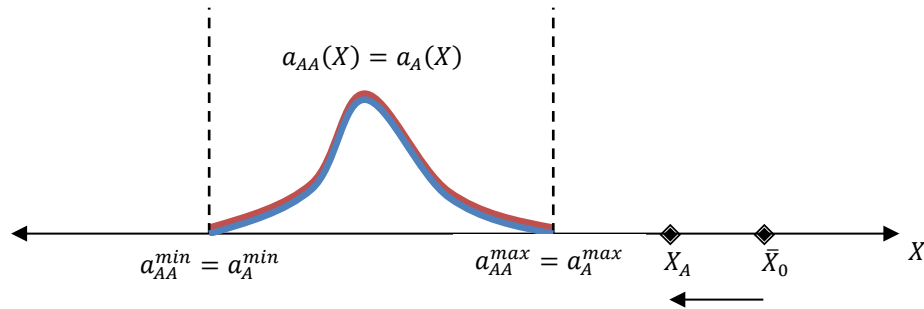
Mathematically, the updating rule is expressed as:

$$a_{AA}^{max} \equiv a_A^{max} |_{F_A=0}^U = \min(a_A^{max}, X_A) \equiv \min(a_A^{max}, X_0') \quad (14)$$

Also note, if X_A takes a value less than or equal to the lower bound a_A^{min} , there is zero probability of Threshold A *not* being crossed during the time period i.e. $Pr(F_A = 0 | X_A \leq a_A^{min}) = 0$. Therefore, given a choice of C_0' that will result in $X_A \leq a_A^{min}$, there is no resulting branch of the decision tree for the condition $F_A = 0$.

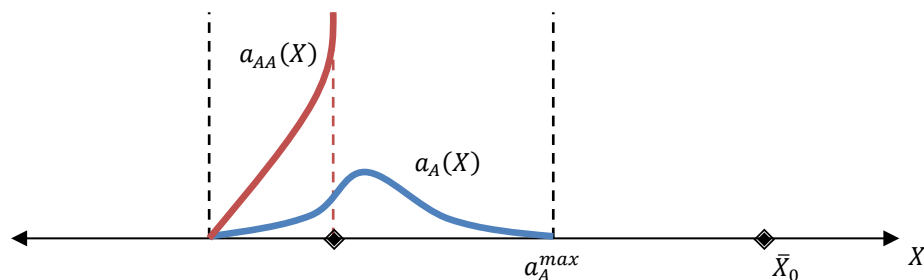
The two possible cases for which $F_A = 0$ are shown graphically below:

- Case 1: $X_A \geq a_A^{max}$



Here, X remains in the ‘safe’ sub-space and no learning occurs. Therefore, $a_{AA}(X) = a_A(X)$, since $a_{AA}^{min} = a_A^{min}$ and $a_{AA}^{max} = a_A^{max}$.

- Case 2: $a_A^{min} < X_A < a_A^{max}$



$$a_{AA}^{min} = a_A^{min} \quad X_A = a_{AA}^{max}$$

←—————

Here, X traverses a risky path, but the threshold is not crossed. Therefore, it is now known that Threshold A is *not* located within the interval $[X_A, a_A^{max}]$. $a_A(X)$ is converted to $a_{AA}(X)$ by truncating at the new upper bound, X_A , and normalising, such that the probability mass of $a_{AA}(X)$ is equal to one.

b. If $F_A = 1$

The system will switch to R_B and only the lower bound (i.e. $a_{j_t}^{min}$) of the a -distribution can potentially be updated. Driving the updating process for $a_{j_t}^{min}$ is whether X_A traversed a path which took it into the risky sub-space, where $a_A^{min} < X_A < a_A^{max}$ (Case 4), or beyond the risky sub-space, where $X_A \leq a_A^{min}$ (Case 3).

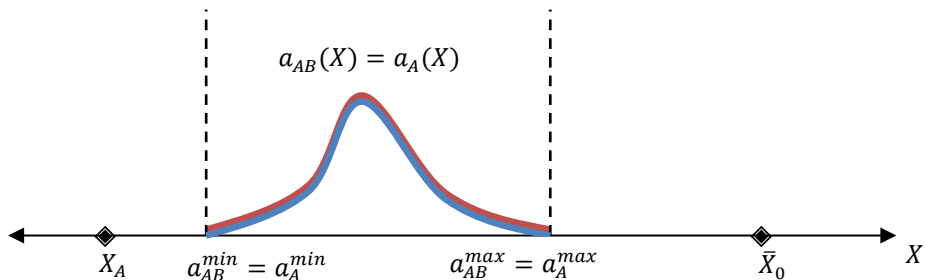
Mathematically, the updating rule is expressed as:

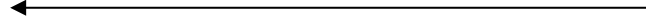
$$a_{AB}^{min} \equiv a_A^{min} \Big|_{F_A=1}^U = \max(a_A^{min}, X_A) = \max(a_A^{min}, X_0') \quad (15)$$

Also note, if X_A takes a value greater than or equal to the upper bound a_A^{max} , there is zero probability of Threshold A being crossed during the time period i.e. $Pr(F_A = 1 | X_A \geq a_A^{max}) = 0$. Therefore, given a choice of C_0' that will result in $X_A \geq a_A^{max}$, there is no resulting branch of the decision tree for the condition $F_A = 1$.

The two possible cases for which $F_A = 1$ are shown graphically below:

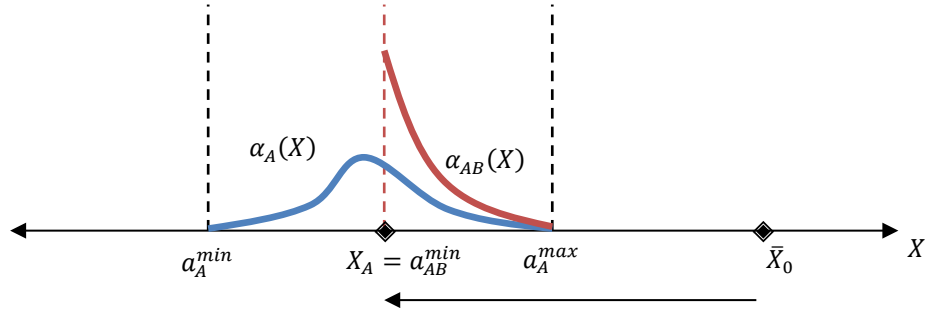
- Case 3: $X_A \leq a_A^{min}$





Here, the path traversed by X crosses the threshold with certainty and no learning occurs. Therefore, $a_{AB}(X) = a_A(X)$, since $a_{AB}^{min} = a_A^{min}$ and $a_{AB}^{max} = a_A^{max}$.

- Case 4: $a_A^{min} < X_A < a_A^{max}$



Here, X traverses a risky path, and the threshold is crossed. Therefore, it is now known that Threshold A is located within the interval $[X_A, a_A^{max}]$. $a_A(X)$ is converted to $a_{AB}(X)$ by truncating at the new lower bound, X_A , and normalising, such that the probability mass of $a_{AB}(X)$ is equal to one.

3.2.3 General updating rules for the four state variables

By combining the four cases described in Section 3.2.2, and the equivalent cases for $b_{j_t}^{max}$ and $b_{j_t}^{min}$, the updating rules for each of the upper and lower bound state variables can be expressed succinctly using piecemeal functions. Each function contains three pieces, which capture all possible scenarios in terms of the threshold-crossing indicator variable F_{j_t} and the system regime at the beginning of the time period, R_{j_t} . Threshold crossings are assumed to occur, and be observed, at the very end of a time period. The updating rules for each of $a_{j_{t+1}}^{max}$, $a_{j_{t+1}}^{min}$, $b_{j_{t+1}}^{max}$ and $b_{j_{t+1}}^{min}$ are shown below:

$$(i) \quad a_{j_{t+1}}^{max} = \begin{cases} \min(a_{j_t}^{max}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = A \\ a_{j_t}^{max} & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = A \\ a_{j_t}^{max} & \text{if } R_{j_t} = B \end{cases} \quad (16)$$

Where

$$a_{j_t}^{max} \leq a_0^{max} \quad \forall t$$

$$(ii) \quad a_{j_{t+1}}^{min} = \begin{cases} a_{j_t}^{min} & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = A \\ \max(a_{j_t}^{min}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = A \\ a_{j_t}^{min} & \text{if } R_{j_t} = B \end{cases} \quad (17)$$

Where

$$a_{j_t}^{min} \geq a_0^{min} \quad \forall t$$

$$(iii) \quad b_{j_{t+1}}^{max} = \begin{cases} b_{j_t}^{max} & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = B \\ \min(b_{j_t}^{max}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = B \\ b_{j_t}^{max} & \text{if } R_{j_t} = A \end{cases} \quad (18)$$

Where

$$b_{j_t}^{max} \leq b_0^{max} \quad \forall t$$

$$(iv) \quad b_{j_{t+1}}^{min} = \begin{cases} \max(b_{j_t}^{min}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = B \\ b_{j_t}^{min} & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = B \\ b_{j_t}^{min} & \text{if } R_{j_t} = A \end{cases} \quad (19)$$

Where

$$b_{j_t}^{min} \geq b_0^{min} \quad \forall t$$

Summary of updating rules for prior bounds

In each of the four cases described above in Section 3.2.2, and the equivalent cases for $b_{j_t}^{max}$ and $b_{j_t}^{min}$, it is important to note that the functional form of $a_{j_t}(X)$ [$b_{j_t}(X)$] does not change internally to the upper and lower bounds of $a_{j_t}^{max}$ [$b_{j_t}^{max}$] and $a_{j_t}^{min}$ [$b_{j_t}^{min}$], respectively. Instead, $a_{j_t}(X)$ [$b_{j_t}(X)$] is merely the initial prior probability distribution, $a_{j_{t=1}}(X)$ [$b_{j_{t=1}}(X)$], truncated from one side, or both sides, as the upper ($a_{j_t}^{max}$ [$b_{j_t}^{max}$]) and/or lower ($a_{j_t}^{min}$ [$b_{j_t}^{min}$]) bounds are updated. This truncated version of $a_{j_{t=1}}(X)$ [$b_{j_{t=1}}(X)$] is then normalised, such that the probability mass of $a_{j_t}(X)$ [$b_{j_t}(X)$] is equal to one.

The general formula for $a_{j_t}(X)$ for any time period t is shown below:

$$a_{j_t}(X) = \begin{cases} 0 & \text{if } X \geq a_{j_t}^{max} \\ \frac{a_1(X)}{1 - \int_{a_0^{min}}^{a_{j_t}^{min}} [a_1(X)]dX - \int_{a_{j_t}^{max}}^{a_0^{max}} [a_1(X)]dX} & \text{if } a_{j_t}^{min} < X < a_{j_t}^{max} \\ 0 & \text{if } X \leq a_{j_t}^{min} \end{cases} \quad (20)$$

Where

$$a_{j_t}(X) = a(X, a_0^{max}, a_0^{min}, a_{j_t}^{max}, a_{j_t}^{min})$$

$$a_1(X) = a_{j_{t=1}}(X) = a(X, a_0^{max}, a_0^{min}, a_{j_{t=1}}^{max} \equiv a_0^{max}, a_{j_{t=1}}^{min} \equiv a_0^{min})$$

Likewise, the general formula for $b_{j_t}(X)$ for any time period t is shown below:

$$b_{j_t}(X) = \begin{cases} 0 & \text{if } X \geq b_{j_t}^{max} \\ \frac{b_1(X)}{1 - \int_{b_0^{min}}^{b_{j_t}^{min}} [b_1(X)]dX - \int_{b_{j_t}^{max}}^{b_0^{max}} [b_1(X)]dX} & \text{if } b_{j_t}^{min} < X < b_{j_t}^{max} \\ 0 & \text{if } X \leq b_{j_t}^{min} \end{cases} \quad (21)$$

Where

$$b_{j_t}(X) = b(X, b_0^{max}, b_0^{min}, b_{j_t}^{max}, b_{j_t}^{min})$$

$$b_1(X) = b_{j_{t=1}}(X) = b(X, b_0^{max}, b_0^{min}, b_{j_{t=1}}^{max} \equiv b_0^{max}, b_{j_{t=1}}^{min} \equiv b_0^{min})$$

3.2.4 Conversion of updating rules for use in simulation model

Suitable rules for the updating of prior beliefs about unknown threshold locations have been described in Sections 3.2.2 and 3.2.3 above. In these examples, updating rules are described when moving in the forward direction through time. However, since the dynamic nature of this problem means it must be solved recursively, the updating rules must be expressed in a form that captures a backward progression through time. In fact, each of these different updating rules represents a different constraint for the optimisation problem.

Due to the complexity of the value functions and constraints in this problem, a numerical solution method is required. An analytical solution cannot be obtained because the problem contains several state and control variables, and the resulting first order conditions cannot be easily solved algebraically. Instead, the value function for each time period must be solved for many combinations of the starting values for each of the state variables (i.e. $X_{j_{t-1}}$, $W_{j_{t-1}}$, $a_{j_t}^{min}$, $a_{j_t}^{max}$, $b_{j_t}^{min}$, $b_{j_t}^{max}$), which are the arguments of the value function. A translog function can then be estimated via regression analysis, using the aforementioned state variables as regressors. This translog estimation is used to interpolate the value function for other combinations of initial values of the state variables.

The first step of this optimisation process is to optimise the two different terminal value functions. The two separate value functions capture the alternative cases of the system being in either Regime A or Regime B for the duration of the terminal time period. Neither of these value functions will have a probabilistic component because the system regime will be (hypothetically) known for the duration of the terminal time period. Also, utility from terminal wealth and the scrap value of the system are independent of the lower and upper bounds of the prior distributions for the unknown locations of Thresholds A and B (i.e. $a_{j_t}^{min}$, $a_{j_t}^{max}$, $b_{j_t}^{min}$, $b_{j_t}^{max}$). Instead, the arguments of the terminal value functions are the levels of the underlying slow variable ($X_{j_{t-1}}$) and the decision-maker's stock of wealth ($W_{j_{t-1}}$) at the end of the problem's penultimate time period.

When solving recursively for all non-terminal time periods, it is necessary to group paths of the underlying slow variable in terms of whether the system regime in the previous time period (i.e. $t - 1$) was Regime A (R_A) or Regime B (R_B). There will be three possible paths for each of the two groups; defined as cases 1-3 for R_A and cases 4-6 for R_B . If considered in the context of a forward progression through time, these three different cases for each group (either cases 1-3 or 4-6) will capture the system (i) remaining in the same regime with certainty, (ii) shifting to the alternative regime with certainty, and (iii) probabilistically remaining in the same regime or shifting to the alternative regime.

It is then necessary to compare the results from each of the cases 1-3 or 4-6. For each combination of starting values for the state variables mentioned above, there will be a different optimal (i.e. maximum) value of the objective function for each of the six cases. For example, if the system was in Regime A in the previous time period (i.e. $t - 1$), cases 1-3 must be compared. Finding the maximum value from these cases identifies the optimal course of action for the decision-maker, given the starting values of the state variables. The possible ‘courses of action’ for the decision-maker correspond to (i), (ii) and (iii) directly above, and the associated objective function value then feeds into the value function for the previous time period $t - 1$. The same logic applies when the system regime in the previous time period is Regime B; however, in terms of cases 4-6, rather than cases 1-3. Each of the six possible cases, and associated constraints, are described in detail below.

When solving recursively, there are three possible ways for the system to be in Regime A in the preceding time period:

1. The system is in R_A in the current period (t), as well as the preceding period ($t - 1$)

For this case, the system is certainly in R_A for both time periods; $t - 1$ and t . This means that nothing new has been learnt about the unknown location of Threshold A during this time. Therefore, $a_{j_{t+1}}^{max}$ will be equal to $a_{j_t}^{max}$, and $a_{j_{t+1}}^{min}$ will be equal to $a_{j_t}^{min}$.

The constraints for Case 1 are as follows:

$$\begin{aligned}
 \text{a. } & X_{j_t} \geq a_{j_t}^{max} \\
 \text{b. } & X_{j_{t-1}} \geq a_{j_t}^{max}
 \end{aligned} \tag{22}$$

Where

the levels of $a_{j_t}^{max}$ and $X_{j_{t-1}}$ are assumed given, and are used as parameter values for the optimisation problem

Constraint (a) ensures that the system certainly remains in R_A in time period t because X_{j_t} is constrained to remain within the ‘safe’ sub-space. Constraint (b) ensures that the system was certainly in R_A in time period $t - 1$. This constraint also ensures that the optimisation for Case 1 is only undertaken for feasible starting levels of $X_{j_{t-1}}$.²⁴

2. The system is in R_B in the current period (t), but was in R_A the preceding period ($t - 1$)

For this case, the system is certainly in R_A for time period $t - 1$, and certainly in R_B for time period t . This means that nothing new has been learnt about the unknown location of Threshold A during this time. Therefore, $a_{j_{t+1}}^{max}$ will be equal to $a_{j_t}^{max}$, and $a_{j_{t+1}}^{min}$ will be equal to $a_{j_t}^{min}$.

The constraints for Case 2 are as follows:

$$\begin{aligned}
 \text{c. } & X_{j_t} \leq a_{j_t}^{min} \\
 \text{d. } & X_{j_{t-1}} \geq a_{j_t}^{max}
 \end{aligned} \tag{23}$$

Where

²⁴ Also, note that the levels of X_{j_t} , $a_{j_{t+1}}^{min} = a_{j_t}^{min}$, and $a_{j_{t+1}}^{max} = a_{j_t}^{max}$ are arguments of the A-value function, $V_{j_{t+1}(A)}$, for time period $t + 1$.

the levels of $a_{j_t}^{min}$, $a_{j_t}^{max}$ and $X_{j_{t-1}}$ are assumed given, and are used as parameter values for the optimisation problem

Constraint (c) ensures that the system certainly shifts to R_B in time period t because X_{j_t} is constrained to pass through the ‘safe’ sub-space, and pass sufficiently through the ‘risky’ sub-space such that the probability of a regime shift occurring is one. Constraint (d) ensures that the system was certainly in R_A in time period $t - 1$. This constraint also ensures that the optimisation for Case 2 is only undertaken for feasible starting levels of $X_{j_{t-1}}$.²⁵

3. The system is probabilistically in both R_A and R_B in the current period (t), but was in R_A the preceding period ($t - 1$)

For this case, the system is certainly in R_A for time period $t - 1$, and probabilistically in R_A and R_B for time period t . The underlying slow variable enters the ‘risky’ sub-space, and something is learnt about the unknown location of Threshold A. This means that, first, the upper bound of the prior probability distribution ($a_{j_t}^{max}$) can be updated if the threshold is not crossed (i.e. the system remains in R_A), and, second, the lower bound of the prior probability distribution ($a_{j_t}^{min}$) can be updated if the threshold is crossed (i.e. the system shifts to R_B).

The constraints for Case 3 are as follows:

$$\begin{aligned} \text{e. } & a_{j_t}^{min} < X_{j_t} < a_{j_t}^{max} \\ \text{f. } & X_{j_{t-1}} \geq a_{j_t}^{max} \end{aligned} \tag{24}$$

Where

the levels of $a_{j_t}^{min}$, $a_{j_t}^{max}$ and $X_{j_{t-1}}$ are assumed given, and are used as parameter values for the optimisation problem

²⁵ Also, note that the levels of X_{j_t} , $a_{j_{t+1}}^{min} = a_{j_t}^{min}$, and $a_{j_{t+1}}^{max} = a_{j_t}^{max}$ are arguments of the B-value function, $V_{j_{t+1}(B)}$, for time period $t + 1$.

Constraint (e) ensures that X_{j_t} enters the ‘risky’ sub-space in time period t , and there is a non-zero and non-one probability of the system shifting regime, or not shifting regime. Constraint (f) ensures that the system was certainly in R_A in time period $t - 1$. This constraint also ensures that the optimisation for Case 3 is only undertaken for feasible starting levels of $X_{j_{t-1}}$.²⁶

When solving recursively, there are three possible ways for the system to be in Regime B in the preceding time period:

4. The system is in R_A in the current period (t), but was in R_B the preceding period ($t - 1$)

For this case, the system is certainly in R_B for time period $t - 1$, and certainly in R_A for time period t . This means that nothing new has been learnt about the unknown location of Threshold B during this time. Therefore, $b_{j_{t+1}}^{max}$ will be equal to $b_{j_t}^{max}$, and $b_{j_{t+1}}^{min}$ will be equal to $b_{j_t}^{min}$.

The constraints for Case 4 are as follows:

- g. $X_{j_t} \geq b_{j_t}^{max}$
 - h. $X_{j_{t-1}} \leq b_{j_t}^{min}$
- (25)

Where

the levels of $b_{j_t}^{min}$, $b_{j_t}^{max}$ and $X_{j_{t-1}}$ are assumed given, and are used as parameter values for the optimisation problem

Constraint (g) ensures that the system certainly shifts to R_A in time period t because X_{j_t} is constrained to pass through the ‘safe’ sub-space, and pass sufficiently through the ‘risky’ sub-space such that the probability of a regime shift occurring is one. Constraint (h) ensures that the system was certainly in R_B in time period $t - 1$.

²⁶ Also, note that the levels of X_{j_t} , $a_{j_{t+1}}^{min}$, and $a_{j_{t+1}}^{max}$ are arguments of the seceding value function for time period $t + 1$. For the A-value function, $V_{j_{t+1}(A)}$, the upper bound of the prior distribution is updated such that $a_{j_{t+1}(A)}^{max} = X_{j_t}$. For the B-value function, $V_{j_{t+1}(B)}$, the lower bound of the prior distribution is updated such that $a_{j_{t+1}(B)}^{min} = X_{j_t}$.

This constraint also ensures that the optimisation for Case 4 is only undertaken for feasible starting levels of $X_{j_{t-1}}$.²⁷

5. The system is in R_B in the current period (t), as well as the preceding period ($t - 1$)

For this case, the system is certainly in R_B for both time periods; $t - 1$ and t . This means that nothing new has been learnt about the unknown location of Threshold B during this time. Therefore, $b_{j_{t+1}}^{max}$ will be equal to $b_{j_t}^{max}$, and $b_{j_{t+1}}^{min}$ will be equal to $b_{j_t}^{min}$.

The constraints for Case 5 are as follows:

$$\begin{aligned} \text{i. } & X_{j_t} \leq b_{j_t}^{min} \\ \text{j. } & X_{j_{t-1}} \leq b_{j_t}^{min} \end{aligned} \tag{26}$$

Where

the levels of $b_{j_t}^{min}$ and $X_{j_{t-1}}$ are assumed given, and are used as parameter values for the optimisation problem

Constraint (i) ensures that the system certainly remains in R_B in time period t because X_{j_t} is constrained to remain within the ‘safe’ sub-space, where a threshold crossing is not possible. Constraint (j) ensures that the system was certainly in R_B in time period $t - 1$. This constraint also ensures that the optimisation for Case 5 is only undertaken for feasible starting levels of $X_{j_{t-1}}$.²⁸

²⁷ Also, note that the levels of X_{j_t} , $b_{j_{t+1}}^{min} = b_{j_t}^{min}$, and $b_{j_{t+1}}^{max} = b_{j_t}^{max}$ are arguments of the A-value function, $V_{j_{t+1}(A)}$, for time period $t + 1$.

²⁸ Also, note that the levels of X_{j_t} , $b_{j_{t+1}}^{min} = b_{j_t}^{min}$, and $b_{j_{t+1}}^{max} = b_{j_t}^{max}$ are arguments of the B-value function, $V_{j_{t+1}(B)}$, for time period $t + 1$.

6. The system is probabilistically in both R_A and R_B in the current period (t), but was in R_B the preceding period ($t - 1$)

For this case, the system is certainly in R_B for time period $t - 1$, and probabilistically in R_A and R_B for time period t . The underlying slow variable enters the ‘risky’ sub-space, and something is learnt about the unknown location of Threshold B. This means that, first, the lower bound of the prior probability distribution ($b_{j_t}^{min}$) can be updated if the threshold is not crossed (i.e. the system remains in R_B), and, second, the upper bound of the prior probability distribution ($b_{j_t}^{max}$) can be updated if the threshold is crossed (i.e. the system shifts to R_A).

The constraints for Case 6 are as follows:

$$\begin{aligned} \text{k. } & b_{j_t}^{min} < X_{j_t} < b_{j_t}^{max} \\ \text{l. } & X_{j_{t-1}} \leq b_{j_t}^{min} \end{aligned} \tag{27}$$

Where

the levels of $b_{j_t}^{min}$, $b_{j_t}^{max}$ and $X_{j_{t-1}}$ are assumed given, and are used as parameter values for the optimisation problem

Constraint (k) ensures that X_{j_t} enters the ‘risky’ sub-space in time period t , and there is a non-zero and non-one probability of the system shifting regime, or not shifting regime. Constraint (l) ensures that the system was certainly in R_B in time period $t - 1$. This constraint also ensures that the optimisation for Case 6 is only undertaken for feasible starting levels of $X_{j_{t-1}}$.²⁹

Summary of updating rules for use in simulation modelling

²⁹ Also, note that the levels of X_{j_t} , $b_{j_{t+1}}^{min}$, and $b_{j_{t+1}}^{max}$ are arguments of the seceding value function for time period $t + 1$. For the A-value function, $V_{j_{t+1}(A)}$, the upper bound of the prior distribution is updated such that $b_{j_{t+1}(A)}^{max} = X_{j_t}$. For the B-value function, $V_{j_{t+1}(B)}$, the lower bound of the prior distribution is updated such that $b_{j_{t+1}(B)}^{min} = X_{j_t}$.

In order to be able to solve this problem recursively, the updating rules previously described in Sections 3.2.2 and 3.2.3 must be rearranged to reflect a backward, rather than a forward, progression through time. These rearranged updating rules then form constraints of the optimisation problem.

A numerical solution method is employed because the large number of state and control variables means an analytical solution, using differential calculus, is not possible. A dynamic programming approach is used to recursively solve value functions for each time period and for many different combinations of starting levels of each state variable. A translog, or flexible form, function is then used to estimate the true value function for each time period and system regime.

These estimated value functions feed into the value functions of the previous time period. This process is repeated many times in order to approximate a problem structure with an infinite time horizon. The solution values obtained are for the decision-maker's optimal course of action in the first time period of the problem (i.e. 'today'), given the initial conditions of the problem. These initial conditions include the level of the underlying slow variable and the prior probability bounds for the unknown threshold location/s.

3.3 Equation of motion (EOM) for wealth

For this model design, there is no requirement for a positive starting amount of wealth (budget). However, for most applications of the model, there will be a positive initial amount of wealth. The source of this wealth and the frequency with which it is subsidised is also likely to be context-specific. It is possible that there is a continuous stream of funding from an external source in the form of a constant per-time-period budget allocation. Alternatively, the only form of external funding might be through a one-off payment (wealth) provided at the start of the problem. Many other specifications are possible for the EOM for wealth; however, it is assumed that the decision-maker is not allowed to engage in any Ponzi game behaviour³⁰. The

³⁰ A Ponzi game, or Ponzi scheme, is one in which each new wave of consumption expenditure is funded by taking on more debt. This new loan is used to repay existing debt and fund new consumption

specification for the EOM for wealth is also likely to differ, conditional on whether the relevant project is primarily publicly or privately funded and/or managed.

For a public project, utility will predominantly be sourced through direct utility. Since such a project is unlikely to generate sufficient wealth to fund its own management, it will be necessary for funding to be provided from an external source (e.g. a government agency) on an ongoing basis. This situation can only be avoided if the ecosystem is able to generate sufficient profits to fund its own management. Consumption (dis)utility in the context of a public project would represent the utility, or welfare, gained by society from putting government funding to an alternative use. In other words, it represents the marginal social opportunity cost (MSOC) of spending the money on managing a particular ecosystem rather than putting it to an alternative use (e.g. funding infrastructure development or education).

For a private project, utility will predominantly be sourced from consumption utility. Funding (wealth) for managing the system will most likely come from profits generated by the system itself. For this reason, no external funding (i.e. external to the private individual or business) would be necessary. However, a positive starting amount of wealth is likely to be required to cover the start-up costs of the project.

The evolution of wealth over time is given by:

$$W_{j_t} = \begin{cases} (1 + r_t) \cdot W_{j_{t-1}} + G_{j_t} - q_{j_t} + \pi_{j_t}^A(C_{j_t}, X_{j_{t-1}}) & \text{if } R_{j_t} = A \\ (1 + r_t) \cdot W_{j_{t-1}} + G_{j_t} - q_{j_t} + \pi_{j_t}^B(C_{j_t}, X_{j_{t-1}}) & \text{if } R_{j_t} = B \end{cases} \quad (28)$$

Where

$(1 + r_t) \cdot W_{j_{t-1}}$ is carried over wealth

G_{j_t} is external funding

q_{j_t} is consumption expenditure paid for by profits from the system

C_{j_t} is a vector of levels of the control variables

expenditure. So long as a sufficient pool of loan funds is available, this spending-borrowing cycle can continue indefinitely. Such a scheme is conceptually equivalent to acquiring a new credit card every time repayments are due on existing debts.

$X_{j_{t-1}}$ is the level of the underlying slow variable

$\pi_{j_t}^i$ is profit from the system, conditional on the system regime i , levels of choice variables and the level of the underlying slow variable

If a regime shift occurs, it occurs at the end of a time period. Therefore, the system remains in a given system regime for the duration of a time period. This means that wealth at the end of a time period is calculated as the sum of carried over wealth from the previous time period (plus interest), any additional external funding and profits acquired from the system in the current time period. Consumption expenditure must be subtracted from this amount to calculate the level of wealth at the end of time period t .

3.4 The value function: the NPV of expected future utility

The value function captures utility acquired in the current time period and all future time periods. It is used as the objective function for the dynamic optimisation problem that determines the optimal level/s of control variable/s and the maximum expected NPV of utility, given a set of initial values of the state variables of the decision problem. Utility can be derived from several sources, which are described below.

Section 3.4.1 introduces four alternative sources of utility. Section 3.4.2 describes the formulation of the value function, which is the dynamic equivalent of an objective function.

3.4.1 Sources of utility

In the most general case of the decision problem, there are four distinct sources of utility, which are:

- (i) Consumption utility

Consumption utility is denoted $U_q(q_{j_t})$ and is a function of the amount of acquired wealth used for expenditure on consumption. It captures utility gained from the goods and services purchased using this consumption expenditure. It is assumed that the marginal utility of consumption utility is strictly positive but decreasing in q_{j_t} , such that $U_q'(q_{j_t}) > 0$, and $U_q''(q_{j_t}) < 0$.

(ii) Direct utility

Direct utility is denoted $U_Y(Y_{j_t})$ and is a function of the level of ‘output’ of the system. It captures utility gained from non-use or non-harvest values, such as aesthetic and existence values. It is assumed that the marginal utility of direct utility is strictly positive but decreasing in Y_{j_t} , such that $U_Y'(Y_{j_t}) > 0$, and $U_Y''(Y_{j_t}) < 0$.

(iii) Utility from terminal wealth

Utility from terminal wealth is denoted $U_{W_T}(W_{j_T})$ and is a function of the level of acquired wealth at the end of the terminal time period T . Utility from terminal wealth is only applicable to the terminal time period T , and none of the preceding time periods. It captures the bequest value of acquired wealth at the end of the problem’s time horizon. Therefore, there is an implicit assumption that society will continue to exist beyond the problem’s finite time horizon. $U_{W_T}(W_{j_T})$ may be thought of as a lump sum utility payment acquired just prior to the termination of the problem. This component is included because the problem is solved over a fixed time horizon, $t = [1, T]$. Utility from terminal wealth assigns a positive, but notional, amount to the utility that could be obtained by future generations using the wealth acquired up to terminal time T . However, optimality in decision making of future generations is not assumed. In the event that terminal wealth is negative, utility from terminal wealth will also be negative. This structure provides a disincentive to engage in Ponzi game behaviour without explicitly preventing such behaviour.

(iv) Scrap value of the system, conditional on its current state

The scrap value of the system is denoted $U_{X_T}(X_{j_T}, R_{j_T})$ and is a function of the level of the underlying slow ecosystem variable and the current system regime at the end of the terminal time period T . For simplicity, it is assumed that the system will remain in regime R_{j_T} and the level of the underlying slow ecosystem variable will be costlessly maintained at X_{j_T} in perpetuity. If the ‘output’ of the system is saleable, the decision-maker will receive $U_q(p.Y_{j_T}(X_{j_T}, R_{j_T}))$ for each subsequent time period, in perpetuity. If the ‘output’ from the system is non-saleable, the decision-maker will receive $U_Y(Y_{j_T}(X_{j_T}, R_{j_T}))$ for each subsequent time period, in perpetuity. Since the utility received from either, or both, of saleable and non-saleable output will be constant for each subsequent time period, the NPV of utility received in each future time period can be characterised as the sum of an infinite geometric series. When expressed in time period T -dollars, the scrap value is given by:

$$U_{X_T}(X_{j_T}, R_{j_T}) = \gamma\bar{U} + \gamma^2\bar{U} + \gamma^3\bar{U} + \dots = \gamma\bar{U}(1 + \gamma + \gamma^2 + \dots) \quad (29)$$

Where

ρ is the pure rate of time preference

$\gamma = (1 + \rho)^{-1}$ is the discount factor

$\bar{U} = [U_q(p.Y_{j_T}(X_{j_T}, R_{j_T})) + U_Y(Y_{j_T}(X_{j_T}, R_{j_T}))]$ is the instantaneous utility function after time period T

If ρ is strictly positive, the scrap value will converge to a positive constant, given by:

$$U_{X_T}(X_{j_T}, R_{j_T}) = \sum_{k=T+1}^{\infty} \bar{U}\gamma^{k-T} = \frac{\bar{U}\gamma}{1 - \gamma} \quad (30)$$

Using this approach, there is no requirement for optimisation beyond the terminal time of the problem. However, some value is still placed on the health (state) of the system beyond the terminal time. In the absence of a scrap value for the system, the optimal course of action would likely be to under-invest in maintaining, or

improving, the level of X_{j_t} during the latter time periods of the problem. This will be the case because without a scrap value of the system, any benefits of such investment that accrue after time T would be unaccounted and, therefore, not impact on the decision-making process.

3.4.2 The value function

The value function captures the expected net present value (NPV) of utility received from all sources in the current time period and all succeeding time periods. Utility sources (iii) and (iv) above are only applicable to the terminal time period T ; however, their influence will extend to all value functions preceding time period T .

Since this problem must be solved recursively, the first value function that must be solved is that of the terminal time period T . The value function for terminal time T will have two slightly different formulations, conditional on whether the starting regime for time period T (i.e. finishing regime for time period $T - 1$) is R_A or R_B . The value function for the terminal time period is denoted by V_{j_T} .

a. If $R_{j_T} = A$

$$V_{j_T(A)} = \max_{c_{j_T}, q_{j_T}, E_{j_T}} \{U_q(q_{j_T}) + U_Y(Y^A(X_{j_T})) + U_{W_T}(W_{j_T}) + U_{X_T}(X_{j_T}, R_A)\} \quad (31)$$

b. If $R_{j_T} = B$

$$V_{j_T(B)} = \max_{c_{j_T}, q_{j_T}, E_{j_T}} \{U_q(q_{j_T}) + U_Y(Y^B(X_{j_T})) + U_{W_T}(W_{j_T}) + U_{X_T}(X_{j_T}, R_B)\} \quad (32)$$

For the terminal time period, utility is sourced from consumption and direct utility, utility from terminal wealth, and the scrap value of the system. For simplicity, it is assumed that if the system is (hypothetically) in Regime i at the beginning of the terminal time period, it will remain in Regime i in perpetuity. From time period $T + 1$ onwards, it is assumed that the level of X is costlessly maintained at a constant level,

and the stream of per-period undiscounted net benefits from the system remains constant.

For all other time periods, the value function is composed of (i) the utility function of the current time period, and (ii) a conditional probability-weighted average of the two possible value functions for the succeeding time period. One value function corresponds to ending the time period in Regime A, and the other in Regime B.

Here, the probability weights assigned to value functions for time period $t + 1$ are functions of the underlying slow variable (X_{j_t}), and the lower and upper bounds for the unknown location of the relevant ecological threshold ($a_{j_t}^{min}$ and $a_{j_t}^{max}$, or $b_{j_t}^{min}$ and $b_{j_t}^{max}$). The levels of all state variables ($X_{j_t}, W_{j_t}, a_{j_t}^{min}, a_{j_t}^{max}, b_{j_t}^{min}, b_{j_t}^{max}$) at the end of time period t are then ‘inherited’ at the beginning of time period $t + 1$, and become the arguments of the corresponding value function; $V_{j_{t+1}(A)}$ or $V_{j_{t+1}(B)}$.

a. If $R_{j_t} = A$

$$\begin{aligned}
& V_{j_t(A)}(X_{j_{t-1}}, W_{j_{t-1}}, a_{j_{t-1}}^{min}, a_{j_{t-1}}^{max}, b_{j_{t-1}}^{min}, b_{j_{t-1}}^{max}) \\
&= \max_{C_{j_t}, q_{j_t}, E_{j_t}} \left\{ U_q(q_{j_t}) + U_Y(Y^A(X_{j_t})) \right. \\
&+ (1 + \rho)^{-1} \left[A_{j_t}(X_{j_t}) \cdot V_{j_{t+1}(A)}(X_{j_t}) + (1 - A_{j_t}(X_{j_t})) \cdot V_{j_{t+1}(B)}(X_{j_t}) \right] \left. \right\} \\
&= \max_{C_{j_t}, q_{j_t}, E_{j_t}} \left\{ U_q(q_{j_t}) + U_Y(Y^A(X_{j_t})) \right. \\
&+ (1 + \rho)^{-1} \left[A(X_{j_t} | a_{j_t}^{min}, a_{j_t}^{max}) \cdot V_{j_{t+1}(A)}(X_{j_t}, W_{j_t}, a_{j_t}^{min}, a_{j_t}^{max}, b_{j_t}^{min}, b_{j_t}^{max}) \right. \\
&+ \left. \left. (1 - A(X_{j_t} | a_{j_t}^{min}, a_{j_t}^{max})) \cdot V_{j_{t+1}(B)}(X_{j_t}, W_{j_t}, a_{j_t}^{min}, a_{j_t}^{max}, b_{j_t}^{min}, b_{j_t}^{max}) \right] \right\} \tag{33}
\end{aligned}$$

b. If $R_{j_t} = B$

$$\begin{aligned}
& V_{j_t(B)}(X_{j_t-1}, W_{j_t-1}, a_{j_t-1}^{min}, a_{j_t-1}^{max}, b_{j_t-1}^{min}, b_{j_t-1}^{max}) \\
&= \max_{C_{j_t}, q_{j_t}, E_{j_t}} \left\{ U_q(q_{j_t}) + U_Y(Y^B(X_{j_t})) \right. \\
&+ (1 + \rho)^{-1} \left[(1 - B_{j_t}(X_{j_t})) \cdot V_{j_{t+1}(A)}(X_{j_t}) + B_{j_t}(X_{j_t}) \cdot V_{j_{t+1}(B)}(X_{j_t}) \right] \left. \right\} \\
&= \max_{C_{j_t}, q_{j_t}, E_{j_t}} \left\{ U_q(q_{j_t}) + U_Y(Y^B(X_{j_t})) \right. \\
&+ (1 + \rho)^{-1} \left[(1 \right. \\
&- B(X_{j_t} | b_{j_t}^{min}, b_{j_t}^{max})) \cdot V_{j_{t+1}(A)}(X_{j_t}, W_{j_t}, a_{j_t}^{min}, a_{j_t}^{max}, b_{j_t}^{min}, b_{j_t}^{max}) \\
&+ B(X_{j_t} | b_{j_t}^{min}, b_{j_t}^{max}) \cdot V_{j_{t+1}(B)}(X_{j_t}, W_{j_t}, a_{j_t}^{min}, a_{j_t}^{max}, b_{j_t}^{min}, b_{j_t}^{max}) \left. \right] \left. \right\} \tag{34}
\end{aligned}$$

Summary of modelling framework

This chapter details a new modelling framework for the management of complicated ecological systems that explicitly models active learning about unknown threshold locations. The decision-maker's objective is to maximise the NPV of expected utility from the system over a finite time horizon, $[1, T]$. The general nature of the modelling framework makes it applicable to ecological systems with reversible or irreversible threshold effects.

Learning about an unknown threshold location is modelled by partitioning the system's state space into regions that could and could not contain the threshold, and postulating a prior probability distribution for the location of the threshold. The prior distribution is updated over time as more information is hypothetically gathered, where the potential for learning is conditional on the management actions undertaken by the decision-maker. A dynamic programming approach is used to solve the problem recursively and determine optimal management decisions, conditional on the initial conditions of the problem.

Chapter 4. Modelling commercial production with uncertain ecological thresholds

The management of ecosystems involves many different types of trade-offs and threshold effects, including trade-offs between use-values, non-use values, and different degrees of reversibility around critical thresholds. Chapter 4 and Chapter 5 each present two detailed case studies of complicated ecological systems that can be managed using the mathematical modelling framework developed in this thesis. Chapter 4 explores the trade-offs between the use values associated with profits from continuing production in a shallow lake system with hysteretic dynamics in comparison with a savannah system with a completely reversible threshold. Chapter 5 focuses on trade-offs associated with environmental quality in terms of use and non-use values, by examining a case study of the trade-offs between harvesting fish whilst maintaining the health of a reef system and a case study of the trade-offs between housing construction versus conserving a koala population. Table 4-1 below provides a summary of the similarities and differences of the two representative case studies discussed in Chapter 4.

Table 4-1 Key features of the shallow lake and savannah case studies

Characteristic	Shallow lake	Savannah
Threshold effect	Hysteretic dynamics	Completely reversible
Underlying slow variable	Phosphorus concentration	Grass cover – fuel load
Trade-offs	Better water quality in lake system vs. Harvesting crops for profit	Non-use values from biodiversity in the rangeland system vs. Grazing livestock for profit
Ecological dynamics	Nutrient cycling, where phosphorus is added via natural and anthropogenic processes and partially assimilated by the system	Grass and tree prevalence modelled using a space implicit model, where trees are the superior competitor
Sources of utility	<p><i>Direct utility:</i> Recreational use of lake – function of water quality</p> <p><i>Consumption utility:</i> Paid for using profits from cropping activities</p>	<p><i>Direct utility:</i> Existence and aesthetic value – function of composition of vegetation types, which serves as a proxy for biodiversity.</p> <p><i>Consumption utility:</i> Paid for using profits from grazing activities</p>

4.1 Shallow lake and adjoining agricultural land

A shallow lake provides benefits to recreational users as they enjoy swimming, fishing, boating and other activities. However, the runoff of phosphorus fertiliser from adjacent agricultural land into the shallow lake system can suddenly turn the lake water from clear (oligotrophic) to dirty (eutrophic) when a critical threshold is crossed. This regime shift adversely affects the utility derived by recreational users, but is a result of the multiple uses of the lake, where the second use is as a waste sink. Utility is also derived from consumption paid for by profits from cropping on land adjacent to the lake, but this comes at a cost of reducing the water quality of, and utility derived from, the lake.

For this case study, the stock of a pollutant, phosphorus, is modelled as a function of phosphorus input resulting from agricultural activities on adjoining land, mean natural input of phosphorus, internal nutrient cycling (feedbacks) and the lake's ability to assimilate the pollutant load (Brozovic and Schlenker 2011). The dynamics and economics of shallow lakes have been studied extensively (Scheffer *et al.* 1993; Carpenter and Cottingham 1997; Scheffer 1997; Carpenter *et al.* 1999; Nævdal 2001; Mäler *et al.* 2003; Peterson *et al.* 2003; Carpenter 2005; Brozovic and Schlenker 2011). Shallow lakes are typified by non-linear dynamics, discontinuities and hysteresis. Shallow lake systems also typically provide competing services as a resource and a waste sink. Shallow lakes can exist in two different regimes. The ecologically desirable regime is the oligotrophic regime, which is typified by clear water and a higher level of ecological services. The ecologically undesirable regime is the eutrophic regime, which is typified by green, turbid water and a lower level of ecological services (Mäler *et al.* 2003).

In this example, crop output and profits are generated through the application of phosphorus-rich fertilisers on agricultural lands adjacent to the lake. Cropping represents a major source of utility, via consumption paid for by profits generated from this activity. Interestingly, the lake serves two competing purposes. First, as a waste sink for phosphorus runoff that flows from the agricultural land into the lake. Second, as a resource in terms of facilitating recreational activities (e.g. swimming,

boating) and providing non-use values (e.g. amenity value). The lake's role as a resource provides direct utility; however, the level of utility provided is conditional on the health of the lake and is adversely affected if the lake shifts from an oligotrophic to a eutrophic regime. It is the responsibility of an ecosystem manager (representative agent) to manage the agricultural land-lake system with the objective of maximising the NPV of utility sourced from it.

4.1.1 Shallow lake: State variables

Table 4-2 below lists and describes the state variables of the shallow lake case study.

Table 4-2 State variables of the shallow lake case study

State variables	Description
Underlying slow ecosystem variable (X)	Stock of phosphorus in the lake – a function of phosphorus runoff from adjacent agricultural land, natural inputs and internal nutrient cycling
Stock of abatement technologies (\bar{I}_t)	Reduces the proportion of phosphorus runoff that makes its way from agricultural land to the lake
Stock of wealth (W)	Wealth acquired via profits generated from the harvest and sale of an agricultural crop
Prior probability bounds (a^{min}, a^{max})	Describes the location of a reversible ecological threshold of unknown location that separates two alternative population growth regimes – although there is a single threshold, the system has hysteretic dynamics because feedbacks naturally push the system away from the threshold when in the eutrophic (undesirable) regime

4.1.2 Shallow lake: Control variables

Table 4-3 below lists and describes the control variables of the shallow lake case study.

Table 4-3 Control variables of the shallow lake case study

Control variable	Description
Phosphorus fertiliser (P_t)	Amount of phosphorus fertiliser applied to the agricultural land adjacent to the lake – crop yield (Y_t) is a monotonically increasing function of phosphorus fertiliser
Investment in abatement technologies (I_t)	Reduces the transfer coefficient (τ_t) of phosphorus into the lake system, where $\tau_t \in [0, 1]$ – phosphorus transfer results from agricultural runoff (O_t) and natural inputs (r)
Crop harvest (H_t)	The entire crop yield is harvested at the end of each time period, such that $H_t = Y_t$
Expenditure on consumption activities (q)	These activities generate consumption utility – paid for using acquired wealth

4.1.3 Shallow lake: Trade-offs

The trade-offs faced in this case study are between consumption utility, which is paid for using profits generated from the harvesting of an agricultural crop (Y) from land adjacent to the lake; direct utility, which is a function of the lake system's current regime (R_i) and level of phosphorus (X); and disutility (opportunity cost) from the use of public funds for the purpose of increasing the stock of abatement technologies rather than on the best alternative use. Direct utility acquired from using the lake for recreational purposes, such as swimming, boating and fishing, is negatively impacted when the lake switches from the oligotrophic to the eutrophic regime and when the stock of phosphorus in the lake increases. Investment in abatement technologies decreases the proportion of phosphorus applied for agricultural purposes that reaches the lake system; however, this investment requires the use of public funding and creates disutility because these funds cannot be spent on the best alternative use.

The transfer coefficient, τ_t , denotes the proportion of agricultural runoff and natural input that makes its way into the lake, compared to the full amount of phosphorus that would make its way into the lake in the absence of any abatement technologies. This coefficient will be less than one if there is a positive stock of abatement technologies because some of this phosphorus is abated prior to it entering the lake. The transfer can be reduced by building up the stock of abatement technologies (Figure 4-1).

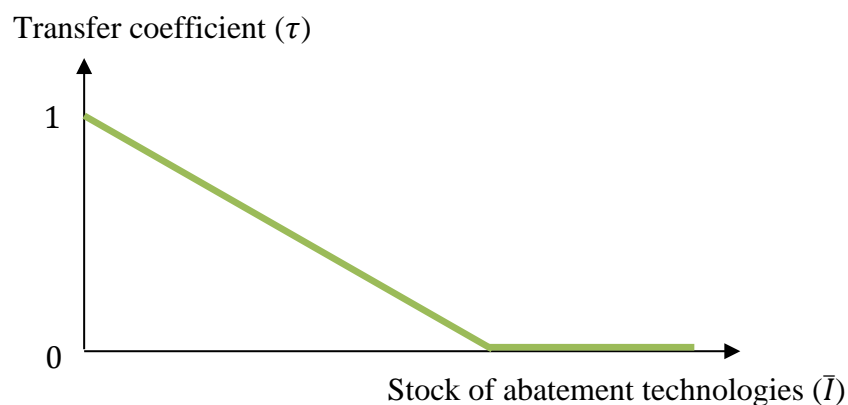


Figure 4-1 Transfer coefficient for agricultural runoff to the lake

Where

$$\tau_t = \max(1 - i \cdot \bar{I}_t, 0)$$

$i > 0$ is an efficiency parameter that measures the effectiveness of abatement technology at reducing the transfer coefficient

The stock of abatement technologies (\bar{I}_t) evolves according to:

$$\bar{I}_t = d \cdot \bar{I}_{t-1} + I_t \quad (35)$$

Where

$d \in [0,1]$ is the proportion of abatement technologies that carry over

Crop yield as a function of phosphorus application is monotonically increasing in its argument but experiences diminishing marginal returns. This relationship is illustrated by the crop response function (Figure 4-2).

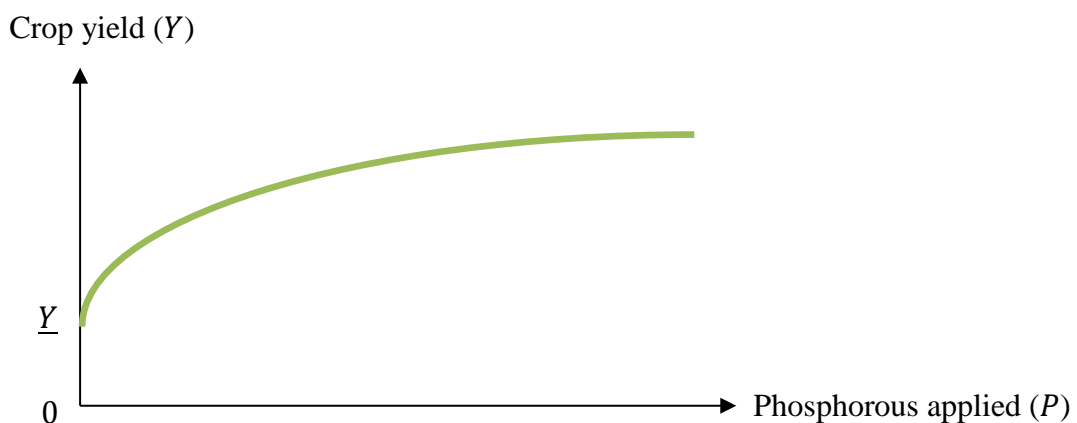


Figure 4-2 Crop response function

Where

$$Y_t = \underline{Y} + p \cdot P_t^\alpha$$

$\underline{Y} \geq 0$ is the crop yield in the absence of phosphorus application

$p > 0$ is a scale parameter for the impact of phosphorus on crop yield

$\alpha \in (0,1)$ determines the curvature of the crop response function

4.1.4 Shallow lake: EOM for stock of phosphorus in the lake (X)

For this case study, it is ultimately the stock of phosphorus within the lake system that determines whether the lake exists in the ecologically desirable oligotrophic regime (R_A) or the eutrophic regime (R_B). The stock of phosphorus is modelled as a function of phosphorus input resulting from agricultural activities on adjoining land, mean natural input of phosphorus, internal nutrient cycling (feedbacks) and the lake's ability to assimilate the pollutant load (Brozovic and Schlenker 2011). It is the internal nutrient cycling (feedbacks) that differs between the two alternative regimes. The dynamics of the lake system are described as hysteretic because, when in the eutrophic regime, these feedbacks naturally push the system in the ecologically undesirable direction (i.e. away from the threshold) at a faster rate than would be the case in the oligotrophic regime. This means that, although a regime shift can be reversed by simply re-crossing the same threshold (in terms of X), it is much more costly (in terms of time and investment in abatement technologies) to engineer a regime shift from the eutrophic to the oligotrophic regime than it would be for a shift from the oligotrophic to the eutrophic regime.

The stock of phosphorus in the lake evolves according to the following regime-dependent (for Regime i) equation:

$$\Delta X_t = \begin{cases} (B - 1)X_{t-1} + \tau_{t-1} \cdot (b + O_{t-1}) & \text{if } X_{t-1} < X^c, i = A \\ (B - 1)X_{t-1} + \tau_{t-1} \cdot (b + O_{t-1}) + r & \text{if } X_{t-1} \geq X^c, i = B \end{cases} \quad (36)$$

Where

$B \in [0,1]$ is the proportion of the pollutant X that carries over from one period to the next (i.e. $(1 - B)X_{t-1}$ is assimilated and the pollutant load is reduced by $(B - 1)X_{t-1}$)

b is the mean natural input of the pollutant to the environmental (lake) system

$r > 0$ is additional pollutant loading when the lake is in the eutrophic regime

$\tau_{t-1} \in [0,1]$ is the transfer coefficient, which determines the proportion of agricultural runoff and natural input of phosphorus that enters the lake system

$O_{t-1} = o \cdot P_{t-1}$ is the amount of agricultural runoff caused by application of phosphorus fertiliser

$o \in (0,1)$ is the proportion of phosphorus fertiliser applied that converts to agricultural runoff

Note that the additional term r is included in the EOM of X when the shallow lake system is in Regime B (eutrophic). It is this term that gives the system its hysteretic dynamics because the term ensures that the process of sufficiently reducing the level of X in order to re-cross the threshold back from the eutrophic to the oligotrophic regime is both costly and time-consuming.

4.1.5 Shallow lake: Critical ecological thresholds

The shallow lake system is typified by two separate system regimes, where one experiences positive feedbacks and additional pollutant loading (eutrophic), and the other does not (oligotrophic). These two regimes are linked by a single threshold in terms of the stock of phosphorus in the lake (X). The system experiences hysteresis in the form of positive feedbacks when in the eutrophic regime, which naturally push the system in the ecologically undesirable direction at a faster rate than would be the case in the oligotrophic regime. When in the eutrophic regime, a lake usually has a greenish look resulting from a dominance of phytoplankton (Carpenter and Cottingham 1997; Scheffer 1997). Additional nutrient loads, especially phosphorus, stimulate the growth of phytoplankton and increase turbidity. High turbidity prevents light from reaching the bottom of the lake, which kills the submerged vegetation and the organisms that graze on it, such as waterfleas (Mäler *et al.* 2003).

In Regime A (R_A), the stock of phosphorus is sufficiently low for the shallow lake to be in the oligotrophic regime. In Regime B (R_B), the stock of phosphorus is sufficiently high for the shallow lake to be in the eutrophic regime. In Figure 4-3, the vertical dashed arrow signifies the crossing of the critical ecological threshold and the resulting shift in system regime. This regime shift is completely reversible along the same path, which is why the arrow is both upward- and downward-pointing. The purpose of the horizontal dotted line is to indicate that the exact location of the threshold is unknown; however, upper and lower bounds can be postulated. The system regime is modelled according to:

$$R_{j_t} = \begin{cases} A & \text{if } X_{j_{t-1}} < X^C \\ B & \text{if } X_{j_{t-1}} \geq X^C \end{cases} \quad (37)$$

Where

R_{j_t} is the system regime for the duration of time period t

$X_{j_{t-1}}$ is the level of the underlying slow variable at the end of the previous time period

X^C is a random variable on a bounded interval $(X^{C^{min}}, X^{C^{max}})$

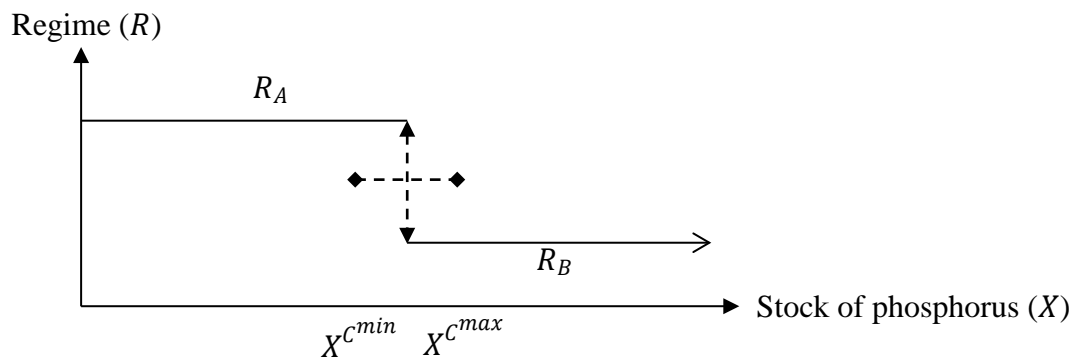


Figure 4-3 Alternative system regimes for the cycling of phosphorus in the shallow lake

4.1.6 Shallow lake: Sources of utility

Utility is gained both from consumption utility, paid for using profits generated from the harvesting of crops (H) from the agricultural land adjoining the lake system, and from direct utility, as a function of the stock of phosphorus in the lake (X) and the current regime of the lake (R_i). The shallow lake system facilitates recreational activities (e.g. swimming, boating) and provides non-use values (e.g. amenity value). The level of direct utility provided is conditional on the health of the lake and is adversely affected if the lake shifts from an oligotrophic to a eutrophic regime. Also, negative utility results from the use of any public funding for investment in building the stock of abatement technologies. This disutility captures the opportunity cost to society of spending public funds on increasing the stock of abatement technologies, and reducing the proportion of phosphorus flows from

agricultural runoff and natural inputs that make their way to the lake system, rather than on the best alternative use.

When in Regime A (oligotrophic), direct utility is acquired according to the following equation:

$$U_A(X_t) = \alpha_A \cdot (X_t + 1)^{\beta_A} + K \quad (38)$$

Where

$$\alpha_A > 0$$

$$\beta_A < 0$$

$$K > \alpha_B > 0$$

Note, when $X_t = 0$, $U_A(X_t) = \alpha_A + K$

When in Regime B (eutrophic), direct utility is acquired according to the following equation:

$$U_B(X_t) = \alpha_B \cdot (X_t + 1)^{\beta_B} \quad (39)$$

Where

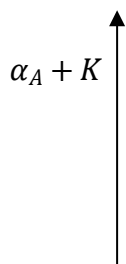
$$\alpha_B > 0$$

$$\beta_B < 0$$

Note, when $X_t = 0$, $U_B(X_t) = \alpha_B$

Direct utility sourced from recreational uses of the shallow lake system is defined by a piecemeal function, where the arguments are the stock of phosphorus in the lake (X) and the current system regime (R_i) (Figure 4-4).

Direct utility ($U_i(X_t)$)



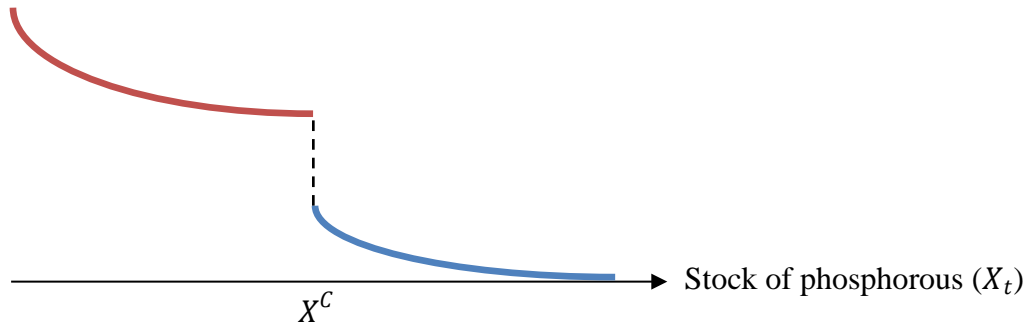


Figure 4-4 Direct utility from recreational use of the shallow lake

The instantaneous total social utility received is given by:

$$U(t) = \gamma_t^A \cdot U_A(X_t) + (1 - \gamma_t^A) \cdot U_B(X_t) + U_q(q_t) - DU(I_t) \quad (40)$$

Where

γ_t^A is an indicator variable, which takes value 1 if the system is in Regime A for the duration of period t , and value 0 if the system is in Regime B for the duration of period t

$$\frac{dU_A}{dX} < 0 ; \frac{d^2U_A}{dX^2} \geq 0 ; \frac{dU_B}{dX} < 0 ; \frac{d^2U_B}{dX^2} \geq 0 ; \frac{dU_q}{dq} > 0 ; \frac{d^2U_q}{dq^2} \leq 0 ; \frac{\partial DU}{\partial I} > 0 ;$$

$$\frac{\partial^2 DU}{\partial I^2} \leq 0$$

Summary of shallow lake case study

A shallow lake system is susceptible to a regime shift from clear conditions (oligotrophic) to green and dirty conditions (eutrophic), if the stock of phosphorus within the lake reaches a critical concentration. This regime shift occurs after crossing a reversible threshold. However, the dynamics of the system are classified as hysteretic because a higher rate of internal nutrient cycling occurs in the undesirable regime, which makes it more difficult for the system to recover to the economically desirable regime.

The trade-off faced by the ecosystem manager is between (i) the additional profits and consumption utility made possible when higher amounts of phosphorus fertiliser are applied to the agricultural land, and (ii) the negative impact that

phosphorus runoff has on the health of the lake system and utility derived from recreational use of the lake.

4.2 Savannah-forest system

A savannah (henceforth alternatively referred to as rangeland) is an ecosystem co-dominated by trees and grasses. One such example is Australia's Northern savannah, which stretches across northern Australia from Broome to Townsville; a distance of roughly 2,600km. Pastoralism occupies approximately 60% of the land area of the rangelands (Lesslie *et al.* 2006). The extensive grasslands systems typical to the rangelands provide important economic and environmental services. Savannahs occur in areas with annual rainfall from 300 to 1800mm and are commonly divided into *dry* and *moist* forms, where 500-700mm of annual rainfall represents the transition zone between the two classifications (Scholes and Walker 1993; Sankaran *et al.* 2005). In *dry* savannahs, grass production is a strongly increasing function of annual rainfall. In *moist* savannahs, this relationship is weak (Accatino *et al.* 2010). This is likely to be because soil moisture is not a limiting factor for grass production in a *moist* savannah. The Australian Northern savannah is on the border of being classified as a *moist* savannah since it receives approximately 700mm of annual rainfall. Rainfall varies from year-to-year and mean rainfall varies slightly across the breadth of the savannah, with some areas receiving mean annual rainfall of approximately 600mm and others 800mm (Bureau of Meteorology 2013).

Australia's Northern savannah is actively used for low-density livestock grazing (Walker and Salt 2006). However, overgrazing can have disastrous effects on this industry. Overgrazing reduces the prevalence of grasses, reduces the intensity of fires, allows more trees to survive to adulthood and significantly increases the likelihood of the system switching regimes from savannah (co-dominance of trees and grasses) to forest (only trees). Adult trees are much less susceptible to fire than are young trees and saplings. Also, as a result of high stocking rates, and especially during periods of low rainfall, grass prevalence will eventually decline to a level that is unable to carry a fire (Walker and Salt 2006). Therefore, a threshold exists in terms

of the minimum amount of grass cover (fuel load) required to be able to carry a fire and, in the process, regulate the prevalence of trees within the savannah system.

4.2.1 Savannah: State variables

Table 4-4 below lists and describes the state variables of the savannah case study.

Table 4-4 State variables of the savannah case study

State variable	Description
Grass cover (G)	A dimensionless variable that measures the proportion of land area occupied by grass cover ($G \in [0,1]$)
Tree cover (T)	A dimensionless variable that measures the proportion of land area occupied by tree cover ($T \in [0,1]$)
Stock of wealth (W)	Wealth possessed by the ecosystem manager, who retains some of the profits from livestock activities
Prior probability bounds (a^{min}, a^{max})	Describes the location of a reversible ecological threshold of unknown location that separates two alternative population growth regimes
Stock size of livestock (\bar{L})	A dimensionless variable that is defined as the proportion of the study area that must be occupied by grass cover in order to feed the livestock population for a single time period – the sum of young and mature cattle populations
Stock of young cattle (L^Y)	A dimensionless variable that measures the feed requirements of the stock of young cattle (0-12 months old) for a one year time period
Stock of mature cattle (L^M)	A dimensionless variable that measures the feed requirements of the stock of mature cattle (12-18 months old) for the six month period that they spend on the savannah

4.2.2 Savannah: Control variables

Table 4-5 below lists and describes the control variables of the savannah case study.

Table 4-5 Control variables of the savannah case study

Control variable	Description
Fire lighting (P)	The number of fires deliberately lit by an ecosystem manager, in addition to naturally occurring fires – fire can be used to reduce tree cover, but also reduces grass cover and feed in the short term
Deliberate clearing of trees (C)	The targeted removal of trees from the savannah system – much more costly than tree removal using fire, but does not impact on grass cover
Re-stocking of young cattle (L_t^Y)	The amount of young cattle added to the total population of cattle
Harvested cattle (H)	Mature cattle are ‘harvested’ and sent to a feedlot for finishing at 18 months of age
Expenditure on consumption activities (q)	These activities generate consumption utility – paid for using acquired wealth

4.2.3 Savannah: Trade-offs

The trade-offs faced in this case study are between consumption utility, which is paid for using profits generated from the harvesting of livestock (H) from the rangeland ecosystem to be sent for finishing in a feedlot; direct utility, which is a function of the composition of vegetation types (i.e. G and T) within the rangeland system and serves as a proxy for the level of non-livestock biodiversity within the rangeland and is acquired from non-use values such as aesthetic and existence values; disutility (opportunity cost) from the use of public funds for the purpose of managing the prevalence of grasses within the rangeland system, rather than on the best alternative use. The prevalence of grasses (and trees) is managed through the deliberate lighting of fires (P), which initially reduces the prevalence of both grasses and trees, but provides grasses with more opportunity to colonise vacant areas of the landscape, and the targeted clearing of trees (C), which initially reduces the prevalence of trees, but leaves grasses unaffected.

The expansion of grass into new areas is restricted to regions that are occupied by neither grass nor trees (i.e. vacant). As such, the expansion of grass is restricted by the current prevalence of trees (T) in the landscape. Figure 4-5, the upper limit is such because it is not feasible for grass (G) to occupy a proportion of the rangeland area greater than $1 - T$. The amount of expansion of grass cover (ΔG) is proportional to both the starting amount of grass cover (G) and the size of the region occupied by neither grass nor trees ($1 - G - T$). At low levels of initial grass cover (G), the amount of expansion (ΔG) will also be low. This is despite a relatively large portion

of the study area being vacant. At intermediate levels of initial grass cover (G), the amount of expansion (ΔG) will be at its highest. This is because there is a higher base amount of grass cover from which expansion can occur, and because there is still a sufficiently sized vacant region ($1 - G - T$) such that there is limited, or no, competition from the expansion of trees (ΔT). At high levels of initial grass cover (G), there is very little vacant land available to be colonised by either grass or trees. Since trees are the superior competitor in this model, the new colonisation of vacant land by grasses is more than offset by the displacement of grasses by trees, and the amount of expansion of grass (ΔG) will be negative.

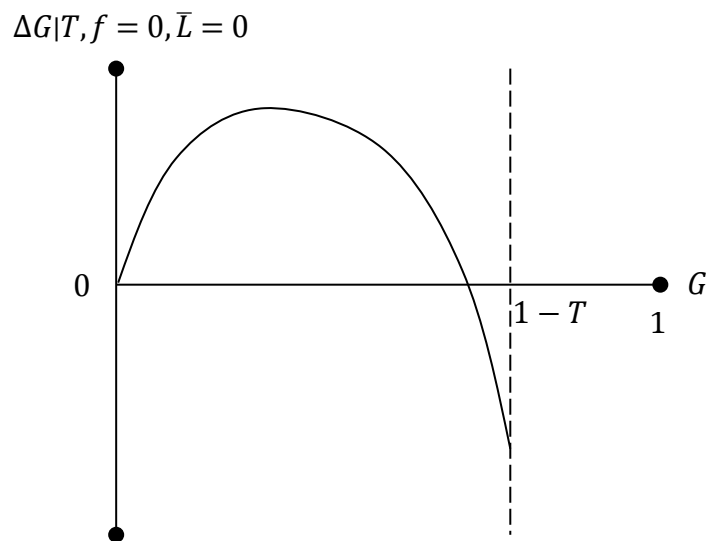


Figure 4-5 Natural growth of grass, net of losses from fire and grazing

Since trees are the superior competitor, the expansion of trees into new areas is restricted only to regions that are not currently occupied by trees. In Figure 4-6, the upper limit on the prevalence of trees is $T = 1$, which represents a savannah system completely dominated by trees (i.e. forest). The amount of expansion of trees (ΔT) is proportional to both the starting amount of tree cover (T) and the size of the region not currently occupied by trees ($1 - T$). At low levels of initial tree cover (T), the amount of expansion (ΔT) will also be low. This is despite a relatively large portion of the study area being absent of trees. At intermediate levels of initial tree cover (T), the amount of expansion (ΔT) will be at its highest. This is because there is a higher base amount of tree cover from which expansion can occur, and because there is still a sufficiently sized region devoid of trees ($1 - T$) upon which to expand. At high levels

of initial tree cover (T), there is very little vacant land available to be colonised by trees, hence expansion (ΔT) will asymptote in the direction of $T = 1$ and expansion (ΔT) will approach zero as T approaches one.

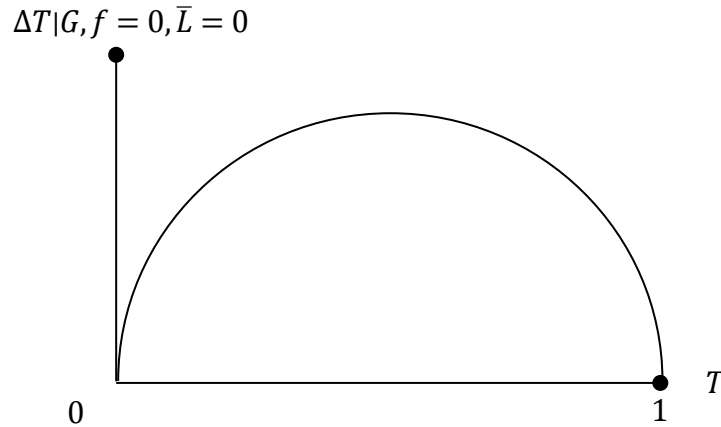


Figure 4-6 Natural growth of trees, net of losses from fire and grazing

The impact of fire is examined explicitly in Figure 4-7. Note that the impact of fire was ignored in Figure 4-5 and Figure 4-6. Fire is a natural driver of savannah ecosystems; however, the incidence of fire can also be increased through deliberate lighting (P). Figure 4-7 applies equally well to losses due to fire of either grass cover (G) or tree cover (T). Since fire is naturally occurring within the rangeland system, a proportion of losses (n) will occur without any deliberate fire-lighting (P). For any subsequent deliberately-lit fires, the fuel load available will be lower and any resulting fire will be of lower intensity. For this reason, each additional deliberately-lit fire results in diminishing marginal losses of vegetation (G or T). The proportional losses from fire, for a chosen vegetation type, cannot exceed one for a given time period because a value of one represents the loss of all vegetation of that type; it is not possible to lose more vegetation than was initially available to lose.

Proportional losses from fire (G or T) | $G, T, \bar{f}, \bar{L} = 0$

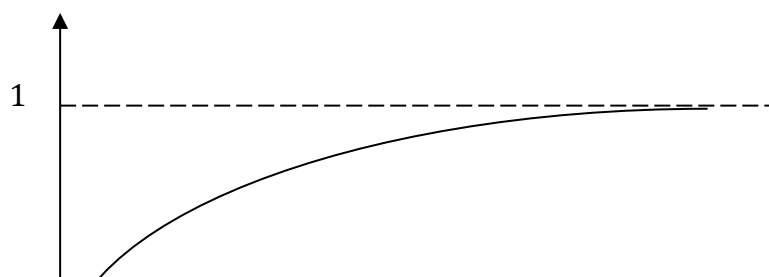


Figure 4-7 Proportional losses of vegetation due to fire

The loss of grass cover ($-\Delta G$) due to grazing is calculated simply as a linear function of the stock size of livestock (\bar{L}); however, the stock of livestock (denoted in units of feed required) cannot exceed the amount of grass cover available (Figure 4-8). Conversely, tree cover (T) is unaffected by livestock grazing.

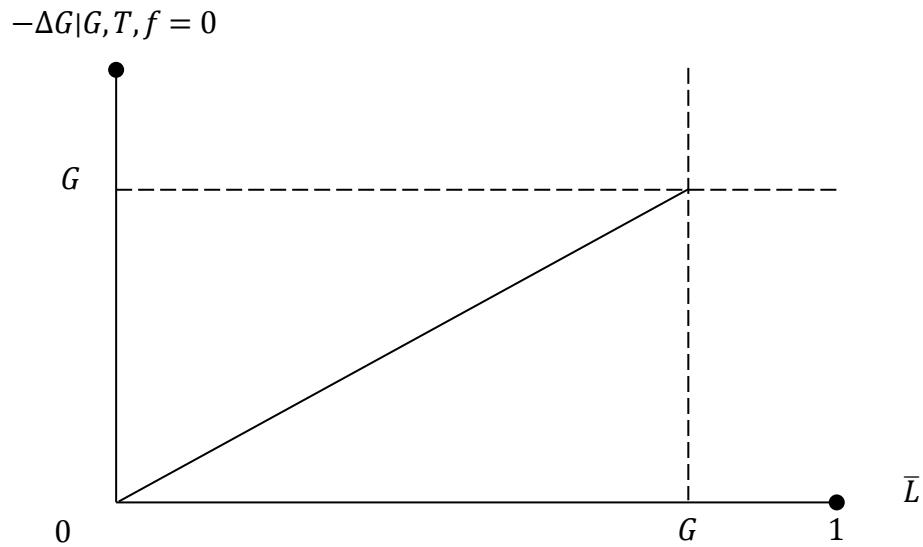


Figure 4-8 Effect of livestock grazing on grass cover

On the other hand, tree cover (T) is impacted by targeted clearing; however, grass cover is unaffected by this action. The loss of tree cover ($-\Delta T$) due to targeted clearing is calculated simply as a linear function of the clearing effort undertaken (C) (Figure 4-9).

$$-\Delta T|G,T,f=0$$

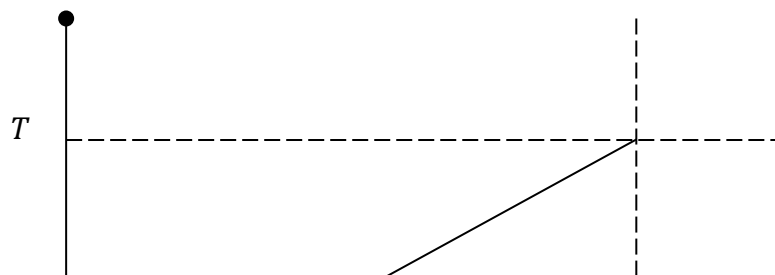


Figure 4-9 Effect of deliberate clearing on tree cover

4.2.4 Savannah: Critical ecological thresholds

The rangeland ecosystem is typified by two separate fire regimes, one where fire is of sufficient intensity to damage trees and another where fire is not of sufficient intensity to damage trees. In Regime A (R_A), there is sufficient grass cover (i.e. fuel load) to produce a fire of sufficient intensity to destroy trees. In Regime B (R_B), there is insufficient grass cover (i.e. fuel load) to produce a fire of sufficient intensity to destroy trees, meaning that trees are free to expand unimpeded. The term ‘sufficiently’, as used above, is defined by a minimum level of grass cover as a proportion of the ecosystem ($G^{critical}$). In Figure 4-10, the vertical dashed arrow signifies the crossing of the critical ecological threshold and the resulting shift in system regime. This regime shift is completely reversible along the same path, which is why the arrow is both upward- and downward-pointing. The purpose of the horizontal dotted line is to indicate that the exact location of the threshold is unknown; however, upper and lower bounds can be postulated. The system regime is modelled according to:

$$R_{j_t} = \begin{cases} A & \text{if } G_{j_{t-1}} \geq G^{critical} \\ B & \text{if } G_{j_{t-1}} < G^{critical} \end{cases} \quad (41)$$

Where

R_{j_t} is the system regime for the duration of time period t

$G_{j_{t-1}}$ is the level of the underlying, slow variable at the end of the previous time period

$G^{critical} \equiv G^c$ is a random variable on a bounded interval (G^{cmin}, G^{cmax})

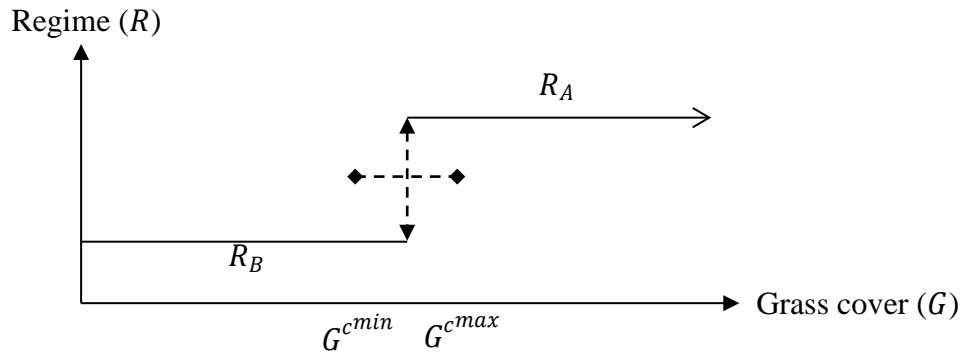


Figure 4-10 Alternative system regimes for prevalence of grass cover

4.2.5 Savannah: Population growth functions

In the space implicit model, introduced by Tilman (1994) and modified by Accatino *et al.* (2010), trees are modelled as the superior competitor and grasses as the inferior competitor. Trees, as the superior competitor, can displace grasses and colonise areas previously occupied by grasses, but grasses can only colonise areas occupied by neither trees nor grasses. This is one explanation for an observed hysteresis effect in terms of the prevalence of grasses. Accatino *et al.* (2010) modify Tilman's model to distinguish between the consumption of grass by fires and grass reduction due to other causes, such as mortality and herbivores (grazing).

For this case study, the model of Accatino *et al.* (2010) has been modified further to include a threshold effect in terms of fire intensity and the propensity of fire to kill trees. In Regime A, there is sufficient grass cover to fuel a fire capable of killing some of the tree cover. In Regime B, there is insufficient grass cover to perform the same function. As a result, trees will continue to grow, unimpeded by fire. It is also assumed that, should fire occur during a time period, it will occur at the very beginning of the time period. This means that the rangeland system has the remainder of the time period to recover (through re-colonisation) and be grazed upon.

The regime-specific (for Regime i) growth function for tree cover is:

$$\Delta T_t = \begin{cases} c_T T'_{t-1}(1 - T'_{t-1}) - \delta_T T'_{t-1} - \delta_F f_t(\bar{f}, P_t) G_{t-1} T_{t-1} & \text{if } i = A \\ c_T T_{t-1}(1 - T_{t-1}) - \delta_T T_{t-1} & \text{if } i = B \end{cases} \quad (42)$$

Where

$T'_{t-1} = T_{t-1} - \delta_F f_t(\bar{f}, P_t) G_{t-1} T_{t-1}$ is tree cover carried over from the time period $t - 1$ net of losses resulting from fire at the beginning of time period t

$c_T \in (0,1)$ is the colonisation rate of trees

$\delta_T \in (0, 1]$ is the 'off-take' rate for trees

δ_F measures the vulnerability of trees to fire

$G_{t-1} \in [0,1]$ is grass cover as a proportion of the study area

$T_{t-1} \in [0,1]$ is tree cover as a proportion of the study area

\bar{f} [in units of $1/t$] is approximated by natural fire frequency

P_{t-1} [in units of $1/t$] is the number of fires that are deliberately lit

f_{t-1} is the total number of fires that occur during the time period

$\delta_F f_{t-1} G_{t-1} T_{t-1}$ is the reduction in above-ground tree cover

$f_{t-1} G_{t-1}$ is the grass fuel load carried over from time period $t - 1$

If there is any grass in the landscape, there is a possibility that a grass-fuelled fire will occur. Unlike with trees, there is no minimum fire intensity required for grass to be damaged by fire; however, grass losses due to fire are a function of the fuel load available. Therefore, there is a single growth function for grass cover. The growth function for grass cover is:

$$\Delta G_t = c_G G'_{t-1}(1 - T'_{t-1} - G'_{t-1}) - c_T T_{t-1} G'_{t-1} - \delta_{GO} G'_{t-1} - \bar{L}_t - f_t(\bar{f}, P_t) G_{t-1} \quad (43)$$

Where

$G'_{t-1} = G_{t-1} - f_t(\bar{f}, P_t) G_{t-1}$ is grass cover carried over from the time period $t - 1$ net of losses resulting from fire at the beginning of time period t

$c_G \in (0, 1]$ is the colonisation rate of grass: $c_G > c_T$

$f_{t-1} G_{t-1}$ is the consumption of grass by fires

$\delta_{GO}G_{t-1}$ is the reduction in grass due to other causes, such as natural mortality

\bar{L}_{t-1} is the reduction in grass due to grazing pressures

Finally, cattle grazing has an adverse effect on the level of grass cover within the rangeland system. The stock of cattle (\bar{L}_t) is expressed in terms of the amount of grass-feed required (as a proportion of the rangeland system that must be occupied by grass cover) to feed the cattle for one time period. Hence, the stock of cattle is constrained such that it cannot exceed the amount of grass-feed that is available for consumption after reductions in grass from other sources (e.g. fire, colonisation by trees) has been taken into account:

$$\begin{aligned} \bar{L}_t \leq & G_{t-1} + c_G G_{t-1}(1 - T_{t-1} - G_{t-1}) - c_T T_{t-1} G_{t-1} - \delta_{GO} G_{t-1} \\ & - f_{t-1}(\bar{f}, P_{t-1}) G_{t-1} \end{aligned} \quad (44)$$

Calves (young cattle) are usually born in Spring (September-November) because the weather conditions give them the greatest chance of surviving through to maturity. Mature cows are usually moved from the rangeland to a feedlot after 18 months for finishing. This means that cows will graze on grasses from the rangeland for a period of 18 months. The time-step used in this model is 12 months (from 1 September to 31 August the following year) and the productive life of a cow on the rangelands will span two consecutive time periods. It is also assumed that a head of cattle requires an equal amount of food from 0 to 12 months of age (12 months) as it does from 12 to 18 months of age (6 months). This is because a ‘mature’ requires a greater amount of feed than a ‘young’ cow.

The stock of ‘mature’ cattle (expressed in terms of the amount of grass-feed required) evolves according to:

$$L_t^M = \gamma \cdot L_{t-1}^Y \quad (45)$$

Where

L_t^M is the stock of ‘mature’ cattle that have carried over from time period $t - 1$
 $\gamma \in (0, 1]$ measures the rate of conversion from young cattle to mature cattle in terms of feed requirements – if there is no mortality from birth to maturity, $\gamma = 1$

L_{t-1}^Y is the stock of ‘young’ cattle at the beginning of time period $t - 1$

The total stock of cattle, both ‘mature’ and ‘young’ (expressed in terms of the amount of grass-feed required), evolves according to:

$$\bar{L}_t = L_t^M + L_t^Y \quad (46)$$

Where

$L_t^Y = \underline{L}_t^Y$ is the level of re-stocking of ‘young’ cattle at the beginning of time period t , which corresponds to the stock of ‘young’ cattle at the beginning of time period t

By the end of each time period, all ‘mature’ cattle will have been ‘harvested’ and sent to a feedlot for finishing. Therefore, no mature cattle will carry over from one time period to the next and the harvest of cattle will be determined by the following equation:

$$H_t = L_t^M \quad (47)$$

The harvesting of cattle reduces grazing pressures (if the harvested cattle are not re-stocked in full) and generates profits from sale of cattle to feedlots for finishing.

4.2.6 Savannah: Sources of utility

Utility is gained both from consumption utility, paid for using profits generated from the harvesting of livestock (H) from the rangeland system, and from direct utility (U_B), as a function of the composition of vegetation types (i.e. G and T) within the rangeland system, which serves as a proxy for the level of non-livestock biodiversity found within the rangeland system. Utility is acquired from non-use values, such as aesthetic and existence values. Also, negative utility results from the use of any public funding for the purpose of managing the prevalence of grasses within the rangeland system. This disutility captures the opportunity cost to society of

spending public funds on managing grass cover (and, subsequently, livestock stocking rates), rather than on the best alternative use. The prevalence of grasses (and trees) is managed through the deliberate lighting of fires (P) and the targeted clearing of trees (C).

The instantaneous total social utility received is given by:

$$U(t) = U_B(G_t, T_t) + U_q(q_t) - DU(P_t, C_t) \quad (48)$$

Where

$$\frac{dU_B}{dG} > 0 ; \frac{d^2U_B}{dG^2} \leq 0 ; \frac{dU_B}{dT} > 0 ; \frac{d^2U_B}{dT^2} \leq 0 ; \frac{dU_q}{dq} > 0 ; \frac{d^2U_q}{dq^2} \leq 0 ; \frac{\partial DU}{\partial P} > 0 ;$$

$$\frac{\partial^2 DU}{\partial P^2} \leq 0 ; \frac{\partial DU}{\partial C} > 0 ; \frac{\partial^2 DU}{\partial C^2} \leq 0$$

Utility acquired from the level of non-livestock biodiversity (U_B), modelled as a function of the composition of vegetation types G and T , is monotonically increasing in its arguments; however, marginal utility is decreasing for both arguments. Likewise, consumption utility is monotonically increasing in its argument but marginal utility is decreasing in its argument. Utility resulting from the use of any public funding on actions intended to actively manage the prevalence of grass and trees within the rangeland system (P and C) is negative and monotonically decreasing in its arguments. Marginal utility from these sources are increasingly negative in their arguments.

Summary of savannah case study

Australia's Northern savannah stretches across a distance of roughly 2,600km and provides important economic and environmental services. A savannah is an ecosystem co-dominated by trees and grasses, where trees are the dominant vegetation type. Utility is derived from non-use values based on the prevalence of trees and grasses in the landscape, as well as from consumption funded by the rearing and selling of beef cattle. Beef cattle are grass-fed, so there is a trade-off between using grasses to feed the cattle or to acquire direct utility via non-use values.

Since trees are the dominant vegetation type, they can only be removed by intense fire or deliberate clearing, which is very costly. A critical threshold exists in terms of the grass fuel load required to create a fire with enough intensity to destroy trees. This threshold is reversible, which means that the proportion of grass cover must simply reach a threshold level before a sufficiently intense fire can burn. The ecosystem manager also has the option of deliberately lighting fires, rather than relying on the natural occurrence of fire, for the purpose of removing trees and promoting grass growth. After each fire, the grass fuel load will be lower, so there are biological limits to the use of deliberately lit fires for the purpose of removing trees from the landscape and promoting future grass growth.

Conclusion

This chapter presents two detailed case studies of complicated ecological systems that can be managed using the mathematical modelling framework developed in this thesis. For both the shallow lake and savannah case studies, profits are the main source of utility from the system. However, the case studies differ most notably in terms of threshold effects and system dynamics. The shallow lake system has hysteretic dynamics, while the savannah system has a completely reversible threshold. When the primary focus is generating profits, to obtain utility from consumption, the decision-maker will optimally incur more risk of crossing an undesirable threshold than for the equivalent case study where utility is primarily obtained from other sources, such as the health of the system.

These findings can be compared to the two other representative case studies described in Chapter 5. For these case studies, utility is sourced primarily from the health of the system. First, in terms of the health of a reef system and, second, in terms of the size of a koala population. When utility is sourced primarily from the health of the system, the decision-maker will incur less risk of crossing an undesirable threshold than for the equivalent case, but where utility from consumption is more important.

Chapter 5. Modelling environmental quality with uncertain ecological thresholds

Chapter 5 follows on from Chapter 4 by presenting a further two detailed case studies of complicated ecological systems that can be managed using the mathematical modelling framework developed in this thesis. Chapter 5 explores the trade-offs associated with environmental quality in terms of use and non-use values, by examining a case study of the trade-offs between harvesting fish whilst maintaining the health of a reef system and a case study of the trade-offs between housing construction versus conserving a koala population. The threshold effect for the reef system is completely reversible, while the population dynamics for koalas include a hysteretic threshold effect. Table 5-1 provides a summary of the similarities and differences of the two representative case studies discussed in Chapter 5.

Table 5-1 Key features of the reef fishery and koala case studies

Characteristic	Reef fishery	Koalas
Threshold effect	Completely reversible	Hysteretic dynamics
Underlying slow variable	Reef-mangrove distance	Habitat quality
Trade-offs	Non-use values from the fish population vs. Harvesting fish for profit	Non-use values from koala population vs. Utility from home ownership
Ecological dynamics	Fish population modelled using a Lotka-Volterra predator-prey model	Koala population modelled using a logistic growth function
Sources of utility	<p><i>Direct utility:</i> Existence and aesthetic value – function of the fish population</p> <p><i>Consumption utility:</i> Paid for using profits from fishing activities</p>	<p><i>Direct utility:</i> Existence and aesthetic value – function of the koala population</p> <p><i>Direct utility:</i> Utility from home ownership</p>

5.1 Reef fishery (reef-mangrove interaction)

The stock size of a harvestable fish species (an amalgamation of yellowfin bream, mores perch and black rabbitfish) within a specified reef ecosystem is modelled as a function of the distance between the boundary of the reef system and

the boundary of an adjacent mangrove system (Olds *et al.* 2012). The mangrove system acts as an important breeding ground because it provides a sheltered and relatively safe environment for young fish. If the mangrove system is within sufficient proximity of the reef system, fish are able to migrate back and forth between the two systems and, ultimately, more successful breeding and a higher equilibrium stock of fish will result in the reef system. The reef ecosystem is typified by two separate population regimes, each with different growth rates of the fish stock and different maximum (equilibrium) populations.

In this example, the mangrove system is protected (i.e. no fishing is allowed) and is therefore considered as exogenous to the problem. However, the proximity of the mangrove system to the reef system is considered of critical importance. Conversely, commercial fishing is allowed within the reef system and represents a major source of utility, via consumption paid for by profits generated from this activity. It is the responsibility of an ecosystem manager (representative agent) to manage the reef-mangrove system with the objective of maximising the NPV of utility sourced from it.

5.1.1 Reef fishery: State variables

Table 5-2 below lists and describes the state variables of the reef fishery case study.

Table 5-2 State variables of the reef fishery case study

State variable	Description
System 'output' (Y)	Stock size of harvestable fish species within the reef system
Underlying slow ecosystem variable (X)	Physical distance between the boundary of the reef system and the boundary of an adjacent mangrove system
Stock of wealth (W)	Wealth acquired via profits generated from the harvest and sale of harvestable fish
Prior probability bounds (a^{min}, a^{max})	Describes the location of a reversible ecological threshold of unknown location that separates two alternative population growth regimes
Stock size of a herbivorous fish species (\bar{C})	Modelled as a function of the stock size of the harvestable fish species (its predator) – this species is not harvested
Stock of coral-damaging pollution within the reef system (\bar{P})	Modelled as a function of the amount of pollution released through harvesting activities, the amount of pollution cleaned up, and the assimilative capacity of the reef ecosystem

5.1.2 Reef fishery: Control variables

Table 5-3 below lists and describes the control variables of the reef fishery case study.

Table 5-3 Control variables of the reef fishery case study

Control variable	Description
Harvest effort (E)	Amount of effort exerted in search of, and harvesting, the harvestable fish species – measured in units of time (e.g. hours)
Introduction of farmed herbivorous fish to the reef system (C)	Herbivorous fish graze on fleshy algae, which compete with the coral for living space, and therefore make it easier for the coral of the reef system to grow or regenerate, if damaged (Olds <i>et al.</i> 2012)
Clean-up of pollution before it is released into the reef system (P')	For example, by installing specialised filters on boat exhausts – will incur a constant marginal cost for each unit of fishing effort
Clean-up of pollution after it has been released into the reef system (P'')	For example, by using specialised machinery to skim pollution off the water's surface – lower pollutant stock is associated with a higher search cost to find and remove it
Consumption activities (q)	These activities generate consumption utility – paid for using acquired wealth

5.1.3 Reef fishery: Trade-offs

The trade-offs faced in this case study are between consumption utility, which is paid for using profits generated from the harvesting of fish (H) from the reef ecosystem; direct utility, which is a function of the stock size of harvestable fish (Y) and is acquired from non-use values, such as aesthetic, existence and bequest values; and disutility (opportunity cost) from the use of public funds for the purpose of reducing the physical distance between the reef and mangrove ecosystems, rather than on the best alternative use. This reduction is achieved through actions (C, P', P'') that improve the health of the reef and increase the rate of its expansion in the direction of the mangrove system, decreasing the physical distance between reef and mangrove.

The harvesting of harvestable fish is assumed to follow a bi-linear catch equation, where harvest in a given time period is a function of the amount of fishing effort undertaken (E) and the stock size (Y) at the beginning of the time period (Figure 5-1).

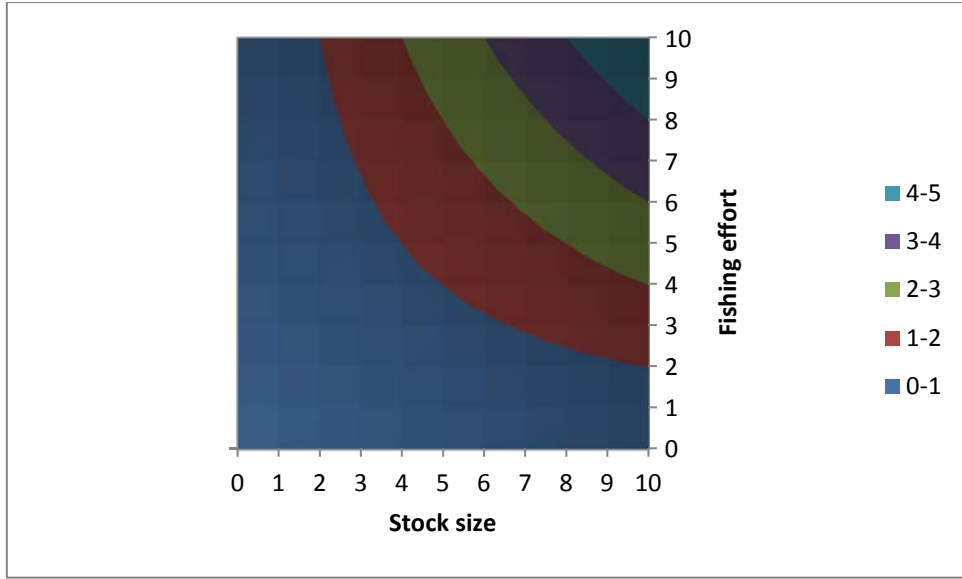


Figure 5-1 Fishing harvest

Where

$$H_t(E_t, Y_{t-1}) = z \cdot E_t \cdot Y_{t-1}$$

z is the catchability of the stock

Although fishing effort generates profits through the harvest and sale of fish, it also adds to the stock of coral-damaging pollution within the reef ecosystem (\bar{P}). The stock of pollution within the reef ecosystem is modelled as a function of the amount of fishing effort undertaken in each period (E), actions taken to clean up pollution before it enters the reef ecosystem (P') and actions taken to clean up pollution after it enters the reef ecosystem (P''). In addition, the reef possesses an ability to naturally assimilate some of the pollution that is deposited into the system. The stock of pollution evolves according to the following equation:

$$\bar{P}_t = \Omega \cdot \bar{P}_{t-1} + (\varepsilon \cdot E_t - P'_t) - Y''(P''_t, \bar{P}_{t-1}) \quad (49)$$

Where

$(1 - \Omega)$ is the proportion of pollution that is naturally assimilated per time period:

$$0 < \Omega < 1$$

$\varepsilon \cdot E_t \geq P'_t$ restricts the amount of pollution filtered to be no more than the amount generated

$\varepsilon > 0$ is a transfer coefficient representing the ratio of fishing effort to pollution generated

$Y''(P''_t, \bar{P}_{t-1}) = v \cdot P''_t \cdot \bar{P}_{t-1}$ is the amount of pollution removed after it enters the reef

$v > 0$ is an efficiency parameter for pollution removal

It is assumed that each unit of fishing effort generates ε units of coral-damaging pollution. To counter this process, special filters (P') can be attached to fishing boats. These filters are capable of capturing one unit of pollution per unit of filter. Each filter has a useful life of one unit of pollution captured. Since it is not possible to capture more pollution during a time period than is generated, the amount of pollution captured before entering the reef system (P'_t) cannot exceed the amount of pollution generated ($\varepsilon \cdot E_t$). Any pollution that is not captured is released into the reef system (Figure 5-2).

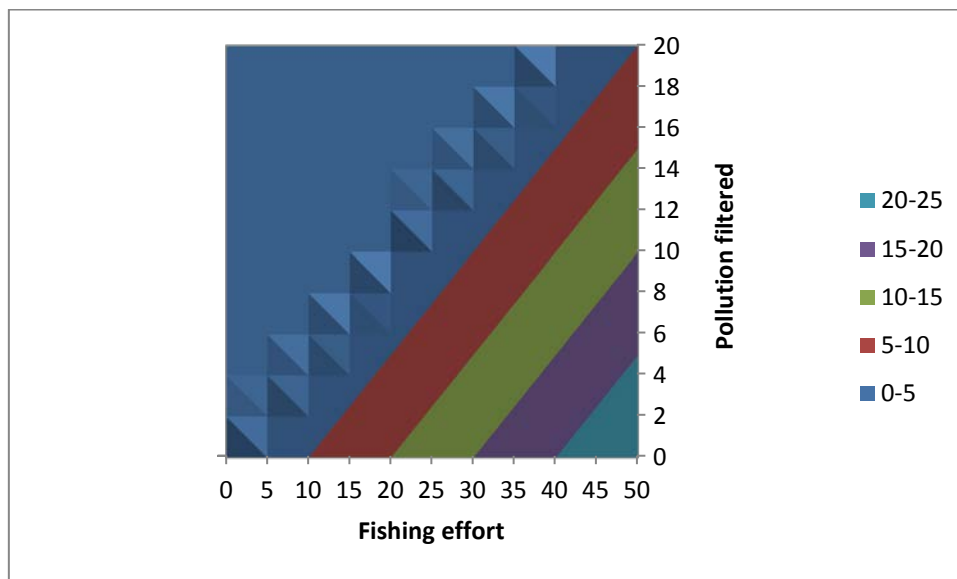


Figure 5-2 Amount of pollution released after filtering

Pollution that is not captured prior to release into the reef ecosystem can be removed via two different mechanisms. First, the reef system can naturally assimilate the stock of pollution at a rate of $(1 - \Omega)$ per time period. Second, pollution can be actively removed from the reef system (e.g. through filtering or surface skimming). Pollution removed via this means is assumed to follow a bi-linear pollution removal

equation, where pollution removed (Y'') during a given time period is a function of the stock of pollution at the beginning of the time period, which is identical to the stock of pollution at the end of the previous time period (\bar{P}_{t-1}), and the amount of clean-up effort undertaken (P_t''), measured in units of time (e.g. hours) (Figure 5-3).

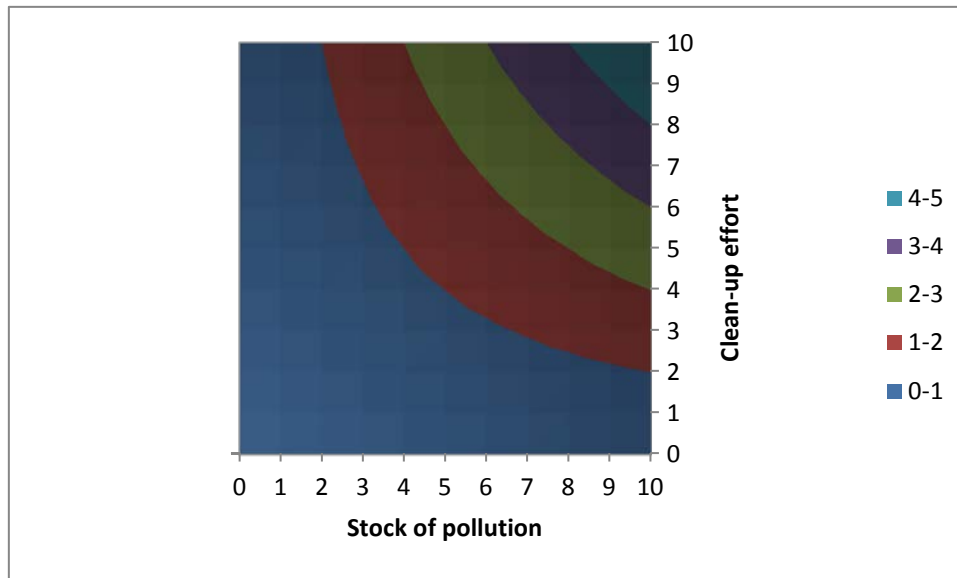


Figure 5-3 Amount of pollution deliberately removed from the system (excl. assimilation)

5.1.4 Reef fishery: EOM for physical distance between reef and mangrove systems (X)

The physical distance between the reef and mangrove systems (X_t) is modelled as a function of the physical distance between the two systems at the end of the previous time period (X_{t-1}), the stock of coral-damaging pollution at the end of the previous time period ($\bar{P}_{j_{t-1}}$) and the stock size of herbivorous fish at the end of the previous time period ($\bar{C}_{j_{t-1}}$). For the purpose of parsimony, it has been assumed that there is a zero or inconsequential amount of nutrient-rich runoff from adjacent agricultural lands into the mangrove and reef ecosystems.

Even in the absence of any impediments to coral growth (e.g. pollution), the reef and mangrove systems will not connect. Instead, differing environmental conditions (including water temperature and salinity concentrations) between the reef and mangrove systems impose a natural physical boundary between the two systems,

since a reef system requires certain environmental conditions to successfully function. As a result, there is a natural physical constraint on the level of X . This minimum distance between the reef and mangrove system is defined as \underline{X} . Also, the growth rate of the reef decreases (i.e. dX becomes a smaller negative number) *ceteris paribus* as the boundary of the reef system approaches the natural physical constraint, \underline{X} .

Herbivorous fish are important because they graze on fleshy algae, which compete with coral for living space (Olds *et al.* 2012). For a high stock of herbivorous fish, fleshy algae are likely to be out-competed and the growth rate of coral *ceteris paribus* will be higher. For a low stock of herbivorous fish, the growth rate of coral will be lower, and for a sufficiently low stock of herbivorous fish, fleshy algae will out-compete the herbivorous fish and the growth rate of coral will be negative (Figure 5-4). The parameter \underline{C} represents the stock of herbivorous fish for which the fleshy algae and herbivorous fish compete equally well and, in the absence of any negative impacts caused by coral-damaging pollution, X will remain stable.

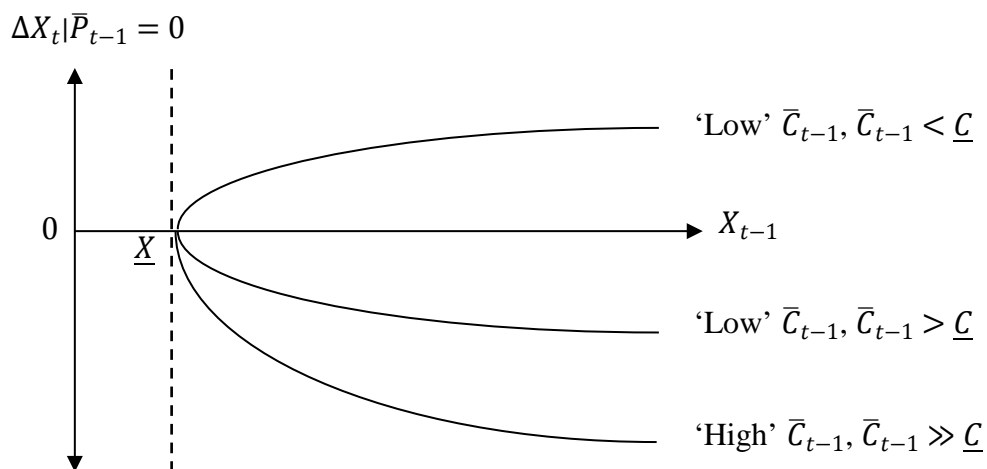


Figure 5-4 Per-period natural growth of reef for various levels of harvestable fish stock, and reef-mangrove distance

Conversely, damage caused to the reef system ($\Delta X > 0$) by the stock of pollution (\bar{P}) is independent of the level of X and associated with increasing marginal damage (Figure 5-5).

The EOM for the physical distance between reef and mangrove systems is given by:

$$\Delta X_{j_t} = X(X_{j_{t-1}}, \bar{P}_{j_{t-1}}, \bar{C}_{j_{t-1}}) = k_1 (\underline{C} - \bar{C}_{j_{t-1}}) (X_{j_{t-1}} - \underline{X})^a + k_2 (\bar{P}_{j_{t-1}})^b \quad (50)$$

Where

$$k_1 > 0$$

$$k_2 > 0$$

$$0 < a < 1$$

$$b \geq 1$$

$$X_{j_{t-1}} \geq \underline{X}$$

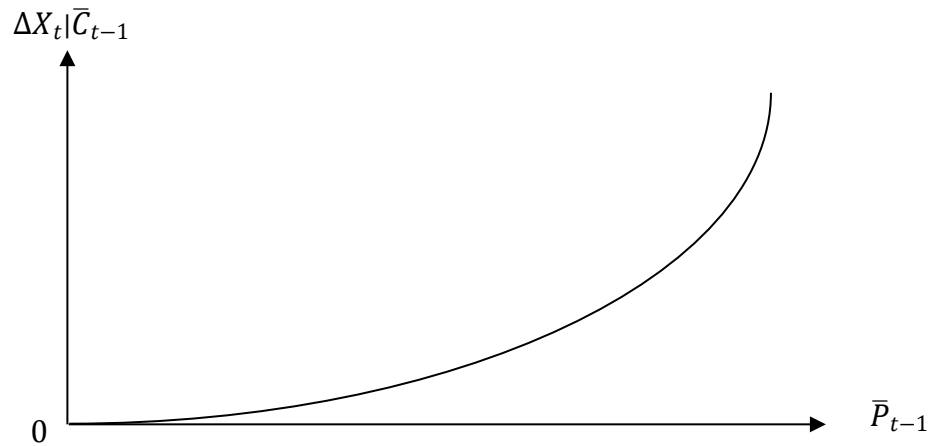


Figure 5-5 Per-period damage caused by accumulated pollution

5.1.5 Reef fishery: Critical ecological thresholds

The reef ecosystem is typified by two separate system regimes, each with different growth rates of the fish stock and different equilibrium populations. In Regime A (R_A), the reef and mangrove ecosystems are sufficiently connected (i.e. their boundaries are sufficiently close together) to allow the migration of fish between the two systems. In Regime B (R_B), the reef and mangrove ecosystems are *not* sufficiently connected (i.e. their boundaries are *not* sufficiently close together) to allow the migration of fish between the two systems. The term ‘sufficiently close’, as used above, is defined by the maximum distance ($X^{critical}$) between the two systems

that still allows successful fish migration to occur between them. Referring to Figure 5-6, the vertical dashed arrow signifies the crossing of the critical ecological threshold and the resulting shift in system regime. This regime shift is completely reversible along the same path, which is why the arrow is both upward- and downward-pointing. The purpose of the horizontal dotted line is to indicate that the exact location of the threshold is unknown; however, upper and lower bounds can be postulated.

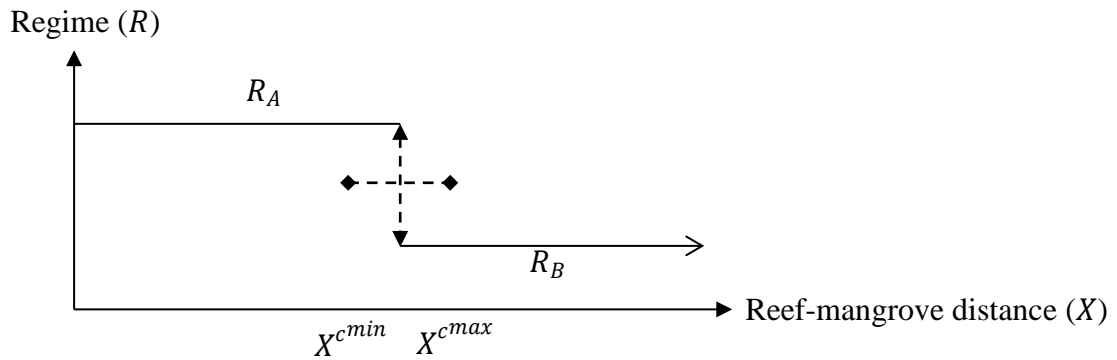


Figure 5-6 Alternative system regimes for harvestable fish population

The system regime is modelled according to:

$$R_{j_t} = \begin{cases} A & \text{if } X_{j_{t-1}} < X^{critical} \\ B & \text{if } X_{j_{t-1}} \geq X^{critical} \end{cases} \quad (51)$$

Where

R_{j_t} is the system regime for the duration of time period t

$X_{j_{t-1}}$ is the level of the underlying, slow variable at the end of the previous time period

$X^{critical} \equiv X^c$ is a random variable on a bounded interval (X^{cmin}, X^{cmax})

5.1.6 Reef fishery: Population growth functions

A simplifying assumption is made that the stock size of neither the harvestable, nor the herbivorous (non-harvestable), fish is affected by the stock of pollution in the reef environment. Instead, the change in the stock of the harvestable

fish (ΔY_t^i) is a function of its own stock size in the previous time period ($Y_{j_{t-1}}$), the stock size of its prey (the herbivorous fish) in the previous time period ($\bar{C}_{j_{t-1}}$) and the number of fish that are harvested during the current time period (H_{j_t}), which itself is a function of the amount of fishing effort (E_{j_t}) undertaken. Also note the i superscript on ΔY ; $i = A$ denotes that the system is in Regime A and $i = B$ denotes that the system is in Regime B. In Regime A, the mangrove and reef systems are sufficiently connected to allow fish migration between the two systems. In Regime B, the mangrove and reef systems are *not* sufficiently connected to allow fish migration between the two systems.

The stock size of harvestable fish (predator) evolves according to a modified Lotka-Volterra predator-prey model with two discrete delays (Ruan 2009):

$$\begin{aligned} \Delta Y_{j_t}^i &= dY \left(Y_{j_{t-1}}, \bar{C}_{j_{t-1}}, H_{j_t}(E_{j_t}) \right) \\ \Delta Y_{j_t}^i &= Y_{j_{t-1}} \left(r_2^i + a_{21}^i \cdot \bar{C}_{j_{t-1-\tau_2}} - a_{22}^i \cdot Y_{j_{t-1}} \right) - H_{j_t} \end{aligned} \quad (52)$$

Where

r_2^i , a_{21}^i and a_{22}^i are regime-specific; for Regime i

$r_2^i > 0$ is the death rate of harvestable fish (predator) in the absence of herbivorous fish (prey)

$a_{21}^i > 0$ is the conversion rate for the harvestable fish (predators); prey to predator

$a_{22}^i \geq 0$ describes the intraspecific competition among harvestable fish (predators)

τ_2 is a positive constant that denotes a discrete time delay

Similarly, the change in the stock of the herbivorous fish ($\Delta \bar{C}_{j_t}^i$) is a function of its own stock size in the previous period ($\bar{C}_{j_{t-1}}$), the stock size of its predator (the harvestable fish) in the previous period ($Y_{j_{t-1}}$) and the number of farmed herbivorous fish that have been introduced to the reef system during the period (C_{j_t}).

The stock size of herbivorous fish (prey) evolves according to a modified Lotka-Volterra predator-prey model with two discrete delays (Ruan 2009):

$$\begin{aligned}\Delta \bar{C}_{j_t} &= \bar{C}(\bar{C}_{j_{t-1}}, Y_{j_{t-1}}, C_{j_t}) \\ \Delta \bar{C}_{j_t} &= \bar{C}_{j_t} \left(r_1^i + a_{11}^i \cdot Y_{j_{t-1}-\tau_1} - a_{12}^i \cdot \bar{C}_{j_t} \right) + C_{j_t}\end{aligned}\tag{53}$$

Where

r_1^i , a_{11}^i and a_{12}^i are regime-specific; for Regime i

$r_1^i > 0$ is the death rate of harvestable fish (predator) in the absence of herbivorous fish (prey)

$a_{11}^i > 0$ is the conversion rate for the harvestable fish (predators); prey to predator

$a_{12}^i \geq 0$ describes the intraspecific competition among harvestable fish (predators)

τ_1 is a positive constant that denotes a discrete time delay

5.1.7 Reef fishery: Sources of utility

Utility is gained both from consumption utility, paid for using profits generated from the harvesting of harvestable fish (H) from the reef ecosystem, and from direct utility, as a function of the stock sizes of harvestable fish (Y) and herbivorous fish (\bar{C}). This direct utility comes in the form of aesthetic, existence and bequest values. Also, negative utility results from the use of any public funding for increasing the connectivity of the reef and mangrove ecosystems. This disutility captures the opportunity cost to society of spending public funds on increasing habitat connectivity, and likely increasing equilibrium (maximum) fish stocks, rather than on the best alternative use. Habitat connectivity can be improved through public spending on introducing farmed herbivorous fish and actively capturing or cleaning up pollution generated through fishing effort.

The instantaneous total social utility is given by:

$$U(t) = U_Y(Y_t) + U_{\bar{C}}(\bar{C}_t) + U_q(q_t) - DU(C_t, P_t', P_t'')\tag{54}$$

Where

$$\frac{dU_Y}{dY} > 0 ; \frac{d^2U_Y}{dY^2} \leq 0 ; \frac{dU_{\bar{C}}}{d\bar{C}} > 0 ; \frac{d^2U_{\bar{C}}}{d\bar{C}^2} \leq 0 ; \frac{dU_q}{dq} > 0 ; \frac{d^2U_q}{dq^2} \leq 0 ; \frac{\partial DU}{\partial C} > 0 ;$$

$$\frac{\partial^2 DU}{\partial C^2} \leq 0 ; \frac{\partial DU}{\partial P'} > 0 ; \frac{\partial^2 DU}{\partial P'^2} \leq 0 ; \frac{\partial DU}{\partial P''} > 0 ; \frac{\partial^2 DU}{\partial P''^2} \leq 0$$

Utility derived from the stock sizes of harvestable and herbivorous fish, and consumption are monotonically increasing in their arguments; however, marginal utility is decreasing for all three sources of utility. Utility resulting from the use of any public funding on actions intended to decrease the physical distance between the reef and mangrove systems (i.e. P' , P'' , C) is negative and monotonically decreasing in its arguments. Marginal utility from these sources are increasingly negative in their arguments.

Summary of reef fishery case study

The stock size of a harvestable fish species within a reef ecosystem is modelled as a function of the distance between the boundary of the reef system and the boundary of an adjacent mangrove system. The fish use the mangrove system as a safe breeding ground, but are only able to do so if the reef and mangrove system are within sufficient proximity. If not, the equilibrium fish population will be lower. Therefore, a reversible threshold exists such that the equilibrium fish population is high when the reef and mangrove systems are close and low when the two systems are far apart.

Utility is derived from non-use values based on the level of the fish stock, and from consumption paid for by any profits collected from the harvest and sale of fish. The physical distance between the reef and mangrove systems is negatively impacted by pollution resulting from fishing effort. This distance can be decreased by reducing the concentration of pollution using mechanical means and by adding herbivorous fish, which remove ecological impediments to coral growth. However, there are decreasing marginal returns to these measures, so the ecosystem manager faces a trade-off between increasing disutility from the use of public funds for this purpose.

5.2 Koala preservation and housing development

Globally, mammal populations are increasingly threatened with population fragmentation. Populations can become fragmented as a result of habitat loss or anthropogenic barriers to gene flow, such as busy roads. Barriers that inhibit the use of important migration or dispersal corridors can effectively isolate adjacent populations, reducing genetic diversity and the species' ability to respond to environmental change. The koala is an example of an iconic and internationally recognisable Australian marsupial. Koalas are biologically unique, since they are the only extant member of their family. Despite this, Australia's koala population has experienced a significant reduction over recent centuries for reasons including, but not limited to, habitat loss and fragmentation (Lee *et al.* 2010).

For this case study, the population of a koala species within a chosen geographical area (study area) is modelled as a function of the koala's habitat 'quality', which is a combined measure of habitat size and connectivity. Habitat quality is important because the koala population is typified by two distinct system regimes, each with different population growth rates and maximum (equilibrium) populations. Two critical ecological thresholds exist in terms of the level of habitat quality. Upon crossing one of these ecological thresholds, the system will shift from the original regime to the alternative regime.

The land area currently occupied by koala habitat may alternatively be used for housing development. The conversion of land from koala habitat to housing development is considered to be effectively irreversible because housing developments are very rarely demolished. Reductions in habitat quality owing to conversions of koala habitat to housing development can be offset by improving existing or developing new wildlife corridors. Better wildlife corridors allow increased access to other geographically-separate koala colonies (populations) and, through cross-colony mating and breeding, the genetic diversity across koalas in the study area will be increased.

5.2.1 Koalas: State variables

Table 5-4 below lists and describes the state variables of the koala case study.

Table 5-4 State variables of the koala case study

State variable	Description
System 'output' (Y)	Population of a koala species (study population) within a chosen geographical area (study area)
Underlying slow ecosystem variable (X)	Habitat 'quality' – combination of habitat size and habitat connectivity – proxy for genetic diversity
Habitat size (Z^S)	Physical size of the study area
Habitat connectivity (Z^C)	Connectivity of the study area with nearby koala colonies
Stock of improvements to habitat connectivity (\bar{C}_t)	Habitat connectivity can be improved by improving existing or developing new wildlife corridors
Prior probability bounds ($a^{min}, a^{max}, b^{min}, b^{max}$)	Describes the locations of two ecological thresholds of unknown location that separate two alternative population growth regimes
Stock of housing development (\bar{H})	Koala habitat within the study area can be converted to housing development

5.2.2 Koalas: Control variables

Table 5-5 below lists and describes the control variables of the koala case study.

Table 5-5 Control variables of the koala case study

Control variable	Description
Effort to increase the study area's habitat quality (C)	Improving existing or developing new wildlife corridors to previously poorly-connected or disconnected koala colonies – cross-colony breeding improves genetic diversity and resilience to negative impacts e.g. disease
Effort to harvest koala habitat and convert it to housing development (E)	Usually, the system output is harvested; however, the underlying slow variable is harvested in this case – any increase in the stock of housing development is offset by an equivalent reduction in habitat size

5.2.3 Koalas: Trade-offs

The trade-offs faced in this case study are between utility derived from non-use values related to the koala population within the study area; utility acquired by individuals owing to living in their own home (this utility is taken to be the net utility gain from living in a house situated within the study area as opposed to living in a

house located elsewhere); and disutility (opportunity cost) from the use of public funds for the purpose of improving the habitat quality of the study area instead of for the best alternative use.

The harvesting of koala habitat for conversion to housing development will have a linear, one-for-one relationship (i.e. one hectare of housing development comes at the expense of one hectare of koala habitat); however, the relationship between housing development and habitat connectivity is likely to be non-linear and conditional on the location of the specific parcel of land within the broader context of the total habitat area. The removal of a parcel of land that is vital for a wildlife corridor is likely to have a much more significant detrimental effect on the koala ecosystem than the removal of habitat from within a large area of habitat that does not perform that same role. Broadly speaking, the smaller the habitat size, the more likely a particular parcel of land forms a vital part of a wildlife corridor. Therefore, the negative impact of housing development on the koala ecosystem will be more significant when the koala habitat is smaller and the stock of housing development is higher. The conversion of land from koala habitat to housing development is considered to be effectively irreversible because housing developments are very rarely demolished.

There is a linear, one-for-one relationship between the reduction in koala habitat due to harvesting and the increase in the stock of housing development (Figure 5-7):

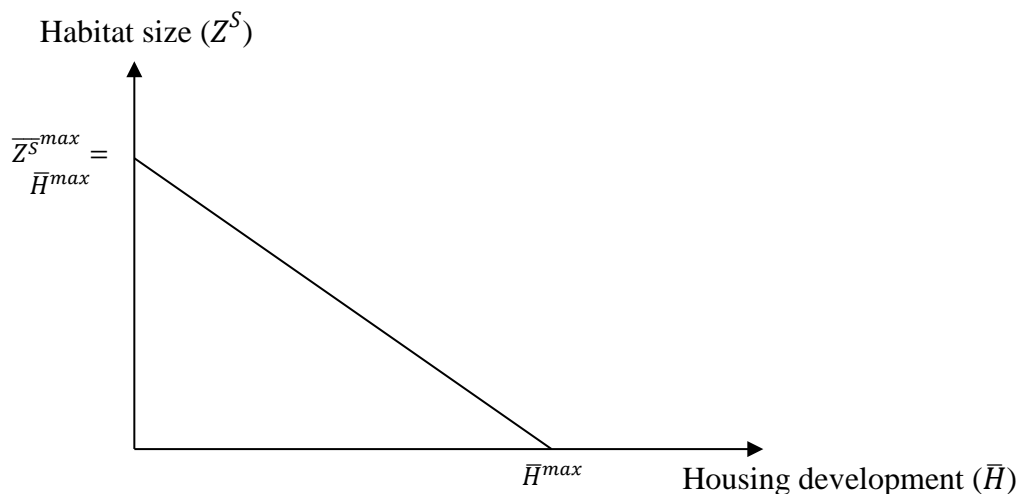


Figure 5-7 Habitat size vs. Housing development

Where

$$Z^S = \overline{Z^S}^{max} - \bar{H} = \bar{H}^{max} - \bar{H}$$

Alternatively,

$$Z^S + \bar{H} = \overline{Z^S}^{max} = \bar{H}^{max}$$

Note that Z^S , \bar{H} , $\overline{Z^S}^{max}$ and \bar{H}^{max} are expressed in hectares. $\overline{Z^S}^{max} = \bar{H}^{max}$ represents the total land area of the study area; the maximum habitat area or, at the other extreme, the maximum stock of housing development. Since the study area has only two potential uses, the sum of habitat area and stock of housing development must equal the total land area of the study area.

The relationship between stock of housing development and habitat connectivity is likely to be non-linear because the removal of one hectare of land from a much larger area of habitat is less likely to significantly adversely affect connections to the remaining hectares of land, compared to the removal of one hectare of land from a much smaller area of habitat (Figure 5-8).

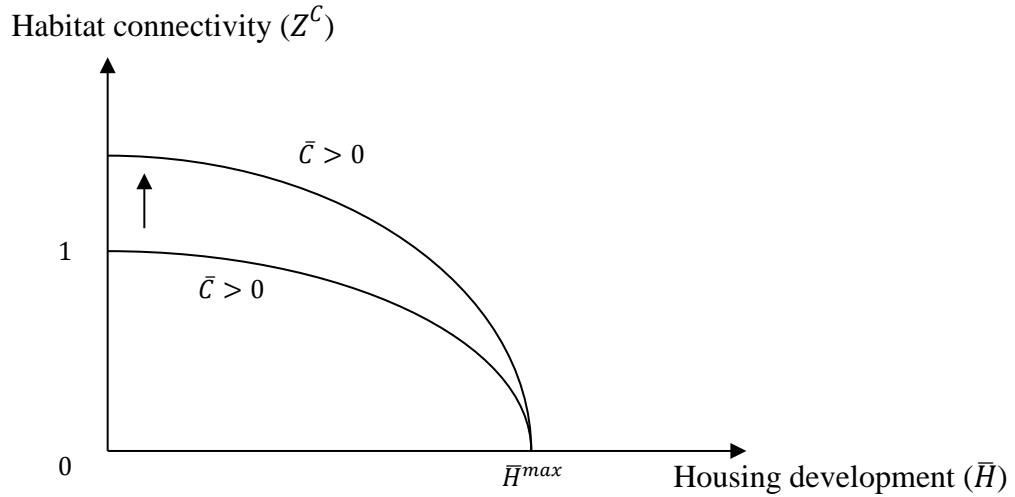


Figure 5-8 Habitat connectivity vs. Housing development

Where

$$Z^C = \varsigma(\bar{C}) \left(1 - \frac{\bar{H}^a}{b} \right)$$

$$a > 1$$

$$b = (\bar{H}^{max})^a$$

$$\zeta(0) = 1$$

$$\frac{d\zeta(\bar{C})}{d\bar{C}} > 0$$

$$\frac{d^2\zeta(\bar{C})}{d\bar{C}^2} \leq 0$$

$\zeta(\bar{C})$ is a scale parameter, which takes value 1 when no improvements have been made to the wildlife corridors that existed at the beginning of the problem, and no new wildlife corridors have been developed. In the case of no improvements to wildlife corridors (i.e. $\bar{C} = 0$) and no housing developments (i.e. $\bar{H} = 0$), Z^C will take a value of 1. Note that Z^C is an index variable and that $Z^C = 1$ has arbitrarily been chosen as its starting value.

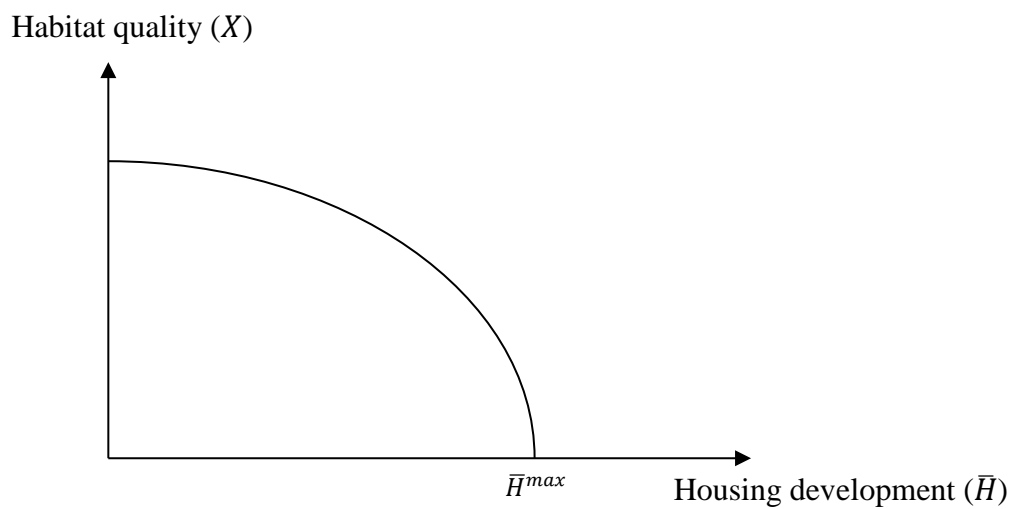


Figure 5-9 Habitat quality vs. Housing development

Where

$$X = mZ^S + (1 - m)Z^C$$

$$0 < m < 1$$

Habitat quality (X) is defined as a convex combination of two variables, habitat size (Z^S) and habitat connectivity (Z^C). Since Z^S is linear in \bar{H} and Z^C is non-linear in \bar{H} , X will also be non-linear in \bar{H} (Figure 5-9).

5.2.4 Koalas: EOM for habitat quality (X)

The maximum land area available for housing development is pre-determined at the beginning of the problem, and will be equivalent to the total land area of koala habitat at the beginning of the problem. This is because it is assumed that no housing development has been undertaken. For this reason, there is a maximum level (land area) of housing development that can occur. However, the quality of koala habitat (X) can be increased while the stock of housing development remains constant, or even increases. This is possible because effort in increasing habitat quality (C), in the form of planting or developing new wildlife corridors to currently unconnected colonies will improve the connectivity of the habitat and the genetic diversity of the study population. Habitat quality (X) serves as a proxy for genetic diversity. The equation of motion for habitat quality is given by:

$$\Delta X = m\Delta Z^S + (1 - m)\Delta Z^C \quad (55)$$

5.2.5 Koalas: Critical ecologic thresholds

The koala ecosystem is susceptible to a threshold effect and regime shift if the habitat quality crosses a critical ecological threshold. Upon crossing a threshold, the system will shift from Regime A to Regime B, or *vice versa*, and the population will evolve according to a different population growth equation. The logic for there being two separate population growth regimes is that a larger and more resilient (higher expected survival rate) population will result when starting from a more diverse genetic base; habitat quality is used as a proxy for the genetic diversity of the population. In Figure 6-10, the vertical dashed arrows signify the crossing of a critical ecological threshold and the resulting shift in system regime. The downward-pointing arrow signifies a shift from Regime A to Regime B and the upward-pointing arrow signifies the opposite case. The purpose of the two horizontal dotted lines is to indicate that the exact locations of these two thresholds are unknown; however, upper and lower bounds can be postulated.

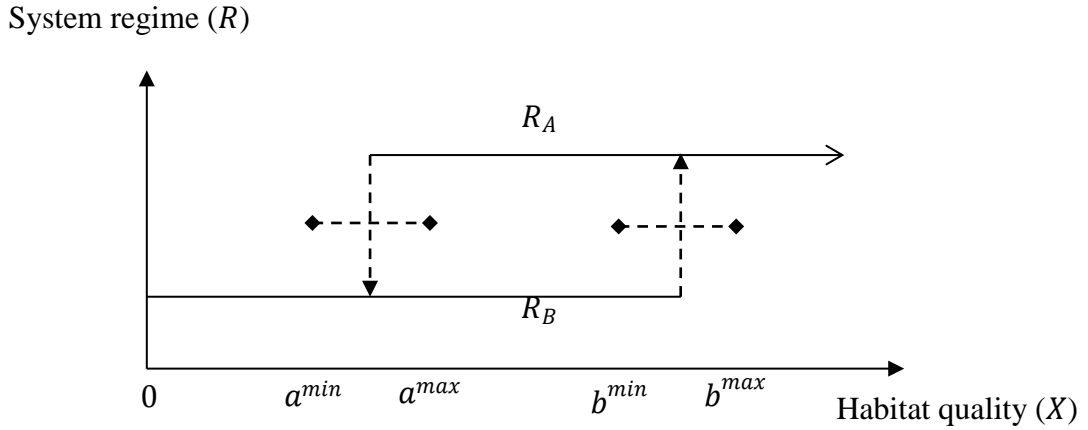


Figure 5-10 Alternative system regimes for koala population

5.2.6 Koalas: Population growth functions

Net population growth is a function of the koala population in the previous period; however, the growth rate is conditional on the current system regime. Also, the carrying capacity within the study area is a function of the habitat size (Z^S), since there is only enough food available per-hectare to feed a certain number of koalas. There is a critical threshold in terms of the habitat quality (X), which acts as a proxy for the level of genetic diversity within the study population. The koala population is modelled to evolve according to a Ricker equation (Ricker 1954). However, the intrinsic growth rate of the population is conditional on the system regime.

Koala population evolves according to:

$$Y_t = Y_{t-1} \cdot \exp\left(r_i \left(1 - \frac{Y_{t-1}}{Y_{t-1}^{max}}\right)\right) \quad (56)$$

Where

r_i is the regime-specific intrinsic growth rate for Regime i ; $i = A, B$

$$r_A > r_B > 0$$

$$Y_{t-1}^{max} = \gamma \cdot Z_{t-1}^S$$

γ is a scale parameter that specifies the maximum number of koalas that can be fed per hectare of koala habitat

Figure 5-11 provides an illustrative example of how the koala population will evolve when starting from a low population base and remaining within the same system regime for the duration of the time horizon. For this example, it is also assumed that the habitat size (Z^S), and therefore carrying capacity (Y^{max}), remains constant. Given sufficiently time, the koala population will converge to the upper limit, $\overline{Y_i^{max}}$, where this upper limit is conditional on the system regime.

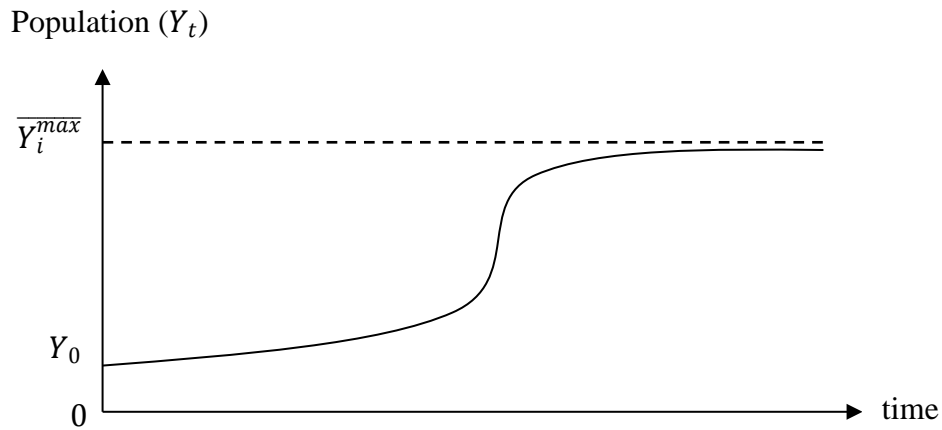


Figure 5-11 An illustrative path of the koala population when habitat size remains constant

Figure 5-12 provides an illustrative example of how the koala population will evolve when starting from a low population base and remaining within the same system regime for the duration of the time horizon. However, the dotted vertical lines signify instances when the study area experiences a discrete reduction in habitat size (Z^S), and therefore carrying capacity (Y^{max}). If the koala population is already close to its carrying capacity, a sufficient reduction in habitat size (and carrying capacity) will cause the population to decrease until it converges to the new carrying capacity.

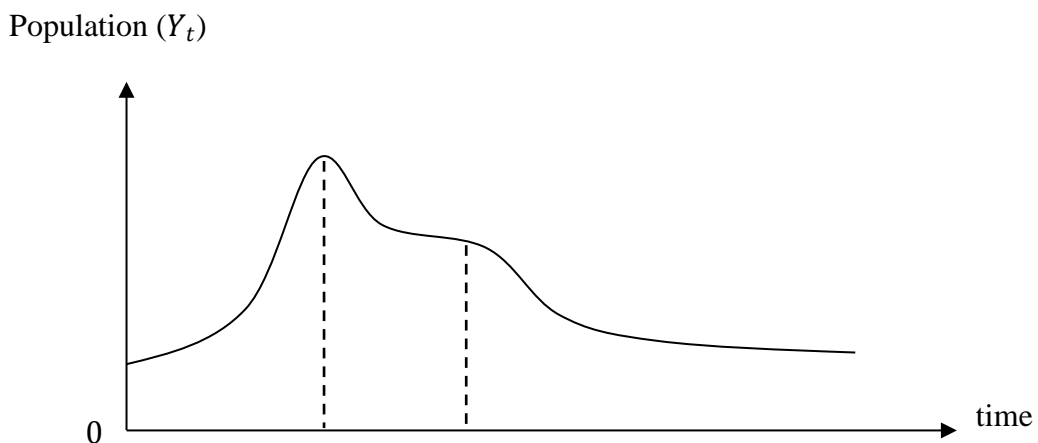


Figure 5-12 An illustrative path of the koala population when habitat size is occasionally reduced

5.2.7 Koalas: Sources of utility

Utility is derived both from direct utility, as a function of the koala population within the chosen geographical area, and from direct utility, as a function of the total stock of housing development; individuals receive utility from owning their own houses. Also, disutility, or negative utility, results from the use of any public funding for increasing the habitat quality. This disutility captures the opportunity cost to society of spending public funds on preserving koala habitat rather than on the best alternative use. Any profits gained by property developers from the sale of new housing developments have not been included in the model because they simply represent transfer payments between different members of society.

The instantaneous total social utility is given by:

$$U(t) = U_Y(Y_t) + U_{\bar{H}}(\bar{H}_t) - DU_C(C_t) \quad (57)$$

Where

$$\frac{dU_Y}{dY} > 0; \frac{d^2U_Y}{dY^2} \leq 0; \frac{dU_{\bar{H}}}{d\bar{H}} > 0; \frac{d^2U_{\bar{H}}}{d\bar{H}^2} \leq 0; \frac{dDU_C}{dC} > 0; \frac{d^2DU_C}{dC^2} \leq 0$$

Utility acquired from the size of the koala population and stock of housing development are both monotonically increasing in their arguments; however, marginal utility is decreasing for both sources of utility. Utility resulting from the use of any public funding for increasing the habitat quality is negative and monotonically decreasing in its argument. Marginal utility from this source is increasingly negative in its argument.

Summary of koala case study

The population of a koala colony and its evolution across time is dependent on the size of the habitat and its connectivity to surrounding koala colonies. These two attributes are combined to form a measure of habitat quality, which serves as a proxy for the genetic diversity within the population. If the habitat quality falls below a

critical level, an undesirable shift occurs to a regime that is characterised by a lower population growth rate and lower equilibrium population. This regime shift is reversible by crossing a different critical threshold i.e. the system has hysteretic dynamics.

The competing use of the land is housing development. An increase in the stock of housing development is associated with an equivalent decrease in the amount of koala habitat. This case study is distinguished by the ability to concurrently increase the stock of housing development and the habitat quality, via increasing habitat connectivity. However, there are decreasing marginal returns to investment in habitat quality, so the ecosystem manager faces a trade-off against increasing disutility from the use of public funds for this purpose. Therefore, there are economic limits to substitutability between housing development and habitat size.

Conclusion

This chapter presents two detailed case studies of complicated ecological systems that can be managed using the mathematical modelling framework developed in this thesis. For both the reef fishery and koala case studies, the health of the system is the main source of utility from the system. However, the case studies differ most notably in terms of threshold effects and system dynamics. The reef fishery has completely reversible dynamics, while the koala system has hysteretic dynamics. When the primary focus is maintaining environmental quality, to obtain utility from the health of the system, the decision-maker will optimally incur less risk of crossing an undesirable threshold than for the equivalent case study where utility is primarily obtained from profits and consumption.

These findings can be compared to the two other representative case studies described in Chapter 4. For these case studies, utility is sourced primarily from profits and consumption. First, profits from crop production and, second, from a cattle grazing enterprise. When utility is sourced primarily from consumption, the decision-maker will incur more risk of crossing an undesirable threshold than for the equivalent case, but where utility from other sources, such as the health of the system, is more important.

Chapter 6. Shallow lake simulation model and results

Phosphorus pollution running off from agricultural land into a shallow lake can suddenly turn the lake water from clear (oligotrophic) to dirty (eutrophic) when a threshold is crossed. This chapter builds on this example of a regime shift presented in Chapter 4, to explicitly simulate the benefits and costs of active learning in a specific case study. Simulation results, with and without the possibility of learning, are compared to explore the implications of the theoretical framework that includes learning within the management of an ecological system with an unknown threshold location (footnote: the theoretical framework is presented in Chapter 3). This component of the research serves to validate the findings of the theoretical model and demonstrate its potential for future applications. For this purpose, an abridged version of the shallow lake model, largely based on that presented by Brozovic and Schlenker (2011), was used to obtain numerical solutions for a range of different model parameter values. The model presented by Brozovic and Schlenker (2011) is extended through the inclusion of a mechanism (similar to Bayes' rule) for updating the prior probability distribution that characterises the location of a critical ecological threshold, where its exact location is unknown.

The model of a shallow lake includes three state variables; the stock of phosphorus in the lake, and the lower and upper bounds for the prior probability distribution of the unknown threshold location. This model has one control variable; the amount of phosphorus fertiliser applied to adjacent agricultural land. Utility is obtained from two sources; (i) utility from the application of fertiliser to grow and sell crops, and (ii) (dis)utility from the build-up of phosphorus in the lake system, which serves as a proxy for the level of ecosystem services provided. As discussed in Chapter 4, utility derived from fertiliser application can be categorised as consumption utility, and (dis)utility from the build-up of phosphorus in the lake can be categorised as direct utility. The simulation model is used to examine the consequences of using alternative prior probability distributions to describe the location of the unknown threshold, as well as the relative contributions of consumption utility and direct utility to total utility. These parameters have been chosen because they are under the control of the decision-maker.

The results of the simulation model demonstrate that (1) the expected NPV of utility from the system will be higher when active learning is considered, (2) the value of information is highest when the prior distribution has low variance and the system state variable is close to the prior distribution, (3) the decision-maker will generally engage in riskier behaviour when active learning is considered, (4) the optimal level of the control variable is generally a function of both the lower and upper bounds of the prior distribution, (5) the decision-maker will be less willing to incur risk of crossing an undesirable threshold when a greater proportion of total utility is sourced from direct utility, compared to consumption utility.

6.1 Simulation model (abridged ‘shallow lake’ model)

What follows is an abridged version of the shallow lake problem described in Chapter 4. The problem is simplified to include three state variables and one control variable. The reason for this simplification is to provide a more straightforward and less ambiguous illustration of how the consideration of active learning affects the level of risk averse behaviour undertaken by a decision-maker. In this example, the level of risk averse behaviour is judged based on the amount of fertiliser applied. A lower application rate is interpreted to mean that a greater amount of risk averse actions have been undertaken. The problem structure is taken from Brozovic and Schlenker (2011) and is described below.

6.1.1 The ‘Learning’ model

In the ‘Learning’ model, the decision-maker is alert to the possibility of learning about unknown threshold locations and factors this possibility into the decision-making process.

The instantaneous total social utility received is given by:

$$U(t) = kP_t - X_t^2 \quad (58)$$

Where

k is a measure of the utility received per unit of phosphorus fertiliser applied

P_t is the amount of phosphorus fertiliser applied, and the control variable

X_t is the stock of phosphorus in the lake at the end of time period t

The two equations of motion for the underlying slow variable (stock of phosphorus) are given by:

$$X_t = \begin{cases} BX_{t-1} + b + P_t & \text{if } R_t = A \\ BX_{t-1} + b + P_t + \Omega & \text{if } R_t = B \end{cases} \quad (59)$$

Where

B is the proportion of phosphorus retained in the lake from one time period to the next; the remainder is assimilated

b is the natural inflow of phosphorus to the lake

Ω is additional phosphorus recycling when the lake is in Regime B

R_t is the current system regime

The other two state variables are the lower and upper bounds of the prior distribution for the unknown threshold location. These state variables are updated in accordance with the updating rules described in Section 3.2 above. A uniform distribution has been used to model the prior distribution for the unknown threshold location. The updating rules for the upper and lower bounds of the prior distribution are conditional on the current system regime and whether or not the critical threshold is crossed. The four possible cases are described below:

a. If $R_{j_t} = A$ and $F_{j_t} = 0$:

$$T_{j_{t+1}} \sim \mathcal{U}(a_{j_{t+1}}^{\min}, a_{j_{t+1}}^{\max}) \quad (60)$$

Where

$$a_{j_{t+1}}^{\min} = \max(a_{j_t}^{\min}, X_{j_t})$$

$$a_{j_{t+1}}^{\max} = a_{j_t}^{\max}$$

b. If $R_{j_t} = A$ and $F_{j_t} = 1$:

$$T_{j_{t+1}} \sim \mathcal{U}(a_{j_{t+1}}^{min}, a_{j_{t+1}}^{max}) \quad (61)$$

Where

$$\begin{aligned} a_{j_{t+1}}^{min} &= a_{j_t}^{min} \\ a_{j_{t+1}}^{max} &= \min(a_{j_t}^{max}, X_{j_t}) \end{aligned}$$

c. If $R_{j_t} = B$ and $F_{j_t} = 0$:

$$T_{j_{t+1}} \sim \mathcal{U}(a_{j_{t+1}}^{min}, a_{j_{t+1}}^{max}) \quad (62)$$

Where

$$\begin{aligned} a_{j_{t+1}}^{min} &= a_{j_t}^{min} \\ a_{j_{t+1}}^{max} &= \min(a_{j_t}^{max}, X_{j_t}) \end{aligned}$$

d. If $R_{j_t} = B$ and $F_{j_t} = 1$:

$$T_{j_{t+1}} \sim \mathcal{U}(a_{j_{t+1}}^{min}, a_{j_{t+1}}^{max}) \quad (63)$$

Where

$$\begin{aligned} a_{j_{t+1}}^{min} &= \max(a_{j_t}^{min}, X_{j_t}) \\ a_{j_{t+1}}^{max} &= a_{j_t}^{max} \end{aligned}$$

Therefore, the conditional probability of the system being in Regime A at the end of time period $t + 1$ is given by:

$$A_{t+1}(X_{t+1}) = \begin{cases} 0 & \text{if } X_{t+1} \leq a_{t+1}^{min} \\ \frac{X_{t+1} - a_{t+1}^{min}}{a_{t+1}^{max} - a_{t+1}^{min}} & \text{if } a_{t+1}^{min} < X_{t+1} < a_{t+1}^{max} \\ 1 & \text{if } X_{t+1} \geq a_{t+1}^{max} \end{cases} \quad (64)$$

And the conditional probability of the system being in Regime B at the end of time period $t + 1$ is given by $1 - A_{t+1}(X_{t+1})$.

Finally, the Bellman equation of the utility function is described by the following equation:

$$V_i(X_t) = \max_{P_t} \left\{ kP_t - X_t^2 + \delta \left(A_{t+1}(X_{t+1}) \cdot V_A(X_{t+1}) + (1 - A_{t+1}(X_{t+1})) \cdot V_B(X_{t+1}) \right) \right\} \quad (65)$$

Where

$$V_A(X_{t+1}) = V_A(BX_t + b + P_t)$$

$$V_B(X_{t+1}) = V_B(BX_t + b + P_t + \Omega)$$

$A_{t+1}(X_{t+1})$ is described above

$i = A, B$

$V_i(X_t)$ is the value function for time period t and Regime i

$A_{t+1}(X_{t+1})$ is the most recently updated conditional probability of the system being in

Regime A in time period $t + 2$; conditional on the stock of phosphorus in the lake

$1 - A_{t+1}(X_{t+1})$ is the most recently updated conditional probability of the system being in Regime B in time period $t + 2$; conditional on the stock of phosphorus in the lake

The system regime (Figure 6-1) is modelled according to:

$$R_t = \begin{cases} A & \text{if } X_{t-1} < X^C \\ B & \text{if } X_{t-1} \geq X^C \end{cases} \quad (66)$$

Where

R_t is the system regime for the duration of time period t

X_{t-1} is the level of the underlying slow variable at the end of the previous time period

X^C is a random variable on a bounded interval (X^{Cmin}, X^{Cmin})

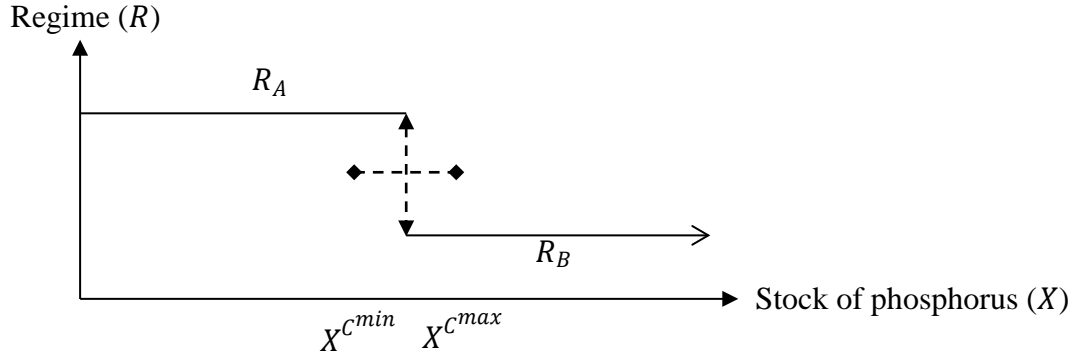


Figure 6-1 Alternative system regimes in abridged 'shallow lake' model

6.1.2 The 'No learning' model

In the 'No learning' model, the decision-maker fails to immediately acknowledge the possibility of learning about unknown threshold locations, and the associated value of information. This means the decision-maker will initially make a sub-optimal decision because the impact of today's actions on the speed of learning is not considered. Primarily, the 'Learning' and 'No learning' models differ because the lower and upper bounds of the prior distribution for threshold location are state variables in the 'Learning' model, whereas, they are parameters in the 'No learning' model.

The Bellman equation of the utility function for the 'No learning' model is identical to that of the 'Learning' model, and is described by the following equation:

$$V_i(X_t) = \max_{P_t} \left\{ kP_t - X_t^2 + \delta \left(A_{t+1}(X_{t+1}) \cdot V_A(X_{t+1}) + (1 - A_{t+1}(X_{t+1})) \cdot V_B(X_{t+1}) \right) \right\} \quad (67)$$

Where

$$V_A(X_{t+1}) = V_A(BX_t + b + P_t)$$

$$V_B(X_{t+1}) = V_B(BX_t + b + P_t + \Omega)$$

$$i = A, B$$

However, the lower and upper bounds of the prior distribution are not updated as hypothetical learning occurs. This means that:

$$\begin{aligned} a_{j_t}^{min} &= a_1^{min} \quad \forall t \\ a_{j_t}^{max} &= a_1^{max} \quad \forall t \\ A_{t+1}(X_{t+1}) &= A_1(X_{t+1}) \quad \forall t \end{aligned}$$

As a result of this failure to acknowledge the potential for learning, a sub-optimal level of the control variable will be perceived as optimal in the first time period (i.e. today). Thereafter³¹, the decision-maker will rely on the correct model specification, the ‘Learning’ model, when determining optimal decisions. Therefore, the value of information captures the cost of the decision-maker making a sub-optimal decision today, and the follow-on effects in future time periods.

6.2 Simulation methodology

This section explains the premise of the MATLAB© code used to solve the simulation model. This string of code forms a recursive solution algorithm that captures the elements of the conceptual model described in Chapter 3. The model is then run to obtain numerical solutions for a range of different model parameter values. The ultimate aim is to determine the level of the control variable ‘today’ that will maximise the expected NPV of utility from the system over an infinite time horizon, and when the benefits of active learning about unknown threshold locations are explicitly considered.

First, the parameter values and initial values of the state variables must be defined. The state variables are the lower and upper bounds of the prior distribution for the unknown threshold location, and the initial concentration of phosphorus in the lake. The model parameter values are the number of discrete time periods modelled, the relative weightings of direct utility and consumption utility in the objective

³¹ From time period 2 onwards.

function, and the parameters that describe the evolution of phosphorus in the lake i.e. the system dynamics.

The computationally intensive nature of the problem necessitated the use of a grid of initial values of the state variables. The grid points were concentrated in regions of the state-space thought most likely to contain optimal solution values, however, the total number of grid points used was restricted by the ‘curse of dimensionality’. A three-dimensional grid containing the lower bound, upper bound and phosphorus stock was then cleaned such that only feasible specifications of the prior distribution were analysed. That is, specifications where the upper bound is greater than the lower bound. This left a grid of 377 points for each iteration. Upon solving the problem for each grid point, the solution values for levels of the state variables that did not form one of the grid points were interpolated using a flexible functional form (translog) regression.³² This process was repeated for each iteration of the model. For each iteration, the optimisation problem was solved using MATLAB’s ‘Global Search’ algorithm, to avoid confusion between local and global optima.

Since the model must be solved recursively, the problem is first optimised for the terminal time period. The value function for the terminal period differs from all other time periods because of the inclusion of a penalty value based on the health of the system, stock of phosphorus, at the end of the terminal time period. The penalty creates a disincentive for the decision-maker to exploit the system in the terminal time period, since a higher final level of phosphorus is associated with a higher penalty. The negative consequences of an undesirable regime shift in the terminal time period would otherwise not be considered because they are not felt until the succeeding time period. The terminal value function is optimised for two alternative specifications, which are spending the terminal time period in either the preferred regime (A) or the less preferred regime (B). One, or both, of these value functions then feed into the value functions of the preceding time period, and the process is repeated many times to approximate an infinite time horizon problem.

³² For all translog regressions corresponding to each iteration, the R^2 value exceeded 0.9. For almost all iterations, the R^2 value exceeded 0.95.

For all non-terminal time periods, the system is modelled such that a regime shift may occur at the end of the time period. For example, the system can be in Regime A at the beginning, and for the duration, of time period $t - 1$ then experience a regime shift to Regime B prior to the beginning of time period t . Therefore, six permutations must be considered, which are modelled as six separate constraints on the level of phosphorus stock in the previous ($t - 1$) and current (t) time periods. These permutations capture (i) the system remaining in the 'safe' region, not crossing the closer bound of the prior, and certainly not experiencing a regime shift, (ii) the system being perturbed beyond the further bound of the prior and certainly experiencing a regime shift, and (iii) the system being perturbed into the 'risky' region, with associated non-zero probabilities of experiencing a regime shift or not experiencing a regime shift. These three possibilities expand to six permutations because the system could possibly have been in either of the two alternative regimes in the previous time period ($t - 1$).

Each of these six permutations represent separate cases to be optimised and are associated with one of the six constraints described above. These six different cases are then divided into two groups, one for when the system is in Regime A in the previous time period ($t - 1$) and another for when the system is in Regime B in the previous time period. Each group contains the three permutations (i), (ii) and (iii) described above that apply to the corresponding system regime, A or B.

The value function is then optimised for each grid point of initial conditions for the state variables, and for each of the three cases in each group. For example, if the system was in Regime A in the previous time period ($t - 1$), it could possibly take any of the three paths (i), (ii) and (iii) described above. Each of these paths is associated with a different level of the value function. A rational decision-maker will choose the path that derives the highest expected NPV of utility. This means that for each combination of the initial conditions of the state variables (grid point), the decision-maker chooses the path that results in the highest expected NPV and only the level of the value function and optimal level of the control variable associated with this particular path is of importance.

It is also possible that one or two of the paths are not feasible, given a particular grid point. For example, if the system is in Regime B in the previous time period ($t - 1$) and the carried-over stock of phosphorus is very large, it may not be ecologically possible for the stock of phosphorus to be reduced sufficiently for a regime shift to occur in the current time period. In this case, the particular system path is infeasible for the chosen initial levels of the state variables and a 'Not a Number' value is recorded for the value function.

The process of solving recursively, or via backward induction, is repeated for a finite number of discrete time periods, with the aim of approximating a problem with an infinite time horizon. This process involves optimising the value function of the current time period (t) for a grid of initial values of the state variables, interpolating these results for values that weren't included in the grid, and feeding these model parameter estimates into the value function of the previous time period ($t - 1$). This iterative process is able to incorporate active learning via the evolution of the lower and upper bound state variables. The final iteration involves optimising the value function, and determining the optimal level of the control variable, for the initial time period of the problem. This corresponds to determining the optimal amount of fertiliser application 'today' that will maximise the expected NPV of utility from the system over an infinite time horizon, and when the benefits of active learning are explicitly considered.

6.3 Simulation results

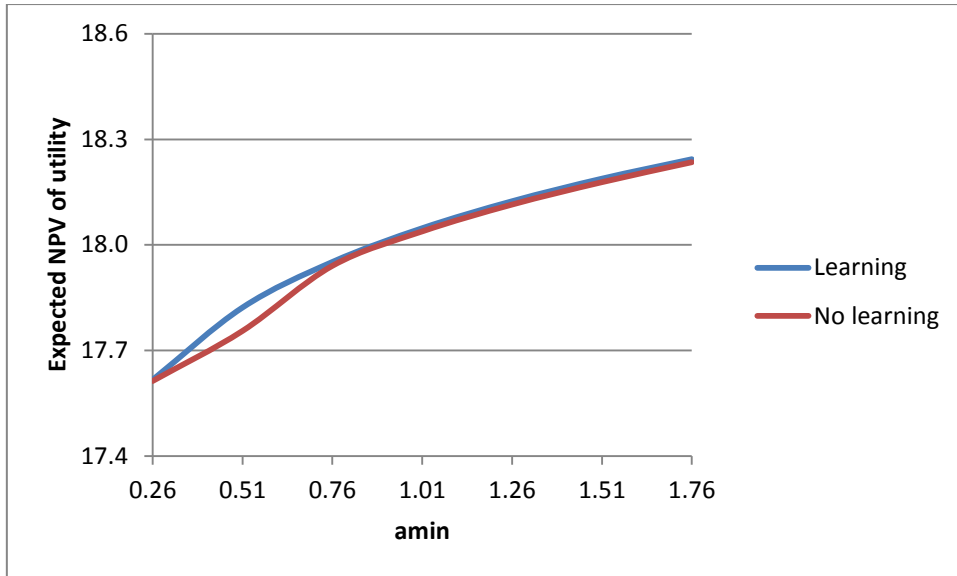
This section presents the results of the model simulation and examines the propositions stated in Chapter 1. There is particular focus on the consequences of using alternative prior probability distributions to describe the unknown location of the threshold, as well as the relative contributions of consumption utility and direct utility to total utility. These parameters have been chosen because they are under the control of the decision-maker. The sensitivity of (i) optimal decisions, (ii) maximum expected NPV of utility, and (iii) the level of the control variable to the initial conditions of the problem are also examined.

6.3.1 Expected net present value calculations and the value of information

A failure to consider the benefits, and costs, of learning means the decision-maker is pursuing an incomplete and incorrect objective. Axiomatically, the maximum expected NPV of utility generated by an ecosystem will be higher when active learning is included in the optimisation. What is perceived to be the optimal solution when active learning is not considered differs from the optimal solution determined using a more complete objective function. Therefore, it is more interesting to examine the magnitude of the increase in the maximum expected NPV of utility and the change in the optimal level of the control variable when active learning is considered. The magnitude of expected gains depends on the subjective choice of both the lower and upper bounds of the prior distribution; however, these gains are always non-negative.

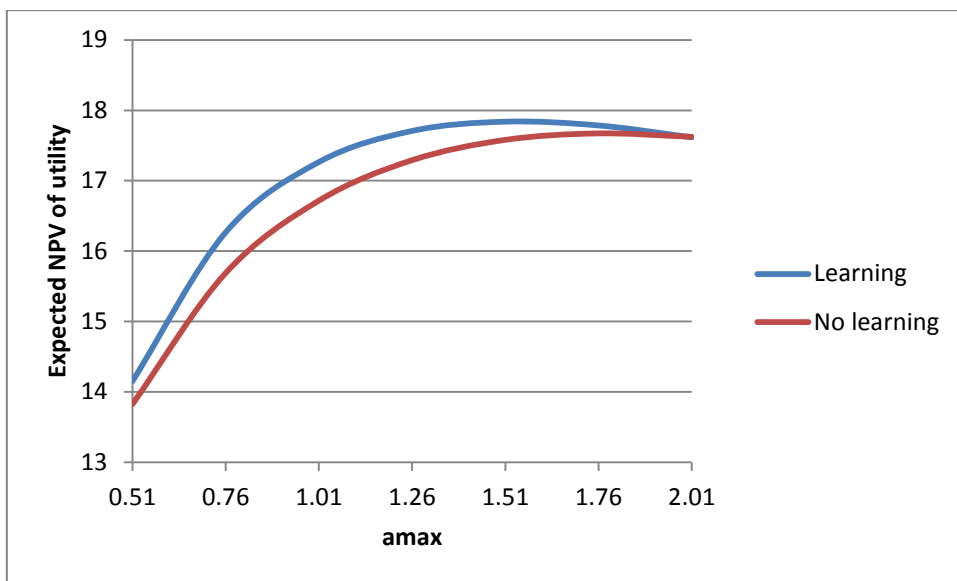
P1.1: The maximum expected NPV of utility generated by an ecosystem will be *higher* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

Figure 6-2 and Figure 6-3 show a comparison of the expected NPV of utility when active learning is considered (Learning) and when active learning is not considered (No learning). For the latter case, the potential for learning is not considered for the initial time period, but is considered for all succeeding time periods. In other words, there is a lag of one time period before the decision-maker realises that the effects of active learning should be considered within the decision problem; however, a sub-optimal decision will have been made in the first time period. If the decision-maker takes longer than one time period to come to this realisation, the value of information will be greater.



Parameters: $X(0) = 0.01$ $a_{max}(0) = 2.01$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-2 Expected NPV of utility when the lower bound is varied



Parameters: $X(0) = 0.01$ $a_{min}(0) = 0.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-3 Expected NPV of utility when the upper bound is varied

For both models, the lake is initially in the preferred regime, and the concentration of phosphorus in the lake is known to be below the threshold level. In Figure 6-2, the upper bound of the prior distribution for the unknown threshold location is held fixed while the lower bound is varied. In Figure 6-3, the lower bound of the prior distribution for the unknown threshold location is held fixed while the upper bound is varied. The level of the other state variable, the stock of phosphorus

carried over from the previous time period, is also held fixed. For both Figure 6-2 and Figure 6-3, one of the prior bounds is held fixed while increasing the variance of the prior distribution, as measured by the distance between the lower and upper bounds of the prior distribution.

As the lower bound (a_{min}) is varied, there is almost no discernible difference between the expected NPV of utility when comparing the 'Learning' and 'No learning' models. This is attributable to the subjective choice of upper bound for the prior distribution, which is far from the initial level of the phosphorus stock. Note that the initial stock of phosphorus (X) is 0.01 and the initial upper bound (a_{max}) is 2.01.

For values of the lower bound (a_{min}) around 0.51, there is a discernible difference between the expected NPV of utility for the two models. For this small range of values, there is an expectation of net gains resulting from the trade-off between incurring extra risk of experiencing an undesirable regime shift and the possibility of favourably updating the prior distribution for the unknown threshold location. A favourable update of the prior distribution involves revising the lower bound (a_{min}) upwards, which means a higher level of fertiliser can be applied in future time periods while knowing that the threshold will not be crossed.

For low values of the lower bound (a_{min}) of the prior distribution, choosing a level of the control variable that results in the lower bound (a_{min}) being crossed is associated with a very low probability of crossing the threshold and learning valuable information about the system. This is because there is a large difference between the lower and upper bound, so the marginal probability of an additional unit of phosphorus stock resulting in the threshold being crossed is quite low. Therefore, there is limited additional incentive to perturb the system beyond the lower bound (a_{min}) when the potential for learning is considered, compared to the case where learning is not considered.

Conversely, for high values of the lower bound (a_{min}), the system must be excessively perturbed in order to learn more about the location of the threshold. This is unlikely to happen because any consumption utility gains from additional

phosphorus application³³ will be more than offset by direct utility losses³⁴. For this reason, the threshold is essentially a non-binding constraint when the initial level of the phosphorus state variable is low and the lower bound (a_{min}) of the prior distribution is sufficiently high.

As the upper bound is varied in the ‘Learning’ and ‘No learning’ models, there is a positive near-monotonic³⁵ relationship between the expected NPV of utility and the subjective choice of upper bound (a_{max}) for the prior distribution of the unknown threshold location. This is expected for the ‘Learning’ model because a consistent instantaneous utility function (one that considers the effects of active learning) is used for all time periods in the model. There is a discernible difference between the two curves in Figure 6-3, which indicates that the first-period level of the control variable is sub-optimal for the ‘No learning’ model. The degree to which this initial sub-optimal decision then affects the expected NPV of utility is dependent on the choice of upper bound (a_{max}) for the prior distribution. For low levels of the upper bound (a_{max}), perturbing the system to a point between the lower (a_{min}) and upper bounds is associated with a high probability of learning about the system and significantly reducing the variance of the prior distribution, as measured by the difference between the lower and upper bounds. Conversely, when the variance of the prior distribution is high, there is a smaller likelihood of learning about the system, and there is a convergence between the expected NPV of utility from the two models.

P1.3: The value of information (i.e. the difference between the expected NPV of utility when active learning is included compared to when it is *not* included) will have a non-monotonic relationship with the initial level of uncertainty about the threshold location (as measured by standard deviation of prior probability distribution).

The value of information is defined as the difference in the expected NPV of utility between the two models. It is naturally expected that the value of information

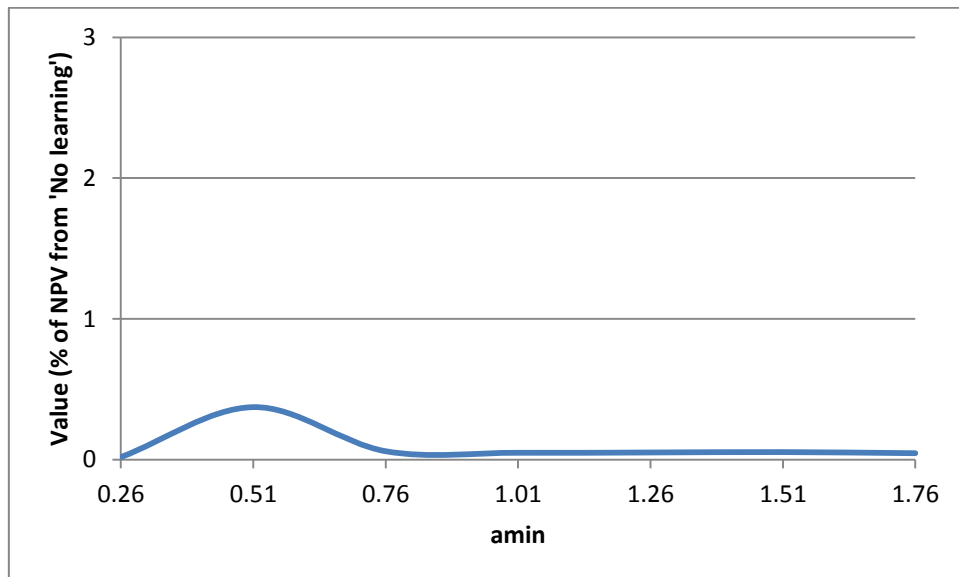
³³ Due to higher crop yields and higher profits.

³⁴ Due to the lake being less suitable for recreational uses e.g. swimming, fishing.

³⁵ This is likely to simply be an artefact of the numerical estimation method used. A monotonic relationship is expected.

will be positive because more information allows a decision-maker to make more accurate decisions. What are of more interest are the particular problem characteristics that make information more, or less, valuable. It is worth noting that, for the ‘No learning’ model, it has been assumed that the decision-maker has only taken one time period to realise that the effects of learning should be considered. If the decision-maker takes more than one time period to realise this, and makes sub-optimal decisions for a greater number of time periods, the value of information will be even higher.

In Figure 6-4, the value of information is shown as the lower bound (a_{min}) of the prior distribution is varied while maintaining a constant level of variance, as measured by the distance between the lower and upper bounds of the prior distribution. In Figure 6-5, the value of information is shown as the upper bound (a_{max}) of the prior distribution is varied while maintaining a constant level of variance, as measured by the distance between the lower and upper bounds of the prior distribution.



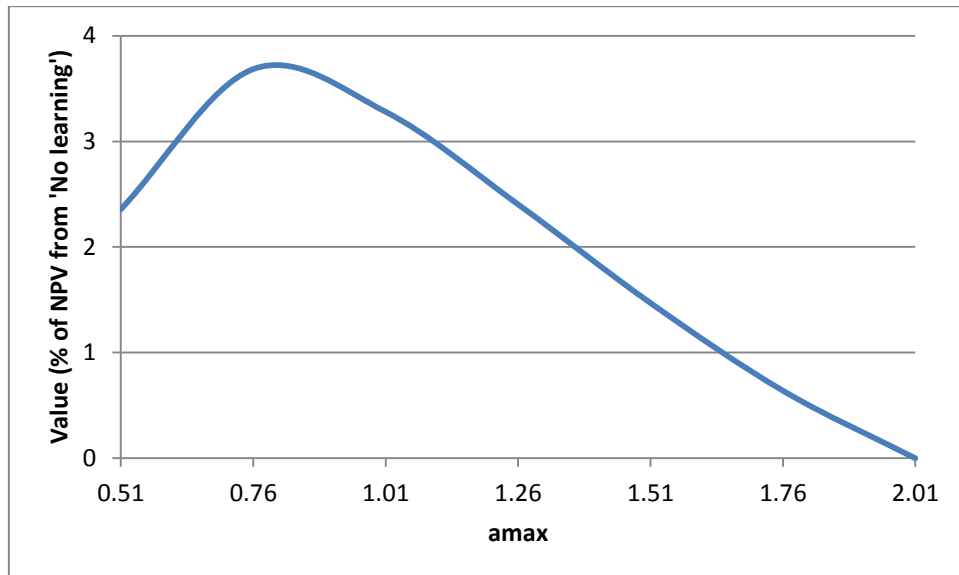
Parameters: $X(0) = 0.01$ $a_{max}(0) = 2.01$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-4 Value of information when the lower bound is varied

Following from the explanation given for Figure 6-2 above, the value of information, as a percentage of the expected NPV of utility from the ‘No learning’ model, is negligible for most values of the lower bound (a_{min}) shown. It is only

around $a_{min} = 0.51$ that there is a beneficial trade-off between incurring extra risk of crossing the threshold in combination with a reasonable probability of learning valuable information about the threshold's location. Although quite small in absolute terms, Figure 6-4 shows a non-monotonic relationship between the value of information and the initial level of uncertainty³⁶ about the threshold location.

The value of information, as a percentage of the expected NPV of utility from the 'No learning' model, is highest for low values of the upper bound (a_{max}), but approaches zero for high values of the upper bound. When both the lower and upper bounds of the prior distribution are low, and therefore the variance is low, perturbing the system between the two bounds is highly likely to result in valuable information being acquired about the unknown threshold location. The explicit consideration of active learning is at its most valuable for intermediate values of the upper bound, around $a_{max} = 0.76$. The value of information then gradually declines as the variance of the prior distribution increases and the likelihood of the decision-maker learning about the threshold location declines. Similar to above, Figure 6-5 shows a non-monotonic relationship between the value of information and the initial level of uncertainty about the threshold location.



Parameters: $X(0) = 0.01$ $a_{min}(0) = 0.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-5 Value of information when the upper bound is varied

³⁶ Measured by assuming Knightian risk, rather than Knightian uncertainty.

6.3.2 Extent of risk averse activity and willingness to incur risk

It is expected that the explicit consideration of active learning will make the decision-maker more willing to incur risk. In this context, incurring more risk means taking actions that result in a higher probability of crossing a threshold. The decision-maker is more likely to incur risk because it is the only way to learn more about a threshold's exact location. The benefits flowing from learning about a threshold's exact location are only considered in a model that explicitly factors in active learning; these benefits are otherwise not considered. The most notable potential benefit results when more information is gathered about the location of a threshold and the prior probability bounds are tightened. A cost saving (alternatively, a benefit) flows from requiring less active control of the system to avoid a threshold crossing than would otherwise have been assumed, if using the original prior distribution. Proposition P1.2 is expected to hold regardless of whether the system has hysteretic dynamics or an irreversible threshold. It is tested as a combination of propositions P2.1 and P2.2 below.

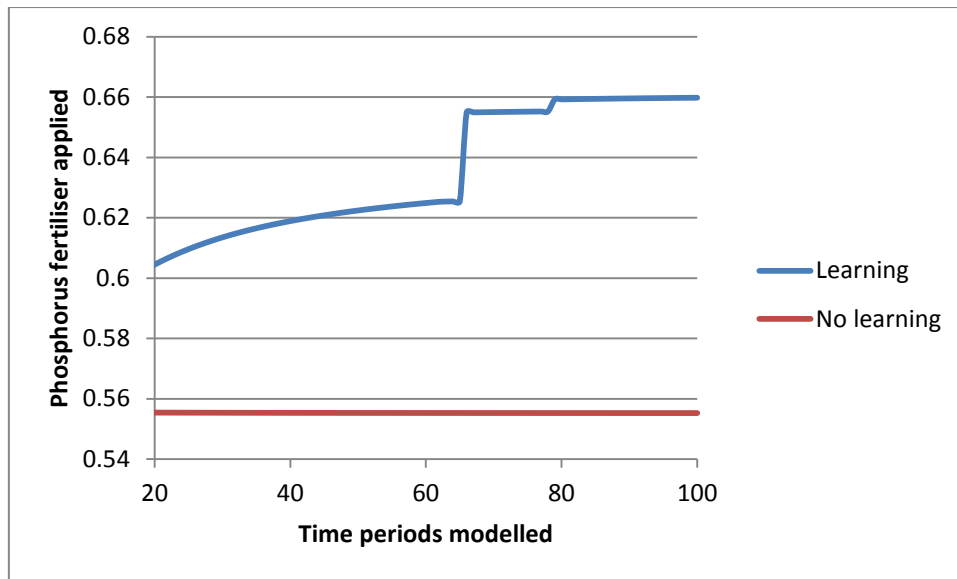
P1.2: For the first time period of the problem, the amount of effort invested into controlling the ecosystem away from an undesirable threshold (i.e. investment in risk averse actions) will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

For this problem, there is only one control variable, which is the amount of fertiliser applied to agricultural land that subsequently flows into the lake system. Therefore, greater investment in 'risk averse actions' is defined as applying a lower level of fertiliser, since a lower stock of phosphorus in the lake is associated with a lower probability of crossing the threshold from the more preferred to the less preferred regime.

P2.1: For a system with hysteretic dynamics, the optimal level of first-period investment in effort to control the system away from an *undesirable* threshold will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

When the initial system regime is the preferred regime, it is expected that the decision-maker will desire to maintain the system in the preferred regime. However, factoring in the potential for learning also acknowledges that there are potential benefits from incurring some risk of crossing the threshold because this means that more information will be gathered about the location of the threshold. Gathering more information means updating the prior probability bounds for the unknown threshold location. If the updated probability bounds are more favourable, the decision-maker is then able to invest a lower amount of effort in controlling the system away from the threshold without incurring any risk of crossing the threshold. In the context of this problem, that means applying a higher level of phosphorus fertiliser without incurring any risk of crossing the threshold.

Figure 6-6 shows a comparison of the optimal first-period levels of the control variable, fertiliser application, when active learning is considered (Learning) and when active learning is not considered (No learning). For both models, the lake is initially in the preferred regime, and the concentration of phosphorus in the lake is known to be below the threshold level. The vertical axis shows the level of fertiliser application, while the horizontal axis shows the number of time periods modelled. The optimal level of the control variable is plotted against the number of time periods modelled to show the solution converging as more time periods are added and an infinite time horizon is better approximated.



Parameters: $X(0) = 0.01$ $a_{min}(0) = 0.51$ $a_{max}(0) = 1.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-6 Optimal first-period level of phosphorus when in the preferred regime

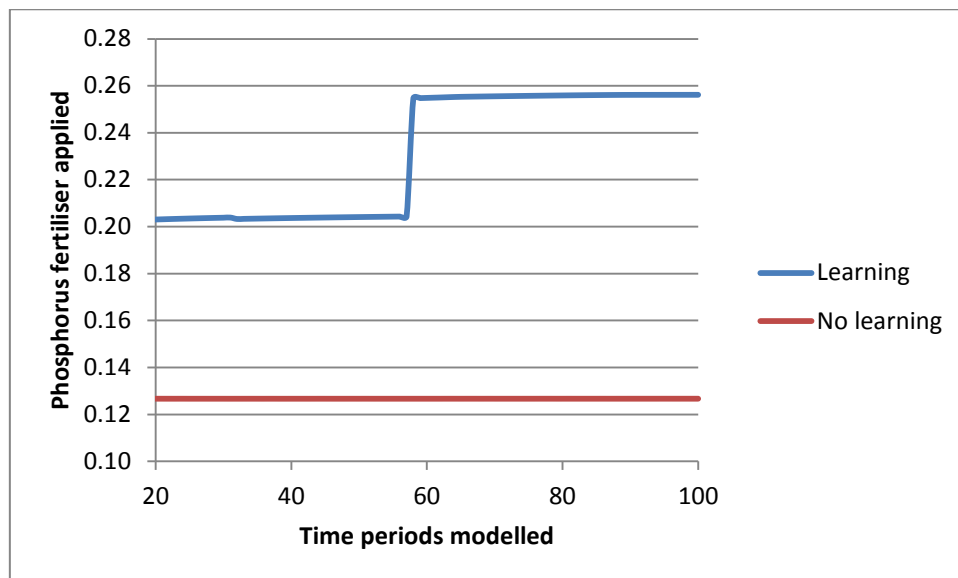
The decision-maker is more inclined to incur risk, by applying a higher amount of fertiliser, when active learning is considered compared to the case when learning is not considered.

P2.2: For a system with hysteretic dynamics, the optimal level of first-period investment in effort to control the system in the direction of a *desirable* threshold will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

When the initial system regime is the less preferred regime, it is expected that the decision-maker will desire to switch the system to the preferred regime. However, the assimilative capacity of the lake system, because of which only a small proportion of the phosphorus stock is retained from one time period to the next, means that the decision-maker is able to learn about the unknown threshold location without engaging in overly risk averse behaviour. This possibility is acknowledged by the decision-maker who considers active learning, and makes them less inclined to forgo short-term utility gains by engaging in overly risk averse behaviour that would result in a slightly higher probability of learning about the threshold's location. This is

because the system itself effectively engages in some risk averse behaviour due to its assimilative capacity.

Figure 6-7 shows a comparison of the optimal first-period levels of the control variable, fertiliser application, when active learning is considered (Learning) and when active learning is not considered (No learning). For both models, the lake is initially in the less preferred regime, and the concentration of phosphorus in the lake is known to be above the threshold level. The vertical axis shows the level of fertiliser application, while the horizontal axis shows the number of time periods modelled. The optimal level of the control variable is plotted against the number of time periods modelled to show the solution converging as more time periods are added and an infinite time horizon is better approximated.



Parameters: $X(0) = 1.76$ $a_{min}(0) = 0.51$ $a_{max}(0) = 1.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

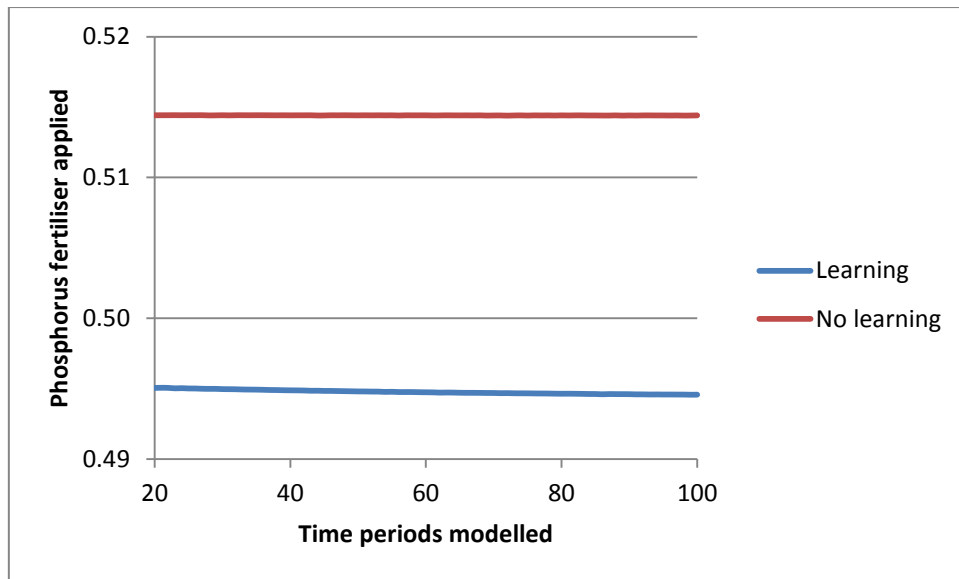
Figure 6-7 Optimal first-period level of phosphorus when in the non-preferred regime

When beginning the problem in the less preferred regime, the decision-maker will apply more fertiliser when active learning is considered compared to the case where learning is not considered. This means that there will be less chance of experiencing a desirable regime shift at the end of the first time period for the case where the effects of learning have been considered by the decision-maker.

P2.3: For a system with an irreversibility, the optimal level of first-period investment in effort to control the system away from an undesirable threshold will be *lower* for the case when active learning is included in the optimisation compared to the case when active learning is *not* included in the optimisation.

When the initial system regime is the preferred regime, and the system has an irreversible threshold, the decision-maker will obviously desire to maintain the system in the preferred regime. However, even at the risk of experiencing an irreversible regime shift, factoring in the possibility of learning also acknowledges that there are potential benefits from incurring some risk of crossing the threshold because this means that more information will be gathered about the location of the threshold. Gathering more information means updating the prior probability bounds for the unknown threshold location. If the updated probability bounds are more favourable, the decision-maker is then able to invest a lower amount of effort in controlling the system away from the threshold without incurring any risk of crossing the threshold. In the context of this problem, that means applying a higher level of phosphorus fertiliser without incurring any risk of crossing the threshold.

Figure 6-8 shows a comparison of the optimal first-period levels of the control variable, fertiliser application, when active learning is considered (Learning) and when active learning is not considered (No learning). For both models, the lake is initially in the preferred regime, and the concentration of phosphorus in the lake is known to be below the threshold level. The vertical axis shows the level of fertiliser application, while the horizontal axis shows the number of time periods modelled. The optimal level of the control variable is plotted against the number of time periods modelled to show the solution converging as more time periods are added and an infinite time horizon is better approximated.



Parameters: $X(0) = 0.01$ $a_{min}(0) = 0.51$ $a_{max}(0) = 1.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-8 Optimal first-period level of phosphorus with an irreversible threshold

Although the optimal first-period level of the control variable is slightly higher for the ‘No learning’ model compared to the ‘Learning’ model, this is most likely explained as a rounding error resulting from the numerical estimation method used. For both models, the lower bound (a_{min}) of the prior distribution is 0.51. The optimal level of the control variable for each model will leave the stock of phosphorus at a level that is marginally below the lower bound of the prior, with zero probability of crossing the undesirable and reversible threshold. In other words, the potential benefits of learning about the threshold location are insufficient to justify incurring any risk of crossing the undesirable threshold.

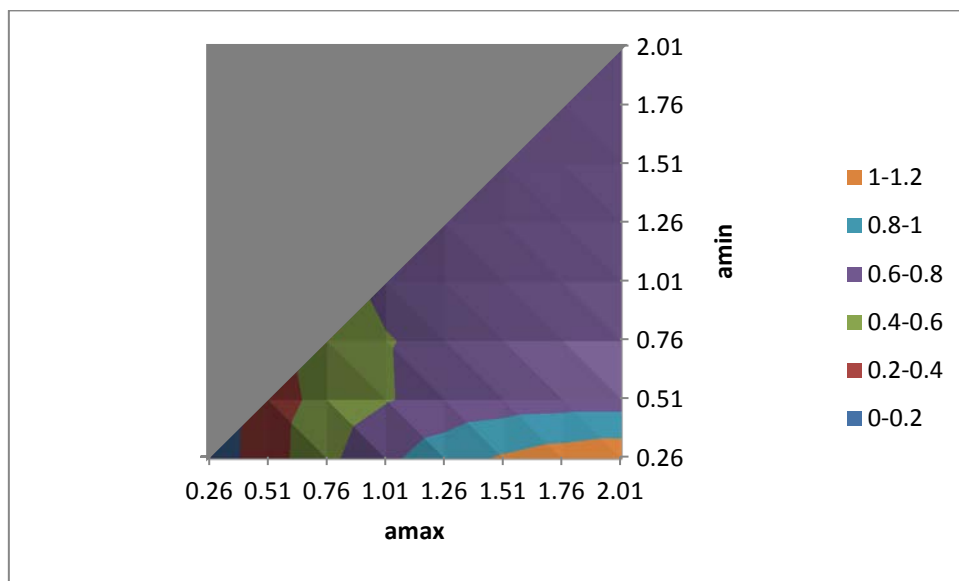
6.3.3 Subjective choice of prior distribution for unknown threshold location

The dynamic optimisation undertaken here relies heavily on the subjective choice of prior distribution for the unknown threshold location. In this section, the sensitivity of optimal decisions to the subjective choice of prior is analysed.

P3.1: For a system with hysteretic dynamics that begins the problem in the *economically-preferred* regime, the optimal level of first-period investment in effort to control the system away from the *undesirable* threshold will be *more*

sensitive to changes in the *closer* bound (i.e. crossing this bound means the probability of a regime shift is positive) than changes in the *further* bound (i.e. crossing this bound means the threshold is crossed with certainty) of the prior probability distribution for the unknown threshold location.

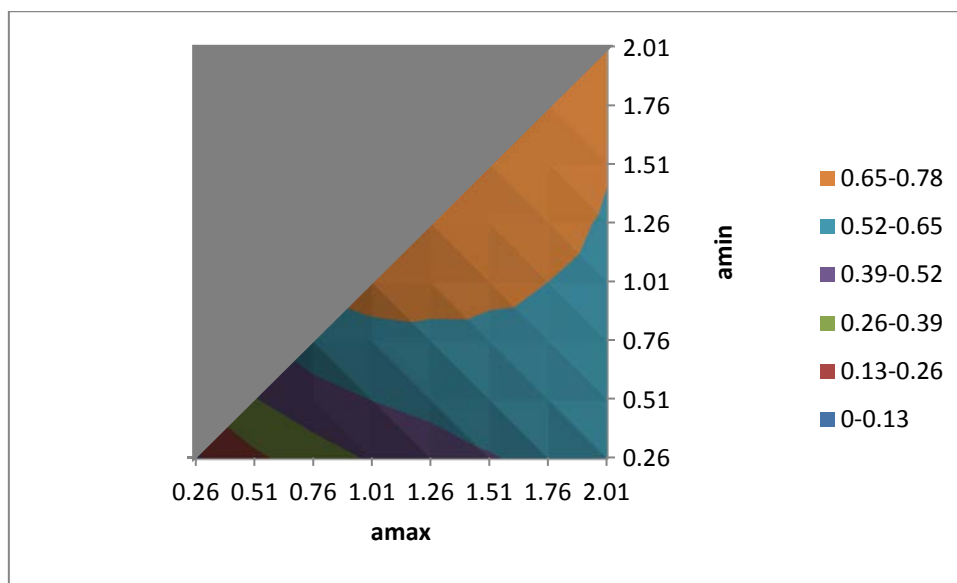
Figure 6-9 and Figure 6-10 show the optimal levels of the control variable when active learning is considered (Learning) and when active learning is not considered (No learning), respectively. For both models, the lake is initially in the preferred regime, and the concentration of phosphorus in the lake is known to be below the threshold level. Both Figure 6-9 and Figure 6-10 are surface plots that show the optimal level of the control variable for different combinations of the lower and upper bounds of the prior distribution. For the ‘Learning’ model, it is expected that the optimal level of the control variable will be more sensitive to changes in the lower bound because the decision-maker can only learn about the threshold location by perturbing the system beyond the lower bound. For the ‘No learning’ model, it is also expected that the optimal level of the control variable will be more sensitive to changes in the lower bound. However, the decision-maker will be inclined to perturb the system in the direction of the threshold, but less intent on crossing the lower bound than would be the case for the ‘Learning’ model.



Parameters: $X(0) = 0.01$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-9 Sensitivity of control variable to choice of prior bounds - 'Learning' model (Preferred regime)

When using the ‘Learning’ model and beginning the problem in the preferred regime, the optimal first-period level of fertiliser application is more sensitive to changes in the upper bound (a_{max}) of the prior distribution for the unknown threshold location than changes in the lower bound (a_{min}). Crossing the lower bound results in a positive probability of crossing the undesirable threshold, while crossing the upper bound means the threshold will be crossed with certainty. Holding the lower bound fixed while increasing the upper bound (moving in a rightward direction) decreases the marginal probability of crossing the threshold for each additional unit of fertiliser applied. Therefore, the decision-maker is willing to apply more fertiliser.



Parameters: $X(0) = 0.01$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

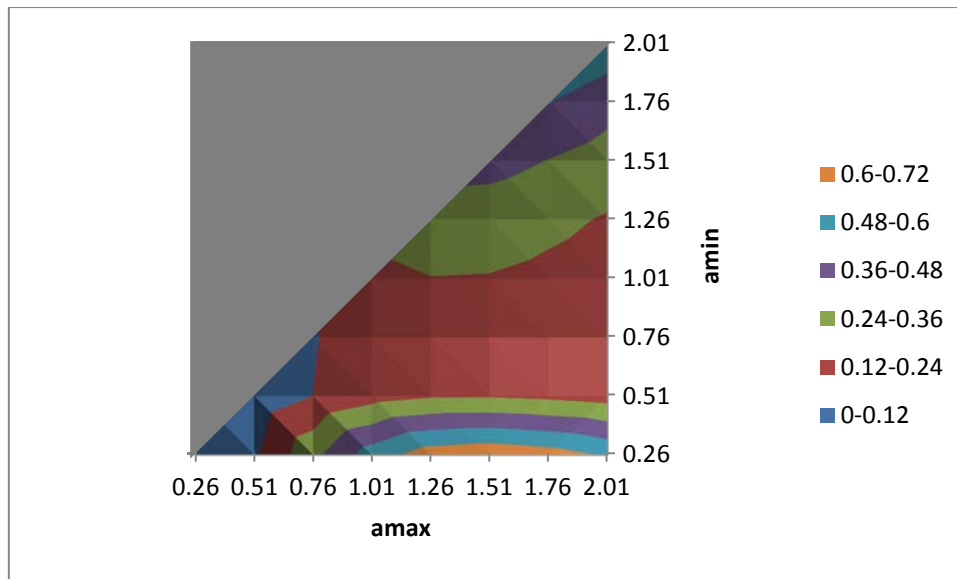
Figure 6-10 Sensitivity of control variable to choice of prior bounds - 'No learning' model (Preferred regime)

When using the ‘No learning’ model and beginning the problem in the preferred regime, the optimal first-period level of fertiliser application is more sensitive to changes in the upper bound (a_{max}) of the prior distribution for the unknown threshold location than changes in the lower bound (a_{min}). However, the optimal level of fertiliser application is monotonically increasing in both the lower (a_{min}) and upper (a_{max}) bounds. This relationship is in contrast to that described for the ‘Learning’ model in Figure 6-9 above, where there is a non-monotonic relationship between the optimal level of fertiliser application and the lower bound (a_{min}).

P3.2: For a system with hysteretic dynamics that begins the problem in the *economically-non-preferred* regime, the optimal level of first-period investment in effort to control the system in the direction of the *desirable* threshold will be *more* sensitive to changes in the *closer* bound (i.e. crossing this bound means the probability of a regime shift is positive) than changes in the *further* bound (i.e. crossing this bound means the threshold is crossed with certainty) of the prior probability distribution for the unknown threshold location.

Figure 6-11 and Figure 6-12 show the optimal levels of the control variable when active learning is considered (Learning) and when active learning is not considered (No learning), respectively. For both models, the lake is initially in the less preferred regime, and the concentration of phosphorus in the lake is known to be above the threshold level. Both Figure 6-11 and Figure 6-12 are surface plots that show the optimal level of the control variable for different combinations of the lower and upper bounds of the prior distribution. For the ‘Learning’ model, it is expected that the optimal level of the control variable will be more sensitive to changes in the upper bound (a_{max}) because the decision-maker can only learn about the threshold location, as well as possibly switch to the more preferred regime, by perturbing the system beyond the upper bound. For the ‘No learning’ model, it is also expected that the optimal level of the control variable will be more sensitive to changes in the upper bound (a_{max}). The decision-maker will be inclined to perturb the system in the direction of the threshold, and more intent on crossing the upper bound than would be the case for the ‘Learning’ model.

When the system is initially in the non-preferred regime, the closer bound will be the upper bound (a_{max}) of the prior distribution. Crossing the threshold will result in a desirable regime shift from the non-preferred to the preferred regime. The optimal level of the control variable is highly sensitive to the subjective choice of lower bound (a_{min}) and also highly sensitive to the subjective choice of upper bound (a_{max}).



Parameters: $X(0) = 2.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

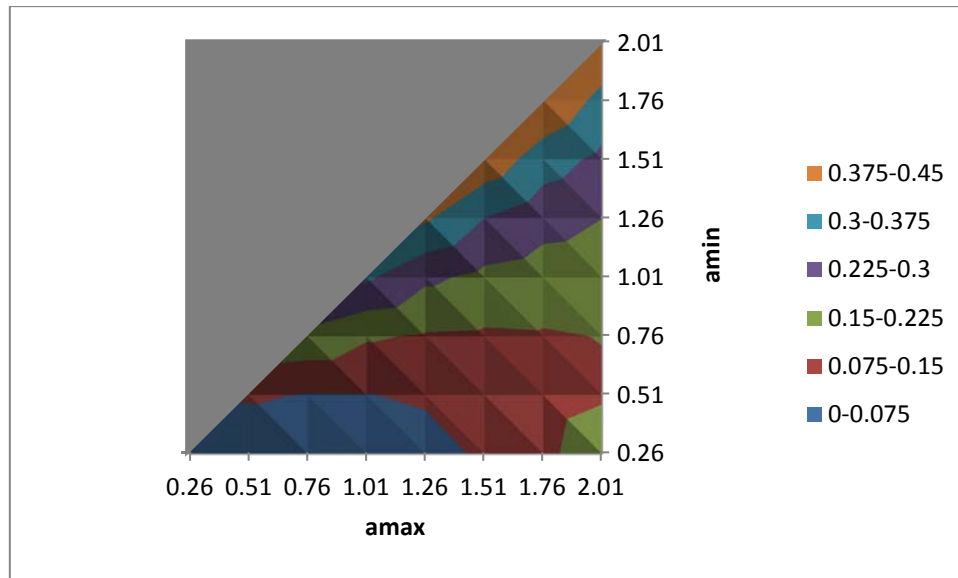
**Figure 6-11 Sensitivity of control variable to choice of prior bound - 'Learning' model
(Non-preferred regime)**

The optimal level of the control variable is sensitive to the subjective choice of both bounds of the prior distribution (a_{min} and a_{max}); however, there is a non-monotonic relationship in both cases. When there is very little uncertainty about the exact location of the threshold (i.e. the lower and upper bounds are close together), yet the threshold location is far from the current level of the state variable (note that $X(0) = 2.26$), there is a strong incentive for the decision-maker to apply a minimal amount of fertiliser, so the desirable threshold can be crossed as quickly as possible (bottom-left corner of Figure 6-11).

When the upper bound of the prior distribution is very close to the current level of the state variable and the level of uncertainty about the unknown threshold location is low, the decision-maker will apply more fertiliser because the assimilative capacity of the system is likely bring about a desirable regime change without additional risk averse actions on the part of the decision-maker (top-right corner of Figure 6-11).

When the variance of the prior distribution is increased, yet the upper bound (a_{max}) is still far from the current stock of phosphorus, the optimal first-period level of the control variable also increases. In contrast, when the variance of the prior

distribution is increased, and the upper bound (a_{max}) is close to the current stock of phosphorus, the optimal first-period level of the control variable decreases. This behaviour is reflective of the high probability³⁷ of experiencing a desirable regime shift if a low level of phosphorus fertiliser is applied.



Parameters: $X(0) = 2.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-12 Sensitivity of control variable to choice of prior bounds - 'No learning' model (Non-preferred regime)

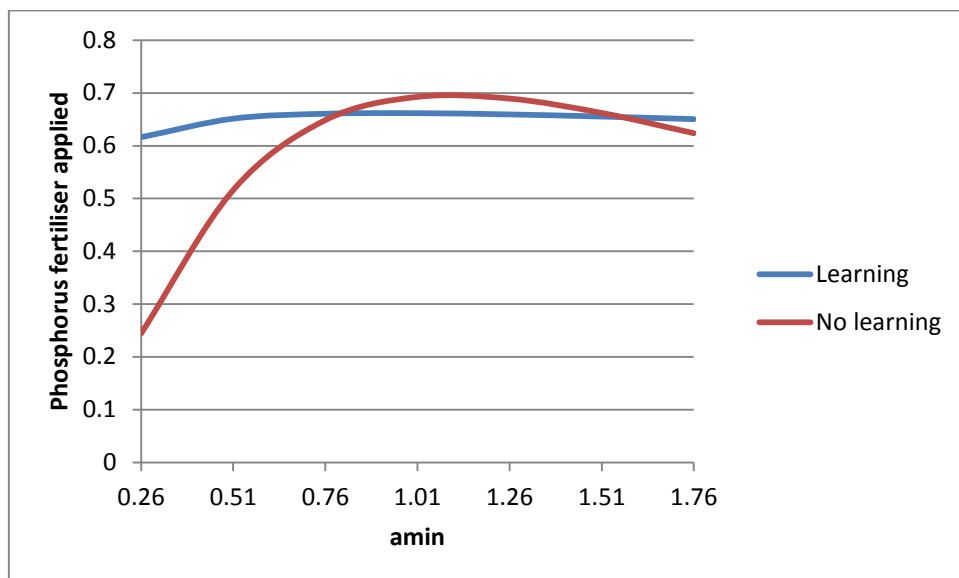
Figure 6-12 shows the sensitivity of the control variable to changes in the prior probability bounds when the 'No learning' model is used. There is a positive monotonic relationship between the optimal level of the control variable and the subjective choice of lower bound (a_{min}). There is also a negative monotonic relationship between the optimal level of the control variable and the subjective choice of upper bound (a_{max}), except for very low values of the upper bound. This is in contrast to the 'Learning' model, where there is a non-monotonic relationship between the optimal level of the control variable and each of the prior probability bounds (a_{min} and a_{max}).

P3.3: For a system with an irreversibility that begins the problem in the *economically-preferred* regime, higher uncertainty about the location of the

³⁷ For some parameter values of the lower and upper bounds, a desirable regime shift will be guaranteed if a sufficiently low level of phosphorus fertiliser is applied.

(sole) *undesirable* threshold will result in a lower optimal level of first-period investment in effort to control the system away from the *undesirable* threshold.

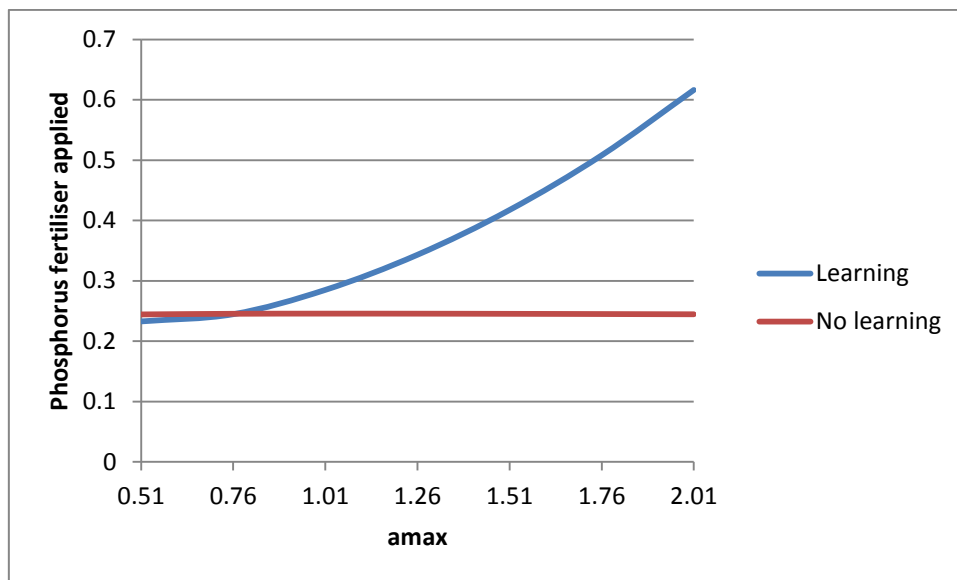
It is expected that higher variance of the prior distribution for an irreversible threshold will be associated with a higher optimal level of the control variable (less risk averse behaviour). This is because greater variance, as measured by the distance between the lower and upper bounds of the prior distribution for the unknown threshold location, means that perturbing the system in the direction of the threshold is associated with a smaller marginal increase in the probability of crossing the threshold, compared to a lower level of variance. In Figure 6-13, the upper bound of the prior distribution for the unknown threshold location is held fixed while the lower bound is varied. In Figure 6-14, the lower bound of the prior distribution for the unknown threshold location is held fixed while the upper bound is varied. The level of the other state variable, the stock of phosphorus carried over from the previous time period, is also held fixed. For both Figure 6-13 and Figure 6-14, one of the prior bounds is held fixed while increasing the variance of the prior distribution, as measured by the distance between the lower and upper bounds of the prior distribution.



Parameters: $X(0) = 0.01$ $a_{max}(0) = 2.01$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

**Figure 6-13 Optimal first-period level of phosphorus when lower bound is varied
(Irreversible threshold)**

Unsurprisingly, the optimal level of phosphorus determined using the ‘No learning’ model will result in the decision-maker incurring zero risk of the undesirable and irreversible threshold being crossed. This is because the lower bound (a_{min}) of the prior distribution will not be crossed, regardless of the subjective choice of lower bound. For the ‘Learning’ model, the decision-maker will incur a positive probability of crossing the threshold for low values of the lower bound (a_{min}). For example, the implied probability of crossing the irreversible threshold, which is a function of the prior distribution and the level of fertiliser applied, is 21% when the lower bound is 0.26 and 10% when the lower bound is 0.51. The decision-maker is willing to incur some risk because, if the threshold is not crossed and the prior distribution is favourably updated, a higher level of fertiliser can be applied in all future time periods while knowing that the threshold will not be crossed.



Parameters: $X(0) = 0.01$ $a_{min}(0) = 0.26$ $k = 1.5$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-14 Optimal first-period level of phosphorus when the upper bound is varied (Irreversible threshold)

Once again, the optimal level of phosphorus determined using the ‘No learning’ model will result in the decision-maker incurring zero risk of the undesirable and irreversible threshold being crossed. This is because the lower bound (a_{min}) of the prior distribution will not be crossed. Instead, the level of the control variable (phosphorus applied) is chosen such that the state variable (phosphorus stock) will remain marginally below the level of the lower bound (a_{min}) of the prior distribution.

As such, the optimal level of the control variable is insensitive to changes in the upper bound (a_{max}) of the prior distribution.

For the 'Learning' model, the decision-maker will incur a positive probability of crossing the threshold for high values of the upper bound (a_{max}). For example, the implied probability of crossing the irreversible threshold, which is a function of the prior distribution and the level of fertiliser applied, is 5% when the upper bound is 1.01, 13% when the upper bound is 1.51, and 21% when the upper bound is 2.01. The decision-maker is willing to incur some risk because, if the threshold is not crossed and the prior distribution is favourably updated, a higher level of fertiliser can be applied in all future time periods while knowing that the threshold will not be crossed.

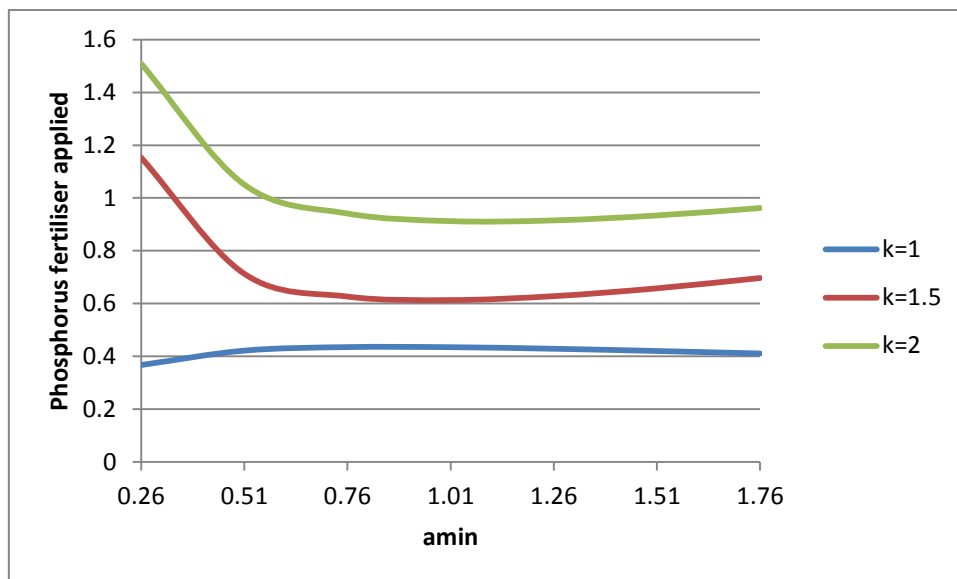
6.3.4 Relative contributions of direct and consumption utility

In the general form of an ecosystem management problem, utility can be sourced in the form of both consumption utility and direct utility. Consumption utility is that utility paid for using any profits acquired from the system. Direct utility is obtained based on the state, or health, of the system. For this case study, consumption utility is sourced from the production and sale of a crop on agricultural land adjacent to the lake. The output of the crop is modelled as a function of the amount of phosphorus fertiliser applied. Direct utility is modelled as disutility; however, could easily be modelled as positive utility relative to a different benchmark level. Direct utility is obtained based on the health of the lake, and captures utility from the aesthetics and recreational use of the lake.

P4.1: For a system where utility is sourced from both consumption and direct utility, the optimal level of first-period investment in effort to control the system away from the *undesirable* threshold will be higher *ceteris paribus* when more weight is placed on direct utility, relative to consumption utility.

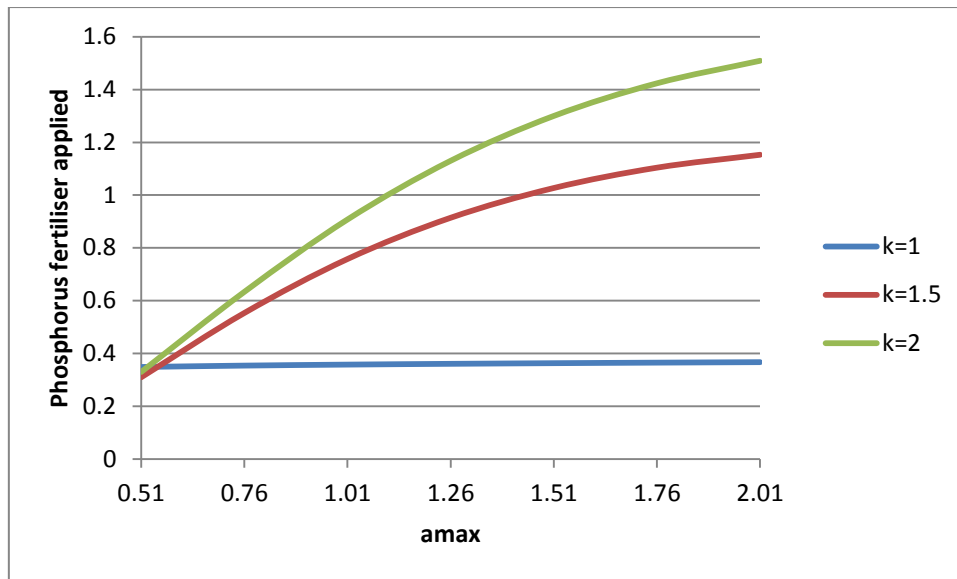
The key difference between consumption utility and direct utility is that consumption utility is much more easily varied over time. For this problem,

consumption utility is simply a function of the level of fertiliser applied. This level can be chosen independently of the current state and previous states of the system. On the other hand, direct utility is a function of the current state of the system, where the current state is a function of the stock of phosphorus in the lake at the end of the previous time period, the current system regime, and the level of phosphorus added in the current time period. Therefore, direct utility is determined as a function of several variables, and is less easily varied than consumption utility. For this reason, it is expected that a decision-maker will be less inclined to risk crossing an undesirable threshold, and experiencing a temporary or prolonged decline in direct utility, when a higher proportion of total utility is sourced from direct utility. Figure 6-15 is used to show the optimal first-period level of phosphorus application for different proportional contributions of consumption and direct utility as the lower bound of the prior distribution is varied, while in Figure 6-16, the upper bound of the prior distribution is varied. The parameter value k assigns a weighting of consumption utility relative to direct utility, where a higher value of k denotes a higher relative weighting of consumption utility.



Parameters: $X(0) = 0.01$ $a_{max}(0) = 2.01$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-15 Optimal first-period level of phosphorus when the relative importance of consumption utility and the lower bound of the prior distribution are varied



Parameters: $X(0) = 0.01$ $a_{min}(0) = 0.26$ $B = 0.1$ $b = 0.02$ $\Omega = 0.2$

Figure 6-16 Optimal first-period level of phosphorus when the relative importance of consumption utility and the upper bound of the prior distribution is varied

Figure 6-15 and Figure 6-16 show the optimal first-period level of phosphorus application as the lower bound and the upper bound of the prior distribution are varied, respectively. In both cases, a higher relative weighting of consumption utility relative to direct utility, as indicated by a higher value of k , results in a higher optimal first-period level of the control variable.³⁸ This means that the decision-maker is willing to incur more risk of crossing the undesirable threshold and experiencing a temporary or prolonged decline in direct utility because the consumption utility obtained is of relatively more importance and is independent of the current system regime of the lake.

Summary of model simulation results

The results of two different dynamic optimisation models have been compared. One model explicitly factors in the possibility of learning about the unknown location of the ecological threshold in the shallow lake system, while the other model does not. First, the expected NPV of utility from the system will be higher when active learning is considered, compared to when it is not considered.

³⁸ The slightly different shapes of the curves in Figure 6-15 are likely to be a result of the numerical estimation method used. A translog, or flexible functional form, function was used to estimate these curves. Of more importance is the observed trend as the parameter k is varied.

Second, the value of information is highest when the prior distribution has low variance and the system state variable is close to the prior distribution. However, the value of information is negligible when the closer bound of the prior distribution for the unknown threshold location is far from the current level of the underlying system variable. In this case, the optimal level of the control variable determined using the 'No learning' model is only slightly sub-optimal. Third, the decision-maker will generally engage in riskier behaviour when active learning is considered because it is only possible to learn about the threshold's location by incurring some risk of crossing it. This is also true for the case of an irreversible threshold. Fourth, the optimal level of the control variable is generally a function of both the lower and upper bounds of the prior distribution, and the associated variance of the prior. Fifth, the decision-maker will be less willing to incur risk of crossing an undesirable threshold when a greater proportion of total utility is sourced from direct utility, compared to consumption utility. This is because direct utility is a function of the current system regime and crossing an undesirable threshold will result in a temporary or prolonged downturn in utility. On the other hand, consumption utility is independent of the system state and is more easily varied.

Finally, it is worthwhile to reiterate the ultimate objective of the decision-maker. The conceptual model is not used to identify paths describing expected time trajectories of state and control variables, or optimal trajectories to a steady state. Instead, the conceptual model is used to determine the decision-maker's optimal actions in the current time period (i.e. 'today'), conditional on the acknowledgement that these actions will impact on the speed of learning about the unknown threshold location. Therefore, the decision-maker undertakes actions that maximise the expected return from the system, having made use of all available information. Following these actions, the decision-maker may learn about the unknown threshold location. This new information is then factored into the decisions made in the following time period, as per the iterative process of learning.

Chapter 7. Conclusions and implications

This thesis aimed to examine (1) how active learning about unknown ecological threshold locations impacts management decisions, (2) the sensitivity of perceived optimal management decisions to the particular dynamics of an ecosystem, that is, the responsiveness of system ‘output’ to changes in system ‘input’, and (3) the sensitivity of perceived optimal management decisions to the subjective choice of prior distribution for unknown threshold locations. To meet these aims, a dynamic optimisation framework was developed to explicitly consider the role of active learning about unknown threshold locations in the modelling and management of ecological systems. The dynamic programming framework was applied to various ecological contexts, including numerical simulations of a shallow lake ecosystem using a wide range of alternative parameter values.

The discussion below is based on comparisons between the results of two different mathematical models, where one model explicitly considers the impact of active learning about unknown threshold locations, and the other model does not consider this impact. Explicit consideration of the value of information, due to active learning, means the decision-maker will generally make decisions that incur a greater risk of crossing the threshold in order to learn about its location. This finding is independent of the initial prior probability distribution used to model threshold location and the type of ecosystem dynamics considered. More specific results, and the implications of these results, are discussed below.

7.1 Impact of active learning on optimal decisions

When managing a complicated ecological system, the decision-maker will frequently be confronted with a scenario where a critical ecological threshold is known, or presumed, to exist, but its exact location is unknown. In this case, the decision-maker can postulate a prior probability distribution for the unknown location of the threshold, based on the current level of understanding about the ecological system. This approach allows the state-space of the system to be partitioned into two

regions, similarly to the risk switching point concept introduced by Nævdal (2006). On one side of the boundary, the threshold could not possibly be located because the system has traversed this path previously without crossing the threshold – it is ‘familiar territory’. Therefore, the threshold must be located on the other side of the boundary. By intelligently perturbing the system, a process known as active learning, the decision-maker is able to learn about the unknown threshold location and refine the prior distribution to encompass any new information. This means that an ‘optimal’ decision made using the initial prior distribution may no longer be perceived as optimal once new information is gathered.

Rather than relying on the same set of information in every future time period, the decision-maker will progressively gather more information about the threshold’s location and use it to make better-informed decisions. However, learning can only occur by leaving the safety of the ‘familiar territory’ and perturbing the system into the region that must contain the threshold. If the decision-maker fails to acknowledge the benefits associated with learning about the unknown threshold location, they have less incentive to perturb the system in its direction. Therefore, a decision-maker who considers the impacts of learning will engage in riskier behaviour by undertaking actions that result in a higher probability of crossing the threshold, but also a higher probability of learning about the threshold’s location.

There are only two cases where the value of information is equal to zero. First, if the locations of ecological thresholds are known with absolute certainty. Second, the value of information will be zero, in the limit, as the magnitude of damages from an undesirable regime shift approaches infinity – otherwise known as catastrophic damages. If either, or both, of these characteristics are present, alertness to learning provides no incentive to perturb the system out of the safety of the ‘familiar territory’. For these two cases, adaptive management is unnecessary and inappropriate, respectively. Conversely, for cases with unknown threshold locations and finite damages, the value of information will be positive. However, its magnitude may be large or small.

Although the risk context considered in this thesis is of a limited nature, the modelling framework, nonetheless, provides insight into how alertness to learning

influences optimal decisions. The risk context considered is one where the locations of ecological thresholds are fixed within the state-space; however, these locations are unknown to the decision-maker. This type of uncertainty is referred to by Brozovic and Schlenker (2011, p. 627) as “threshold uncertainty”. Real-world systems involve several other sources of risk, such as unknown costs of reversion following an undesirable regime shift, unknown or stochastic dynamics in an alternative regime, or the presence of stochastic thresholds. None of these additional sources of risk are considered here, but nor does this absence detract from the general result that alertness to the value of learning leads to riskier behaviour.

7.2 Sensitivity of optimal decisions to particular ecological dynamics

A complicated ecological system may have a reversible threshold or an irreversible threshold, or different combinations of the two. The degree of reversibility of threshold effects has a significant bearing on the size of benefits resulting from active learning and the optimal level of risk incurred by the decision-maker in order to learn about threshold locations. In almost all cases, the optimal management decision is determined after having considered how this decision will impact on the process of learning about the unknown threshold location. The exception to this statement is a system with an undesirable and irreversible threshold that has already been crossed. Beginning the decision problem in the less preferred regime presents the decision-maker with a standard unconstrained optimisation problem because the system will remain in the less preferred regime in perpetuity, regardless of the decision-maker’s actions. Therefore, the decision-maker will not need to consider the possibility of crossing a threshold or learning about its location.

In the case of a system with an undesirable irreversible threshold that hasn’t been crossed, considering the benefits of learning will lead the decision-maker to incur more risk than would otherwise be perceived as optimal. If the system is perturbed into the ‘risky’ part of the state-space but the threshold is not crossed, the decision-maker will know that they can continue to operate in what is likely to be a more profitable region with zero probability of experiencing an undesirable regime shift. In the case of a reversible threshold, these same benefits will apply, but the

system is also able to recover to the preferred regime in the event of the undesirable threshold being crossed. Therefore, a decision-maker who considers the benefits of learning will incur even more risk when managing a system with a reversible threshold *ceteris paribus*.

For a system that has already crossed an undesirable desirable threshold and is currently in the less preferred regime, the optimal actions of the decision-maker are less generalisable. The decision-maker faces a trade-off between (i) actions that increase the probability of remaining in the less preferred regime, but generate high immediate returns³⁹ and (ii) actions that increase the probability of recovering the system to the economically preferred regime, but require sacrificing immediate returns. The optimal decision is a function of the difference in returns between the two alternative regimes and the ease with which the system can be recovered to the preferred regime.

The impact of different ecosystem dynamics on optimal decisions is also dependent on which sources of utility are held in higher regard. Utility can be a direct function of the health of the system, direct utility, or result from any profits generated by the system, consumption utility. For example, direct utility could be sourced from the recreational value of a lake based on its water quality. Consumption utility is sourced from consumption paid for by any profits from the system. For example, from profits generated by selling a wheat crop grown on agricultural land that would become saline if poorly managed.

When direct utility is more important, the decision-maker will be less inclined to risk damaging the health of the system. Therefore, they will choose management actions that are less likely to result in an undesirable threshold being crossed. Conversely, if consumption utility is more important, the decision-maker will choose riskier management actions *ceteris paribus*. This is because saved profits can be used to smooth consumption even if the system experiences an undesirable regime shift. The decision-maker can 'ride out the storm' until the system recovers to the preferred regime.

³⁹ These returns are 'high' in terms of what is achievable in the economically less preferred regime, but are lower than the returns that could be generated in the economically preferred regime.

7.3 Sensitivity of optimal decisions to the subjective choice of prior distribution

The sensitivity of optimal decisions to changes in the prior distribution for the unknown threshold location can be viewed from two different perspectives. First, in terms of sensitivity to changes in the amount of risk associated with the unknown threshold location, as measured by the variance of the prior distribution. Second, in terms of sensitivity to changes in the expected proximity of the system to the unknown threshold location. This is measured as the physical distance between the underlying system variable and the closest possible location of the threshold (closer bound), as defined by the prior distribution for the unknown threshold location. If this closer bound is crossed by the underlying variable, the probability of experiencing an undesirable regime shift increases from zero to a positive amount. In other words, this closer bound represents the boundary of the current data set about where the threshold is *not* located.

If the expected distance between the underlying system variable and the threshold is increased, while the degree of uncertainty about the threshold location⁴⁰ is held fixed, the decision-maker will undertake actions that incur less risk of crossing the undesirable threshold. When the threshold is thought to be located far from the underlying system variable, it will act as a less binding constraint on ‘production’. Therefore, the potential benefits of learning about the threshold’s location are lower and there is less incentive to risk crossing the undesirable threshold.

If the minimum distance to the threshold (i.e. closer bound) is held fixed, but the degree of uncertainty about the threshold location is increased⁴¹, there is a non-monotonic relationship between the optimal amount of risk incurred and the degree of uncertainty about the threshold. First, when the degree of uncertainty about the threshold location is low, the benefits associated with learning are also low, but the likelihood of crossing the threshold by perturbing the system into the risky region is

⁴⁰ That is, the variance of the prior distribution for the unknown threshold location. For the model simulation reported in Chapter 6, a uniform distribution was used for the prior probability distribution associated with the unknown threshold location. This means the variance of the prior is a function of the distance between the lower and upper bounds of the prior distribution, where greater distance between the bounds implies a higher degree of uncertainty about the threshold location.

⁴¹ That is, the variance of the prior probability distribution is increased.

high. Therefore, the decision-maker is less willing to incur risk. Second, when the degree of uncertainty about the threshold location is high, the benefits associated with learning are high, but the likelihood of crossing the threshold by perturbing the system into the risky region is low. Therefore, the decision-maker is more willing to incur risk. Third, when the degree of uncertainty about the threshold location⁴² is extremely high, the benefits associated with learning are low, and the likelihood of crossing the threshold by perturbing the system into the risky region is very low. This is because perturbing the system further and further in the direction of the undesirable threshold is associated with decreasing marginal returns from the system and, therefore, decreasing marginal benefits of learning.

Brozovic and Schlenker (2011) also observed a non-monotonic relationship between the uncertainty of the decision-maker about the unknown threshold, modelled as a stochastic threshold, and risk averse behaviour. In addition, Brozovic and Schlenker (2011) observed a non-monotonic relationship between the degree of risk averse behaviour and the degree of natural variability of the system. These findings imply that the effects of considering the benefits of learning and considering the natural variability of the system work in opposite directions. The first effect provides an incentive for the decision-maker to engage in riskier behaviour, while the second effect encourages risk averse behaviour.

Finally, for a system with an irreversible threshold that is currently in the preferred regime, a decision-maker who considers the benefits of learning will also be willing to incur more risk of crossing the undesirable threshold. The exception to this rule is when the system is thought to already be very close to the irreversible threshold and there is a low amount of uncertainty about the threshold's exact location. Given these circumstances, the undesirable threshold will almost certainly be crossed if the system is perturbed in its direction. Even when the benefits of learning are considered, the decision-maker is unlikely to incur any additional risk of crossing the threshold, compared to the case where these benefits are not considered.

⁴² That is, the variance of the prior probability distribution that characterises the unknown threshold location.

7.4 Avenues for future research

The modelling framework developed in this thesis makes a significant contribution, to a specific class of management problem, by way of explicitly including within a decision framework the benefits of learning about the unknown locations of thresholds. However, the model includes only one source of uncertainty, which is uncertainty on the part of the decision-maker about the exact locations of critical thresholds. All other elements of the decision problem have been assumed deterministic. This approach has been taken to maintain tractability of the conceptual model and parsimony when introducing a new modelling approach concerning the management of complicated systems. Logical extensions of this model involve introducing other forms of uncertainty. The most notable of these are (i) uncertainty regarding the actual dynamics, or production function, for a system when within a particular system regime, (ii) the possibility of a stochastic threshold location, and (iii) uncertainty about the path of the underlying system variable.

It is entirely possible that the dynamics of a system will be well understood within a small range of values of the underlying system variable. However, this degree of precision is unlikely to apply to the entire state-space of the underlying variable. Therefore, the modelling of some parts of the system may be associated with a substantial margin of error, which could be captured using a stochastic production function, such as that used by Brozovic and Schlenker (2011). These authors describe this concept as natural variability, or uncertainty embedded in the natural system.

In the absence of risk aversion on the part of the decision-maker, only the expected returns from the system are of importance. Conversely, if the decision-maker is risk averse, it will be the expected utility from the system that is important. It is expected that the decision-maker will undertake more risk averse behaviour if at least part of the ecological system is modelled using a stochastic production function. This is because crossing a threshold and experiencing a regime shift from a familiar to an unfamiliar regime, for which the dynamics are less well understood, increases the uncertainties associated with the environmental effects of management activities and the variability of economic returns (Arrow *et al.* 1995). However, the decision-maker

will still have an incentive to perturb the system into regions of the state-space where the system dynamics are poorly understood because such behaviour will allow learning to occur and may result in future benefits.

Second, the location of a threshold within the state-space may be stochastic, rather than deterministic. This means that the state-space can no longer be easily partitioned into regions that could and could not contain the threshold. Instead, potentially all points within the state-space will be associated with a positive probability of containing a threshold. In this case, the prior probability distribution for the location of the threshold can be updated using a standard application of Bayes' rule, rather than the updating rules used in this thesis.

Although the threshold location may be viewed as stochastic from the point of view of the decision-maker, this will actually be the result of an incompletely specified model, rather than any stochasticity of the threshold location. Instead, the location of the threshold may be a function of multiple variables, rather than a single underlying variable. For example, in this thesis, the critical threshold in a shallow lake system is modelled as a function of a single underlying variable, phosphorus concentration. However, the threshold location is likely to also be a function of other omitted variables, such as a nitrogen concentration and other measures of water quality.

Third, the path of the underlying system variable may be poorly understood. This, too, is likely to be the result of an incompletely specified model. Drawing on the same shallow lake example, the stock of phosphorus in the lake is modelled as a function of the phosphorus stock retained from the previous time period and the amount of phosphorus fertiliser added in the current time period. However, the process of nutrient cycling in the lake is likely to be a function of several other variables, such as water temperature, the composition of organisms in the lake, and the presence of other nutrients. Therefore, it might also be necessary to model changes in the level of the underlying variable as being stochastic.

Conclusion

This thesis aimed to examine (1) how active learning about unknown ecological threshold locations impacts management decisions, (2) the sensitivity of perceived optimal management decisions to the particular dynamics of an ecosystem, that is, the responsiveness of system ‘output’ to changes in system ‘input’, and (3) the sensitivity of perceived optimal management decisions to the subjective choice of prior distribution for unknown threshold locations. These aims were examined by developing a new mathematical modelling approach that factors into the decision-making process the impact of active learning about threshold locations. The mathematical framework was applied to various ecological contexts, including numerical simulations of a shallow lake ecosystem, and used to demonstrate the role of learning.

First, considering the impacts of active learning about unknown threshold locations means the decision-maker will generally make decisions that incur a greater risk of crossing the threshold in order to learn about its location. This finding is independent of the initial prior probability distribution used to model threshold location and the type of ecosystem dynamics considered; namely, whether threshold effects are reversible or irreversible. Second, the decision-maker will undertake management actions that incur more risk of crossing an undesirable threshold if the threshold effect is reversible rather than irreversible *ceteris paribus*. This is because the ability of the system to recover provides an additional source of economic value compared to a system with an irreversible threshold.

Finally, there is a non-monotonic relationship between the optimal amount of risk incurred and the degree of uncertainty about the threshold location. When uncertainty about the threshold location is either low or very high, the decision-maker will incur less risk of crossing the threshold. For intermediate levels of uncertainty, the decision-maker will incur more risk of crossing the threshold because this case provides the most favourable trade-off between the potential benefits and costs of engaging in active learning. By explicitly modelling the value of information, this thesis better demonstrates the nature of optimal decision-making in the adaptive management of ecological systems.

References

- Accatino, F., De Michele, C., Vezzoli, R., Donzelli, D. and Scholes, R.J. (2010). Tree-grass co-existence in savanna: Interactions of rain and fire, *Journal of Theoretical Biology* 267, 235-242.
- Anselin, L., Bongiovanni, R. and Lowenberg-DeBoer, J. (2004). A Spatial Econometric Approach to the Economics of Site-Specific Nitrogen Management in Corn Production, *American Journal of Agricultural Economics* 86, 675-687.
- Ansink, E. and Wesseler, J. (2009). Quantifying Type I and Type II Errors in Decision-Making under Uncertainty: The Case of GM Crops, *Letters in Spatial and Resource Sciences* 2, 61-66.
- Arrow, K., Bolin, B., Costanza, R., Dasgupta, P., Folke, C., Holling, C.S., Jansson, B.O., Levin, S., Mäler, K.G., Perrings, C. and Pimentel, D. (1995). Economic growth, carrying capacity, and the environment, *Ecological Economics* 15, 91-95.
- Arrow, K.J., Dasgupta, P. and Mäler, K.G. (2003). Evaluating projects and assessing sustainable development in imperfect economies, *Environmental & Resource Economics* 26, 647-685.
- Arrow, K.J. and Fisher, A.C. (1974). Environmental Preservation, Uncertainty, and Irreversibility, *Quarterly Journal of Economics* 88, 312-319.
- Balikcioglu, M., Fackler, P.L. and Pindyck, R.S. (2011). Solving Optimal Timing Problems in Environmental Economics, *Resource and Energy Economics* 33, 761-768.
- Baskett, M.L. and Salomon, A.K. (2010). Recruitment facilitation can drive alternative states on temperate reefs, *Ecology* 91, 1763-1773.
- Bellman, R.E. (1957). *Dynamic Programming*. Princeton University Press, Princeton, NJ.
- Brock, W.A. (2006). Profiling problems with partially identified structure, *Economic Journal* 116, F427-F440.
- Brozovic, N. and Schlenker, W. (2011). Optimal Management of an Ecosystem with an Unknown Threshold, *Ecological Economics* 70, 627-640.
- Bureau of Meteorology (2013). Commonwealth of Australia.

- Carpenter, S.R. (2005). Eutrophication of aquatic ecosystems: Bistability and soil phosphorus, *Proceedings of the National Academy of Sciences of the United States of America* 102, 10002-10005.
- Carpenter, S.R. and Cottingham, K.L. (1997). Resilience and restoration of lakes, *Conservation Ecology (online)* 1.
- Carpenter, S.R., Kitchell, J.F. and Hodgson, J.R. (1985). Cascading Trophic Interactions and Lake Productivity, *Bioscience* 35, 634-639.
- Carpenter, S.R., Ludwig, D. and Brock, W.A. (1999). Management of eutrophication for lakes subject to potentially irreversible change, *Ecological Applications* 9, 751-771.
- Casini, M., Hjelm, J., Molinero, J.C., Lovgren, J., Cardinale, M., Bartolino, V., Belgrano, A. and Kornilovs, G. (2009). Trophic cascades promote threshold-like shifts in pelagic marine ecosystems, *Proceedings of the National Academy of Sciences of the United States of America* 106, 197-202.
- Cicchetti, C.J. and Freeman, A.M., III (1971). Option Demand and Consumer Surplus: Further Comment, *Quarterly Journal of Economics* 85, 528-539.
- Clark, C.W. (1976). *Mathematical bioeconomics: the optimal management of renewable resources*. John Wiley and Sons, Inc., New York,.
- Clark, J.S., Bell, D.M., Hersh, M.H., Kwit, M.C., Moran, E., Salk, C., Stine, A., Valle, D. and Zhu, K. (2011). Individual-scale variation, species-scale differences: inference needed to understand diversity, *Ecology Letters* 14, 1273-1287.
- Collie, J.S., Richardson, K. and Steele, J.H. (2004). Regime shifts: can ecological theory illuminate the mechanisms?, *Progress in Oceanography* 60, 281-302.
- Cropper, M.L. (1976). Regulating activities with catastrophic environmental effects, *Journal of Environmental Economics and Management* 3, 1-15.
- Dasgupta, P. and Mäler, K.G. (2001). Wealth as a Criterion for Sustainable Development, *World Economics* 2, 19-44.
- Dietz, S. and Neumayer, E. (2007). Weak and strong sustainability in the SEEA: Concepts and measurement, *Ecological Economics* 61, 617-626.
- Ding, W. and Lenhart, S. (2009). Optimal Harvesting of a Spatially Explicit Fishery Model, *Natural Resource Modeling* 22, 173-211.
- Dobbelaere, S. and Mairesse, J. (2008). Panel Data Estimates Of The Production Function And Product And Labor Market Imperfections.

- Doyle, J.C. and Csete, M. (2011). Architecture, constraints, and behavior, *Proceedings of the National Academy of Sciences of the United States of America* 108, 15624-15630.
- Eldridge, M.D.B. (2010). *Marsupial Population and Conservation Genetics*. Springer.
- Gozlan, P., Rivas, J. and Schlag, K.H. (2006). Minimax regret in practice - Four examples on treatment choice, *Mimeo*. European University Institute.
- Ewald, C.-O. and Wang, W.-K. (2010). Sustainable Yields in Fisheries: Uncertainty, Risk-Aversion, and Mean-Variance Analysis, *Natural Resource Modeling* 23, 303-323.
- Farmer, M.C. (2001). Getting the safe minimum standard to work in the real world: a case study in moral pragmatism, *Ecological Economics* 38, 209-226.
- Farzin, Y.H. (1996). Optimal pricing of environmental and natural resource use with stock externalities, *Journal of Public Economics* 62, 31-57.
- Freeman, A.M., III (2003). *The measurement of environmental and resource values: Theory and methods*. Second edition. Washington, D.C.: Resources for the Future.
- Friedel, M.H. (1991). Range Condition Assessment and the Concept of Thresholds - A Viewpoint, *Journal of Range Management* 44, 422-426.
- Gelman, A., Carlin, J.B., Stern, H.S. and Rubin, D.B. (2003). *Bayesian Data Analysis*. Chapman & Hall/CRC.
- Gil-Romera, G., Lamb, H.F., Turton, D., Sevilla-Callejo, M. and Umer, M. (2010). Long-term resilience, bush encroachment patterns and local knowledge in a Northeast African savanna, *Global Environmental Change-Human and Policy Dimensions* 20, 612-626.
- Gollier, C. and Treich, N. (2003). Decision-making under scientific uncertainty: The economics of the precautionary principle, *Journal of Risk and Uncertainty* 27, 77-103.
- Grune, L., Kato, M. and Semmler, W. (2005). Solving ecological management problems using dynamic programming, *Journal of Economic Behavior & Organization* 57, 448-473.
- Hanley, N. and Barbier, E.B. (2009). *Pricing Nature: Cost-Benefit Analysis and Environmental Policy*. Cheltenham, U.K. and Northampton, Mass.: Elgar.
- Hanley, N., Shogren, J.F. and White, B. (2007). *Environmental Economics: In Theory and Practice*. Palgrave Macmillan, New York, N.Y.

- Hannesson, R. (2002). The Economics of Marine Reserves, *Natural Resource Modeling* 15, 273-290.
- Hart, A.G. (1942). Risk, Uncertainty, and the Unprofitability of Compounding Probabilities. in Lange, O., McIntyre, F. and Yntema, T.O. (eds.), *Studies in Mathematical Economics and Econometrics*. University of Chicago Press.
- Hertzler, G. (2007). Adapting to climate change and managing climate risks by using real options, *Australian Journal of Agricultural Research* 58, 985-992.
- Hirano, K. and Porter, J.R. (2008). Asymptotics for statistical treatment rules, *Mimeo*. University of Arizona.
- Holling, C.S. (1973). Resilience and stability of ecological systems, *Annual Rev Ecol Syst* 4, 1-23.
- Hurley, T.M., Oishi, K. and Malzer, G.L. (2005). Estimating the Potential Value of Variable Rate Nitrogen Applications: A Comparison of Spatial Econometric and Geostatistical Models, *Journal of Agricultural and Resource Economics* 30, 231-249.
- Johnson, S. and Mengersen, K. (2012). Integrated Bayesian network framework for modeling complex ecological issues, *Integrated Environmental Assessment and Management* 8, 480-490.
- Kamien, M.I. and Schwartz, N.L. (1991). *Dynamic optimization: The calculus of variations and optimal control in economics and management*. Second edition. Advanced Textbooks in Economics, vol. 31. Amsterdam; London and Tokyo: North-Holland; distributed in the U.S. and Canada by Elsevier Science, New York.
- Karp, L. and Zhang, J. (2006). Regulation with anticipated learning about environmental damages, *Journal of Environmental Economics and Management* 51, 259-279.
- Keller, K., Bolker, B.M. and Bradford, D.F. (2004). Uncertain climate thresholds and optimal economic growth, *Journal of Environmental Economics and Management* 48, 723-741.
- Kelly, D.L. and Kolstad, C.D. (1999). Bayesian learning, growth, and pollution, *Journal of Economic Dynamics & Control* 23, 491-518.
- Kemp, M.C. (1976). *Three topics in the theory of international trade--distribution, welfare, and uncertainty / Murray C. Kemp*. North-Holland Pub. Co. ; American Elsevier Pub. Co, Amsterdam : New York.

- Knight, F.H. (1921). *Risk, uncertainty and profit*. Houghton Mifflin Company, Boston.
- Kolmogorov, A.N. (1956). *Foundations of the Theory of Probability*. Chelsea Publishing Company, New York.
- Lee, T., Zenger, K.R., Close, R.L., Jones, M. and Phalen, D.N. (2010). Defining spatial genetic structure and management units for vulnerable koala (*Phascolarctos cinereus*) populations in the Sydney region, Australia, *Wildlife Research* 37, 156-165.
- Lesslie, R., Hill, M., Woldendorp, G., Dawson, S. and Smith, J. (2006). *Towards sustainability for Australia's rangelands; analysing the options*. Bureau of Rural Sciences, Canberra.
- Levin, S.A. (1998). Ecosystems and the biosphere as complex adaptive systems, *Ecosystems* 1, 431-436.
- Li, L. (2010). Estimation of Production Function in U.S. Food Manufacturing Industries, *Empirical Economics Letters* 9, 749-755.
- Lizal, L. and Galuscak, K. (2012). The Impact of Capital Measurement Error Correction on Firm-Level Production Function Estimation. William Davidson Institute at the University of Michigan, William Davidson Institute Working Papers Series: wp1026.
- Lucas, R.E. (1971). Optimal Management of a Research and Development Project, *Management Science Series a-Theory* 17, 679-697.
- Mäler, K.G., Li, C.Z. and Destouni, G. (2007). Pricing Resilience in a Dynamic Economy-Environment System: A Capital-Theoretical Approach, *Beijer Discussion Papers* 208. the Royal Swedish Academy of Sciences.
- Mäler, K.G., Xepapadeas, A. and de Zeeuw, A. (2003). The Economics of Shallow Lakes, *Environmental and Resource Economics* 26, 603-624.
- Malzer, G.L., Hurley, T.M. and Kilian, B. (2004). Estimating Site-Specific Nitrogen Crop Response Functions: A Conceptual Framework and Geostatistical Model.
- Manski, C.F. (2004). Statistical Treatment Rules for Heterogeneous Populations, *Econometrica* 72, 1221-1246.
- Manski, C.F. (2005). *Social Choice with Partial Knowledge of Treatment Response*. Princeton University Press.

- Manski, C.F. (2006). Search Profiling with Partial Knowledge of Deterrence, *Economic Journal* 116, F385-F401.
- Manski, C.F. (2007a). Minimax-Regret Treatment Choice with Missing Outcome Data, *Journal of Econometrics* 139, 105-115.
- Manski, C.F. (2007b). Identification for Prediction and Decision. Harvard University Press.
- Manski, C.F. (2008). Diversified treatment under ambiguity, *Mimeo*. Northwestern University.
- Margolis, M. and Nævdal, E. (2008). Safe minimum standards in dynamic resource problems: Conditions for living on the edge of risk, *Environmental & Resource Economics* 40, 401-423.
- May, R.M. (1977). Thresholds and Breakpoints in Ecosystems with a Multiplicity of Stable States, *Nature* 269, 471-477.
- Milnor, J. (1964). Games against nature. in Shubik, M. (ed.), *Game Theory and Related Approaches to Social Behavior*. Wiley, New York, pp 20-31.
- Nævdal, E. (2001). Optimal regulation of eutrophying lakes, fjords, and rivers in the presence of threshold effects, *American Journal of Agricultural Economics* 83, 972-984.
- Nævdal, E. (2003). Optimal Regulation of Natural Resources in the Presence of Irreversible Threshold Effects, *Natural Resource Modeling* 16, 305-333.
- Nævdal, E. (2006). Dynamic Optimisation in the Presence of Threshold Effects When the Location of the Threshold Is Uncertain--With an Application to a Possible Disintegration of the Western Antarctic Ice Sheet, *Journal of Economic Dynamics and Control* 30, 1131-1158.
- Nævdal, E. and Oppenheimer, M. (2007). The Economics of the Thermohaline Circulation - A Problem with Multiple Thresholds of Unknown Locations, *Resource and Energy Economics* 29, 262-283.
- Noymeir, I. (1995). Interactive Effects of Fire and Graxing on Structure and Diversity of Mediterranean Grasslands, *Journal of Vegetation Science* 6, 701-710.
- Olds, A.D., Connolly, R.M., Pitt, K.A. and Maxwell, P.S. (2012). Habitat connectivity improves reserve performance, *Conservation Letters* 5, 56-63.
- Palmini, D. (1999). Uncertainty, risk aversion, and the game theoretic foundations of the safe minimum standard: a reassessment, *Ecological Economics* 29, 463-472.

- Perrings, C. and Pearce, D. (1994). Threshold Effects and Incentives for the Conservation of Biodiversity, *Environmental and Resource Economics* 4, 13-28.
- Peterson, G.D., Carpenter, S.R. and Brock, W.A. (2003). Uncertainty and the management of multistate ecosystems: An apparently rational route to collapse, *Ecology* 84, 1403-1411.
- Petraitis, P.S. and Dudgeon, S.R. (2004). Detection of alternative stable states in marine communities, *Journal of Experimental Marine Biology and Ecology* 300, 343-371.
- Pindyck, R.S. and Rubinfeld, D.L. (2005). *Microeconomics*. Pearson Prentice Hall, Upper Saddle River, NJ.
- Prato, T. (2005). Bayesian adaptive management of ecosystems, *Ecological Modelling* 183, 147-156.
- Randall, A. (1991). The Value of Biodiversity, *Ambio* 20, 64-68.
- Randall, A. (2011). *Risk and Precaution*. Cambridge and New York: Cambridge University Press.
- Ready, R.C. and Bishop, R.C. (1991). Endangered Species and the Safe Minimum Standard, *American Journal of Agricultural Economics* 73, 309-312.
- Ricker, W.E. (1954). Stock and Recruitment, *Journal of the Fisheries Research Board of Canada* 11, 559-623.
- Rolfe, J.C. (1995). Ulysses Revisited - A Closer Look at the Safe Minimum Standard Rule, *Australian Journal of Agricultural Economics* 39, 55-70.
- Ruan, S. (2009). On Nonlinear Dynamics of Predator-Prey Models with Discrete Delay, *Mathematical Modelling of Natural Phenomena* 4, 140-188.
- Sabine, E., Schreiber, G., Bearlin, A.R., Nicol, S.J. and Todd, C.R. (2004). Adaptive management: a synthesis of current understanding and effective application, *Ecological Management & Restoration* 5, 177-182.
- Sankaran, M., Hanan, N.P., Scholes, R.J., Ratnam, J., Augustine, D.J., Cade, B.S., Gignoux, J., Higgins, S.I., Le Roux, X., Ludwig, F., Ardo, J., Banyikwa, F., Bronn, A., Bucini, G., Caylor, K.K., Coughenour, M.B., Diouf, A., Ekaya, W., Feral, C.J., February, E.C., Frost, P.G.H., Hiernaux, P., Hrabar, H., Metzger, K.L., Prins, H.H.T., Ringrose, S., Sea, W., Tews, J., Worden, J. and Zambatis, N. (2005). Determinants of woody cover in African savannas, *Nature* 438, 846-849.

- Sarkar, S. (2009). Optimal Fishery Harvesting Rules under Uncertainty, *Resource and Energy Economics* 31, 272-286.
- Scheffer, M. (1997). *Ecology of Shallow Lakes*. Chapman and Hall, New York.
- Scheffer, M., Carpenter, S., Foley, J.A., Folke, C. and Walker, B. (2001). Catastrophic shifts in ecosystems, *Nature* 413, 591-596.
- Scheffer, M., Carpenter, S.R., Lenton, T.M., Bascompte, J., Brock, W., Dakos, V., van de Koppel, J., van de Leemput, I.A., Levin, S.A., van Nes, E.H., Pascual, M. and Vandermeer, J. (2012). Anticipating Critical Transitions, *Science* 338, 344-348.
- Scheffer, M., Hosper, S.H., Meijer, M.L., Moss, B. and Jeppesen, E. (1993). Alternative Equilibria in Shallow Lakes, *Trends in Ecology & Evolution* 8, 275-279.
- Schlag, K.H. (2003). How to minimize maximum regret under repeated decisionmaking, *Mimeo*. European University Institute.
- Schlag, K.H. (2006). Minimax Regret with Reduced Variance - The Correlated Binomial Average Rule - A Technical. European University Institute.
- Scholes, R.J. and Walker, B.H. (1993). *An African savanna: synthesis of the Nylsvley study*. Cambridge University Press.
- Seierstad, A. and Sydsaeter, K. (1987). *Optimal control theory with economic applications*. Amsterdam; New York; Oxford and Tokyo: Elsevier Science, North-Holland; distributed in North America by Elsevier Science, New York.
- Shenton, W., Hart, B.T. and Brodie, J. (2010). A Bayesian network model linking nutrient management actions in the Tully catchment (northern Queensland) with Great Barrier Reef condition, *Marine and Freshwater Research* 61, 587-595.
- Smith, R.S. and Doyle, J.C. (1992). Model validation: a connection between robust control and identification, *Automatic Control, IEEE Transactions on* 37, 942-952.
- Stoye, J. (2007). Minimax Regret Treatment Choice with Incomplete Data and Many Treatments, *Econometric Theory* 23, 190-199.
- Stoye, J. (2009a). Minimax regret treatment choice with finite samples, *Journal of Econometrics* 151, 70-81.
- Stoye, J. (2009b). Partial Identification and Robust Treatment Choice: An Application to Young Offenders, *Journal of Statistical Theory and Practice* 3, 239-254.

- Suding, K.N. and Hobbs, R.J. (2009). Threshold models in restoration and conservation: a developing framework, *Trends in Ecology & Evolution* 24, 271-279.
- Sutherland, J.P. (1974). Multiple Stable Points in Natural Communities, *American Naturalist* 108, 859-873.
- Tilman, D. (1994). Competition and Biodiversity in Spatially Structured Habitats, *Ecology* 75, 2-16.
- Tsur, Y. and Zemel, A. (1995). Uncertainty and Irreversibility in Groundwater Management, *Journal of Environmental Economics and Management* 29, 149-161.
- Tsur, Y. and Zemel, A. (1996). Accounting for global warming risks: Resource management under event uncertainty, *Journal of Economic Dynamics & Control* 20, 1289-1305.
- Tumusiime, E., Brorsen, B.W., Mosali, J., Johnson, J., Locke, J. and Biermacher, J.T. (2011). Determining Optimal Levels of Nitrogen Fertilizer Using Random Parameter Models, *Journal of Agricultural and Applied Economics* 43, 541-552.
- van den Belt, H. (2003). Debating the precautionary principle: "Guilty until proven innocent" or "innocent until proven guilty"?, *Plant Physiology* 132, 1122-1126.
- Walker, B., Holling, C.S., Carpenter, S.R. and Kinzig, A. (2004). Resilience, adaptability and transformability in social-ecological systems, *Ecology and Society* 9.
- Walker, B. and Meyers, J.A. (2004). Thresholds in ecological and social-ecological systems: a developing database, *Ecology and Society* 9.
- Walker, B., Pearson, L., Harris, M., Mäler, K.G., Li, C.Z., Biggs, R. and Baynes, T. (2010). Incorporating Resilience in the Assessment of Inclusive Wealth: An Example from South East Australia, *Environmental & Resource Economics* 45, 183-202.
- Walker, B.H. and Salt, D. (2006). *Resilience thinking: sustaining ecosystems and people in a changing world*. Island Press, Washington, D.C.
- Walters, C. (1986). *Adaptive management of renewable resources*. Macmillan, New York.

- Weitzman, M.L. (2003). *Income, wealth, and the maximum principle*. Cambridge and London: Harvard University Press.
- Zemel, A., David, I. and Mehrez, A. (2001). On Conducting Simultaneous Versus Sequential Engineering Activities in Risky R&D, *International Transactions in Operational Research* 8, 585-601.
- Zhou, K., Doyle, J.C. and Glover, K. (1996). *Robust and Optimal Control*. Prentice Hall, Englewood Cliffs, NJ.

Appendix A: Reverse framing of updating rules for prior distribution

$$(i) \quad a_{j_{t+1}}^{max} = \begin{cases} a_{j_t}^{max} & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = A \\ \min(a_{j_t}^{max}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = A \\ a_{j_t}^{max} & \text{if } R_{j_t} = B \end{cases}$$

Where

$$a_{j_t}^{max} \leq a_0^{max} \quad \forall t$$

$$(ii) \quad a_{j_{t+1}}^{min} = \begin{cases} \max(a_{j_t}^{min}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = A \\ a_{j_t}^{min} & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = A \\ a_{j_t}^{min} & \text{if } R_{j_t} = B \end{cases}$$

Where

$$a_{j_t}^{min} \geq a_0^{min} \quad \forall t$$

$$(iii) \quad b_{j_{t+1}}^{max} = \begin{cases} \min(b_{j_t}^{max}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = B \\ b_{j_t}^{max} & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = B \\ b_{j_t}^{max} & \text{if } R_{j_t} = A \end{cases}$$

Where

$$b_{j_t}^{max} \leq b_0^{max} \quad \forall t$$

$$(iv) \quad b_{j_{t+1}}^{min} = \begin{cases} b_{j_t}^{min} & \text{if } F_{j_t} = 0 \text{ and } R_{j_t} = B \\ \max(b_{j_t}^{min}, X_{j_t} \equiv X'_{j_{t-1}}) & \text{if } F_{j_t} = 1 \text{ and } R_{j_t} = B \\ b_{j_t}^{min} & \text{if } R_{j_t} = A \end{cases}$$

Where

$$b_{j_t}^{min} \geq b_0^{min} \quad \forall t$$

Appendix B: MATLAB code for 'Learning' model

```
%%% Define parameter values and initial values

% problem time horizon
T = 100;

% absolute lower bound of A-distribution
amin0 = 0.26;

% absolute upper bound of A-distribution
amax0 = 2.01;

% initial level of wealth
%W0 = 0;

% initial concentration of phosphorus in the lake
X0 = 0;

% initial stock of abatement technologies
%Ibar0 = 0;

% pure rate of time preference
rho = 0.02;

% real interest rate (probably set to zero)
r = 0;

% marginal cost of harvest effort
%h = 1;

% price per unit of harvested crop
%p = 3;

% marginal cost per unit of fertiliser applied
%n = 1;

% marginal cost per unit of abatement technology acquired
%k = 0.3;

% etc., etc.
B = 0.1;
bb = 0.02;
omega = 0.2; % equivalent to 'r' in Brozovic and Schlenker
k = 1.5; % utility received from fertiliser use
```

```

penalty = 0.01; % penalty for terminal stock of X
bonus = 10; % to ensure that terminal value function is positive
%o = 0.5; % transfer coefficient of phosphorus from farm to lake
%muP = 1;
%gammaP = 0.3;
%muA = 5;
%gammaA = -0.25;
%muB = 5;
%gammaB = -0.25;
%muq = 0.5;
%gammaq = 0.5;
%muI = 1;
%gammaI = -0.5;
%muX = 1;
%gammaX = -0.5;
%muW = 1;
%gammaW = 1;
%K = 3; % difference in direct utility from Ra and Rb when X=0
%d = 0.8; % depreciation rate for abatement technologies
%Ybar = 1; % crop yield when P=0
%scale = 2; % scale factor to make W positive

```

```

%%% For reference

```

```

%%% Define EOMs for state variables (excluding A-distribution)

```

```

% phosphorus concentration (i.e. X) when in Regime A

```

```

% XAcurr = B*Xprev + bb + Pcurr;

```

```

% phosphorus concentration (i.e. X) when in Regime B

```

```

% XAcurr = B*Xprev + bb + Pcurr + omega;

```

```

% specify bounds and step size for ndgrid of starting values for

```

```

% 'inherited' state variables

Xlow = 0.01;
Xstep = 0.25;
Xhigh = 3.01;

% specify bounds and step size for ndgrid of values for amin and amax
astep = 0.25;
aminlow = amin0;
amaxhigh = amax0;

% create ndgrid that includes all combinations of Xprev, Wprev, Ibarprev,
% amin and amax
sets = {Xlow:Xstep:Xhigh, aminlow:astep:amaxhigh-astep,
aminlow+astep:astep:amaxhigh};
[x amin amax] = ndgrid(sets{:});
StateAbounds = [x(:) amin(:) amax(:)];

% total number of combinations of inherited state and Abounds variables - max. #
iterations
numrowsStateAbounds = length(StateAbounds(:,1));

% create indicator variable for whether amin/amax combination is feasible
StateAbounds_test = zeros(numrowsStateAbounds,1);
for i = 1:numrowsStateAbounds;
    if StateAbounds(i,2) < StateAbounds(i,3);
        StateAbounds_test(i,1) = 1;
    else
        StateAbounds_test(i,1) = 0;
    end
end

% horizontally concatenate StateAbounds and StateAbounds_test
StateAbounds_test2 = horzcat(StateAbounds,StateAbounds_test);

```

```

% delete rows where indicator variable (column 4) takes value zero
indices = find(StateAbounds_test2(:,4)==0);
StateAbounds_test2(indices,:) = [];

% redefine StateAbounds
clear StateAbounds
StateAbounds = StateAbounds_test2(:,1:3);

% redefine numRowsStateAbounds
numrowsStateAbounds = length(StateAbounds(:,1));

% separate StateAbounds into its individual variables for use as regressors
Xreg = StateAbounds(:,1);
aminreg = StateAbounds(:,2);
amaxreg = StateAbounds(:,3);

% Define regressors for terminal value functions
Xregression = [ones(size(Xreg)) log(Xreg) log(Xreg).^2];

% Define regressors for all other value functions
XAregression = [ones(size(Xreg)) log(Xreg) log(aminreg) log(amaxreg) log(Xreg).^2
log(aminreg).^2 log(amaxreg).^2 log(Xreg).*log(aminreg) log(Xreg).*log(amaxreg)
log(aminreg).*log(amaxreg)];

%%% Solve for optimal value of objective function in Terminal time period
%%% and when in Regime A

% create empty matrix to contain optimal values of UAT
% with dimensions (numrowsInheritState x 1)

MaxUAT = zeros(numrowsStateAbounds,1);

% create matrix to contain the optimal levels of choice variables

```



```

ChoiceAT = zeros(numrowsStateAbounds,1);
ChoiceAT(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);

% define optimisation problem and options
% x(1)=Pcurr
gs = GlobalSearch('StartPointsToRun','bounds-ineqs');
opts = optimset('Display','off','Algorithm','interior-point');
TerminalUA = @(x)(-(k*x(1) - (B*kX + bb + x(1))^2 - penalty*(B*kX + bb +
x(1))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',TerminalUA,'Aineq',[-
1],'bineq',[0],'options',opts);
[xmintermA, fmintermA, flagtermA, outputtermA, manyminstermA] =
run(gs,problem);

MaxUAT(i,1) = -fmintermA;
ChoiceAT(i,:) = xmintermA';

end

logMaxUAT = log(MaxUAT + bonus); %%% bonus has been added to ensure value
function is positive, so it can be logged

% estimate regression for MaxUAT(Xprev)
TermAregression = regress(logMaxUAT,Xregression);

%% Solve for optimal value of objective function in Terminal time period
%% and when in Regime B

% create empty matrix to contain optimal values of UBT
% with dimensions (numrowsInheritState x 1)

```

```

MaxUBT = zeros(numrowsStateAbounds,1);

% create matrix to contain the optimal levels of choice variables

ChoiceBT = zeros(numrowsStateAbounds,1);
ChoiceBT(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);

% define optimisation problem and options
% x(1)=Pcurr

TerminalUB = @(x)(-(k*x(1) - (B*kX + bb + x(1) + omega)^2 - penalty*(B*kX + bb
+ x(1) + omega)));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',TerminalUB,'Aineq',[-
1],'bineq',[0],'options',opts);
[xmintermB, fmintermB, flagtermB, outputtermB, manyminstermB] =
run(gs,problem);

MaxUBT(i,1) = -fmintermB;
ChoiceBT(i,:) = xmintermB;

end

logMaxUBT = log(MaxUBT + bonus); %%% bonus has been added to ensure value
function is positive, so it can be logged

% estimate regression for MaxUBT(Xprev)
TermBregression = regress(logMaxUBT,Xregression);

%% above are the value functions / instantaneous utility functions from

```

```

%%% the terminal time period when system is in Regime A or Regime B

%%% link the utility functions of two consecutive (earlier and later) time
%%% periods via the A-distribution - determine the corresponding value function

%%% when system is in Regime A during earlier time period

% case 1: earlier = Regime A, later = Regime A
% constrained optimisation where  $X_{end}(\text{formula}) \leq \text{amin\_earlier}$ 

MaxUAAflag = zeros(numrowsStateAbounds,1);
MaxUAA = zeros(numrowsStateAbounds,1);
ChoiceAA = zeros(numrowsStateAbounds,1);
ChoiceAA(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

    % reset flag
    flagAA = -10;

    % define optimisation problem and options
    %  $x(1) = P_{curr}$ 
    %  $XA_{end} = (B * kX + bb + P_{curr})$ 

    if kX > amin;
        MaxUAA(i,1) = NaN;
    end
end

```

```

else UAA = @(x)-((k*x(1) - (B*kX + bb + x(1))^2) + 1/(1 + rho)*(-bonus +
exp(TermAregression(1,1) + TermAregression(2,1)*log(B*kX + bb + x(1)) +
TermAregression(3,1)*(B*kX + bb + x(1))^2)));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAA,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint1(x,amin,B,kX,bb),'options',opts);
[xminAA, fminAA, flagAA, outputAA, manyminsAA] = run(gs,problem);

```

```

MaxUAAflag(i,1) = flagAA;

```

```

if flagAA > 0;

```

```

    MaxUAA(i,1) = -fminAA;

```

```

else MaxUAA(i,1) = NaN;

```

```

end

```

```

if flagAA > 0;

```

```

    ChoiceAA(i,:) = xminAA;

```

```

else ChoiceAA(i,:) = NaN;

```

```

end

```

```

end

```

```

end

```

```

% case 2: earlier = Regime A, later = Regime B

```

```

% constrained optimisation where Xend(formula) >= amax_earlier

```

```

MaxUABflag = zeros(numrowsStateAbounds,1);

```

```

MaxUAB = zeros(numrowsStateAbounds,1);

```

```

ChoiceAB = zeros(numrowsStateAbounds,1);

```

```

ChoiceAB(:, :) = NaN;

```

```

for i = 1:numrowsStateAbounds;

```

```

kX = StateAbounds(i,1);
amin = StateAbounds(i,2);
amax = StateAbounds(i,3);

% reset flag
flagAB = -10;

% define optimisation problem and options
% x(1)=Pcurr
% XAend = (B*kX + bb + Pcurr)

if kX > amin;
    MaxUAB(i,1) = NaN;
else UAB = @(x)(-(k*x(1) - (B*kX + bb + x(1))^2) + 1/(1 + rho)*(-bonus +
exp(TermBregression(1,1) + TermBregression(2,1)*log(B*kX + bb + x(1)) +
TermBregression(3,1)*(B*kX + bb + x(1))^2)));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAB,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint2(x,amax,B,kX,bb),'options',opts);
[xminAB, fminAB, flagAB, outputAB, manyminsAB] = run(gs,problem);

MaxUABflag(i,1) = flagAB;

if flagAB > 0;
    MaxUAB(i,1) = -fminAB;
else MaxUAB(i,1) = NaN;

end

if flagAB > 0;
ChoiceAB(i,:) = xminAB';
else ChoiceAB(i,:) = NaN;

end

```

end

end

% case 3: earlier = Regime A, later = Regime A or B

% constrained optimisation where $\text{amin_earlier} \leq \text{Xend}(\text{formula}) \leq \text{amax_earlier}$

MaxUA_AorBflag = zeros(numrowsStateAbounds,1);

MaxUA_AorB = zeros(numrowsStateAbounds,1);

ChoiceA_AorB = zeros(numrowsStateAbounds,1);

ChoiceA_AorB(:, :) = NaN;

for i = 1:numrowsStateAbounds;

 kX = StateAbounds(i,1);

 amin = StateAbounds(i,2);

 amax = StateAbounds(i,3);

% reset flag

flagA_AorB = -10;

% define optimisation problem and options

% x(1)=Pcurr

% XAend = (B*kX + bb + Pcurr)

if kX > amin;

 MaxUA_AorB(i,1) = NaN;

else UA_AorB = @(x)(-((k*x(1) - (B*kX + bb + x(1)))^2) + 1/(1 + rho)*(-bonus + ((max(min((amax - (B*kX + bb + x(1)))/(amax - amin),1),0))))*exp(TermAregression(1,1) + TermAregression(2,1)*log(B*kX + bb + x(1) + TermAregression(3,1)*(B*kX + bb + x(1))^2) + (max(min(((B*kX + bb + x(1)) - amin)/(amax - amin),1),0))*exp(TermBregression(1,1) + TermBregression(2,1)*log(B*kX + bb + x(1) + TermBregression(3,1)*(B*kX + bb + x(1))^2))));

```

problem = createOptimProblem('fmincon','x0',[0.1],'objective',UA_AorB,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint3(x,amin,amax,B,kX,bb),'options',opts);
[xminA_AorB, fminA_AorB, flagA_AorB, outputA_AorB, manyminsA_AorB] =
run(gs,problem);

```

```

MaxUA_AorBflag(i,1) = flagA_AorB;

```

```

if flagA_AorB > 0;

```

```

    MaxUA_AorB(i,1) = -fminA_AorB;

```

```

else MaxUA_AorB(i,1) = NaN;

```

```

end

```

```

if flagA_AorB > 0;

```

```

    ChoiceA_AorB(i,:) = xminA_AorB';

```

```

else ChoiceA_AorB(i,:) = NaN;

```

```

end

```

```

end

```

```

end

```

```

% take supremum of cases 1-3

```

```

supremumAT = zeros(numrowsStateAbounds,1);

```

```

for i = 1:numrowsStateAbounds;

```

```

    maxintermediate = max(MaxUAA(i,1),MaxUAB(i,1));

```

```

    maxvalue = max(maxintermediate,MaxUA_AorB(i,1));

```

```

    supremumAT(i,1) = maxvalue;

```

```

end

```

```
logsupremumAT = log(supremumAT + bonus); %%% bonus has been added to  
ensure value function is positive, so it can be logged
```

```
% estimate regression for supremumAT(Xprev,Wprev,Ibarprev,amin,amax)  
supremumATregression = regress(logsupremumAT,XAregression);
```

```
% save levels of choice variables that correspond to optimal scenario (1-3)
```

```
ChoiceApenult = zeros(numrowsStateAbounds,1);
```

```
% check supremum for isnan
```

```
isnan_supremumAT = isnan(supremumAT);
```

```
for count = 1:numrowsStateAbounds
```

```
    if isnan_supremumAT(count,1) == 1;
```

```
        ChoiceApenult(count,:) = NaN;
```

```
    elseif MaxUAA(count,1) == supremumAT(count,1);
```

```
        ChoiceApenult(count,:) = ChoiceAA(count,:);
```

```
    elseif MaxUAB(count,1) == supremumAT(count,1);
```

```
        ChoiceApenult(count,:) = ChoiceAB(count,:);
```

```
    elseif MaxUA_AorB(count,1) == supremumAT(count,1);
```

```
        ChoiceApenult(count,:) = ChoiceA_AorB(count,:);
```

```
    end
```

```
end
```

```
%% when system is in Regime B during earlier time period
```

```
% case 4: earlier = Regime B, later = Regime A
```

```
% constrained optimisation where Xend(formula) <= amin_earlier
```

```
MaxUBAflag = zeros(numrowsStateAbounds,1);
```

```
MaxUBA = zeros(numrowsStateAbounds,1);
```

```
ChoiceBA = zeros(numrowsStateAbounds,1);
```

```
ChoiceBA(:, :) = NaN;
```



```

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagBA = -10;

% define optimisation problem and options
% x(1)=Pcurr
% XBend = (B*kX + bb + Pcurr + omega)

if kX < amax;
    MaxUBA(i,1) = NaN;
else UBA = @(x)(-((k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-bonus +
exp(TermAregression(1,1) + TermAregression(2,1)*log(B*kX + bb + x(1) + omega)
+ TermAregression(3,1)*(B*kX + bb + x(1) + omega)^2)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBA,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint4(x,amin,B,kX,bb,omega),'options',opts);
[xminBA, fminBA, flagBA, outputBA, manyminsBA] = run(gs,problem);

MaxUBAflag(i,1) = flagBA;

if flagBA > 0;
    MaxUBA(i,1) = -fminBA;
else MaxUBA(i,1) = NaN;

end

if flagBA > 0;
ChoiceBA(i,:) = xminBA';
else ChoiceBA(i,:) = NaN;

```

```

end

end

end

% case 5: earlier = Regime B, later = Regime B
% constrained optimisation where Xend(formula) >= amax_earlier

MaxUBBflag = zeros(numrowsStateAbounds,1);
MaxUBB = zeros(numrowsStateAbounds,1);
ChoiceBB = zeros(numrowsStateAbounds,1);
ChoiceBB(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

    % reset flag
    flagBB = -10;

    % define optimisation problem and options
    % x(1)=Pcurr
    % XBend = (B*kX + bb + Pcurr + omega)

    if kX < amax;
        MaxUBB(i,1) = NaN;
    else UBB = @(x)(-((k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-bonus +
    exp(TermBregression(1,1) + TermBregression(2,1)*log(B*kX + bb + x(1) + omega)
    + TermBregression(3,1)*(B*kX + bb + x(1) + omega)^2)))));
    problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBB,'Aineq',[-
    1],'bineq',[0],'nonlcon',@(x)constraint5(x,amax,B,kX,bb,omega),'options',opts);
    [xminBB, fminBB, flagBB, outputBB, manyminsBB] = run(gs,problem);

```

```
MaxUBBflag(i,1) = flagBB;
```

```
if flagBB > 0;
```

```
    MaxUBB(i,1) = -fminBB;
```

```
else MaxUBB(i,1) = NaN;
```

```
end
```

```
if flagBB > 0;
```

```
    ChoiceBB(i,:) = xminBB';
```

```
else ChoiceBB(i,:) = NaN;
```

```
end
```

```
end
```

```
end
```

```
% case 6: earlier = Regime B, later = Regime A or B
```

```
% constrained optimisation where amin_earlier <= Xend(formula) <= amax_earlier
```

```
MaxUB_AorBflag = zeros(numrowsStateAbounds,1);
```

```
MaxUB_AorB = zeros(numrowsStateAbounds,1);
```

```
ChoiceB_AorB = zeros(numrowsStateAbounds,1);
```

```
ChoiceB_AorB(:, :) = NaN;
```

```
for i = 1:numrowsStateAbounds;
```

```
    kX = StateAbounds(i,1);
```

```
    amin = StateAbounds(i,2);
```

```
    amax = StateAbounds(i,3);
```

```
% reset flag
```

```
flagB_AorB = -10;
```

```

% define optimisation problem and options
% x(1)=Pcurr
% XBend = (B*kX + bb + Pcurr + omega)

if kX < amax;
    MaxUB_AorB(i,1) = NaN;
else UB_AorB = @(x)(-(k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-
bonus + ((max(min((amax - (B*kX + bb + x(1) + omega))/(amax -
amin),1),0))) * exp(TermAregression(1,1) + TermAregression(2,1)*log(B*kX + bb +
x(1) + omega) + TermAregression(3,1)*(B*kX + bb + x(1) + omega)^2) +
(max(min(((B*kX + bb + x(1) + omega) - amin)/(amax -
amin),1),0))) * exp(TermBregression(1,1) + TermBregression(2,1)*log(B*kX + bb +
x(1) + omega) + TermBregression(3,1)*(B*kX + bb + x(1) + omega)^2)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UB_AorB,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint6(x,amin,amax,B,kX,bb,omega),'options',opts);
[xminB_AorB, fminB_AorB, flagB_AorB, outputB_AorB, manyminsB_AorB] =
run(gs,problem);

MaxUB_AorBflag(i,1) = flagB_AorB;

if flagB_AorB > 0;
    MaxUB_AorB(i,1) = -fminB_AorB;
else MaxUB_AorB(i,1) = NaN;

end

if flagB_AorB > 0;
ChoiceB_AorB(i,:) = xminB_AorB';
else ChoiceB_AorB(i,:) = NaN;

end

end

```

```

end

% take supremum of cases 4-6

supremumBT = zeros(numrowsStateAbounds,1);

for i = 1:numrowsStateAbounds;
    maxintermediate = max(MaxUBA(i,1),MaxUBB(i,1));
    maxvalue = max(maxintermediate,MaxUB_AorB(i,1));
    supremumBT(i,1) = maxvalue;
end

logsupremumBT = log(supremumBT + bonus); %%% bonus has been added to
ensure value function is positive, so it can be logged

% estimate regression for supremumAT(Xprev,Wprev,Ibarprev,amin,amax)
supremumBTregression = regress(logsupremumBT,XAregression);

% save levels of choice variables that correspond to optimal scenario (4-6)
ChoiceBpenult = zeros(numrowsStateAbounds,3);
% check supremum for isnan
isnan_supremumBT = isnan(supremumBT);

for count = 1:numrowsStateAbounds
    if isnan_supremumBT(count,1) == 1;
        ChoiceBpenult(count,:) = NaN;
    elseif MaxUBA(count,1) == supremumBT(count,1);
        ChoiceBpenult(count,:) = ChoiceBA(count,:);
    elseif MaxUBB(count,1) == supremumBT(count,1);
        ChoiceBpenult(count,:) = ChoiceBB(count,:);
    elseif MaxUB_AorB(count,1) == supremumBT(count,1);
        ChoiceBpenult(count,:) = ChoiceB_AorB(count,:);
    end
end

```

```

end

end

%%%%%%%% rather than linking only two consecutive time periods, it is
%%%%%%%% necessary to link three consecutive time periods
%%%%%%%% the new information learnt at the end of the 1st time period cannot
%%%%%%%% be put to use until the 2nd time period and does not feed into a
%%%%%%%% value function until the 3rd period

%%%%%%%% what were previously referred to as cases will now be referred to as
%%%%%%%% scenarios; to differentiate between linking only two time periods and
%%%%%%%% linking three time periods

% scenario 1: earlier = Regime A, later = Regime A (certain)
% only feasible if  $kX \leq a_{min}$ 
% constrained optimisation where  $X_{end}(\text{formula}) \leq a_{min\_earlier}$ 

%%% notation: started 1st period in Regime A, then scenario 1 occurred
MaxUAS1flag = zeros(numrowsStateAbounds,1);
MaxUAS1 = zeros(numrowsStateAbounds,1);
ChoiceAS1 = zeros(numrowsStateAbounds,1);
ChoiceAS1(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagAS1 = -10;

% define optimisation problem and options
%  $x(1) = P_{curr}$ 

```

```

% XAend = (B*kX + bb + Pcurr)

% XAregression = [ones(size(Xreg)) log(Xreg) log(aminreg) log(amaxreg)
log(Xreg).^2 log(aminreg).^2 log(amaxreg).^2 log(Xreg).*log(aminreg)
log(Xreg).*log(amaxreg) log(aminreg).*log(amaxreg)]

if kX > amin;
    MaxUAS1(i,1) = NaN;
else UAS1 = @(x)(-((k*x(1) - (B*kX + bb + x(1))^2) + 1/(1 + rho)*(-bonus +
exp(supremumATregression(1,1) + supremumATregression(2,1)*log(B*kX + bb +
x(1)) + supremumATregression(3,1)*log(amin) +
supremumATregression(4,1)*log(amax) + supremumATregression(5,1)*(log(B*kX +
bb + x(1))^2) + supremumATregression(6,1)*(log(amin)^2) +
supremumATregression(7,1)*(log(amax)^2) +
supremumATregression(8,1)*log(B*kX + bb + x(1))*log(amin) +
supremumATregression(9,1)*log(B*kX + bb + x(1))*log(amax) +
supremumATregression(10,1)*log(amin)*log(amax)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAS1,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint1(x,amin,B,kX,bb),'options',opts);
[xminAS1, fminAS1, flagAS1, outputAS1, manyminsAS1] = run(gs,problem);

MaxUAS1flag(i,1) = flagAS1;

if flagAS1 > 0;
    MaxUAS1(i,1) = -fminAS1;
else MaxUAS1(i,1) = NaN;

end

if flagAS1 > 0;
ChoiceAS1(i,:) = xminAS1';
else ChoiceAS1(i,:) = NaN;

end

```

end

end

% scenario 2: earlier = Regime A, later = Regime B

% only feasible if $kX \leq \text{amin}$

% constrained optimisation where $X_{\text{end}}(\text{formula}) \geq \text{amax_earlier}$

MaxUAS2flag = zeros(numrowsStateAbounds,1);

MaxUAS2 = zeros(numrowsStateAbounds,1);

ChoiceAS2 = zeros(numrowsStateAbounds,1);

ChoiceAS2(:, :) = NaN;

for i = 1:numrowsStateAbounds;

 kX = StateAbounds(i,1);

 amin = StateAbounds(i,2);

 amax = StateAbounds(i,3);

% reset flag

flagAS2 = -10;

% define optimisation problem and options

% $x(1) = P_{\text{curr}}$

% $XA_{\text{end}} = (B * kX + bb + P_{\text{curr}})$

if $kX > \text{amin}$;

 MaxUAS2(i,1) = NaN;

else UAS2 = @(x)(-((k*x(1) - (B*kX + bb + x(1)))^2) + 1/(1 + rho)*(-bonus +
exp(supremumBTregression(1,1) + supremumBTregression(2,1)*log(B*kX + bb +
x(1)) + supremumBTregression(3,1)*log(amin) +
supremumBTregression(4,1)*log(amax) + supremumBTregression(5,1)*(log(B*kX +
bb + x(1))^2) + supremumBTregression(6,1)*(log(amin)^2) +
supremumBTregression(7,1)*(log(amax)^2)


```

supremumBTregression(8,1)*log(B*kX + bb + x(1))*log(amin) +
supremumBTregression(9,1)*log(B*kX + bb + x(1))*log(amax) +
supremumBTregression(10,1)*log(amin)*log(amax)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAS2,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint2(x,amax,B,kX,bb),'options',opts);
[xminAS2, fminAS2, flagAS2, outputAS2, manyminsAS2] = run(gs,problem);

MaxUAS2flag(i,1) = flagAS2;

if flagAS2 > 0;
    MaxUAS2(i,1) = -fminAS2;
else MaxUAS2(i,1) = NaN;

end

if flagAS2 > 0;
ChoiceAS2(i,:) = xminAS2';
else ChoiceAS2(i,:) = NaN;

end

end

end

end

% scenario 3: earlier = Regime A, later = Regime A or B (uncertain)
% only feasible if kX <= amin
% constrained optimisation where amin_earlier <= Xend(formula) <= amax_earlier

MaxUAS3flag = zeros(numrowsStateAbounds,1);
MaxUAS3 = zeros(numrowsStateAbounds,1);
ChoiceAS3 = zeros(numrowsStateAbounds,1);
ChoiceAS3(:, :) = NaN;

```

```

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagAS3 = -10;

% define optimisation problem and options
% x(1)=Pcurr
% XAend = (B*kX + bb + Pcurr)

% for 2nd period, XAend becomes amin for Va and amax for Vb

if kX > amin;
    MaxUAS3(i,1) = NaN;
else UAS3 = @(x)(-((k*x(1) - (B*kX + bb + x(1)))^2) + 1/(1 + rho)*(-bonus +
((max(min((amax - (B*kX + bb + x(1)))/(amax -
amin),1),0))) * exp(supremumATregression(1,1) +
supremumATregression(2,1)*log(B*kX + bb + x(1)) +
supremumATregression(3,1)*log(B*kX + bb + x(1)) +
supremumATregression(4,1)*log(amax) + supremumATregression(5,1)*(log(B*kX +
bb + x(1))^2) + supremumATregression(6,1)*(log(B*kX + bb + x(1))^2) +
supremumATregression(7,1)*(log(amax)^2) +
supremumATregression(8,1)*log(B*kX + bb + x(1))*log(B*kX + bb + x(1)) +
supremumATregression(9,1)*log(B*kX + bb + x(1))*log(amax) +
supremumATregression(10,1)*log(B*kX + bb + x(1))*log(amax)) +
(max(min(((B*kX + bb + x(1)) - amin)/(amax -
amin),1),0)) * exp(supremumBTregression(1,1) +
supremumBTregression(2,1)*log(B*kX + bb + x(1)) +
supremumBTregression(3,1)*log(amin) + supremumBTregression(4,1)*log(B*kX +
bb + x(1)) + supremumBTregression(5,1)*(log(B*kX + bb + x(1))^2) +
supremumBTregression(6,1)*(log(amin)^2) +
supremumBTregression(7,1)*(log(B*kX + bb + x(1))^2) +

```

```

supremumBTregression(8,1)*log(B*kX + bb + x(1))*log(amin) +
supremumBTregression(9,1)*log(B*kX + bb + x(1))*log(B*kX + bb + x(1)) +
supremumBTregression(10,1)*log(amin)*log(B*kX + bb + x(1))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAS3,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint3(x,amin,amax,B,kX,bb),'options',opts);
[xminAS3, fminAS3, flagAS3, outputAS3, manyminsAS3] = run(gs,problem);

```

```

MaxUAS3flag(i,1) = flagAS3;

```

```

if flagAS3 > 0;

```

```

    MaxUAS3(i,1) = -fminAS3;

```

```

else MaxUAS3(i,1) = NaN;

```

```

end

```

```

if flagAS3 > 0;

```

```

ChoiceAS3(i,:) = xminAS3;

```

```

else ChoiceAS3(i,:) = NaN;

```

```

end

```

```

end

```

```

end

```

```

% take supremum of scenarios 1-3

```

```

supremumAother = zeros(numrowsStateAbounds,1);

```

```

for i = 1:numrowsStateAbounds;

```

```

    maxintermediate = max(MaxUAS1(i,1),MaxUAS2(i,1));

```

```

    maxvalue = max(maxintermediate,MaxUAS3(i,1));

```

```

    supremumAother(i,1) = maxvalue;

```

```

end

```

```
logsupremumAother = log(supremumAother + bonus); %%% bonus has been added
to ensure value function is positive, so it can be logged
```

```
% estimate regression for supremumAT(Xprev,Wprev,Ibarprev,amin,amax)
supremumAotherregression = regress(logsupremumAother,XAregression);
```

```
% save levels of choice variables that correspond to optimal scenario (1-3)
```

```
ChoiceApenpen = zeros(numrowsStateAbounds,1);
```

```
% check supremum for isnan
```

```
isnan_supremumAother = isnan(supremumAother);
```

```
for count = 1:numrowsStateAbounds
```

```
    if isnan_supremumAother(count,1) == 1;
```

```
        ChoiceApenpen(count,:) = NaN;
```

```
    elseif MaxUAS1(count,1) == supremumAother(count,1);
```

```
        ChoiceApenpen(count,:) = ChoiceAS1(count,:);
```

```
    elseif MaxUAS2(count,1) == supremumAother(count,1);
```

```
        ChoiceApenpen(count,:) = ChoiceAS2(count,:);
```

```
    elseif MaxUAS3(count,1) == supremumAother(count,1);
```

```
        ChoiceApenpen(count,:) = ChoiceAS3(count,:);
```

```
    end
```

```
end
```

```
P_logChoiceApenpen = log(ChoiceApenpen(:,1) + 0.01);
```

```
% estimate regression for choice 1 (i.e. P)
```

```
PAPenpenregression = regress(P_logChoiceApenpen,XAregression);
```

```
% scenario 4: earlier = Regime B, later = Regime A (certain)
```

```
% only feasible if  $kX \geq amax$ 
```

```

% constrained optimisation where Xend(formula) <= amin_earlier

MaxUBS4flag = zeros(numrowsStateAbounds,1);
MaxUBS4 = zeros(numrowsStateAbounds,1);
ChoiceBS4 = zeros(numrowsStateAbounds,1);
ChoiceBS4(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagBS4 = -10;

% define optimisation problem and options
% x(1)=Pcurr
% XBend = (B*kX + bb + Pcurr + omega)

if kX < amax;
    MaxUBS4(i,1) = NaN;
else UBS4 = @(x)(-((k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-bonus
+ exp(supremumATregression(1,1) + supremumATregression(2,1)*log(B*kX + bb +
x(1) + omega) + supremumATregression(3,1)*log(amin) +
supremumATregression(4,1)*log(amax) + supremumATregression(5,1)*(log(B*kX +
bb + x(1) + omega)^2) + supremumATregression(6,1)*(log(amin)^2) +
supremumATregression(7,1)*(log(amax)^2)
+ supremumATregression(8,1)*log(B*kX + bb + x(1) + omega)*log(amin) +
supremumATregression(9,1)*log(B*kX + bb + x(1) + omega)*log(amax) +
supremumATregression(10,1)*log(amin)*log(amax)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBS4,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint4(x,amin,B,kX,bb,omega),'options',opts);
[xminBS4, fminBS4, flagBS4, outputBS4, manyminsBS4] = run(gs,problem);

```

```

MaxUBS4flag(i,1) = flagBS4;

if flagBS4 > 0;
    MaxUBS4(i,1) = -fminBS4;
else MaxUBS4(i,1) = NaN;

end

if flagBS4 > 0;
ChoiceBS4(i,:) = xminBS4';
else ChoiceBS4(i,:) = NaN;

end

end

end

end

% scenario 5: earlier = Regime B, later = Regime B (certain)
% only feasible if kX >= amax
% constrained optimisation where Xend(formula) >= amax_earlier

MaxUBS5flag = zeros(numrowsStateAbounds,1);
MaxUBS5 = zeros(numrowsStateAbounds,1);
ChoiceBS5 = zeros(numrowsStateAbounds,1);
ChoiceBS5(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagBS5 = -10;

```

```

% define optimisation problem and options
% x(1)=Pcurr
% XBend = (B*kX + bb + Pcurr + omega)

if kX < amax;
    MaxUBS5(i,1) = NaN;
else UBS5 = @(x)(-(k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-bonus
+ exp(supremumBTregression(1,1) + supremumBTregression(2,1)*log(B*kX + bb +
x(1) + omega) + supremumBTregression(3,1)*log(amin) +
supremumBTregression(4,1)*log(amax) + supremumBTregression(5,1)*(log(B*kX +
bb + x(1) + omega)^2) + supremumBTregression(6,1)*(log(amin)^2) +
supremumBTregression(7,1)*(log(amax)^2) +
supremumBTregression(8,1)*log(B*kX + bb + x(1) + omega)*log(amin) +
supremumBTregression(9,1)*log(B*kX + bb + x(1) + omega)*log(amax) +
supremumBTregression(10,1)*log(amin)*log(amax))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBS5,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint5(x,amax,B,kX,bb,omega),'options',opts);
[xminBS5, fminBS5, flagBS5, outputBS5, manyminsBS5] = run(gs,problem);

MaxUBS5flag(i,1) = flagBS5;

if flagBS5 > 0;
    MaxUBS5(i,1) = -fminBS5;
else MaxUBS5(i,1) = NaN;

end

if flagBS5 > 0;
ChoiceBS5(i,:) = xminBS5';
else ChoiceBS5(i,:) = NaN;

end

```

end

end

% scenario 6: earlier = Regime B, later = Regime A or B (uncertain)

% only feasible if $kX \geq \text{amin}$

% constrained optimisation where $\text{amin_earlier} \leq X_{\text{end}}(\text{formula}) \leq \text{amax_earlier}$

MaxUBS6flag = zeros(numrowsStateAbounds,1);

MaxUBS6 = zeros(numrowsStateAbounds,1);

ChoiceBS6 = zeros(numrowsStateAbounds,1);

ChoiceBS6(:, :) = NaN;

for i = 1:numrowsStateAbounds;

 kX = StateAbounds(i,1);

 amin = StateAbounds(i,2);

 amax = StateAbounds(i,3);

% reset flag

flagBS6 = -10;

% define optimisation problem and options

% $x(1) = P_{\text{curr}}$

% $XB_{\text{end}} = (B * kX + bb + P_{\text{curr}} + \omega)$

% for 2nd period, XB_{end} becomes amax for Va and amin for Vb

if $kX < \text{amax}$;

 MaxUBS6(i,1) = NaN;

else UBS6 = @(x)(-((k*x(1) - (B*kX + bb + x(1) + omega))^2) + 1/(1 + rho)*(-bonus
+ ((max(min((amax - (B*kX + bb + x(1) + omega))/(amax -
amin),1),0))) * exp(supremumATregression(1,1) +
supremumATregression(2,1)*log(B*kX + bb + x(1) + omega) +
supremumATregression(3,1)*log(amin) + supremumATregression(4,1)*log(B*kX +


```

bb + x(1) + omega) + supremumATregression(5,1)*(log(B*kX + bb + x(1) +
omega)^2) + supremumATregression(6,1)*(log(amin)^2) +
supremumATregression(7,1)*(log(B*kX + bb + x(1) + omega)^2) +
supremumATregression(8,1)*log(B*kX + bb + x(1) + omega)*log(amin) +
supremumATregression(9,1)*log(B*kX + bb + x(1) + omega)*log(B*kX + bb + x(1)
+ omega) + supremumATregression(10,1)*log(amin)*log(B*kX + bb + x(1) +
omega) + (max(min(((B*kX + bb + x(1) + omega) - amin)/(amax -
amin),1),0))*exp(supremumBTregression(1,1) +
supremumBTregression(2,1)*log(B*kX + bb + x(1) + omega) +
supremumBTregression(3,1)*log(B*kX + bb + x(1) + omega) +
supremumBTregression(4,1)*log(amax) + supremumBTregression(5,1)*(log(B*kX +
bb + x(1) + omega)^2) + supremumBTregression(6,1)*(log(B*kX + bb + x(1) +
omega)^2) + supremumBTregression(7,1)*(log(amax)^2) +
supremumBTregression(8,1)*log(B*kX + bb + x(1) + omega)*log(B*kX + bb + x(1)
+ omega) + supremumBTregression(9,1)*log(B*kX + bb + x(1) + omega)*log(amax)
+ supremumBTregression(10,1)*log(B*kX + bb + x(1) + omega)*log(amax)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBS6,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint6(x,amin,amax,B,kX,bb,omega),'options',opts);
[xminBS6, fminBS6, flagBS6, outputBS6, manyminsBS6] = run(gs.problem);

```

```

MaxUBS6flag(i,1) = flagBS6;

```

```

if flagBS6 > 0;

```

```

    MaxUBS6(i,1) = -fminBS6;

```

```

else MaxUBS6(i,1) = NaN;

```

```

end

```

```

if flagBS6 > 0;

```

```

    ChoiceBS6(i,:) = xminBS6';

```

```

else ChoiceBS6(i,:) = NaN;

```

```

end

```

```

end

end

% take supremum of scenarios 4-6

supremumBother = zeros(numrowsStateAbounds,1);

for i = 1:numrowsStateAbounds;
    maxintermediate = max(MaxUBS4(i,1),MaxUBS5(i,1));
    maxvalue = max(maxintermediate,MaxUBS6(i,1));
    supremumBother(i,1) = maxvalue;
end

logsupremumBother = log(supremumBother + bonus); %%% bonus has been added
to ensure value function is positive, so it can be logged

% estimate regression for supremumBT(Xprev,Wprev,Ibarprev,amin,amax)
supremumBotherregression = regress(logsupremumBother,XAregression);

% save levels of choice variables that correspond to optimal scenario (4-6)
ChoiceBpenpen = zeros(numrowsStateAbounds,1);
% check supremum for isnan
isnan_supremumBother = isnan(supremumBother);

for count = 1:numrowsStateAbounds
    if isnan_supremumBother(count,1) == 1;
        ChoiceBpenpen(count,:) = NaN;
    elseif MaxUBS4(count,1) == supremumBother(count,1);
        ChoiceBpenpen(count,:) = ChoiceBS4(count,:);
    elseif MaxUBS5(count,1) == supremumBother(count,1);
        ChoiceBpenpen(count,:) = ChoiceBS5(count,:);
    elseif MaxUBS6(count,1) == supremumBother(count,1);
        ChoiceBpenpen(count,:) = ChoiceBS6(count,:);
    end
end

```

```

    end

end

P_logChoiceBpenpen = log(ChoiceBpenpen(:,1) + 0.01);

% estimate regression for choice 1 (i.e. P)
PBpenpenregression = regress(P_logChoiceBpenpen,XAregression);

%%% Repeat the above steps a further T-3 times, to reach the 1st time
%%% period of the problem

% ensure required number of iterations are performed
itertotal = T-3;

% iteration counter
iter = 0;

% create matrix of zeros to save values of sumpremumAotherregression at
% each iteration
Aregpara = zeros(10,itertotal);

% create matrix of zeros to save values of sumpremumBotherregression at
% each iteration
Bregpara = zeros(10,itertotal);

% create matrices for parameter values of choice variable regressions
PAregpara = zeros(10,itertotal);
PBregpara = zeros(10,itertotal);

for s = 1:itertotal;

```

```

% scenario 1: earlier = Regime A, later = Regime A (certain)
% only feasible if kX <= amin
% constrained optimisation where Xend(formula) <= amin_earlier

%%% notation: started 1st period in Regime A, then scenario 1 occurred
MaxUAS1othflag = zeros(numrowsStateAbounds,1);
MaxUAS1oth = zeros(numrowsStateAbounds,1);
ChoiceAS1oth = zeros(numrowsStateAbounds,1);
ChoiceAS1oth(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagAS1oth = -10;

% define optimisation problem and options
% x(1)=qcurr, x(2)=Pcurr, x(3)=Icurr
% XAend = (B*kX + ((d*kI + x(3) + 1)^gammaI)*(bb + o*x(2)))
% WAend = ((1 + r)*kW + (p - h)*(Ybar + muP*x(2)^gammaP) - n*x(2) - x(1) -
% k*x(3))
% IbarAend = (d*kI + x(3))

if kX > amin;
    MaxUAS1oth(i,1) = NaN;
else UAS1oth = @(x)(-(k*x(1) - (B*kX + bb + x(1))^2) + 1/(1 + rho)*(-bonus +
exp(supremumAotherregression(1,1) + supremumAotherregression(2,1)*log(B*kX +
bb + x(1)) + supremumAotherregression(3,1)*log(amin) +
supremumAotherregression(4,1)*log(amax) +
supremumAotherregression(5,1)*(log(B*kX + bb + x(1))^2) +
supremumAotherregression(6,1)*(log(amin)^2) +
supremumAotherregression(7,1)*(log(amax)^2)

```

```

supremumAotherregression(8,1)*log(B*kX + bb + x(1))*log(amin) +
supremumAotherregression(9,1)*log(B*kX + bb + x(1))*log(amax) +
supremumAotherregression(10,1)*log(amin)*log(amax)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAS1oth,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint1(x,amin,B,kX,bb),'options',opts);
[xminAS1oth, fminAS1oth, flagAS1oth, outputAS1oth, manyminsAS1oth] =
run(gs,problem);

```

```

MaxUAS1othflag(i,1) = flagAS1oth;

```

```

if flagAS1oth > 0;

```

```

    MaxUAS1oth(i,1) = -fminAS1oth;

```

```

else MaxUAS1oth(i,1) = NaN;

```

```

end

```

```

if flagAS1oth > 0;

```

```

    ChoiceAS1oth(i,:) = xminAS1oth;

```

```

else ChoiceAS1oth(i,:) = NaN;

```

```

end

```

```

end

```

```

end

```

```

% scenario 2: earlier = Regime A, later = Regime B

```

```

% only feasible if kX <=amin

```

```

% constrained optimisation where Xend(formula) >= amax_earlier

```

```

MaxUAS2othflag = zeros(numrowsStateAbounds,1);

```

```

MaxUAS2oth = zeros(numrowsStateAbounds,1);

```

```

ChoiceAS2oth = zeros(numrowsStateAbounds,1);

```

```

ChoiceAS2oth(:, :) = NaN;

```

```

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagAS2oth = -10;

% define optimisation problem and options
% x(1)=qcurr, x(2)=Pcurr, x(3)=Icurr
% XAend = (B*kX + ((d*kI + x(3) + 1)^gammaI)*(bb + o*x(2)))
% WAend = ((1 + r)*kW + (p - h)*(Ybar + muP*x(2)^gammaP) - n*x(2) - x(1) -
% k*x(3))
% IbarAend = (d*kI + x(3))

if kX > amin;
    MaxUAS2oth(i,1) = NaN;
else UAS2oth = @(x)(-((k*x(1) - (B*kX + bb + x(1)))^2) + 1/(1 + rho)*(-bonus +
exp(supremumBotherregression(1,1) + supremumBotherregression(2,1)*log(B*kX +
bb + x(1)) + supremumBotherregression(3,1)*log(amin) +
supremumBotherregression(4,1)*log(amax) +
supremumBotherregression(5,1)*(log(B*kX + bb + x(1))^2) +
supremumBotherregression(6,1)*(log(amin)^2) +
supremumBotherregression(7,1)*(log(amax)^2) +
supremumBotherregression(8,1)*log(B*kX + bb + x(1))*log(amin) +
supremumBotherregression(9,1)*log(B*kX + bb + x(1))*log(amax) +
supremumBotherregression(10,1)*log(amin)*log(amax))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAS2oth,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint2(x,amax,B,kX,bb),'options',opts);
[xminAS2oth, fminAS2oth, flagAS2oth, outputAS2oth, manyminsAS2oth] =
run(gs,problem);

MaxUAS2othflag(i,1) = flagAS2oth;

```

```

if flagAS2oth > 0;
    MaxUAS2oth(i,1) = -fminAS2oth;
else MaxUAS2oth(i,1) = NaN;

end

if flagAS2oth > 0;
ChoiceAS2oth(i,:) = xminAS2oth';
else ChoiceAS2oth(i,:) = NaN;

end

end

end

end

% scenario 3: earlier = Regime A, later = Regime A or B (uncertain)
% only feasible if kX <= amin
% constrained optimisation where amin_earlier <= Xend(formula) <= amax_earlier

MaxUAS3othflag = zeros(numrowsStateAbounds,1);
MaxUAS3oth = zeros(numrowsStateAbounds,1);
ChoiceAS3oth = zeros(numrowsStateAbounds,1);
ChoiceAS3oth(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagAS3oth = -10;

```

```

% define optimisation problem and options
% x(1)=qcurr, x(2)=Pcurr, x(3)=Icurr
% XAend = (B*kX + ((d*kI + x(3) + 1)^gammaI)*(bb + o*x(2)))
% WAend = ((1 + r)*kW + (p - h)*(Ybar + muP*x(2)^gammaP) - n*x(2) - x(1) -
% k*x(3))
% IbarAend = (d*kI + x(3))

% for 2nd period, XAend becomes amin for Va and amax for Vb

if kX > amin;
    MaxUAS3oth(i,1) = NaN;
else UAS3oth = @(x)(-(k*x(1) - (B*kX + bb + x(1))^2) + 1/(1 + rho)*(-bonus +
((max(min((amax - (B*kX + bb + x(1)))/(amax -
amin),1),0))) * exp(supremumAotherregression(1,1) +
supremumAotherregression(2,1)*log(B*kX + bb + x(1)) +
supremumAotherregression(3,1)*log(B*kX + bb + x(1)) +
supremumAotherregression(4,1)*log(amax) +
supremumAotherregression(5,1)*(log(B*kX + bb + x(1))^2) +
supremumAotherregression(6,1)*(log(B*kX + bb + x(1))^2) +
supremumAotherregression(7,1)*(log(amax)^2) +
supremumAotherregression(8,1)*log(B*kX + bb + x(1))*log(B*kX + bb + x(1)) +
supremumAotherregression(9,1)*log(B*kX + bb + x(1))*log(amax) +
supremumAotherregression(10,1)*log(B*kX + bb + x(1))*log(amax)) +
(max(min(((B*kX + bb + x(1)) - amin)/(amax -
amin),1),0)) * exp(supremumBotherregression(1,1) +
supremumBotherregression(2,1)*log(B*kX + bb + x(1)) +
supremumBotherregression(3,1)*log(amin) +
supremumBotherregression(4,1)*log(B*kX + bb + x(1)) +
supremumBotherregression(5,1)*(log(B*kX + bb + x(1))^2) +
supremumBotherregression(6,1)*(log(amin)^2) +
supremumBotherregression(7,1)*(log(B*kX + bb + x(1))^2) +
supremumBotherregression(8,1)*log(B*kX + bb + x(1))*log(amin) +
supremumBotherregression(9,1)*log(B*kX + bb + x(1))*log(B*kX + bb + x(1)) +
supremumBotherregression(10,1)*log(amin)*log(B*kX + bb + x(1)))));

```



```

problem = createOptimProblem('fmincon','x0',[0.1],'objective',UAS3oth,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint3(x,amin,amax,B,kX,bb),'options',opts);
[xminAS3oth, fminAS3oth, flagAS3oth, outputAS3oth, manyminsAS3oth] =
run(gs,problem);

```

```

MaxUAS3othflag(i,1) = flagAS3oth;

```

```

if flagAS3oth > 0;

```

```

    MaxUAS3oth(i,1) = -fminAS3oth;

```

```

else MaxUAS3oth(i,1) = NaN;

```

```

end

```

```

if flagAS3oth > 0;

```

```

    ChoiceAS3oth(i,:) = xminAS3oth;

```

```

else ChoiceAS3oth(i,:) = NaN;

```

```

end

```

```

end

```

```

end

```

```

% scenario 4: earlier = Regime B, later = Regime A (certain)

```

```

% only feasible if kX >= amax

```

```

% constrained optimisation where Xend(formula) <= amin_earlier

```

```

MaxUBS4othflag = zeros(numrowsStateAbounds,1);

```

```

MaxUBS4oth = zeros(numrowsStateAbounds,1);

```

```

ChoiceBS4oth = zeros(numrowsStateAbounds,1);

```

```

ChoiceBS4oth(:, :) = NaN;

```

```

for i = 1:numrowsStateAbounds;

```

```

    kX = StateAbounds(i,1);

```

```

    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagBS4oth = -10;

% define optimisation problem and options
% x(1)=qcurr, x(2)=Pcurr, x(3)=Icurr
% XBend = (B*kX + ((d*kI + x(3) + 1)^gammaI)*(bb + o*x(2)) + omega)
% WBend = ((1 + r)*kW + (p - h)*(Ybar + muP*x(2)^gammaP) - n*x(2) - x(1) -
% k*x(3))
% IbarBend = (d*kI + x(3))

if kX < amax;
    MaxUBS4oth(i,1) = NaN;
else UBS4oth = @(x)(-((k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-
bonus          +          exp(supremumAotherregression(1,1)          +
supremumAotherregression(2,1)*log(B*kX + bb + x(1) + omega) +
supremumAotherregression(3,1)*log(amin)          +
supremumAotherregression(4,1)*log(amax)          +
supremumAotherregression(5,1)*(log(B*kX + bb + x(1) + omega)^2) +
supremumAotherregression(6,1)*(log(amin)^2)          +
supremumAotherregression(7,1)*(log(amax)^2)          +
supremumAotherregression(8,1)*log(B*kX + bb + x(1) + omega)*log(amin) +
supremumAotherregression(9,1)*log(B*kX + bb + x(1) + omega)*log(amax) +
supremumAotherregression(10,1)*log(amin)*log(amax)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBS4oth,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint4(x,amin,B,kX,bb,omega),'options',opts);
[xminBS4oth, fminBS4oth, flagBS4oth, outputBS4oth, manyminsBS4oth] =
run(gs,problem);

MaxUBS4othflag(i,1) = flagBS4oth;

if flagBS4oth > 0;

```

```

    MaxUBS4oth(i,1) = -fminBS4oth;
else MaxUBS4oth(i,1) = NaN;

end

if flagBS4oth > 0;
ChoiceBS4oth(i,:) = xminBS4oth';
else ChoiceBS4oth(i,:) = NaN;

end

end

end

end

% scenario 5: earlier = Regime B, later = Regime B (certain)
% only feasible if kX >= amax
% constrained optimisation where Xend(formula) >= amax_earlier

MaxUBS5othflag = zeros(numrowsStateAbounds,1);
MaxUBS5oth = zeros(numrowsStateAbounds,1);
ChoiceBS5oth = zeros(numrowsStateAbounds,1);
ChoiceBS5oth(:, :) = NaN;

for i = 1:numrowsStateAbounds;
    kX = StateAbounds(i,1);
    amin = StateAbounds(i,2);
    amax = StateAbounds(i,3);

% reset flag
flagBS5oth = -10;

% define optimisation problem and options
% x(1)=qcurr, x(2)=Pcurr, x(3)=Icurr

```

```

% XBend = (B*kX + ((d*kI + x(3) + 1)^gammaI)*(bb + o*x(2)) + omega)
% WBend = ((1 + r)*kW + (p - h)*(Ybar + muP*x(2)^gammaP) - n*x(2) - x(1) -
% k*x(3))
% IbarBend = (d*kI + x(3))

if kX < amax;
    MaxUBS5oth(i,1) = NaN;
else UBS5oth = @(x)(-(k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-
bonus          +          exp(supremumBotherregression(1,1)          +
supremumBotherregression(2,1)*log(B*kX + bb + x(1) + omega) +
supremumBotherregression(3,1)*log(amin)          +
supremumBotherregression(4,1)*log(amax)          +
supremumBotherregression(5,1)*(log(B*kX + bb + x(1) + omega)^2) +
supremumBotherregression(6,1)*(log(amin)^2)          +
supremumBotherregression(7,1)*(log(amax)^2)          +
supremumBotherregression(8,1)*log(B*kX + bb + x(1) + omega)*log(amin) +
supremumBotherregression(9,1)*log(B*kX + bb + x(1) + omega)*log(amax) +
supremumBotherregression(10,1)*log(amin)*log(amax))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBS5oth,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint5(x,amax,B,kX,bb,omega),'options',opts);
[xminBS5oth, fminBS5oth, flagBS5oth, outputBS5oth, manyminsBS5oth] =
run(gs,problem);

MaxUBS5othflag(i,1) = flagBS5oth;

if flagBS5oth > 0;
    MaxUBS5oth(i,1) = -fminBS5oth;
else MaxUBS5oth(i,1) = NaN;

end

if flagBS5oth > 0;
ChoiceBS5oth(i,:) = xminBS5oth';
else ChoiceBS5oth(i,:) = NaN;

```

end

end

end

% scenario 6: earlier = Regime B, later = Regime A or B (uncertain)

% only feasible if $kX \geq \text{amin}$

% constrained optimisation where $\text{amin_earlier} \leq X_{\text{end}}(\text{formula}) \leq \text{amax_earlier}$

MaxUBS6othflag = zeros(numrowsStateAbounds,1);

MaxUBS6oth = zeros(numrowsStateAbounds,1);

ChoiceBS6oth = zeros(numrowsStateAbounds,1);

ChoiceBS6oth(:, :) = NaN;

for i = 1:numrowsStateAbounds;

 kX = StateAbounds(i,1);

 amin = StateAbounds(i,2);

 amax = StateAbounds(i,3);

% reset flag

flagBS6oth = -10;

% define optimisation problem and options

% $x(1)=q_{\text{curr}}$, $x(2)=P_{\text{curr}}$, $x(3)=I_{\text{curr}}$

% $XB_{\text{end}} = (B \cdot kX + ((d \cdot kI + x(3) + 1)^{\text{gammaI}}) \cdot (bb + o \cdot x(2)) + \text{omega})$

% $WB_{\text{end}} = ((1 + r) \cdot kW + (p - h) \cdot (Y_{\text{bar}} + \mu P \cdot x(2)^{\text{gammaP}}) - n \cdot x(2) - x(1) -$

% $k \cdot x(3))$

% $I_{\text{barBend}} = (d \cdot kI + x(3))$

% for 2nd period, XB_{end} becomes amax for V_a and amin for V_b

if $kX < \text{amax}$;

```

MaxUBS6oth(i,1) = NaN;
else UBS6oth = @(x)(-(k*x(1) - (B*kX + bb + x(1) + omega)^2) + 1/(1 + rho)*(-
bonus + ((max(min((amax - (B*kX + bb + x(1) + omega))/(amax -
amin),1),0))) * exp(supremumAotherregression(1,1)
supremumAotherregression(2,1)*log(B*kX + bb + x(1) + omega) +
supremumAotherregression(3,1)*log(amin)
supremumAotherregression(4,1)*log(B*kX + bb + x(1) + omega) +
supremumAotherregression(5,1)*(log(B*kX + bb + x(1) + omega)^2) +
supremumAotherregression(6,1)*(log(amin)^2)
supremumAotherregression(7,1)*(log(B*kX + bb + x(1) + omega)^2) +
supremumAotherregression(8,1)*log(B*kX + bb + x(1) + omega)*log(amin) +
supremumAotherregression(9,1)*log(B*kX + bb + x(1) + omega)*log(B*kX + bb +
x(1) + omega) + supremumAotherregression(10,1)*log(amin)*log(B*kX + bb + x(1)
+ omega) + (max(min(((B*kX + bb + x(1) + omega) - amin)/(amax -
amin),1),0))) * exp(supremumBotherregression(1,1)
supremumBotherregression(2,1)*log(B*kX + bb + x(1) + omega) +
supremumBotherregression(3,1)*log(B*kX + bb + x(1) + omega) +
supremumBotherregression(4,1)*log(amax)
supremumBotherregression(5,1)*(log(B*kX + bb + x(1) + omega)^2) +
supremumBotherregression(6,1)*(log(B*kX + bb + x(1) + omega)^2) +
supremumBotherregression(7,1)*(log(amax)^2)
supremumBotherregression(8,1)*log(B*kX + bb + x(1) + omega)*log(B*kX + bb +
x(1) + omega) + supremumBotherregression(9,1)*log(B*kX + bb + x(1) +
omega)*log(amax) + supremumBotherregression(10,1)*log(B*kX + bb + x(1) +
omega)*log(amax)))));
problem = createOptimProblem('fmincon','x0',[0.1],'objective',UBS6oth,'Aineq',[-
1],'bineq',[0],'nonlcon',@(x)constraint6(x,amin,amax,B,kX,bb,omega),'options',opts);
[xminBS6oth, fminBS6oth, flagBS6oth, outputBS6oth, manyminsBS6oth] =
run(gs,problem);

```

```

MaxUBS6othflag(i,1) = flagBS6oth;

```

```

if flagBS6oth > 0;

```

```

    MaxUBS6oth(i,1) = -fminBS6oth;

```

```

else MaxUBS6oth(i,1) = NaN;

end

if flagBS6oth > 0;
ChoiceBS6oth(i,:) = xminBS6oth';
else ChoiceBS6oth(i,:) = NaN;

end

end

end

end

% take supremum of scenarios 1-3

supremumAother = zeros(numrowsStateAbounds,1);

for i = 1:numrowsStateAbounds;
    maxintermediate = max(MaxUAS1oth(i,1),MaxUAS2oth(i,1));
    maxvalue = max(maxintermediate,MaxUAS3oth(i,1));
    supremumAother(i,1) = maxvalue;
end

logsupremumAother = log(supremumAother + bonus); %%% bonus has been added
to ensure value function is positive, so it can be logged

% estimate regression for supremumAT(Xprev,Wprev,Ibarprev,amin,amax)
supremumAotherregression = regress(logsupremumAother,XAregression);

% take supremum of scenarios 4-6

supremumBother = zeros(numrowsStateAbounds,1);

```

```

for i = 1:numrowsStateAbounds;
    maxintermediate = max(MaxUBS4oth(i,1),MaxUBS5oth(i,1));
    maxvalue = max(maxintermediate,MaxUBS6oth(i,1));
    supremumBother(i,1) = maxvalue;
end

logsupremumBother = log(supremumBother + bonus); %%% bonus has been added
to ensure value function is positive, so it can be logged

% estimate regression for supremumBT(Xprev,Wprev,Ibarprev,amin,amax)
supremumBotherregression = regress(logsupremumBother,XAregression);

% save parameter estimates for supremumAotherregression
Aregpara(:,itertotal-s+1) = supremumAotherregression;

% save parameter estimates for supremumBotherregression
Bregpara(:,itertotal-s+1) = supremumBotherregression;

% save levels of choice variables that correspond to optimal scenario (1-3)
ChoiceAoth = zeros(numrowsStateAbounds,1);
% check supremum for isnan
isnan_supremumAother = isnan(supremumAother);

for count = 1:numrowsStateAbounds
    if isnan_supremumAother(count,1) == 1;
        ChoiceAoth(count,:) = NaN;
    elseif MaxUAS1oth(count,1) == supremumAother(count,1);
        ChoiceAoth(count,:) = ChoiceAS1oth(count,:);
    elseif MaxUAS2oth(count,1) == supremumAother(count,1);
        ChoiceAoth(count,:) = ChoiceAS2oth(count,:);
    elseif MaxUAS3oth(count,1) == supremumAother(count,1);
        ChoiceAoth(count,:) = ChoiceAS3oth(count,:);
    end
end

```



```

end

P_logChoiceAoth = log(ChoiceAoth(:,1) + 0.01);

% estimate regression for choice 1 (i.e. P)
PAotherregression = regress(P_logChoiceAoth,XAregression);

% save parameter estimates for PAotherregression
PAregpara(:,itertotal-s+1) = PAotherregression;

% save levels of choice variables that correspond to optimal scenario (4-6)
ChoiceBoth = zeros(numrowsStateAbounds,1);
% check supremum for isnan
isnan_supremumBother = isnan(supremumBother);

for count = 1:numrowsStateAbounds
    if isnan_supremumBother(count,1) == 1;
        ChoiceBoth(count,:) = NaN;
    elseif MaxUBS4oth(count,1) == supremumBother(count,1);
        ChoiceBoth(count,:) = ChoiceBS4oth(count,:);
    elseif MaxUBS5oth(count,1) == supremumBother(count,1);
        ChoiceBoth(count,:) = ChoiceBS5oth(count,:);
    elseif MaxUBS6oth(count,1) == supremumBother(count,1);
        ChoiceBoth(count,:) = ChoiceBS6oth(count,:);
    end
end

end

P_logChoiceBoth = log(ChoiceBoth(:,1) + 0.01);

% estimate regression for choice 1 (i.e. P)

```

```
PBotherregression = regress(P_logChoiceBoth,XAregression);
```

```
% save parameter estimates for PBotherregression
```

```
PBregpara(:,itertotal-s+1) = PBotherregression;
```

```
% iteration counter
```

```
iter = iter + 1
```

```
end
```

```
% end of for s = 1, T-4;
```