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Hidden symmetry breaking in quantum spin systems with applications to measurement-based quantum computation

Dominic Else

Supervisor: A/Prof. Andrew Doherty
Associate supervisor: Prof. Stephen Bartlett

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Centre for Engineered Quantum Systems
School of Physics
Faculty of Science

Abstract.

We extend the hidden symmetry breaking picture, first proposed by Kennedy and Tasaki in the context of the Haldane phase, to a wider class of symmetry-protected topological (SPT) phases. We construct a generalization of the Kennedy-Tasaki transformation that transforms SPT phases into symmetry-breaking phases and relates long-range order in the latter to the more subtle “string order” in the former. In doing so we directly connect the form of the Kennedy-Tasaki transformation to the modern formulation of SPT order.

We apply our generalized Kennedy-Tasaki transformation to solve the following problem in quantum information theory. We consider the 2-D cluster state, a simple “toy model” of a locally interacting system whose ground state is a universal resource for MBQC. We prove that, in the presence of a perturbation to the interaction Hamiltonian, the perturbed ground state remains a universal resource. We do this by using the generalized Kennedy-Tasaki transformation to prove that, if we employ the techniques of fault-tolerant quantum computation, the ground states of models in an appropriate SPT phases can serve as universal resources for MBQC provided that the symmetry-breaking is sufficiently strong in the symmetry-breaking phase obtained through the generalized Kennedy-Tasaki transformation.

Publications

This thesis comprises the following two articles:

Paper 1 (*Chapter 2*). Dominic V. Else, Stephen D. Bartlett, and Andrew C. Doherty, “Symmetry protection of measurement-based quantum computation in ground states”, *New J. Phys.* (2012) **14** 113016.

Paper 2 (*Chapter 3*). Dominic V. Else, Stephen D. Bartlett, and Andrew C. Doherty, “The hidden symmetry breaking picture of symmetry protected topological order”, *Phys. Rev. B* (2013) **88** 085114.

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Statement of contribution

Paper 1 is based in large part on work which I previously did for my Honours degree. However, the preparation of this research into a form that is publishable in an international research journal required the development of alternate derivations and new material. In particular, the derivation in Section II of the effective noise model construction for finitely correlated states is new, although most of the the key steps in the argument carry over from my Honours thesis without modification. The other portions that constitute original work are:

- The discussion of finite temperature in Section III E.
- The derivations presented in Appendices C and D.
- The connection with the Kennedy-Tasaki transformation, as discussed in Appendix B.

My Honours work contained some contributions from other people, but all the new material is entirely my own.

The generalized Kennedy-Tasaki transformation constructed in Paper 2 is essentially equivalent to (although not the same as) the duality transformation discussed in Paper 1 (which originated from my Honours work). Apart from this, all the research presented in Paper 2 is new and was carried out by myself, except for the verification of equivalence between our generalized Kennedy-Tasaki transformation and that of [1] for the $SO(2k + 1)$ Haldane phase [Section 3.7.1], which is due to SDB. SDB and ACD suggested the general idea of writing a paper establishing the properties of the generalized Kennedy-Tasaki transformation as an object in its own right, and, in particular, deriving a generalization of the link between string order in the SPT phase and symmetry-breaking order [Section 3.6].

For both papers, the actual writing (as opposed to the research) was done in collaboration with my co-authors, SDB and ACD.

Acknowledgments

I would especially like to thank my supervisors, Andrew Doherty and Stephen Bartlett, for introducing me to research questions which had such interesting answers lying just below the surface, and for giving me exactly the right amount of help along the way. The year and a half I spent working with them turned out to be as as fun and rewarding as I could possibly have hoped for.

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Chapter 1

Introduction and Background

1.1 Measurement-based quantum computation

One of the main goals of the field of quantum information is to construct a working *universal quantum computer*. The usefulness of such a device would be due to its ability to efficiently solve computational problems (most notably integer factorization) that cannot be efficiently solved on a classical computer. There are a variety of candidate architectures for universal quantum computation, but what they all have in common is that they rely in some way on large-scale *quantum entanglement*. The reason why constructing a quantum computer has proven so challenging experimentally is the difficulty in sustaining this large-scale quantum entanglement in the quantum computer in the face of the disrupting influence of the environment.

In this thesis, our focus will be on a particular formulation of quantum computation, interesting both from a theoretical and a practical standpoint, known as *measurement-based quantum computation* (MBQC) [2, 3]. In MBQC, the process of quantum computation is divided into two stages. In the first stage (the preparation stage), a problem-*independent* highly entangled many-body resource state is prepared. In the second stage (the measurement stage), single particle measurements are performed according to some problem-dependent measurement protocol. Because the second stage involves only single-particle operations, which cannot create entanglement, all the entanglement needed for an arbitrary quantum computation must be present in the original resource state. The question of what properties this resource state must have in order to permit universal quantum computation in the measurement stage is therefore of considerable research interest, and could shed light on the origins of quantum computers' mysterious efficiency for certain problems.

From a practical point of view, the relevance of MBQC is due to the relative ease, in many experimental implementations of quantum information, of performing single-particle operations, as opposed to entangling operations. This is, of course, of little use if we still need to use those entangling operations to construct the resource state. However, there is another intriguing possibility: the necessary entanglement could be obtained in the *thermal equilibrium* state of a locally-interacting quantum spin system [4, 5]. Indeed, a number of (albeit somewhat artificial) exactly-solvable local interaction Hamiltonians on spin systems have thermal equilibrium states which are universal resources for MBQC, at least at zero temperature (see also [6] for some finite-temperature examples).

In order for this approach to be practical, it would need to be insensitive to slight variations in the interactions, which after all could never be controlled exactly. However, determining whether the method is in fact robust in this way poses a very substantial theoretical challenge. All of the models currently known to be universal resources in thermal equilibrium have straightforward exact solutions from which allow the effects of measurements to be determined analytically [2, 7–10]. However, introducing a perturbation will usually prevent such a method of analysis, causing us to enter unknown territory without any obvious plan of attack. Quantum spin

systems have, of course, been the subject of extensive research in the field of condensed matter physics; however, the question of whether a system is or is not a universal resource for quantum computation is very different from the sorts of questions that have traditionally been studied in that field. Furthermore, even for the latter type of questions, it has often been necessary to resort to numerical techniques such as Quantum Monte Carlo or the Density Matrix Renormalization Group; such techniques are out of the question for our purposes, as it is not possible to simulate a large-scale quantum computation on a classical computer.

In this thesis, the approach we take is to exploit existing results on *fault-tolerant quantum computation* (FTQC) [11–13]. The techniques of FTQC were originally developed to deal with imperfect operations and noise in the context of the standard “circuit model” model of quantum computation. The idea is that by encoding the quantum information into a physical system in an appropriate way, the effects of noise can, through an active detection and correction process, be reduced to manageable levels such that quantum computation can proceed. Because protocols for MBQC typically simulate the circuit model in some way, FTQC can also be used to account for noise acting on the resource state [14]. However, when the resource state is obtained as the thermal equilibrium state of a quantum spin system, the effect on the resource state of a change in the interactions cannot be described by a simple noise model.

In this thesis, we resolve the aforementioned difficulties for a class of perturbations to a “toy model” of a quantum spin system for which the zero-temperature thermal equilibrium state (i.e. the ground state) is a universal resource state for MBQC. That this turns out to be possible is due to the fact that the unperturbed system is in a phase of matter characterized by *symmetry-protected topological order* (SPTO), which is the subject of the next Section.

1.2 Symmetry-protected topological order

A remarkable property of condensed matter systems is their propensity to form *ordered phases* of matter, which are inherently distinct from the trivial, disordered phase, and cannot be continuously deformed into the trivial phase without crossing a phase transition. In the traditional “Landau paradigm”, order in condensed matter systems has been viewed as related to, or indeed synonymous with, the spontaneous breaking of a symmetry. For example, in a ferromagnetic material, the rotational symmetry is broken when spins align and a net magnetic polarization is formed. Since the discovery that the order in the fractional quantum Hall effect cannot be related to the spontaneous breaking of a symmetry, much effort has been devoted to identifying ordered phases beyond the Landau paradigm. The *symmetry-protected topological order* (SPTO) [15–17] to which this section is devoted is one such example.

In this thesis, we will be concerned only with quantum spin systems at zero temperature (hence in their ground state). Let us first discuss what we mean by spontaneous symmetry-breaking in this context. Consider, for example, the one-dimensional ferromagnetic quantum Ising model with no external field, with Hamiltonian

$$H = - \sum_i \sigma_i^z \sigma_{i+1}^z \quad (1.1)$$

(where σ_i^z is the Pauli z operator). This has a two-fold degenerate ground state subspace, spanned by the “up” and “down” states $|\cdots \uparrow\uparrow\uparrow \cdots\rangle$ and $|\cdots \downarrow\downarrow\downarrow \cdots\rangle$. The important thing to note about these ground states is that they break the spin-flip symmetry $\prod_i \sigma_i^x$ even though the original Hamiltonian does not, and that the breaking of the symmetry is detectable locally, for example through the expectation value of $\langle \sigma_z \rangle$ (which gives the net magnetization, and must be zero whenever the spin-flip symmetry is respected). These are the hallmarks of symmetry-breaking order.

By contrast, consider the Heisenberg antiferromagnetic spin-1 chain, with Hamiltonian

$$H = - \sum_i \mathbf{S}_i \cdot \mathbf{S}_{i+1}, \quad (1.2)$$

where $\mathbf{S} = (S^x, S^y, S^z)$ are the spin component operators for spin-1 particles. This model is believed to lie in the so-called *Haldane phase* [18, 19], which implies the following properties. With periodic boundary conditions, the ground state is non-degenerate. With *open* boundary conditions, it has (in the thermodynamic limit), a four-fold degenerate ground state subspace; however, the degenerate ground states can only be distinguished near the edges; in the bulk, they are completely indistinguishable and no symmetry-breaking can be detected. Therefore, this model clearly does not have the traditional symmetry-breaking order. Nevertheless, the four-fold degeneracy of the ground state is a generic property of the Haldane phase, and cannot be destroyed by any rotationally-invariant perturbation without passing through a phase transition. Therefore, the spin-1 Heisenberg antiferromagnet still has a kind of order, which (since it is distinct from the trivial order only when the appropriate symmetry is enforced) has come to be known as *symmetry-protected topological (SPT) order*.

A link between symmetry-breaking and SPT order in the context of the Haldane phase was given by Kennedy and Tasaki in [20]. They constructed a non-local unitary transformation that maps systems in the Haldane phase, with SPT order, into a symmetry-breaking phase. Thus, the somewhat abstruse SPT order is related to the more straightforward symmetry-breaking order. A major component of this thesis relates to the development of a generalization of the Kennedy-Tasaki transformation and the study of its properties.

Apart from the light this sheds on SPT order as a phenomenon in its own right, this generalized Kennedy-Tasaki transformation turns out to be of particular relevance to MBQC. The reason is that, since the unperturbed model that we consider is in an SPT-ordered phase, so will be the perturbed model, provided that the perturbation is sufficiently small and respects the appropriate symmetry. The adaptive measurement protocol required for performing MBQC turns out to map in a particularly natural way under the generalized Kennedy-Tasaki transformation. Whereas the original measurement protocol depends on highly non-local properties of the system (whence the difficulty in studying it using traditional condensed matter techniques), the performance of the transformed protocol can be expressed purely in terms of local properties, allowing for a much more straightforward analysis of the effect of perturbations.

Chapter 2

Symmetry protection of measurement-based quantum computation in ground states

D.V. Else, S.D. Bartlett and A.C. Doherty, *New J. Phys.* (2012) **14** 113016.

Abstract. The two-dimensional cluster state, a universal resource for measurement-based quantum computation, is also the gapped ground state of a short-ranged Hamiltonian. Here, we examine the effect of perturbations to this Hamiltonian. We prove that, provided the perturbation is sufficiently small and respects a certain symmetry, the perturbed ground state remains a universal resource. We do this by characterizing the operation of an adaptive measurement protocol throughout a suitable symmetry-protected quantum phase, relying on generic properties of the phase rather than any analytic control over the ground state.

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2.1 Introduction

A quantum computer relies on quantum entanglement to achieve computational speedups. In the traditional, circuit-based model for quantum computation, the required entanglement is built up throughout the course of the computation through application of entangling gates coupling two or more qubits at a time. Alternatively, in the model of *measurement-based quantum computation* (MBQC) [2, 3], universal quantum computation is achieved solely through single-particle operations (specifically, single-particle measurements) on a fixed entangled resource state, independent of the quantum algorithm being performed.

Since the initial discovery that the 2-D cluster state is a universal resource for MBQC [2], much effort has been devoted to characterizing other universal resource states. Many of the universal resource states so far identified [2, 7–10] have been *projected entangled pair states* (PEPS) [21] of small bond dimension. The tensor network structure of these states facilitates the analysis of measurements, which might otherwise be an intractable problem. Another advantage of such states is that under appropriate conditions [22], they are unique (possibly gapped) ground states of local frustration-free Hamiltonians on spin lattices. This suggests a method of constructing the resource state by cooling an appropriate interacting spin system [4, 5].

However, if we wish to adopt this viewpoint of the resource state for MBQC as the ground state of a quantum spin system, it would be too restrictive to confine ourselves to states in which the effect of measurements can be determined analytically from the tensor-network structure. A generic local Hamiltonian, or even an arbitrarily small generic local perturbation to a PEPS

parent Hamiltonian, will not have such a property. Therefore, it is desirable to develop an understanding of MBQC in ground states of spin systems that does not rely on analytic control of the ground state. For this reason, there has been an interest in relating MBQC to forms of *quantum order* which, as parameters of the Hamiltonian are varied, can disappear only at a quantum phase transition [23–25].

In this paper, we will use such a connection between MBQC and quantum order to give a precise characterization of the operation of MBQC in the ground states of a large class of perturbations to the 2-D cluster model. This will allow us to give a rigorous proof that such perturbed ground states remain universal resources for MBQC provided that the perturbation is sufficiently small. Our proof relies in part on an extension of the relationship introduced in [25] between MBQC and *symmetry-protected topological (SPT) order* [15–17], a form of quantum order characterizing quantum systems which cannot be smoothly deformed into a product state while a certain symmetry is enforced. If the perturbation to the 2-D cluster model respects an appropriate symmetry, then the perturbed ground state will still possess non-trivial SPT order, and we will show that this gives us sufficient information about the ground state to characterize the implications of the perturbation for MBQC. Our result therefore holds independently of any analytic solution for the perturbed ground state.

Our proof of universality is in the same spirit as [14]. There, it was shown that, whereas measurements on the cluster state simulate quantum circuits, measurements on a noisy cluster state simulate the same circuits, but with added noise. Here, our task is complicated by the highly correlated nature of the “errors” in the resource state that from result a change in the Hamiltonian. Nevertheless, we will show how to exploit the additional structure resulting from SPT order to establish an effective noise model for ground states of appropriate perturbed cluster models. Therefore, universal quantum computation can be achieved (for sufficiently small perturbations, corresponding to sufficiently weak noise in the effective circuit model) by choosing a measurement protocol which simulates a *fault-tolerant* quantum circuit. The universality is then a consequence of the *threshold theorem* [11] for fault-tolerant quantum computation with noisy quantum circuits.

2.1.1 Summary of results

Our ultimate goal in this paper is to prove the universality for a MBQC of a class of perturbations of the 2-D cluster state. However, in order to reach this goal, most of this paper will be devoted to a further elucidation of the relationship between SPT order and MBQC. For simplicity of presentation, we will first explore this relationship in one-dimensional systems. It has already been shown that in a class of quantum phases characterized by SPT order, the structure implied by SPT order leads to the perfect operation of the identity gate in MBQC [25]. Here, we consider the 1-D cluster model, which lies in the simplest of the SPT phases considered in [25], and characterize the operation of non-trivial (i.e., not the identity) gates in the presence of a perturbation which respects the symmetry protecting this SPT phase. We obtain the following:

Theorem 1 (Effective noise model in one dimension). *Consider a measurement protocol which in the exact 1-D cluster model would simulate a sequence of gates. In the perturbed resource state, the same measurement protocol simulates the same gate sequence, but with additional noise associated with each non-trivial gate. So long as the non-trivial gates are sufficiently separated from each other by identity gates, this effective noise has no correlations between different time steps, i.e. it is Markovian.*

The proof of Theorem 1 will be divided into two stages. First, in Section 2.2 we will establish Theorem 1 for ground states which are *pure finitely-correlated states* (pFCS), a special case of *matrix-product states* (MPS). For such states, both the manifestations of SPT order [16, 17], and the effect of measurements [7] can be understood straightforwardly in terms of the tensor-network structure. The ideas leading to Theorem 1 can thus be understood most directly in this

context. Second, in Section 2.3 we will prove Theorem 1 for arbitrary ground states within the SPT phase.

The extension of these ideas to the 2-D cluster model will be considered in Section 2.4. We will construct an appropriate symmetry group, such that the following result is satisfied for symmetry-respecting perturbations.

Theorem 2 (Effective noise model in two dimensions). *Consider a measurement protocol which in the exact 2-D cluster model would simulate a sequence of gates. In the perturbed resource state, the same measurement protocol simulates the same gate sequence, but with additional noise associated with each gate. So long as the non-trivial gates are sufficiently separated from each other by identity gates, this effective noise has no correlations between different time steps, or between different gates taking place at the same time step, i.e., it is local and Markovian.*

Combined with the existing results on fault tolerance in the circuit model [11], Theorem 2 will imply the main result of this paper:

Theorem 3. *For sufficiently small symmetry-respecting perturbations, the perturbed ground state remains a universal resource for measurement-based quantum computation.*

2.2 The effective noise model construction: finitely-correlated states

In this section, we will prove our effective noise model result, Theorem 1, for a restricted class of ground states of infinite one-dimensional chains. Specifically, we consider *pure finitely-correlated states* (pFCS) [26, 27]. A pFCS can be considered as the thermodynamic limit of the translationally-invariant matrix-product states (MPS) $|\Psi_N\rangle$ generated by the MPS tensor A , on finite chains of N sites with periodic boundary conditions, e.g.

$$|\Psi_5\rangle = \begin{array}{c} \boxed{} \\ | \quad | \quad | \quad | \quad | \\ \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \\ | \quad | \quad | \quad | \quad | \\ \boxed{} \end{array} \quad (2.1)$$

(here and throughout this paper, we use a graphical notation to represent the contraction of tensors, e.g. see [28, 29]). The MPS tensor A must satisfy an additional condition known as *injectivity*, which is related to the exponential decay of the correlation functions; each of the finite-chain states $|\Psi_N\rangle$ (for sufficiently large N) is then the unique gapped ground state of a local frustration-free Hamiltonian [30].

We have several motivations for considering this class of ground states. First, it is widely believed that pFCS capture the essential physics of gapped ground states of infinite one-dimensional translationally-invariant spin chains in general. (Note that, although the theorem regarding the efficient approximation of ground states of finite spin chains by MPS [31, 32] could be regarded as supporting this belief, we cannot use this theorem to draw any rigorous conclusions for our purposes here, since it does not hold that the MPS tensor A can be kept fixed independently of the system size for a constant accuracy.) Second, the ideas leading to our effective noise model result find their simplest and most physically meaningful expression in this context. Finally, the proof presented here will play a dual role in our paper, as it can also be applied to arbitrary quantum states, provided that they satisfy a few extra criteria in common with pFCS. Thus, in order to establish the effective noise model result for general ground states, which we do in Section 2.3, it will suffice to provide a separate proof of these criteria.

The outline of this section is as follows. We begin in Secs. 2.2.1 and 2.2.2 by reviewing the properties of the 1-D cluster model and the nature of the SPT phase in which it is contained. In Sec. 2.2.3, we review the results of [25] regarding the structure shared by pFCS ground states throughout the whole SPT phase. In Sec. 2.2.4, we prove a key result: the standard adaptive

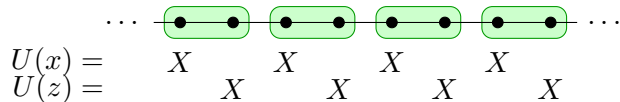


Figure 2.1: The generators of the on-site $Z_2 \times Z_2$ symmetry of the 1-D cluster model. Each dot is a qubit, and the shaded areas constitute two-qubit sites.

measurement protocol acting on a ground state in the phase is equivalent to a non-adaptive *dual process* acting on a ‘topologically disentangled’ version of the ground state, which we refer to as the *dual state*. In Sec. 2.2.5, we give a characterization of the dual process in the case that the original resource state is the exact cluster state. Finally, in Sec. 2.2.6 we exploit the short range of the correlations in pFCS to construct the effective noise model for any pFCS ground states within the SPT phase, establishing Theorem 1 for the case of pFCS ground states.

2.2.1 The 1-D cluster model in the absence of perturbations

Here we recall the properties of the 1-D cluster model in the absence of perturbations. The Hamiltonian is

$$H = - \sum_i Z_{i-1} X_i Z_{i+1}, \quad (2.2)$$

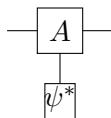
where X_i denotes the Pauli X operator acting on the i -th site, and similarly for Z_i . With appropriate boundary conditions, the system has a unique ground state (the *cluster state*), and an energy gap of 2, independent of the system size.

This model has a global $Z_2 \times Z_2$ symmetry generated by the symmetry operations $\prod_{i \text{ even}} X_i$ and $\prod_{i \text{ odd}} X_i$. We consider this symmetry to be *on-site*, which is to say it acts on states as a unitary representation $U(g)$ of the symmetry group $G = Z_2 \times Z_2 = \{1, x, y, z\}$ (with $y = xz$), such that $U(g)$ acts as $U(g) = [u(g)]^{\otimes N}$, where N is the number of sites (we group qubits into two-qubit sites in order to ensure this condition is satisfied; see Fig. 2.1). As we will see in Sec. 2.2.2, the cluster model lies in a nontrivial SPT phase with respect to this symmetry, so that the cluster state cannot be smoothly deformed into a product state without breaking the symmetry [33].

The 1-D cluster state can be represented as a pFCS [7]. For our purposes we will take the MPS tensor A_C to have the form

$$A_C[++] = \mathbb{I}, \quad A_C[+-] = X, \quad A_C[-+] = Z, \quad A_C[--] = XZ = -iY. \quad (2.3)$$

This is expressed with respect to a particular basis for a two-qubit site, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. Here, and throughout this paper, we use the notation $A[\psi]$ to refer to the linear operator obtained from the MPS tensor A by interpreting



as a linear operator (acting on states from the right), where



denotes the tensor obtained by complex conjugation from the rank-1 tensor corresponding to the state $|\psi\rangle$.

The MPS representation for the cluster state plays a crucial role in the correlation space picture [7] for the operation of the cluster state as a quantum computational wire [34]. When

a projective measurement is performed on a site, giving the outcome $|\psi\rangle$, this is interpreted as inducing an evolution $A[\psi]$ on a “correlation system”. In the case of the cluster state, for any qubit rotation U about the x - or z -axis, one can find a product basis $\{|\alpha\rangle\}$ for a two-qubit site, such that

$$A_C[\alpha] = B_\alpha U, \quad (2.4)$$

where B_α is an outcome-dependent unitary byproduct operator. This byproduct can be accounted for by adjusting the basis for future measurements depending on the outcome of the current one.

2.2.2 Symmetry-protected topological order in finitely-correlated states

Here, we will review the results of [16, 17] on the manifestation of SPT order in pFCS, and demonstrate that the 1-D cluster model indeed lies in a nontrivial SPT phase with respect to the $Z_2 \times Z_2$ symmetry.

Consider some ground state which is invariant under the on-site representation $U(g) = [u(g)]^{\otimes N}$ of some symmetry group G , and which can be represented as a pFCS, as in Eq. (2.1). The tensor A can be taken to obey a symmetry condition [17, 35]

$$A[u(g)^\dagger|\psi] = \beta(g)W(g)^\dagger A[\psi]W(g), \quad (2.5)$$

where $\beta(g)$ is a one-dimensional linear representation of the symmetry group G , and $W(g)$ is a *projective* unitary representation of the symmetry group G . This means that

$$W(g_1)W(g_2) = \omega(g_1, g_2)W(g_1g_2) \quad (2.6)$$

for some function ω , called the *factor system* of the projective representation, which maps pairs of group elements to phase factors. By blocking sites, we can ensure that $\beta(g) = 1$ (however, for simplicity we will assume that $\beta(g) = 1$ without blocking). Eq. (2.5) can then be represented graphically as

$$\text{---} \boxed{A} \text{---} = \text{---} \boxed{W(g)} \boxed{A} \boxed{W(g)^\dagger} \text{---} \quad (2.7)$$

Observe that $W(g)$ can be multiplied by a g -dependent phase factor without affecting Eq. (2.7); a set of factor systems related by such a transformation is referred to as a *cohomology class*. The arguments of [16, 17] show that two such pFCS ground states correspond to the same cohomology class if and only if they are in the same symmetry-protected phase. Nontrivial cohomology classes [those not containing the trivial factor system $\omega(g_1, g_2) = 1$] correspond to phases with nontrivial SPT order.

As an example, consider the cluster model, and its $Z_2 \times Z_2$ symmetry. The on-site representation $u(g)$ of the symmetry is generated by

$$u(x) = X \otimes \mathbb{I}, \quad (2.8)$$

$$u(z) = \mathbb{I} \otimes X. \quad (2.9)$$

and the MPS tensor is given by Eq. (2.3). It can be shown that the symmetry condition Eq. (2.7) is satisfied with the projective representation $W = V_{\mathcal{P}}$, where $V_{\mathcal{P}}$ is the *Pauli representation*

$$V_{\mathcal{P}}(1) = \mathbb{I}, \quad V_{\mathcal{P}}(x) = X, \quad V_{\mathcal{P}}(z) = Z, \quad V_{\mathcal{P}}(y) = Y. \quad (2.10)$$

This projective representation has nontrivial cohomology class, so that the cluster model lies in a non-trivial symmetry-protected phase.

2.2.3 Symmetry-respecting perturbations to the cluster state

Suppose we now consider a perturbation to the cluster Hamiltonian Eq. (2.2), such that the perturbed model still respects the $Z_2 \times Z_2$ symmetry and admits a pFCS ground state. Unless the perturbation is large enough to induce a phase transition, the MPS tensor A corresponding to the perturbed pFCS ground state should still satisfy the symmetry condition Eq. (2.7), for some projective representation $W(g)$ with the same factor system as the Pauli projective representation V [Eq. (2.10)].

The general form of the MPS satisfying these symmetry conditions was established in [25]. Here we will briefly review the relevant results from [25]. We observe that the Pauli representation satisfies a property which we refer to as *maximal non-commutativity*:

Definition 1. A projective representation $W(g)$ of an abelian group G is called *maximally non-commutative* if the subgroup $Z_W(G) \equiv \{g \in G : W(g) \text{ commutes with } W(h) \text{ for all } h \in G\}$ (which we can think of as the “projective centre” of G) is the trivial subgroup.

Notice that the subgroup $Z_W(G)$ is actually determined by the factor system ω , since $W(g)W(h) = \omega(g, h)W(gh) = \omega(g, h)\omega(h, g)^{-1}W(h)W(g)$. Furthermore, it is the same for all factor systems within a given cohomology class. Much of the discussion in this paper can be applied to any SPT phase characterized by a *finite* abelian symmetry group and a maximally non-commutative cohomology class.

An important consequence of maximal non-commutativity of a factor system is [25]

Lemma 1. For each maximally non-commutative factor system ω of a finite abelian group G , there exists a unique (up to unitary equivalence) irreducible projective representation with factor system ω . The dimension of this irreducible representation is $\sqrt{|G|}$.

Specifically, the Pauli representation $V_{\mathcal{P}}$ of $Z_2 \times Z_2$ is the unique irreducible projective representation corresponding to its factor system. In general, throughout this paper, we will use $V(g)$ to denote the unique irreducible projective representation for the current factor system. A consequence of Lemma 1 is that, for a tensor satisfying the symmetry condition Eq. (2.7), the bond space decomposes as a tensor product of a $\sqrt{|G|}$ -dimensional *protected subsystem* in which $W(g)$ acts irreducibly as $V(g)$ and a *junk subsystem* in which $W(g)$ acts trivially, i.e.

$$W(g) = V(g) \otimes \mathbb{I} \quad (2.11)$$

Thus the tensor A appearing in the MPS representation of ground states in the symmetry-protected phase satisfies the symmetry condition

$$\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \\ \boxed{u(g)} \end{array} = \begin{array}{c} \text{---} \\ | \\ \boxed{V(g)} \\ | \\ \boxed{A} \\ | \\ \boxed{V(g)^\dagger} \\ | \\ \text{---} \end{array} \quad (2.12)$$

Here we use a thick line (—) to represent the protected subsystem, and a dashed line (---) to represent the junk subsystem. The protected subsystem enjoys several nice properties for storing and manipulating logical information in a quantum computation, as we now show.

Suppose we perform a projective measurement on one site in a simultaneous eigenbasis $\{|i\rangle\}$ (which is $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$ for the $Z_2 \times Z_2$ cluster state symmetry), and obtain the outcome $|i\rangle$. Then the resulting state on the remaining sites is found by replacing the original

MPS tensor at the measured site by



$$(2.13)$$

Now we make use of another consequence of maximal non-commutativity [25]:

Lemma 2. *Let $u(g)$ be a linear on-site representation of a finite abelian symmetry group G , and let ω be a maximally non-commutative factor system of G . Then for each basis element $|i\rangle$ in a simultaneous eigenbasis $\{|i\rangle\}$ of $u(g)$, there exists a group element g_i such that*

$$\chi_i(g)V(g) = V(g_i)V(g)V(g_i)^\dagger, \quad \forall g \in G \quad (2.14)$$

for any projective representation $V(g)$ with factor system ω , where $\chi_i(g)$ is the scalar representation of G such that $u(g)|i\rangle = \chi_i(g)|i\rangle$.

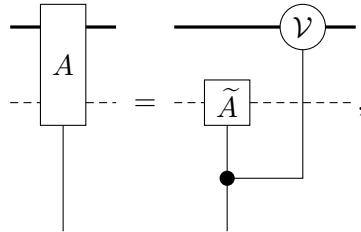
For the example of the cluster state symmetry, we have $g_{++} = 1$, $g_{+-} = x$, $g_{-+} = z$, $g_{--} = y$, as can readily be verified directly.

As was shown in [25], Lemma 2 in conjunction with the symmetry condition Eq. (2.12) implies the decomposition $A|i\rangle = V(g_i) \otimes \tilde{A}|i\rangle$, represented graphically as



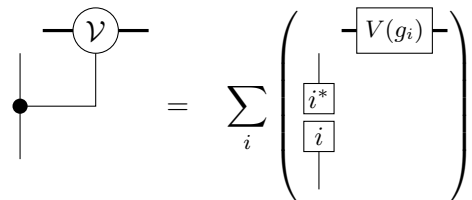
$$(2.15)$$

for some tensor \tilde{A} . Another way of writing this result is that



$$(2.16)$$

where we have defined the tensor



$$(2.17)$$

Note that, from a quantum circuit perspective, this tensor can also be interpreted as a unitary controlled operation $\sum_i |i\rangle\langle i| \otimes V(g_i)$ coupling a site to an ancilla particle; hence the choice of notation. Conversely, any MPS tensor of the form Eq. (2.16) for some tensor \tilde{A} will satisfy the symmetry condition Eq. (2.12). Following [36], we refer to the tensor \tilde{A} as the *degeneracy tensor*; and to the tensor of Eq. (2.17), which is determined entirely by the symmetry, as the *structural tensor*.

From Eq. (2.15), we see that, in the correlation space picture, measuring in a simultaneous eigenbasis $\{|i\rangle\}$ leads to an evolution on the protected subsystem of correlation space given by

an outcome-dependent unitary $V(g_i)$; this evolution is determined by the symmetry (hence the same throughout the SPT phase), and decoupled from the junk subsystem. Viewing the unitaries $V(g_i)$ as outcome-dependent byproducts of the measurements, which can straightforwardly be accommodated in a deterministic evolution using the standard techniques of measurement-based quantum computation, we say that the identity gate operates perfectly throughout the SPT phase. However, the result of measurement in any other basis is not fixed by the symmetry, and in general leads to the protected subsystem being coupled to the junk subsystem, so that the operation of other measurement-based gates will not be a robust property of the symmetry-protected phase.

2.2.4 The dual picture for MBQC on a 1D resource state

In order to deal with the randomness of measurement outcomes, the measurement protocol for MBQC with the cluster state needs to be *adaptive*: the outcome of the measurement on one site will affect the measurement basis on other sites arbitrarily far away. In analysing the effect of this protocol when acting on a *perturbed* resource state, we would like to make an argument based on the locality of the perturbed Hamiltonian, but the non-local adaptivity of the measurement protocol poses a difficulty. Therefore, in this section, we develop an alternate characterization of the effect of the cluster state adaptive measurement protocol acting on a ground state in the symmetry-protected phase. We will show that this protocol is equivalent to a *dual process* acting on a related state, which we call the *dual state*. We will show that this dual process simply consists of a sequence of unitary interactions between selected sites (those corresponding to the locations of non-trivial gates) and an ancilla particle, with no adaptivity.

In our discussion of the dual process, we will represent a pFCS ground state on an infinite chain as a formal tensor network

$$|\Psi\rangle = \cdots \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \cdots . \quad (2.18)$$

This is not, of course, the mathematically rigorous way to describe pFCS, but we find it useful for facilitating understanding. In Appendix 2.A, we will discuss how to formulate similar arguments in the rigorous pFCS framework. Later on (in Section 2.3), we will also be interested in finite chains; in that case, the arguments of this subsection can be applied more directly, given appropriate boundary conditions [specifically, the boundary conditions at the right edge should be as depicted in Eq. (2.50)].

The usefulness of the exact 1-D cluster state (with MPS tensor A_C) as a quantum computational wire results from the fact that, for each gate U in a certain set, there exists a basis $\{|\alpha\rangle\}$ such that

$$\text{---} \boxed{A_C} \text{---} = \text{---} \boxed{U} \text{---} \boxed{B_\alpha} \text{---} \quad (2.19)$$

where B_α is the outcome-dependent unitary byproduct operator. When we measure one site projectively and obtain the outcome $|\alpha\rangle$, the original MPS tensor A is replaced at the measured site by Eq. (2.19) in the tensor-network description of the resultant state.

In the case of the exact cluster state, the effect of the byproduct operator can be accounted for by adjusting the measurement basis for future measurements. This fact turns out to be closely related to the nontrivial SPT order, as we now demonstrate. Our discussion relies on the observation that, in the cluster state, the byproduct operators are Pauli operators. That is to say, it is always the case that B_α is a scalar multiple of $V(g_\alpha)$ for some $g_\alpha \in Z_2 \times Z_2$. Hence,

we can make use of the symmetry condition [which can be derived from Eq. (2.12)]

$$\begin{array}{c} \text{---} \boxed{B_\alpha} \text{---} \boxed{A_C} \text{---} \\ | \\ \text{---} \boxed{A_C} \text{---} \boxed{B_\alpha} \text{---} \\ | \\ \boxed{b_\alpha} \end{array} = \text{---} \boxed{A_C} \text{---} \boxed{B_\alpha} \text{---} , \quad (2.20)$$

where $b_\alpha = u(g_\alpha)$. Applying this condition repeatedly shows that the byproduct operator can be displaced arbitrarily far to the right. In our formal tensor-network picture for an infinite chain, we consider that this process is continued indefinitely, so that the byproduct operator “disappears out to infinity”, and is replaced with b_α acting on all sites to the right of the one on which the measurement took place, i.e.

$$\begin{array}{c} \text{---} \boxed{B_\alpha} \text{---} \boxed{A_C} \text{---} \boxed{A_C} \text{---} \boxed{A_C} \text{---} \boxed{A_C} \text{---} \cdots \\ | \quad | \quad | \quad | \\ \text{---} \boxed{A_C} \text{---} \boxed{A_C} \text{---} \boxed{A_C} \text{---} \boxed{A_C} \text{---} \cdots \\ | \quad | \quad | \quad | \\ \boxed{b_\alpha} \quad \boxed{b_\alpha} \quad \boxed{b_\alpha} \quad \boxed{b_\alpha} \quad \cdots \end{array} . \quad (2.21)$$

Hence, whenever we obtain the “wrong” outcome for a measurement (i.e. the corresponding byproduct operator B_α is not the identity), we can recover the “correct” resultant state by applying the correction b_α^\dagger to all the remaining sites on the right (equivalently, we can simply adjust the measurement basis for measurements on those sites).

Let us now examine what happens when we perform the same adaptive measurement protocol on a resource state that is not the exact cluster state. Consider a pFCS ground state contained with the same SPT phase as the cluster state, characterised by the Pauli representation of the group $Z_2 \times Z_2$. We will keep using the same measurement protocol as for the exact cluster state. (Our argument could be generalised to any pFCS ground state contained within any SPT phase characterized by a finite abelian symmetry group G and a maximally non-commutative cohomology class, so long as the the adaptive correction appearing in the measurement protocol takes the same form as for the cluster state, i.e. application of $u(g_\alpha)$ to the sites on the right for some group elements $g_\alpha \in G$.) The resource state is then of the form

$$|\Psi\rangle = \cdots \begin{array}{c} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \boxed{A} \text{---} \cdots \\ | \quad | \quad | \quad | \quad | \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} , \quad (2.22)$$

with the MPS tensor A of the form Eq. (2.16). We now repeat the above argument, in reverse. We make use of the symmetry condition Eq. (2.12) in the form

$$\begin{array}{c} \text{---} \boxed{A} \text{---} \\ | \\ \boxed{b_\alpha^\dagger} \end{array} = \begin{array}{c} \text{---} \boxed{B_\alpha^\dagger} \text{---} \boxed{A} \text{---} \boxed{B_\alpha} \text{---} \\ | \\ \text{---} \text{---} \end{array} , \quad (2.23)$$

from which we obtain

$$\dots \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \begin{array}{c} \boxed{\alpha^*} \\ \boxed{\alpha} \end{array} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \boxed{b_\alpha^\dagger} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \boxed{b_\alpha^\dagger} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \boxed{b_\alpha^\dagger} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \boxed{b_\alpha^\dagger} \end{array} \right] \dots = \dots \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \begin{array}{c} \boxed{\alpha^*} \\ \boxed{\alpha} \end{array} \end{array} \right] \left[\begin{array}{c} \boxed{B_\alpha^\dagger} \\ \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \dots \quad (2.24)$$

Therefore, we have shown that the process we actually perform, i.e. applying the measurement-dependent correction to the sites on the right of the one measured, is equivalent to a different process, in which the measurement-dependent correction is applied in the internal bond space of the MPS, as depicted in the right-hand side of Eq. (2.24). In a physical system, of course, we do not have direct access to the internal bonds of a tensor network state, so we could never perform the latter process directly; nevertheless, the two are equivalent.

Following the measurement and the adaptive correction, which we think of as being performed internally, as in the right-hand side of Eq. (2.24), the outcome of the measurement can be “forgotten”, i.e. we describe the resultant state of the system as the mixture of the right-hand side of Eq. (2.24) for all possible measurement outcomes. Without affecting the reduced state on the remaining unmeasured sites, for notational convenience we replace this mixture with a coherent superposition, i.e.

$$\dots \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \blacklozenge_k \end{array} \right] \left(\begin{array}{c} \text{---} \\ | \\ \boxed{\gamma^\dagger} \\ | \\ \text{---} \end{array} \right) \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \text{---} \end{array} \right] \dots, \quad (2.25)$$

where we have defined the tensor

$$\left(\begin{array}{c} \text{---} \\ | \\ \boxed{\gamma^\dagger} \\ | \\ \blacklozenge_k \end{array} \right) = \sum_{\alpha} \left(\begin{array}{c} \boxed{B_\alpha^\dagger} \\ \text{---} \\ | \\ \begin{array}{c} \boxed{\alpha^*} \\ \boxed{\alpha} \end{array} \end{array} \right), \quad (2.26)$$

which we can also interpret as a unitary coupling $\sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes B_{\alpha}^{\dagger} = \sum_{\alpha} |\alpha\rangle\langle\alpha| \otimes V(g_{\alpha})^{\dagger}$ coupling a site to an ancilla particle. We now use the k index throughout the paper to distinguish the “ \mathcal{G} ”s resulting from different measurement operations. (The label k refers to the site at which the measurement is being performed; we include this label to reflect the dependence on the measurement basis $\{|\alpha\rangle\}$ and byproduct operators B_{α} , which will in general be different for each site at which a measurement is performed.)

Now, using the expression Eq. (2.16) for the MPS tensor A , we can write

$$\left[\begin{array}{c} \text{---} \\ | \\ \boxed{A} \\ | \\ \blacklozenge_k \end{array} \right] \left(\begin{array}{c} \text{---} \\ | \\ \boxed{\gamma^\dagger} \\ | \\ \text{---} \end{array} \right) = \left[\begin{array}{c} \text{---} \\ | \\ \boxed{\tilde{A}} \\ | \\ \text{---} \end{array} \right] \left(\begin{array}{c} \boxed{\mathcal{G}_k} \\ \text{---} \\ | \\ \text{---} \end{array} \right), \quad (2.27)$$

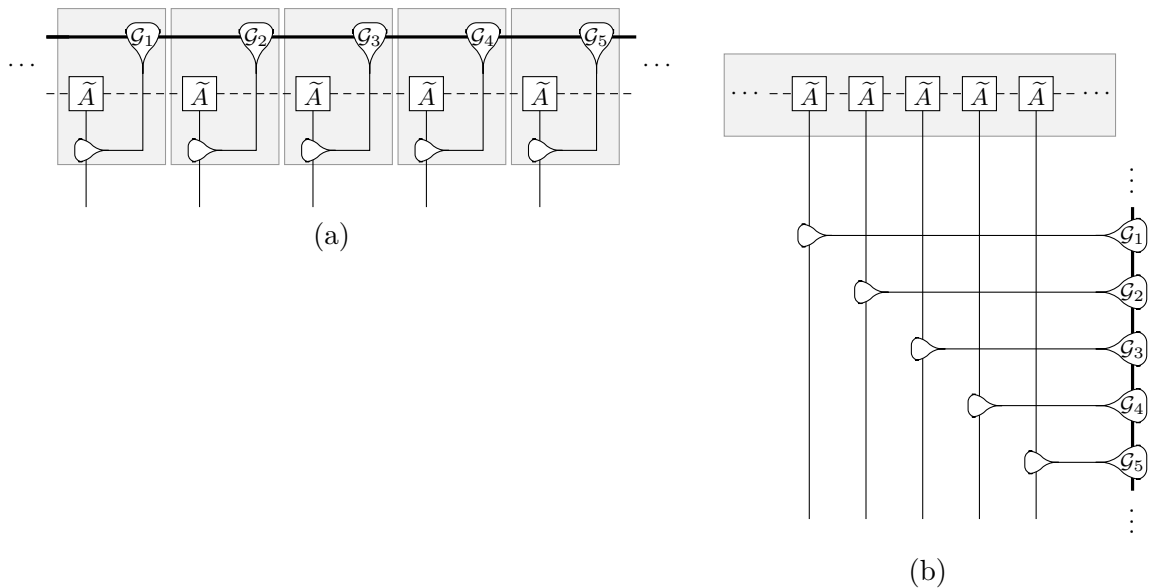


Figure 2.2: (a) After the adaptive measurement sequence, we can treat the resultant state as having the form shown. This is equivalent (b) to building $|\Psi\rangle$ from the dual state $|\tilde{\Psi}\rangle$ [the shaded box; see Eq. (2.29)] by unitary couplings to an ancilla particle.

where we have defined the tensor \mathcal{G}_k (which can also be interpreted as a unitary coupling between a site and the ancilla particle) according to

$$\mathcal{G}_k = \begin{array}{c} \text{---} \mathcal{G}_k \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \mathcal{V} \quad \mathcal{V}^\dagger \text{---} \\ | \quad | \\ \bullet \quad \blacklozenge_k \end{array} . \quad (2.28)$$

We are now in a position to define our dual process. Suppose we perform a sequence of such adaptive measurements at successive sites, which at each site is described by the insertion of the tensor Eq. (2.26), as in Eq. (2.25). There will be a different coupling \mathcal{G}_k associated with each site k . As shown in Figure 2.2, we find that the original adaptive measurement process, applied to the resource state $|\Psi\rangle$, is equivalent to a *dual process* applied to the *dual state* $|\tilde{\Psi}\rangle$. The dual state $|\tilde{\Psi}\rangle$ is defined to be the state built from the degeneracy tensor \tilde{A} , with the structural tensor discarded:

$$|\tilde{\Psi}\rangle = \cdots \text{---} \begin{array}{c} \tilde{A} \\ | \end{array} \text{---} \begin{array}{c} \tilde{A} \\ | \end{array} \text{---} \begin{array}{c} \tilde{A} \\ | \end{array} \text{---} \begin{array}{c} \tilde{A} \\ | \end{array} \text{---} \cdots . \quad (2.29)$$

The dual process comprises a series of consecutive unitary interactions \mathcal{G}_k between individual sites k and an ancilla particle.

There are several reasons why this “dual picture” is a useful way to understand the operation of MBQC in one-dimensional ground states. First, the dual process lacks the long-range measurement adaptivity which is a characteristic of the original adaptive measurement protocol. Second, the perfect operation of the identity gate is automatically built in, because, for sites k at which the adaptive measurement process at the given site is the one that corresponds in the exact cluster state to the identity gate, [i.e. the measurement basis is the simultaneous eigenbasis $\{|i\rangle\}$ of the symmetry, and the byproduct operators are $B_i = V(g_i)$, where the g_i are the group elements appearing in Eq. (2.15)], the corresponding coupling is trivial, $\mathcal{G}_k = \mathbb{I}$.

The final motivation for the dual picture is that the dual state on which it is based has some physical significance in its own right, and retains some key properties of the the original resource state. For example, if the original MPS tensor A generates a pFCS, then so does \tilde{A} , and the respective correlation lengths obey the inequality $\xi \leq \xi$ (see Appendix 2.A). Additionally, in

Appendix 2.B we will show how our dual state can be obtained from the original ground state through a generalization of the unitary that was introduced by Kennedy and Tasaki [20] to transform the SPT Haldane phase [18, 19] into a local symmetry-breaking phase; this unitary has recently been described as a “topological disentangler” [37], and in some sense we can think of the dual state as being a topologically disentangled version of the original resource state.

The dual process for initialization and readout in the 1-D cluster state

Above we only discussed measurement sequences corresponding to unitary gates in correlation space. A complete scheme for using a 1-D resource state as a quantum computational wire also includes measurement sequences corresponding to *initialization* (i.e. discarding the current state in correlation space and replacing it with a fixed state), and *readout* (i.e. making the state in correlation space available as the physical state of one qubit). We now describe briefly how the measurement protocols used on the 1-D cluster state for these purposes can be accommodated in our framework.

Initialization.—The initialization procedure involves measuring a site in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. In this basis, the MPS tensor A_C for the exact cluster state takes the form

$$A_C[00] = |0\rangle\langle 0|, \quad (2.30)$$

$$A_C[01] = |1\rangle\langle 0|, \quad (2.31)$$

$$A_C[10] = |0\rangle\langle 1|, \quad (2.32)$$

$$A_C[11] = -|1\rangle\langle 1|. \quad (2.33)$$

The randomness of measurement outcomes can therefore be accounted for by applying the appropriate outcome-dependent correction operator in correlation space following the measurement: $B_{00}^\dagger = B_{10}^\dagger = \mathbb{I}$, $B_{01}^\dagger = B_{11}^\dagger = X$. Since the correction operators are Pauli operators, the above discussion applies without change.

Readout.—The standard readout procedure for the cluster state involves measuring the *second* qubit of a two-qubit site in the computational basis, then applying an outcome-dependent correction operator to the first qubit, which acts as the output. Provided that we are only interested in the final state of the output qubit, this procedure is equivalent to a coherent correction operator coupling the two qubits in the site (specifically, it is a controlled-Z gate $C_Z = |0\rangle\langle 0| \otimes \mathbb{I} + |1\rangle\langle 1| \otimes Z$). Carrying through a similar argument to that given above for unitary gates, Sec. 2.2.4, we obtain the same result, but with the interaction \mathcal{G}_k in the dual process between the site k in question and the ancilla particle given by

$$\mathcal{G}_k = \begin{array}{c} \text{---} \mathcal{G}_k \text{---} \\ \diagdown \quad \diagup \\ \text{---} \end{array} = \begin{array}{c} \text{---} \mathcal{V} \text{---} \\ | \\ \bullet \\ | \\ \text{---} C_Z \text{---} \end{array}. \quad (2.34)$$

2.2.5 MPS of minimal bond dimension and the dual picture

As an example of the general formalism introduced in Sec. 2.2.4, here we will examine the form of the couplings \mathcal{G}_k appearing in the dual process [Eq. (2.28)], in the particular case that the resource state is an MPS with bond dimension $D = \sqrt{|G|}$, where G is the symmetry group characterizing the symmetry-protected phase. Given that the dimension of the protected subsystem is $\sqrt{|G|}$ (by Lemma 1), this is the smallest possible value of D , and corresponds to the absence of a junk subsystem (or, more precisely, a junk subsystem of dimension 1). In particular, the 1D cluster state is of this type. In general, the MPS tensor A for such an MPS must be of the form

$$A[i] = \tilde{A}[i]V(g_i), \quad (2.35)$$

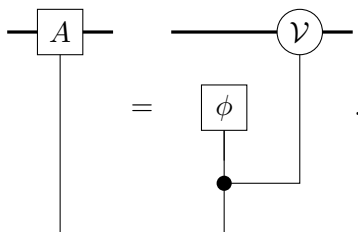
where the $\tilde{A}[i]$ here are *scalars*. It follows that the dual of such a state is a product state, with each site in the state $|\phi\rangle = \sum_i \tilde{A}[i]|i\rangle$. (We choose the normalization for the MPS tensor A so that $\langle\phi|\phi\rangle = 1$.) Therefore, the effect of the dual process acting on the dual state results from a series of independent interactions of the form


(2.36)

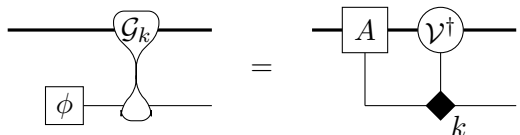
We recall that, in the correlation space picture of quantum computational wires, a quantum state can serve as a resource for executing a unitary gate U if there exists some basis $\{|\alpha\rangle\}$ such that

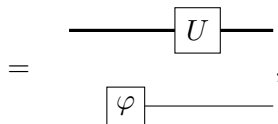
$$A[\alpha] = \beta_\alpha B_\alpha U, \tag{2.37}$$

for some set of unitary byproduct operators B_α and scalars β_α . We will now show how this property manifests itself in the dual picture, for the class of states considered here. We make use of the representation for the MPS tensor A as


(2.38)

It follows that, at a site k measured in the basis $\{|\alpha\rangle\}$, with the byproduct operators B_α , we have


(2.39)


(2.40)

where $|\varphi\rangle = \sum_\alpha \beta_\alpha |\alpha\rangle$. (It can be shown that our choice of normalization ensures that $\langle\varphi|\varphi\rangle = 1$.) Here the first equality follows from Eq. (2.38) and the definition of \mathcal{G} ; and the second inequality follows by Eq. (2.37). Thus, we have shown that in the dual picture the gate U simply acts on the ancilla particle.

Next we will do a similar analysis for the initialization and readout procedures specific to the 1-D cluster state.

Initialization.— Recall the discussion of initialization in Sec. 2.2.4. We make use of the form of the MPS tensor A_C in the computational basis, Eqs. (2.30–2.33), multiplied by the appropriate normalization factor as discussed above. Thus, for the site k at which initialization takes place,

we find that

(2.41)

(2.42)

where $\Gamma = \frac{1}{\sqrt{2}} (|00\rangle\langle 0| + |01\rangle\langle 0| + |10\rangle\langle 1| - |11\rangle\langle 1|)$ (thanks to our choice of normalization, we find that Γ is an isometry, i.e. $\Gamma^\dagger\Gamma = \mathbb{I}$). Therefore, applying the measurement sequence for initialization leads to the ancilla system getting initialized in the state $|0\rangle$, as we would expect.

Readout.— From the definition of the operator \mathcal{G}_k in the case of sites k at which readout takes place [Eq. (2.34)], we find that (here we separate a site into its two constituent qubits, each denoted by a thick line)

(2.43)

(2.44)

where $|I\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ is the canonical maximally-entangled state. Thus the state of the ancilla qubit indeed gets transferred onto the output qubit.

2.2.6 MBQC on a perturbed resource state simulates a noisy quantum circuit

In the previous subsection, we saw how measurements on an MPS of minimal bond dimension correspond to quantum gates. Now we will consider what happens when we perform the same measurement sequences on a *perturbed* resource state, assuming that the perturbed state remains within the same SPT phase. We will find that measurements on such a perturbed cluster state simulate the same quantum circuit, but with *noisy* gates. The noise is described by application of a completely positive, trace preserving (CPTP) noise superoperator following each gate.

In Section 2.2.5, we were able to treat each gate independently in the case of the unperturbed cluster state because the dual state was a product state, $|\tilde{\Psi}\rangle = |\phi\rangle^{\otimes N}$. This will no longer be true once we introduce perturbations, but we still want to treat gates independently. Towards this end, we recall that for a site for which the corresponding sequence is that for the identity gate, the associated coupling \mathcal{G}_k in the dual picture between that site and the ancilla particle is trivial. Therefore, such a site can be traced out from the beginning without affecting the final state of the output qubit. That is to say, we only need to consider the reduced state $\tilde{\rho} = \text{Tr}_{\text{trivial sites}} |\tilde{\Psi}\rangle\langle\tilde{\Psi}|$ on the remaining sites, which are those corresponding to non-identity gates (we refer to these as the *non-trivial* sites). We are free to choose our measurement protocol to ensure that the distance between any two non-trivial sites is much greater than the correlation length $\tilde{\xi}$. For pFCS, it is then straightforward to show that $\tilde{\rho}$ is approximately a product state $\tilde{\rho}_{\text{prod}} = \bigotimes_k \tilde{\rho}_k$ over the non-trivial sites, or more precisely

$$\|\tilde{\rho} - \tilde{\rho}_{\text{prod}}\|_1 \leq mf(R), \quad (2.45)$$

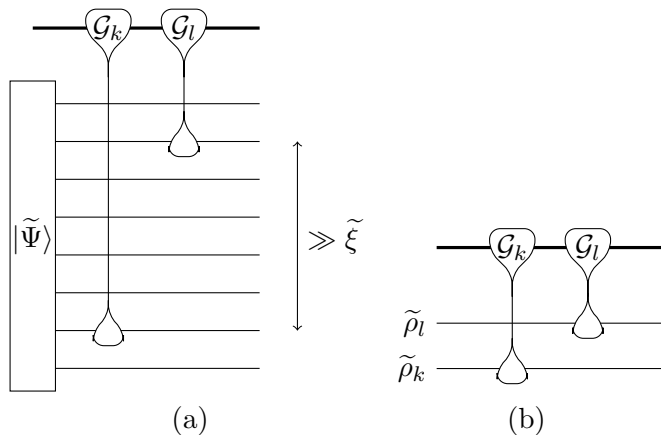


Figure 2.3: (a) As long as the locations of non-trivial gates (sites k and l in this diagram) are separated by a distance much greater than the correlation length $\tilde{\xi}$, the reduced state on those locations will be approximately a product state, and the dual process then reduces (b) to a sequence of independent interactions.

where $\|\cdot\|_1$ denotes the trace norm, m is the number of non-trivial sites, R is the minimum distance between any two non-trivial sites, and $f(R)$ is a function related to the transfer channel of the pFCS, behaving asymptotically as $f(R) = \mathcal{O}[\exp(-R/\tilde{\xi})]$ for large R , where $\tilde{\xi}$ is the correlation length associated with the pFCS.

We first consider the case where $\tilde{\rho} = \tilde{\rho}_{\text{prod}}$ exactly. Because $\tilde{\rho}$ is then a product state, we find, as in the previous subsection, that the dual process acting on the dual state is again effectively a sequence of independent interactions, this time of the form

(2.46)

(see Fig. 2.3). Let us suppose that \mathcal{G}_k results from the measurement sequence corresponding to a unitary gate U_k . Then, after tracing out the physical site, Eq. (2.46) corresponds to an evolution on the ancilla qubit described by the CPTP map

$$\mathcal{A}_k(\sigma) = \text{Tr}_{\text{physical site}} \left(\mathcal{G}_k(\sigma \otimes \tilde{\rho}_k) \mathcal{G}_k^\dagger \right) \quad (2.47)$$

As we saw in Sec. 2.2.5, in the absence of perturbations to the cluster state, $\tilde{\rho}_k = |\phi\rangle\langle\phi|$ and $\mathcal{A}_k = \mathcal{U}_k$, where $\mathcal{U}_k(\sigma) = U_k \sigma U_k^\dagger$. In general we can write $\mathcal{A}_k = \mathcal{E}_k \circ \mathcal{U}_k$, where \mathcal{E}_k is a noise superoperator for which it is straightforward to show that

$$\|\mathcal{E}_k - \mathcal{I}\|_\diamond \leq \|\tilde{\rho}_k - |\phi\rangle\langle\phi|\|_1, \quad (2.48)$$

where $\|\cdot\|_\diamond$ is the diamond norm on superoperators [38]. The cases when \mathcal{G}_k corresponds to initialization or readout are analogous. Therefore we have shown (in the case $\tilde{\rho} = \tilde{\rho}_{\text{prod}}$) that the measurement protocol on the perturbed cluster state reproduces the desired quantum circuit, except that each gate (as well as the initialization and readout steps) is accompanied by some associated noise. Furthermore, if the perturbation is sufficiently small, then the reduced states $\tilde{\rho}_k$ will be close to $|\phi\rangle\langle\phi|$ (see Appendix D for the proof), so that the noise will be weak, in the sense that \mathcal{E}_k is close to the identity superoperator in the diamond norm.

In the general case, in which $\tilde{\rho}$ and $\tilde{\rho}_{\text{prod}}$ are not equal, but are ϵ -close in the trace distance, we just need to observe that the reduced state of the output qubit following the dual process can be obtained from $\tilde{\rho}$ by application of some CPTP superoperator, which we call \mathcal{B} . From the

contractivity property of the trace distance, it follows that $\|\mathcal{B}(\tilde{\rho}) - \mathcal{B}(\tilde{\rho}_{\text{prod}})\|_1 \leq \|\tilde{\rho} - \tilde{\rho}_{\text{prod}}\|_1 \leq \epsilon$. Therefore, the effective noisy quantum circuit description correctly describes the final state of the output qubit up to an accuracy ϵ . Note that, because the bound Eq. (2.45) depends on the number of non-trivial gates m , it will be necessary to have the separation R scale with m in order to obtain a fixed accuracy ϵ , but only logarithmically; indeed, the minimum separation required to achieve an accuracy ϵ scales like $R_{\text{min}}/\xi = \mathcal{O}[\log(m/\epsilon)]$.

2.2.7 Summary of Section 2.2

In Section 2.2, we have presented, within the context of pure finitely-correlated states, the main ideas leading to our effective noise model construction. Our discussion has hinged around the “dual state” which we associated with each ground state carrying the appropriate SPT order. Loosely speaking, we can think of the entanglement in SPT-ordered ground states as comprising “topological” and “non-topological” components intertwined. The topological component is fixed throughout the phase and is responsible for the distinctive characteristics of the SPT phase, such as the degeneracy in the entanglement spectrum [39], the diverging localizable entanglement length [40, 41], and the perfect operation of the identity gate. One can think of the dual state as being obtained from the original ground state by a topological disentangler, “separating out” the topological component of the entanglement and leaving only the non-topological component [37].

In this paper, the importance of the dual state is due to the following fact, which we established in Sec. 2.2.4: the cluster state adaptive measurement protocol, when applied to an SPT-ordered ground state, couples in a natural way to the topological component of the entanglement, and the effect is thus equivalent to a “dual process” (with a simpler structure) acting on the dual state. This result gives rise to an effective quantum circuit description describing the outcome of the measurement protocol applied to any SPT-ordered ground state [e.g. see Fig. 2.3(a)]. The action of non-trivial gates is determined by an interaction with a single site in the dual state, and perturbations to the dual state give rise to noisy gates. As long as the locations of non-trivial gates are sufficiently separated, the reduced state on the sites relevant for the gate operation will be a product state, and this corresponds to independent noise acting on each gate in the effective circuit description.

2.3 The effective noise model construction: general ground states

In this section, we will extend our characterization of the effective noise model to any ground state within the symmetry-protected phase, without reference to finitely correlated states. Instead of starting from scratch, we will build on the results of Section 2.2, as follows. We formulate a condition which we believe (on physical grounds) to be satisfied for any system within the symmetry-protected phase. We will show that this condition leads to a construction for the dual state of any ground state in the SPT phase, independently of the pFCS formalism. Furthermore, given an exact MPS representation for the dual state (which always exists, albeit possibly with a bond dimension exponentially large in the system size), we show that one can construct a corresponding MPS representation for the original ground state, such that the arguments of Section 2.2 can be applied without significant change. In order to establish the approximate factorization condition Eq. (2.45) in the case of general ground states, we will show that the dual state is (like the original resource state) the gapped ground state of a local Hamiltonian, which can be constructed in a straightforward way from the original Hamiltonian. This will allow us to establish the approximate factorization condition Eq. (2.45) without assuming that the dual state has a pFCS structure.

2.3.1 Symmetry-protected topological order and boundary conditions

Because we are considering general ground states, we can no longer make direct use of the characterization of SPT order in finitely-correlated states of [16, 17]. Instead, we adopt the perspective in which SPT order is related to the fractionalized edge modes associated with open boundary conditions [19]. Our discussion will, out of necessity, be physically motivated rather than mathematically rigorous, but will suggest the formulation of the precise assumptions under which the rigorous results of this paper can be proven.

Consider a 1-D chain with open boundary conditions, with symmetry-respecting interactions such that, in the bulk, there is no symmetry-breaking and a finite energy gap for excitations. It is still possible that the energy gap for *edge* excitations may be much smaller than the bulk gap (or even zero). We denote by \mathcal{P} the subspace comprising the low-lying edge states. If the chain is sufficiently long, we expect that the gap in the bulk should ensure that the edges are non-interacting, so that \mathcal{P} decomposes a tensor product of degrees of freedom associated with the left and right edges respectively, $\mathcal{P} = \mathcal{P}_l \otimes \mathcal{P}_r$, and the restriction $H_{\mathcal{P}}$ of the Hamiltonian H onto this subspace is a sum of terms acting on each edge: $H_{\mathcal{P}} = h_l \otimes \mathbb{I} + \mathbb{I} \otimes h_r$.

For our purposes, we are interested in how the edge states transform under the symmetry. Because \mathcal{P} is spanned by a set of energy eigenspaces, and the Hamiltonian commutes with the representation of the symmetry $U(g)$, it follows that \mathcal{P} must be an invariant subspace for $U(g)$. We write the operation of $U(g)$ restricted to \mathcal{P} as $U_{\mathcal{P}}(g)$. We expect that, for sufficiently long chains, the symmetry should act independently on the respective edge states, i.e.

$$U_{\mathcal{P}}(g) = V_l(g) \otimes V_r(g). \quad (2.49)$$

By assumption, $U(g)$, and hence $U_{\mathcal{P}}(g)$, is a linear representation of the symmetry group G . It follows that $V_l(g)$ and $V_r(g)$ are in general *projective* representations of G , and if $V_l(g)$ has factor system ω , then $V_r(g)$ must have factor system ω^{-1} . We are free to transform $V_r(g) \rightarrow \beta(g)V_r(g)$, $V_l(g) \rightarrow \beta^{-1}(g)V_l(g)$ for any g -dependent phase factors $\beta(g)$ without affecting Eq. (2.49), but the cohomology class $[\omega]$ is uniquely determined. Furthermore, it is intuitively clear that any continuous symmetry-respecting variation in the Hamiltonian cannot change the cohomology class $[\omega]$, except at a phase transition (where the gap closes in the bulk, and the left- and right- edge modes need no longer be non-interacting). Therefore, we have an alternative characterization of the SPT phase corresponding to a cohomology class $[\omega]$: it comprises the systems where a left edge is associated with emergent edge states transforming projectively under the symmetry with cohomology class $[\omega]$. This is a generalization of the well-known observation that systems in the $SO(3)$ -invariant Haldane phase have emergent spin-1/2 degrees of freedom at the edges [42, 43].

In non-trivial SPT phases, the edge interactions h_l and h_r (and therefore the overall Hamiltonian H) will always have degenerate ground states, due to the fact that non-trivial projective representations cannot be one-dimensional. On the other hand, we expect, at least in the case that the symmetry group G is abelian, that a non-degenerate ground state can be recovered by introducing terminating particles at the left and right edges, transforming projectively under the symmetry with factor systems ω^{-1} and ω respectively (see Figure 2.4). This is because, loosely speaking, these terminating particles can couple to the edge modes, with the composite system at each edge transforming under a linear representation (and therefore, in the case of an abelian symmetry group, generically having a non-degenerate ground state). For example, the ground state of a spin chain in the Haldane phase can be made non-degenerate through coupling to spin-1/2 particles at the edges. Conversely, if the terminating particles do not transform with the cohomology classes $[\omega^{-1}]$ and $[\omega]$ respectively, then the degeneracy cannot be removed completely because there is still a non-trivial projective symmetry transformation at each edge.

Thus, we have arrived at yet another characterization of SPT order, which we state as a conjecture in the absence of a rigorous proof:

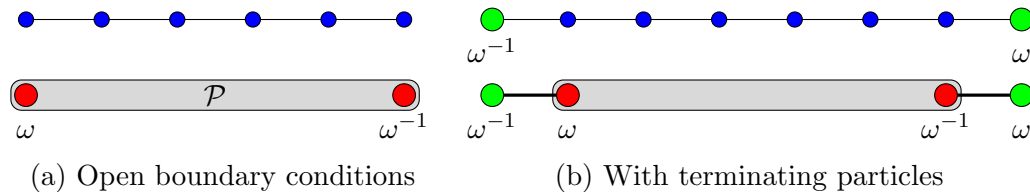


Figure 2.4: (a) The low-lying energy subspace \mathcal{P} of a 1-D chain with open boundary conditions, in an SPT phase characterized by the cohomology class $[\omega]$, decomposes as a tensor product of “emergent edge modes” (red) associated with each end, transforming projectively under the symmetry. (b) The degeneracy can be removed by coupling terminating particles (green) at each end, leading to an effective coupling to the edge modes.

Conjecture 1. *A 1-D chain respecting an on-site representation of an abelian symmetry group G is in the SPT phase characterized by cohomology class $[\omega]$ if and only if the following condition is satisfied:*

Condition 1. The finite-chain ground state can be made non-degenerate and gapped by the inclusion of symmetry-respecting interactions coupling the left and right edges of the chain to terminating particles transforming projectively under the symmetry, with factor systems ω and ω^{-1} respectively.

In any case, in the remainder of this section, we will consider systems satisfying Condition 1. Specifically, all the results will apply to finite chains with the appropriate edge couplings imposed to ensure a non-degenerate gapped ground state. This will prove convenient for our analysis, but the properties of the system in the bulk should not, of course, depend on the boundary conditions.

Note also that, in the case of a system with the interactions governed by the parent Hamiltonian of a pFCS [generated by an MPS tensor satisfying the symmetry condition Eq. (2.7) corresponding to the symmetry-protected phase], Condition 1 can easily be established directly. Furthermore, the stability theorem of [44] ensures that Condition 1 remains true for sufficiently small symmetry-respecting perturbations of such models, regardless of the validity of Conjecture 1.

2.3.2 The general construction for the dual state; exact MPS representation of SPT-ordered ground states

Recall that in Section 2.2.4, we defined the dual state in the context of pFCS. Here, we will give an analogous construction for the dual state corresponding to a general ground state within a symmetry-protected phase, provided that the phase is characterized by a finite abelian group G and a maximally non-commutative cohomology class $[\omega]$. The construction applies to a finite chain, with the appropriate boundary conditions as discussed in Section 2.3.1. This construction will then allow us to express the original ground state as an MPS, with the MPS tensors satisfying an appropriate symmetry condition.

We consider a finite chain coupled to terminating particles, such that the overall system is invariant under the symmetry $U(g) = V^*(g) \otimes [u(g)]^{\otimes N} \otimes V(g)$. Here we have taken the right terminating particle to transform under $V(g)$, the unique irreducible projective representation with factor system ω ; and the left terminating particle under $V^*(g)$ [$V^*(g)$ is the operator obtained from $V(g)$ by complex conjugation of the matrix elements in some basis; observe that $V^*(g)$ is a projective representation of G with factor system ω^{-1}].

The natural analogues in the current setting (finite chains, with the specific choice of boundary conditions) of the pFCS ground states which we considered in Section 2.2 are states of the

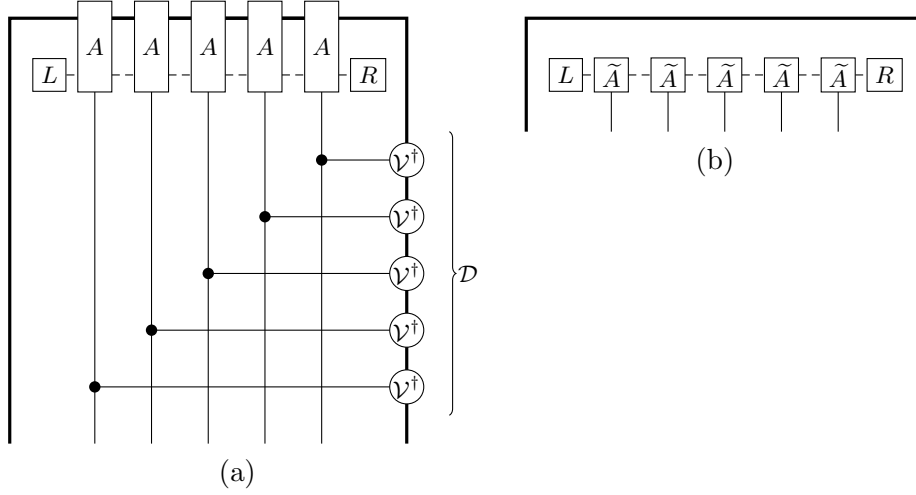


Figure 2.5: The “topological disentangler” \mathcal{D} applied to the original ground state (a) turns it into the dual state (b), leaving the terminating particles maximally entangled.

form

$$|\Psi\rangle = \left[\begin{array}{c} \boxed{L} - \boxed{A} - \boxed{A} - \boxed{A} - \boxed{A} - \boxed{A} - \boxed{R} \\ | \\ | \\ | \\ | \\ | \end{array} \right], \quad (2.50)$$

for some end vectors $|L\rangle$ and $\langle R|$, and where the MPS tensor A satisfies the symmetry condition Eq. (2.12) [which can be shown to ensure the invariance of the state under $U(g)$]. Given the decomposition Eq. (2.16) for the MPS tensor A , it follows that the dual state can be obtained from the original ground state by a sequence of unitary interactions between individual sites and the terminating particle on the right (see Fig. 2.5); we can think of the overall unitary transformation \mathcal{D} as a “topological disentangler”. Specifically, we have $\mathcal{D}|\Psi\rangle = |\tilde{\Psi}\rangle \otimes |I\rangle$, where $|I\rangle = \sum_{k=1}^D |k\rangle|k\rangle$ is the canonical maximally-entangled state between the terminating particles.

We will now show that, for a general gapped symmetry-respecting ground state $|\Psi\rangle$ [not necessarily in the MPS form Eq. (2.50)], it remains the case that $\mathcal{D}|\Psi\rangle = |\tilde{\Psi}\rangle \otimes |I\rangle$ for some state $|\tilde{\Psi}\rangle$ on the non-terminating sites; this will serve as the definition of the dual state $|\tilde{\Psi}\rangle$ for general ground states.

We observe that the original ground state $|\Psi\rangle$ must be invariant under the global symmetry operation $U(g)$, i.e.

$$U(g)|\Psi\rangle = \alpha(g)|\Psi\rangle. \quad (2.51)$$

(Without loss of generality, we can set $\alpha(g) = 1$ by absorbing it into into the definition of the symmetry¹.) This implies that the state $\mathcal{D}|\Psi\rangle$ is invariant under $\mathcal{D}U(g)\mathcal{D}^\dagger$. Let us examine what this ‘dual’ symmetry looks like. We observe that

$$\mathcal{D}_1[u(g) \otimes V(g)]\mathcal{D}_1^\dagger = \sum_i \chi_i(g)|i\rangle\langle i| \otimes V(g_i)^\dagger V(g)V(g_i) \quad (2.52)$$

$$= \sum_i |i\rangle\langle i| \otimes V(g) \quad (2.53)$$

$$= \mathbb{I} \otimes V(g), \quad (2.54)$$

¹Specifically, we replace the action of the symmetry $V(g)$ on the right terminating particle with its equivalent under a unitary transformation (by Lemma 1 of Ref. [25]), $\alpha^{-1}(g)V(g)$.

where

$$\mathcal{D}_1 = \begin{array}{c} \textcircled{\mathcal{V}^\dagger} \\ | \\ \bullet \end{array} = \sum_i |i\rangle\langle i| \otimes V(g_i)^\dagger \quad (2.55)$$

is the interaction from which \mathcal{D} is built; to get to Eq. (2.53), we made use of Eq. (2.14). From this, one can show that

$$\mathcal{D}U(g)\mathcal{D}^\dagger = V^*(g) \otimes \mathbb{I}^{\otimes N} \otimes V(g). \quad (2.56)$$

It is straightforward to show (using the irreducibility of V) that invariance of a state under the right-hand side of Eq. (2.56) implies that it must be of the form $\mathcal{D}|\Psi\rangle = |\tilde{\Psi}\rangle \otimes |I\rangle$ for some state $|\tilde{\Psi}\rangle$, as required.

It is now straightforward to construct an appropriate MPS representation for a general ground state. Indeed, let us consider an MPS representation for the dual state $|\tilde{\Psi}\rangle$ of the form

$$|\tilde{\Psi}\rangle = \begin{array}{c} \boxed{\tilde{A}_1} \text{---} \boxed{\tilde{A}_2} \text{---} \boxed{\tilde{A}_3} \text{---} \boxed{\tilde{A}_4} \text{---} \boxed{\tilde{A}_5} \\ | \quad | \quad | \quad | \quad | \end{array}. \quad (2.57)$$

We choose this representation to be *exact*; this may require the bond dimension to be very large (scaling exponentially in the system size), but that is of no importance to us. Then we have

$$\mathcal{D}|\Psi\rangle = |I\rangle \otimes |\tilde{\Psi}\rangle \quad (2.58)$$

$$= \begin{array}{c} \boxed{\tilde{A}_1 \text{---} \tilde{A}_2 \text{---} \tilde{A}_3 \text{---} \tilde{A}_4 \text{---} \tilde{A}_5} \\ | \quad | \quad | \quad | \quad | \end{array}. \quad (2.59)$$

Now we can apply the inverse transformation \mathcal{D}^\dagger to obtain

$$|\Psi\rangle = \begin{array}{c} \textcircled{\mathcal{V}} \text{---} \textcircled{\mathcal{V}} \text{---} \textcircled{\mathcal{V}} \text{---} \textcircled{\mathcal{V}} \text{---} \textcircled{\mathcal{V}} \\ | \quad | \quad | \quad | \quad | \\ \boxed{\tilde{A}_1} \text{---} \boxed{\tilde{A}_2} \text{---} \boxed{\tilde{A}_3} \text{---} \boxed{\tilde{A}_4} \text{---} \boxed{\tilde{A}_5} \\ | \quad | \quad | \quad | \quad | \end{array}. \quad (2.60)$$

This is a representation of $|\Psi\rangle$ as an MPS, with each of the shaded regions corresponding to an MPS tensor A of the form Eq. (2.16), and hence satisfying the symmetry condition corresponding to the symmetry-protected phase. In addition, we should take note of the boundary conditions at the right edge. These boundary conditions ensure that the arguments of Section 2.2.4 apply without any need to invoke an infinite-system limit.

2.3.3 The dual state as the ground state of a local Hamiltonian

In the previous subsection, we have constructed the dual state for any ground state in the symmetry-protected phase. The original ground state is, by assumption, the gapped ground state of a local Hamiltonian. In this subsection we will show that this is also true of the dual state. That is, starting from the original Hamiltonian H , we construct another local Hamiltonian \tilde{H} for which the dual state is the gapped ground state.

We start by proving a useful fact about the unitary transformation \mathcal{D} introduced in the previous section: although it is in general non-local, it maps symmetry-respecting local observables (i.e. those supported on a small set of sites of finite size) to local observables. Indeed, let us

consider some local observable h ; we will show that $\mathcal{D}h\mathcal{D}^\dagger$ is also local. For concreteness, we suppose that h acts on two adjacent sites somewhere in the bulk. Now, observe that

$$(2.61)$$

where

$$(2.62)$$

By means of Eqs. (2.62) and (2.54), it can be verified that if h commutes with the symmetry, i.e.

$$(2.63)$$

then

$$(2.64)$$

Since $V(g)$ is an irreducible projective representation, Eq. (2.64) implies (by Schur's Lemma) that \tilde{h} acts trivially on the terminating particle, i.e.

$$(2.65)$$

Now, using Eqs. (2.61) and (2.65), we find that $\mathcal{D}h\mathcal{D}^\dagger = \tilde{h}$, where \tilde{h} acts on the same two sites as h (see Fig. 2.6). Thus, although the duality transformation \mathcal{D} is non-local, we have shown that it maps local symmetry-respecting operators to local operators, as promised. The exception is operators h at the left edge, which act non-trivially on the left terminating particle; in that case, the above argument breaks down, but we can observe directly from the structure of \mathcal{D} that $\mathcal{D}h\mathcal{D}^\dagger$ is supported on the union of the support of h and the right terminating particle. For operators h acting non-trivially on the *right* terminating particle, the argument must be adjusted, but the conclusion that $\mathcal{D}h\mathcal{D}^\dagger$ is supported on the support of h still holds.

We are now in a position to construct the Hamiltonian for which the dual state $|\tilde{\Psi}\rangle$ is the gapped ground state. We observe that $\mathcal{D}H\mathcal{D}^\dagger$ has $|I\rangle \otimes |\tilde{\Psi}\rangle$ as its gapped ground state; however, it includes terms acting non-trivially on the terminating particles. We define a Hamiltonian acting only on the intermediate sites according to $\tilde{H} = \langle I | \mathcal{D}H\mathcal{D}^\dagger | I \rangle \equiv \mathcal{F}(H)$; by the locality result proven above, each local interaction term in H corresponds to a local term in \tilde{H} supported on the same set of sites. It can be shown that $|\tilde{\Psi}\rangle$ is the unique ground state of \tilde{H} , and that the gap is at least as large as that of $\mathcal{D}H\mathcal{D}^\dagger$, or equivalently H .

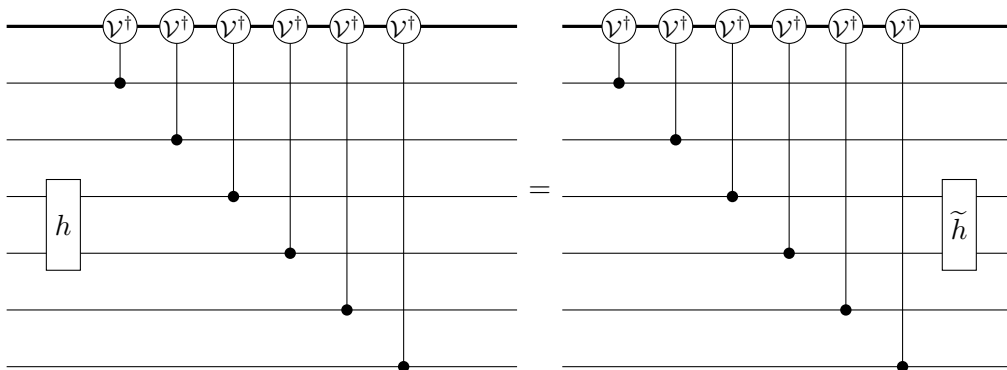


Figure 2.6: From Eqs. (2.61) and (2.65), we get the pictured equality. This shows that $\mathcal{D}h = \tilde{h}\mathcal{D}$, or equivalently $\mathcal{D}h\mathcal{D}^\dagger = \tilde{h}$.

2.3.4 The factorization condition for general ground states

Recall that the other condition that needed to be satisfied in order to apply the arguments of Section 2.2 for general ground states was that the factorization condition for the reduced density operator $\tilde{\rho}$ on the non-trivial sites in the dual state,

$$\tilde{\rho} \approx \bigotimes_k \tilde{\rho}_k \equiv \tilde{\rho}_{\text{prod}} \quad (2.66)$$

should be satisfied when the non-trivial sites are sufficiently separated from each other. Recall that, for the case of pFCS, one can prove the bound

$$\|\tilde{\rho} - \tilde{\rho}_{\text{prod}}\|_1 \leq mf(R), \quad (2.67)$$

with m the number of non-trivial sites, and $f(R)$ a function scaling asymptotically as $f(R) = \mathcal{O}[\exp(-R/\tilde{\xi})]$, where $\tilde{\xi}$ is the correlation length in the dual state. We conjecture that Eq. (2.67) should be a general property of all gapped ground states of local Hamiltonians. However, we have only been able to rigorously prove the weaker bound

$$\|\tilde{\rho} - \tilde{\rho}_{\text{prod}}\|_1 \leq md^{2m}f(R), \quad (2.68)$$

where $f(R)$ is as before, and d is the dimension of the Hilbert space at each site; see Appendix 2.C for the proof.

Note that if we assume only the weaker bound Eq. (2.68), then the separation between non-trivial sites will need to scale more rapidly with the number of gates m ; we find that the minimum separation R_{min} required for an accuracy ϵ scales like

$$R_{\text{min}}/\tilde{\xi} = \mathcal{O}(m) + \mathcal{O}[\log(1/\epsilon)]. \quad (2.69)$$

This still implies that the number of measurements need scale only as a polynomial in the number of non-trivial gates.

2.3.5 Nonzero temperature

The formulation of the dual state as the ground state of a dual Hamiltonian extends naturally to nonzero temperature: under the topological disentangler \mathcal{D} , the thermal state of the original Hamiltonian H maps to the thermal state of a dual Hamiltonian \tilde{H} . Furthermore, it can be shown that an appropriate adaptive measurement protocol acting on the thermal state of the original Hamiltonian is equivalent to a non-adaptive dual process (of the same form as in the zero-temperature case), acting on the thermal state of the dual Hamiltonian.

However, it does not appear possible to construct a Markovian effective noise model for nonzero temperature using the same techniques as for zero temperature. The reason is that our arguments were based on the assumption that the reduced state $\tilde{\rho}_k$ on each of the non-trivial sites in the dual state does not differ greatly from its value in the dual of the unperturbed resource state. This is indeed the case for small local perturbations to the Hamiltonian (as we prove in Appendix 2.D), but it need not be true for nonzero temperature. For example, consider the one-dimensional Ising model, with Hamiltonian

$$-\sum_i Z_i Z_{i+1} + Z_1 \quad (2.70)$$

(we include the Z_1 term to select out a unique ground state). In this model, it can be shown (e.g. using the transfer matrix method) that the reduced state on a single spin changes discontinuously as soon as the temperature is switched on (this is closely related to the disappearance of the magnetic order in the 1-D Ising model at nonzero temperature). Given the structure of the dual Hamiltonian as discussed in Appendix 2.B, there is good reason to believe that it will exhibit a similar phenomenon.

The difficulty of treating thermal states in our framework should not be surprising, as the dual process has the perfect operation of the identity gate built in, whereas the cluster model is not expected to have a long-range identity gate at nonzero temperature. On the other hand, there exists a measurement protocol for a *three*-dimensional cluster model which retains the perfect operation of the identity gate at sufficiently small nonzero temperatures [45]. Therefore, if the dual process description could be extended to measurement protocols such as this one, then it might be expected that the dual Hamiltonian would possess an ordered phase that persists at nonzero temperature, such that the local reduced state varies continuously with temperature up to the phase transition.

2.4 Two-dimensional systems and fault tolerance

The equivalence we demonstrated in Sections 2.2 and 2.3, between MBQC on perturbed resource states and noisy quantum circuits, opens up the possibility of exploiting the results in the literature on fault-tolerant quantum computation with noisy quantum circuits. Here, we will extend the results of the previous sections to the 2D cluster model, which, unlike the 1D models considered previously, is a universal resource for quantum computation. We will again find that, provided the perturbation to the Hamiltonian respects a certain symmetry, MBQC using the perturbed ground state as a resource is equivalent to a noisy quantum circuit. We will show that the noise in this effective circuit description has no correlations in time (as in the previous section), nor any correlations in space. This reduction to local, Markovian noise will allow us to invoke the threshold theorem to deduce that, provided the perturbation respects the symmetry and is sufficiently small, the perturbed ground state remains a universal resource for MBQC.

It should be emphasized that, although we make use of the theory of fault-tolerant quantum computation, our final result cannot be described as a fault tolerance result for MBQC, since it applies only to symmetry-respecting perturbations, and we assume noiseless operation of the measurement protocol.

2.4.1 The ‘quasi-1D’ model

Here, we make a first attempt at generalizing the 1D results to a 2D model which is universal for quantum computation. The ground state of the model we introduce here is not strictly a universal resource for MBQC unless we allow non-single-qubit measurements; however the discussion here will serve as a stepping stone for consideration of the 2D cluster model in Sec. 2.4.2.

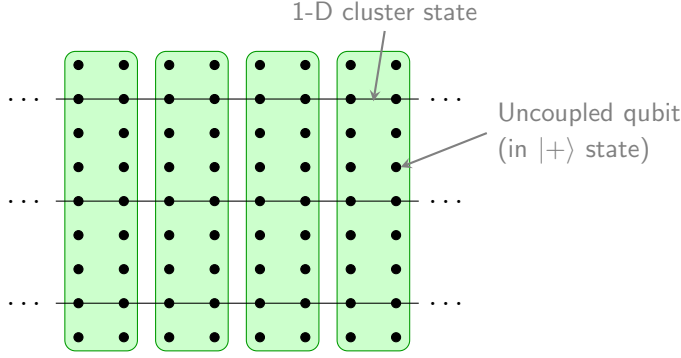


Figure 2.7: The first step in generalizing the 1D results to 2D models involves consideration of a ‘quasi-1D’ model, which consists of a 1D cluster Hamiltonian acting on each of N qubit chains arranged in the vertical dimension, as well as a term favouring the $|+\rangle$ states on the uncoupled qubits. The model has a $(Z_2 \times Z_2)^{\times N}$ symmetry, arising from the $Z_2 \times Z_2$ symmetry associated with each of the N chains. We can treat this model as ‘quasi-1D’ by defining our sites (shown as green shaded areas) so that they span the vertical dimension.

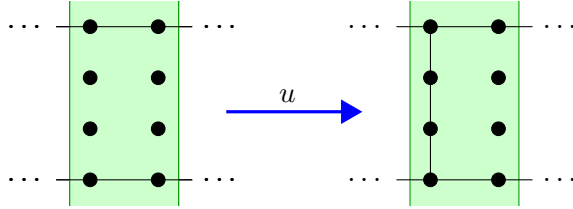


Figure 2.8: In the ‘quasi-1D’ model, two-qubit gates in correlation space cannot be done with single-qubit measurements. However, after applying controlled- Z gates between neighbouring qubits in order to couple two chains, an entangling gate can be performed in correlation space by means of single-qubit measurements.

In the absence of perturbations, the 2D model we consider involves N uncoupled 1D cluster states arranged in the second dimension, as shown in Fig. 2.7. The Hamiltonian acting on each chain is simply the 1D cluster Hamiltonian. For generality we also assume the existence of some uncoupled qubits, each with an associated term $-X$ in the Hamiltonian (i.e. the ground state is $|+\rangle$). In order to treat this 2D model within the same framework which we have developed for 1D systems, we will consider an entire $N_v \times 2$ block (where N_v is the extent in the vertical direction) to be a single ‘site’, as shown in Fig. 2.7(a); hence we can consider the lattice to comprise a 1D chain of such ‘sites’. The unperturbed ground state, which we denote $|\Psi_Q\rangle$, then has an MPS representation which is essentially a tensor product of several copies of the 1D cluster state MPS representation, with a correlation system comprising N qubits. Each chain contributes a separate $Z_2 \times Z_2$ symmetry, so that the model is invariant under a symmetry group $G = (Z_2 \times Z_2)^{\times N} = \{(g_1, \dots, g_N) | g_1, \dots, g_N \in Z_2 \times Z_2\}$. The projective representation of this symmetry in correlation space is the N -qubit generalization of the Pauli representation, namely

$$V((g_1, \dots, g_N)) = V_{\mathcal{P}}(g_1) \otimes \dots \otimes V_{\mathcal{P}}(g_N), \quad (2.71)$$

where $V_{\mathcal{P}}$ is the single-qubit Pauli representation of $Z_2 \times Z_2$, given by Eq. (2.10). It can be checked that this projective representation is maximally noncommutative, and because it has dimension $2^N = \sqrt{|G|}$, it must be the unique irreducible projective representation corresponding to its factor system (by Lemma 1 from Sec. 2.2.3).

Now, it is easy to see that, where \mathcal{S} is the set of gates which can be executed in correlation space by a single-site measurement (up to Pauli byproducts) in the 1D cluster state, we can

execute any tensor product

$$s_1 \otimes s_2 \otimes \cdots \otimes s_N, \quad s_1, \dots, s_N \in \mathcal{S} \quad (2.72)$$

(up to Pauli byproducts) in correlation space by a single-site measurement in our 2D model; we just do the corresponding measurements on each chain separately. We can also find a measurement basis for a columnar site which induces entangling gates between two qubits in correlation space; however, this measurement basis clearly cannot correspond to single-qubit measurements, since the two chains would then remain uncoupled. For reasons that will become clear when we consider the relation of the present model to the 2-D cluster state in Sec. 2.4.2, we will only consider entangling gates between nearest-neighbour qubits in correlation space, for which we construct the measurement basis in a particular way, as follows.

We define the on-site unitary u , which involves applying controlled-Z gates between neighbouring qubits to turn our original resource state $|\Psi_Q\rangle$ into another graph state $|\Psi'_Q\rangle$ in which the two chains of interest are coupled, as shown in Figure 2.8. If $A[\cdot]$ is the MPS tensor for $|\Psi_Q\rangle$ at the given site, then $A'[\cdot] = A[u^\dagger(\cdot)]$ is the MPS tensor for $|\Psi'_Q\rangle$. Using the measurement sequences described in [46], it can be shown that there exists a measurement basis $\{|\alpha\rangle\}$ for a columnar site, corresponding to *single-qubit* measurements, such that $A'[\alpha] = B_\alpha U$, where U is an entangling two-qubit gate, and the B_α are outcome-dependent Pauli byproducts. It follows that this two-qubit gate can be performed in correlation space (up to the same Pauli byproducts) by measuring in the basis $\{u^\dagger|\alpha\rangle\}$.

From the above considerations, we see that the model we are discussing can be considered as a generalization of the 1-D cluster state in which N qubits can be propagated in correlation space, acted on by entangling gates between nearest neighbour qubits as well as single-qubit gates. In the presence of symmetry-respecting perturbations to the Hamiltonian, the arguments of Sections 2.2 and 2.3 can still be applied for any finite N . However, if we want to exploit the locality of the perturbation in the vertical direction as well as the horizontal, we need to make some additional arguments. First, we observe that (by Lemma 1) the protected subsystem of correlation space (which corresponds to the ancilla system appearing in the dual picture of MBQC) will have dimension 2^N , and by identifying the action of the symmetry within the protected subsystem with Eq. (2.71), we can decompose the protected subsystem into N qubits, one associated with each chain.

Our argument now hinges on two observations. First, the dual Hamiltonian of which the dual state is the gapped ground state, as constructed in Sec. 2.3.3, is in fact a sum of interactions that are local on the original two-dimensional lattice. Second, the unitary couplings \mathcal{G}_k appearing in the dual process, which *a priori* could couple an entire columnar site to the entire N -qubit ancilla system, in fact acts trivially outside an appropriately localized area (see Fig. 2.10). These observations both follow from the form of the interaction

$$\mathcal{D}_1 = \sum_i |i\rangle\langle i| \otimes V(g_i) \quad (2.73)$$

between a columnar site and the ancilla system. (Recall that \mathcal{D}_1 and its inverse appeared in the development of the dual picture in Sec. 2.2.4, as well as in the construction of the duality transformation \mathcal{D} from which the dual Hamiltonian \mathcal{H} was obtained in Sec. 2.3.3.) It is easily seen that in the present quasi-1D setup, \mathcal{D}_1 simply comprises a number of applications of the corresponding operator $\mathcal{D}_1^{(1)}$ for the one-dimensional cluster chain (see Fig. 2.9).

Now, consider a quantum circuit comprising a sequence of gates, and let Q_k be the sets of physical (not ancilla) qubits acted on by the corresponding couplings $\tilde{\mathcal{G}}_k$ in the dual process. Just as in the one-dimensional case, we expect that if $R \equiv \min_{k_1, k_2} \text{dist}(Q_{k_1}, Q_{k_2})$ is much larger than the correlation length $\tilde{\xi}$ for the dual state, then the reduced state $\text{Tr}_{(\cup_k Q_k)^c} |\tilde{\Psi}\rangle\langle\tilde{\Psi}|$ on $\cup_k Q_k$ should be approximately a product state over the Q_k 's. Thus, arguing as in the one-dimensional case (see Sec. 2.2.6), we find that performing the measurement sequence on a perturbed resource

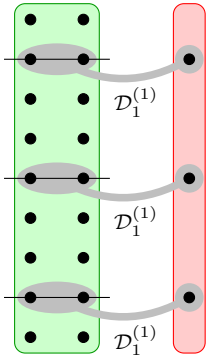


Figure 2.9: The unitary operator \mathcal{D}_1 , which couples a columnar site and the N -qubit ancilla system.

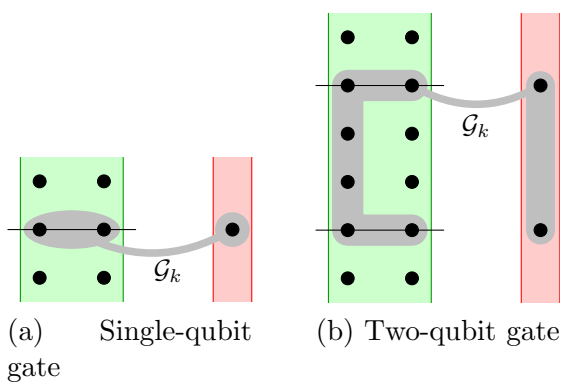


Figure 2.10: The unitary couplings \mathcal{G}_k appearing in the dual process corresponding to single- and two-qubit gates.

state corresponds to a noisy quantum circuit, with the noise described by a noise superoperator \mathcal{E}_k following each gate. Furthermore, \mathcal{E}_k acts non-trivially only on the same qubits that were acted on by the corresponding gate in the original noiseless quantum circuit. The strength of the noise, as given by $\|\mathcal{E}_k - \mathcal{I}\|_\diamond$, is determined by the deviation (in the trace norm) of the reduced density operator on Q_k from its unperturbed value [see Eq. (2.48) in Sec. 2.2.6], which should be small for small perturbations.

Let us now estimate the required scale-up in the size of the resource state. We only consider in detail the case of local quantum circuits (i.e. containing only gates acting between nearest-neighbour qubits). As in the one-dimensional case (Sec. 2.3.4), according to the rigorous factorization result proved in Appendix 2.C, the minimum separation R_{\min} between any of the Q_k 's required for an accuracy ϵ scales like

$$R_{\min}/\tilde{\xi} = \mathcal{O}(m) + \mathcal{O}[\log(1/\epsilon)]. \quad (2.74)$$

The required scale-up can be expressed in terms of R_{\min} , as follows. First, we must ensure that, at each time step, all non-trivial gates are separated by a distance of at least R_{\min} . This leads to a scale-up by a factor of $\sim R_{\min}$ in the number of time steps. Then, the buffering between horizontal locations at which nontrivial gates take place implies another factor of R_{\min} scale-up. Hence, the total scale-up factor is $s \sim R_{\min}^2$. On the other hand, if the quantum circuit that we want to simulate is not already local, then translating it into a local circuit will introduce additional overhead (still scaling at worst polynomially in the number of qubits in the quantum circuit).

2.4.2 The 2D cluster model

Now we will return to the model we are actually interested in: the 2D cluster model on a square lattice. Investigations of the effect on this model of perturbations [23, 47–49] have demonstrated a variety of results depending on the perturbation. Here, we will focus on perturbations respecting an appropriate symmetry. When this symmetry is enforced, the cluster model lies in a robust SPT phase, within which the identity gate is protected and the effective noise model construction of this paper can be applied.

In order to achieve our goal, we will establish an equivalence between the 2D cluster model and a ‘quasi-1D’ model of the type considered in the previous section. The basic idea is to define a (local) duality transformation \mathcal{U} (not the same as the duality transformation \mathcal{D} which we have considered previously) which relates the two models. Specifically, we define

$$\mathcal{U} = \prod_{(i,j) \in L} (CZ)_{ij}, \quad (2.75)$$

where $(CZ)_{ij}$ is the controlled-Z gate acting on qubits i and j , and the product is over an appropriate set L comprising nearest-neighbour pairs of qubits. By an appropriate choice of L , we can ensure that applying \mathcal{U} to the 2D cluster Hamiltonian turns it into a model of the type we considered in the previous section.

Now, let H be a perturbation to the 2-D cluster Hamiltonian. Then $\mathcal{U}H\mathcal{U}^\dagger$ is a perturbation to the quasi-1D model, and the arguments of the preceding section can be applied provided that the perturbation respects the appropriate symmetry. Furthermore, the result (in terms of statistics of measurement outcomes) of performing the adaptive measurement protocol described in Section 2.4.1 on the ground state of $\mathcal{U}H\mathcal{U}^\dagger$, involving measuring the observables \hat{o} , must be the same as the effect of performing the same protocol on the ground state of H , but measuring the observables $\mathcal{U}^\dagger \hat{o} \mathcal{U}$. We will now examine in detail this corresponding measurement protocol for perturbations of the 2-D cluster model.

In the quasi-1D resource state, there is a set of ‘redundant’ qubits which never need to be measured. It turns out to be convenient to assume, however, that we do measure those qubits,

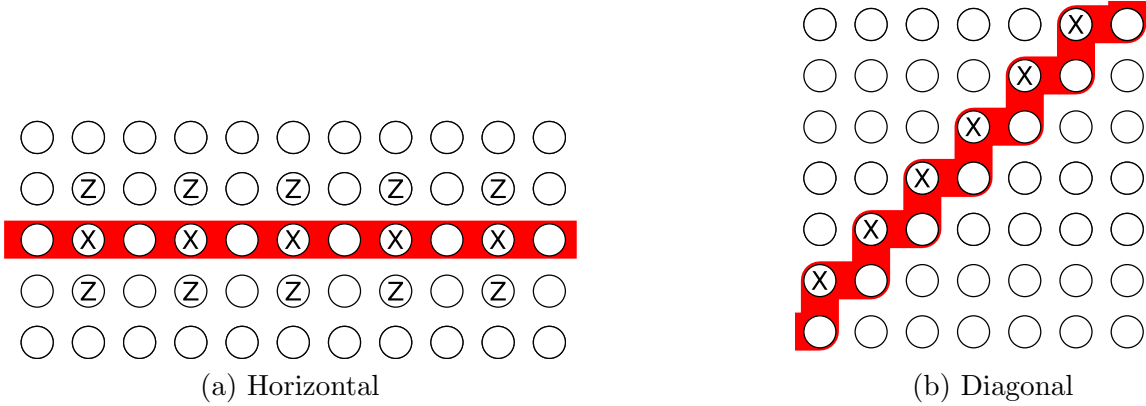


Figure 2.11: Two possible layouts for the 1D chains (red lines) on a 2D lattice. Each layout is associated with a measurement protocol for MBQC on the 2D cluster state, and with a symmetry group (a representative generator of which is shown in each case). For any layout, we can construct an effective noise model corresponding to performing the associated measurement protocol on a perturbed cluster state, provided that the perturbation respects the associated symmetry.

in the z basis, and that we do this before any other measurements. We observe that all the measurements that are performed on the quasi-1D state (after applying controlled- Z gates to couple chains where we want to perform a two-qubit gate) are all single-qubit, and are either in the z basis, i.e. measuring Z , or in the x - y plane, i.e. measuring $\sigma_\theta = (\cos\theta)X + (\sin\theta)Y$ for some angle θ . Hence, the corresponding observables to measure in the 2D cluster state are either of the form Z_j (for some qubit j), or

$$\sigma_{\theta_j}^{(j)} \prod_{k \in \mathcal{N}_j} Z_k \quad (2.76)$$

for some qubit j and angle θ_j , and where \mathcal{N}_j is some set of neighbouring redundant qubits. But, since we measured the redundant qubits first, they are all now in eigenstates of Z . Therefore, labelling the measured values of Z on the redundant qubits by $\{z_j\}$, we see that measuring $\mathcal{U}\hat{\sigma}_j\mathcal{U}^\dagger$ is equivalent to measuring $\sigma_{\theta_j}^{(j)} \left(\prod_{k \in \mathcal{N}_j} z_k \right)$, which in turn is equivalent to measuring $\sigma_{\theta_j}^{(j)}$ and reinterpreting the measurement outcomes based on the value of $\prod_{k \in \mathcal{N}_j} z_k$. Therefore, we have shown that the measurement protocol on the 2D cluster state can be implemented using only single-qubit measurements and adaptivity. It can be checked that the measurement protocol so constructed is essentially the same as the usual one for the 2D cluster state on a square lattice, which is described, e.g. in [2, 46].

Finally, let us discuss the required symmetry. The duality transformation \mathcal{U} can be used to relate the $(Z_2 \times Z_2)^{\times N}$ symmetry which protects the quasi-1D model to a corresponding one in the 2D cluster model. The form of the generators of the latter symmetry is shown in Figure 2.11(a). Let us remark that we can also make similar arguments in the case that the 1D chains are arranged on the 2D square lattice in an unconventional way, for example diagonally [23, 50] as shown in Figure 2.11(b). The advantage of the diagonal layout is that the symmetry [one of the generators of which is depicted in Figure 2.11(b)] takes a particularly simple form, due to the fact that every non-chain qubit neighbours an even number of chain qubits, and so the Z 's that would normally appear on non-chain qubits all cancel. In particular, this symmetry commutes with an especially simple and physically meaningful perturbation, namely a uniform magnetic field in the x direction, i.e. $V = B \sum_i X_i$. (The effect of such a perturbation has been studied

numerically in [49, 51]; the SPT cluster phase persists up to a first-order phase transition at $|B| = 1$.)

2.4.3 Perturbed ground states are universal resources

Let us summarize the conclusions which we obtain from the considerations in Sections 2.4.1 and 2.4.2 by stating them as a theorem. In combination with the threshold theorem of fault-tolerant quantum computation, this theorem will allow us to deduce that sufficiently small symmetry-respecting perturbations to the 2-D cluster Hamiltonian retain ground states which can serve as universal resources for MBQC.

We consider perturbations to the exact cluster Hamiltonian H_C on the 2-D square lattice, which we can take to be a sum of local commuting terms, with an energy gap to the first excited state of 2. Suppose now we consider a perturbed Hamiltonian $H = H_C + V$, and V is a perturbation of the form

$$V = \sum_{u \in \Lambda} V_u, \quad (2.77)$$

where Λ is the set of all lattice sites, and each V_u is an interaction term supported on the set $\mathcal{B}(u, r)$ of sites within some fixed distance r (more generally, interactions decaying exponentially with distance would not present an obstacle to our arguments). We define the local strength of the perturbation by

$$J \equiv \max_u \|V_u\|. \quad (2.78)$$

The cluster Hamiltonian H_C belongs to a class of Hamiltonians for which it has been shown [52] that the gap is stable to local perturbations, i.e. there exists a threshold $\eta > 0$ (depending only on r), such that the gap of the perturbed Hamiltonian is at least 1, provided that $J \leq \eta$.

Let us assume that the perturbation V respects an appropriate symmetry group, constructed according to the procedure described in Sec. 2.4.2 (such as the one of the symmetry groups depicted in Fig. 2.11). Suppose then we want to use the perturbed ground state $|\Psi\rangle$ to simulate a local quantum circuit containing N qubits, T time steps, and m gates, with the gates drawn from the gate set \mathcal{S} comprising single-qubit rotations, a two-qubit entangling gate (as constructed in Sec. 2.4.1), and the non-unitary RESTART gate (which corresponds to the reinitialization of a qubit). We obtain the result

Theorem 4. *Provided $J \leq \eta$, we can find an appropriate measurement protocol on the ground state $|\Psi\rangle$ such that the final reduced state on the output qubits is ϵ -close in the trace norm to the outcome of the quantum circuit, with added noise. In each time step t of the equivalent circuit process, the appropriate gates are applied, followed by a noise process described by a superoperator \mathcal{E}_t . This superoperator can be written as a tensor product $\mathcal{E}_t = \bigotimes_A \mathcal{E}_{t,A}$, where the product is over ‘locations’, i.e. sets of qubits coupled by a gate in the time step t (each qubit not coupled by a gate in the time step t also counts as a location, but $\mathcal{E}_{t,A} = \mathcal{I}$ in that case). Thus, the noise has no correlations in space (other than those due to gates acting between qubits) or time. Furthermore, the noise operator $\mathcal{E}_{t,A}$ at each location and time is close to the identity superoperator in the diamond norm:*

$$\|\mathcal{E}_{t,A} - \mathcal{I}\|_{\diamond} \leq cJ, \quad (2.79)$$

for some constant c (dependent only on r). The number of qubits measured n satisfies

$$n = NT \{ \mathcal{O}(m) + \mathcal{O}[\log(1/\epsilon)] \}. \quad (2.80)$$

Proof. The only aspect that we have not previously discussed is the bound Eq. (2.79). Following the same argument as in the one-dimensional case (Sec. 2.2.6), we find [using the analogue of Eq. (2.48)] that the deviation $\|\mathcal{E}_{t,A} - \mathcal{I}\|_{\diamond}$ is bounded above by $\Delta_X \equiv \|\tilde{\rho}_X - \tilde{\rho}_{X,0}\|_1$, where X is

the set of qubits in the 2-D lattice that affect the operation of the gate in question, and $\tilde{\rho}_X$ and $\tilde{\rho}_{X,0}$ are the reduced states on X of the perturbed dual state $|\tilde{\Psi}\rangle$ and the unperturbed dual state $|\tilde{\Psi}_0\rangle$ respectively. Physically, it should be clear that Δ_X will be small for small perturbations; in Appendix 2.D we demonstrate that, so long as $J \leq \eta$, the inequality $\Delta_X \leq cJ$ holds for some constant c depending only on r . \square

Now that we have shown that perturbations in the Hamiltonian correspond to noisy quantum circuits, we can invoke the threshold theorem of fault-tolerant quantum computation [11–13]. For our purposes, the most suitable version is Theorem 13 of Ref. [11], which we can state as follows:

Theorem 5. *Let us assume a noise model as described in Theorem 4. Then there exists a threshold $\eta' > 0$ and a constant α such that, so long as $\|\mathcal{E}_{t,A} - \mathcal{I}\|_{\diamond} \leq \eta'$ for all A, t , then the following properties hold. For any $\epsilon > 0$, and any local quantum circuit \mathcal{C} made from gates drawn from \mathcal{S} (with N qubits, T time steps, and m gates), there exists another local circuit \mathcal{C}' with gates drawn from \mathcal{S} , such that \mathcal{C}' with noise produces the same result (in terms of the probability distribution for the final readout, and up to an error ϵ) as \mathcal{C} without noise. The scale-up factors for the number of qubits, the number of time steps, and the number of gates are all bounded by $(\text{const.}) \times \log^{\alpha}(m/\epsilon)$.*

Combining Theorems 4 and 5, we obtain:

Theorem 6. Consider the perturbed model $H = H_C + V$ as described above. Then there exists some threshold $\eta'' = \min\{\eta, \eta'/c\} > 0$ (depending only on r) with the following property. Provided that $J < \eta''$, then for any local quantum circuit \mathcal{C} (with N qubits, T time steps, and m gates), with gates drawn from \mathcal{S} , we can find an appropriate measurement protocol on the perturbed ground state $|\Psi\rangle$ such that the result is equivalent (in terms of the probability distribution for the final readout, and up to an error ϵ which can be made arbitrarily small) to the outcome of the original quantum circuit. As $m \rightarrow \infty$ with ϵ held fixed, the number of measured qubits n satisfies

$$n \leq NT \times \mathcal{O}(m \log^{3\alpha} m). \quad (2.81)$$

This is sufficient to show that the perturbed ground states remain universal resources, which is Theorem 3 as stated in Sec. 2.1.1; it is the main result of this paper.

2.5 Conclusion

In this paper, we have developed a framework to characterize the effectiveness of measurement protocols for MBQC with SPT-ordered ground states of quantum spin systems. This has allowed us to prove the universality for MBQC of the ground states of perturbed versions of the 2-D cluster Hamiltonian, provided that the perturbation is sufficiently small and respects an appropriate symmetry.

The type of SPT order that we have presented here is that present in one-dimensional systems, which is related to a nontrivial factor system (also known as a 2-cocycle). It is for this reason that, in order to establish universality in two-dimensional systems, we had to treat them as ‘quasi-one-dimensional’ and assume an extensive symmetry group $(Z_2 \times Z_2)^{\times N}$, which grows with the vertical extent of the system. For standard, non-extensive symmetries in two dimensions, SPT orders can be related to 3-cocycles [53, 54], but it remains to be seen whether similar connections can be drawn between such two-dimensional SPT order and MBQC.

Finally, we note that if MBQC in ground states of quantum spin systems is to be a robust form of quantum computation, then it must be possible in the presence of arbitrary (not necessarily symmetry-respecting) local perturbations to the Hamiltonian, as well as at nonzero temperature. Non-symmetry-respecting perturbations break the symmetry that is essential to our argument; the difficulty of extending our treatment to nonzero temperature was discussed in Sec. 2.3.5. Nor have we considered the effect of non-ideal measurements, or of decoherence of the resource state taking place during the course of the measurement protocol. Therefore, it remains an open question whether fault-tolerant MBQC is possible with such imperfections.

Acknowledgements

We thank I. Schwarz for bringing to our attention the connection between SPT order and the nature of the adaptive measurement protocol to correct for “wrong” outcomes (see Sec. 2.2.4). We acknowledge support from the ARC via the Centre of Excellence in Engineered Quantum Systems (EQuS), project number CE110001013.

2.A The dual finitely correlated state

A pure finitely-correlated state (pFCS) [26, 27] is the thermodynamic limit of the translationally invariant MPS generated by a fixed MPS tensor A . The nature of the correlations can be expressed through the *transfer channel*

$$\mathcal{A}(\sigma) = \sum_k A[k]\sigma A[k]^\dagger \quad (2.82)$$

(here the sum is over some basis $\{|k\rangle\}$ for the site Hilbert space; it can be shown that this definition of \mathcal{A} is independent of the choice of basis). In its canonical form, a pure FCS is further characterized by the following properties:

- (a) \mathcal{A} is unital, i.e. $\mathcal{A}(\mathbb{I}) = \mathbb{I}$.
- (b) There exists a density operator Λ such that $\mathcal{A}^\dagger(\Lambda) = \Lambda$.
- (c) Defining a to be the largest magnitude eigenvalue of \mathcal{A}^\dagger other than the one corresponding to the eigenvector Λ , we have that $|a|$ is strictly less than 1.

The correlation length is then defined by $\xi \equiv -1/\log|a|$, and the eigenvalues of Λ correspond to the entanglement spectrum obtained from a cut in an infinite chain.

Now we restrict ourselves to pFCS generated by tensors A satisfying the decomposition Eq. (2.16). We define the CPTP superoperator \mathcal{V}_g according to $\mathcal{V}_g(\sigma) = [V(g) \otimes \mathbb{I}]\sigma[V(g) \otimes \mathbb{I}]^\dagger$; it can be checked that \mathcal{V}_g is a linear representation (in the space of superoperators) of the symmetry group G , and that it commutes with \mathcal{A}^\dagger for all $g \in G$. Therefore, since Λ is the unique eigenvector of \mathcal{A}^\dagger with eigenvalue 1, it must satisfy $\mathcal{V}_g(\Lambda) = \chi(g)\Lambda$ for some scalars $\chi(g)$. The fact that \mathcal{V}_g is trace preserving ensures that $\chi(g) = 1$. Hence we find that Λ commutes with $V(g) \otimes \mathbb{I}$; it follows by Schur’s Lemma that Λ factorizes as $\Lambda = \Omega \otimes \tilde{\Lambda}$ for some density operator $\tilde{\Lambda}$, and where $\Omega = \mathbb{I}/\sqrt{|G|}$ is the maximally-mixed state on the protected subsystem (recall that the $\sqrt{|G|}$ is the dimension of the protected subsystem). It follows that there is a $\sqrt{|G|}$ -fold degeneracy in the entanglement spectrum throughout the SPT phase, generalizing the 2-fold degeneracy in the $(Z_2 \times Z_2)$ -protected Haldane phase [39].

We can define the transfer channel corresponding to the dual FCS (generated by \tilde{A}) according to

$$\tilde{\mathcal{A}}(\rho) = \sum_k \tilde{A}[k]\rho\tilde{A}[k]^\dagger. \quad (2.83)$$

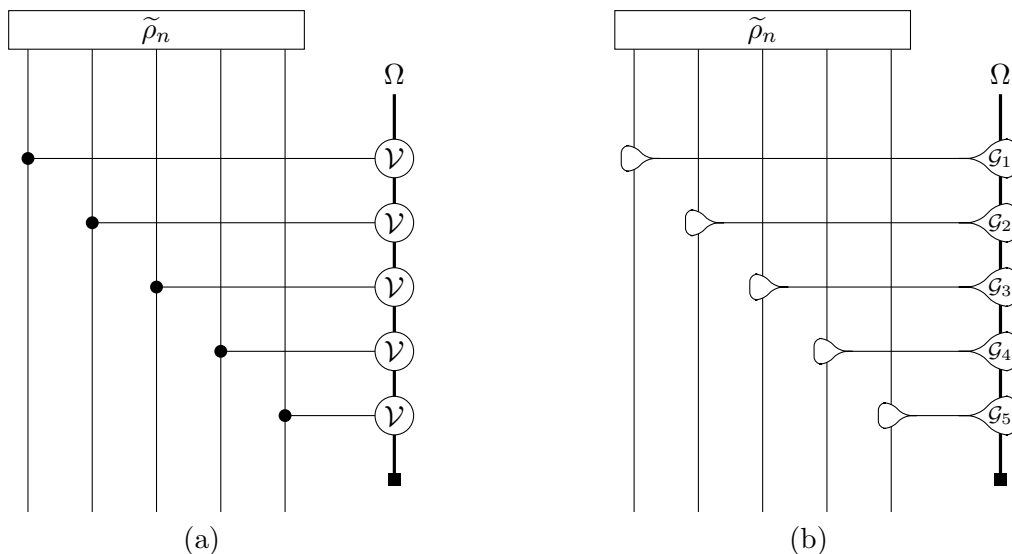


Figure 2.12: (a) The quantum circuit generating the reduced state on n sites for the original FCS. The ancilla particle is initially in the maximally-mixed state Ω , and the symbol \blacksquare indicates that it should be traced out at the end. (b) The adaptive measurement protocol acting on the original FCS is equivalent to a dual process acting on the dual state, comprising a sequence of unitary interactions \mathcal{G}_k coupling the dual state to an ancilla particle.

Observe that \mathcal{A} unital implies that $\tilde{\mathcal{A}}$ is also unital, and that

$$\mathcal{A}^\dagger(\Omega \otimes \sigma) = \Omega \otimes \tilde{\mathcal{A}}^\dagger(\sigma) \quad (2.84)$$

for any operator σ acting on the junk subsystem. Hence, any eigenvalue of $\tilde{\mathcal{A}}$ must also be an eigenvalue of \mathcal{A} . From this we can see that the dual FCS is also a pure FCS, and $\tilde{\Lambda}$ is the unique fixed point of $\tilde{\mathcal{A}}^\dagger$; thus, the entanglement spectrum of the dual state is the same as that of the original state, but with the $\sqrt{|G|}$ -fold degeneracy removed². The respective correlation lengths obey the inequality $\tilde{\xi} \leq \xi$.

We now outline how the arguments of Section 2.2.4 can be put on a rigorous footing within the pFCS formalism. Given everything that we have established so far, it can be shown that the reduced state ρ_n of the original FCS on a block of n adjacent sites can be obtained from the corresponding reduced state $\tilde{\rho}_n$ of the dual FCS according to the quantum circuit shown in Fig. 2.12(a). Assuming that we choose n large enough that all measurements take place within this block of n sites, we can then make arguments similar to those of Sec. 2.2.4, and we find that the result of the adaptive measurement protocol is equivalent to a sequence of interactions between the dual state and an ancilla particle as shown in Fig. 2.12(b), with the unitary interactions \mathcal{G}_k defined as they were previously.

2.B Connection with the Kennedy-Tasaki transformation

The Kennedy-Tasaki (KT) transformation [20] is a non-local unitary transformation which transforms a spin-1 chain in the $Z_2 \times Z_2$ symmetry-protected Haldane phase into a system where the symmetry is spontaneously broken in the bulk. In this section, adapting Ref. [37], we will define a generalized version of the Kennedy-Tasaki transformation, which can be applied to any

²A similar property was found numerically for the Kennedy-Tasaki transformation in [37]. We will discuss the connection between that transformation and our “dual state” in Appendix 2.B.

system in the non-trivial SPT phase with respect to an on-site representation $U(g) = [u(g)]^{\otimes N}$ of the symmetry group $G = Z_2 \times Z_2$ (like the original KT transformation, our generalization is defined for finite chains with open boundary conditions and no terminating particles). We will show that, when the ground state of the original system can be expressed as a pFCS, the ground state of the KT-transformed system is essentially the same as the state which we have referred to throughout this paper as the “dual state”. We expect that for general ground states the situation should be qualitatively similar.

Observe that the symmetry group is generated by two commuting elements x and z ; hence for any $g \in G$, we can write $g = x^{m(g)}z^{n(g)}$ for some $m(g)$ and $n(g)$ taking values of 0 or 1. We will choose to write the unique non-trivial irreducible projective representation as $V(g) = X^{m(g)}Z^{n(g)}$, where X and Z are the appropriate Pauli operators. This is a rephasing of the Pauli representation $V_{\mathcal{P}}$ defined in Eq. (2.10); thus the factor system is different to, but in the same cohomology class as, that of $V_{\mathcal{P}}$. (The construction of the dual state does depend on the specific choice of representative factor system for a cohomology class, although in a fairly trivial way; the present choice is the one that will ensure that the Kennedy-Tasaki transformation reproduces the dual state exactly.)

The generalized Kennedy-Tasaki transformation \mathcal{D}_{KT} is then defined as follows:

$$\mathcal{D}_{KT} = \prod_{k < l} D_{kl}, \quad (2.85)$$

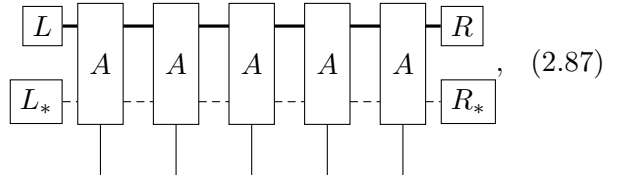
where D_{kl} is a unitary coupling the two sites k and l according to

$$D_{kl} \equiv \sum_i |i\rangle\langle i| \otimes u(x^{m(g_i)}). \quad (2.86)$$

Notice that all the operators D_{kl} appearing in the product Eq. (2.85) commute. In the case that the particles are spin-1’s, with the $Z_2 \times Z_2$ rotation symmetry consisting of π rotations about the x , y , and z particles (the simultaneous eigenbasis of the symmetry for a single site is then $\{|x\rangle, |y\rangle, |z\rangle\}$, where $|\alpha\rangle$, $\alpha = x, y, z$ is the 0 eigenstate of the spin component operator S_α), the transformation \mathcal{D}_{KT} reduces to the standard Kennedy-Tasaki transformation. Specifically, Eqs. (2.85) and (2.86) correspond to Eqs. (4) and (5) in Ref. [37].

For open boundary conditions, we expect there to be an approximate (becoming exact in the limit as the length of the chain goes to infinity) four-fold degeneracy, due to the two-fold degeneracy associated with each edge. An appropriate analogue of the SPT pFCS for this choice of boundary conditions is as follows: the low-energy subspace is spanned by states of the form

$$|\Psi(L, R)\rangle =$$

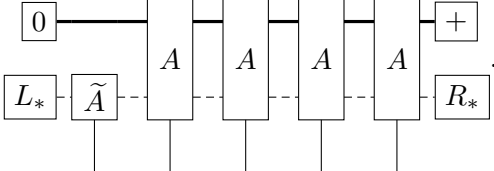


$$, \quad (2.87)$$

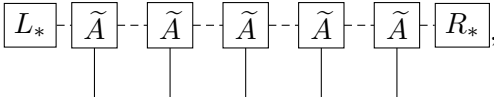
where the MPS tensor A obeys the usual symmetry condition, where $|L_*\rangle$ and $\langle R_*|$ are *fixed* end vectors, while $|L\rangle$ and $\langle R|$ are allowed to vary (within the two-dimensional irrep space) in order to generate the four-dimensional low-energy subspace.

Now we want to examine what happens to a state of the form Eq. (2.87) under \mathcal{D}_{KT} . Suppose we set $\langle R| = \langle +|$, $|L\rangle = |0\rangle$. It is then straightforward to show [using the symmetry condition Eq. (2.12) and the decomposition Eq. (2.15)] that applying all the pairwise interactions D_{1l}

involving the first site gives the result

$$\left(\prod_{l>1} D_{l1} \right) |\Psi(0, +)\rangle =$$

(2.88)

Continuing this process, we find that

$$\mathcal{D}_{KT} |\Psi(0, +)\rangle =$$

(2.89)

which is the dual state. To obtain the other states within the low-energy subspace, it is sufficient to observe that, using the symmetry condition on the tensor A , we have

$$|\Psi(1, +)\rangle = [u(x)]^{\otimes N} |\Psi(0, +)\rangle, \quad (2.90)$$

$$|\Psi(0, -)\rangle = [u(z)]^{\otimes N} |\Psi(0, +)\rangle, \quad (2.91)$$

$$|\Psi(1, -)\rangle = [u(xz)]^{\otimes N} |\Psi(0, +)\rangle, \quad (2.92)$$

and that \mathcal{D}_{KT} commutes with $u(g)$ for all $g \in Z_2 \times Z_2$. Hence, the other basis states for the Kennedy-Tasaki transformed system can be obtained from the dual state by application of a symmetry operation $[u(g)]^{\otimes N}$ for some $g \in G$. The dual state thus represents one of the four distinct symmetry-breaking states within the low-energy subspace of the transformed system.

Let us also note that, although they relate to different boundary conditions, the unitary transformation \mathcal{D} that we introduced in Sec. 2.3.2 is equivalent to the KT transformation \mathcal{D}_{KT} , in the following sense. Indeed, an important property of \mathcal{D}_{KT} is that for any local symmetry-respecting observable \hat{o} , $\mathcal{D}_{KT} \hat{o} \mathcal{D}_{KT}^\dagger$ remains local and symmetry-respecting. It turns out that this also holds for \mathcal{D} , in the case of observables \hat{o} acting in the bulk (the fact that $\mathcal{D} \hat{o} \mathcal{D}^\dagger$ is local was established in Sec. 2.3.3; it can be shown that $\mathcal{D} \hat{o} \mathcal{D}^\dagger$ still respects the on-site symmetry as well). Thus, like \mathcal{D}_{KT} , the transformation \mathcal{D} can be applied to yield a local, symmetry-respecting Hamiltonian \tilde{H} in the bulk. It can be shown \tilde{H} is precisely the KT transformed Hamiltonian [55]. Thus, in line with the results for pFCS described above, we expect \tilde{H} to have four degenerate, locally distinguishable symmetry-breaking ground states in the bulk. On the other hand, when the transformation \mathcal{D} is applied to the edge interactions (those which couple the ends of the chain to the terminating particles), the result need not respect the symmetry. Therefore, including the edge interactions favours one of the symmetry-breaking ground states over the others and leads to the non-degeneracy of the dual state $|\tilde{\Psi}\rangle$.

2.C The factorization condition for ground states of a local Hamiltonian

In this Appendix, we will show how to derive the approximate factorization result Eq. (2.68) for a non-degenerate gapped ground state $|\Psi\rangle$ of a local Hamiltonian. Our main tool is the existing theorem on the exponential decay of correlation functions for such a ground state [56–58]. This theorem states that there exists a correlation length ξ and a function $f(x)$, with

$f(x) = \mathcal{O}[\exp(-x/\xi)]$ as $x \rightarrow \infty$, such that for any sets of lattice sites X and Y , and observables A_X and B_Y supported on X and Y respectively, we have

$$\frac{|\langle A_X B_Y \rangle_\Psi - \langle A_X \rangle_\Psi \langle B_Y \rangle_\Psi|}{\|A_X\| \|B_Y\|} \leq f(\text{dist}(X, Y)) \min\{|X|, |Y|\}, \quad (2.93)$$

where $\langle \cdot \rangle_\Psi$ denotes the expectation value of an observable with respect to $|\Psi\rangle$, and $|X|$ denotes the number of sites contained in the set X .

Suppose that instead of two sets of lattice sites, we have m disjoint sets X_1, \dots, X_m . Let $N = \sum_{k=1}^m |X_k|$ be the total number of lattice sites contained within all of the X_k 's. We can obtain the following corollary:

Lemma 3. *For any observables A_{X_1}, \dots, A_{X_m} supported on the respective sets,*

$$\frac{|\langle A_{X_1} \cdots A_{X_m} \rangle_\Psi - \langle A_{X_1} \rangle_\Psi \cdots \langle A_{X_m} \rangle_\Psi|}{\|A_{X_1}\| \cdots \|A_{X_m}\|} \leq f(R)N, \quad (2.94)$$

where R is the smallest distance between any two of the X_k 's, i.e. $R \equiv \min_{k \neq l} \text{dist}(X_k, X_l)$.

Proof. Without loss of generality, we can assume that $\|A_{X_k}\| = 1$ for all k . Observing that $|\langle A_{X_1} \rangle| \leq \|A_{X_1}\| = 1$, we have that

$$\begin{aligned} & |\langle A_{X_1} \cdots A_{X_m} \rangle_\Psi - \langle A_{X_1} \rangle_\Psi \cdots \langle A_{X_m} \rangle_\Psi| \\ & \leq |\langle A_{X_1} \cdots A_{X_m} \rangle_\Psi - \langle A_{X_1} \rangle_\Psi \langle A_{X_2} \cdots A_{X_m} \rangle_\Psi| \\ & \quad + |\langle A_{X_2} \cdots A_{X_m} \rangle_\Psi - \langle A_{X_2} \rangle_\Psi \cdots \langle A_{X_m} \rangle_\Psi|. \end{aligned} \quad (2.95)$$

The two-body result implies that the first term in the right-hand side of Eq. (2.95) is bounded by $|X_1|f(R)$. Continuing in this way, the lemma follows by induction. \square

Now we want to show that the reduced state on the union of the X_k 's is close to a product state. To do this, we make use of the following lemma. We remind the reader that here we make use of both the trace norm $\|A\|_1 = \text{Tr} \sqrt{A^\dagger A}$ and the spectral norm $\|A\| = \max_{\langle \psi | \psi \rangle = 1} \|A|\psi\rangle\|$. They are both distinct from the norm induced by the Hilbert-Schmidt inner product.

Lemma 4. *Consider a vector space of dimension D . Then there exists a basis $\{E_i | i = 1, \dots, D^2\}$ for the space of linear operators supported on the site, orthonormal with respect to the Hilbert-Schmidt inner product $\langle A, B \rangle = \text{Tr}(A^\dagger B)$, and comprising Hermitian operators E_i such that such that $\|E_i\|_1 \|E_i\| = 1$.*

Proof. Given a basis $\{|m\rangle\}$, $m = 1, \dots, d$, such a set can be constructed comprising the operators $|m\rangle\langle m|$, and $(1/\sqrt{2})(|m\rangle\langle n| + |n\rangle\langle m|)$ and $(i/\sqrt{2})(|m\rangle\langle n| - |n\rangle\langle m|)$ for $m \neq n$. \square

Now we are ready to prove the main result.

Theorem 7. *Let $|\Psi\rangle$ be the non-degenerate gapped ground state of a local Hamiltonian. Let ρ be the reduced state of $|\Psi\rangle$ on $\bigcup_{k=1}^m X_k$, and let ρ_k be the reduced state on X_k . Then*

$$\|\rho - \rho_{\text{prod}}\|_1 \leq Nf(R)d^{2N}, \quad (2.96)$$

where $\rho_{\text{prod}} = \bigotimes_k \rho_k$.

Proof. Recall that for a linear operator P , $\|P\|_1 = \max_{\|A\|=1} |\text{Tr}(AP)|$, where the maximization is over all linear operators A with unit spectral norm. Now, we can expand

$$A = \sum_{i_1, \dots, i_m} \alpha_{i_1, \dots, i_m} E_{i_1} \otimes \dots \otimes E_{i_m}, \quad (2.97)$$

where the tensor product is over the sets X_1, \dots, X_m ; the E_i are as constructed in Lemma 4; and the scalars α_{i_1, \dots, i_m} are determined by $\alpha_{i_1, \dots, i_m} = \text{Tr}[A^\dagger(E_{i_1} \otimes \dots \otimes E_{i_m})]$, which implies (since $\|A\| = 1$) that

$$|\alpha_{i_1, \dots, i_m}| \leq \|E_{i_1}\|_1 \dots \|E_{i_m}\|_1. \quad (2.98)$$

Now, notice that

$$\text{Tr}(A\rho) = \sum_{i_1, \dots, i_m} \alpha_{i_1, \dots, i_m} \langle E_{i_1} \otimes \dots \otimes E_{i_m} \rangle_\rho, \quad (2.99)$$

$$\text{Tr}(A\rho_{\text{prod}}) = \sum_{i_1, \dots, i_m} \alpha_{i_1, \dots, i_m} \langle E_{i_1} \rangle_{\rho_1} \dots \langle E_{i_m} \rangle_{\rho_m}. \quad (2.100)$$

Hence, by Lemma 3, we find that

$$|\text{Tr}[A(\rho - \rho_{\text{prod}})]| \quad (2.101)$$

$$\leq Nf(R) \sum_{i_1, \dots, i_m} |\alpha_{i_1, \dots, i_m}| \|E_{i_1}\| \dots \|E_{i_m}\| \quad (2.102)$$

$$\leq Nf(R) \sum_{i_1, \dots, i_m} (\|E_{i_1}\|_1 \dots \|E_{i_m}\|_1) (\|E_{i_1}\| \dots \|E_{i_m}\|) \quad (2.103)$$

$$= Nf(R)d^{2N}, \quad (2.104)$$

since $\|E_k\|_1 \|E_k\| = 1$ for all k . \square

2.D Local perturbations perturb continuously

Physically, it should be clear that small perturbations in a gapped local Hamiltonian lead to small variations in the reduced state obtained from the ground state on a finite region of the lattice. Here we will give a rigorous proof of this fact, as follows:

Theorem 8. *Let $H(s)$ be a differentiable path of Hamiltonians of the form*

$$H(s) = \sum_{u \in \Lambda} H_u(s), \quad (2.105)$$

where the sum is over all the lattice sites u in a finite-dimensional lattice Λ , and $H_u(s)$ is supported on the set $\mathcal{B}(u, r)$ of sites within some fixed distance r of u . Suppose that for $0 \leq s \leq S$, the Hamiltonian $H(s)$ has a unique ground state $|\Psi(s)\rangle$, and there is a uniform lower bound $\gamma > 0$ on the gap. Then there exists a constant c (dependent only on the lattice geometry, on r , and on γ) such that for any set X of lattice sites, we have

$$\|\rho_X(S) - \rho_X(0)\|_1 \leq c|X|J'S, \quad (2.106)$$

where $\rho_X(s)$ is the reduced state on X , i.e. $\rho_X(s) = \text{Tr}_{X^c} |\Psi(s)\rangle \langle \Psi(s)|$, and $J' \equiv \max_{u \in \Lambda, s \in [0, S]} \|\partial_s H_u(s)\|$.

Proof. The proof relies on the following consequence of the theory of quasiadiabatic continuation [59, 60]: under the given assumptions, there exists a family of Hamiltonians $\mathcal{H}(s)$ such that

$$i \frac{d}{ds} |\Psi(s)\rangle = \mathcal{H}(s) |\Psi(s)\rangle, \quad (2.107)$$

where $\mathcal{H}(s)$ can be written as

$$\mathcal{H}(s) = \sum_{u \in \Lambda} \mathcal{H}_u(s), \quad (2.108)$$

such that for any site u , $\mathcal{H}_u(s)$ can be approximated by an observable $\overline{\mathcal{H}_u}(s)$ supported on X^c (the complement of X), with error

$$\|\overline{\mathcal{H}_u}(s) - \mathcal{H}_u(s)\| \leq J' f \left(\frac{\text{dist}(u, X)}{\gamma} \right) \quad (2.109)$$

where f is a rapidly decaying function (dependent only on r).

Now, for $s \in [0, S]$, we have (where $\rho(s) \equiv |\Psi(s)\rangle\langle\Psi(s)|$)

$$\partial_s \rho_X(s) = i \text{Tr}_{X^c}[\mathcal{H}(s), \rho(s)]. \quad (2.110)$$

Hence

$$\|\partial_s \rho_X(s)\|_1 = \max_{\|A_X\|} |\text{Tr}(A_X \text{Tr}_{X^c}[\mathcal{H}(s), \rho(s)])| \quad (2.111)$$

$$= \max_{\|A_X\|=1} |\text{Tr}(A_X[\mathcal{H}(s), \rho(s)])| \quad (2.112)$$

$$= \max_{\|A_X\|=1} |\text{Tr}([A_X, \mathcal{H}(s)]\rho(s))| \quad (2.113)$$

$$\leq \max_{\|A_X\|=1} \|[A_X, \mathcal{H}(s)]\|. \quad (2.114)$$

$$\leq \max_{\|A_X\|=1} \sum_{u \in \Lambda} \|[A_X, \mathcal{H}_u(s)]\| \quad (2.115)$$

Here the maximization is over all linear operators A_X supported on X with unit spectral norm. We have made use of the fact that for any linear operator P , $\|P\|_1 = \max_{\|A\|=1} |\text{Tr}(AP)|$, with $\|A\|$ the spectral norm.

Now, for any operator A_X supported on X , we have that $[A_X, \overline{\mathcal{H}_u}(s)] = 0$ since A_X and $\overline{\mathcal{H}_u}(s)$ are supported on disjoint subsets. Hence, using Eq. (2.109) (and the fact that $\|A_X\| = 1$), we find that

$$\|[A_X, \mathcal{H}_u(s)]\| \leq 2\|A_X\| \|\overline{\mathcal{H}_u}(s) - \mathcal{H}_u(s)\| \quad (2.116)$$

$$\leq 2J' f \left(\frac{\text{dist}(u, X)}{\gamma} \right), \quad (2.117)$$

so that

$$\|\partial_s \rho_X(s)\|_1 \leq 2J' \sum_{u \in \Lambda} f \left(\frac{\min_{x \in X} \text{dist}(u, x)}{\gamma} \right). \quad (2.118)$$

We can bound the sum according to

$$\sum_{u \in \Lambda} f \left(\frac{\text{dist}(u, X)}{\gamma} \right) \quad (2.119)$$

$$= \sum_{u \in \Lambda} f \left(\frac{\min_{x \in X} \text{dist}(u, x)}{\gamma} \right) \quad (2.120)$$

$$\leq \sum_{u \in \Lambda} \sum_{x \in X} f \left(\frac{\text{dist}(u, x)}{\gamma} \right) \quad (2.121)$$

$$= \sum_{x \in X} \sum_{u \in \Lambda} f \left(\frac{\text{dist}(u, x)}{\gamma} \right) \quad (2.122)$$

$$\leq |X| \max_{x \in X} \sum_{u \in \Lambda} f \left(\frac{\text{dist}(u, x)}{\gamma} \right) \quad (2.123)$$

$$\leq |X|c/2, \quad (2.124)$$

where in the last step the rapid decay of f ensures that the sum is bounded by a constant $c/2$ dependent on the lattice geometry, γ , and r .

Hence, by the triangle inequality for the trace norm, we have

$$\|\rho_X(S) - \rho_X(0)\|_1 \leq \int_0^S \|\partial_s \rho_X(s)\|_1 ds \quad (2.125)$$

$$\leq c|X|J'S, \quad (2.126)$$

as required. □

Chapter 3

The hidden symmetry-breaking picture of symmetry-protected topological order

D.V. Else, S.D. Bartlett and A.C. Doherty, Phys. Rev. B (2013) **88** 085114.

Abstract. We generalize the hidden symmetry-breaking picture of symmetry-protected topological (SPT) order developed by Kennedy and Tasaki in the context of the Haldane phase. Our generalization applies to a wide class of SPT phases in one-dimensional spin chains, protected by an on-site representation of a finite abelian group. This generalization takes the form of a non-local unitary map that relates local symmetry-respecting Hamiltonians in an SPT phase to local Hamiltonians in a symmetry-broken phase. Using this unitary, we establish a relation between the two-point correlation functions that characterize fully symmetry-broken phases with the string-order correlation functions that characterise the SPT phases, therefore establishing the perspective in these systems that SPT phases are characterised by *hidden* symmetry-breaking. Our generalization is also applied to systems with continuous symmetries, including $SO(2k + 1)$ and $SU(k)$.

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3.1 Introduction

In the traditional Landau paradigm, order in condensed matter systems is viewed as synonymous with the spontaneous breaking of a symmetry. However, it is now well-established that at zero temperature there exist topologically ordered phases, such as those of the fractional quantum Hall effect, which cannot be understood through the Landau paradigm. Additionally, one can also consider *symmetry-protected topological (SPT)* phases [61], which are not topologically ordered in the conventional sense, yet remain distinct from the trivial phase in the presence of an appropriate symmetry. A well-known example of an SPT phase is the Haldane phase of antiferromagnetic spin-1 chains, which is protected by the $D_2 \cong Z_2 \times Z_2$ symmetry comprising π rotations about a set of orthogonal axes. The “topological” nature of the Haldane phase is manifested in a number of ways, such as the long-range string order [62], fractionalized edge modes [43], degenerate entanglement spectrum [39], and long-range localizable entanglement [63, 64]. It is now known that many of the interesting properties of the Haldane phase extend in general to SPT phases of one-dimensional spin chains protected by a unitary “on-site” representation of an arbitrary symmetry group G (for which the Haldane phase, with $G = Z_2 \times Z_2$, is the simplest non-trivial example), which have been fully classified [16, 17].

An early and influential characterization of the Haldane phase was provided by Kennedy and Tasaki [20] (see also Ref. [19]). They constructed a non-local unitary [which we refer to

as the *Kennedy-Tasaki (KT) transformation*] to transform the Haldane phase to a conventional symmetry-breaking phase. Although the transformation is non-local, for any local Hamiltonian H that respects the D_2 rotation symmetry, the KT transformation yields another D_2 -symmetric local Hamiltonian \tilde{H} . It turns out that if H is in the SPT Haldane phase, then \tilde{H} will have a space of four degenerate locally-distinguishable ground states corresponding to the spontaneous breaking of the D_2 rotation symmetry, i.e. \tilde{H} is in the maximal symmetry-breaking phase for the D_2 symmetry. Thus, the ordering in the SPT phase is interpreted as “hidden symmetry-breaking”. Furthermore, the long-ranged string order in the Haldane phase is related by the KT transformation to conventional long-ranged order in the symmetry breaking phase.

The Haldane phase is also a special case of the SPT phases that were classified in Refs. [16, 17] through the symmetry properties of a matrix-product state ansatz for the ground state. The general result is that the distinct SPT phases with respect to an on-site representation of a symmetry group G are classified by the second cohomology classes of the projective representations of the symmetry group. This classification appears to be the most fundamental and general approach to SPT phases for one-dimensional chains with an on-site symmetry. However, the connection with the original hidden symmetry-breaking picture of Kennedy and Tasaki has not been explored. As a result, it remains unclear to what extent other SPT phases can also be understood to arise from a similar hidden symmetry-breaking mechanism (although see Ref. [1] for one example).

In this paper, we extend the hidden-symmetry breaking picture to any SPT phase protected in a one-dimensional spin chain by an on-site representation of a finite abelian group, provided that the cohomology class describing the phase satisfies a condition called *maximal non-commutativity*. (We say that a cohomology class is maximally non-commutative if, in the corresponding projective representations, for the matrix representation of any non-trivial group element there exists at least one other matrix in the representation with which it does not commute [25].) We achieve this by constructing a suitable generalization of the KT transformation (presented in Sec. 3.4), expressed explicitly in terms of the appropriate cohomology class of the symmetry group, to transform the SPT phase into a conventional symmetry-breaking phase. The generalized KT transformation is essentially equivalent to the duality transformation introduced by us in the context of quantum computation [65], and some of its properties were already discussed in the appendices of that paper; however, our treatment here will be self-contained.

Where it can be applied, our generalized KT transformation affords a different perspective on properties of the SPT phase. For example, for abelian symmetry groups, SPT phases (and indeed, all symmetric phases) can be identified from a pattern of string order (as we will show, based in part on the results of Ref. [66]). For an SPT phase corresponding to a maximally non-commutative cohomology class, this pattern of string order can be understood in a natural way through the generalized KT transformation, which relates it to the long-range order characterizing the symmetry-breaking phase (just as the original KT transformation does in the case of the Haldane phase). We explore this perspective in Sec. 3.6. We remark that, although we are only able to consider finite abelian symmetry groups, these groups can arise as subgroups for systems with a larger symmetry. For SPT phases in systems that have a $SO(2k + 1)$ or $SU(k)$ symmetry, in Sec. 3.7 we will exhibit an appropriate finite abelian subgroup that allows the hidden symmetry breaking to be identified.

We note that a closely related investigation, Ref. [67], appeared shortly after our work and contains similar results to ours. Where relevant, we will remark on some of the similarities and differences between the two works. In particular, our generalized KT transformation coincides with that of Ref. [67] for the specific case of a $Z_N \times Z_N$ symmetry group.

3.2 The Kennedy-Tasaki transformation

Let us recall the definition of the unitary \mathcal{D}_{KT} that effects the Kennedy-Tasaki transformation for a chain of N spin-1's with open boundary conditions. It can be written as [68]

$$\mathcal{D}_{KT} = \prod_{j < k} \exp(i\pi S_j^z S_k^x), \quad (3.1)$$

where S_j^a ($a = x, y, z$) denotes the appropriate spin component operator for the j -th spin. This unitary is non-local, but, for any local observable A that respects the D_2 symmetry operations $\prod_j \exp(i\pi S_j^a)$ ($a = x, y, z$), the transformed observable $\mathcal{D}_{KT} A \mathcal{D}_{KT}^\dagger$ remains local and symmetry-respecting. Therefore, for any Hamiltonian H that is the sum of local symmetry-respecting interactions, one can generate the dual Hamiltonian $\tilde{H} = \mathcal{D}_{KT} H \mathcal{D}_{KT}^\dagger$. If H is in the SPT phase with respect to the D_2 symmetry, then \tilde{H} is expected to be in a conventional symmetry-breaking phase with respect to the symmetry, with the four-fold degenerate edge states mapping under \mathcal{D}_{KT} to the four locally distinguishable symmetry-breaking ground states.

In order to see that \mathcal{D}_{KT} is a special case of the generalized KT transformation to be defined later, we will want to express \mathcal{D}_{KT} in terms of the single-site basis $\{|x\rangle, |y\rangle, |z\rangle\}$ (where $|a\rangle$ is the zero eigenstate of S^a for $a = x, y, z$), that is the simultaneous eigenbasis of the on-site representation of the symmetry. Observe that

$$\exp(i\pi S_z \otimes S_x)(|a_1\rangle \otimes |a_2\rangle) = (-1)^{\mu(a_1)\nu(a_2)} |a_1\rangle \otimes |a_2\rangle, \quad (3.2)$$

where $\mu(a) = 1 - \delta_{a,z}$ and $\nu(a) = 1 - \delta_{a,x}$. Hence we can write

$$\mathcal{D}_{KT} = \sum_{a_1, \dots, a_N} (-1)^{\sum_{j < k} \mu(a_j)\nu(a_k)} |a_1, \dots, a_N\rangle \langle a_1, \dots, a_N|. \quad (3.3)$$

3.3 Classification of SPT phases by cohomology classes

Throughout this paper, we will assume a chain of N spins, such that the Hamiltonian commutes with the on-site representation $[u(g)]^{\otimes N}$ of a symmetry group G . We will assume open boundary conditions unless otherwise stated. According to the general classification of SPT phases for on-site symmetries in one-dimensional systems [16, 17], the different SPT phases for this symmetry can be classified by the second cohomology group $H^2(G, U(1))$, which is related to the projective representations $V(g)$ for the group G , as we now describe. (One interpretation of these projective representations is that they describe the action of the symmetry on the fractionalized edge mode associated with each edge for open boundary conditions; we note that there are several subtleties with the formal treatment of such edge modes, and we refer the reader to Sec. 3.1 of Ref. [65] for a careful discussion.) By definition, a projective representation $V(g)$ must satisfy

$$V(g_1)V(g_2) = \omega(g_1, g_2)V(g_1g_2), \quad \forall g_1, g_2 \in G, \quad (3.4)$$

where ω is a function mapping pairs of group elements to complex phase factors, known as the *factor system* of the projective representation. The associativity of matrix multiplication implies that the factor system must satisfy the 2-cocycle condition

$$\omega(g_1, g_2)\omega(g_1g_2, g_3) = \omega(g_2, g_3)\omega(g_1, g_2g_3), \quad \forall g_1, g_2, g_3 \in G. \quad (3.5)$$

Conversely, any ω satisfying Eq. (3.5) is the factor system for some projective representation [69]. Furthermore, given any projective representation, it is trivial to generate another one by rephasing of the operators $V(g)$, i.e. $V(g) \rightarrow \beta(g)V(g)$, where β is a function that sends group elements to phase factors. The effect on the factor system is

$$\omega(g_1, g_2) \rightarrow \beta(g_1g_2)^{-1}\beta(g_1)\beta(g_2)\omega(g_1, g_2). \quad (3.6)$$

Two factor systems related by a transformation of the form Eq. (3.6) are said to be in the same *cohomology class*, and the second cohomology group $H^2(G, \mathbb{U}(1))$ comprises all the distinct cohomology classes for the group G . We will denote by $[\omega]$ the cohomology class containing a given factor system ω .

In the case of the Haldane phase for spin-1 chains, the relevant symmetry group is $D_2 = \{1, x, y, z\}$ (where $y = xz$), with the on-site representation $u(a) = \exp(i\pi S_a)$ (for $a = x, y, z$). The Haldane phase corresponds to the unique non-trivial cohomology class for the symmetry group D_2 . We can specify a representative factor system ω for this cohomology class by giving an example of a projective representation for which ω is the factor system, namely

$$V(1) = \mathbb{I}, \quad V(x) = \sigma_x, \quad V(z) = \sigma_z, \quad V(y) = \sigma_x \sigma_z, \quad (3.7)$$

where σ_x and σ_z are the respective Pauli spin matrices. We define $V(y)$ as above, rather than the more symmetrical $V(y) = \sigma_y$ (which would correspond to a different factor system within the same cohomology class), because the factor system of Eq. (3.7) will turn out to be closely connected to the conventional formulation of the KT transformation.

3.4 The generalized Kennedy-Tasaki transformation

In this section, we will define our generalized KT transformation, for an SPT phase characterized by a cohomology class $[\omega]$ and an on-site symmetry representation of a group G . We will require that G be finite and abelian, and that the cohomology class $[\omega]$ be maximally non-commutative (to be defined below). A special property of an abelian symmetry is that the irreps are one-dimensional; therefore, the on-site representation $u(g)$ must decompose as

$$u(g) = \bigoplus_{\chi} \chi(g) \mathbb{I}_{m_{\chi}} \quad (3.8)$$

where the sum is over the one-dimensional representations (characters) χ of G . For simplicity of presentation, we assume that none of the multiplicities m_{χ} are greater than 1; thus, we can write

$$u(g) = \sum_{\chi} \chi(g) |\chi\rangle \langle \chi| \quad (3.9)$$

where the $\{|\chi\rangle\}$ form an orthonormal basis, and the sum is over those χ such that $m_{\chi} > 0$. However, all the results of this paper can easily be generalized to the case of multiplicities greater than 1. For the Haldane phase, this basis $\{|\chi\rangle\}$ is the basis $\{|x\rangle, |y\rangle, |z\rangle\}$ discussed in Sec. 3.2.

Our generalized construction applies for any SPT phase with respect to the aforementioned symmetry, so long as the corresponding cohomology class $[\omega]$ is *maximally-noncommutative*, which is to say that the subgroup $G(\omega) = \{g \in G : V_{\omega}(g)V_{\omega}(h) = V_{\omega}(h)V_{\omega}(g) \forall h \in G\}$ is trivial. This property does not depend on the choice of representative factor system for the cohomology class. (Throughout this section, we will use V_{ω} to denote some projective representation of G with factor system ω ; it does not matter how the projective representation is chosen because we only use the multiplicative relations between the matrices $V_{\omega}(g)$, and these are determined by ω .) As follows from Refs. [69, 70], a finite abelian group G will have at least one maximally non-commutative factor system if and only if it is of ‘‘symmetric type’’, i.e. $G \cong H \times H$ for some group H . Of course, even if the full symmetry group is not of this form, then it might still have a subgroup of symmetric type, for which our method could be applied.

An important property of a maximally non-commutative factor system is the following. Any cohomology class for an abelian group can be considered to induce a homomorphism φ_{ω} from G to G^* (where G^* is the *character group* of G , i.e. the group of one-dimensional projective representations of G under multiplication), according to

$$\varphi_{\omega}(g) = \chi_g^{\omega}, \quad (3.10)$$

where χ_g^ω is the one-dimensional representation of G such that

$$V_\omega(g')V_\omega(g)V_\omega(g')^\dagger = \chi_g^\omega(g')V_\omega(g). \quad (3.11)$$

Observe that from Eq. (3.11) one can prove both that $\chi_g^\omega(g'_1)\chi_g^\omega(g'_2) = \chi_g^\omega(g'_1g'_2)$ (i.e. $\chi_g^\omega = \varphi_\omega(g)$ is in G^*) and that $\chi_{g_1}^\omega\chi_{g_2}^\omega = \chi_{g_1g_2}^\omega$ (i.e. φ_ω is a homomorphism). For the particular case of a maximally non-commutative projective representation of a finite abelian group, the kernel of φ_ω [which is equal to $G(\omega)$ in general] is trivial, and therefore φ_ω is an *isomorphism*; that is, for any $\chi \in G^*$ there is a unique $g \in G$ such that $\chi_g^\omega = \chi$.

We construct the unitary \mathcal{D}_ω corresponding to the generalized Kennedy-Tasaki transformation (as we will see later, it maps *from* the maximal symmetry-breaking phase into the SPT phase), acting on a chain of N sites with open boundary conditions, according to

$$\mathcal{D}_\omega = \sum_{\mathbf{x}} \Omega_\omega(\varphi_\omega^{-1}(\mathbf{x})) |\mathbf{x}\rangle \langle \mathbf{x}|, \quad (3.12)$$

where we use the abbreviations $\mathbf{x} = (\chi_1, \dots, \chi_N)$, $\varphi_\omega^{-1}(\mathbf{x}) = (\varphi_\omega^{-1}(\chi_1), \dots, \varphi_\omega^{-1}(\chi_N))$, and $\Omega_\omega(\mathbf{g})$ is the phase factor defined such that

$$V_\omega(g_N) \cdots V_\omega(g_1) = \Omega_\omega(\mathbf{g}) V_\omega(g_N \cdots g_1). \quad (3.13)$$

(here, as in Eq. (3.8), the sum is over the characters χ that appear in the representation.)

For the case of a spin-1 chain with D_2 symmetry, one can check directly that the choice of factor system ω defined by the projective representation Eq. (3.7) gives

$$V(a_N) \cdots V(a_1) = (-1)^{\sum_{j < k} \mu(a_j) \nu(a_k)} V(a_N \cdots a_1), \quad (3.14)$$

(where, loosely, one obtains a phase factor of -1 for every $V(z) = \sigma_z$ operator to the left of a $V(x) = \sigma_x$ operator). Therefore, Eq. (3.12) reduces to the standard Kennedy-Tasaki transformation Eq. (3.3) if we choose this factor system. Note that the definition of \mathcal{D}_ω is not the same for different factor systems ω within the same cohomology class. However, the difference is not very significant; see Appendix 3.A.

Due to the way the unitary \mathcal{D}_ω is defined, we can immediately derive the basic property that, although it is a non-local transformation, for any *symmetry-respecting* observable A supported on a block of n sites, the transformed observable $\mathcal{D}A\mathcal{D}^\dagger$ is still supported on the same block. We will use the notation $\mathbf{x} = (\mathbf{x}_l, \mathbf{x}_b, \mathbf{x}_r)$, corresponding to grouping the sites in the chain according to whether they are, respectively, to the left of, within, or to the right of the block containing the support of A . Thus the matrix element $\langle \mathbf{x}_l, \mathbf{x}_b, \mathbf{x}_r | A | \xi_l, \xi_b, \xi_r \rangle$ can be nonzero only if $\mathbf{x}_l = \xi_l$, $\mathbf{x}_r = \xi_r$, and $\prod_{j=1}^n \chi_{b,j} = \prod_{j=1}^n \xi_{b,j}$ (the last condition comes from the assumption that A commutes with the symmetry). As a result, it is easy to show from the definition of Ω [Eq. (3.13)] that $\Omega_\omega(\varphi_\omega^{-1}(\mathbf{x})) \Omega_\omega(\varphi_\omega^{-1}(\xi))^{-1} = \Omega_\omega^{(n)}(\varphi_\omega^{-1}(\mathbf{x}_b)) \Omega_\omega^{(n)}(\varphi_\omega^{-1}(\xi_b))^{-1}$, and hence that $\mathcal{D}_\omega A \mathcal{D}_\omega^\dagger = \mathcal{D}_\omega^{(n)} A \mathcal{D}_\omega^{(n)\dagger}$, where $\Omega_\omega^{(n)}$ and $\mathcal{D}_\omega^{(n)}$ are defined as Ω_ω and \mathcal{D}_ω would be if the n sites in the block constituted the entire chain.

3.5 Action of the generalized KT transformation on a generalized AKLT state

Although we have focussed on the transformation of the Hamiltonian under the generalized KT transformation \mathcal{D}_ω , for illustrative purposes we will consider in this section a particular Hamiltonian within the SPT phase described by cohomology class $[\omega]$, for which the ground state subspace can be found analytically. We calculate explicitly how this subspace transforms under $\mathcal{D}_\omega^\dagger$, and show that the transformed ground state subspace reflects the spontaneous breaking of

the symmetry in the bulk. The definition of \mathcal{D}_ω [Eqs. (3.12) and (3.13)] arises naturally out of this discussion.

Affleck, Kennedy, Lieb and Tasaki (AKLT) [71] constructed a system in the Haldane phase for which the ground state can be represented exactly as a “valence-bond solid”, or (in more modern language) a *matrix-product state* (MPS) [30]. We will now define a generalization of the AKLT ground state for the SPT phase with cohomology class $[\omega]$. We will write it for open boundary conditions, which means we have to define a subspace \mathcal{P} of states corresponding to the degenerate ground-state subspace. The states in this subspace are of the MPS form

$$\sum_{\mathbf{x}} \text{Tr}(A_{\chi_N} \cdots A_{\chi_1} B) |\mathbf{x}\rangle, \quad (3.15)$$

where we set $A_\chi = V_\omega(\varphi_\omega^{-1}(\chi))$ (where V_ω is an irreducible projective representation with factor system ω), and the subspace comprises the states obtained from all possible $D \times D$ matrices B (with D the dimension of V_ω). The theory of MPS parent Hamiltonians [30] allows one to construct a local frustration-free Hamiltonian for which \mathcal{P} is the D^2 -fold degenerate ground state subspace. From the classification of SPT order in matrix-product states [16, 17], one can show [25] that H indeed lies in the SPT phase described by cohomology class $[\omega]$.

We will only consider the state $|\Psi\rangle$ resulting from setting $B = \frac{1}{D} \sum_g V(g)^\dagger$, as it turns out that applying $[u(g)]^{\otimes N}$ to $|\Psi\rangle$ for group elements g generates a basis for \mathcal{P} . This gives

$$|\Psi\rangle = \sum_{\mathbf{x}} \Omega_\omega(\varphi_\omega^{-1}(\mathbf{x})) |\mathbf{x}\rangle, \quad (3.16)$$

where we have used the fact that $\text{Tr}[V_\omega(h)V_\omega(g)^\dagger] = D\delta_{g,h}$ (which follows from the fact that $\text{Tr} V_\omega(g) = 0$ for $g \neq 1$, a consequence of maximal non-commutativity). This means that

$$\mathcal{D}_\omega^\dagger |\Psi\rangle = \sum_{\mathbf{x}} |\mathbf{x}\rangle = |\phi\rangle^{\otimes N}, \quad (3.17)$$

where $|\phi\rangle = \sum_{\chi} |\chi\rangle$. Since $[u(g)]^{\otimes N}$ commutes with $\mathcal{D}_\omega^\dagger$, a basis for the transformed subspace $\mathcal{D}_\omega^\dagger \mathcal{P}$ comprises the states $\{[u(g)]^{\otimes N} |\phi\rangle^{\otimes N}, g \in G\}$. Thus the transformed Hamiltonian under $\mathcal{D}_\omega^\dagger$ indeed has a set of locally distinguishable symmetry-breaking ground states, as we expect.

3.6 String order

A key property of the Kennedy-Tasaki transformation is that it relates two-particle correlations (which are expected to be long-ranged in the maximal symmetry-breaking phase for the $Z_2 \times Z_2$ symmetry) to the string correlation functions that characterize the Haldane phase. Here, we will establish a similar correspondence for our general construction. This property will also allow us to determine how the generalized Kennedy-Tasaki transformation maps between different quantum phases.

3.6.1 Symmetry-breaking phases and two-particle correlations

Let us first give a general discussion of the two-particle correlations that we expect to see in the maximal symmetry-breaking phase (i.e. where the subgroup of symmetry operations that are *not* spontaneously broken in the bulk is trivial) for an on-site abelian symmetry. A system in this phase will have a collection of degenerate symmetry-breaking ground states. Traditionally, the symmetry-breaking is detected through the nonzero value of an *order parameter*, which is the expectation value of a single-site observable A such that $\langle A \rangle = 0$ for any symmetry-*respecting*

state. For example, in the case of the quantum transverse-field Ising model on spin-1/2's, with Hamiltonian

$$H = - \sum_i \sigma_i^z \sigma_{i+1}^z + \lambda \sum_i \sigma_i^x, \quad (3.18)$$

the appropriate order parameter is $\langle \sigma_z \rangle$. The fact that $\sigma_x \sigma_z \sigma_x = -\sigma_z$ ensures that $\langle \sigma_z \rangle$ must be zero for any state respecting the spin-flip symmetry $\prod_i \sigma_i^x$. In the general case, we can consider an observable A such that

$$u(g')Au(g')^\dagger = \chi(g')A \quad (3.19)$$

for some $\chi \in G^*$. By a similar argument as before, for $\chi \neq 1$ we find that $\langle A \rangle = 0$ for any symmetry-respecting state. As another example, in the case of a spin-1 chain with the D_2 rotation symmetry, the spin-component operators S_a ($a = x, y, z$) satisfy Eq. (3.19) for appropriate choices of χ .

Denote the space of operators A satisfying Eq. (3.19) by \mathcal{A}_χ . For a given symmetry-breaking state, a given operator in \mathcal{A}_χ could still have zero expectation value by accident. However, we will now argue that, for a given maximal symmetry-breaking state, for *every* non-trivial $\chi \in G^*$ a generic choice of $A \in \mathcal{A}_\chi$ will reveal the symmetry-breaking through its nonzero expectation value. Indeed, let ρ be the reduced state density operator on a single site. It suffices to show that the subspace $\mathcal{B}_\chi = \{A \in \mathcal{A}_\chi | \text{Tr}(A\rho) = 0\}$ is a proper subspace (i.e. $\mathcal{B}_\chi \neq \mathcal{A}_\chi$). Suppose by way of contradiction that $\mathcal{B}_\chi = \mathcal{A}_\chi$. Then *every* $A \in \mathcal{A}_\chi$ must satisfy $\text{Tr}(A\rho) = 0$. But since the set $\{|\chi'\rangle\langle\chi'\chi| : \chi' \in G^*\}$ comprises a basis for \mathcal{A}_χ , this would imply that $\langle\chi'\chi|\rho|\chi'\rangle = 0$ for all χ' . In the maximal symmetry breaking phase, the ground state has no residual symmetry, and hence there is no constraint on the reduced state ρ that would force all of these matrix elements to be zero [whereas if the ground state were invariant under the symmetry operation corresponding to the group element g , then this would force $\langle\chi_1|\rho|\chi_2\rangle = 0$ for all $\chi_1, \chi_2 \in G^*$ such that $\chi_1(g) \neq \chi_2(g)$.] Certainly, for a *generic* state in the maximal symmetry-breaking phase, these matrix elements would not all be zero.

An alternative measure of the symmetry-breaking is the two-particle correlation function $\langle C_n(A, B) \rangle$ (for $A, B \in \mathcal{A}_\chi$), where

$$C_n(A, B) \equiv A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B. \quad (3.20)$$

Because each of the symmetry-breaking ground states should be short-range correlated, the correlation function $\langle C_n(A, B) \rangle$ converges to $\langle A \rangle^* \langle B \rangle$ as $n \rightarrow \infty$. The expectations here are taken with respect to a particular choice of symmetry-breaking ground state, but notice that $C_n(A, B)$ commutes with the symmetry, and therefore its expectation is independent of this choice.

3.6.2 SPT phases and string correlation functions

A key feature of the Haldane phase and its generalizations is that there is no locally-detectable symmetry breaking in the bulk, and consequently all the two-particle correlations $\langle C_n(A, B) \rangle$ decay exponentially as $n \rightarrow \infty$. Nevertheless, such SPT phases still have a more subtle form of long-range order detectable through *string* correlations. As a result, we are led to consider the following generalization of Eq. (3.20) (reducing to it when $g = 1$):

$$C_n(A, B; g) = A^\dagger \otimes [u(g)]^{\otimes(n-2)} \otimes B. \quad (3.21)$$

(Recall that $u(g)$ is the unitary on-site action of the symmetry.) In particular, the den Nijs-Rommelse string operators [62] for the Haldane phase,

$$S^\alpha \otimes [e^{i\pi S^\alpha}]^{\otimes(n-2)} \otimes S^\alpha = C_n(S^\alpha, S^\alpha; \alpha), \quad \alpha = x, y, z, \quad (3.22)$$

with S^α the spin-component operators, are of this form. In the Haldane phase, the den Nijs-Rommelse string correlations are long-ranged, i.e. $\lim_{n \rightarrow \infty} \langle C_n(S^\alpha, S^\alpha; \alpha) \rangle \neq 0$.

Traditionally, long-ranged string correlations have been viewed as evidence of non-trivial order. However, there is a need for caution: the limiting string correlation functions $\lim_{n \rightarrow \infty} \langle C_n(A, B; g) \rangle$ are nonzero for generic choices of A and B whenever the symmetry is unbroken in the bulk, and need not reflect any non-trivial SPT order [35, 66]. Therefore, in order to obtain useful criteria for identifying SPT phases, we must restrict ourselves to restricted classes of A and B . Indeed, it turns out to be useful to require, as in the symmetry-breaking case, $A, B \in \mathcal{A}_\chi$ for some linear character χ . In that case, the selection rule discussed in Ref. [66] forces $\lim_{n \rightarrow \infty} \langle C_n(A, B; g) \rangle = 0$ when $\varphi_\omega(g) \neq \chi$, where φ_ω is the homomorphism induced by the cohomology class [Eq. (3.10)] (see Appendix 3.C for the proof). In the case $\varphi_\omega(g) = \chi$, there is no such selection rule and so we expect that the corresponding string correlation will generically be long-ranged. (It can be checked that the latter case is the relevant one for the den Nijs-Rommelse string correlations in the Haldane phase.) Thus, the pattern of long-ranged string orders of the form considered is a useful way of identifying phases; we will make this idea more precise in Section 3.6.4.

3.6.3 Mapping of correlation functions under the generalized Kennedy-Tasaki transformation

We have established that the string operators $C_n(A, B; g)$ are useful tools for identifying phases. Therefore, it makes sense to calculate how these operators transform under the generalized KT transformation \mathcal{D}_ω , where $[\omega]$ is a maximally non-commutative cohomology class. This calculation is done in Appendix 3.B; the result is (for $A, B \in \mathcal{A}_\chi$)

$$\mathcal{D}_\omega C_n(A, B; g) \mathcal{D}_\omega^\dagger = C_n(\tilde{A}, \tilde{B}; \tilde{g}) \quad (3.23)$$

with

$$\tilde{A} = AW_\omega(\chi)^\dagger, \quad \tilde{B} = BW'_\omega(\chi)^\dagger, \quad \tilde{g} = g\varphi_\omega^{-1}(\chi), \quad (3.24)$$

where

$$W_\omega(\chi) = \sum_{\chi'} \omega(\varphi_\omega^{-1}(\chi), \varphi_\omega^{-1}(\chi')) |\chi'\rangle \langle \chi'|, \quad (3.25)$$

$$W'_\omega(\chi) = \sum_{\chi'} \omega(\varphi_\omega^{-1}(\chi'), \varphi_\omega^{-1}(\chi)) |\chi'\rangle \langle \chi'|, \quad (3.26)$$

(observe that $A, B \in \mathcal{A}_\chi$ implies $\tilde{A}, \tilde{B} \in \mathcal{A}_\chi$ as well, since W_ω and W'_ω commute with the symmetry).

To see the significance of this result, suppose that $[\omega]$ is a maximally non-commutative cohomology class. Recall that, in a system with maximal symmetry breaking, for any $\chi \neq 1$ we expect to be able to find $A, B \in \mathcal{A}_\chi$ such that $\lim_{n \rightarrow \infty} \langle C_n(A, B; 1) \rangle \neq 0$. But then, by Eq. (3.23), this implies that in the transformed system obtained from the original one by \mathcal{D}_ω , we will have $\lim_{n \rightarrow \infty} \langle C_n(\tilde{A}, \tilde{B}; \varphi_\omega^{-1}(\chi)) \rangle \neq 0$, i.e. there are long-ranged string correlations of precisely the form that we expect to get in the SPT phase characterized by cohomology class $[\omega]$. In the next subsection, we will turn this into a proof that the transformed system is indeed in that phase.

3.6.4 Patterns of string order as a “signature” for quantum phases

We have already seen that the long-range behavior of string correlations of the form $\langle C(A, B; g) \rangle$ (with $A, B \in \mathcal{A}_\chi$ for some character χ) is a useful probe for identifying different kinds of ordering in systems with a finite abelian on-site symmetry. In Appendix 3.C, we go further, and show that this long-range behavior *uniquely* identifies all possible quantum phases that result from symmetry-respecting Hamiltonians. (The general classification of such phases was given in Ref. [17, 72]; it includes conventional symmetry-breaking phases, SPT phases with no

symmetry-breaking in the bulk, as well as other examples in which SPT and symmetry-breaking orders combine.)

The result of Appendix 3.C is expressed in terms of the following “signature” function M acting on $G^* \times G$ to measure which of the string correlations are long-ranged:

$$M(\chi, g) = \begin{cases} 1 & \text{if } \lim_{n \rightarrow \infty} \langle C_n(A, B, g) \rangle \neq 0 \text{ generically} \\ & \text{in the phase, when } A, B \in \mathcal{A}_\chi, \\ 0 & \text{otherwise} \end{cases} \quad (3.27)$$

We have included the word “generically”, because it is possible that there might be specific points in the phase and/or choices of $A, B \in \mathcal{A}_\chi$ such that the limiting correlation is “accidentally” zero. [For example, in the case $g = 1$, we derived in Sec. 3.6.1 the condition for a given state in the maximal symmetry-breaking phase to satisfy $\lim_{n \rightarrow \infty} C_n(A, B; 1) = 0$ for all $A, B \in \mathcal{A}_\chi$, even though generically we expect these two-body correlations to be long-ranged.] The result of Appendix 3.C is then that each possible phase in the general classification has a distinct signature M .

Combining this result with that of the previous subsection 3.6.3 allows us to definitively establish in general how different phases are transformed into each other by the generalized KT transformation \mathcal{D}_ω (with $[\omega]$ a maximally non-commutative cohomology class). Indeed, suppose we start from a phase described by signature M . Then, by Eqs. (3.23) and (3.24), the transformed phase resulting from application of \mathcal{D}_ω has signature

$$M'(\chi, g) = M(\chi, g[\varphi_\omega^{-1}(\chi)]^{-1}). \quad (3.28)$$

In particular, we can consider the case that the starting phase is the maximal symmetry breaking phase (all the symmetries broken in the bulk). The arguments of Sec. 3.6.1 show that for such a phase, $M(\chi, 1) = 1$ for all χ . Furthermore, it was shown in Ref. [66] that string order of the form $\langle C_n(A, B; g) \rangle$ can be long-ranged only when the symmetry corresponding to g is unbroken in the bulk. Thus, the maximal symmetry-breaking phase has signature $M(\chi, g) = 1 \Leftrightarrow g = 1$. It follows that the transformed phase resulting from applying \mathcal{D}_ω has signature $M'(\chi, g) = 1 \Leftrightarrow g = \varphi_\omega^{-1}(\chi)$. From the discussion of Appendix 3.C, we see that this is precisely the signature of the SPT phase with cohomology class $[\omega]$, as expected.

Note that, although in this paper we have concentrated on the duality between pure SPT order and maximal symmetry-breaking order, Eq. (3.28) can be used to determine in general how the generalized KT transformation relates \mathcal{D}_ω different symmetric phases to each other, including combined symmetry-breaking/SPT phases. For example, see Ref. [67] for a discussion of the $Z_N \times Z_N$ case. (In this case, our generalized KT transformation reduces to the one defined in Ref. [67], or a variant thereof, depending on which cohomology class $[\omega]$ and factor system representative ω one uses in the construction).

3.7 The Kennedy-Tasaki transformation for continuous symmetries

We stress that our assumption of a maximally-noncommutative cohomology class of a finite abelian group might not be as restrictive as it sounds. Indeed, an SPT phase characterized by an arbitrary group could still be identified as part of a maximally non-commutative SPT phase with respect to a finite abelian subgroup. As an example, here we will discuss how our framework allows us to apply the concept of hidden symmetry breaking to some generalizations of the Haldane phase.

Just as the Haldane phase is motivated by $SO(3)$ -invariant antiferromagnets, these generalized Haldane phases contain systems that are invariant under an $SO(2k+1)$ or $SU(k)$ symmetry.

However, in each case, we will identify a finite abelian subgroup (analogous to D_2 for the Haldane phase), which will turn out to be the relevant one for identifying the hidden symmetry breaking. In each case, this finite abelian subgroup will turn out to be sufficient to classify the phases, since imposing the full continuous symmetry does not separate any phases that could not already be distinguished through this subgroup. This suggests that, even when the full continuous symmetry is present, we should describe the SPT order in terms of the hidden breaking of the finite abelian subgroup.

3.7.1 The $\text{SO}(2k + 1)$ Haldane phase

For systems invariant under an on-site $\text{SO}(2k + 1)$ symmetry, there is exactly one non-trivial SPT phase [1, 73], which we can think of as a generalization of the Haldane phase (reducing to it in the case $k = 1$). The corresponding cohomology class is that of the spinor representations of $\text{SO}(2k + 1)$ (which are, in fact, *projective* representations).

Identifying $\text{SO}(2k + 1)$ with its representation in terms of $(2k + 1) \times (2k + 1)$ orthogonal matrices with unit determinant, we define $G_k \equiv \{A \in \text{SO}(2k + 1) : A \text{ is diagonal in the standard basis}\}$, which constitutes a finite abelian subgroup. We can construct a minimal set of generators $\{u^{(l)}, l = 1, \dots, 2k\}$ with matrix elements

$$u_{i,j}^{(l)} = (-1)^{1 - \delta_{i,l}} \delta_{i,j} \quad (3.29)$$

(we do not include $u^{(2k+1)}$ in our of minimal set of generators because it is not independent of the rest; indeed, $u^{(2k+1)} = \prod_{l=1}^{2k} u^{(l)}$). This shows that $G_k \cong Z_2^{\times 2k}$. It can be shown, by considering the restriction of the spinor representations of $\text{SO}(2k + 1)$ to the subgroup G_k , that the cohomology class of G_k for systems in the non-trivial SPT phase with respect to $\text{SO}(2k + 1)$ is that of the projective representation generated by

$$V(u^{(l)}) = \Gamma_l, \quad (3.30)$$

where the $2k$ matrices Γ_l obey the anti-commutation relations $\{\Gamma_a, \Gamma_b\} = 2\delta_{a,b}$. It is straightforward to show that this cohomology class is maximally non-commutative.

Thus, we can use our general prescription [Eq. (3.12)] to construct a generalized KT transformation for systems in the non-trivial SPT phase with respect to $\text{SO}(2k + 1)$. In analogy to the original KT transformation for the Haldane phase (which breaks the full rotation symmetry, preserving only the discrete subgroup D_2), the resulting transformed system will only have the discrete $Z_2^{\times 2k}$ symmetry instead of the full $\text{SO}(2k + 1)$. Furthermore, the transformed system will be in a maximal local symmetry-breaking phase for this discrete symmetry. In this sense, the non-trivial $\text{SO}(2k + 1)$ SPT phase can be understood as a result of the ‘‘hidden breaking’’ of the $Z_2^{\times 2k}$ symmetry. Note that, if we make a particular choice of factor system within the appropriate cohomology class, it can be shown that the generalized Kennedy-Tasaki transformation constructed according to our general prescription [see Eq. (3.12)] coincides with the one constructed in Ref. [1].

We remark that, since for the $\text{SO}(2k + 1)$ symmetry group there is only one nontrivial SPT phase, it can already be distinguished from the trivial phase via a $Z_2 \times Z_2$ subgroup. Although one could therefore construct the generalized KT transformation \mathcal{D}_ω based on the $Z_2 \times Z_2$ subgroup, as in Ref. [67], we prefer to construct it based on $Z_2^{\times 2k}$. This ensures that the transformed phase is maximally symmetry-breaking. If one instead uses only the $Z_2 \times Z_2$ subgroup to construct \mathcal{D}_ω , it can be shown (using similar arguments to Sec. 3.6.4 and Appendices 3.B and 3.C) that the resulting phase breaks $Z_2 \times Z_2$, but is still SPT-ordered with respect to the remaining $Z_2^{\times 2(k-1)}$.

In general, a useful way to ensure that the transformed phase has no residual SPT order is by counting the ground state degeneracy: if the degeneracy of the original SPT phase is fully explained by the symmetry-breaking in the transformed phase, then there cannot be any residual

SPT order. For instance, consider the $SO(2k+1)$ case. We can assume that the fractionalized representation of the symmetry on the edge is the fundamental spinor representation, as it suffices to confirm the lack of residual SPT order at a single point in the phase. There is then a 2^k -fold degeneracy associated with each edge, which agrees with the 2^{2k} -fold degeneracy we expect for a phase that maximally breaks a $Z_2^{\times 2k}$ symmetry. A similar property also holds for the $SU(k)$ example considered in the next section.

3.7.2 SPT phases for $SU(k)$

Instead of thinking of the Haldane phase as invariant under an $SO(3)$ symmetry, we can also think of it as invariant under $PSU(2) \equiv SU(2)/\{+1, -1\}$. Of course, $PSU(2) \cong SO(3)$, but this suggests an alternative generalization of the Haldane phase: one that is invariant under an on-site representation of $PSU(k) \equiv SU(k)/C_k$, where $C_k = \{\exp(2i\pi l/k) : l = 0, \dots, k-1\}$.

Given the definition of $PSU(k)$, we can construct a finite abelian subgroup by identifying a subgroup of $SU(k)$ that is abelian up to phase factors (i.e. up to elements of C_k). The discrete Heisenberg-Weyl group is such a subgroup; it is the group generated [in the standard representation of $SU(k)$] by the two operators

$$X = \frac{1}{W} \sum_{l=0}^{k-1} |(l+1) \bmod k\rangle \langle l|, \quad (3.31)$$

$$Z = \frac{1}{W} \sum_{l=0}^{k-1} w^l |l\rangle \langle l| \quad (3.32)$$

where w is a primitive k -th root of unity, and we have included the normalization factor $W = w^{(k-1)/2}$ to ensure that $\det X = \det Z = 1$. The fact that the subgroup generated by X and Z is abelian up to phases follows from the relation $ZX = wXZ$. The abelian subgroup of $PSU(k)$ corresponding to the Heisenberg-Weyl group is isomorphic to $Z_k \times Z_k$. The cohomology group for $Z_k \times Z_k$ is $H^2(Z_k \times Z_k, U(1)) \cong Z_k \cong H^2(PSU(k), U(1))$, and it can be shown[67] that the k cohomology classes of $PSU(k)$ correspond exactly to the k cohomology classes of the $Z_k \times Z_k$ subgroup. Thus, the $Z_k \times Z_k$ subgroup is sufficient to characterize all the SPT phases even in the presence of the full $PSU(k)$ symmetry. In order to apply our generalized Kennedy-Tasaki transformation, we need a maximally non-commutative cohomology class; if we let $[\omega_0]$ be a generator for the cohomology group $H^2(Z_k \times Z_k, U(1))$, then it can be shown that $[\omega_0^l]$ is maximally non-commutative if and only if l and k are coprime.

3.7.3 Other continuous symmetry groups

We leave it as an open question whether a similar analysis to the above for $SO(2k+1)$ and $SU(k)$ holds for other continuous symmetry groups. In Ref. [67], it is shown that a subgroup of the form $Z_N \times Z_N$ can be found for all the cases involving classical Lie groups. However, as we have shown with the $SO(2k+1)$ example in Sec. 3.7.1, this is not the whole story, especially if the aim is to identify a relevant generalised KT transformation that can ‘fully’ remove the SPT order, i.e., that can relate a SPT-ordered phase to a maximally symmetry-breaking phase.

3.8 The topological disentangler

In this section, we will briefly outline a physical interpretation of the resulting correspondence between the SPT ground states and the symmetry-breaking ground states, in terms of the entanglement structure of the ground states.

When we group sites together in blocks of size $\gg \xi$, with ξ the correlation length, any gapped ground state starts to resemble (up to on-site unitary rotations on the blocked sites) a ‘dimer

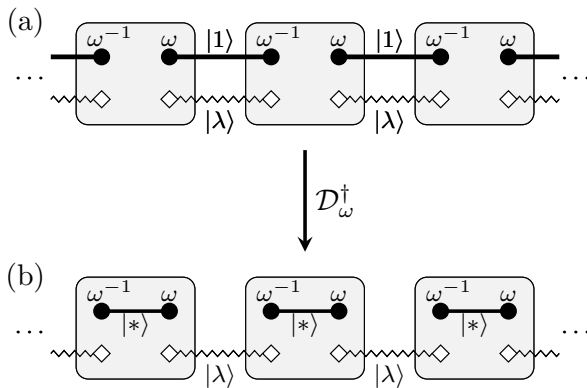


Figure 3.1: (a) The “dimer state” renormalization fixed point for the SPT phase corresponding to a maximally commutative cohomology class $[\omega]$; (b) The result of applying $\mathcal{D}_\omega^\dagger$, for a particular choice of boundary conditions. Each shaded area represents one coarse-grained site. The black dots transform under irreducible projective representations with factor systems ω and ω^{-1} under the symmetry, and the diamonds do not transform at all under the symmetry. Note: Two adjacent black dots transform linearly under the symmetry; therefore, we can introduce the simultaneous eigenbasis $\{|\chi\rangle\}$ of the symmetry (with the states labelled by linear characters χ ; from Schur’s Lemma it follows that they must be maximally entangled). For $\chi = 1$ this gives the state $|1\rangle$ appearing in (a). In (b), we have defined $|*\rangle = \sum_\chi |\chi\rangle$. The state $|\lambda\rangle$ is not universal and depends on the specific point in the phase.

state” which can be viewed as a renormalization fixed point [74]. If the ground state is in an SPT phase characterized by a maximally non-commutative cohomology class, this dimer state will take the form shown in Fig. 3.1(a); this is a consequence of the fact that a maximally non-commutative factor system corresponds to a unique irreducible projective representation [69, 70]. We see that the entanglement between two halves of the chain has two origins: the universal “topological” entanglement represented by the maximally entangled state $|1\rangle$, and the “non-topological” entanglement represented by the state $|\lambda\rangle$. (The “dot” and “diamond” particles correspond to the “protected” and “junk” subsystems discussed in Ref. [25].)

It is therefore instructive to consider what happens under $\mathcal{D}_\omega^\dagger$ to a ground state of the form shown in Fig. 3.1(a). The calculation required is similar to that of Sec. 3.5. (Indeed, when written in a matrix-product state form, the state of Fig. 3.1(a) reduces to Eq. (3.15) when the “extra” particles in the $|\lambda\rangle$ state are absent.) Here we just state the result: for a suitable choice of boundary conditions, the resulting state is as depicted in Fig. 3.1(b); note that this resulting state is no longer invariant under the symmetry, and the orbit of this state under the symmetry is the set of symmetry-breaking ground states for the transformed system. We see that the topological component of the entanglement has been eliminated, with the non-topological part of the state remaining untouched. In this sense, we can think of the generalized KT transformation $\mathcal{D}_\omega^\dagger$ as a “topological disentangler” [37].

3.9 Discussion

We have presented a generalization of the Kennedy-Tasaki transformation, which maps certain one-dimensional models with SPT order to ones with traditional symmetry-breaking. This formulation further expands the characterization of SPT order as a form of hidden symmetry breaking to a broad class of models, specifically those for which the SPT order can be related to a *maximally non-commutative* factor system of a finite abelian group that acts on the system

through an on-site unitary representation. Whether any analogous results hold in higher dimensions, or for other kinds of symmetries (e.g. time reversal), remains an open question. We point out, however, that in two dimensions and higher there is a different kind of duality that holds for any on-site unitary representation of a finite group G : between SPT phases and topological lattice gauge theories with gauge group G [75, 76].

We have also interpreted the action of our generalized KT transformation as a *topological disentangler* [37], removing the topological component of the entanglement from the ground state. Transformations that remove entanglement from a quantum many-body system have found use in numerical methods such as the multiscale entanglement renormalisation ansatz (MERA) [77, 78], and so may the generalized KT transformation presented here.

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3.A Choice of representative factor system

Here we will discuss the difference between \mathcal{D}_ω and $\mathcal{D}_{\omega'}$, where

$$\omega'(g, h) = \beta(g)\beta(h)\beta(gh)^{-1}\omega(g, h), \quad (3.33)$$

[for some set of phase factors $\beta(g)$] is another factor system in the same cohomology class as ω . The important thing to consider is the transformed Hamiltonians resulting from the respective transformations. Thus we will only consider the way \mathcal{D}_ω acts on symmetry-respecting observables A . First of all we observe that the isomorphism φ_ω only depends on the cohomology class, so that $\varphi_\omega = \varphi_{\omega'}$. It is then straightforward to show that

$$\mathcal{D}_{\omega'} A \mathcal{D}_{\omega'}^\dagger = b_\beta^{\otimes N} \mathcal{D}_\omega A \mathcal{D}_\omega^\dagger b_\beta^{\otimes N \dagger}, \quad (3.34)$$

where we have defined

$$b_\beta = \sum_\chi \beta(\varphi_\omega^{-1}(\chi)) |\chi\rangle\langle\chi|. \quad (3.35)$$

Therefore, the two transformed Hamiltonians differ only by a rephasing of the basis on each site. Clearly, this does not change the nature of the resulting transformed phase.

3.B Proof of Eq. (3.23)

For simplicity of notation, we use the isomorphism φ_ω to label our site basis by group elements instead of group characters, i.e. we define

$$|g\rangle \equiv |\varphi_\omega(g)\rangle. \quad (3.36)$$

For some choice of character χ_* , let $A, B \in \mathcal{A}_{\chi_*}$, and define $g_* = \varphi_\omega^{-1}(\chi_*)$. This implies that

$$A^\dagger = \sum_g \mu_g |g\rangle\langle gg_*| \quad (3.37)$$

$$B = \sum_g \nu_g |gg_*\rangle\langle g|. \quad (3.38)$$

for some scalars $\{\mu_g\}$ and $\{\nu_g\}$. Thus,

$$A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B = \sum_{g_1, \dots, g_n} \mu_{g_1} \nu_{g_n} |g_1, g_2, \dots, g_n g_*\rangle\langle g_1 g_*, \dots, g_{n-1}, g_n|. \quad (3.39)$$

Therefore

$$\mathcal{D}_\omega(A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B)\mathcal{D}_\omega^\dagger = \sum_{g_1, \dots, g_n} \mu_{g_1} \nu_{g_n} \Gamma_{g_1, \dots, g_n} |g_1, g_2, \dots, g_n g_*\rangle \langle g_1 g_*, \dots, g_{n-1}, g_n|, \quad (3.40)$$

where Γ_{g_1, \dots, g_n} is the phase factor such that

$$V_\omega(g_n g_*) V_\omega(g_{n-1}) \cdots V_\omega(g_1) = \Gamma_{g_1, \dots, g_n} V_\omega(g_n) \cdots V_\omega(g_2) V_\omega(g_1 g_*). \quad (3.41)$$

This gives

$$\Gamma_{g_1, \dots, g_n} = \omega(g_n, g_*)^{-1} \alpha_\omega(g_*, g_{n-1}) \cdots \alpha_\omega(g_*, g_2) \omega(g_*, g_1), \quad (3.42)$$

where $\alpha_\omega(g, h)$ is the phase factor such that

$$V_\omega(g) V_\omega(h) = \alpha_\omega(g, h) V_\omega(h) V_\omega(g). \quad (3.43)$$

However, now comparing with Eq. (3.11), we find that

$$u(g)|h\rangle = \alpha_\omega(g, h)|h\rangle. \quad (3.44)$$

Therefore, we can conclude that

$$\mathcal{D}_\omega(A^\dagger \otimes \mathbb{I}^{\otimes(n-2)} \otimes B)\mathcal{D}_\omega^\dagger = [W_\omega(g_*) A^\dagger] \otimes [u(g_*)]^{\otimes(n-2)} \otimes [B W'_\omega(g_*)^\dagger], \quad (3.45)$$

which leads to Eq. (3.23), where we have defined

$$W_\omega(g_*) = \sum_g \omega(g_*, g) |g\rangle \langle g|, \quad W'_\omega(g_*) = \sum_g \omega(g, g_*) |g\rangle \langle g|. \quad (3.46)$$

3.C Identifying phases from patterns of string order

Symmetry-breaking phases and SPT phases are two different kinds of phases that can arise in one-dimensional systems invariant under an on-site symmetry. As was shown in Ref. [17, 72], the most general kind of phase for such systems combines both aspects. A general symmetric phase for a symmetry group G is characterized by a subgroup H (corresponding to the symmetries that are unbroken in the bulk) and a cohomology class $[\omega]$ for H , such that each of the degenerate symmetry-breaking ground states is in the SPT phase $[\omega]$ with respect to the subsymmetry. In this Appendix, we will show that, in the case of finite abelian symmetry groups G , each distinct phase gives rise to a distinct pattern of long-range string correlations, as defined through the “signature” function M of Eq. (3.27).

Let us first consider the case of pure SPT phases (i.e. $H = G$, and the phases are classified by cohomology classes of G). It was argued in Ref. [35] that, if the operators A and B are chosen at random, then generically one finds that $\langle C_n(A, B; g) \rangle \neq 0$ is nonzero for any ground state that is invariant under the symmetry in the bulk (which will be the case for any pure SPT phase as well as the trivial phase). Suppose, however, that we instead choose $A, B \in \mathcal{A}_\chi$ for some character χ . Then, whenever $\chi \neq \varphi_\omega(g)$, we must be able to find some $g' \in G$ such that $\chi(g') \neq [\varphi_\omega(g)](g')$. Recalling the definition of φ_ω [Eq. (3.10)], and of \mathcal{A}_χ [Section 3.6.1], this implies that $\alpha_1 \neq \alpha_2$, where α_1, α_2 are the scalars such that

$$V_\omega(g) V_\omega(g') = \alpha_1 V_\omega(g') V_\omega(g), \quad (3.47)$$

$$A u(g') = \alpha_2 u(g') A \quad (3.48)$$

As shown in Ref. [66], there is a selection rule that prevents the string correlation $\langle C_n(A, B; g) \rangle$ from being long-ranged when $\alpha_1 \neq \alpha_2$. On the other hand, if $\chi = \varphi_\omega(g)$ then there is no such

selection rule and we expect that $\langle C_n(A, B; g) \rangle$ will generically be long-ranged even with the constraint $A, B \in \mathcal{A}_\chi$. In summary, therefore, the pure SPT phase has the signature $M(\chi, g) = 1 \Leftrightarrow \chi = \varphi_\omega(g)$. For example, if the cohomology class is maximally non-commutative, then φ_ω is invertible and thus, for each character χ , there is a unique g such that $M(\chi, g) = 1$, and vice versa. At the other extreme, if the cohomology class is trivial, then $\varphi_\omega(g) = 1$ for all g , and thus $M(\chi, g) = 1 \Leftrightarrow \chi = 1$.

The above arguments show that the possible signature functions M are in one-to-one correspondence with the homomorphisms φ_ω . In order to establish that distinct SPT phases correspond to different signature functions, it only remains to show that if ω_1 and ω_2 have different cohomology classes, then $\varphi_{\omega_1} \neq \varphi_{\omega_2}$. Since φ_ω is linear in ω , it suffices to prove that if $\varphi_\omega = 1$ (the trivial homomorphism), then ω has trivial cohomology class. Indeed, $\varphi_\omega = 1$ implies, by definition of φ_ω , that $V_\omega(g)V_\omega(g')V_\omega(g)^\dagger = V_\omega(g')$ for all $g, g' \in G$, which is to say all the elements $V_\omega(g)$ commute. If we choose V_ω to be irreducible, then Schur's Lemma implies that $V_\omega(g) = \beta(g)$ for some scalar phase factors $\beta(g)$. Therefore, the projective representation V_ω has trivial cohomology class.

Now let us return to the general case, where phases are classified by a subgroup $H \leq G$ and a cohomology class $[\omega]$ of H . Because all the order parameters we are considering are expectation values of symmetry-respecting operators, we just need to determine their value for a single symmetry-breaking ground state. Since each of these symmetry-breaking ground states lies in an SPT phase with respect to the subsymmetry H , we find that, for $h \in H$, $M(\chi, h) = 1$ if and only if $\chi_H = \varphi_\omega(h)$, where χ_H is the restriction χ onto the subgroup H . On the other hand, if $g \notin H$ then $M(\chi, g) = 0$ for any χ (because the symmetry operation corresponding to h is broken in the bulk [35]). In summary, the signature of a general symmetric phase is $M(\chi, g) = 1 \Leftrightarrow [g \in H \text{ and } \chi_H = \varphi_\omega(g)]$. Notice that, for $H \neq G$, there will exist non-trivial characters χ such that $\chi_H = 1$, which implies that $M(\chi, 1) = 1$. Recall that this corresponds to nonzero values of

$$\lim_{n \rightarrow \infty} \langle A^\dagger \otimes \mathbb{I}^{\otimes n} \otimes B \rangle \quad (3.49)$$

(for $A, B \in \mathcal{A}_\chi$), which is what we expect, since for $H \neq G$ there is partial symmetry-breaking, and therefore there should also be long-range order.

We will now prove that no two symmetric phases can have the same signature. Arguing as in the pure SPT case, it is easy to see that for a *fixed* H all distinct phases have different signatures. To complete the proof, we will now show that the subgroup H can be recovered from the signature, and therefore two phases with different H must have different signatures. To do this we make use of the following result:

Lemma 5. *Let G be a finite abelian group, and let H be a subgroup. Then any linear character acting on H can be extended to a linear character on G . That is, for any $\xi \in H^*$, there exists $\chi \in G^*$ such that $\chi_H = \xi$.*

Proof. Define the homomorphism $\psi : G^* \rightarrow H^*$, $\chi \mapsto \chi_H$. Observe that $\ker \psi \cong (G/H)^*$, and therefore $|\ker \psi| = |(G/H)^*| = |G/H| = |G|/|H|$. But $\psi(G^*) \cong G^*/\ker \psi$, so $|\psi(G^*)| = |G^*|/|\ker \psi| = |G|/|\ker \psi| = |H| = |H^*|$. It follows that $\psi(G^*) = H^*$, i.e. ψ is surjective. \square

Lemma 5 ensures that, for any $h \in H$, we can find a character $\chi \in G^*$ such that $\chi_H = \varphi_\omega(h)$, and hence $M(\chi, h) = 1$. By contrast, if $g \notin H$ then we found above that $M(\chi, g) = 0$ for all $\chi \in G^*$. Therefore, the subgroup H can be recovered from the signature according to

$$H = \{h \in G | M(\chi, h) = 1 \text{ for some } \chi \in G^*\}. \quad (3.50)$$

This completes the proof that distinct phases have distinct signatures.

Chapter 4

Discussion and conclusion

In this thesis, we have put the Kennedy-Tasaki transformation on a more systematic footing, showing how it and its properties arise directly out of the fundamental characterization of SPT order in the Haldane phase. Furthermore, we have seen that the Kennedy-Tasaki transformation can be generalized to a whole class of SPT phases which share the essential characteristics of the Haldane phase (i.e. protected by a unitary on-site symmetry, and a maximally non-commutative cohomology class). This allowed us to derive an alternative “hidden symmetry-breaking” picture of SPT phases, in which the SPT order is related to traditional symmetry-breaking through the generalized Kennedy-Tasaki transformation.

Whether the connection between SPT order and symmetry-breaking order will prove to be a useful tool in condensed matter theory research into SPT phases remains to be determined. However, as we have seen, for the particular quantum information application that was the original motivation, the correspondence was crucial in our discussion. In general, it is not well understood what properties a resource state must have in order to be useful for MBQC. However, in the case we considered of ground states in an appropriate SPT phase, the discussion in Chapter 2 essentially amounts to a demonstration that a particular system in the SPT phase has a ground state that is useful for MBQC whenever the transformed system (under the generalized Kennedy-Tasaki transformation) is sufficiently ordered (in the Landau sense). Thus, in this case, at least, the Kennedy-Tasaki transformation leads to a drastic simplification of the analysis.

This simplification allowed us to give a rigorous proof that the effect of perturbations to a particular toy model does not destroy the property of universality for MBQC provided that the perturbation respects the appropriate symmetry and is small. We do not expect the result itself to be any practical relevance, as the model was not very realistic even in the absence of perturbations and there is no physical reason why the symmetry we specified should be respected. Nevertheless, so far as we know, this is the first time that a ground state has been proven to be a universal resource in the absence of an explicit tensor-network representation for the ground state. We hope that the techniques that we have developed will provide inspiration for attempts to extend our results to more realistic situations. One possible avenue of inquiry would be whether there is any analogue of our results for genuine two-dimensional SPT order [53, 54], as opposed the ‘quasi-1D’ SPT order that we employed throughout.

Bibliography

- [1] H.-H. Tu, G.-M. Zhang, and T. Xiang, “String order and hidden topological symmetry in the $SO(2n + 1)$ symmetric matrix product states,” *J. Phys. A: Math. Theor.* **41**, 415201 (2008), arXiv:0804.1685.
- [2] R. Raussendorf and H. J. Briegel, “A one-way quantum computer,” *Phys. Rev. Lett.* **86**, 5188 (2001).
- [3] H. J. Briegel, D. E. Browne, W. Dur, R. Raussendorf, and M. Van den Nest, “Measurement-based quantum computation,” *Nat. Phys.* **5**, 19 (2009), arXiv:0910.1116.
- [4] F. Verstraete and J. I. Cirac, “Valence-bond states for quantum computation,” *Phys. Rev. A* **70**, 060302 (2004), arXiv:quant-ph/0311130.
- [5] D. Jennings, A. Dragan, S. D. Barrett, S. D. Bartlett, and T. Rudolph, “Quantum computation via measurements on the low-temperature state of a many-body system,” *Phys. Rev. A* **80**, 032328 (2009), arXiv:0906.3553.
- [6] R. Raussendorf, J. Harrington, and K. Goyal, “A fault-tolerant one-way quantum computer,” *Annals of Physics* **321**, 2242 (2006), arXiv:quant-ph/0510135; R. Raussendorf, J. Harrington, and K. Goyal, “Topological fault-tolerance in cluster state quantum computation,” *New Journal of Physics* **9**, 199 (2007), arXiv:quant-ph/0703143.
- [7] D. Gross and J. Eisert, “Novel schemes for measurement-based quantum computation,” *Phys. Rev. Lett.* **98**, 220503 (2007), arXiv:quant-ph/0609149; D. Gross, J. Eisert, N. Schuch, and D. Perez-Garcia, “Measurement-based quantum computation beyond the one-way model,” *Phys. Rev. A* **76**, 052315 (2007), arXiv:0706.3401.
- [8] X. Chen, B. Zeng, Z.-C. Gu, B. Yoshida, and I. L. Chuang, “Gapped two-body hamiltonian whose unique ground state is universal for one-way quantum computation,” *Phys. Rev. Lett.* **102**, 220501 (2009), arXiv:0812.4067.
- [9] T.-C. Wei, I. Affleck, and R. Raussendorf, “Affleck-Kennedy-Lieb-Tasaki state on a honeycomb lattice is a universal quantum computational resource,” *Phys. Rev. Lett.* **106**, 070501 (2011), arXiv:1102.5064; A. Miyake, “Quantum computational capability of a 2D valence bond solid phase,” *Ann. Phys.* **326**, 1656 (2011), arXiv:1009.3491.
- [10] A. S. Darmawan, G. K. Brennen, and S. D. Bartlett, “Measurement-based quantum computation in a two-dimensional phase of matter,” *New J. Phys.* **14**, 013023 (2012), arXiv:1108.4741.
- [11] D. Aharonov and M. Ben-Or, “Fault-tolerant quantum computation with constant error rate,” (1999), arXiv:quant-ph/9906129.
- [12] P. Aliferis, D. Gottesman, and J. Preskill, “Quantum accuracy threshold for concatenated distance-3 codes,” *Quant. Inf. Comput.* **6**, 97 (2006), arXiv:quant-ph/0504218.

- [13] D. Aharonov, A. Kitaev, and J. Preskill, “Fault-tolerant quantum computation with long-range correlated noise,” *Phys. Rev. Lett.* **96**, 050504 (2006), arXiv:quant-ph/0510231.
- [14] M. A. Nielsen and C. M. Dawson, “Fault-tolerant quantum computation with cluster states,” *Phys. Rev. A* **71**, 042323 (2005), arXiv:quant-ph/0405134.
- [15] Z.-C. Gu and X.-G. Wen, “Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order,” *Phys. Rev. B* **80**, 155131 (2009), arXiv:0903.1069.
- [16] X. Chen, Z.-C. Gu, and X.-G. Wen, “Classification of gapped symmetric phases in one-dimensional spin systems,” *Phys. Rev. B* **83**, 035107 (2011), arXiv:1008.3745.
- [17] N. Schuch, D. Pérez-García, and I. Cirac, “Classifying quantum phases using matrix product states and projected entangled pair states,” *Phys. Rev. B* **84**, 165139 (2011), arXiv:1010.3732.
- [18] F. D. M. Haldane, “Continuum dynamics of the 1-D Heisenberg antiferromagnet: Identification with the $O(3)$ nonlinear sigma model,” *Phys. Lett. A* **93**, 464 (1983); F. D. M. Haldane, “Nonlinear field theory of large-spin Heisenberg antiferromagnets: Semiclassically quantized solitons of the one-dimensional easy-axis Néel state,” *Phys. Rev. Lett.* **50**, 1153–1156 (1983).
- [19] F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, “Symmetry protection of topological phases in one-dimensional quantum spin systems,” *Phys. Rev. B* **85**, 075125 (2012), arXiv:0909.4059.
- [20] T. Kennedy and H. Tasaki, “Hidden $Z_2 \times Z_2$ symmetry breaking in Haldane-gap antiferromagnets,” *Phys. Rev. B* **45**, 304–307 (1992); T. Kennedy and H. Tasaki, “Hidden symmetry breaking and the Haldane phase in $S = 1$ quantum spin chains,” *Comm. Math. Phys.* **147**, 431 (1992).
- [21] F. Verstraete and J. I. Cirac, “Renormalization algorithms for quantum-many body systems in two and higher dimensions,” (2004), arXiv:cond-mat/0407066.
- [22] D. Perez-Garcia, F. Verstraete, and J. I. Cirac, “PEPS as unique ground states of local hamiltonians,” *Quant. Inf. Comput.* **8**, 650 (2008), arXiv:0707.2260.
- [23] A. C. Doherty and S. D. Bartlett, “Identifying phases of quantum many-body systems that are universal for quantum computation,” *Phys. Rev. Lett.* **103**, 020506 (2009), arXiv:0802.4314.
- [24] S. D. Bartlett, G. K. Brennen, A. Miyake, and J. M. Renes, “Quantum computational renormalization in the Haldane phase,” *Phys. Rev. Lett.* **105**, 110502 (2010), arXiv:1004.4906.
- [25] D. V. Else, I. Schwarz, S. D. Bartlett, and A. C. Doherty, “Symmetry-protected phases for measurement-based quantum computation,” *Phys. Rev. Lett.* **108**, 240505 (2012), arXiv:1201.4877.
- [26] M. Fannes, B. Nachtergaele, and R. Werner, “Finitely correlated states on quantum spin chains,” *Comm. Math. Phys.* **144**, 443 (1992).
- [27] M. Fannes, B. Nachtergaele, and R. Werner, “Finitely correlated pure states,” *Journal of Functional Analysis* **120**, 511 (1994).
- [28] Y.-Y. Shi, L.-M. Duan, and G. Vidal, “Classical simulation of quantum many-body systems with a tree tensor network,” *Phys. Rev. A* **74**, 022320 (2006), arXiv:quant-ph/0511070.

- [29] N. Schuch, I. Cirac, and D. Prez-Garca, “PEPS as ground states: Degeneracy and topology,” *Annals of Physics* **325**, 2153 (2010), arXiv:1001.3807.
- [30] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, “Matrix product state representations,” *Quant. Inf. Comput.* **7**, 401 (2007), arXiv:quant-ph/0608197.
- [31] F. Verstraete and J. I. Cirac, “Matrix product states represent ground states faithfully,” *Phys. Rev. B* **73**, 094423 (2006), arXiv:cond-mat/0505140.
- [32] M. B. Hastings, “An area law for one-dimensional quantum systems,” *J. Stat. Mech.: Theor. Exp.*, P08024(2007), arXiv:0705.2024.
- [33] W. Son, L. Amico, R. Fazio, A. Hamma, S. Pascazio, and V. Vedral, “Quantum phase transition between cluster and antiferromagnetic states,” *Europhys. Lett.* **95**, 50001 (2011), arXiv:1103.0251.
- [34] D. Gross and J. Eisert, “Quantum computational webs,” *Phys. Rev. A* **82**, 040303 (2010), arXiv:0810.2542.
- [35] D. Pérez-García, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac, “String order and symmetries in quantum spin lattices,” *Phys. Rev. Lett.* **100**, 167202 (2008), arXiv:0802.0447.
- [36] S. Singh, R. N. C. Pfeifer, and G. Vidal, “Tensor network decompositions in the presence of a global symmetry,” *Phys. Rev. A* **82**, 050301 (2010), arXiv:0907.2994.
- [37] K. Okunishi, “Topological disentangler for the valence-bond-solid chain,” *Phys. Rev. B* **83**, 104411 (2011), arXiv:1011.3277.
- [38] D. Aharonov, A. Kitaev, and N. Nisan, “Quantum circuits with mixed states,” (1998), arXiv:quant-ph/9806029.
- [39] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, “Entanglement spectrum of a topological phase in one dimension,” *Phys. Rev. B* **81**, 064439 (2010), arXiv:0910.1811.
- [40] M. Popp, F. Verstraete, M. A. Martin-Delgado, and J. I. Cirac, “Localizable entanglement,” *Phys. Rev. A* **71**, 042306 (2005), arXiv:quant-ph/0411123.
- [41] L. Venuti and M. Roncaglia, “Analytic relations between localizable entanglement and string correlations in spin systems,” *Phys. Rev. Lett.* **94**, 207207 (2005), arXiv:cond-mat/0503021.
- [42] M. Hagiwara, K. Katsumata, I. Affleck, B. I. Halperin, and J. P. Renard, “Observation of $S = 1/2$ degrees of freedom in an $S = 1$ linear-chain Heisenberg antiferromagnet,” *Phys. Rev. Lett.* **65**, 3181 (1990).
- [43] E. Polizzi, F. Mila, and E. S. Sørensen, “ $S = 1/2$ chain-boundary excitations in the Haldane phase of one-dimensional $S = 1$ systems,” *Phys. Rev. B* **58**, 2407 (1998).
- [44] S. Michalakis and J. Pytel, “Stability of frustration-free Hamiltonians,” (2011), arXiv:1109.1588.
- [45] R. Raussendorf, S. Bravyi, and J. Harrington, “Long-range quantum entanglement in noisy cluster states,” *Phys. Rev. A* **71**, 062313 (2005), arXiv:quant-ph/0407255.
- [46] R. Raussendorf, D. E. Browne, and H. J. Briegel, “Measurement-based quantum computation on cluster states,” *Phys. Rev. A* **68**, 022312 (2003), arXiv:quant-ph/0301052.

- [47] S. D. Barrett, S. D. Bartlett, A. C. Doherty, D. Jennings, and T. Rudolph, “Transitions in the computational power of thermal states for measurement-based quantum computation,” *Phys. Rev. A* **80**, 062328 (2009), arXiv:0807.4797.
- [48] D. Klagges and K. P. Schmidt, “Constraints on measurement-based quantum computation in effective cluster states,” *Phys. Rev. Lett.* **108**, 230508 (2012), arXiv:1111.5945.
- [49] H. Kalis, D. Klagges, R. Orus, and K. P. Schmidt, “Fate of the cluster state on the square lattice in a magnetic field,” arXiv:1205.5185.
- [50] T. Chung, S. D. Bartlett, and A. C. Doherty, “Characterizing measurement-based quantum gates in quantum many-body systems using correlation functions,” *Can. J. Phys.* **87**, 219 (2009), arXiv:0904.2609.
- [51] R. Orús, A. C. Doherty, and G. Vidal, “First order phase transition in the anisotropic quantum orbital compass model,” *Phys. Rev. Lett.* **102**, 077203 (2009), arXiv:0809.4068.
- [52] S. Bravyi, M. B. Hastings, and S. Mikhalkin, “Topological quantum order: Stability under local perturbations,” *J. Math. Phys.* **51**, 093512 (2010), arXiv:1001.0344; S. Bravyi and M. B. Hastings, “A short proof of stability of topological order under local perturbations,” (2010), arXiv:1001.4363.
- [53] X. Chen, Z.-X. Liu, and X.-G. Wen, “Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations,” *Phys. Rev. B* **84**, 235141 (2011), arXiv:1106.4752.
- [54] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, “Symmetry-protected topological orders in interacting bosonic systems,” *Science* **338**, 1604 (2012), arXiv:1106.4772.
- [55] D. V. Else, S. D. Bartlett, and A. C. Doherty(2013), in preparation.
- [56] M. B. Hastings, “Lieb-Schultz-Mattis in higher dimensions,” *Phys. Rev. B* **69**, 104431 (2004).
- [57] B. Nachtergaele and R. Sims, “Lieb-Robinson bounds and the exponential clustering theorem,” *Comm. Math. Phys.* **265**, 119 (2006).
- [58] M. B. Hastings, “Locality in quantum systems,” (2010), arXiv:1008.5137.
- [59] M. B. Hastings and X.-G. Wen, “Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance,” *Phys. Rev. B* **72**, 045141 (2005), arXiv:cond-mat/0503554.
- [60] T. J. Osborne, “Simulating adiabatic evolution of gapped spin systems,” *Phys. Rev. A* **75**, 032321 (2007), arXiv:quant-ph/0601019.
- [61] X. Chen, Z.-C. Gu, and X.-G. Wen, “Local unitary transformation, long-range quantum entanglement, wave function renormalization, and topological order,” *Phys. Rev. B* **82**, 155138 (2010), arXiv:1004.3835.
- [62] M. den Nijs and K. Rommelse, “Preroughening transitions in crystal surfaces and valence-bond phases in quantum spin chains,” *Phys. Rev. B* **40**, 4709 (1989).
- [63] F. Verstraete, M. A. Martín-Delgado, and J. I. Cirac, “Diverging entanglement length in gapped quantum spin systems,” *Phys. Rev. Lett.* **92**, 087201 (2004), arXiv:0311087.

- [64] L. C. Venuti and M. Roncaglia, “Analytic relations between localizable entanglement and string correlations in spin systems,” Phys. Rev. Lett. **94**, 207207 (2005), arXiv:cond-mat/0503021.
- [65] D. V. Else, S. D. Bartlett, and A. C. Doherty, “Symmetry protection of measurement-based quantum computation in ground states,” New J. Phys. **14**, 113016 (2012), arXiv:1207.4805.
- [66] F. Pollmann and A. M. Turner, “Detection of symmetry-protected topological phases in one dimension,” Phys. Rev. B **86**, 125441 (2012), arXiv:1204.0704.
- [67] K. Duivenvoorden and T. Quella, “From symmetry-protected topological order to Landau order,” (2013), arXiv:1304.7234.
- [68] M. Oshikawa, “Hidden $Z_2 \times Z_2$ symmetry in quantum spin chains with arbitrary integer spin,” J. Phys.: Condens. Matter **4**, 7469 (1992).
- [69] Y. G. Berkovich and E. M. Zhmud', *Characters of Finite Groups*, Vol. 1 (American Mathematical Society, Providence, Rhode Island, 1998).
- [70] R. Frucht, “Über die Darstellung endlicher abelscher Gruppen durch Kollineationen,” Journal für die reine und angewandte Mathematik **1932**, 16 (1932).
- [71] I. Affleck, T. Kennedy, E. Lieb, and H. Tasaki, “Valence bond ground states in isotropic quantum antiferromagnets,” Comm. Math. Phys. **115**, 477 (1988); I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, “Rigorous results on valence-bond ground states in antiferromagnets,” Phys. Rev. Lett. **59**, 799 (1987).
- [72] X. Chen, Z.-C. Gu, and X.-G. Wen, “Complete classification of one-dimensional gapped quantum phases in interacting spin systems,” Phys. Rev. B **84**, 235128 (2011), arXiv:1103.3323.
- [73] K. Duivenvoorden and T. Quella, “On topological phases of spin chains,” (2012), arXiv:1206.2462.
- [74] F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, “Renormalization-group transformations on quantum states,” Phys. Rev. Lett. **94**, 140601 (2005), arXiv:quant-ph/0410227.
- [75] M. Levin and Z.-C. Gu, “Braiding statistics approach to symmetry-protected topological phases,” Phys. Rev. B **86**, 115109 (2012), arXiv:1202.3120.
- [76] L.-Y. Hung and X.-G. Wen, “Quantized topological terms in weakly coupled gauge theories and their connection to symmetry protected topological phases,” (2012), arXiv:1211.2767.
- [77] G. Vidal, “Entanglement renormalization,” Phys. Rev. Lett. **99**, 220405 (2007), arXiv:cond-mat/0512165.
- [78] G. Vidal, “Class of quantum many-body states that can be efficiently simulated,” Phys. Rev. Lett. **101**, 110501 (2008), arXiv:quant-ph/0610099.