Integral Basis Theorem of cyclotomic Khovanov-Lauda-Rouquier Algebras of type A

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Contents

Introduction

Khovanov and Lauda [**13, 12**] and Rouquier [**25**] have introduced a remarkable new family of algebras $\mathscr{R}_n$, the **quiver Hecke algebras**, for each oriented quiver, and they showed that it can categorify the positive part of the enveloping algebras of the corresponding quantum groups. The algebras $\mathscr{R}_n$ are naturally $\mathbb{Z}$-graded. Varagnolo and Vasserot [**26**] proved that, under this categorification, the canonical basis corresponds to the image of the projective indecomposable modules of the Grothendieck rings of the quiver Hecke algebras when the Cartan matrix is symmetric.

The algebra $\mathscr{R}_n$ is infinite dimensional and for every highest weight vector in the corresponding Kac-Moody algebra there is an associated finite dimensional 'cyclotomic quotient' $\mathscr{R}_n^\Lambda$ of $\mathscr{R}_n$. The cyclotomic quiver algebras $\mathscr{R}_n^\Lambda$ were originally defined by Khovanov and Lauda [**13, 12**] and Rouquier [**25**] who conjectured that these algebras should categorify the irreducible representations of the corresponding quantum group. Lauda and Vazirani [**19**] proved that, up to shift, the simple $\mathscr{R}_n$-modules are indexed by the vertices of the corresponding crystal graph, and Kang and Kashiwara [**11**] proved the full conjecture by showing that the images of the projective irreducible modules in the Grothendieck ring $Rep(\mathscr{R}_n^\Lambda)$ correspond to the canonical basis of the corresponding highest weight module. Prior to this work, Brundan and Stropple [**6**] proved this conjecture in the special the case when Λ is a dominat weight of level 2 and Γ is the linear quiver and Brundan and Kleshchev [**4**] established the conjecture for all Λ when Γ is a quiver of type A.

Brundan and Kleshchev [**3**] proved that every degenerate and non-degenerate cyclotomic Hecke algebra H_n^Λ of type $G(r,1,n)$ over a field is isomorphic to a cyclotomic quiver Hecke algebra $\mathscr{R}_n^\Lambda$ of type A. They did this by constructing an explicit isomorphisms between these two algebras.

The algebras $\mathscr{R}_n^\Lambda$ are defined by generators and relations and so these algebras are defined over any integral domain. Let Γ be the quiver of type A_e, for $e \in \{0,2,3,4,\dots\}$. Hu and Mathas [**9**] defined a homogeneous basis $\{\psi_{\mathsf{st}}\}$ of the cyclotomic quiver algebras $\mathscr{R}_n^\Lambda$ (see Theorem 1.4.5 below), and they showed that $\mathscr{R}_n^\Lambda$ is $\mathbb{Z}$-free whenever $e = 0$ or e is invertible in the ground ring. They asked whether the algebra $\mathscr{R}_n^\Lambda$ is always $\mathbb{Z}$-free. Kleshchev-Mathas-Ram [**14**] defined $\mathbb{Z}$-free Specht modules for the cyclotomic KLR algebras of type A (and the affine KLR algebras of type A), but that the existence of these modules does not imply that the cyclotomic KLR algebras are torsion free. The main result of this thesis shows that this is always the case. More precisely, we prove the following.

0.0.1. Theorem. *Let $\mathscr{R}_n^\Lambda(\mathbb{Z})$ be a cyclotomic Khovanov-Lauda-Rouquier algebra of type A over $\mathbb{Z}$, where Λ is a dominant weight of height ℓ. Then $\mathscr{R}_n^\Lambda(\mathbb{Z})$ is a graded cellular algebra, with respect to the dominance order, with homogeneous cellular basis $\{\psi_{\mathsf{st}} \mid \lambda \in \mathscr{P}_n^\Lambda$ and $\mathsf{s},\mathsf{t} \in \mathrm{Std}(\lambda)\}$. In particular, $\mathscr{R}_n^\Lambda(\mathbb{Z})$ is $\mathbb{Z}$-free of rank $\ell^n n!$.*

If $\mathscr{O}$ is any integral domain then $\mathscr{R}_n^\Lambda(\mathscr{O}) \cong \mathscr{R}_n^\Lambda(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathscr{O}$, so it follows that $\mathscr{R}_n^\Lambda(\mathscr{O})$ is free over $\mathscr{O}$.

The proof of our main theorem is long and technical, requiring a delicate multistage induction. Fortunately, by [**9**, Theorem 5.14] we may assume that $e \neq 2$. Even though our arguments should apply in this case, being able to assume that $e \neq 2$ dramatically simplifies our arguments because the quiver of type A_e is simply laced when $e \neq 2$.

The starting point for our arguments is the observation that the definition of Hu and Mathas' the homogeneous elements ψ_{st} makes sense over any ring. Consequently, the linearly independents elements $\{\psi_{\mathsf{st}}\}$ span a $\mathbb{Z}$-free submodule R_n^Λ of $\mathscr{R}_n^\Lambda$. To prove our Main Theorem it is therefore enough to show that R_n^Λ is closed multiplication by the generators of $\mathscr{R}_n^\Lambda$ and that the identity element of $\mathscr{R}_n^\Lambda$ belongs to R_n^Λ.

The algebra $\mathscr{R}_n^\Lambda$ is generated by elements y_r, ψ_s and $e(\mathbf{i})$, where $1 \leq r \leq n$, $1 \leq s < n$ and $\mathbf{i} \in (Z/e\mathbb{Z})^n$. These three classes of generators must all be treated separately. The cellular basis element ψ_{st} is indexed by two standard λ-tableaux where λ is a multipartition of n; the definitions of these terms are recalled in Chapter 1. We argue by simultaneous induction on n, and on the lexicographic orderings on the set of multipartitions, to show that multiplication by the KLR generators always sends ψ_{st} to a $\mathbb{Z}$-linear combination of terms ψ_{uv} which are larger in the lexicographic order. Multiplication by y_r is the hardest case, partly because once this case is understood it can be used to understand the action of ψ_r and $e(\mathbf{i})$ on the ψ-basis of $\mathscr{R}_n^\Lambda$.

After we have proved Theorem 0.0.1, we obtain a graded cellular basis of $\mathscr{R}_n^\Lambda$. We then extend it to obtain a graded cellular basis of $\mathscr{R}_n$, which indicates that $\mathscr{R}_n$ is an affine graded cellular algebra. Hence we can use similar argument to Graham-Lehrer [**7**] to give a complete set of non-isomorphic graded irreducible $\mathscr{R}_n$-modules. Koenig and Xi [**18**] introduced the notion of affine cellular algebras and they have shown that the affine Hecke algebra of type A is affine cellular. They gave a different approach to classify the irreducible representation of affine Hecke algebras.

Finally, we work with the Jucys-Murphy elements of cyclotomic Hecke algebras of type A for $e > 0$ and $p > 0$ in both degenerate and non-degenerate cases. We have known that the cyclotomic KLR algebras are isomorphic to cyclotomic Hecke algebras of type A, our first task is to express $e(\mathbf{i})$'s in $\mathscr{R}_n^\Lambda$ using Jucys-Murphy elements in explicit form, and then we show that the Jucys-Murphy elements have certain periodic property, i.e. we can find n and d such that $x_r^n = x_r^{n+d}$, and we give information about the minimal values of n and d.

In more detail, this thesis is organized as follows. In Chapter 1 we summarise the background material from the representation theory of the cyclotomic Khovanov-Lauda-Rouquier algebras that we need, including the theory of (graded) cellular algebras and the combinatorics of multipartitions and tableaux. In Chapter 2 considers the special case where λ is a multicomposition which has at most two rows. Once this case is understood we are able to show for an arbitary multipartition λ that $\psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda} y_r$ is a $\mathbb{Z}$-linear combination of higher terms, where t^λ is the 'initial' λ-tableau. Chapter 3 begins by proving, again by induction, that $\psi_{\mathsf{st}} y_r$ is a linear combination of bigger terms in R_n^Λ. By considering the Garnir tableau of two-rowed multipartition we then show that $\psi_{\mathsf{st}}\psi_r$ can be written in the required form. This result is then extended to multipartitions of arbitrary shape. Finally, we deduce that $e(\mathbf{i}) \in R_n^\Lambda$, for all $\mathbf{i} \in (\mathbb{Z}/e\mathbb{Z})^n$, which completes the proof of our main result. In Chapter 4 we define a sequence of weights $(\Lambda^{(k)})$ and using it to extend the graded cellular basis of $\mathscr{R}_\alpha^\Lambda$ to $\mathscr{R}_\alpha$ and hence generate a graded cellular basis for $\mathscr{R}_n$. Then using similar arguments as Graham-Lehrer [**7**] we give a complete set of non-isomorphic graded simple $\mathscr{R}_n$-modules. In Chapter 5 first we give an expression of $e(\mathbf{i})$ using Jucys-Murphy elements and then simplify the expression to an explicit form. Finally using the nilpotency properties of y_r's in $\mathscr{R}_n^\Lambda$ and our explicit form of $e(\mathbf{i})$'s we prove the periodic property of the Jucys-Murphy elements in both degenerate and non-degenerate cases.

Finally, we remark that the calculations in Chapters 2 and 3 gives an algorithm inductively for multiplying y_r and ψ_s to ψ_{st}.

Acknowledgements. I would like to thank my supervisor Andrew Mathas for his support and proofreading my thesis. I also want to thank Matthew Gibson, who gave me advice on writing thesis in English. Further thanks go to Jun Hu, Jonathan Brundan and Alexander Kleshchev for many helpful discussions with me about cyclotomic Khovanov-Lauda-Rouquier algebras.

Declaration of originality. The work presented in this thesis is original except where stated otherwise. No part of this thesis has been submitted for the award of any other degree or diploma at this or any other university.

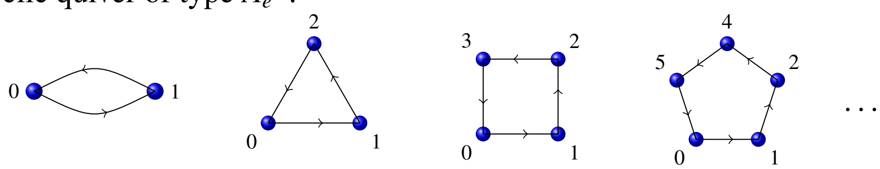
Chapter 1

Khovanov-Lauda-Rouquier Algebras

In this chapter we are going to introduce the necessary background for our work. First we will define our principal object of study — the (cyclotomic) Khovanov-Lauda-Rouquier algebras $\mathscr{R}_n^\Lambda$. Then we give a brief introduction to (graded) cellular algebras and symmetric groups. Finally after explaining tableaux combinatorics we describe a graded cellular basis for the cyclotomic KLR algebra, found by Hu and Mathas [**9**].

1.1. The cyclotomic Khovanov-Lauda-Rouquier algebras

Fix an integer $e \in \{0, 2, 3, 4 \dots\}$ and $I = \mathbb{Z}/e\mathbb{Z}$. Let Γ_e be the oriented quiver with vertex set I and directed edges $i \to i+1$, for $i \in I$. Thus, Γ_e is the quiver of type A_∞ if $e = 0$ and if $e \geq 2$ then it is a cyclic quiver of type $A_e^{(1)}$:



$e = 2$ $\qquad$ $e = 3$ $\qquad$ $e = 4$ $\qquad$ $e = 5$

Let $(a_{i,j})_{i,j \in I}$ be the symmetric Cartan matrix associated with Γ_e, so that

$$a_{i,j} = \begin{cases} 2, & \text{if } i = j, \\ 0, & \text{if } i \neq j \pm 1, \\ -1, & \text{if } e \neq 2 \text{ and } i = j \pm 1, \\ -2, & \text{if } e = 2 \text{ and } i = j + 1. \end{cases}$$

To the quiver Γ_e attach the standard Lie theoretic data of a Cartan matrix $(a_{ij})_{i,j \in I}$, fundamental weights $\{\Lambda_i | i \in I\}$, positive weights $P_+ = \sum_{i \in I} \mathbb{N}\Lambda_i$, positive roots $Q_+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ and let $(\cdot, \cdot)$ be the bilinear form determined by

$$(\alpha_i, \alpha_j) = a_{ij} \qquad \text{and} \qquad (\Lambda_i, \alpha_j) = \delta_{ij}, \qquad \text{for } i, j \in I.$$

Fix a **weight** $\Lambda = \sum_{i \in I} a_i \Lambda_i \in P_+$. Then Λ is a weight of **level** $l(\Lambda) = \ell = \sum_{i \in I} a_i$. A **multicharge** for Λ is a sequence $\kappa_\Lambda = (\kappa_1, \dots, \kappa_\ell) \in I^\ell$ such that

$$(\Lambda, \alpha_i) = a_i = \#\{\, 1 \leq s \leq \ell \mid \kappa_s \equiv i \pmod e \,\}$$

for any $i \in I$.

The following algebras were introduced by Khovanov and Lauda and Rouquier who defined KLR algebras for arbitrary oriented quivers.

1.1.1. Definition (Khovanov and Lauda [**13, 12**] and Rouquier [**25**]). *Suppose $\mathscr{O}$ is an integral ring and n is a positive integer. The **Khovanov-Lauda–Rouquier algebra**, $\mathscr{R}_n(\mathscr{O})$ of type Γ_e is the unital associative $\mathscr{O}$-algebra with generators*

$$\{\hat{\psi}_1, \dots, \hat{\psi}_{n-1}\} \cup \{\hat{y}_1, \dots, \hat{y}_n\} \cup \{\, \hat{e}(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$$

and relations

$$\hat{e}(\mathbf{i})\hat{e}(\mathbf{j}) = \delta_{\mathbf{ij}}\hat{e}(\mathbf{i}), \qquad \textstyle\sum_{\mathbf{i}\in I^n}\hat{e}(\mathbf{i}) = 1, \tag{1.1.2}$$

$$\hat{y}_r\hat{e}(\mathbf{i}) = \hat{e}(\mathbf{i})y_r, \qquad \hat{\psi}_r\hat{e}(\mathbf{i}) = \hat{e}(s_r{\cdot}\mathbf{i})\hat{\psi}_r, \qquad \hat{y}_r\hat{y}_s = \hat{y}_s\hat{y}_r, \tag{1.1.3}$$

$$\hat{\psi}_r\hat{y}_s = \hat{y}_s\hat{\psi}_r, \qquad \text{if } s \neq r, r+1, \tag{1.1.4}$$

$$\hat{\psi}_r\hat{\psi}_s = \hat{\psi}_s\hat{\psi}_r, \qquad \text{if } |r-s| > 1, \tag{1.1.5}$$

$$\hat{\psi}_r\hat{y}_{r+1}\hat{e}(\mathbf{i}) = \begin{cases} (\hat{y}_r\hat{\psi}_r + 1)\hat{e}(\mathbf{i}), & \text{if } i_r = i_{r+1},\\ \hat{y}_r\hat{\psi}_r\hat{e}(\mathbf{i}), & \text{if } i_r \neq i_{r+1}\end{cases} \tag{1.1.6}$$

$$\hat{y}_{r+1}\hat{\psi}_r\hat{e}(\mathbf{i}) = \begin{cases} (\hat{\psi}_r\hat{y}_r + 1)\hat{e}(\mathbf{i}), & \text{if } i_r = i_{r+1},\\ \hat{\psi}_r\hat{y}_r\hat{e}(\mathbf{i}), & \text{if } i_r \neq i_{r+1}\end{cases} \tag{1.1.7}$$

$$\hat{\psi}_r^2\hat{e}(\mathbf{i}) = \begin{cases} 0, & \text{if } i_r = i_{r+1},\\ \hat{e}(\mathbf{i}), & \text{if } i_r \neq i_{r+1} \pm 1,\\ (\hat{y}_{r+1} - \hat{y}_r)\hat{e}(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r + 1,\\ (\hat{y}_r - \hat{y}_{r+1})\hat{e}(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+1} = i_r - 1,\\ (\hat{y}_{r+1} - \hat{y}_r)(\hat{y}_r - \hat{y}_{r+1})\hat{e}(\mathbf{i}), & \text{if } e = 2 \text{ and } i_{r+1} = i_r + 1 \end{cases} \tag{1.1.8}$$

$$\hat{\psi}_r\hat{\psi}_{r+1}\hat{\psi}_r\hat{e}(\mathbf{i}) = \begin{cases} (\hat{\psi}_{r+1}\hat{\psi}_r\hat{\psi}_{r+1} + 1)\hat{e}(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+2} = i_r = i_{r+1} - 1,\\ (\hat{\psi}_{r+1}\hat{\psi}_r\hat{\psi}_{r+1} - 1)\hat{e}(\mathbf{i}), & \text{if } e \neq 2 \text{ and } i_{r+2} = i_r = i_{r+1} + 1,\\ (\hat{\psi}_{r+1}\hat{\psi}_r\hat{\psi}_{r+1} + \hat{y}_r & \\ \qquad -2\hat{y}_{r+1} + \hat{y}_{r+2})\hat{e}(\mathbf{i}), & \text{if } e = 2 \text{ and } i_{r+2} = i_r = i_{r+1} + 1,\\ \hat{\psi}_{r+1}\hat{\psi}_r\hat{\psi}_{r+1}\hat{e}(\mathbf{i}), & \text{otherwise.} \end{cases} \tag{1.1.9}$$

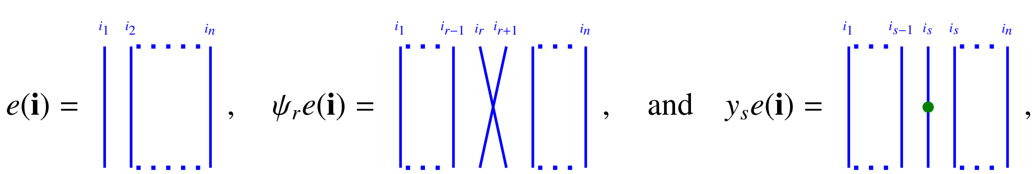
for $\mathbf{i}, \mathbf{j} \in I^n$ *and all admissible* r *and* s. *Moreover,* $\mathscr{R}_n(\mathscr{O})$ *is naturally* $\mathbb{Z}$*-graded with degree function determined by*

$$\deg \hat{e}(\mathbf{i}) = 0, \qquad \deg \hat{y}_r = 2 \qquad \textit{and} \qquad \deg \hat{\psi}_s\hat{e}(\mathbf{i}) = -a_{i_s,i_{s+1}},$$

for $1 \le r \le n$, $1 \le s < n$ *and* $\mathbf{i} \in I^n$.

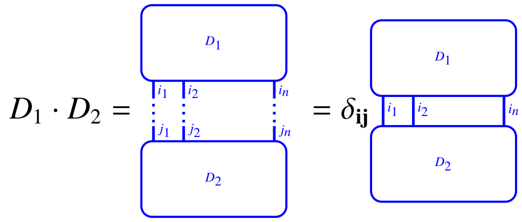
Notice that the relations depend on the quiver Γ_e. By [**9**, Theorem 5.14], if $\mathscr{O}$ is a commutative integral domain and suppose either $e = 0$, e is non-zero prime, or that $e{\cdot}1_{\mathscr{O}}$ is invertible in $\mathscr{O}$, $\mathscr{R}_n^\Lambda(\mathscr{O})$ is an $\mathscr{O}$-free algebra.

Following Khovanov and Lauda [**13**], we will frequently use diagrammatic analogues of the relations of $\mathscr{R}_n(\mathscr{O})$ in order to simplify our calculations. To do this we associate to each generator of $\mathscr{R}_n(\mathscr{O})$ an I-labelled decorated planar diagram on $2n$ points in the following way:



for $\mathbf{i} \in I^n$, $1 \le r < n$ and $1 \le s \le n$. The r-th string of the diagram is the string labelled with i_r.

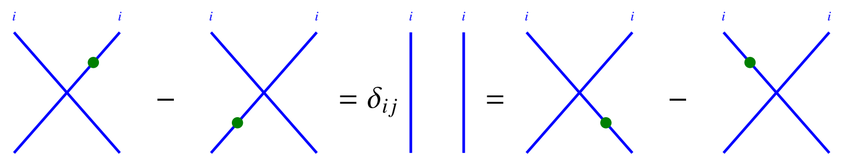
Diagrams are considered up to isotopy, and multiplication of diagrams is given by concatenation, subject to the relations (1.1.2)–(1.1.9). In more detail, if D_1 and D_2 are two diagrams then the diagrammatic analogue of the relation $e(\mathbf{i})e(\mathbf{j}) = \delta_{\mathbf{ij}}e(\mathbf{i})$ is



That is, $D_1 \cdot D_2 = 0$ unless the labels of the strings on the bottom of D_1 match the corresponding labels on the top of the strings in D_2 in which case we just concatenate the two diagrams.

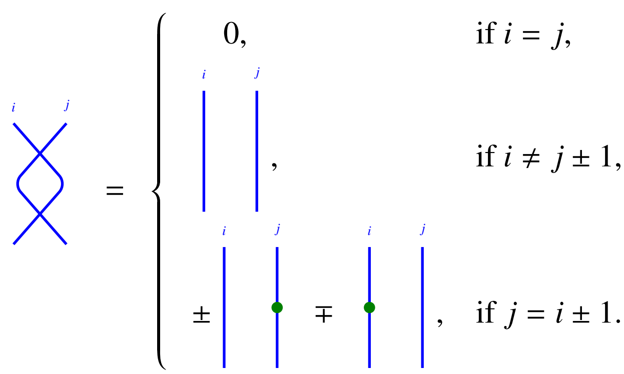
Multiplication by y_r simply adds a decorative dot to the r-th string, reading left to right, so relations (1.1.3)–(1.1.5) become self when written in terms of diagrams. Ignoring the extraneous strings on the left and right, and setting $i = i_r$ and $j = i_{r+1}$, the diagrammatic analogue of relations (1.1.6) and (1.1.7) is

(1.1.10)



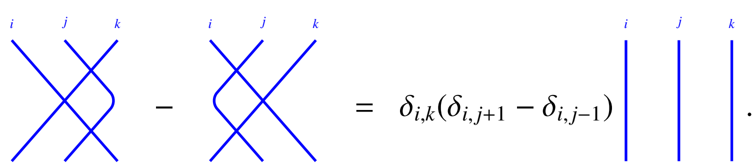
Similarly, if $e \neq 2$ then relation (1.1.8) becomes

(1.1.11)



and if $e \neq 2$ then the diagrammatic analogue of relation (1.1.9) is

(1.1.12)

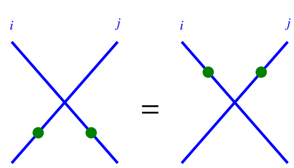


Using the relations in $\mathscr{R}_n(\mathscr{O})$ it is easy to verify the following identity which we record for future use:

$$\hat{e}(\mathbf{i})\hat{y}_r^k\hat{y}_{r+1}^k\hat{\psi}_r = \hat{e}(\mathbf{i})\hat{\psi}_r\hat{y}_r^k\hat{y}_{r+1}^k \tag{1.1.13}$$

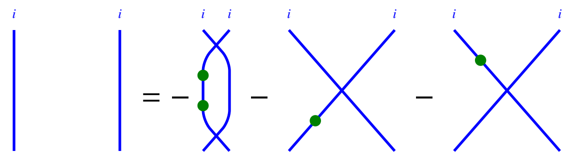
for any $\mathbf{i}$. Clearly it is enough to prove this relation when $k = 1$ when, diagrammatically, this identity takes the form

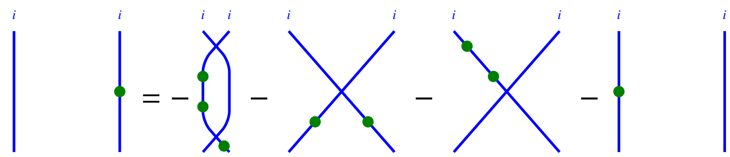
(1.1.14)

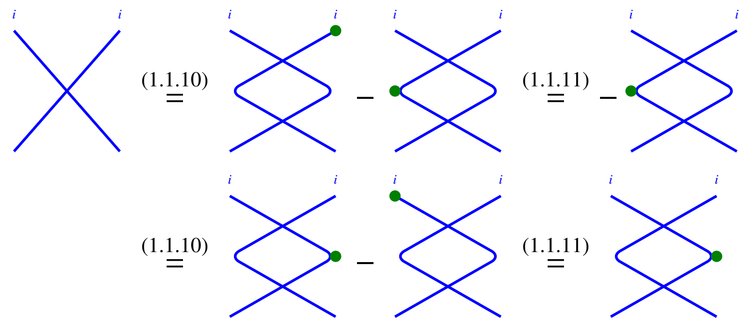


locally on the r and $r + 1$-th strings and where we set $i = i_r$ and $j = i_{r+1}$.

Three more easy, and very useful, consequences of the relations are the following:

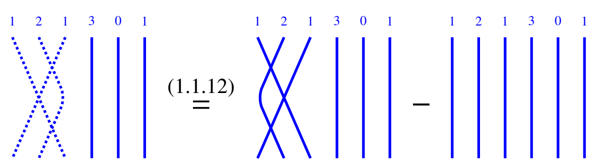
(1.1.15) 

(1.1.16) 

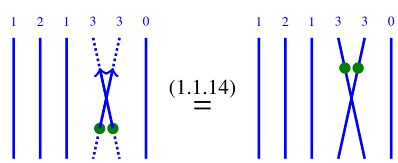
(1.1.17) 

Note that (1.1.16) follows by multiplying (1.1.15) by y_{r+1} and expanding.

In the rest of the thesis we will play around with these diagrammatic notations a lot. In order to make the reader easy to follow our calculation we will use dotted strands to represent moving strands and arrows to represent moving dots. If we are going to move a dot then we will also write the strand which the dot is on dotted so the reader can see the arrow clearly. For example, we will write

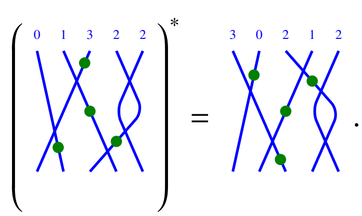


to signify the application of relation (1.1.12) and



to signify the application of relation (1.1.14).

We can define a linear map $*: \mathscr{R}_n \longrightarrow \mathscr{R}_n$ by swapping the diagrams of $\mathscr{R}_n$ up-side-down. For example,



It is obvious that $*$ is an anti-isomorphism and it preserves the generators of $\mathscr{R}_n$.

Fix a weight $\Lambda = \sum_{i\in I} a_i\Lambda_i$ with $a_i \in \mathbb{N}$. Let $N_n^\Lambda(\mathscr{O})$ be the two-sided ideal of $\mathscr{R}_n$ generated by the elements with form $e(\mathbf{i})y_1^{(\Lambda,\alpha_{i_1})}$. We can now define the main object of study in this thesis, the cyclotomic Khovanov-Lauda-Rouquier algebras, which were introduced by Khovanov and Lauda [**13**, Section 3.4].

1.1.18. Definition. *The **cyclotomic Khovanov-Lauda-Rouquier algebras** of weight Λ and type Γ_e is the algebra $\mathscr{R}_n^\Lambda(\mathscr{O}) = \mathscr{R}_n(\mathscr{O})/N_n^\Lambda(\mathscr{O})$.*

Therefore, if we write $e(\mathbf{i}) = \hat{e}(\mathbf{i}) + N_n^\Lambda(\mathscr{O})$, $y_r = \hat{y}_r + N_n^\Lambda(\mathscr{O})$ and $\psi_s = \hat{\psi}_s + N_n^\Lambda(\mathscr{O})$, the algebra $\mathscr{R}_n^\Lambda(\mathscr{O})$ is the unital $\mathscr{O}$-algebra generated by

$$\{\psi_1, \dots, \psi_{n-1}\} \cup \{y_1, \dots, y_n\} \cup \{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$$

subject to the relations (1.1.2)–(1.1.9) of $\mathscr{R}_n(\mathscr{O})$ together with the additional relation

$$e(\mathbf{i})y_1^{(\Lambda,\alpha_{i_1})} = 0, \qquad \text{for each } \mathbf{i} \in I^n. \tag{1.1.19}$$

1.2. The (graded) cellular algebras and the symmetric groups

Following Graham and Lehrer [**7**], we now introduce the graded cellular algebras. Reader may also refer to Hu-Mathas [**9**]. Let $\mathscr{O}$ be a commutative ring with 1 and let A be a unital $\mathscr{O}$-algebra.

1.2.1. Definition. *A **graded cell datum** for A is a triple $(\Lambda, T, C, \deg)$ where $\Lambda = (\Lambda, >)$ is a poset, either finite or infinite, and $T(\lambda)$ is a finite set for each $\lambda \in \Lambda$, $\deg$ is a function from $\coprod_\lambda T(\lambda)$ to $\mathbb{Z}$, and*

$$C\colon \coprod_{\lambda\in\Lambda} T(\lambda) \times T(\lambda) \longrightarrow A$$

is an injective map which sends (s,t) to a_{st}^λ such that:

(a) $\{\, a_{\mathsf{st}}^\lambda \mid \lambda \in \Lambda, s, t \in T(\lambda)\,\}$ is an $\mathscr{O}$-free basis of A;

(b) for any $r \in A$ and $\mathsf{t} \in T(\lambda)$, there exists scalars $c_{\mathsf{t}}^{\mathsf{v}}(r)$ such that, for any $\mathsf{s} \in T(\lambda)$,

$$a_{\mathsf{st}}^\lambda \cdot r \equiv \sum_{\mathsf{v}\in T(\lambda)} c_{\mathsf{t}}^{\mathsf{v}}(r) a_{\mathsf{sv}}^\lambda \mod A^{>\lambda}$$

where $A^{>\lambda}$ is the $\mathscr{O}$-submodule of A spanned by $\{\, a_{\mathsf{xy}}^\mu \mid \mu > \lambda, \mathsf{x}, \mathsf{y} \in T(\mu)\,\}$;

(c) the $\mathscr{O}$-linear map $*: A \longrightarrow A$ which sends a_{st}^λ to a_{ts}^λ, for all $\lambda \in \Lambda$ and $\mathsf{s}, \mathsf{t} \in T(\lambda)$, is an anti-isomorphism of A.

(d) each basis element a_{st}^λ is homogeneous of degree $\deg a_{\mathsf{st}}^\lambda = \deg(\mathsf{s}) + \deg(\mathsf{t})$, for $\lambda \in \Lambda$ and all $\mathsf{s}, \mathsf{t} \in T(\lambda)$.

If a graded cell datum exists for A then A is a **graded cellular algebra**. Similarly, by forgetting the grading we can define a **cell datum** and hence a **cellular algebra**.

Suppose A is a graded cellular algebra with graded cell datum $(\Lambda, T, C, \deg)$. For any $\lambda \in \Lambda$, define $A^{\geq\lambda}$ to be the $\mathscr{O}$-submodule of A spanned by

$$\{\, c_{\mathsf{st}}^\mu \mid \mu \geq \lambda, \mathsf{s}, \mathsf{t} \in T(\mu)\,\}.$$

Then $A^{>\lambda}$ is an ideal of $A^{\geq\lambda}$ and hence $A^{\geq\lambda}/A^{>\lambda}$ is a A-module. For any $\mathsf{s} \in T(\lambda)$ we define C_{s}^λ to be the A-submodule of $A^{\geq\lambda}/A^{>\lambda}$ with basis $\{\, a_{\mathsf{st}}^\lambda + A^{>\lambda} \mid \mathsf{t} \in T(\lambda)\,\}$. By the cellularity of A we have $C_{\mathsf{s}}^\lambda \cong C_{\mathsf{t}}^\lambda$ for any $\mathsf{s}, \mathsf{t} \in T(\lambda)$.

1.2.2. Definition. *Suppose $\lambda \in \mathscr{P}_n^\Lambda$. Define the **cell module** of A to be $C^\lambda = C_{\mathsf{s}}^\lambda$ for any $\mathsf{s} \in T(\lambda)$, which has basis $\{\, a_{\mathsf{t}}^\lambda \mid \mathsf{t} \in T(\lambda)\,\}$ and for any $r \in A$,*

$$a_{\mathsf{t}}^\lambda \cdot r = \sum_{\mathsf{u}\in T(\lambda)} c_{\mathsf{u}}^r a_{\mathsf{u}}^\lambda$$

where c^r_{u} are determined by

$$a^\lambda_{\mathsf{st}}\cdot r = \sum_{\mathsf{u}\in T(\lambda)} c^r_{\mathsf{u}} a^\lambda_{\mathsf{su}} + A^{>\lambda}.$$

We can define a bilinear map $\langle\cdot,\cdot\rangle : C^\lambda \times C^\lambda \longrightarrow \mathbb{Z}$ such that

$$\langle a^\lambda_{\mathsf{s}}, a^\lambda_{\mathsf{t}}\rangle a^\lambda_{\mathsf{uv}} = a^\lambda_{\mathsf{us}} a^\lambda_{\mathsf{tv}} + A^{>\lambda}$$

and let $\operatorname{rad} C^\lambda = \{\, \mathsf{s} \in C^\lambda \mid \langle \mathsf{s}, \mathsf{t}\rangle = 0 \text{ for all } \mathsf{t} \in C^\lambda \,\}$. The $\operatorname{rad} C^\lambda$ is a graded A-submodule of C^λ.

1.2.3. Definition. *Suppose $\lambda \in \mathscr{P}^\Lambda_n$. Let $D^\lambda = C^\lambda / rad\, C^\lambda$ as a graded A-module.*

Exactly as in the ungraded case [**7**, Theorem 3.4] or [**9**, Theorem 2.10], we obtain the following:

1.2.4. Theorem. *The set $\{\, D^\lambda\langle k\rangle \mid \lambda \in \Lambda, D^\lambda \neq 0, k \in \mathbb{Z}\,\}$ is a complete set of pairwise non-isomorphic graded simple A-modules.*

We give an example of graded cellular algebras here, which is called the cyclotomic Hecke algebras.

Let $\mathbb{F}_p$ be a fixed field of characteristic $p \geq 0$ with $q \in \mathbb{F}^\times_p$. Let e be the smallest positive integer such that $1 + q + \ldots + q^{e-1} = 0$ and setting $e = 0$ if no such integer exists. Then define $I = \mathbb{Z}/e\mathbb{Z}$ if $e > 0$ and $I = \mathbb{Z}$ if $e = 0$.

For $n \geq 0$, assume that $q = 1$. Let H_n be the **degenerate affine Hecke algebra**, working over $\mathbb{F}_p$. So H_n has generators

$$\{x_1, \ldots, x_n\} \cup \{s_1, \ldots, s_{n-1}\}$$

subject to the following relations

$$\begin{aligned} x_r x_s &= x_s x_r; \\ s_r x_{r+1} &= x_r s_r + 1, \qquad s_r s_x = x_s s_r \quad \text{if } s \neq r, r+1 \\ s_r^2 &= 1; \\ s_r s_{r+1} s_r &= s_{r+1} s_r s_{r+1}, \qquad s_r s_t = s_t s_r \quad \text{if } |r-t| > 1 \end{aligned}$$

Now we assume that $q \neq 1$ and H_n be the **non-degenerate affine Hecke algebra** over $\mathbb{F}_p$. So H_n has generators

$$\{X_1^{\pm1}, \ldots, X_n^{\pm1}\} \cup \{T_1, \ldots, T_{n-1}\}$$

subject to the following relations

$$\begin{aligned} X_r^{\pm1} X_s^{\pm1} &= X_s^{\pm1} X_r^{\pm1}, \qquad && X_r X_r^{-1} = 1; \\ T_r X_r T_r &= q X_{r+1}, \qquad && T_r X_s = X_s T_r \quad \text{if } s \neq r, r+1; \\ T_r^2 &= (q-1)T_r + q; && \\ T_r T_{r+1} T_r &= T_{r+1} T_r T_{r+1}, \qquad && T_r T_s = T_s T_r \quad \text{if } |r-s| > 1. \end{aligned}$$

Then for any $\Lambda \in P_+$, we define

$$H^\Lambda_n = \begin{cases} H_n / \langle \prod_{i\in I} (X_1 - q^i)^{(\Lambda,\alpha_i)} \rangle, & \text{if } q \neq 1, \\ H_n / \langle \prod_{i\in I} (x_1 - i)^{(\Lambda,\alpha_i)} \rangle, & \text{if } q = 1. \end{cases} \tag{1.2.5}$$

and we call H^Λ_n the **degenerate cyclotomic Hecke algebra** if $q = 1$ and **non-degenerate cyclotomic Hecke algebra** if $q \neq 1$.

By the definitions, degenerate and non-degenerate cyclotomic Hecke algebras are similar with some minor difference. In order to minimize their difference we define

$$q_i = \begin{cases} i, & \text{if } q = 1, \\ q^i, & \text{if } q \neq 1. \end{cases} \tag{1.2.6}$$

and use x_r instead of X_r when we don't have to distinguish which case we are working with. Hence we can re-write (1.2.5) as

$$H_n^\Lambda = H_n/\langle\prod_{i\in I}(x_1 - q_i)^{(\Lambda,\alpha_i)}\rangle. \tag{1.2.7}$$

Murphy [**24**] gave a set of cellular basis for H_n^Λ which shows that H_n^Λ is a cellular algebra. Brundan and Kleshchev [**3**] proved the remarkable result that every H_n^Λ over $\mathbb{F}_p$ is isomorphic to $\mathscr{R}_n^\Lambda(\mathbb{F}_p)$ introduced in Definition 1.1.1, where in both algebras Λ and e are the same. Therefore when H_n^Λ is over a field it is a graded cellular algebra.

In the rest of the section we will introduce a special case of H_n^Λ, which is also an important object we are going to need for the rest of the thesis.

The symmetric group $\mathfrak{S}_n$ is the group of permutations on $1, 2, \dots, n$. For $i = 1, 2, \dots, n-1$, let s_i be the transposition $(i, i+1)$. The following result is well-known; see for example, [**20**, Exercise 1.1].

1.2.8. Definition. *The symmetric group $\mathfrak{S}_n$ is generated by $s_1, s_2, \dots, s_{n-1}$ subject only to the relations:*

$$\begin{aligned} s_i^2 &= 1, && \textit{for } i = 1, 2, \dots, n-1,\\ s_is_j &= s_js_i, && \textit{for } 1 \le i < j-1 \le n-2,\\ s_is_{i+1}s_i &= s_{i+1}s_is_{i+1}, && \textit{for } i = 1, 2, \dots, n-2. \end{aligned}$$

It is easy to see that $H_n^\Lambda = \mathbb{F}_p\mathfrak{S}_n$ when $q = 1$ and $\Lambda = \Lambda_i$ for some $i \in I$. Therefore $\mathbb{F}_p\mathfrak{S}_n$ is a graded cellular algebras as well.

Suppose w is an element of $\mathfrak{S}_n$ and $w = s_{i_1}s_{i_2}\dots s_{i_m}$. If m is minimal we say that w has **length** m and write $l(w) = m$. In this case we say $s_{i_1}s_{i_2}\dots s_{i_m}$ is a **reduced expression** of w. In general an element of $\mathfrak{S}_n$ has more than one reduced expressions. For example, we have $w = s_1s_2s_1 = s_2s_1s_2$. Nonetheless, all the reduced expression of an element have the same length.

In this thesis we let $\mathfrak{S}_n$ act on $\{1, 2, \dots, n\}$ from right. For example, $(i)s_is_{i+1} = (i+1)s_{i+1} = i+2$.

The following result is well-known. See, for example, [**20**, Corollary 1.4].

1.2.9. Proposition. *Suppose that $w \in \mathfrak{S}_n$. For $i = 1, 2, \dots, n-1$,*

$$l(s_iw) = \begin{cases} l(w) + 1, & \textit{if } (i)w^{-1} < (i+1)w^{-1},\\ l(w) - 1, & \textit{if } (i)w^{-1} > (i+1)w^{-1}. \end{cases}$$

We recall the definition of the **Bruhat order** $\le$ on $\mathfrak{S}_n$. For $u, w \in \mathfrak{S}_n$ define $u \le w$ if $u = s_{r_{a_1}}s_{r_{a_2}}\dots s_{r_{a_b}}$ for some $1 \le a_1 < a_2 < \dots < a_b \le m$, where $w = s_{r_1}s_{r_2}\dots s_{r_m}$ is a reduced expression for w.

1.3. Tableaux combinatorics

In this section we recall the combinatorics of (multi)partitions and (multi)tableaux that we will need in this thesis.

Let n be a positive integer. A **composition** of n is an ordered sequence of nonnegative integers $\lambda = (\lambda_1, \lambda_2, \dots)$ and $|\lambda| = \sum_{i=1}^{\infty}\lambda_i = n$. We say λ is a **partition** of n if $\lambda = (\lambda_1, \lambda_2, \dots)$ is a composition and $\lambda_1 \ge \lambda_2 \ge \lambda_3 \ge \dots$. We can then identify λ with a sequence $(\lambda_1, \dots, \lambda_k)$ whenever $\lambda_i = 0$ for $i > k$.

As we now recall, there is a natural partial ordering on the set of compositions of n. Suppose $\lambda = (\lambda_1, \lambda_2, \dots,)$ and $\mu = (\mu_1, \mu_2, \dots)$ are compositions of n. Then λ **dominates** μ, and we write

$\lambda \trianglerighteq \mu$, if

$$\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i$$

for any k. We write $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$. The dominance ordering can be extended to a total ordering $\geq$, called the **lexicographic ordering**. We write $\lambda > \mu$ if we can find some k, such that $\lambda_i = \mu_i$ for all $i < k$ and $\lambda_k > \mu_k$. Define $\lambda \geq \mu$ if $\lambda > \mu$ or $\lambda = \mu$. Then $\lambda \trianglerighteq \mu$ implies $\lambda \geq \mu$.

A **multicomposition** of n of **level** ℓ is an ordered sequence $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ of compositions such that $\sum_{i=1}^{\ell} |\lambda^{(i)}| = n$. Similarly, a **multipartition** of level ℓ is multicomposition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ of n such that each $\lambda^{(i)}$ is a partition. We will identify multicompositions and multipartitions of level 1 with compositions and partitions in the obvious way.

Let $\mathscr{C}_n^\Lambda$ be the set of all multicomposition of n and $\mathscr{P}_n^\Lambda$ be the set of all multipartitions of n. We can extend the dominance ordering to $\mathscr{C}_n^\Lambda$ by defining $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{k-1} |\lambda^{(i)}| + \sum_{j=1}^{s} \lambda_j^{(k)} \geq \sum_{i=1}^{k-1} |\mu^{(i)}| + \sum_{j=1}^{s} \mu_j^{(k)}$$

for any $1 \leq k \leq \ell$ and all $s \geq 1$. Again, we write $\lambda \triangleright \mu$ if $\lambda \trianglerighteq \mu$ and $\lambda \neq \mu$. Similarly, we extend the lexicographic ordering $\lambda > \mu$ and $\lambda \geq \mu$ to $\mathscr{C}_n^\Lambda$ in the obvious way way.

The **Young diagram** of a multicomposition λ of level ℓ is the set

$$[\lambda] = \{(r, c, l) \mid 1 \leq c \leq \lambda_r^{(l)}, r \geq 0 \text{ and } 1 \leq l \leq \ell\}$$

which we think of as an ordered ℓ-tuple of the diagrams of the partitions $\lambda^{(1)}, \ldots, \lambda^{(\ell)}$. The triple $(r, c, l) \in [\lambda]$ is **node** of λ in row r, column c and component l. A **λ-tableau** is any bijection $\mathsf{t} : [\lambda] \longrightarrow \{1, 2, \ldots, n\}$. We identify a λ-tableau t with a labeling of the diagram of λ. That is, we label the node $(r, c, l) \in [\lambda]$ with the integer $\mathsf{t}(r, c, l)$. For example,

$$\left(\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & \\ 8 & & & \end{array} \middle| \begin{array}{cc} 9 & 10 \\ 11 & 12 \\ 13 & \end{array} \middle| \begin{array}{ccc} 14 & 15 & 16 \end{array} \right)$$

is a $(4, 3, 1|2^2, 1|3)$-tableaux. If t is a λ-tableau then the **shape** of t is the multicomposition λ and we write $\mathrm{Shape}(\mathsf{t}) = \lambda$.

If $\mathsf{t} \in \mathrm{Std}(\lambda)$ and $1 \leq k \leq n$ define $\mathsf{t}|_k$ to be the subtableau of t obtained by removing all the nodes containing an entry greater than k. We define an analogue of the dominance ordering for standard tableaux by defining $\mathsf{t} \trianglerighteq \mathsf{s}$ if $\mathrm{Shape}(\mathsf{t}|_k) \trianglerighteq \mathrm{Shape}(\mathsf{s}|_k)$, for $1 \leq k \leq n$. As with the dominance ordering, if $\mathsf{t} \trianglerighteq \mathsf{s}$ then we write $\mathsf{s} \trianglelefteq \mathsf{t}$ and if $\mathsf{s} \neq \mathsf{t}$ then write $\mathsf{t} \triangleright \mathsf{s}$ and $\mathsf{s} \triangleleft \mathsf{t}$. We also define $(\mathsf{s}, \mathsf{t}) \triangleright (\mathsf{u}, \mathsf{v})$ if $\mathsf{s} \trianglerighteq \mathsf{u}$, $\mathsf{t} \trianglerighteq \mathsf{v}$ and $(\mathsf{s}, \mathsf{t}) \neq (\mathsf{u}, \mathsf{v})$.

For any multicomposition λ, define t^λ to be the unique λ-tableau such that $\mathsf{t}^\lambda \trianglerighteq \mathsf{t}$ for all standard λ-tableau t. For example, if $\lambda = (4, 3, 1|2^2, 1|3)$ then t^λ is the tableau displayed above.

The symmetric group acts on the set of all λ-tableaux. Let t be a λ-tableau, then $\mathsf{t} \cdot s_r$ is the tableau obtained by exchanging the entries r and $r + 1$ in t, i.e. $(r)\mathsf{t}^{-1} = (r + 1)(\mathsf{t} \cdot s_r)^{-1}$, $(r + 1)\mathsf{t}^{-1} = (r)(\mathsf{t} \cdot s_r)^{-1}$, and $(k)\mathsf{t}^{-1} = (k)(\mathsf{t} \cdot s_r)^{-1}$ for $k \neq r, r + 1$. Then for each λ-tableau t let $d(\mathsf{t})$ be the permutation in $\mathfrak{S}_n$ such that $\mathsf{t}^\lambda \cdot d(\mathsf{t}) = \mathsf{t}$.

Recall the Bruhat order $\leq$ on $\mathfrak{S}_n$ from section 1.1. The following result, which goes back to work of Ehresmann and James, is part of the folklore for these algebras. The proof for level 1 can be found from [**20**, Lemma 3.7]. The higher level cases follow easily.

1.3.1. Lemma. *Suppose $\lambda \in \mathscr{P}_n^\Lambda$ and s and t are standard λ-tableaux. Then $\mathsf{s} \trianglerighteq \mathsf{t}$ if and only if $d(\mathsf{s}) \leq d(\mathsf{t})$.*

Suppose λ is a multicomposition and $\gamma = (r, c, l) \in [\lambda]$ and recall from Section 1.1 that $\kappa_\Lambda = (\kappa_1, \kappa_2, \ldots, \kappa_\ell)$ is a fixed multicharge of Λ. The **residue** of γ associate to κ_Λ is

$$\operatorname{res}(\gamma) \equiv r - c + \kappa_l \pmod{e}.$$

If t is a standard λ-tableau and the **residue sequence** of t is $\operatorname{res}(\mathsf{t}) = \mathbf{i}_t = (i_1, i_2, i_3, \ldots, i_n)$, where $i_k = \operatorname{res}(\gamma_k)$ and γ_k is the unique node in $[\lambda]$ such that $\mathsf{t}(\gamma_k) = k$. In particular, we write $\mathbf{i}_{\mathsf{t}^\lambda} = \mathbf{i}_\lambda$ and $\operatorname{res}_{\mathsf{t}}(k) = \operatorname{res}(\gamma_k)$.

Suppose that t is a λ-tableau. Then t is **standard** if $\lambda = \operatorname{Shape}(\mathsf{t})$ is a multipartition and if, in each component, the entries increase along each row and down each column. More precisely, if $(r, c, l) \in [\lambda]$ then $\mathsf{t}(r, c, l) < \mathsf{t}(r+1, c, l)$ whenever $(r+1, c, l) \in [\lambda]$ and $\mathsf{t}(r, c, l) < \mathsf{t}(r, c+1, l)$ whenever $(r, c+1, l) \in [\lambda]$. Let $\operatorname{Std}(\lambda)$ be the set of all standard λ-tableaux and $\operatorname{Std}(> \lambda)$ be the set of all standard μ-tableaux with $\mu > \lambda$. We can define $\operatorname{Std}(\geq \lambda)$ similarly. Note that if t is standard then so is $\mathsf{t}|_k$ for $1 \leq k \leq n$.

Recall that for each standard tableau t, we can define a permutation $d(\mathsf{t}) \in \mathfrak{S}_n$ such that $\mathsf{t} = \mathsf{t}^\lambda \cdot d(\mathsf{t})$. For each permutation we may have more than one reduced expression. Here we fix a choice of the reduced expression of $d(\mathsf{t})$.

For any standard λ-tableau t, define $\mathsf{t}^{(i)}$ to be a standard λ-tableau where $\mathsf{t}^{(i)}|_k = \mathsf{t}^\lambda|_k$ for any $1 \leq k < i$, and ${\mathsf{t}^{(i)}}^{-1}(k) = \mathsf{t}^{-1}(k)$ for any $i \leq k \leq n$. In particular, $\mathsf{t}^{(1)} = \mathsf{t}$ and $\mathsf{t}^{(n+1)} = \mathsf{t}^\lambda$. Therefore we have a series of standard λ-tableau

$$\mathsf{t}^\lambda = \mathsf{t}^{(n+1)}, \mathsf{t}^{(n)}, \mathsf{t}^{(n-1)}, \ldots, \mathsf{t}^{(2)}, \mathsf{t}^{(1)} = \mathsf{t}.$$

Then define w_i to be the unique permutation in $\mathfrak{S}_n$ such that $\mathsf{t}^{(i+1)} w_i = \mathsf{t}^{(i)}$ and define $w_n w_{n-1} \ldots w_2 w_1$ to be the **standard expression** of $d(\mathsf{t})$. Obviously this is a reduced expression of $d(\mathsf{t})$. In the rest of this thesis, we fix $d(\mathsf{t})$ to be its standard expression.

1.3.2. Remark. For each w_i, if $w_i \neq 1$, we can write $w_i = s_{a_i} s_{a_i+1} s_{a_i+2} \ldots s_{i-2} s_{i-1}$ for some $a_i \leq i - 1$. Notice that

$$(k)(\mathsf{t}^{(i+1)})^{-1} = \begin{cases} (i)(\mathsf{t}^{(i)})^{-1}, & \text{if } k = a_i, \\ (k-1)(\mathsf{t}^{(i)})^{-1}, & \text{if } a_i < k \leq i, \\ (k)(\mathsf{t}^{(i)})^{-1}, & \text{otherwise.} \end{cases}$$

and $l(w_i)$ is always greater than or equal to the length of the row containing i in $\mathsf{t}^{(i+1)}$.

Also for each i, if $\operatorname{Shape}(\mathsf{t}^{(i)}|_{i-1}) = \lambda$, then $\mathsf{t}^{(i)}|_{i-1} = \mathsf{t}^\lambda$.

1.3.3. **Example** Suppose $\mathsf{t}^{(11)} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&12\\\hline 5&6&7&11\\\cline{1-4} 8&9&10&13\\\cline{1-4} 14&15\\\cline{1-2}\end{array}$ and $\mathsf{t}^{(10)} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&10&12\\\hline 4&5&6&11\\\cline{1-4} 7&8&9&13\\\cline{1-4} 14&15\\\cline{1-2}\end{array}$. Therefore we have $w_{10} = s_4 s_5 s_6 s_7 s_8 s_9$ such that $\mathsf{t}^{(11)} \cdot w_{10} = \mathsf{t}^{(10)}$.

Notice that in this case, $i = 10$ and $a_{10} = 4$. So

$$(k)(\mathsf{t}^{(11)})^{-1} = \begin{cases} (10)(\mathsf{t}^{(10)})^{-1}, & \text{if } k = a_i = 4, \\ (k-1)(\mathsf{t}^{(10)})^{-1}, & \text{if } 4 = a_i < k \leq i = 10, \\ (k)(\mathsf{t}^{(10)})^{-1}, & \text{otherwise.} \end{cases}$$

Furthermore, $\mathsf{t}^{(11)}|_{10} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6&7\\\cline{1-3} 8&9&10\\\cline{1-3}\end{array} = \mathsf{t}^{(4,3,3)}$ and $\mathsf{t}^{(10)}|_9 = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5&6\\\hline 7&8&9\\\hline\end{array} = \mathsf{t}^{(3,3,3)}$. ◇

1.3.4. Lemma. *Suppose* t *is a standard* λ*-tableau and* $d(\mathsf{t}) = s_{r_1} \ldots s_{r_m}$ *is the standard expression. For any* $1 \leq k \leq m$, *define* $\mathsf{s} = \mathsf{t}^\lambda \cdot s_{r_1} s_{r_2} \ldots s_{r_k}$. *Then* s *is a standard* λ*-tableau.*

Proof. The proof is trivial by the definition of the standard expression. □

1.3.5. **Example** Suppose $\mathsf{t} = \begin{array}{|c|c|c|c|c|}\hline 1&2&4&6&7\\\hline 3&5\\\cline{1-2}\end{array}$. Then we have $d(\mathsf{t}) = s_5 s_6 \cdot s_4 s_5 \cdot s_3$. Then

$$\begin{aligned}
\mathsf{t}^\lambda \cdot s_5 &= \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&6\\\hline 5&7\\\cline{1-2}\end{array},\\
\mathsf{t}^\lambda \cdot s_5 s_6 &= \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&7\\\hline 5&6\\\cline{1-2}\end{array},\\
\mathsf{t}^\lambda \cdot s_5 s_6 s_4 &= \begin{array}{|c|c|c|c|c|}\hline 1&2&3&5&7\\\hline 4&6\\\cline{1-2}\end{array},\\
\mathsf{t}^\lambda \cdot s_5 s_6 s_4 s_5 &= \begin{array}{|c|c|c|c|c|}\hline 1&2&3&6&7\\\hline 4&5\\\cline{1-2}\end{array},
\end{aligned}$$

and the above tableaux are all standard. ◇

1.4. Graded cellular basis of KLR algebras over a field

Suppose $\mathscr{O}$ is a field, Hu and Mathas [**9**, Theorem 5.8] have found a homogeneous basis of $\mathscr{R}_n^\Lambda(\mathscr{O})$. Here we give an equivalent definition of their basis. For any multicomposition λ, recall t^λ to be the unique standard λ-tableau such that $\mathsf{t}^\lambda \trianglerighteq \mathsf{t}$ for all standard λ-tableau t, and $\mathbf{i}_\lambda$ is the residue sequence of t^λ. We define $\hat{e}_\lambda = \hat{e}(\mathbf{i}_\lambda)$.

Suppose λ is a multicomposition. A node (r, c, l) is an **addable node** of λ if $(r, c, l) \notin [\lambda]$ and $[\lambda] \cup \{(r, c, l)\}$ is the Young diagram of a multipartition. Similarly, a node (r, c, l) is a **removable node** of λ if $(r, c, l) \in [\lambda]$ and $[\lambda] \backslash \{(r, c, l)\}$ is the Young diagram of a multipartition. Given two nodes $\alpha = (r, c, l)$ and $\beta = (s, t, m)$ then α is **below** β if either $l > m$, or $l = m$ and $r > s$.

Suppose that $\mathsf{s} \in \operatorname{Std}(\lambda)$. Let $\mathscr{A}_\mathsf{s}(k)$ be the set of addable nodes of the multicomposition $\operatorname{Shape}(\mathsf{s}|_k)$ which are below $\mathsf{s}^{-1}(k)$ and let

$$\mathscr{A}_\mathsf{s}^\Lambda(k) = \{\alpha \in \mathscr{A}_\mathsf{s}(k) \mid \operatorname{res}(\alpha) = \operatorname{res}_\mathsf{t}(k)\}.$$

Similarly as in [**9**, Definition 4.12], define

$$\hat{y}_\lambda = \prod_{k=1}^{n} \hat{y}_k^{|\mathscr{A}_{\mathsf{t}^\lambda}^\Lambda(k)|} \in \mathscr{R}_n(\mathscr{O}).$$

For example, if $\lambda = (3, 1|4^2, 2|5, 1)$, $e = 4$ and $\Lambda = 3\Lambda_0$ then

$$\mathsf{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 5&6&7&8\\\hline 9&10&11&12\\\hline 13&14\\\cline{1-2}\end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 16&17&18&19\\\hline 20\\\cline{1-1}\end{array} \right)$$

and $\hat{y}_\lambda = \hat{y}_1^2 \hat{y}_5 \hat{y}_8 \hat{y}_{10} \hat{y}_{12} \hat{y}_{18}$. Therefore,

$$\hat{e}_\lambda \hat{y}_\lambda = \hat{e}(0123012330122012303) \hat{y}_1^2 \hat{y}_5 \hat{y}_8 \hat{y}_{10} \hat{y}_{12} \hat{y}_{18}.$$

We define a particular kind of element in $\mathscr{R}_n(\mathscr{O})$. Suppose $w \in \mathfrak{S}_n$ has length ℓ and $s_{i_1} s_{i_2} \ldots s_{i_\ell}$ is a reduced expression for w in $\mathfrak{S}_n$. Recall that $\mathscr{R}_n(\mathscr{O})$ has a unique anti-isomorphism $*$ which fixes all of the KLR generators. Define

$$\hat{\psi}_w = \hat{\psi}_{i_1} \hat{\psi}_{i_2} \ldots \hat{\psi}_{i_\ell} \in \mathscr{R}_n(\mathscr{O}) \quad \text{and} \quad \hat{\psi}_w^* = \hat{\psi}_{i_\ell} \hat{\psi}_{i_{\ell-1}} \ldots \hat{\psi}_{i_2} \hat{\psi}_{i_1} \in \mathscr{R}_n(\mathscr{O}).$$

Notice that $\hat{\psi}_w$ and $\hat{\psi}_w^*$ depend on the choice of the reduced expression of w, even though in $\mathfrak{S}_n$ all reduced expressions of w are the same. For example, $s_1 s_2 s_1$ and $s_2 s_1 s_2$ are equal to the same element of $\mathfrak{S}_n$, but in general $\hat{\psi}_1\hat{\psi}_2\hat{\psi}_1 \neq \hat{\psi}_2\hat{\psi}_1\hat{\psi}_2$ in $\mathscr{R}_n(\mathscr{O})$. Define $l(\hat{\psi}_w) = l(\hat{\psi}^*_w) = l(w)$ for any standard tableau t. Similarly we can define

$$\psi_w = \psi_{i_1}\psi_{i_2}\ldots\psi_{i_\ell} \in \mathscr{R}^\Lambda_n(\mathscr{O}) \quad \text{and} \quad \psi^*_w = \psi_{i_\ell}\psi_{i_{\ell-1}}\ldots\psi_{i_2}\psi_{i_1} \in \mathscr{R}^\Lambda_n(\mathscr{O})$$

and ψ_w and ψ^*_w depends on the choice of reduced expressions of w as well.

Suppose $l(d(\mathsf{t})) = \ell$ and $d(\mathsf{t}) = s_{i_1}s_{i_2}\ldots s_{i_\ell}$ is the standard expression of $d(\mathsf{t})$ where $\mathsf{t}^\lambda \cdot d(\mathsf{t}) = \mathsf{t}$. Define $\hat{\psi}_{d(\mathsf{t})} = \hat{\psi}_{i_1}\hat{\psi}_{i_2}\ldots\hat{\psi}_{i_\ell}$ and $\hat{\psi}^*_{d(\mathsf{t})} = \hat{\psi}_{i_\ell}\hat{\psi}_{i_{\ell-1}}\ldots\hat{\psi}_{i_2}\hat{\psi}_{i_1}$.

1.4.1. Definition. *Suppose $\Lambda \in P_+$, $\lambda \in \mathscr{P}^\Lambda_n$ and s, t are two standard λ-tableaux. We define*

$$\hat{\psi}^{\mathscr{O}}_{\mathsf{st}} = \hat{\psi}^*_{d(\mathsf{s})}\hat{e}_\lambda\hat{y}_\lambda\hat{\psi}_{d(\mathsf{t})} \in \mathscr{R}_n(\mathscr{O}),$$

and hence

$$\psi^{\mathscr{O}}_{\mathsf{st}} = \hat{\psi}^{\mathscr{O}}_{\mathsf{st}} + N^\Lambda_n \in \mathscr{R}^\Lambda_n(\mathscr{O}).$$

1.4.2. Remark. Notice that Hu and Mathas [**9**, Definition 5.1] defined $\psi^{\mathscr{O}}_{\mathsf{st}}$ differently. Actually if we define e_λ, y_λ and ψ_w in $\mathscr{R}^\Lambda_n(\mathscr{O})$ as analogues of $\hat{e}_\lambda$, $\hat{y}_\lambda$ and $\hat{\psi}_w$, and define $\psi^{\mathscr{O}}_{\mathsf{st}} = \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda\psi_{d(\mathsf{t})}$ for $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$, it is equivalent to Definition 1.4.1. We define $\psi^{\mathscr{O}}_{\mathsf{st}}$ as in Definition 1.4.1 because we need to work in $\mathscr{R}_n(\mathscr{O})$ later.

1.4.3. Remark. By construction, then this $\psi^{\mathscr{O}}_{\mathsf{st}}$ is well defined as an element of $\mathscr{R}^\Lambda_n(\mathscr{O})$ for any ring $\mathscr{O}$. Many of the calculations in this thesis depend heavily on the choice of $\mathscr{O}$ so we write $\psi^{\mathscr{O}}_{\mathsf{st}}$ to emphasize that $\psi^{\mathscr{O}}_{\mathsf{st}}$ is an element of $\mathscr{R}^\Lambda_n(\mathscr{O})$. Most of the time, however, we will work in $\mathscr{R}^\Lambda_n(\mathbb{Z})$ so for convenience we set $\psi_{\mathsf{st}} = \psi^{\mathbb{Z}}_{\mathsf{st}}$.

1.4.4. Lemma (Hu and Mathas [**9**, Lemma 5.2] [**10**, Corollary 3.11,3.12])**.** *Suppose $\mathscr{O}$ is a field and s and t are standard λ-tableaux and $1 \le r \le n$,*

$$\psi_{\mathsf{st}}\psi_r = \begin{cases} \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{t}\cdot s_r \text{ is not standard} \\ & \quad \text{or } d(\mathsf{t})\cdot s_r \text{ is not reduced,} \\ \psi_{\mathsf{sv}} + \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{v} = \mathsf{t}\cdot s_r \text{ standard and } d(\mathsf{t})\cdot s_r = d(\mathsf{v}). \end{cases}$$

for $c_{\mathsf{uv}} \in \mathscr{O}$, and $c_{\mathsf{uv}} \neq 0$ only if $\mathrm{res}(\mathsf{s}) = \mathrm{res}(\mathsf{u})$ *and* $\mathrm{res}(\mathsf{t}\cdot s_r) = \mathrm{res}(\mathsf{v})$. *Similarly, we have*

$$\psi^{\mathscr{O}}_{\mathsf{st}}y_r = \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi^{\mathscr{O}}_{\mathsf{uv}}$$

for $c_{\mathsf{uv}} \in \mathscr{O}$, and $c_{\mathsf{uv}} \neq 0$ only if $\mathrm{res}(\mathsf{s}) = \mathrm{res}(\mathsf{u})$ *and* $\mathrm{res}(\mathsf{t}) = \mathrm{res}(\mathsf{v})$.

1.4.5. Theorem (Hu and Mathas [**9**, Theorem 5.14])**.** *Suppose $\mathscr{O}$ is an integral domain and that either $e = 0$, e is a prime or e is a non-zero non-prime integer such that $e \cdot 1_{\mathscr{O}}$ is invertible in $\mathscr{O}$. Then*

$$\{\psi^{\mathscr{O}}_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in Std(\lambda) \text{ for } \lambda \in \mathscr{P}^\Lambda_n\}$$

is a graded cellular basis of $\mathscr{R}^\Lambda_n(\mathscr{O})$. In particular, $\mathscr{R}^\Lambda_n(\mathscr{O})$ is free as an $\mathscr{O}$-module of rank $\ell^n n!$.

The main purpose of this thesis is to prove that $\mathscr{R}^\Lambda_n(\mathbb{Z})$ is free of rank $\ell^n n!$. To do this we will show that $\{\psi^{\mathbb{Z}}_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$ for $\lambda \in \mathscr{P}^\Lambda_n\}$ is a homogeneous basis of $\mathscr{R}^\Lambda_n(\mathbb{Z})$.

We define some notation for future use.

1.4.6. Definition. *Suppose λ is a multipartition of $\mathscr{P}^\Lambda_n$. Define:*

$$\begin{aligned} R^\Lambda_n &= \langle \psi_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in Std(\mu) \text{ for } \mu \in \mathscr{P}^\Lambda_n \rangle_{\mathbb{Z}}, \\ R^{\ge\lambda}_n &= \langle \psi_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in Std(\mu) \text{ and } \mu \ge \lambda \text{ for } \mu \in \mathscr{P}^\Lambda_n \rangle_{\mathbb{Z}}, \\ R^{>\lambda}_n &= \langle \psi_{\mathsf{st}} \mid \mathsf{s}, \mathsf{t} \in Std(\mu) \text{ and } \mu > \lambda \text{ for } \mu \in \mathscr{P}^\Lambda_n \rangle_{\mathbb{Z}}. \end{aligned}$$

where $R^{>\lambda}_n \subseteq R^{\ge\lambda}_n \subseteq R^\Lambda_n \subseteq \mathscr{R}^\Lambda_n(\mathbb{Z})$

This section closes with an important Proposition:

Khovanov and Lauda[**13**][**12**] have found a basis of $\mathscr{R}_n(\mathscr{O})$

$$\{\, \hat{e}(\mathbf{i})\hat{y}_1^{\ell_1}\hat{y}_2^{\ell_2}\dots\hat{y}_n^{\ell_n}\hat{\psi}_w \mid \mathbf{i}\in I^n, w\in\mathfrak{S}_n, \ell_1,\ell_2,\dots,\ell_n\ge 0\,\} \tag{1.4.7}$$

for any ring $\mathscr{O}$.

Consider the quiver Hecke algebra $\mathscr{R}_n(\mathbb{Q})$ defined over the rational field $\mathbb{Q}$. We have $\mathscr{R}_n(\mathbb{Q})\cong\mathscr{R}_n(\mathbb{Z})\otimes\mathbb{Q}$ and we can define a linear map $f:\mathscr{R}_n(\mathbb{Z})\longrightarrow\mathscr{R}_n(\mathbb{Q})$ by sending $x\in\mathscr{R}_n(\mathbb{Z})$ to $x\otimes 1$.

1.4.8. Lemma. *The linear map $f:\mathscr{R}_n(\mathbb{Z})\longrightarrow\mathscr{R}_n(\mathbb{Q})$ is an injection.*

Proof. By (1.4.7), $\{\,\hat{e}(\mathbf{i})^{\mathbb{Z}}(\hat{y}_1^{\mathbb{Z}})^{\ell_1}(\hat{y}_2^{\mathbb{Z}})^{\ell_2}\dots(\hat{y}_n^{\mathbb{Z}})^{\ell_n}\hat{\psi}_w^{\mathbb{Z}} \mid w\in\mathfrak{S}_n,\ell_1,\ell_2,\dots,\ell_n\ge 0\,\}$ is a basis of $\mathscr{R}_n(\mathbb{Z})$ and similarly, $\{\,\hat{e}(\mathbf{i})^{\mathbb{Q}}(\hat{y}_1^{\mathbb{Q}})^{\ell_1}(\hat{y}_2^{\mathbb{Q}})^{\ell_2}\dots(\hat{y}_n^{\mathbb{Q}})^{\ell_n}\hat{\psi}_w^{\mathbb{Q}} \mid w\in\mathfrak{S}_n,\ell_1,\ell_2,\dots,\ell_n\ge 0\,\}$ is a basis of $\mathscr{R}_n(\mathbb{Q})$. Because f sends the basis elements $\hat{e}(\mathbf{i})^{\mathbb{Z}}(\hat{y}_1^{\mathbb{Z}})^{\ell_1}(\hat{y}_2^{\mathbb{Z}})^{\ell_2}\dots(\hat{y}_n^{\mathbb{Z}})^{\ell_n}\hat{\psi}_w^{\mathbb{Z}}$ of $\mathscr{R}_n(\mathbb{Z})$ to $\hat{e}(\mathbf{i})^{\mathbb{Q}}(\hat{y}_1^{\mathbb{Q}})^{\ell_1}(\hat{y}_2^{\mathbb{Q}})^{\ell_2}\dots(\hat{y}_n^{\mathbb{Q}})^{\ell_n}\hat{\psi}_w^{\mathbb{Q}}$, the basis elements of $\mathscr{R}_n(\mathbb{Q})$, it is sufficient to prove that f is an injection. □

From the definitions, it is evident that $f(N_n^\Lambda(\mathbb{Z}))\subseteq N_n^\Lambda(\mathbb{Q})$. Hence, f induces a homomorphism,

$$f:\mathscr{R}_n^\Lambda(\mathbb{Z})\longrightarrow\mathscr{R}_n^\Lambda(\mathbb{Q});\, x+N_n^\Lambda(\mathbb{Z})\mapsto f(x)+N_n^\Lambda(\mathbb{Q}),$$

which by abuse of notation we also denote by f. In particular, observe that $f(\psi_{\mathsf{st}}^{\mathbb{Z}})=\psi_{\mathsf{st}}^{\mathbb{Q}}$. The main Theorem of this thesis is equivalently to prove that $f:\mathscr{R}_n^\Lambda(\mathbb{Z})\longrightarrow\mathscr{R}_n^\Lambda(\mathbb{Q})$ is an injection.

We then introduce an important special case where we already know that f is injective.

1.4.9. Proposition. *The homomorphism $f:\mathscr{R}_n^\Lambda(\mathbb{Z})\longrightarrow\mathscr{R}_n^\Lambda(\mathbb{Q})$ restricts to an injective map from R_n^Λ to $\mathscr{R}_n^\Lambda(\mathbb{Q})$.*

Proof. As we have already noted above, $f(\psi_{\mathsf{st}}^{\mathbb{Z}})=\psi_{\mathsf{st}}^{\mathbb{Q}}$ for all $\mathsf{s},\mathsf{t}\in\operatorname{Std}(\lambda)$ and $\lambda\in\mathscr{P}_n^\Lambda$. Hence, Theorem 1.4.5 implies the result. □

1.4.10. Corollary. *The elements $\{\psi_{\mathsf{st}}^{\mathbb{Z}}\mid \mathsf{s},\mathsf{t}\in Std(\lambda)\text{ for }\lambda\in\mathscr{P}_n^\Lambda\}$ are a linearly independent subset of $\mathscr{R}_n^\Lambda(\mathbb{Z})$.*

1.4.11. Remark. Proposition 1.4.9 is quite crucial. In this thesis we prove that $\psi_{\mathsf{st}}^{\mathbb{Z}}\cdot\psi_r\in R_n^\Lambda$ whenever $d(\mathsf{t})\cdot s_r$ is not reduced or $\mathsf{t}\cdot s_r$ is not standard in $\mathscr{R}_n^\Lambda(\mathbb{Z})$. We can only have

$$\psi_{\mathsf{st}}^{\mathbb{Z}}\cdot\psi_r=\sum_{\mathsf{u},\mathsf{v}}c_{\mathsf{uv}}^{\mathbb{Z}}\psi_{\mathsf{uv}}^{\mathbb{Z}}.$$

In $\mathscr{R}_n^\Lambda(\mathbb{Q})$, however, by Lemma 1.4.4, under these conditions we have

$$\psi_{\mathsf{st}}^{\mathbb{Q}}\cdot\psi_r=\sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})}c_{\mathsf{uv}}^{\mathbb{Q}}\psi_{\mathsf{uv}}^{\mathbb{Q}}$$

for some $c_{\mathsf{uv}}^{\mathbb{Q}}\in\mathbb{Q}$, where $(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})$ if $\mathsf{u}\trianglerighteq\mathsf{s}$, $\mathsf{v}\trianglerighteq\mathsf{t}$ and $(\mathsf{u},\mathsf{v})\ne(\mathsf{s},\mathsf{t})$. Therefore, $c_{\mathsf{uv}}^{\mathbb{Q}}=c_{\mathsf{uv}}^{\mathbb{Z}}$ by Proposition 1.4.9 and we see that $c_{\mathsf{uv}}^{\mathbb{Z}}\ne 0$ only if $(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})$. In such case we have much more information about u and v with $c_{\mathsf{uv}}^{\mathbb{Z}}\ne 0$. Similar remarks apply to the products $\psi_{\mathsf{st}}^{\mathbb{Z}}\cdot y_r$.

Chapter 2

Integral Basis Theorem I

In the next two chapters we will prove that $\mathscr{R}_n^\Lambda$ is $\mathbb{Z}$-free. The essence of our argument is that we will verify that the following three properties hold.

(1). $e(\mathbf{i}_{\lambda_-} \vee k)y_{\lambda_-}y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}$.
(2). $\psi_{\mathsf{st}}\mathsf{y}_r \in R_n^\Lambda$.
(3). $\psi_{\mathsf{st}}\psi_r \in R_n^\Lambda$.

for any $\lambda \in \mathscr{P}_n^\Lambda$. We will define a partial ordering $\prec$ on $\mathscr{P} = \cup_\Lambda \cup_n \mathscr{P}_n^\Lambda$. Our proof proceeds by induction on multipartitions using $\prec$. The main result of this chapter is to prove that if for any $\mu \succ \lambda$, μ has above three properties, then λ will have the first property. This result is crucial for showing that $e(\mathbf{i}) \in R_n^\Lambda$ for any $\mathbf{i} \in I^n$.

In the rest of this thesis we write $\mathscr{R}_n(\mathbb{Z})$ as $\mathscr{R}_n$ and $\mathscr{R}_n^\Lambda(\mathbb{Z})$ as $\mathscr{R}_n^\Lambda$. Fix a weight Λ, a multicharge $\kappa_\Lambda = (\kappa_1, \dots, \kappa_l)$ corresponding to Λ and an integer $e > 2$. In this and the next chapter we mainly work with the algebra $\mathscr{R}_n^\Lambda$.

2.1. The base step of the induction

In this section we set up the notations and inductive machinery that we use in the next two chapters to prove our main theorem. We then consider the base case of our induction which is when $\lambda = (n|\emptyset|\dots|\emptyset)$. Finally we develop some technical Lemmas which will be useful later.

2.1.1. Definition. *Suppose that λ is a multipartition of n. Let λ^+ be the multicomposition of $n+1$ obtained by adding a node at the end of the last non-empty row of λ, and $\lambda_- = \lambda|_{n-1}$ be the multipartition of $n-1$ obtained by removing the last node from λ.*

For example, if $\lambda = (4,3|3,3)$ then $\lambda^+ = (4,3|3,4)$ and $\lambda_- = (4,3|3,2)$. Notice that in general, λ^+ will be a multicomposition rather than a multipartition.

For $k \in I$ and $\lambda \in \mathscr{P}_n^\Lambda$, define $\mathscr{A}_{\mathsf{t}^\lambda}^k = \{\alpha \in \mathscr{A}_{\mathsf{t}^\lambda}(n) \mid \operatorname{res}(\alpha) = k\}$. Recall $\mathbf{i}_\lambda = \operatorname{res}(\mathsf{t}^\lambda)$ and $e_\lambda = e(\mathbf{i}_\lambda)$ from section 1.4.

2.1.2. Definition. *Suppose that $\lambda \in \mathscr{P}_n^\Lambda$ and $k \in I$. Define the integer b_k^λ by*

$$b_k^\lambda = \begin{cases} |\mathscr{A}_{\mathsf{t}^\lambda}^k| + 1, & \text{if } \lambda^+ \text{ is a multipartition and } i_n + 1 = k, \\ |\mathscr{A}_{\mathsf{t}^\lambda}^k|, & \text{otherwise.} \end{cases}$$

If $\mathbf{i} = (i_1, i_2, \dots, i_n) \in I^n$ and $k \in I$ then define $\mathbf{i} \vee k = (i_1, i_2, \dots, i_n, k) \in I^{n+1}$.

2.1.3. Lemma. *Suppose that $\lambda \in \mathscr{P}_n^\Lambda$ and $k \in I$. Then for each integer b with $0 \le b < b_k^\lambda$, there exists a multipartition $\nu = \nu(b)$ such that $e_\nu y_\nu = e(\mathbf{i}_\lambda \vee k)y_\lambda y_{n+1}^b$.*

Proof. The definitions of λ and b_k^λ ensure that there are b_k^λ addable nodes of residue k below $(\mathsf{t}^\lambda)^{-1}(n)$. Suppose those nodes are $(r_1, c_1, l_1), (r_2, c_2, l_2), \dots, (r_{b_k^\lambda}, c_{b_k^\lambda}, l_{b_k^\lambda})$, where $l_1 \ge l_2 \ge l_3 \ge \dots \ge l_{b_k^\lambda}$, and if $l_i = l_{i+1}$ then $r_i \ge r_{i+1}$. In another word, (r_i, c_i, l_i) is a node below $(r_{i+1}, c_{i+1}, l_{i+1})$.

For any b with $0 \le b < b_k^\lambda$, we define ν to be the multipartition obtained by adding the node $(r_{b+1}, c_{b+1}, l_{b+1})$ on to λ. Then $y_\nu = y_\lambda y_{n+1}^b$ and $e_\nu = e(\mathbf{i}_\lambda \vee k) = e(\mathbf{i} \vee k)$. This completes the proof. □

2.1.4. **Example** Suppose that $\lambda = (4,3|2,1|0|0)$ with $e = 4$ and $\kappa_\Lambda = (0,0,2,1)$. Then $e(\mathbf{i}_\lambda) = e(0123301013)$ and $y_\lambda = y_1y_2y_3y_4y_6y_7y_9$. Then $b_0^\lambda = 1$, $b_1^\lambda = 1$, $b_2^\lambda = 2$ and $b_3^\lambda = 0$ and the proof of Lemma 2.1.3 shows that:

$$\begin{aligned} e(01233010130)y_1y_2y_3y_4y_6y_7y_9 &= e_{\mu_1}y_{\mu_1}, \\ e(01233010131)y_1y_2y_3y_4y_6y_7y_9 &= e_{\mu_2}y_{\mu_2}, \\ e(01233010132)y_1y_2y_3y_4y_6y_7y_9 &= e_{\mu_3}y_{\mu_3}, \\ e(01233010132)y_1y_2y_3y_4y_6y_7y_9y_{11} &= e_{\mu_4}y_{\mu_4}, \end{aligned}$$

where $\mu_1 = (4,3|2,2|\emptyset|\emptyset)$, $\mu_2 = (4,3|2,1|\emptyset|1)$, $\mu_3 = (4,3|2,1|1|\emptyset)$ and $\mu_4 = (4,3|2,1,1|\emptyset|\emptyset)$.

◇

2.1.5. Definition. *Let $\mathscr{P}^\Lambda = \cup_{n\geq 0}\mathscr{P}_n^\Lambda$. Define three sets $\mathscr{P}_I^\Lambda$, $\mathscr{P}_y^\Lambda$ and $\mathscr{P}_\psi^\Lambda$ of multipartitions by:*

$$\mathscr{P}_I^\Lambda = \{\lambda \in \mathscr{P}^\Lambda \mid |\lambda| = n \text{ and } e(\mathbf{i}_{\lambda_-} \vee k)y_{\lambda_-}y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda} \text{ for all } k \in I\},$$

$$\mathscr{P}_y^\Lambda = \{\lambda \in \mathscr{P}^\Lambda \mid |\lambda| = n \text{ and } \psi_{\mathsf{st}}y_r \in R_n^\Lambda \text{ whenever } \mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda) \text{ and } 1 \leq r \leq n\},$$

$$\mathscr{P}_\psi^\Lambda = \{\lambda \in \mathscr{P}^\Lambda \mid |\lambda| = n \text{ and } \psi_{\mathsf{st}}\psi_r \in R_n^\Lambda \text{ whenever } \mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda) \text{ and } 1 \leq r < n\}.$$

2.1.6. Remark. Notice that if for some $\mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda)$ and $1 \leq r \leq n$ we have $\psi_{\mathsf{st}}y_r \in R_n^\Lambda$, then $y_r\psi_{\mathsf{st}} \in R_n^\Lambda$ as well. Similar property holds for $\psi_{\mathsf{st}}\psi_r$. Therefore we can write

$$\mathscr{P}_y^\Lambda = \{\lambda \in \mathscr{P}^\Lambda \mid |\lambda| = n \text{ and } y_r\psi_{\mathsf{st}} \in R_n^\Lambda \text{ whenever } \mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda) \text{ and } 1 \leq r \leq n\},$$

$$\mathscr{P}_\psi^\Lambda = \{\lambda \in \mathscr{P}^\Lambda \mid |\lambda| = n \text{ and } \psi_r\psi_{\mathsf{st}} \in R_n^\Lambda \text{ whenever } \mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda) \text{ and } 1 \leq r < n\}$$

as well.

By Proposition 1.4.9 if one of $e(\mathbf{i}_\nu)y_{\lambda_-}y_n^n$, $\psi_{\mathsf{st}}y_r$ or $\psi_{\mathsf{st}}\psi_r$ belongs to R_n^Λ then it can be written in a unique way as an (integral) linear combination of the ψ-basis elements. In particular, these linear combinations must satisfy the restrictions imposed by Lemma 1.4.4.

We note also that our main theorem is equivalent to the claim that

$$\mathscr{P}_n^\Lambda \subseteq \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda.$$

We prove this by considering each of these three sets separately, beginning with $\mathscr{P}_I^\Lambda$.

Suppose λ and μ are two multipartitions, not necessarily of the same integer. Define $\mu \prec \lambda$ if $|\mu| < |\lambda|$, or $|\mu| = |\lambda|$ and $l(\mu) < l(\lambda)$, or $|\mu| = |\lambda|$, $l(\mu) = l(\lambda)$ and $\lambda < \mu$.

2.1.7. Definition. *Define $\mathscr{S}_n^\Lambda = \{\lambda \in \mathscr{P}_n^\Lambda \mid \mu \in \mathscr{P}_I^{\Lambda'} \cap \mathscr{P}_y^{\Lambda'} \cap \mathscr{P}_\psi^{\Lambda'}$ whenever $\mu \in \mathscr{P}_m^{\Lambda'}$ and $\mu \prec \lambda\}$*

Now we can state the main result of this chapter.

2.1.8. Theorem. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$. Then we have $\lambda \in \mathscr{P}_I^\Lambda$.*

As we mentioned before we are going to apply induction on λ to prove the main Theorem. Lemma 2.1.9, Corollary 2.1.10 and Corollary 2.1.11 give the base case of the induction. Recall that $e \neq 2$.

2.1.9. Lemma. *Suppose that $n \geq 1$ and $\lambda = (n|0|\dots|0) \in \mathscr{P}_n^\Lambda$. Then $e(\mathbf{i}_{\lambda_-} \vee k)y_{\lambda_-}y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}$ for any $k \in I$.*

Proof. As λ is the maximal element of $\mathscr{P}_n^\Lambda$, $R_n^{>\lambda} = \{0\}$. Therefore the Lemma is equivalent to the claim that $e(\mathbf{i}_{\lambda_-} \vee k)y_{\lambda_-}y_n^{b_k^{\lambda_-}} = 0$. We prove this by induction on n.

If $n = 1$ then it is easy to see that $b_k^{\lambda_-} = (\Lambda, \alpha_{i_1})$. Therefore, $e(\mathbf{i}_{\lambda_-} \vee k)y_{\lambda_-}y_n^{b_k^{\lambda_-}} = e(k)y_1^{(\Lambda,\alpha_{i_1})} = 0$ by (1.1.19).

Suppose now that the Lemma holds for any $n' < n$. Notice that for any $m \geq 1$, set $\gamma = (m|0|\dots|0)$ then $|\mathscr{A}^k_{\mathfrak{t}^\gamma}|$ is independent to the value of m. For the rest of the proof we set $a_k = |\mathscr{A}^k_{\mathfrak{t}^{\lambda_-}}|$.

In order to simplify the notations, for the rest of the proof we will omit $i_1 i_2 \dots i_{n-3}$ and simply write $e(\mathbf{i}) = e(i_{n-2}, i_{n-1}, i_n)$. We will also suppress y_ν, where $\nu = \lambda|_{n-3}$.

We consider four cases separately, depending on the value of k.

Case 2.1.9a: $k = i_{n-1}$**.** Then $e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e(\mathbf{i}|_{n-3}, k-1, k, k) y_{\lambda|_{n-3}} y_{n-2}^{a_{k-1}} y_{n-1}^{a_k} y_n^{a_k}$. In this case we have

$$\begin{aligned} e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k} y_n^{a_k} &\overset{(1.1.6)}{=} -e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k+1} y_n^{a_k} \psi_{n-1} + e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k} y_n^{a_k} \psi_{n-1} y_n \\ &\overset{(1.1.13)}{=} \psi_{n-1} e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k} y_n^{a_k} y_n, \end{aligned}$$

where $e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k+1} y_n^{a_k} = 0$ by induction. Therefore,

$$\begin{aligned} e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k} y_n^{a_k} &= \psi_{n-1} e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k} y_n^{a_k} y_n \\ &= \psi_{n-1}^2 e(k-1,k,k) y_{n-2}^{a_{k-1}} y_{n-1}^{a_k} y_n^{a_k} y_n^2 = 0 \end{aligned}$$

by relation (1.1.8).

Case 2.1.9b: $k = i_{n-1} + 1$. Now, $e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e(\mathbf{i}|_{n-3}, k-2, k-1, k) y_{\lambda|_{n-3}} y_{n-2}^{a_{k-2}} y_{n-1}^{a_{k-1}} y_n^{a_k+1}$. Therefore,

$$\begin{aligned} & e(k-2,k-1,k) y_{n-2}^{a_{k-2}} y_{n-1}^{a_{k-1}} y_n^{a_k+1} \\ \overset{(1.1.8)}{=} \; & e(k-2,k-1,k) y_{n-2}^{a_{k-2}} y_{n-1}^{a_{k-1}+1} y_n^{a_k} + e(k-2,k-1,k) y_{n-2}^{a_{k-2}} y_{n-1}^{a_{k-1}} y_n^{a_k} \psi_{n-1}^2 \\ \overset{\substack{(1.1.6)\\(1.1.7)}}{=} \; & e(k-2,k-1,k) y_{n-2}^{a_{k-2}} y_{n-1}^{a_{k-1}+1} y_n^{a_k} + \psi_{n-1} e(k-2,k,k-1) y_{n-2}^{a_{k-2}} y_{n-1}^{a_k} y_n^{a_{k-1}} \psi_{n-1} = 0, \end{aligned}$$

where the last equality follows by induction.

Case 2.1.9c: $k = i_{n-1} - 1$**.** If $n = 2$ then $e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e(k, k-1) y_1^{a_k} y_2^{a_{k-1}}$. Then $a_{k-1} \geq 1$. Therefore,

$$e(k,k-1) y_1^{a_k} y_2^{a_{k-1}} \overset{(1.1.8)}{=} e(k,k-1) y_1^{a_k+1} y_2^{a_{k-1}-1} - \psi_1 e(k-1,k) y_1^{a_{k-1}-1} y_2^{a_k} \psi_1 = 0,$$

using relation (1.1.19) and induction. Hence, the lemma follows in this case when $n = 2$.

If $n > 2$ then $e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e(\mathbf{i}|_{n-3}, k, k+1, k) y_{\lambda|_{n-3}} y_{n-2}^{a_k} y_{n-1}^{a_{k+1}} y_n^{a_k}$. Hence,

$$\begin{aligned} e(k,k+1,k) y_{n-2}^{a_k} y_{n-1}^{a_{k+1}} y_n^{a_k} &\overset{(1.1.9)}{=} \psi_{n-2}\psi_{n-1}\psi_{n-2} e(k,k+1,k) y_{n-2}^{a_k} y_{n-1}^{a_{k+1}} y_n^{a_k} \\ &\qquad - \psi_{n-1}\psi_{n-2}\psi_{n-1} e(k,k+1,k) y_{n-2}^{a_k} y_{n-1}^{a_{k+1}} y_n^{a_k} \\ &= \psi_{n-2}\psi_{n-1} e(k+1,k,k) y_{n-2}^{a_{k+1}} y_{n-1}^{a_k} y_n^{a_k} \psi_{n-2} \\ &\qquad - \psi_{n-1}\psi_{n-2} e(k,k,k+1) y_{n-2}^{a_k} y_{n-1}^{a_k} y_n^{a_{k+1}} \psi_{n-1} \\ &= 0, \end{aligned}$$

where the last equality follows by induction.

Case 2.1.9d: $|k - i_{n-1}| > 1$**.** Because $i_{n-2} = i_{n-1} - 1$, we have $i_{n-2} \neq k$. Therefore we have

$$\begin{aligned} e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} &= e(\mathbf{i}|_{n-3}, i_{n-2}, i_{n-1}, k) y_{\lambda|_{n-3}} y_{n-2}^{a_{i_{n-2}}} y_{n-1}^{a_{i_{n-1}}} y_n^{a_k} \\ &\overset{(1.1.8)}{=} \psi_{n-1} e(\mathbf{i}|_{n-3}, i_{n-2}, k, i_{n-1}) y_{\lambda|_{n-3}} y_{n-2}^{a_{i_{n-2}}} y_{n-1}^{a_k} y_n^{a_{i_{n-1}}} \psi_{n-1} = 0 \end{aligned}$$

by induction. This completes the proof. □

Lemma 2.1.9 has two immediate Corollaries:

2.1.10. Corollary. *Suppose $n \geq 2$ and $\lambda = (n|0|\dots|0)$. Then $e_\lambda y_\lambda y_r \in R_n^{>\lambda}$ for any $1 \leq r \leq n$.*

2.1.11. Corollary. *Suppose that $n \geq 2$ and $\lambda = (n|0|\dots|0)$. Then $e_\lambda y_\lambda \psi_r \in R_n^{>\lambda}$, for any $1 \leq r \leq n-1$.*

Proof. Write $y_\lambda = y_1^{l_1} y_2^{l_2} \dots y_n^{l_n}$ and $\mathbf{i}_\lambda = (i_1 i_2 \dots i_n)$,

$$e(i_1 \dots i_n) y_1^{l_1} \dots y_n^{l_n} \psi_r = \psi_r e(i_1 \dots i_{r-1} i_{r+1} i_r \dots i_n) y_1^{l_1} \dots y_{r-1}^{l_{r-1}} y_r^{l_{r+1}} y_{r+1}^{l_r} \dots y_n^{l_n} = 0,$$

by Lemma 2.1.9. □

The results in the rest of the section will be used frequently in the later proofs.

Recall that for any multipartition λ, $R_n^{>\lambda}$ is the subspace of $\mathscr{R}_n^\Lambda$ spanned by all of the elements ψ_{st}, where $\mathrm{Shape}(\mathsf{s}) = \mathrm{Shape}(\mathsf{t}) > \lambda$.

2.1.12. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$. Then $R_n^{>\lambda}$ is a two-sided ideal of $\mathscr{R}_n^\Lambda$. More precisely, $R_n^{>\mu}$ is a two-sided ideal of $\mathscr{R}_n^\Lambda$ whenever $\mu \prec \lambda$.*

Proof. The Lemma follows directly from the definition of the set $\mathscr{S}_n^\Lambda$, $\mathscr{P}_y^\Lambda$, $\mathscr{P}_\psi^\Lambda$ and Remark 2.1.6. □

In order to simplify the notation, for each $i \in I$ define $\theta_i : \mathscr{R}_n^\Lambda \longrightarrow \mathscr{R}_{n+1}^\Lambda$ to be the unique $\mathbb{Z}$-linear map which sends $e(\mathbf{i})$ to $e(\mathbf{i} \vee i)$, y_r to y_r and ψ_r to ψ_r. It is easy to see that θ_i respects the relations in $\mathscr{R}_n^\Lambda$, so θ_i is a $\mathbb{Z}$-algebra homomorphism.

2.1.13. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$ and $\mathsf{u}, \mathsf{v} \in \mathrm{Std}(\mu)$, where $\mu \in \mathscr{P}_m^\Lambda$ with $m < n$ such that $\mu > \lambda|_m$. Let $\sigma = \lambda|_{m+1} \in \mathscr{P}_{m+1}^\Lambda$. Then $\theta_i(\psi_{\mathsf{uv}}) \in R_n^{>\sigma}$, for any $i \in I$.*

Proof. Write $\mu = (\mu^{(1)}, \dots, \mu^{(\ell)})$ and $\mu^{(\ell)} = (\mu_1^{(\ell)}, \dots, \mu_k^{(\ell)})$ and define $\gamma = (\mu^{(1)}, \dots, \mu^{(\ell-1)}, \gamma^{(\ell)})$ where

$$\gamma^{(\ell)} = \begin{cases} (\mu_1^{(\ell)}, \dots, \mu_{k-1}^{(\ell)}, \mu_k^{(\ell)} + 1), & \text{if } \mu_{k-1}^{(\ell)} > \mu_k^{(\ell)}, \\ (\mu_1^{(\ell)}, \dots, \mu_{k-1}^{(\ell)}, \mu_k^{(\ell)}, 1), & \text{if } \mu_{k-1}^{(\ell)} = \mu_k^{(\ell)}. \end{cases}$$

Then γ is a multipartition of $m+1$ and $\gamma|_m = \mu$. Since $m < n$, if $m = n-1$, then $\gamma|_{n-1} = \mu > \lambda_-$, so that $\gamma > \lambda$. On the other hand, if $m < n-1$ then $|\gamma| = m+1 < n = |\lambda|$. So we always have $\gamma \prec \lambda$. Therefore, $\gamma \in \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$ because $\lambda \in \mathscr{S}_n^\Lambda$.

As $\gamma|_m = \mu$, we have $\theta_i(\psi_{\mathsf{uv}}) = \theta_i(\psi_{d(\mathsf{u})}^* e_\mu y_\mu \psi_{d(\mathsf{v})}) = \psi_{d(\mathsf{u})}^* e(\mathbf{i}_{\gamma|_m} \vee i) y_\mu \psi_{d(\mathsf{v})}$. First suppose that $b_i^\mu = 0$. Then using the definition of $\mathscr{P}_I^\Lambda$, we have $e(\mathbf{i}_{\gamma|_m} \vee i) y_\mu \in R_n^{>\gamma} \subseteq R_n^{>\sigma}$. Hence, by Lemma 2.1.12, we have $\theta_i(\psi_{\mathsf{uv}}) = \psi_{d(\mathsf{u})}^* e(\mathbf{i}_{\gamma|_m} \vee i) y_\mu \psi_{d(\mathsf{v})} \in R_n^{>\sigma}$.

Now supose that $b_i^\mu > 0$. By Lemma 2.1.3 we can find a multipartition ν with $\nu|_m = \mu$ such that $e(\mathbf{i}_{\gamma|_m} \vee i) y_\mu = e_\nu y_\nu$. Further, as $\nu|_m = \mu$, we can find two standard ν-tableaux s and t such that $\mathsf{s}|_m = \mathsf{u}$ and $\mathsf{t}|_m = \mathsf{v}$. That is, $d(\mathsf{s}) = d(\mathsf{u})$ and $d(\mathsf{t}) = d(\mathsf{v})$. Therefore,

$$\theta_i(\psi_{\mathsf{uv}}) = \psi_{d(\mathsf{u})}^* e(\mathbf{i}_{\gamma|_m} \vee i) y_\mu \psi_{d(\mathsf{v})} = \psi_{d(\mathsf{s})}^* e_\nu y_\nu \psi_{d(\mathsf{t})} = \psi_{\mathsf{st}} \in R_n^{\geq \nu} \subseteq R_n^{>\sigma}$$

because $\nu > \sigma$. This completes the proof. □

If $\mathbf{i} = (i_1, \dots, i_n) \in I^n$ and $1 \leq m \leq n$ let $\mathbf{i}_m = (i_1 \dots i_m)$.

2.1.14. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$, $m \leq n$ and $\sigma = \lambda|_m$. For any $\mathbf{i} = (i_1, i_2, \dots, i_{n-m})$ we have $\mathscr{R}_n^\Lambda \theta_{\mathbf{i}}(R_n^{>\sigma}) \mathscr{R}_n^\Lambda \subseteq R_n^{>\lambda}$.*

Proof. Suppose $r \in R_n^{>\sigma}$, we have that

$$r = \sum_{\mathsf{u},\mathsf{v} \in \mathrm{Std}(>\sigma)} c_{\mathsf{uv}} \psi_{\mathsf{uv}}$$

for some $c_{\mathsf{uv}} \in \mathbb{Z}$. For any $i \in I$,

$$\theta_i(r) = \sum_{\mathsf{u},\mathsf{v} \in \mathrm{Std}(>\sigma)} c_{\mathsf{uv}} \theta_i(\psi_{\mathsf{uv}}).$$

By Lemma 2.1.13, $\theta_i(\psi_{\mathsf{uv}}) \in R_n^{>\lambda|_{m+1}}$. Hence $\theta_i(r) \in R_n^{>\lambda|_{m+1}}$. By induction we have $\theta_{\mathbf{i}}(r) \in R_n^{>\lambda}$. By Lemma 2.1.12, $R_n^{>\lambda}$ is an ideal. Therefore $\mathscr{R}_n^\Lambda \theta_{\mathbf{i}}(R_n^{>\sigma})\mathscr{R}_n^\Lambda \subseteq R_n^{>\lambda}$ which completes the proof. □

2.2. The action of y_r on two-rowed partitions

Recall that the main result of this chapter is to prove that if $\lambda \in \mathscr{S}_n$, then

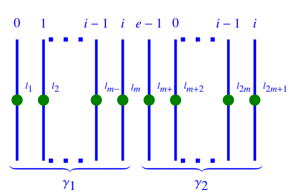
$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

In the inductive process we consider different types of multipartitions λ and a residue $k \in I$. We will consider the more difficult case first, namely when $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_l^{(\ell)}, 1) \neq \emptyset$ with $l \geq 2$, $\lambda_{l-1}^{(\ell)} = \lambda_l^{(\ell)} = m$ and $k \equiv \kappa_\ell - l + m + 1 \pmod{e}$. In this section we assume that $\ell = 1$ and $l = 2$. We will extend the result to the general case in the next section. Notice that in this case $\lambda = (m, m, 1)$ for some integer m and $k \equiv \kappa_1 - 1 + m \pmod{e}$. Then $e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e_\gamma y_\gamma$ where $\gamma = (m, m+1)$. It is very hard to prove that $e_\gamma y_\gamma \in R_n^{>\lambda}$ directly, so we are going to work with γ which is in a more general form.

In this section we fix $\Lambda = \Lambda_j$ for some $j \in I$, $\gamma = (\gamma_1, \gamma_2)$ and $\lambda = (\gamma_1, \gamma_2 - 1, 1)$ with $\gamma_2 > 1$ and $\gamma_2 - \gamma_1 \equiv 1 \pmod{e}$. We will prove that if $\gamma_1 + 1 = \gamma_2$ and $\lambda \in \mathscr{S}_n^\Lambda$ then $e_\gamma y_\gamma \in R_n^{>\gamma}$.

Without loss of generality we can assume that $\Lambda = \Lambda_0$. Define $i \equiv \gamma_2 - 2 \pmod{e}$, which is the residue of $(2, \gamma_2, 1)$. Because $\gamma_2 \equiv \gamma_1 + 1 \pmod{e}$, it is also the residue of the node $(1, \gamma_1, 1)$. In diagrammatic notation, we have

$$e_\gamma y_\gamma =$$

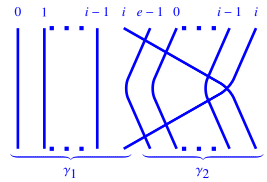


where $\mathbf{i}_\gamma = (i_1, i_2, \dots, i_n)$ and $l_k = |\mathscr{A}_{\mathfrak{t}^\gamma|_k}^{i_k}|$ is the multiplicity of the green dot on the k-th string. For the rest of this section, for clarity we will omit extraneous dots when they do not play an important role in the argument.

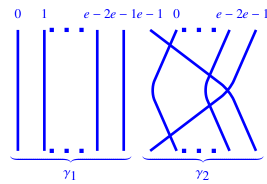
Next we introduce an important equivalent relation $=_\gamma$. For $\gamma \in \mathscr{S}_n^\Lambda$, and $r_1, r_2 \in \mathscr{R}_n^\Lambda$, we write $r_1 =_\gamma r_2$ if $r_1 \pm r_2 \in R_n^{>\gamma}$. It is clearly an equivalent relation. Moreover, by Lemma 2.1.12, for any $r \in \mathscr{R}_n^\Lambda$ we have $r_1 \cdot r =_\gamma r_2 \cdot r$ if $r_1 =_\gamma r_2$. This will be helpful for us to simplify the notations and calculations.

Recall that $\gamma_2 > 1$. We can write $\gamma_2 = k \cdot e + t$ for some nonnegative integer k and $2 \leq t \leq e+1$. We will first prove

(2.2.1) $e_\gamma y_\gamma =_\gamma$ {



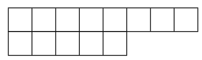
, if $i \neq e-1$,

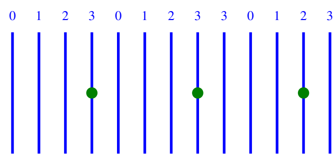


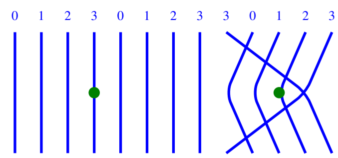
, if $i = e-1$.

by induction on k, which can imply $e_\gamma y_\gamma \in R_n^{>\lambda}$ easily.

In order to clarify the meaning of the diagrams in (2.2.1), let us give two examples below. In these examples for convenience we fix $e = 4$.

2.2.2. **Example** Suppose $\gamma = (8, 5)$, then $\gamma =$  and $i = 3$. Then we are trying to prove that

$e_\gamma y_\gamma =$ 

$=_\gamma$ 

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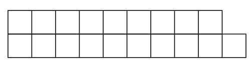
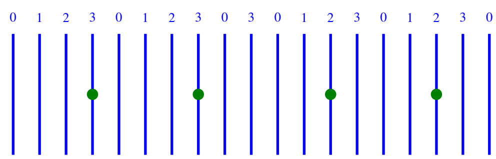
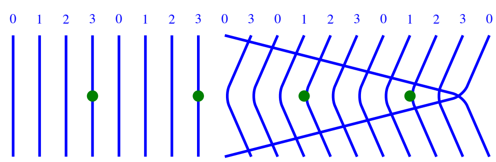
◇

2.2.3. **Example** Suppose $\gamma = (9, 10)$, then $\gamma =$  and $i = 0$. We are trying to prove that

$e_\gamma y_\gamma =$ 

$=_\gamma$ 

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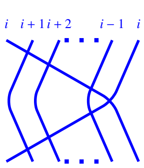
◇

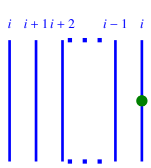
The next Proposition is the base case of the induction. When $k = 0$, we have $2 \leq \gamma_2 \leq e + 1$.

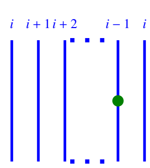
2.2.4. Proposition. *Suppose $\gamma = (\gamma_1, \gamma_2) \in \mathscr{C}_n^\Lambda$ with $\gamma_2 > 1$ and $\gamma_2 - \gamma_1 \equiv 1 \pmod{e}$ and $\lambda = (\gamma_1, \gamma_2 - 1, 1) \in \mathscr{S}_n^\Lambda$. Define i to be the residue of the node at position $(1, \gamma_1, 1)$ or $(2, \gamma_2, 1)$. When $2 \leq \gamma_2 \leq e + 1$, (2.2.1) holds.*

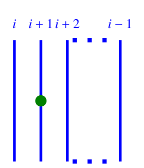
Before proving Proposition 2.2.4 we first give a useful lemma.

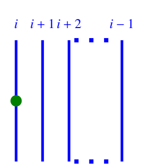
2.2.5. Lemma. *For any $i \in I$, we have*

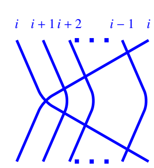


$=$ 

$-$ 

$-$ 

$+$ 

$+$ 

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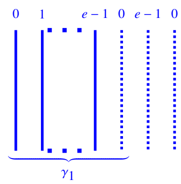
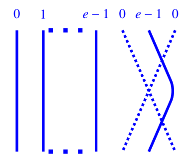
Proof. The Lemma follows by directly applying braid relations on the left hand side of the equation. □

Now we are ready to prove Proposition 2.2.4.

Proof. We prove the Proposition by considering four different cases depending upon the value of i. Notice that in this Proposition, we have $\gamma_1 \geq \gamma_2 - 1$ because $2 \leq \gamma_2 \leq e+1$ and $\gamma_2 - \gamma_1 \equiv 1 \pmod{e}$.

Case 2.2.4a: $i = 0$, i.e. $\gamma_2 = 2$.

$e_\gamma y_\gamma$ =

$\overset{(1.1.12)}{=}$

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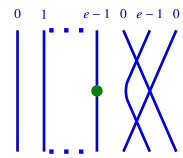
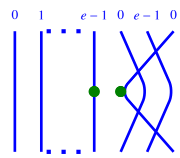


$\overset{(1.1.17)}{=}$ −

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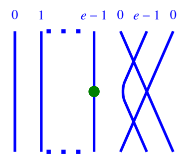

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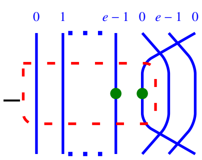
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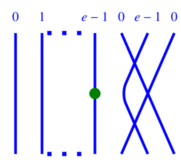
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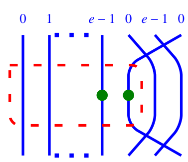
Because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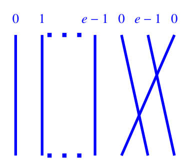
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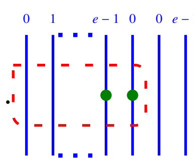
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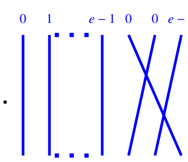
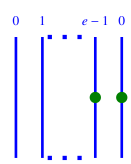
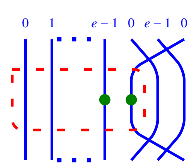


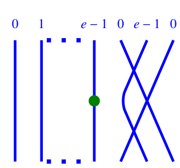
and if we define $\nu = (\gamma_1, 1) = \lambda|_{\gamma_1+1}$, then as $\lambda \in \mathscr{S}_n^\Lambda$ and $|\nu| = \gamma_1 + 1 < n = \gamma_1 + \gamma_2 = |\lambda|$, $\nu \in \mathscr{P}_I^\Lambda$. Moreover as $b_0^{\nu_-} = 1$,


$= e(\mathbf{i}_{\nu_-} \vee 0) y_{\nu_-} y_{|\nu|}^1 \in R_n^{>\nu}$.

Then by Lemma 2.1.14,


$\in R_n^{>\gamma}$.

Therefore,

$e_\gamma y_\gamma =_\gamma$

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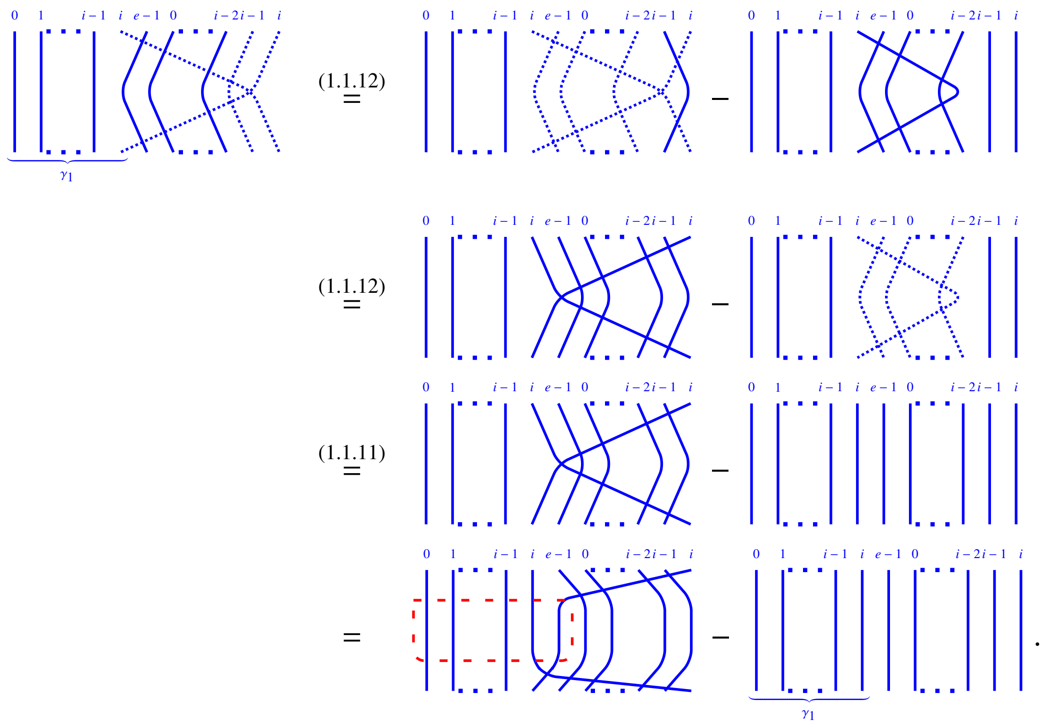
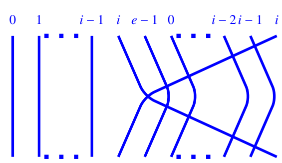
which gives the proposition in this case.

Case 2.2.4b: $1 \le i \le e-3$, i.e. $3 \le \gamma_2 \le e-1$.



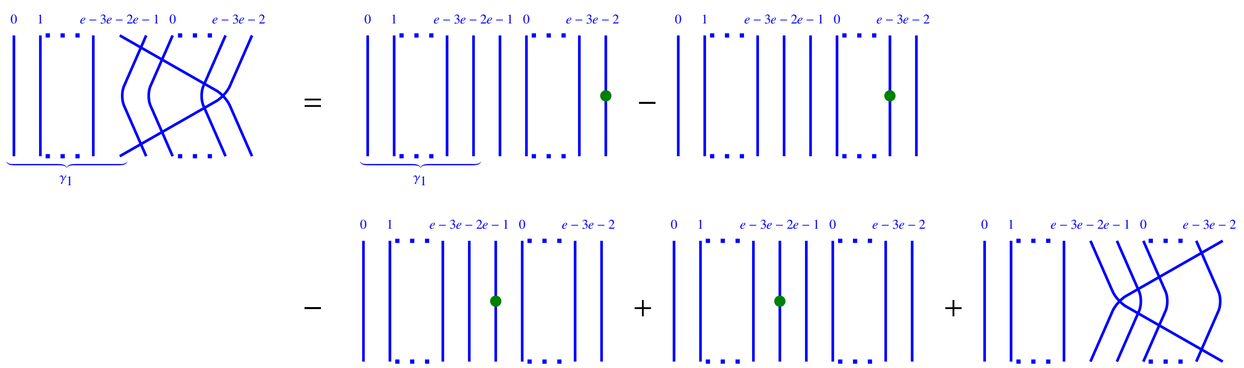
For the same reason as in Case 2.2.4a,



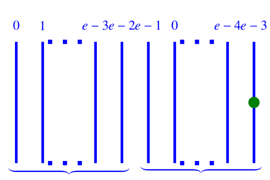
$$\in R_n^{>\gamma},$$

which implies the proposition in this case.

Case 2.2.4c: $i = e-2$, i.e. $\gamma_2 = e$. By Lemma 2.2.5,

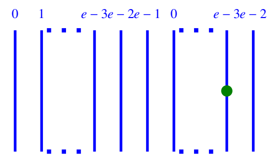


Set $\nu = (\gamma_1, \gamma_2 - 1) = \gamma|_{n-1}$. As $\gamma_1 \ge \gamma_2 - 1$, we have $\nu \in \mathscr{P}^{\Lambda}_{n-1}$. As $\lambda \in \mathscr{S}^{\Lambda}_n$ and $|\nu| < |\lambda|$, we have $\nu \in \mathscr{P}^{\Lambda}_I$. It is not hard to see that $b^{\nu_-}_{e-3} = 1$. Hence

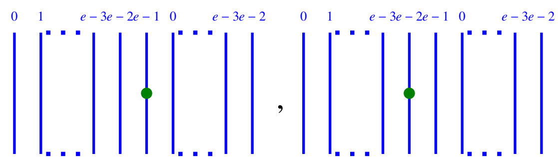


$$= e(\mathbf{i}_{\nu_-} \vee e-3) y_{\nu_-} y^1_{n-1} \in R_n^{>\nu}.$$

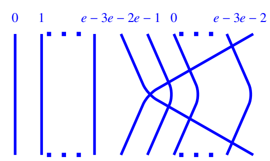
Then by Lemma 2.1.14,

 $\in R_n^{>\gamma}$.

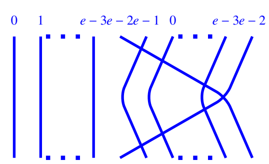
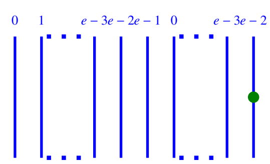
Similarly, we have

 $\in R_n^{>\gamma}$,

and for the similar method as in Case 2.2.4a, we have

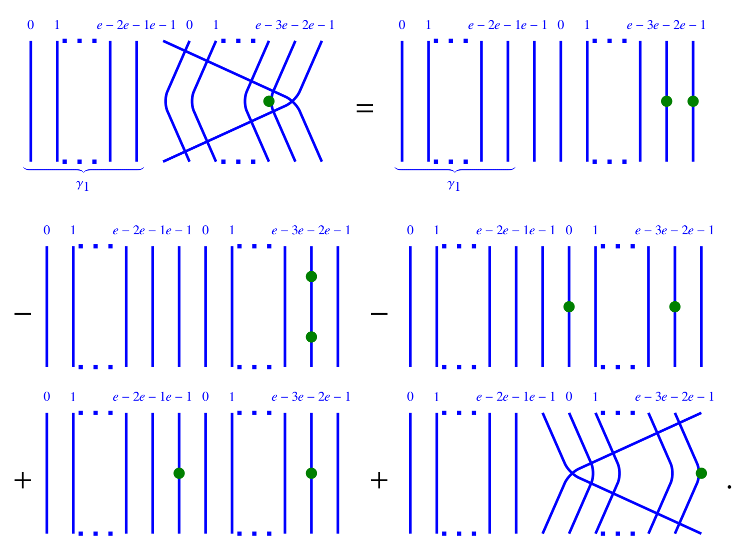
 $\in R_n^{>\gamma}$.

Therefore,

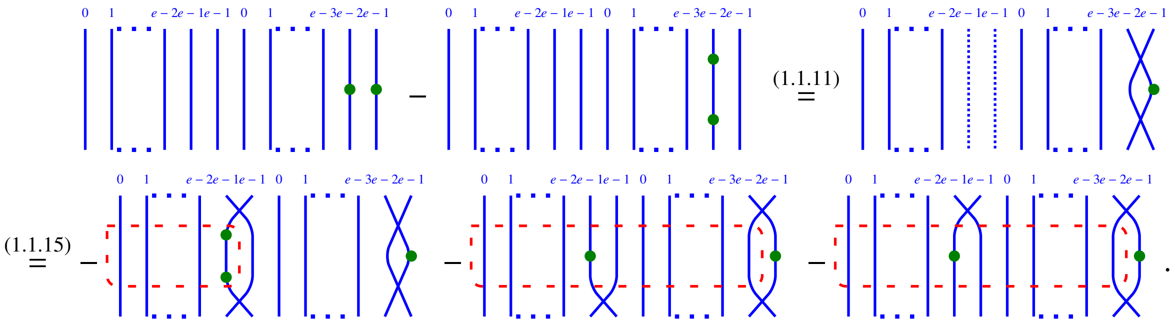
 $=_\gamma$  $= e_\gamma y_\gamma$,

which follows the Proposition.

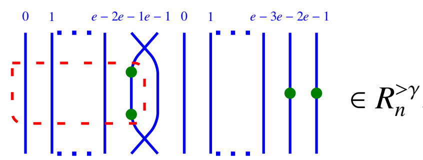
Case 2.2.4d: $i = e-1$, i.e. $\gamma_2 = e+1$. By Lemma 2.2.5,

(2.2.6) 

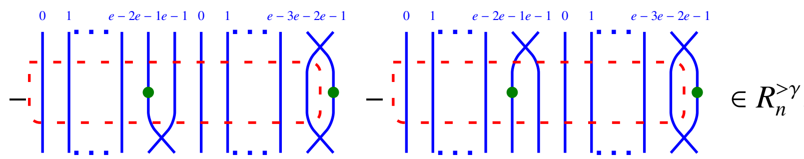
For the first two terms of (2.2.6),



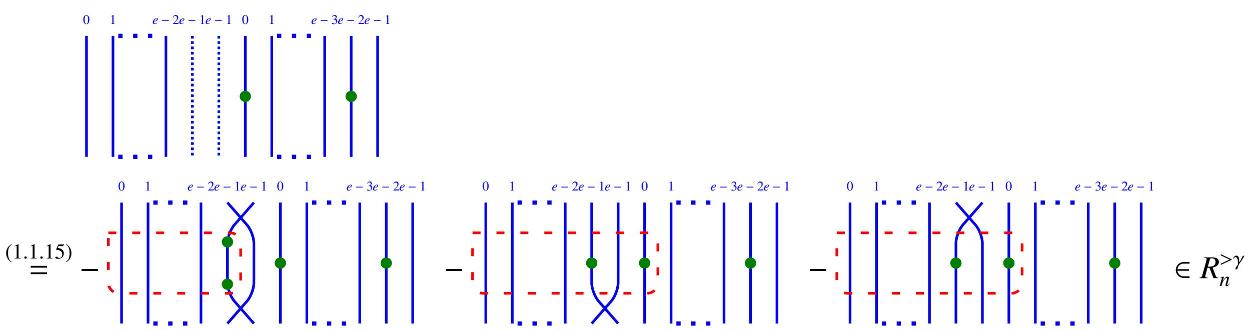
Using the similar method as in Case 2.2.4a,



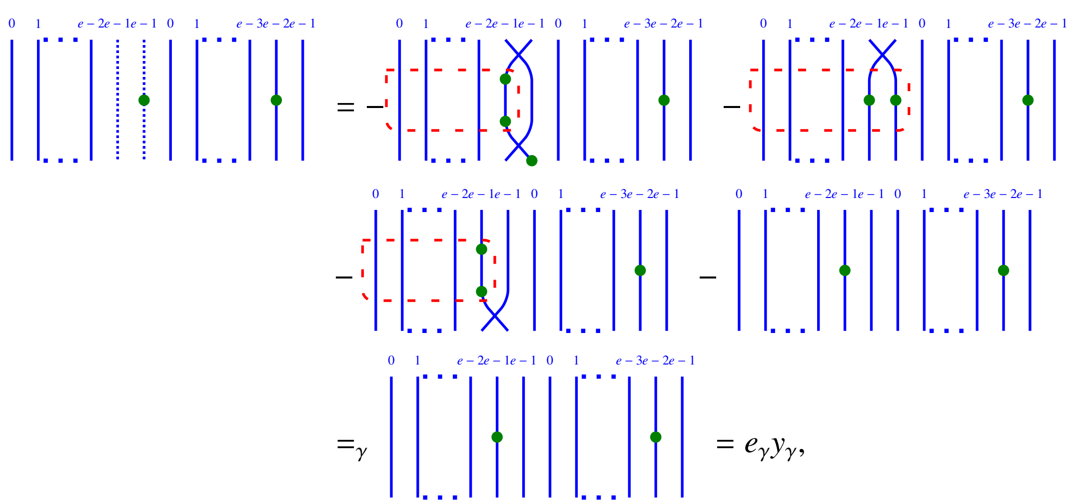
and using the similar method in Case 2.2.4c,



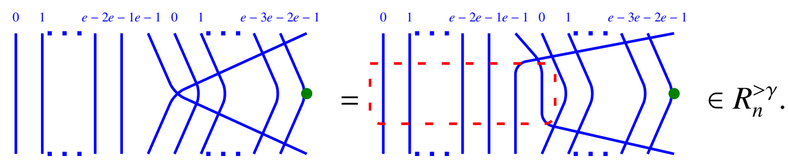
Hence the first two terms of (2.2.6) is in $R_n^{>\gamma}$. For the third term of (2.2.6),



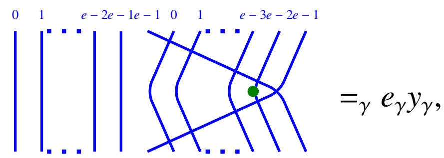
For the fourth term of (2.2.6), by (1.1.16),



and for the last term of (2.2.6),



Combine the result above, we have



which completes the proof.

□

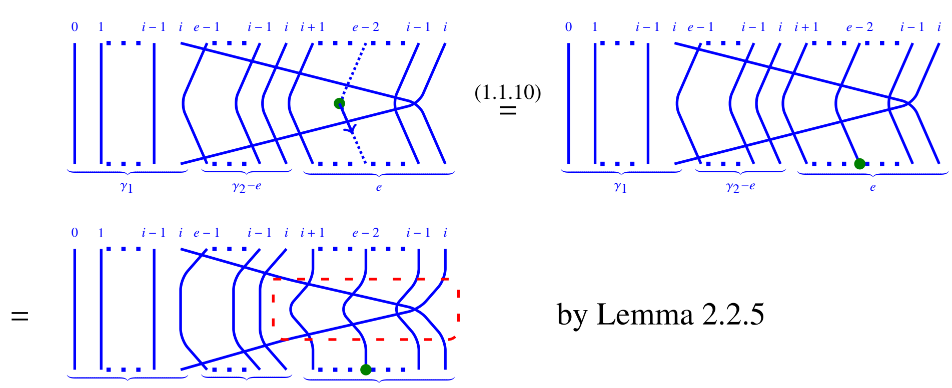
2.2.7. Remark. The technique of applying Lemma 2.1.14 in proving Proposition 2.2.4 will be used many times in the rest of the thesis. Although the process is straightforward, individual details will vary from case to case, thus in order to simplify the process we will omit details in the future.

Recall $\gamma_2 = k{\cdot}e + t$ where k is a nonnegative integer and $2 \leq t \leq e+1$. Now we remove the restriction on γ_2 by applying the induction on k.

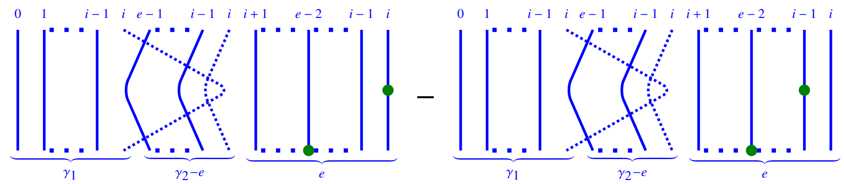
2.2.8. Proposition. *Suppose* $\gamma = (\gamma_1, \gamma_2) \in \mathscr{C}_n^{\Lambda}$ *with* $\gamma_2 > 1$ *and* $\gamma_2 - \gamma_1 \equiv 1 \pmod e$ *and* $\lambda = (\gamma_1, \gamma_2 - 1, 1) \in \mathscr{S}_n^{\Lambda}$. *Define i to be the residue of the node at position* $(1, m, 1)$. *Then (2.2.1) holds.*

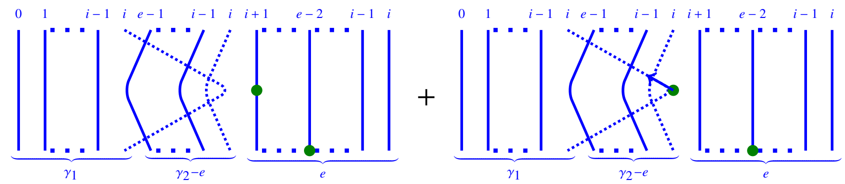
Proof. We prove this Proposition by induction. As we claimed before that we can write $\gamma_2 = k{\cdot}e + t$ with $2 \leq t \leq e+1$ and we will apply induction on k. Proposition 2.2.4 implies that for $k = 0$ the Proposition holds. Assume that for $k \leq k'$ the Proposition holds. For $k = k'$, we consider two different cases, which are $i = e-2$, $i = e-1$ and $i \neq e-2, e-1$. Recall that i is the residue of the node at $(1, m, 1)$ or $(2, m+1, 1)$.

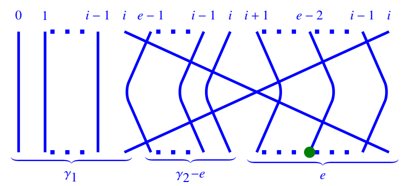
Case 2.2.8a: $i \neq e-2, e-1$.

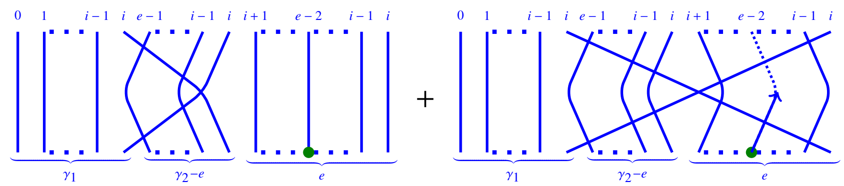


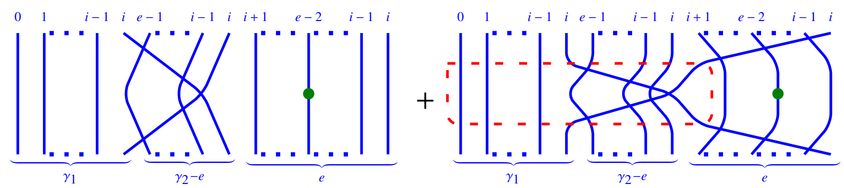
by Lemma 2.2.5

$=$ 

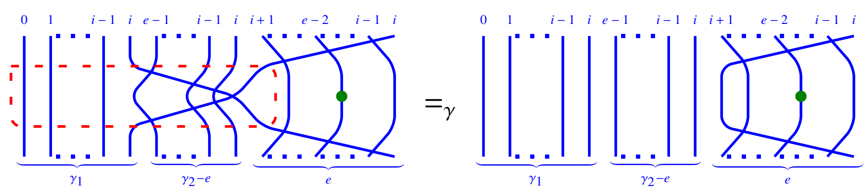
$-$ 

$+$ 

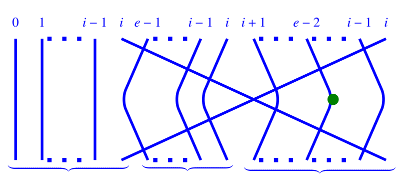
$\overset{(1.1.11)}{\underset{}{\overset{(1.1.17)}{=}}}$ 

$\overset{(1.1.10)}{=}$  .

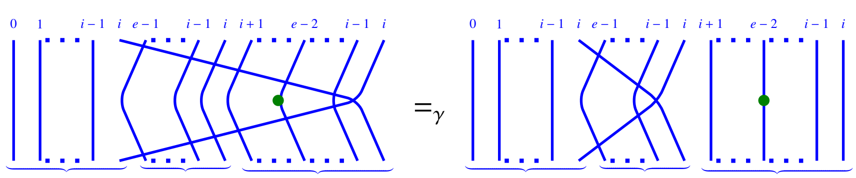
Then by induction and Lemma 2.1.14, we have

 $= 0$,

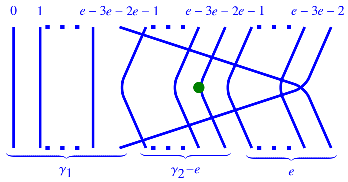
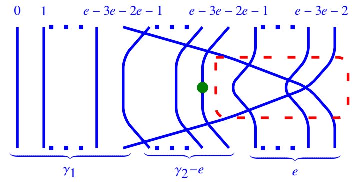
which implies that

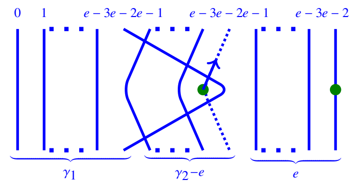
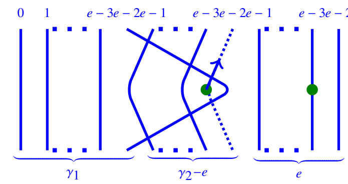
 $\in R_n^{>\gamma}$.

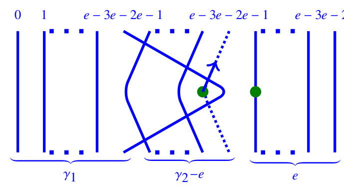
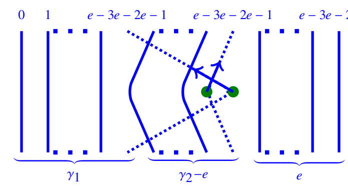
Hence by induction and Lemma 2.1.14

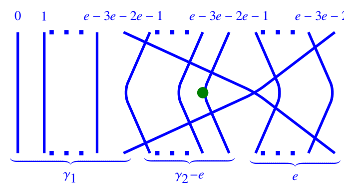
 $=_\gamma e_\gamma y_\gamma$.

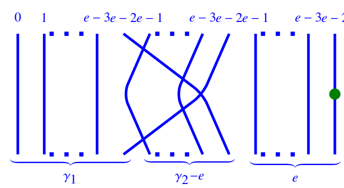
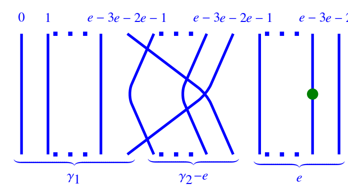
Case 2.2.8b: $i = e - 2$.

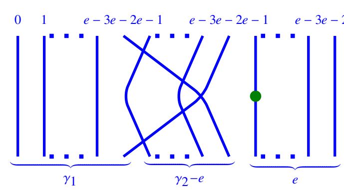
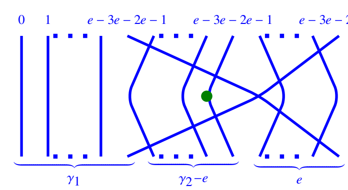
 $=$  by Lemma 2.2.5

$=$  $-$ 

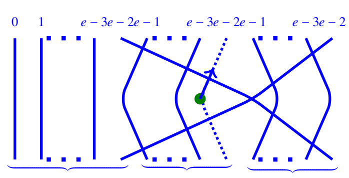
$-$  $+$ 

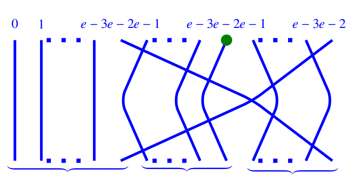
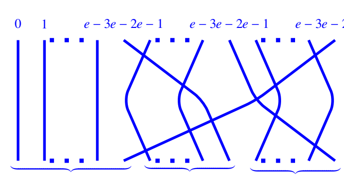
$+$ 

(2.2.9) $\overset{(1.1.10)}{\underset{(1.1.11)}{=}}$ $-$  $+$ 

$+$  $+$  .

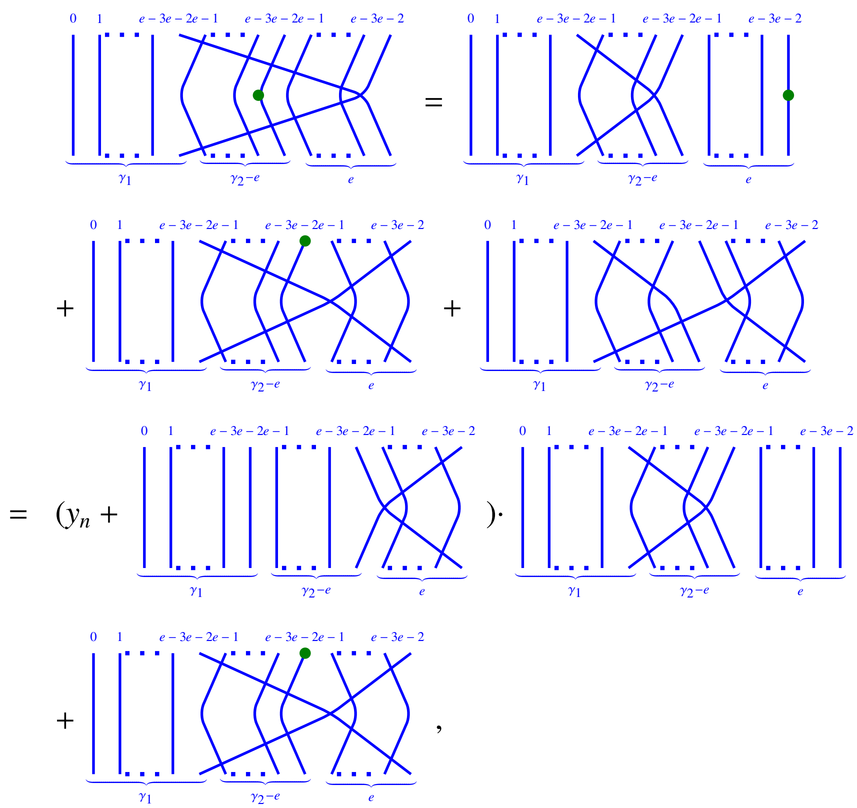
By induction and Lemma 2.1.14, the second and the third terms of (2.2.9) are both in $R_n^{>\gamma}$. Now for the last term.



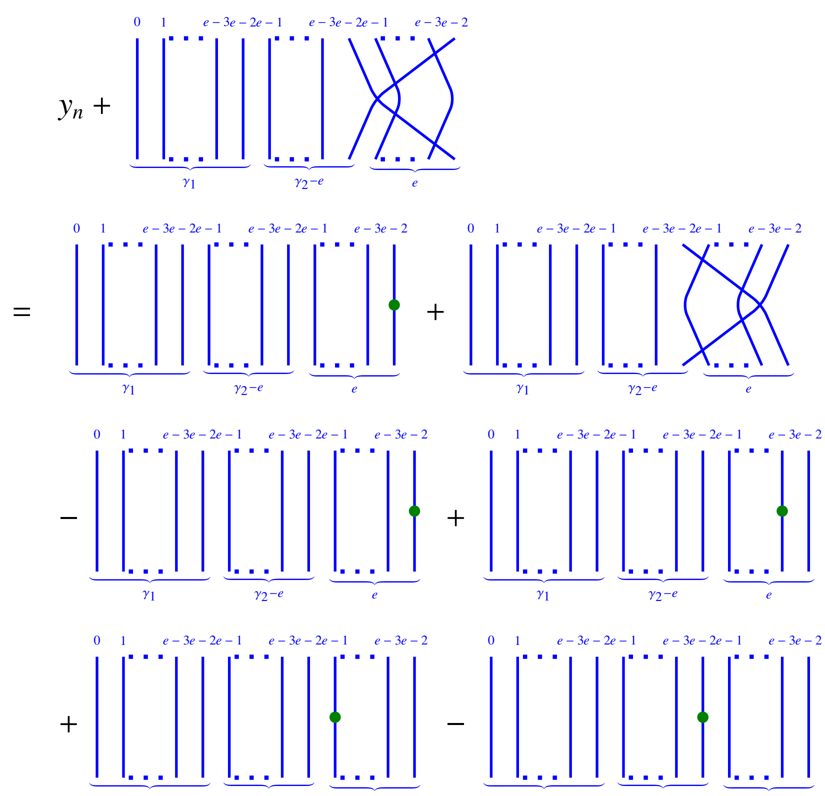
(2.2.10) $\overset{(1.1.10)}{=}$  $-$  .

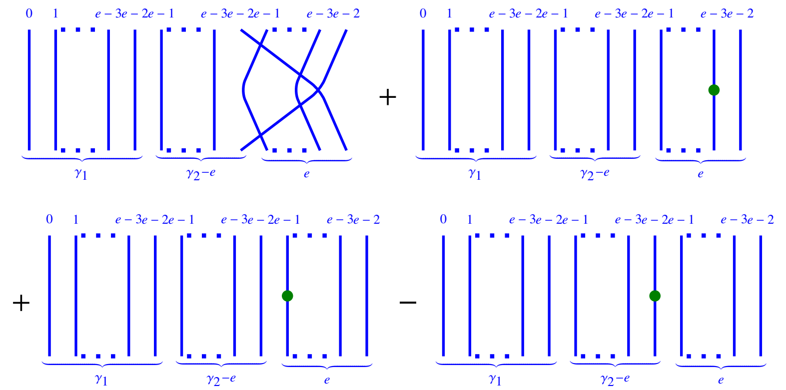
Substitute (2.2.10) to (2.2.9), let $n = \gamma_1 + \gamma_2$, we have

(2.2.11)

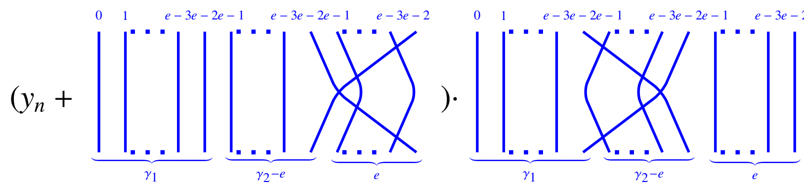


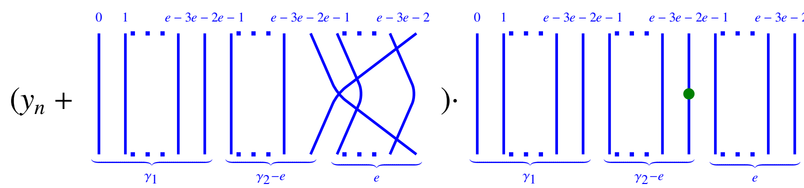
where by Lemma 2.2.5

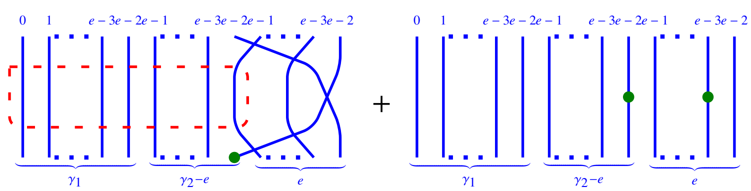


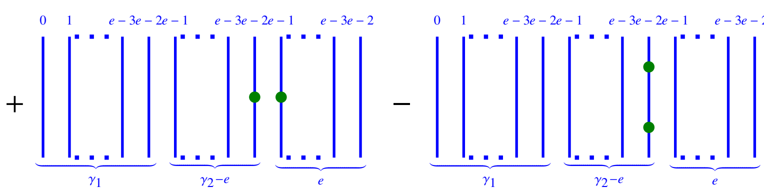
(2.2.12) =


Then by $\lambda \in \mathscr{S}_n^{\Lambda}$ and Lemma 2.1.14, for the first term of (2.2.11),

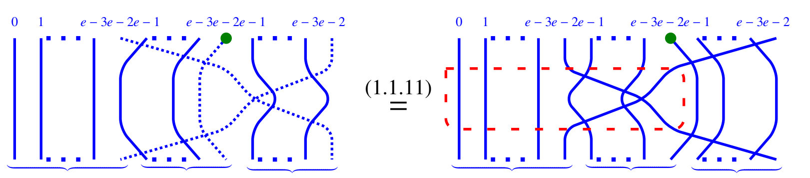

by induction

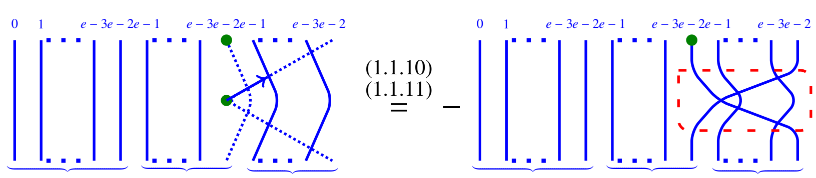
$=_\gamma$

by (2.2.12)

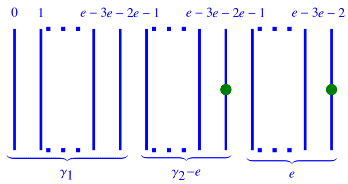
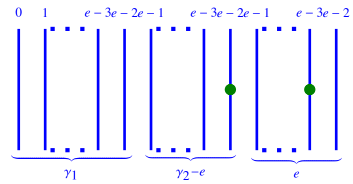
$=$

by Lemma 2.1.14

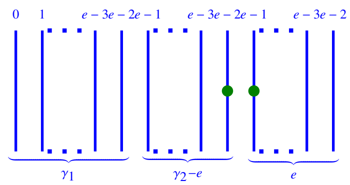
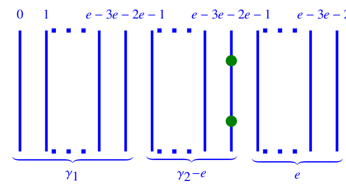
(2.2.13)

$\in R_n^{>\gamma}$.

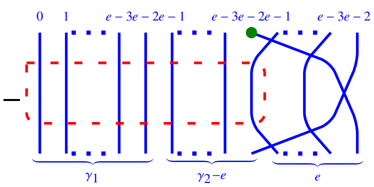
For the second term of (2.2.11), by induction, Lemma 2.2.5, $\lambda \in \mathscr{S}_n^{\Lambda}$ and Lemma 2.1.14,

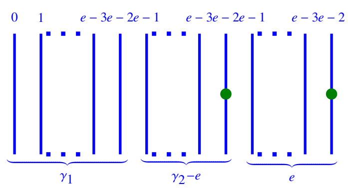

by induction

$=_\gamma$

by Lemma 2.2.5

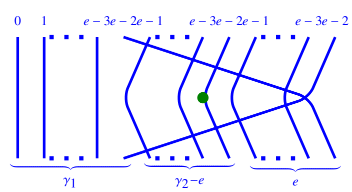
$=$  $-$ 

$-$  $+$ 

$-$  by Lemma 2.1.14

(2.2.14) $=_\gamma$  $= e_\gamma y_\gamma.$

Substitute the results of (2.2.13) and (2.2.14) to (2.2.11), we have

 $=_\gamma e_\gamma y_\gamma.$

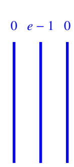
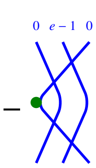
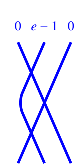
Case 2.2.8c: $i = e-1$. The method to prove this is the same as for Case 2.2.8a so it is left as an exercise. Then by induction, this completes the proof. □

Finally, we can use (2.2.1) to prove our main result of this section.

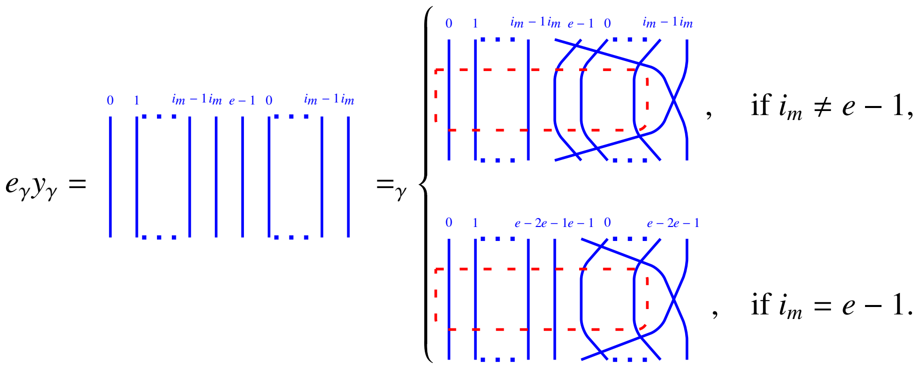
2.2.15. Proposition. *Suppose m is a positive integer, $\lambda = (m, m, 1) \in \mathscr{S}_n^\Lambda$ and $\gamma = (\gamma_1, \gamma_2) = (m, m+1)$. Recall $\lambda_- = (m, m)$. Write $\mathbf{i}_{\lambda_-} = (i_1, i_2, \dots, i_{n-1})$. If $k = i_{n-1} + 1 \in I$, we have*

$$e_\gamma y_\gamma = e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

Proof. Without loss of generally we assume $\Lambda = \Lambda_0$. When $m = 1$, then $\gamma = (1, 2)$ and

$e_\gamma y_\gamma =$  $\overset{(1.1.12)}{\underset{(1.1.17)}{=}} -$  $-$  $\in N_3^{\Lambda_0} \subseteq R_n^{>\lambda}.$

When $m > 1$, write $\mathbf{i}_{(\gamma_1)} = (i_1, i_2, \dots, i_m)$. Set $\sigma = (\gamma_1 - 1, \gamma_2) = (m-1, m)$. Then by Proposition 2.2.8, we have



In both cases, the parts bounded by square are both $e_\sigma y_\sigma$. As $|\sigma| = n-2$ and $\lambda \in \mathscr{S}_n^\Lambda$, by induction, $e_\sigma y_\sigma \in R_n^{\rhd\sigma}$. By the definition of σ and Lemma 1.4.4, it forces that $e_\sigma y_\sigma \in R_n^{\rhd\lambda|_{n-2}}$. Then by Lemma 2.1.14, we have $e_\gamma y_\gamma \in R_n^{\rhd\lambda}$. □

2.3. Final part of y-problem

In the last section we have proved that if $\lambda = (m, m, 1) \in \mathscr{S}_n^\Lambda$, then

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{\rhd\lambda}$$

with $k = i_n + 1$. In this section we will gradually remove the restrictions on λ and k. First we are going to introduce a useful homomorphism and use it to prove a few more properties of $\mathscr{R}_n$ and $\mathscr{R}_n^\Lambda$. After that we are going to show that if $\lambda \in \mathscr{P}_I^\Lambda$, then we can extend λ to a $\ell + 1$ multipartition by adding an $\emptyset$ at the end and thus the new multipartition is in $\mathscr{P}_I^{\Lambda+\Lambda_i}$ for any $i \in I$. Analogous results are also true for $\mathscr{P}_y^\Lambda$ and $\mathscr{P}_\psi^\Lambda$. These will allow us to extend the result to an arbitrary multipartition λ.

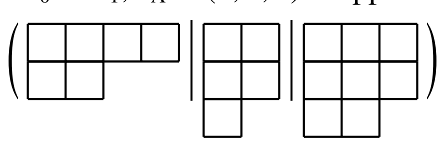
For any $\mathbf{j} \in I^m$, we can define a linear map $\hat{\theta}_\mathbf{j} : \mathscr{R}_n \longrightarrow \mathscr{R}_{n+m}^\Lambda$ sending $e(\mathbf{i})$ to $e(\mathbf{j} \vee \mathbf{i})$, y_r to y_{r+m} and ψ_r to ψ_{r+m}. This map $\hat{\theta}_\mathbf{j}$ works as embedding from $\mathscr{R}_n$ to $\mathscr{R}_{n+m}$ followed by the projection onto $\mathscr{R}_{n+m}^\Lambda$.

2.3.1. Lemma. *For $\mathbf{j} \in I^m$, the map $\hat{\theta}_\mathbf{j}$ is a homomorphism.*

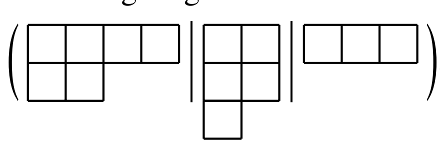
Proof. The map is defined to be linear. Hence we only have to check the relations. Since the relations of $\mathscr{R}_n$ and $\mathscr{R}_{n+m}^\Lambda$ from Definition 1.1.1 are independent of the value of r, we can see that $\hat{\theta}_\mathbf{j}$ is a homomorphism. □

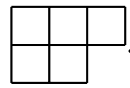
It will be necessary to cut a multicomposition λ into one multicomposition μ and a composition γ for our later work. Note that in our work we will mainly set μ to be a multipartition and γ to be a partition, but generally we don't have such restriction.

2.3.2. **Example** Fix $e = 4$, $\Lambda = 2\Lambda_0 + \Lambda_1$, $\kappa_\Lambda = (0, 1, 0)$. Suppose $\lambda = (4, 2|2^2, 1|3^2, 2)$. So

$[\lambda] =$ .

We want to divide the last partition of λ after the first row. This is called the cut row of λ. This gives us a multipartition μ with Young diagram

$[\mu] =$ , and a partition γ with diagram

$[\gamma] =$  .

We call μ and γ the cut part and the remaining part, respectively.

Moreover we want to preserve the following data. The value $|\mu|$ is called the cut of λ which is 14 in this case. The residue of the top left node of γ as a subdiagram of $\lambda^{(3)}$ is called the cut residue, which in this case is 3. $\diamond$

Now we give a formal definition.

2.3.3. Definition. *Suppose $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell)}) \in \mathscr{C}_n^\Lambda$ with $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_{k_\ell}^{(\ell)})$ and a is an integer such that $0 \le a < k_\ell$. We call m a **cut** of λ and a the **cut row** associated to m where $m = \sum_{i=1}^{\ell-1} |\lambda^{(i)}| + \sum_{j=1}^{a} \lambda_j^{(\ell)}$. Define $\Lambda' = \Lambda_s$, where $s = \kappa_\ell + 1 - (a+1) = \kappa_\ell - a$, the residue of the node at position $(a+1, 1, \ell)$. We call s to be **cut residue** associated to m and Λ' to be **cut weight** associated to m. We then define $\mu = \lambda|_m \in \mathscr{C}_m^\Lambda$ and $\gamma = (\lambda_{a+1}^{(\ell)}, \lambda_{a+2}^{(\ell)}, \dots, \lambda_{k_\ell}^{(\ell)}) \in \mathscr{C}_{n-m}^{\Lambda'}$ and call μ and γ to be **cut part** and **remaining part** of λ associated to m, respectively.*

Note we can either remove a portion of the last tableau, or cut out the whole partition.

We will start to work with $\hat{\theta}_{\mathbf{i}}$, which involving elements in both $\mathscr{R}_n$ and $\mathscr{R}_n^\Lambda$. Recall that $\hat{e}(\mathbf{i})$, $\hat{y}_r$, $\hat{\psi}_s$ and $\hat{\psi}_{\mathsf{st}}$ are elements from $\mathscr{R}_n$ and $e(\mathbf{i})$, y_r, ψ_s and ψ_{st} are elements from $\mathscr{R}_n^\Lambda$.

2.3.4. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$. Let m be a cut of λ with $m < n-1$, $\nu = \lambda|_m$ and Λ' be the cut weight associated to m. Consider $N_{n-m}^{\Lambda'} \subseteq \mathscr{R}_{n-m}$. If $\hat{\theta}_{\mathbf{i}_\nu} : \mathscr{R}_{n-m} \longrightarrow \mathscr{R}_n^\Lambda$, then $\hat{\theta}_{\mathbf{i}_\nu}(N_{n-m}^{\Lambda'}) y_\nu \subseteq R_n^{>\lambda}$.*

Proof. Consider $r \in N_{n-m}^{\Lambda'}$. Then by (1.4.7),

$$r = \sum_{\mathbf{j}=(j_1,j_2,\dots,j_{n-m}) \in I^{n-m}} c_{\mathbf{j}} R'_{\mathbf{j}} \hat{e}(\mathbf{j}) \hat{y}_1^{(\Lambda',\alpha_{j_1})} R_{\mathbf{j}},$$

where $R_{\mathbf{j}}$ and $R'_{\mathbf{j}}$ are some elements in $\mathscr{R}_{n-m}$ and $c_{\mathbf{j}} \in \mathbb{Z}$. Therefore

$$\begin{aligned}
\hat{\theta}_{\mathbf{i}_\nu}(r) y_\nu &= \sum_{\mathbf{j}=(j_1,j_2,\dots,j_{n-m}) \in I^{n-m}} c_{\mathbf{j}} \hat{\theta}_{\mathbf{i}_\nu}(R'_{\mathbf{j}}) \hat{\theta}_{\mathbf{i}_\nu}(\hat{e}(\mathbf{j}) \hat{y}_1^{(\Lambda',\alpha_{j_1})}) \hat{\theta}_{\mathbf{i}_\nu}(R_{\mathbf{j}}) y_\nu \\
&= \sum_{\mathbf{j}=(j_1,j_2,\dots,j_{n-m}) \in I^{n-m}} c_{\mathbf{j}} \hat{\theta}_{\mathbf{i}_\nu}(R'_{\mathbf{j}}) e(\mathbf{i}_\nu \vee j_1 \vee j_2 j_3 \dots j_{n-m}) y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})} \hat{\theta}_{\mathbf{i}_\nu}(R_{\mathbf{j}}) \\
&= \sum_{\mathbf{j}=(j_1,j_2,\dots,j_{n-m}) \in I^{n-m}} c_{\mathbf{j}} \hat{\theta}_{\mathbf{i}_\nu}(R'_{\mathbf{j}}) \theta_{(j_2,j_3,\dots,j_{n-m})}(e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})}) \hat{\theta}_{\mathbf{i}_\nu}(R_{\mathbf{j}}).
\end{aligned}$$

Next we consider $e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})} \in \mathscr{R}_{m+1}^\Lambda$.

Recall that we can write $\nu = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(\ell-1)}, \nu^{(\ell)})$ and $\nu^{(\ell)} = (\nu_1^{(\ell)}, \dots, \nu_l^{(\ell)})$. Let $\mu = \lambda|_{m+1}$. As $m < n-1$, $|\mu| = m+1 < n = |\lambda|$, and $\lambda \in \mathscr{S}_n^\Lambda$, we have $\mu \in \mathscr{P}_I^\Lambda$. Write $\mathbf{i}_\nu = (i_1, i_2, \dots, i_m)$. Notice that $(\Lambda', \alpha_{j_1}) = |\mathscr{A}_{\mathfrak{t}^\nu}^{j_1}|$. We consider two cases.

Suppose $j_1 = i_m + 1$ and ν^+ is a multipartition. By Definition 2.1.2 we have $|\mathscr{A}_{\mathfrak{t}^\nu}^{j_1}| = b_{j_1}^\nu - 1$. Then by Lemma 2.1.3 we have $e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})} = e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{b_{j_1}^\nu - 1} = e_{\nu^+} y_{\nu^+}$. Because m is a cut of λ and $\nu = \lambda|_m$, $\mu = \lambda|_{m+1}$, we must have $\nu^+ > \mu$. Therefore $e_{\nu^+} y_{\nu^+} \in R_n^{>\mu}$. So

$$e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})} \in R_n^{>\mu}.$$

Otherwise, by Definition 2.1.2 we have $|\mathscr{A}_{\mathfrak{t}^\nu}^{j_1}| = b_{j_1}^\nu$. Then by $\mu \in \mathscr{P}_I^\Lambda$ and the definition of $\mathscr{P}_I^\Lambda$, for any $j_1 \in I$, we have $e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{b_{j_1}^\nu} \in R_n^{>\mu}$ because $\nu = \mu_-$. Therefore

$$e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})} = e(\mathbf{i}_\nu \vee j_1) y_\nu y_{m+1}^{b_{j_1}^\nu} \in R_n^{>\mu}.$$

Therefore for any $j_1 \in I$ we have $e(\mathbf{i}_\nu \vee j_1)y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})} \in R_n^{>\mu}$. Hence by Lemma 2.1.14 and Lemma 2.1.12,

$$\hat{\theta}_{\mathbf{i}_\nu}(R'_{\mathbf{j}})\theta_{(j_2,j_3,\ldots,j_{n-m})}(e(\mathbf{i}_\nu \vee j_1)y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})})\hat{\theta}_{\mathbf{i}_\nu}(R_{\mathbf{j}}) \subseteq R_n^{>\lambda}.$$

Therefore $\hat{\theta}_{\mathbf{i}_\nu}(r)y_\nu = \sum_{\mathbf{j}=(j_1,j_2,\ldots,j_{n-m})\in I^{n-m}} c_{\mathbf{j}}\hat{\theta}_{\mathbf{i}_\nu}(R'_{\mathbf{j}})\theta_{(j_2,j_3,\ldots,j_{n-m})}(e(\mathbf{i}_\nu \vee j_1)y_\nu y_{m+1}^{(\Lambda',\alpha_{j_1})})\hat{\theta}_{\mathbf{i}_\nu}(R_{\mathbf{j}}) \subseteq R_n^{>\lambda}$. □

2.3.5. Definition. *Suppose λ is a multicomposition of m and μ is a composition. If we can find a multicomposition γ such that λ and μ are cut part and remaining part of γ associated m, we write $\gamma = \lambda \vee \mu$ and say γ is the* ***concatenation*** *of λ and μ.*

For example, suppose $\lambda = (2^2, 1|3^3|2)$ and $\mu = (4, 2)$, then $\gamma = \lambda \vee \mu = (2^2, 1|3^3|2, 4, 2)$. Notice that in general γ is not a multipartition.

The following Corollaries follows by the definition of $\lambda \vee \mu$.

2.3.6. Corollary. *Suppose λ is a multipartition of n and μ, γ are partitions of m. Then $\mu > \gamma$ if and only if $\lambda \vee \mu > \lambda \vee \gamma$.*

2.3.7. Corollary. *Suppose λ is a multipartition of n and μ is a partition of m. If $\gamma = \lambda \vee \mu$, $\hat{\theta}_{\mathbf{i}_\lambda}(\hat{e}_\mu \hat{y}_\mu)y_\lambda = e_\gamma y_\gamma$.*

2.3.8. Corollary. *Suppose λ and μ are multipartitions and γ is a partition such that $\lambda = \mu \vee \gamma$. If u and v are standard γ-tableaux, we can find standard λ-tableaux $\dot{\mathsf{u}}$ and $\dot{\mathsf{v}}$ such that $\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}})y_\mu = \psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}$.*

Proof. Suppose $\lambda \in \mathscr{P}_n^\Lambda$ and $\mu \in \mathscr{P}_m^\Lambda$. By Definition 2.3.3, μ is the cut part of λ associated to m. Let a be the cut row associated to m. Define $\dot{\mathsf{u}}$ to be the standard λ-tableau such that $\dot{\mathsf{u}}|_m = \mathsf{t}^\mu$, and for any $k > m$, if $\dot{\mathsf{u}}^{-1}(k) = (r_1, c_1, \ell_1)$ and $\mathsf{u}^{-1}(k-m) = (r_2, c_2, 1)$, then

$$c_1 = c_2,$$
$$r_1 = r_2 + a.$$

Define $\dot{\mathsf{v}}$ similarly. It is trivial that $\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{d(\mathsf{u})}) = \psi_{d(\dot{\mathsf{u}})}$ and $\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{d(\mathsf{v})}) = \psi_{d(\dot{\mathsf{v}})}$. Therefore by Corollary 2.3.7,

$$\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}}) = \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}^*_{d(\mathsf{u})})\hat{\theta}_{\mathbf{i}_\mu}(\hat{e}_\gamma \hat{y}_\gamma)\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{d(\mathsf{v})}) = \psi^*_{d(\dot{\mathsf{u}})}e_\lambda y_\lambda \psi_{d(\dot{\mathsf{v}})} = \psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}.$$

□

2.3.9. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$ and $\mu \in \mathscr{C}_n^\Lambda$ with $\mu > \lambda$. If $\mu_- \neq \lambda_-$, then $e_\mu y_\mu \in R_n^{>\lambda}$.*

Proof. As $\mu > \lambda$ and $\mu_- \neq \lambda_-$, we can find $m < n$ such that $\mu|_m > \lambda|_m$ and $\mu|_{m-1} = \lambda|_{m-1}$. Set $\nu = \mu|_m$. If $\nu \in \mathscr{P}_m^\Lambda$, we have $e_\nu y_\nu = \psi_{\mathsf{t}^\nu \mathsf{t}^\nu} \in R_n^{>\lambda|_m}$, so by Lemma 2.1.14 we have $e_\mu y_\mu \in R_n^{>\lambda}$.

If $\nu \notin \mathscr{P}_m^\Lambda$, because $\lambda \in \mathscr{S}_n^\Lambda$ and $|\nu| = m < n$, we have $\nu \in \mathscr{P}_I^\Lambda$. Notice that if we write $\nu = (\nu^{(1)}, \ldots, \nu^{(l)}, \emptyset, \ldots, \emptyset)$ with $\nu^{(l)} = (\nu_1^{(l)}, \ldots, \nu_{k-1}^{(l)}, \nu_k^{(l)})$, because $\nu|_{m-1} = \mu|_{m-1} = \lambda|_{m-1} \in \mathscr{P}_{m-1}^\Lambda$ and $\nu \notin \mathscr{P}_m^\Lambda$, we must have $\nu_{k-1}^{(l)} + 1 = \nu_k^{(l)}$. Therefore if we write $\mathbf{i}_\nu = (i_1, i_2, \ldots, i_m)$, we have

$$e_\nu y_\nu = e(\mathbf{i}_{\nu_-} \vee i_m)y_{\nu_-}y_m^{b_{i_m}^{\nu_-}} \in R_n^{>\nu} \subseteq R_n^{>\lambda|_m}.$$

Then by Lemma 2.1.14, we have $e_\mu y_\mu \in R_n^{>\lambda}$. This completes the proof. □

Now we are ready to start proving the main result of this chapter. We start by proving two more specialized Propositions. After that we will introduce a Proposition which removes these restrictions and leads to the main Theorem of this chapter.

2.3.10. Proposition. *Suppose* $\lambda \in \mathscr{S}_n^\Lambda$ *and* $\lambda_- = (\lambda^{(1)}, \dots, \lambda_-^{(\ell)})$ *with* $\lambda_-^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_{l-1}^{(\ell)}, \lambda_l^{(\ell)}) \neq \emptyset$. *Write* $\mathbf{i}_{\lambda_-} = (i_1, i_2, \dots, i_{n-1})$. *For* $k \in I$, *if* $k \neq i_{n-1} + 1$, *or* $k = i_{n-1} + 1$ *and* $\lambda_{l-1}^{(\ell)} > \lambda_l^{(\ell)}$, *we have*

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

Proof. For convenience set $m = \lambda_l^{(\ell)}$ and $\mu = \lambda|_{n-m-1}$. Therefore $\mu = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \mu^{(\ell)})$ where

$$\mu^{(\ell)} = \begin{cases} (\lambda_1^{(\ell)}, \dots, \lambda_{l-1}^{(\ell)}), & \text{if } l > 1, \\ \emptyset, & \text{if } l = 1. \end{cases}$$

Suppose i is the residue of node $(l, 1, \ell)$ in λ and $\Lambda' = \Lambda_i$. Define $\gamma = (m+1) \in \mathscr{P}_{m+1}^{\Lambda'}$. Notice that $\lambda_- = \mu \vee \gamma_-$. Because $k \neq i_{n-1} + 1$ or $k = i_{n-1} + 1$ and $\lambda_{k-1}^{(\ell)} > \lambda_k^{(\ell)}$, we have $b_k^{\gamma_-} = b_k^{\lambda_-}$. By Lemma 2.1.9, in $\mathscr{R}_{m+1}^{\Lambda'}$ we have $e(\mathbf{i}_{\gamma_-} \vee k) y_{\gamma_-} y_{m+1}^{b_k^{\gamma_-}} \in R_n^{>\gamma}$. This implies that in $\mathscr{R}_{m+1}$, $\hat{e}(\mathbf{i}_{\gamma_-} \vee k) \hat{y}_{\gamma_-} \hat{y}_{m+1}^{b_k^{\gamma_-}} \in N_{m+1}^{\Lambda'}$. Then let $\hat{\theta}_{\mathbf{i}_\mu} : \mathscr{R}_{m+1} \longrightarrow \mathscr{R}_n^\Lambda$, by Lemma 2.3.4,

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e(\mathbf{i}_\mu \vee \mathbf{i}_{\gamma_-} \vee k) y_{\lambda_-} y_n^{b_k^{\gamma_-}} = \hat{\theta}_{\mathbf{i}_\mu}(\hat{e}(\mathbf{i}_{\gamma_-} \vee k) \hat{y}_{\gamma_-} \hat{y}_{m+1}^{b_k^{\gamma_-}}) y_\mu \in \hat{\theta}_{\mathbf{i}_\mu}(N_{m+1}^{\Lambda'}) y_\mu \subseteq R_n^{>\lambda},$$

which completes the proof. □

2.3.11. Proposition. *Suppose* $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{S}_n^\Lambda$ *with* $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_{l-1}^{(\ell)}, \lambda_l^{(\ell)}, 1)$ *and* $l \geq 2$, *where* $\lambda_{l-1}^{(\ell)} = \lambda_l^{(\ell)}$. *Write* $\mathbf{i}_{\lambda_-} = (i_1, i_2, \dots, i_{n-1})$. *Suppose* $k \in I$ *and* $k \equiv i_{n-1} + 1 \pmod e$. *Then*

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

Proof. For convenience set $m = \lambda_{l-1}^{(\ell)} = \lambda_l^{(\ell)}$, and $\mu = \lambda|_{n-2m-1}$. Therefore $\mu = (\lambda^{(1)}, \dots, \lambda^{(\ell-1)}, \mu^{(\ell)})$ where

$$\mu^{(\ell)} = \begin{cases} (\lambda_1^{(\ell)}, \dots, \lambda_{l-2}^{(\ell)}), & \text{if } l > 2, \\ \emptyset, & \text{if } l = 2. \end{cases}$$

Suppose i is the residue of node $(l-1, 1, \ell)$ in λ and $\Lambda' = \Lambda_i$. Define $\gamma = (m, m+1) \in \mathscr{P}_{2m+1}^{\Lambda'}$. Notice that $\lambda_- = \mu \vee \gamma_-$. Because $k \equiv i_{n-1} + 1 \pmod e$, we have $b_k^{\gamma_-} = b_k^{\lambda_-}$ and $e(\mathbf{i}_{\gamma_-} \vee k) y_{\gamma_-} y_{2m+1}^{b_k^{\gamma_-}} = e_\gamma y_\gamma$. By Proposition 2.2.15, we have $e_\gamma y_\gamma \in R_n^{>\gamma}$. Therefore we can write $e_\gamma y_\gamma = \sum_{\substack{\mathsf{u},\mathsf{v} \in \mathrm{Std}(\sigma) \\ \sigma > \gamma}} c_{\mathsf{uv}} \psi_{\mathsf{uv}}$ with $\sigma = (\sigma_1, \sigma_2)$ where $\sigma_2 \geq 0$ and $\sigma_1 > \gamma_1 = m$. Therefore in $\mathscr{R}_{2m+1}$, we have

$$\hat{e}_\gamma \hat{y}_\gamma = \sum_{\substack{\mathsf{u},\mathsf{v} \in \mathrm{Std}(\sigma) \\ \sigma > \gamma}} c_{\mathsf{uv}} \hat{\psi}_{\mathsf{uv}} + r,$$

with $r \in N_{2m+1}^{\Lambda'}$ and $c_{\mathsf{uv}} \in \mathbb{Z}$. Therefore

$$\begin{aligned} e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} &= e(\mathbf{i}_\mu \vee \mathbf{i}_{\gamma_-} \vee k) y_{\lambda_-} y_n^{b_k^{\gamma_-}} = \hat{\theta}_{\mathbf{i}_\mu}(\hat{e}(\mathbf{i}_{\gamma_-} \vee k) \hat{y}_{\gamma_-} \hat{y}_{2m+1}^{b_k^{\gamma_-}}) y_\mu = \hat{\theta}_{\mathbf{i}_\mu}(\hat{e}_\gamma \hat{y}_\gamma) y_\mu \\ &= \sum_{\substack{\mathsf{u},\mathsf{v} \in \mathrm{Std}(\sigma) \\ \sigma > \gamma}} c_{\mathsf{uv}} \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}}) y_\mu + \hat{\theta}_{\mathbf{i}_\mu}(r). \end{aligned} \tag{2.3.12}$$

For the first term of (2.3.12), define $\alpha = \mu \vee \sigma \in \mathscr{C}_n^\Lambda$. Because $\sigma > \gamma$, by Corollary 2.3.6 we have $\alpha = \mu \vee \sigma > \mu \vee \gamma > \lambda$. Therefore

$$\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}}) y_\mu = \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}^*_{d(\mathsf{u})}) \hat{\theta}_{\mathbf{i}_\mu}(\hat{e}_\sigma \hat{y}_\sigma) y_\mu \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{d(\mathsf{v})}) = \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}^*_{d(\mathsf{u})}) e_{\mu\vee\sigma} y_{\mu\vee\sigma} \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{d(\mathsf{v})}) = \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}^*_{d(\mathsf{u})}) e_\alpha y_\alpha \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{d(\mathsf{v})})$$

Because $\sigma = (\sigma_1, \sigma_2) > \gamma = (m, m+1)$, we must have $\sigma_1 > m$. Therefore $\alpha_- = \mu \vee \sigma_- \neq \mu \vee \gamma_- = \lambda_-$. Then by Lemma 2.3.9, $e_\alpha y_\alpha \in R_n^{>\lambda}$. By Lemma 2.1.12, we have $\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}}) y_\mu = \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}^*_{d(\mathsf{u})}) e_\alpha y_\alpha \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{d(\mathsf{v})}) \in R_n^{>\lambda}$ which yields $\sum_{\substack{\mathsf{u},\mathsf{v} \in \mathrm{Std}(\sigma) \\ \sigma > \gamma}} c_{\mathsf{uv}} \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}}) y_\mu \in R_n^{>\lambda}$.

For the second term of (2.3.12), by Lemma 2.3.4, we have $\hat{\theta}_{\mathbf{i}_\mu}(r) \in R_n^{>\lambda}$. Therefore

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} \in R_n^{>\lambda}.$$

□

Suppose $\lambda \in \mathscr{S}_n^\Lambda$. If $\lambda_- = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell-1)}, \lambda_-^{(\ell)})$ with $\lambda_-^{(\ell)} \neq \emptyset$ by Proposition 2.3.10 and Proposition 2.3.11 we have $\lambda \in \mathscr{P}_I^\Lambda$. In the rest of the section we are going to prove the result is still true if $\lambda_-^{(\ell)} = \emptyset$.

Suppose $\mu = (\mu^{(1)}, \ldots, \mu^{(\ell)}) \in \mathscr{P}_n^\Lambda$ and $\kappa_\Lambda = (\kappa_1, \ldots, \kappa_\ell)$, if $\mu^{(\ell)} = \emptyset$, we define $\bar{\Lambda} = \Lambda - \Lambda_{\kappa_\ell}$, $\kappa_{\bar{\Lambda}} = (\kappa_1, \ldots, \kappa_{\ell-1})$ and $\bar{\mu} = (\mu^{(1)}, \ldots, \mu^{(\ell-1)}) \in \mathscr{P}_n^{\bar{\Lambda}}$. Suppose u, v are two standard μ-tableaux, define $\bar{\mathsf{u}}$ and $\bar{\mathsf{v}}$ to be standard $\bar{\mu}$-tableaux obtained by removing the $\emptyset$ at the end of u and v respectively. Write $k = \kappa_\ell$ for convenience. If $\mathbf{i} = (i_1, \ldots, i_n) \in I^n$, define

$$y_{\mathbf{i},k} = y_1^{\delta_{i_1,k}} y_2^{\delta_{i_2,k}} \ldots y_n^{\delta_{i_n,k}}.$$

2.3.13. Lemma. *Suppose the notations are defined as above and $\mathbf{i}_\mathsf{v}$ is the residue sequence of v, then*

$$\psi_{\mathsf{uv}} = \psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}} y_{\mathbf{i}_\mathsf{v},k},$$

where ψ_{uv} is an element in $\mathscr{R}_n^\Lambda$ and $\psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}}$ is an element in $\mathscr{R}_n^{\bar{\Lambda}}$.

Proof. Without loss of generality, assume $\mathsf{u} = \mathfrak{t}^\mu$. By the definition of μ and $\bar{\mu}$, writing $\mathbf{i}_\mu = (i_1, i_2, \ldots, i_n)$, we have $y_\mu = y_{\bar{\mu}} y_1^{\delta_{i_1,k}} y_2^{\delta_{i_2,k}} \ldots y_n^{\delta_{i_n,k}} = y_{\bar{\mu}} y_{\mathbf{i}_\mu,k}$.

Now for any residue sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ and any r, If $i_r \neq i_{r+1}$

$$\begin{aligned} e(\mathbf{i}) y_1^{\delta_{i_1,k}} y_2^{\delta_{i_2,k}} \ldots y_n^{\delta_{i_n,k}} \psi_r &= (e(\mathbf{i}) y_r^{\delta_{i_r,k}} y_{r+1}^{\delta_{i_{r+1},k}} \psi_r) y_1^{\delta_{i_1,k}} \ldots y_{r-1}^{\delta_{i_{r-1},k}} y_{r+2}^{\delta_{i_{r+1},k}} \ldots y_n^{\delta_{i_n,k}} \\ &= e(\mathbf{i}) \psi_r y_r^{\delta_{i_{r+1},k}} y_{r+1}^{\delta_{i_r,k}} y_1^{\delta_{i_1,k}} \ldots y_{r-1}^{\delta_{i_{r-1},k}} y_{r+2}^{\delta_{i_{r+1},k}} \ldots y_n^{\delta_{i_n,k}} \\ &= e(\mathbf{i}) \psi_r y_1^{\delta_{s_r(i_1),k}} \ldots y_n^{\delta_{s_r(i_n),k}} = e(\mathbf{i}) \psi_r y_{\mathbf{i} \cdot s_r,k}. \end{aligned}$$

If $i_r = i_{r+1}$, then by relation (1.1.13), as $\delta_{i_r,k} = \delta_{i_{r+1},k}$, we have the same result.

Hence

$$\begin{aligned} \psi_{\mathfrak{t}^\mu \mathsf{v}} &= e_\mu y_\mu \psi_{d(\mathsf{v})} = e(\mathbf{i}_\mu) y_{\bar{\mu}} y_{\mathbf{i}_\mu,k} \psi_{d(\mathsf{v})} \\ &= e(\mathbf{i}_\mu) y_{\bar{\mu}} \psi_{d(\mathsf{v})} y_{\mathbf{i}_\mu \cdot d(\mathsf{v}),k} = e_\mu y_{\bar{\mu}} \psi_{d(v)} y_{\mathbf{i}_\mathsf{v},k}. \end{aligned}$$

As $e_\mu = e_{\bar{\mu}}$ and $\psi_{d(\mathsf{v})} = \psi_{d(\bar{\mathsf{v}})}$, this completes the proof. □

2.3.14. Proposition. *Suppose μ, κ_Λ, $\bar{\mu}$, and $\kappa_{\bar{\Lambda}}$ are defined as above. Then $\bar{\mu} \in \mathscr{P}_I^{\bar{\Lambda}} \cap \mathscr{P}_y^{\bar{\Lambda}} \cap \mathscr{P}_\psi^{\bar{\Lambda}}$ implies $\mu \in \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$.*

Proof. We are only going to prove that $\bar{\mu} \in \mathscr{P}_I^{\bar{\Lambda}}$ implies $\mu \in \mathscr{P}_I^\Lambda$. The other two cases are similar.

Suppose $\bar{\mu} \in \mathscr{P}_I^{\bar{\Lambda}}$. Then for any $s \in I$, by the definition of $\mathscr{P}_I^{\bar{\Lambda}}$,

$$e(\mathbf{i}_{\bar{\mu}_-} \vee s) y_{\bar{\mu}_-} y_n^{b_s^{\bar{\mu}_-}} = \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}},$$

where $\mathbf{i}_{\bar{\mathsf{v}}} = \mathbf{i}_{\bar{\mu}_-} \vee s = \mathbf{i}_{\mu_-} \vee s$ and $c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \in \mathbb{Z}$.

Also we have $e(\mathbf{i}_{\bar{\mu}_-} \vee s) y_{\bar{\mu}_-} y_n^{b_s^{\bar{\mu}_-}} = \theta_s(\psi_{\mathfrak{t}^{\bar{\mu}_-}\mathfrak{t}^{\bar{\mu}_-}}) y_n^{b_s^{\bar{\mu}_-}}$. Therefore we have

$$\theta_s(\psi_{\mathfrak{t}^{\bar{\mu}_-}\mathfrak{t}^{\bar{\mu}_-}}) y_n^{b_s^{\bar{\mu}_-}} = \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}}.$$

Notice that $\mathfrak{t}^{\bar{\mu}_-} = \overline{\mathfrak{t}^{\mu_-}}$. Recall $k = \kappa_\ell$, the last term of the multicharge κ_Λ. We consider two cases, $s \neq k$ and $s = k$ in the rest of the proof.

If $s \neq k$, then $b_s^{\mu_-} = b_s^{\bar{\mu}_-}$. Hence by Lemma 2.3.13

$$\begin{aligned} e(\mathbf{i}_{\mu_-} \vee s) y_{\mu_-} y_n^{b_s^{\mu_-}} &= \theta_s(\psi_{\mathfrak{t}^{\mu_-}\mathfrak{t}^{\mu_-}}) y_n^{b_s^{\mu_-}} \\ &= \theta_s(\psi_{\bar{\mathfrak{t}}^{\mu_-}\bar{\mathfrak{t}}^{\mu_-}} y_{\mathbf{i}_{\mu_-},k}) y_n^{b_s^{\bar{\mu}_-}} \\ &= \theta_s(\psi_{\mathfrak{t}^{\bar{\mu}_-}\mathfrak{t}^{\bar{\mu}_-}}) y_n^{b_s^{\bar{\mu}_-}} y_{\mathbf{i}_{\mu_-},k} \\ &= \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}} y_{\mathbf{i}_{\mu_-},k}, \end{aligned}$$

and as $s \neq k$, $\delta_{s,k} = 0$. Hence $y_{\mathbf{i}_{\mu_-},k} = y_{\mathbf{i}_{\mu_-}\vee s,k} = y_{\mathbf{i}_{\bar{\mathsf{v}}},k} = y_{\mathbf{i}_{\mathsf{v}},k}$. By Lemma 2.3.13,

$$e(\mathbf{i}_{\mu_-} \vee s) y_{\mu_-} y_n^{b_s^{\mu_-}} = \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}} y_{\mathbf{i}_{\mathsf{v}},k} = \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\mathsf{u}\mathsf{v}} \in R_n^{>\lambda},$$

because $\bar{\mathsf{u}}, \bar{\mathsf{v}} \in \mathrm{Std}(> \bar{\mu})$ implies $\mathsf{u}, \mathsf{v}\,\mathrm{Std}(> \mu)$.

If $s = k$, then $b_s^{\mu_-} = b_s^{\bar{\mu}_-} + 1$. Hence by Lemma 2.3.13

$$\begin{aligned} e(\mathbf{i}_{\mu_-} \vee s) y_{\mu_-} y_n^{b_s^{\mu_-}} &= \theta_s(\psi_{\mathfrak{t}^{\mu_-}\mathfrak{t}^{\mu_-}}) y_n^{b_s^{\mu_-}} \\ &= \theta_s(\psi_{\bar{\mathfrak{t}}^{\mu_-}\bar{\mathfrak{t}}^{\mu_-}} y_{\mathbf{i}_{\mu_-},k}) y_n^{b_s^{\bar{\mu}_-}} y_n \\ &= \theta_s(\psi_{\mathfrak{t}^{\bar{\mu}_-}\mathfrak{t}^{\bar{\mu}_-}}) y_n^{b_s^{\bar{\mu}_-}} y_{\mathbf{i}_{\mu_-},k} y_n \\ &= \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}} y_{\mathbf{i}_{\mu_-},k} y_n, \end{aligned}$$

and as $s = k$, $\delta_{s,k} = 1$. Hence $y_{\mathbf{i}_{\mu_-},k} y_n = y_{\mathbf{i}_{\mu_-}\vee s,k} = y_{\mathbf{i}_{\bar{\mathsf{v}}},k} = y_{\mathbf{i}_{\mathsf{v}},k}$. By Lemma 2.3.13

$$e(\mathbf{i}_{\mu_-} \vee s) y_{\mu_-} y_n^{b_s^{\mu_-}} = \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\bar{\mathsf{u}}\bar{\mathsf{v}}} y_{\mathbf{i}_{\mathsf{v}},k} = \sum_{\bar{\mathsf{u}},\bar{\mathsf{v}} \in \mathrm{Std}(>\bar{\mu})} c_{\bar{\mathsf{u}}\bar{\mathsf{v}}} \psi_{\mathsf{u}\mathsf{v}} \in R_n^{>\lambda}.$$

These implies that $\mu \in \mathscr{P}_I$. □

Now we are ready to prove Theorem 2.1.8.

Proof of Theorem 2.1.8. Write $\mu = \lambda_- = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell-1)}, \mu^{(\ell)})$. If $\mu^{(\ell)} \neq \emptyset$, by Proposition 2.3.10 and Proposition 2.3.11, we have $\lambda \in \mathscr{P}_I^{\Lambda}$.

If $\mu^{(\ell)} = \emptyset$, write $\lambda^{(\ell-1)} = (\lambda_1^{(\ell-1)}, \lambda_2^{(\ell-1)}, \ldots, \lambda_{k_{\ell-1}}^{(\ell-1)})$ and define $\gamma = (\lambda^{(1)}, \ldots, \lambda^{(\ell-2)}, \gamma^{(\ell-1)}, \emptyset) \in \mathscr{P}_n^{\Lambda}$ with $\gamma^{(\ell-1)} = (\lambda_1^{(\ell-1)}, \lambda_2^{(\ell-1)}, \ldots, \lambda_{k_{\ell-1}}^{(\ell-1)}, 1)$ and $\bar{\gamma} = (\lambda^{(1)}, \ldots, \lambda^{(\ell-2)}, \gamma^{(\ell-1)})$. As $l(\bar{\gamma}) < l(\lambda) = \ell$, by the definition of $\mathscr{S}_n^{\Lambda}$, $\bar{\gamma} \in \mathscr{P}_I^{\bar{\Lambda}}$. Then by Proposition 2.3.14 we have $\gamma \in \mathscr{P}_I^{\Lambda}$. Since $\gamma_- = \mu = \lambda_-$ and $\gamma > \lambda$, for any $k \in I$,

$$e(\mathbf{i}_{\lambda_-} \vee k) y_{\lambda_-} y_n^{b_k^{\lambda_-}} = e(\mathbf{i}_{\gamma_-} \vee k) y_{\gamma_-} y_n^{b_k^{\gamma_-}} \in R_n^{>\gamma} \subseteq R_n^{>\lambda},$$

which yields that $\lambda \in \mathscr{P}_I$. This completes the proof.

The following Corollary is directly implied by Theorem 2.1.8. It will contribute to proving the ψ-problem.

2.3.15. Corollary. *Suppose $\lambda \in \mathscr{S}_n^{\Lambda}$ and $\mu \in \mathscr{C}_n^{\Lambda}$ where $\mu > \lambda$. Then we have $e_\mu y_\mu \in R_n^{>\lambda}$.*

Proof. If $\mu_- \neq \lambda_-$, using Lemma 2.3.9 we have $e_\mu y_\mu \in R_n^{>\lambda}$. Suppose then that $\mu_- = \lambda_-$. If $\mu \in \mathscr{P}_n^{\Lambda}$, then $e_\mu y_\mu = \psi_{\mathfrak{t}^\mu\mathfrak{t}^\mu} \in R_n^{>\lambda}$. Finally, suppose that $\mu \notin \mathscr{P}_n^{\Lambda}$. If we write $\mu = (\mu^{(1)}, \ldots, \mu^{(l)}, \emptyset, \ldots, \emptyset)$ with $\mu^{(l)} = (\mu_1^{(l)}, \ldots, \mu_{k-1}^{(l)}, \mu_k^{(l)})$, we must have $\mu_{k-1}^{(l)} + 1 = \mu_k^{(l)}$. If we write $\mathbf{i}_\mu = (i_1, i_2, \dots, i_n)$, we have $e_\mu y_\mu = e(\mathbf{i}_{\mu_-} \vee i_n) y_{\mu_-} y_n^{b_{i_n}^{\mu_-}}$. By Theorem 2.1.8, as $\lambda \in \mathscr{S}_n^\Lambda$, we have $\lambda \in \mathscr{P}_I^\Lambda$. Since $\lambda_- = \mu_-$,

$$e_\mu y_\mu = e(\mathbf{i}_{\mu_-} \vee i_n) y_{\mu_-} y_n^{b_{i_n}^{\mu_-}} = e(\mathbf{i}_{\lambda_-} \vee i_n) y_{\lambda_-} y_n^{b_{i_n}^{\lambda_-}} \in R_n^{>\lambda}.$$

□

Chapter 3

Integral Basis Theorem II

In this chapter our main purpose is to prove that $\mathscr{R}_n^\Lambda = R_n^\Lambda$ by proving that $\psi_{\mathsf{st}} y_r$ and $\psi_{\mathsf{st}}\psi_r$ are both in R_n^Λ. We first define an integer m_λ such that if $\mathsf{t} \in \mathrm{Std}(\lambda)$ and $l(d(\mathsf{t})) < m_\lambda$, we have $\psi_{\mathsf{st}} y_r \in R_n^\Lambda$ and $\psi_{\mathsf{st}}\psi_r \in R_n^\Lambda$. Our first step is to show that $m_\lambda > 0$. Then we prove if $l(d(\mathsf{t})) \le m_\lambda$, we will have $\psi_{\mathsf{st}} y_r \in R_n^\Lambda$ and $\psi_{\mathsf{st}}\psi_r \in R_n^\Lambda$ as well. By induction we will show that for any $\mathsf{t} \in \mathrm{Std}(\lambda)$, $l(d(\mathsf{t})) < m_\lambda$, which indicates that $\psi_{\mathsf{st}} y_r \in R_n^\Lambda$ and $\psi_{\mathsf{st}}\psi_r \in R_n^\Lambda$ for any $\mathsf{s},\mathsf{t} \in \mathrm{Std}(\lambda)$. Finally combining the results from the last chapter, we can prove that $\mathscr{R}_n^\Lambda = R_n^\Lambda$.

3.1. Base case of induction

In this chapter we fix $\lambda \in \mathscr{S}_n^\Lambda$. First we will give a proper definition of m_λ.

3.1.1. Definition. *Define m_λ to be the smallest nonnegative integer such that for any standard λ-tableau t with $l(d(\mathsf{t})) < m_\lambda$ we have*

$$\psi_{\mathsf{st}} y_r = \sum_{(\mathsf{u},\mathsf{v}) \rhd (\mathsf{s},\mathsf{t})} c_{\mathsf{uv}} \psi_{\mathsf{uv}},$$

$$\psi_{\mathsf{st}}\psi_r = \begin{cases} \psi_{\mathsf{sw}} + \sum_{(\mathsf{u},\mathsf{v}) \rhd (\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{w} = \mathsf{t}\cdot s_r \text{ is standard and } d(\mathsf{u})\cdot s_r \text{ is reduced,} \\ \sum_{(\mathsf{u},\mathsf{v}) \rhd (\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{u}\cdot s_r \text{ is not standard or } d(\mathsf{u})\cdot s_r \text{ is not reduced.} \end{cases}$$

for some $c_{\mathsf{uv}} \in \mathbb{Z}$.

We will use induction to prove that for any $\mathsf{t} \in \mathrm{Std}(\lambda)$, $l(d(\mathsf{t})) < m_\lambda$ in this chapter. In this section we will prove that $m_\lambda > 0$, which is the base case of the induction.

3.1.2. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$. For any $1 \le r \le n$, $e_\lambda y_\lambda y_r = \psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda} y_r \in R_n^{>\lambda}$.*

Proof. If $r < n$, write $\mu = \lambda|_r$. As $\lambda \in \mathscr{S}_n^\Lambda$ we have $\mu \in \mathscr{P}_y^\Lambda$. Therefore $e_\mu y_\mu y_r \in R_n^{>\mu}$. By Lemma 2.1.14, we have $e_\lambda y_\lambda y_r \in R_n^{>\lambda}$.

If $r = n$, write $\mathbf{i}_\lambda = (i_1, i_2, \ldots, i_n)$. We can find a positive integer b such that $e_\lambda y_\lambda = e(\mathbf{i}_{\lambda_-} \vee i_n) y_{\lambda_-} y_n^b$. By the definition of $b_{i_n}^{\lambda_-}$ we have $b < b_{i_n}^{\lambda_-}$. If $b+1 < b_{i_n}^{\lambda_-}$, by Lemma 2.1.3 we can find $\nu \in \mathscr{P}_n^\Lambda$ such that

$$e_\lambda y_\lambda y_n = e(\mathbf{i}_{\lambda_-} \vee i_n) y_{\lambda_-} y_n^{b+1} = e_\nu y_\nu,$$

and it is trivial that $\nu > \lambda$. Therefore $e_\lambda y_\lambda y_n \in R_n^{>\lambda}$. If we have $b+1 = b_{i_n}^{\lambda_-}$, by Theorem 2.1.8 we have

$$e_\lambda y_\lambda y_n = e(\mathbf{i}_{\lambda_-} \vee i_n) y_{\lambda_-} y_n^{b+1} = e(\mathbf{i}_{\lambda_-} \vee i_n) y_{\lambda_-} y_n^{b_{i_n}^{\lambda_-}} \in R_n^{>\lambda},$$

which completes the proof. □

3.1.3. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$. For any $1 \le r < n$, $e_\lambda y_\lambda \psi_r = \psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda}\psi_r \in R_n^{\ge\lambda}$.*

Proof. Suppose $\mathsf{t}^\lambda \cdot s_r = \mathsf{t}$ is standard, we have $e_\lambda y_\lambda \psi_r = \psi_{\mathsf{t}^\lambda\mathsf{t}} \in R_n^{\ge\lambda}$. So we consider the case that $\mathsf{t}^\lambda \cdot s_r = \mathsf{t}$ is not standard. If $r < n-1$, as $\lambda \in \mathscr{S}_n^\Lambda$, we have $\mu = \lambda_- \in \mathscr{P}_\psi^\Lambda$. Because $\mathsf{t}^\mu \cdot s_r = \mathsf{t}|_{n-1}$ which is not standard, $e_\mu y_\mu \psi_r \in R_n^{>\mu}$. Then by Lemma 2.1.14, we have $e_\lambda y_\lambda \psi_r \in R_n^{>\lambda}$. If $r = n-1$

and write $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(l)}, \emptyset, \ldots, \emptyset)$ with $\lambda^{(l)} = (\lambda^{(l)}_1, \ldots, \lambda^{(l)}_{k-1}, \lambda^{(l)}_k)$, we must have either $\lambda^{(l)}_k \geq 2$ or $\lambda^{(l)}_{k-1} = \lambda^{(l)}_k = 1$. Then set $\mu = (\lambda^{(1)}, \ldots, \mu^{(l)}, \emptyset, \ldots, \emptyset)$ with

$$\mu^{(l)} = \begin{cases} (\lambda^{(l)}_1, \ldots, \lambda^{(l)}_{k-1}), & \text{if } \lambda^{(l)}_k \geq 2 \text{ and } k > 1, \\ (\lambda^{(l)}_1, \ldots, \lambda^{(l)}_{k-2}), & \text{if } \lambda^{(l)}_{k-1} = \lambda^{(l)}_k = 1 \text{ and } k > 2, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Suppose i is the residue of node $(k, 1, l)$ in λ or the residue of node $(k-1, 1, l)$ in λ, $\Lambda' = \Lambda_i$, $m = \lambda^{(l)}_k$ or $m = 2$ and $\gamma = (m) \in \mathscr{P}^{\Lambda'}_m$ or $\gamma = (1, 1) \in \mathscr{P}^{\Lambda'}_m$ if $\lambda^{(l)}_k \geq 2$ or $\lambda^{(l)}_{k-1} = \lambda^{(l)}_k = 1$ respectively. Therefore $\lambda = \mu \vee \gamma$. Because $\lambda \in \mathscr{S}^{\Lambda}_n$, we have $\gamma \in \mathscr{P}^{\Lambda}_{\psi}$. Hence because $\mathsf{t}^{\gamma} \cdot s_{m-1}$ is not standard, we have $e_{\gamma} y_{\gamma} \psi_{m-1} \in R^{>\gamma}_n = N^{\Lambda'}_m$. Then by Lemma 2.3.4,

$$e_{\lambda} y_{\lambda} \psi_r = e(\mathbf{i}_{\mu} \vee \mathbf{i}_{\gamma}) y_{\lambda} \psi_r = \hat{\theta}_{\mathbf{i}_{\mu}}(e_{\gamma} y_{\gamma} \psi_{m-1}) y_{\mu} \in R^{>\lambda}_n,$$

which completes the proof. □

3.1.4. Corollary. *For $\lambda \in \mathscr{S}^{\Lambda}_n$, we have $m_{\lambda} > 0$.*

Proof. Combining the above two Lemmas, Lemma 1.4.4 and Proposition 1.4.9, the Corollary follows. □

3.2. Completion of the y-problem

In this section we are going to prove that for any $\mathsf{t} \in \mathrm{Std}(\lambda)$, if $l(d(\mathsf{t})) \leq m_{\lambda}$, then for any $1 \leq r \leq n-1$ and any $\mathsf{s} \in \mathrm{Std}(\lambda)$, if $\mathsf{t} \cdot s_r$ is standard and $d(\mathsf{t}) \cdot s_r$ is reduced, $\psi_{\mathsf{st}} \psi_r \in R^{\geq\lambda}_n$ and for any $1 \leq r \leq n$, $\psi_{\mathsf{st}} y_r \in R^{\geq\lambda}_n$.

First we introduce the following Lemma.

3.2.1. Lemma. *Suppose m is a positive integer such that $m \leq m_{\lambda}$, then*

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \ldots \psi_{r_m} =_{\lambda} \sum_{\substack{\mathsf{v} \in Std(\lambda) \\ l(d(\mathsf{v})) \leq m}} c_{\mathsf{t}^{\lambda}\mathsf{v}} \psi_{t^{\lambda}\mathsf{v}}.$$

Proof. We apply induction on m. Suppose $m = 0$ then there is nothing to prove. Assume for any $m' < m$ the Lemma holds. Therefore $e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \ldots \psi_{r_{m-1}} =_{\lambda} \sum_{\substack{\mathsf{u} \in \mathrm{Std}(\lambda) \\ l(d(\mathsf{u})) \leq m-1}} c_{\mathsf{t}^{\lambda}\mathsf{u}} \psi_{t^{\lambda}\mathsf{u}}$ which yields

$$e_{\lambda} y_{\lambda} \psi_{r_1} \psi_{r_2} \ldots \psi_{r_{m-1}} \psi_{r_m} =_{\lambda} \sum_{\substack{\mathsf{u} \in \mathrm{Std}(\lambda) \\ l(d(\mathsf{u})) \leq m-1}} c_{\mathsf{t}^{\lambda}\mathsf{u}} \psi_{t^{\lambda}\mathsf{u}} \psi_{r_m}.$$

For $\mathsf{u} \in \mathrm{Std}(\lambda)$ and $l(d(\mathsf{u})) \leq m-1 < m_{\lambda}$, if $\mathsf{u} \cdot s_r$ is standard and $s_{d(\mathsf{v})} \cdot s_{r_m}$ is reduced, by the definition of m_{λ},

$$\psi_{\mathsf{t}^{\lambda}\mathsf{u}} \psi_{r_m} = \psi_{\mathsf{t}^{\lambda}, \mathsf{u} \cdot s_r} + \sum_{(\mathsf{x},\mathsf{y}) \rhd (\mathsf{t}^{\lambda}, \mathsf{u})} c_{\mathsf{xy}} \psi_{\mathsf{xy}} =_{\lambda} \psi_{\mathsf{t}^{\lambda}, \mathsf{u} \cdot s_r} + \sum_{\mathsf{v} \rhd \mathsf{u}} c_{\mathsf{t}^{\lambda}\mathsf{v}} \psi_{\mathsf{t}^{\lambda}\mathsf{v}},$$

where $l(d(\mathsf{u} \cdot s_r)) = 1 + l(d(\mathsf{u})) \leq m$ and $l(d(\mathsf{v})) < l(d(\mathsf{u})) < m$ as $\mathsf{v} \rhd \mathsf{u}$. Hence

$$\psi_{\mathsf{t}^{\lambda}\mathsf{u}} \psi_{r_m} =_{\lambda} \sum_{\substack{\mathsf{v} \in \mathrm{Std}(\lambda) \\ l(d(\mathsf{v})) \leq m}} c_{\mathsf{v}} \psi_{t^{\lambda}\mathsf{v}}.$$

If $\mathsf{u} \cdot s_r$ is not standard or $s_{d(\mathsf{v})} \cdot s_{r_m}$ is not reduced, we have

$$\psi_{\mathsf{t}^{\lambda}\mathsf{u}} \psi_{r_m} = \sum_{(\mathsf{x},\mathsf{y}) \rhd (\mathsf{t}^{\lambda}, \mathsf{u})} c_{\mathsf{xy}} \psi_{\mathsf{xy}} =_{\lambda} \sum_{\mathsf{v} \rhd \mathsf{u}} c_{\mathsf{t}^{\lambda}\mathsf{v}} \psi_{\mathsf{t}^{\lambda}\mathsf{v}},$$

where $l(d(\mathsf{v})) < l(d(\mathsf{u})) \leq m-1 < m$ as $\mathsf{v} \rhd \mathsf{u}$. Hence

$$\psi_{\mathsf{t}^\lambda\mathsf{u}}\psi_{r_m} =_\lambda \sum_{\substack{\mathsf{v}\in \mathrm{Std}(\lambda)\\ l(d(\mathsf{v}))\leq m}} c_{\mathsf{t}^\lambda\mathsf{v}}\psi_{t^\lambda\mathsf{v}},$$

which completes the proof. □

Now we can start to prove that $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$ when $\mathsf{t}\cdot s_r$ is standard and $d(\mathsf{t})\cdot s_r$ is reduced.

3.2.2. Lemma. *Suppose* t *is a standard λ-tableau with* $d(\mathsf{t}) = s_{r_1}s_{r_2}\ldots s_{r_l}$ *where* $l \leq m_\lambda$, *and* $d'(t) = s_{t_1}s_{t_2}\ldots s_{t_l}$ *is another reduced decomposition of* $d(\mathsf{t})$, *then*

$$e_\lambda y_\lambda \psi_{d(\mathsf{t})} - e_\lambda y_\lambda \psi_{d'(\mathsf{t})} = \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{t}^\lambda\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}.$$

Proof. By [**5**, Proposition 2.5], we have

$$y_\lambda e_\lambda \psi_{d(\mathsf{t})} - y_\lambda e_\lambda \psi_{d'(\mathsf{t})} = \sum_{u<d(\mathsf{t})} c_{u,f}\, y_\lambda e_\lambda f(y)\psi_u,$$

where $f(y)$ is a polynomial in y_r's and u is a word in $\mathfrak{S}_n$. If $f(y) \neq 1$, by Lemma 3.1.2 we have $e_\lambda y_\lambda f(y) \in R_n^{>\lambda}$. Hence $y_\lambda e_\lambda f(y)\psi_u \in R_n^{>\lambda}$. If $f(y) = 1$, as $u < d(\mathsf{t})$ then $l(u) < l \leq m_\lambda$, by Lemma 3.2.1 we have $e_\lambda y_\lambda \psi_u \in R_n^{\geq\lambda}$. Henceforth

$$y_\lambda e_\lambda \psi_w - y_\lambda e_\lambda \psi_{w'} \in R_n^{\geq\lambda}.$$

Then by Proposition 1.4.9 and [**9**, Lemma 5.7], we complete the proof. □

The following Corollary is straightforward by Lemma 3.2.2 which explains the action of ψ_r to ψ_{st} from right when $\mathsf{t}\cdot s_r$ is standard and $d(\mathsf{t})\cdot s_r$ is reduced.

3.2.3. Corollary. *Suppose* t *is a standard λ-tableau with* $l(d(\mathsf{t})) \leq m_\lambda$, *if* $\mathsf{w} = \mathsf{t}\cdot s_r$ *is standard and* $d(\mathsf{t})\cdot s_r$ *is reduced,*

$$\psi_{\mathsf{st}}\psi_r = \psi_{\mathsf{sw}} + \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}.$$

Now we start to prove that $\psi_{\mathsf{st}}y_r \in R_n^{\geq\lambda}$.

3.2.4. Lemma. *Suppose* t *is a standard λ-tableau with* $l(d(\mathsf{t})) < m_\lambda$. *For any* $1 \leq k \leq n$, $1 \leq r \leq n-1$ *and any standard λ-tableau* s, *we have*

$$\psi_{\mathsf{st}}y_k\psi_r \in R_n^{\geq\lambda}.$$

Proof. As $l(d(\mathsf{t})) < m_\lambda$, we have

$$\psi_{\mathsf{st}}y_k = \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}} = \sum_{\mathsf{v}\rhd\mathsf{t}} c_{\mathsf{sv}}\psi_{\mathsf{sv}} + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\lambda)} c_{\mathsf{uv}}\psi_{\mathsf{uv}}.$$

For Shape$(\mathsf{v}) = \lambda$, since $\mathsf{v} \rhd \mathsf{t}$, $l(d(\mathsf{v})) < l(d(\mathsf{t})) < m_\lambda$. Then we have $\psi_{\mathsf{sv}}\psi_r \in R_n^{\geq\lambda}$.

For $\mathsf{u},\mathsf{v} \in \mathrm{Std}(>\lambda)$, $\psi_{\mathsf{uv}} \in R_n^{>\lambda}$. As $\lambda \in \mathscr{P}_n^\Lambda$, $R_n^{>\lambda}$ is an ideal by Lemma 2.1.12. Hence $\psi_{\mathsf{uv}}\psi_{r_l} \in R_n^{>\lambda}$ and this completes the proof. □

3.2.5. Proposition. *Suppose* t *is a standard λ-tableau with* $l(d(\mathsf{t})) \leq m_\lambda$, *for any* $1 \leq r \leq n$ *and any standard λ-tableau* s, *we have*

$$\psi_{\mathsf{st}}y_r = \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}.$$

Proof. Write $d(\mathsf{t}) = s_{r_1}s_{r_2}\dots s_{r_{l-1}}s_{r_l}$ and $\mathsf{w} = \mathsf{t}\cdot s_{r_l} = s_{r_1}s_{r_2}\dots s_{r_{l-1}}$. We prove this Proposition by considering different values of r.

If $r \notin \{r_l, r_l+1\}$, then ψ_{r_l} and y_r commute. Hence

$$\psi_{\mathsf{st}}y_r = \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{t})}y_r = \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{w})}y_r\psi_{r_l} = \psi_{\mathsf{sw}}y_r\psi_{r_l}.$$

As $l(d(\mathsf{w})) = l(d(\mathsf{t})) - 1 < m_\lambda$, by Lemma 3.2.4 we have $\psi_{\mathsf{st}}y_r \in R_n^{\geq\lambda}$.

If $r = r_l$, let $\mathbf{j}$ be a sequence such that $e(\mathbf{i}_\lambda)\psi_{d(\mathsf{t})} = \psi_{d(\mathsf{w})}e(\mathbf{j})\psi_{r_l}$. We separate this case further into $j_{r_l} \neq j_{r_l+1}$ and $j_{r_l} = j_{r_l+1}$. First suppose $j_{r_l} \neq j_{r_l+1}$, then

$$\psi_{\mathsf{st}}y_r = \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{t})}y_r = \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{w})}y_{r+1}\psi_{r_l} = \psi_{\mathsf{sw}}y_{r+1}\psi_{r_l}.$$

Hence as $l(d(\mathsf{w})) < m_\lambda$, by Lemma 3.2.4 we have $\psi_{\mathsf{st}}y_r \in R_n^{\geq\lambda}$ when $j_{r_l} \neq j_{r_l+1}$. Now suppose $j_{r_l} = j_{r_l+1}$, we have

$$\psi_{\mathsf{st}}y_r = \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{t})}\mathsf{y}_r = \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{w})} + \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{w})}y_{r+1}\psi_{r_l} = \psi_{\mathsf{sw}} + \psi_{\mathsf{sw}}y_{r+1}\psi_{r_l}.$$

As $l(d(\mathsf{w})) < m_\lambda$, by Lemma 3.2.4 we have $\psi_{\mathsf{sw}}y_{r+1}\psi_{r_l} \in R_n^{\geq\lambda}$. As $\psi_{\mathsf{sw}} \in R_n^{\geq\lambda}$ as well, we have $\psi_{\mathsf{st}}y_r \in R_n^{\geq\lambda}$. So for $r = r_l$, we have $\psi_{\mathsf{st}}y_r \in R_n^{\geq\lambda}$.

If $r = r_l + 1$, the method is the same as $r = r_l$.

Therefore in all the cases, we have $\psi_{\mathsf{st}}y_r \in R_n^{\geq\lambda}$. So

$$\psi_{\mathsf{st}}y_r = \sum_{\mathsf{u},\mathsf{v}\in \mathrm{Std}(\geq\lambda)} c_{\mathsf{uv}}\psi_{\mathsf{uv}},$$

and by Proposition 1.4.9 we complete the proof. □

3.3. Properties of m_λ

In the rest of this chapter we will prove that if $\mathsf{t} \in \mathrm{Std}(\lambda)$ and $l(d(\mathsf{t})) \leq m_\lambda$, then for any $1 \leq r \leq n-1$ and any $\mathsf{s} \in \mathrm{Std}(\lambda)$, we have $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. In this section we will give some properties for m_λ which will be used in proving the above argument.

3.3.1. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$. For any permutation $w \in \mathfrak{S}_n$ with reduced expression $w = s_{r_1}s_{r_2}\dots s_{r_{m-1}}s_{r_m}$ and $r = min\{r_1, r_2, \dots, r_m\}$, if we write*

$$e_\lambda y_\lambda \psi_w = e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = \sum_{\mathsf{u},\mathsf{v}\in \mathrm{Std}(\geq\lambda)} c_{\mathsf{uv}}\psi_{\mathsf{uv}},$$

then $c_{\mathsf{uv}} \neq 0$ implies $\mathsf{v}|_k \trianglerighteq \mathsf{t}^{\lambda|_k}$ for any $k < r$.

Proof. We prove the Lemma by induction. If $m = 1$, then $r_1 = r$.

$$e_\lambda y_\lambda \psi_r = \begin{cases} \psi_{\mathsf{t}^\lambda\mathsf{v}}, & \text{if } \mathsf{v} = \mathsf{t}^\lambda\cdot s_r \text{ is standard,} \\ \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{t}^\lambda,\mathsf{t}^\lambda)} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{otherwise.} \end{cases}$$

If $\mathsf{v} = \mathsf{t}^\lambda\cdot s_r$ is standard, then by the definition of v, $\mathsf{v}|_k = \mathsf{t}^\lambda|_k = \mathsf{t}^{\lambda|_k}$ for $k < r$. If it is the other case, as $\mathsf{v} \rhd \mathsf{t}^\lambda$, then $\mathsf{v}|_k \trianglerighteq \mathsf{t}^\lambda|_k = \mathsf{t}^{\lambda|_k}$.

Assume for any $m' < m$ the Corollary holds. Then

$$e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_{m-1}} = \sum_{\mathsf{u}_1,\mathsf{v}_1\in \mathrm{Std}(\geq\lambda)} c_{\mathsf{u}_1\mathsf{v}_1}\psi_{\mathsf{u}_1\mathsf{v}_1},$$

where $\mathsf{v}_1|_k \trianglerighteq \mathsf{t}^{\lambda|_k}$ for $k < r$. Then

$$e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_{m-1}}\psi_{r_m} = \sum_{\mathsf{u}_1,\mathsf{v}_1\in \mathrm{Std}(\geq\lambda)} c_{\mathsf{u}_1\mathsf{v}_1}\psi_{\mathsf{u}_1\mathsf{v}_1}\psi_{r_m}.$$

Since

$$\psi_{\mathsf{u}_1\mathsf{v}_1}\psi_{r_m} = \begin{cases}\psi_{\mathsf{u}_1\mathsf{v}} + \sum_{(\mathsf{u},\mathsf{v}_2)\triangleright(\mathsf{u}_1,\mathsf{v}_1)} c_{\mathsf{u}\mathsf{v}_2}\psi_{\mathsf{u}\mathsf{v}_2}, & \text{if } \mathsf{v} = \mathsf{v}_1\cdot s_{r_m} \text{ is standard and } d(\mathsf{v}_1)\cdot s_{r_m} \text{ is reduced,}\\ \sum_{(\mathsf{u},\mathsf{v})\triangleright(\mathsf{u}_1,\mathsf{v}_1)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}}, & \text{otherwise.}\end{cases}$$

If $\mathsf{v} = \mathsf{v}_1\cdot s_{r_m}$ is standard and $d(\mathsf{v}_1)\cdot s_{r_m}$ is reduced, recall $\mathsf{v}_1|_k \trianglerighteq \mathsf{t}^{\lambda|_k}$ for $k < r$, as $\mathsf{v} = \mathsf{v}_1\cdot s_{r_m}$, $\mathsf{v}|_k = \mathsf{v}_1|_k \trianglerighteq \mathsf{t}^{\lambda|_k}$ for $k < r \leq r_m$. For $\mathsf{v}_2 \triangleright \mathsf{v}_1$, we have $\mathsf{v}_2|_k \triangleright \mathsf{v}_1|_k \trianglerighteq \mathsf{t}^{\lambda|_k}$ for $k < r$.

If it is of the other case, as $\mathsf{v} \triangleright \mathsf{v}_1$, $\mathsf{v}|_k \triangleright \mathsf{v}_1|_k \trianglerighteq \mathsf{t}^{\lambda|_k}$. Therefore

$$e_\lambda y_\lambda \psi_w = e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = \sum_{\mathsf{u},\mathsf{v}\in \mathrm{Std}(\geq\lambda)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}},$$

and $c_{\mathsf{u}\mathsf{v}} \neq 0$ implies $\mathsf{v}|_k \trianglerighteq \mathsf{t}^{\lambda|_k}$ for any $k < r$. This completes the proof. □

3.3.2. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda \cap (\mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda)$. Then for any $1 \leq r_1, r_2, \dots, r_m \leq n-1$*

$$e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = \sum_{\mathsf{v}\in \mathrm{Std}(\lambda)} c_{\mathsf{t}^\lambda\mathsf{v}}\psi_{\mathsf{t}^\lambda\mathsf{v}} + \sum_{\substack{\mathsf{u},\mathsf{v}\in \mathrm{Std}(>\lambda)\\ \mathsf{u}\triangleright\mathsf{t}^\lambda}} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}}.$$

Proof. When $m = 1$, we have

$$e_\lambda y_\lambda \psi_{r_1} = \begin{cases}\psi_{\mathsf{t}^\lambda\mathsf{v}}, & \text{if } \mathsf{v} = \mathsf{t}^\lambda\cdot s_{r_1} \text{ is standard,}\\ \sum_{(\mathsf{u},\mathsf{v})\triangleright(\mathsf{t}^\lambda,\mathsf{t}^\lambda)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}} = \sum_{\substack{\mathsf{u},\mathsf{v}\in \mathrm{Std}(>\lambda)\\ \mathsf{u}\triangleright\mathsf{t}^\lambda}} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}}, & \text{if } \mathsf{v} = \mathsf{t}^\lambda\cdot s_{r_1} \text{ is not standard.}\end{cases}$$

which follows the Lemma.

Suppose for $m' < m$ the Lemma holds. Then by induction

$$e_\lambda y_\lambda \psi_{r_1}\dots\psi_{r_{m-1}}\psi_{r_m} = \sum_{\mathsf{v}_1\in \mathrm{Std}(\lambda)} c_{\mathsf{t}^\lambda\mathsf{v}_1}\psi_{\mathsf{t}^\lambda\mathsf{v}_1}\psi_{r_m} + \sum_{\substack{\mathsf{u}_1,\mathsf{v}_1\in \mathrm{Std}(>\lambda)\\ \mathsf{u}_1\triangleright\mathsf{t}^\lambda}} c_{\mathsf{u}_1\mathsf{v}_1}\psi_{\mathsf{u}_1\mathsf{v}_1}\psi_{r_m}. \tag{3.3.3}$$

For $\mathsf{v}_1 \in \mathrm{Std}(\lambda)$, as $\lambda \in \mathscr{P}_\psi^\Lambda$,

$$\psi_{\mathsf{t}^\lambda\mathsf{v}_1}\psi_{r_m} = \begin{cases}\psi_{\mathsf{t}^\lambda\mathsf{v}_2} + \sum_{(\mathsf{u}_2,\mathsf{v}_2)\triangleright(\mathsf{t}^\lambda,\mathsf{v}_1)} c_{\mathsf{u}_2\mathsf{v}_2}\psi_{\mathsf{u}_2\mathsf{v}_2}, & \text{if } \mathsf{v}_2 = \mathsf{v}_1\cdot s_{r_m} \text{ is standard}\\ & \quad\text{and } d(\mathsf{v}_2) = d(\mathsf{v}_1)\cdot s_{r_m} \text{ is reduced,}\\ \sum_{(\mathsf{u}_2,\mathsf{v}_2)\triangleright(\mathsf{t}^\lambda,\mathsf{v}_1)} c_{\mathsf{u}_2\mathsf{v}_2}\psi_{\mathsf{u}_2\mathsf{v}_2}, & \text{if } \mathsf{v}_2 = \mathsf{v}_1\cdot s_{r_m} \text{ is not standard}\\ & \quad\text{or } d(\mathsf{v}_2) = d(\mathsf{v}_1)\cdot s_{r_m} \text{ is not reduced.}\end{cases}$$

where in both cases, we can write

$$\psi_{\mathsf{t}^\lambda\mathsf{v}_1}\psi_{r_m} = \sum_{\mathsf{v}_2\in \mathrm{Std}(\lambda)} c_{\mathsf{t}^\lambda\mathsf{v}_2}\psi_{\mathsf{t}^\lambda\mathsf{v}} + \sum_{\substack{\mathsf{u}_2,\mathsf{v}_2\in \mathrm{Std}(>\lambda)\\ \mathsf{u}_2\triangleright\mathsf{t}^\lambda}} c_{\mathsf{u}_2\mathsf{v}_2}\psi_{\mathsf{u}_2\mathsf{v}_2}. \tag{3.3.4}$$

For $\mathsf{u}_1, \mathsf{u}_2 \in \mathrm{Std}(>\lambda)$,

$$\psi_{\mathsf{u}_1\mathsf{v}_1}\psi_{r_m} = \begin{cases}\psi_{\mathsf{u}_1\mathsf{v}_2} + \sum_{(\mathsf{u}_2,\mathsf{v}_2)\triangleright(\mathsf{u}_1,\mathsf{v}_1)} c_{\mathsf{u}_2\mathsf{v}_2}\psi_{\mathsf{u}_2\mathsf{v}_2}, & \text{if } \mathsf{v}_2 = \mathsf{v}_1\cdot s_{r_m} \text{ is standard}\\ & \quad\text{and } d(\mathsf{v}_2) = d(\mathsf{v}_1)\cdot s_{r_m} \text{ is reduced,}\\ \sum_{(\mathsf{u}_2,\mathsf{v}_2)\triangleright(\mathsf{u}_1,\mathsf{v}_1)} c_{\mathsf{u}_2\mathsf{v}_2}\psi_{\mathsf{u}_2\mathsf{v}_2}, & \text{if } \mathsf{v}_2 = \mathsf{v}_1\cdot s_{r_m} \text{ is not standard}\\ & \quad\text{or } d(\mathsf{v}_2) = d(\mathsf{v}_1)\cdot s_{r_m} \text{ is not reduced.}\end{cases}$$

where since $\mathsf{u}_1 \rhd \mathsf{t}^\lambda$, we can always write

$$\psi_{\mathsf{u}_1\mathsf{v}_1}\psi_{r_m} = \sum_{\substack{\mathsf{u}_2,\mathsf{v}_2\in \mathrm{Std}(>\lambda)\\ \mathsf{u}_2\rhd \mathsf{t}^\lambda}} c_{\mathsf{u}_2\mathsf{v}_2}\psi_{\mathsf{u}_2\mathsf{v}_2}. \tag{3.3.5}$$

Therefore, substitute (3.3.4) and (3.3.5) back to (3.3.3), we have

$$e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = \sum_{\mathsf{v}\in \mathrm{Std}(\lambda)} c_{\mathsf{t}^\lambda\mathsf{v}}\psi_{\mathsf{t}^\lambda\mathsf{v}} + \sum_{\substack{\mathsf{u},\mathsf{v}\in \mathrm{Std}(>\lambda)\\ \mathsf{u}\rhd \mathsf{t}^\lambda}} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}},$$

which completes the proof. □

3.3.6. Lemma. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$ and $r_1, r_2, \dots, r_m$ are positive integers such that $r_1, \dots, r_m < n-1$. Then*

$$e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} \in R_n^{\geq\lambda}.$$

Proof. Define $\mu = \lambda|_{n-1}$. As $\lambda \in \mathscr{S}_n^\Lambda$, $\mu \in \mathscr{S}_{n-1}^\Lambda \cap (\mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda)$. Define $i \in I$ such that $\mathbf{i}_\lambda = \mathbf{i}_\mu \vee i$. As $r_1, r_2, \dots, r_m < n-1$, we have

$$e_\lambda y_\lambda \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = \theta_i(e_\mu y_\mu \psi_{r_1}\psi_{r_2}\dots\psi_{r_m}),$$

where

$$e_\mu y_\mu \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = \sum_{\dot{\mathsf{v}}\in \mathrm{Std}(\mu)} c_{\mathsf{t}^\mu\dot{\mathsf{v}}}\psi_{\mathsf{t}^\mu\dot{\mathsf{v}}} + \sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in \mathrm{Std}(>\mu)} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}.$$

As $\sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in \mathrm{Std}(>\mu)} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}} \in R_n^{>\mu} = R_n^{>\lambda|_{n-1}}$, by Lemma 2.1.14, $\theta_i(\sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in \mathrm{Std}(>\mu)} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{>\lambda}$.

For $\dot{\mathsf{v}} \in \mathrm{Std}(\mu) = \mathrm{Std}(\lambda|_{n-1})$ and $\mathbf{i}_\mu \vee i = \mathbf{i}_\lambda$, define v to be the standard λ-tableau with $\mathsf{v}|_{n-1} = \dot{\mathsf{v}}$. Hence $\theta_i(\psi_{\mathsf{t}^\mu\dot{\mathsf{v}}}) = \psi_{\mathsf{t}^\lambda\mathsf{v}}$. Therefore

$$\theta_i(\sum_{\dot{\mathsf{v}}\in \mathrm{Std}(\mu)} c_{\mathsf{t}^\mu\dot{\mathsf{v}}}\psi_{\mathsf{t}^\mu\dot{\mathsf{v}}}) = \sum_{\mathsf{v}\in \mathrm{Std}(\lambda)} c_{\mathsf{t}^\mu\dot{\mathsf{v}}}\psi_{\mathsf{t}^\lambda\mathsf{v}} \in R_n^{\geq\lambda}.$$

So

$$e_\mu y_\mu \psi_{r_1}\psi_{r_2}\dots\psi_{r_m} = \theta_i(\sum_{\dot{\mathsf{v}}\in \mathrm{Std}(\mu)} c_{\mathsf{t}^\mu\dot{\mathsf{v}}}\psi_{\mathsf{t}^\mu\dot{\mathsf{v}}}) + \theta_i(\sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in \mathrm{Std}(>\mu)} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{\geq\lambda}.$$

□

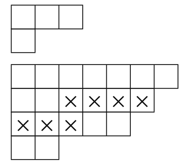
3.4. Garnir tableaux

In the following sections we will prove that $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$ for $l(d(\mathsf{t})) \leq m_\lambda$. Generally, if $\mathsf{t}{\cdot}s_r$ is standard and $d(\mathsf{t}){\cdot}s_r$ is reduced or $l(d(\mathsf{t})){\cdot}s_r$ is not reduced, it is comparatively easy to prove that $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. Our main difficulty is to prove that when $\mathsf{t}{\cdot}s_r$ is not standard then $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. In order to prove this we consider different types of t. Among these cases the hardest part is that when t is a special kind of tableaux which is called the Garnir tableau and $\mathsf{t}{\cdot}s_r$ is not standard. In this section we will prove that in such case $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$.

The method of proving the argument in this section is first assuming that Shape(t) is a partition of two rows, and using the similar argument we used in the last chapter to extend the result to general multipartitions. First we give a detailed definition of garnir tableaux.

We introduce a special kind of tableaux, the Garnir tableaux, which was first introduced by Murphy [**24**]. Let (a, b, m) be a node of λ such that $(a+1, b, m)$ is also a node of λ. The (a, b, m)-**Garnir belt** of λ consists of the nodes (a, c, m) for $b \leq c \leq \lambda_a^{(m)}$ and the nodes $(a+1, g, m)$ for

$1 \le g \le b$. For example here is a picture of the $(2,3,2)$-Garnir belt for $\lambda = (3,1|7,6,5,2)$.



The (a,b,m)**-Garnir tableau** of shape λ is the unique maximal standard λ-tableau with respect to the Bruhat order ($\rhd$) among the standard λ-tableaux which agree with t^λ outside the (a,b,m)-Garnir belt. For example the following is the $(2,3,2)$-Garnir tableau for $\lambda = (3,1|7,6,5,2)$.

$$\left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \\ & & \\ & & \end{array} \middle| \begin{array}{ccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 13 & 16 & 18 & 19 & 20 & \\ 14 & 15 & 17 & 21 & 22 & & \\ 23 & 24 & & & & & \end{array} \right)$$

Suppose $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda^{(\ell)}_1, \dots, \lambda^{(\ell)}_k)$. Let $(a,b,m) = (k-1, \lambda^{(\ell)}_k, \ell)$ and t be the (a,b,m)-Garnir tableau. Let the entry in node (a,b,m) of t be r. Then $\mathsf{t}{\cdot}s_r$ is not standard.

3.4.1. Definition. *Suppose $\lambda \in \mathscr{P}^\Lambda_n$ with $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda^{(\ell)}_1, \dots, \lambda^{(\ell)}_k)$. If $k \ge 2$, and t is the $(k-1, \lambda^{(\ell)}_k, \ell)$-Garnir tableau, then we call t the* ***last Garnir tableau*** *of shape λ, and $r = \mathsf{t}(k-1, \lambda^{(\ell)}_k, \ell)$ the* ***last Garnir entry*** *of t.*

For example

$$\left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \\ & & \end{array} \middle| \begin{array}{ccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 13 & 16 & 18 & 19 & 20 & \\ 14 & 15 & 17 & & & & \end{array} \right)$$

is the last $(2,3,2)$-Garnir tableau, and

$$\left(\begin{array}{ccc} 1 & 2 & 3 \\ 4 & & \\ & & \\ & & \end{array} \middle| \begin{array}{ccccccc} 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 12 & 13 & 16 & 18 & 19 & 20 & \\ 14 & 15 & 17 & 21 & 22 & & \\ 23 & 24 & & & & & \end{array} \right)$$

is not the last one. Notice that $\mathsf{t}{\cdot}s_r$ is not standard.

Because we are going to play around with $\psi_{d(\mathsf{t})}$ a lot, we then introduce more defailed notation for these elements in the next Lemma.

3.4.2. Lemma. *Suppose $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$. Let t be a (a,b,m)-Garnir tableau of shape λ and $\lambda^{(m)} = (\lambda^{(m)}_1, \dots, \lambda^{(m)}_k)$. Suppose*

$$\begin{cases} \mathsf{t}^\lambda(a,b,m) & = l, \\ \mathsf{t}^\lambda(a, \lambda^{(m)}_a, m) & = s, \\ \mathsf{t}^\lambda(a+1,b,m) & = t. \end{cases}$$

Then $l \le s < t$. Write $t - s = c$,

$$\psi_{d(\mathsf{t})} = \psi_s\psi_{s+2}\dots\psi_{t-1}\cdot\psi_{s-1}\psi_s\dots\psi_{t-2}\cdot\dots\cdot\psi_{l+1}\psi_{l+2}\dots\psi_{l+c}\cdot\psi_l\psi_{l+1}\dots\psi_{l+c-2}$$

where

$$l(\psi_s\psi_{s+1}\dots\psi_{t-1}) = l(\psi_{s-1}\psi_s\dots\psi_{t-2}) = \dots = l(\psi_{l+1}\psi_{l+2}\dots\psi_{l+c}) = c$$

and

$$l(\psi_l\psi_{l+1}\dots\psi_{l+c-2}) = c - 1.$$

Proof. The Lemma follows by direct calculation. □

3.4.3. **Example** Suppose $\lambda = (3,1|7,6,5,2)$ and $(a,b,m) = (2,3,2)$. Let $\mathfrak{t}$ be the $(2,3,2)$-Garnir tableau of shape λ. Then

$$\mathfrak{t} = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \;\middle|\; \begin{array}{|c|c|c|c|c|c|c|}\hline 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 12 & 13 & 16 & 18 & 19 & 20 \\ \cline{1-6} 14 & 15 & 17 & 21 & 22 \\ \cline{1-5} 23 & 24 \\ \cline{1-2} \end{array} \right),$$

and

$$\mathfrak{t}^\lambda = \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1} \end{array} \;\middle|\; \begin{array}{|c|c|c|c|c|c|c|}\hline 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline 12 & 13 & 14 & 15 & 16 & 17 \\ \cline{1-6} 18 & 19 & 20 & 21 & 22 \\ \cline{1-5} 23 & 24 \\ \cline{1-2} \end{array} \right).$$

Then

$$\begin{aligned} \mathfrak{t}^\lambda(a,b,m) &= \mathfrak{t}^\lambda(2,3,2) &&= 14, \\ \mathfrak{t}^\lambda(a,\lambda_a^{(m)},m) &= \mathfrak{t}^\lambda(2,6,2) &&= 17, \\ \mathfrak{t}^\lambda(a+1,b,m) &= \mathfrak{t}^\lambda(3,3,2) &&= 20, \end{aligned}$$

and $c = t - s = 3$. Therefore

$$\psi_{d(\mathfrak{t})} = \psi_{17}\psi_{18}\psi_{19}\cdot\psi_{16}\psi_{17}\psi_{18}\cdot\psi_{15}\psi_{16}\psi_{17}\cdot\psi_{14}\psi_{15}$$

with

$$l(\psi_{17}\psi_{18}\psi_{19}) = l(\psi_{16}\psi_{17}\psi_{18}) = l(\psi_{15}\psi_{16}\psi_{17}) = 3 = c$$

and

$$l(\psi_{14}\psi_{15}) = 2 = c - 1.$$

◇

3.4.4. Remark. For $a \le b-1$, we will write $\psi_{a,b} = \psi_a\psi_{a+1}\psi_{a+2}\dots\psi_{b-2}\psi_{b-1}$ and $\psi_{b,a} = \psi^*_{a,b}$ in order to simplify our notations.

Our first step is to prove that when λ is a partition with two rows and $\mathfrak{t}$ is a last Garnir tableau of shape λ with r as its last Garnir entry, then $\psi_{\mathfrak{s}\mathfrak{t}}\psi_r \in R_n^{\ge\lambda}$ for any $\mathfrak{s} \in \mathrm{Std}(\lambda)$. We set $\lambda = (\lambda_1, \lambda_2)$ and without loss of generality, set $\Lambda = \Lambda_0$. Therefore $\lambda \in \mathscr{P}_n^\Lambda$ with $n = \lambda_1 + \lambda_2$. Also we set $\mu = (\lambda_1, \lambda_2 - 1, 1)$, $\dot\lambda = (\lambda_1 - 1, \lambda_2)$ and $\dot\mu = (\lambda_1 - 1, \lambda_2 - 1, 1)$. Furthermore, let $i = \mathrm{res}(\gamma_1)$, $j = \mathrm{res}(\gamma_2)$, where $\gamma_1 = (1, \lambda_1, 1)$ and $\gamma_2 = (2, \lambda_2, 1)$.

First we prove a few useful Lemmas.

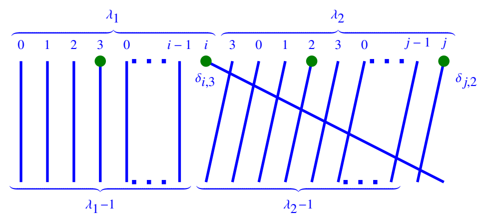
3.4.5. Lemma. *Suppose λ, $\dot\lambda$, $\dot\mu$, i and j are defined as above, we have*

$$e_\lambda y_\lambda \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1} =_\lambda \begin{cases} \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda}y_n \\ \qquad -\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda}, & \text{if } i = e-1,\ j \ne e-1, \\ \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda}y_n \\ \qquad -\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda} \\ \qquad\qquad -\psi_{\lambda_1}\dots\psi_{n-2}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda}, & \text{if } i = j = e-1, \\ \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda} \\ \qquad +\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}e(\mathbf{i}_{\dot\mu}\vee i)y_{\dot\mu}, & \text{if } i = j = e-2, \\ \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda}, & \text{otherwise.} \end{cases}$$

Proof. In order to make the notations and the diagrams clearer we set $e = 4$. For the other choices of e one can check that the method is the same because the proof doesn't depend on the value of e.

By using the diagrammatic notation we have

$$e_\lambda y_\lambda \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1} =$$



.

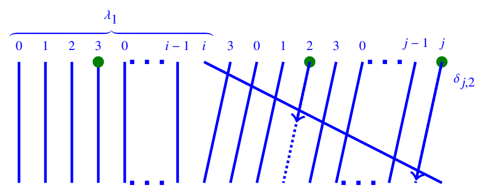
This Lemma is equivalent to move all the dots from top to bottom. In order to do this we have to consider several cases.

Case 3.4.5a: $i \neq e-2, e-1$.

We set $e = 4$ so in this case we have $i \neq 2, 3$. As $i \neq 3$, $\delta_{i,3} = 0$. Therefore there are no dots on the strand labelled by i. And as $i \neq 2$, by relation 1.1.10, we have

$$e_\lambda y_\lambda \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}$$

=



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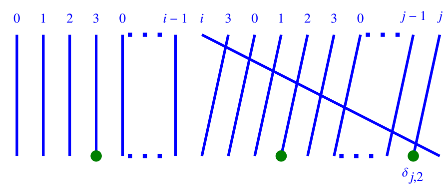
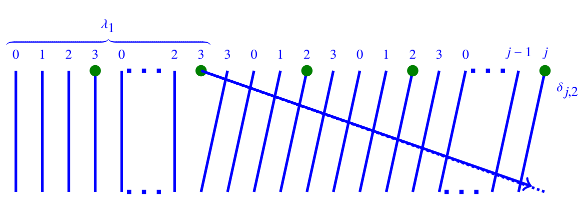


$$= \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_\lambda \vee i)y_\lambda.$$

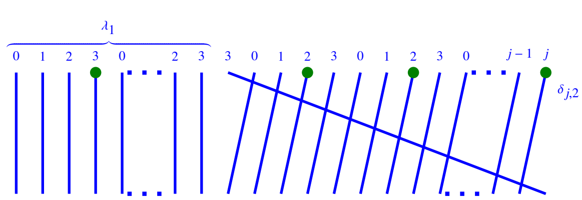
Case 3.4.5b: $i = e-1$ and $j \neq e-1$.

We set $e = 4$ so in this case we have $i = 3$ and $j \neq 3$. Then $\delta_{i,3} = 1$. Hence

$$e_\lambda y_\lambda \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1} =$$



$\overset{(1.1.10)}{=}$ −



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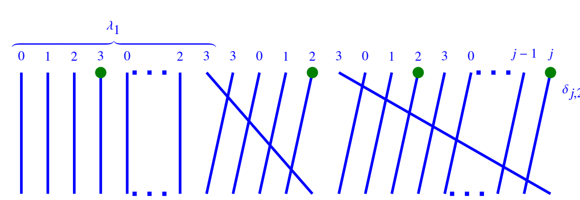

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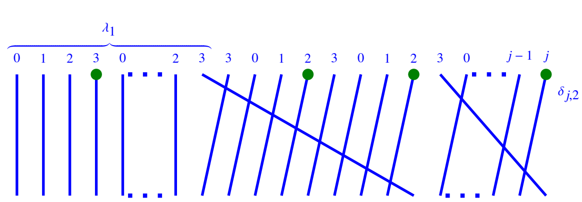
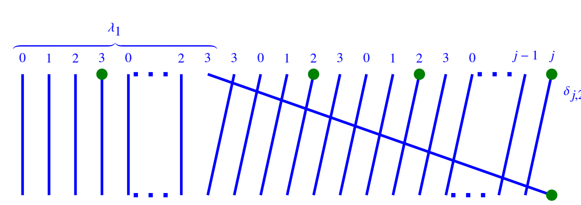
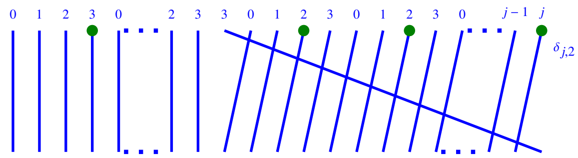
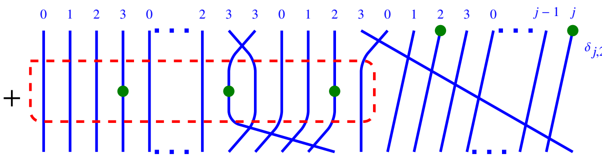
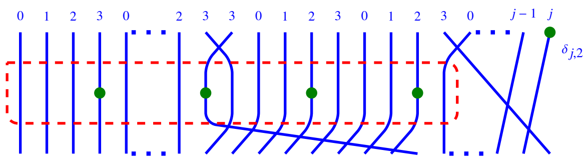
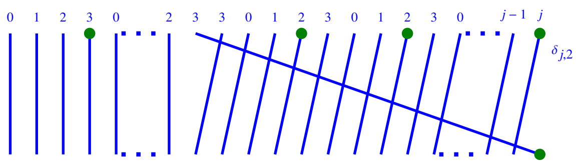


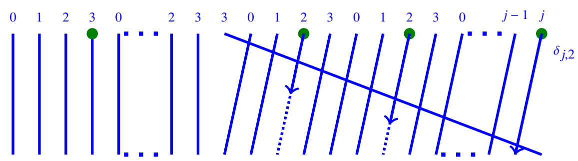
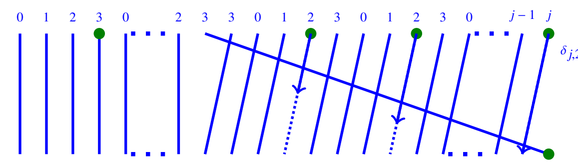
− … +

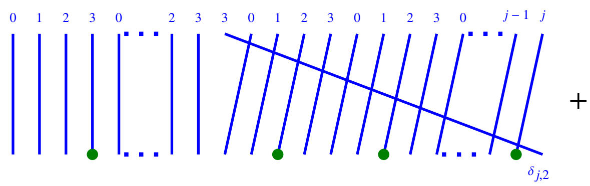
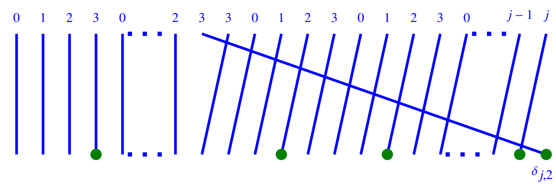


$\overset{(1.1.17)}{=}$ $-$  $+$ 

$+$  $+$ .... by Lemma 2.1.14

$+$ 

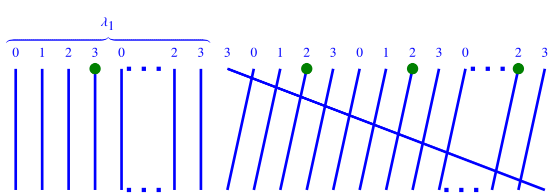
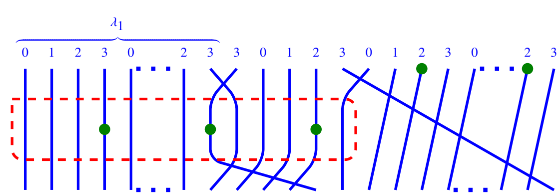
$=_\lambda$ $-$  $+$ 

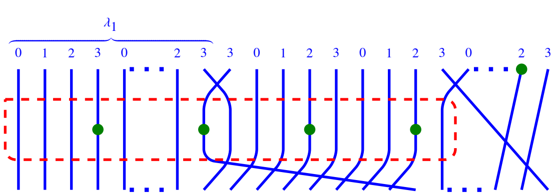
$\overset{(1.1.10)}{=}$ $-$  $+$ 

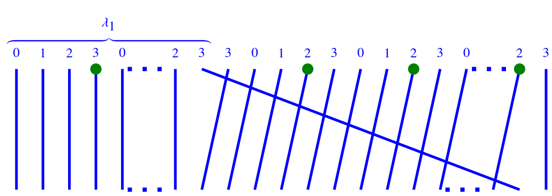
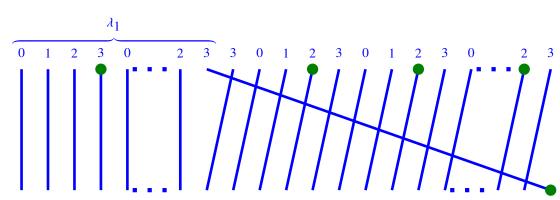
$$= -\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_\lambda \vee i)y_\lambda + \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_\lambda \vee i)y_\lambda y_n.$$

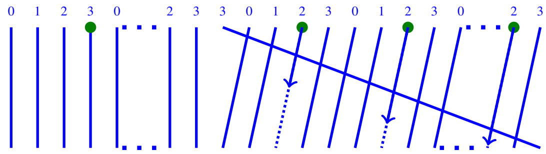
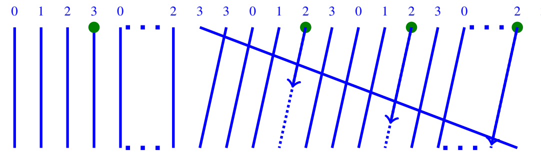
Case 3.4.5c: $i = j = e - 1$.
We set $e = 4$ so in this case we have $i = j = 3$. Similarly as in Case 3.4.5b, we have

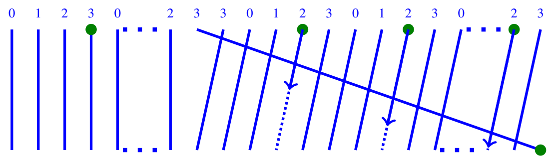
$e_\lambda y_\lambda \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}$

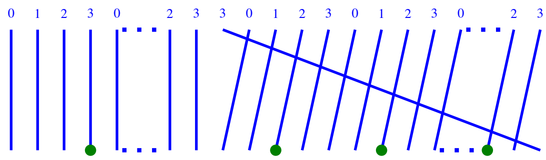
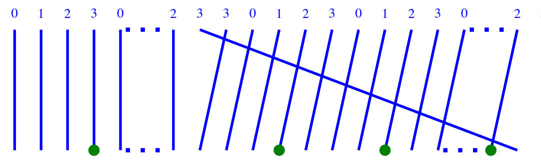
$=$ $-$  $-$ 

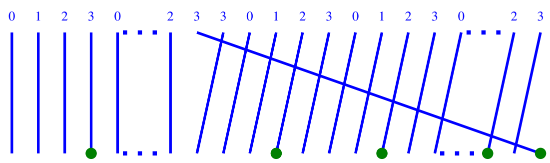
$-$  $+$ .... by Lemma 2.1.14

$-$  $+$ 

$=_\lambda$ $-$  $-$ 

$+$ 

$=$ $-$  $-$ 

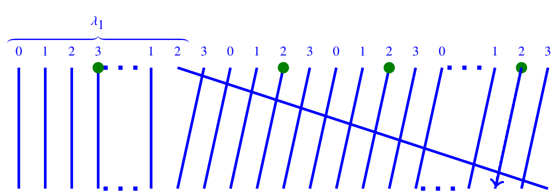
$+$ 

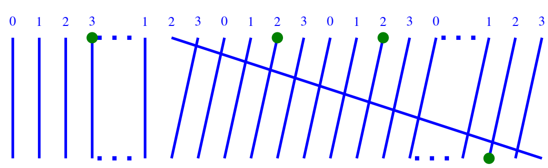
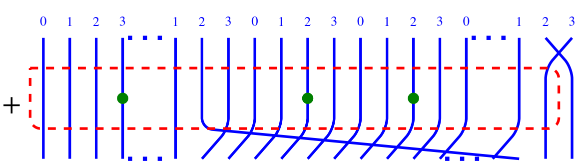
$$= \quad -\psi_{\lambda_1+1}\ldots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda} - \psi_{\lambda_1}\ldots\psi_{n-2}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda} + \psi_{\lambda_1}\psi_{\lambda_1+1}\ldots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda}y_n.$$

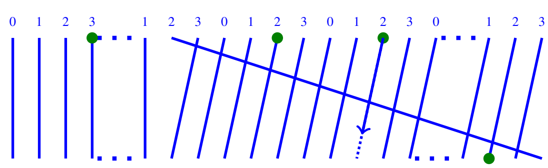
Case 3.4.5d: $i = e-2$ and $j \neq e-2$.

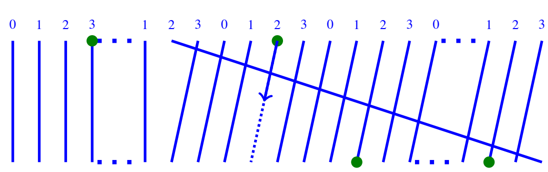
We set $e = 4$ so in this case we have $i = 2$ and $j \neq 2$. Similarly as we set $e = 4$, we set $j = 3$ in this case in order to make the diagram clearer. For the other j with $j \neq 2$ the method is similar. By Lemma 2.1.14,

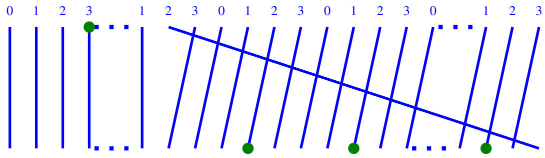
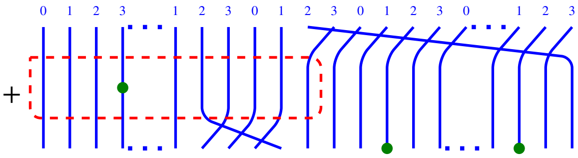
$$e_\lambda y_\lambda \psi_{\lambda_1}\psi_{\lambda_1+1}\ldots\psi_{n-2}\psi_{n-1}$$

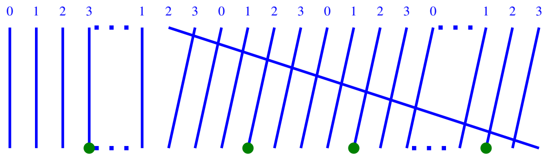
$=$ 

$\overset{(1.1.10)}{=}$  $+$ 

$=_\lambda$  $=$ ....

$=_\lambda$ 

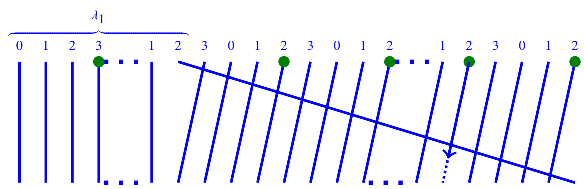
$\overset{(1.1.10)}{=}$  + 

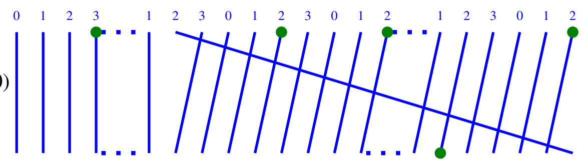
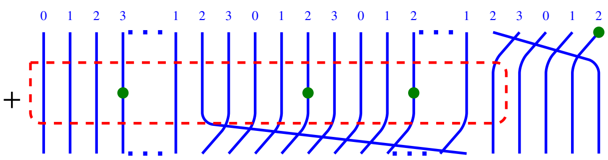
$=_\lambda$  $= \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda} \vee i)y_{\dot\lambda}.$

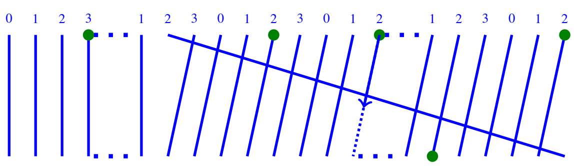
Case 3.4.5e: $i = j = e - 2$.

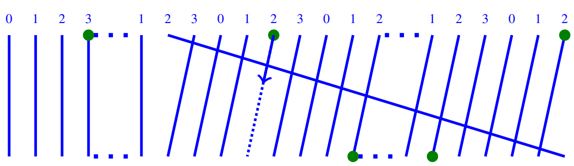
We set $e = 4$ so in this case we have $i = j = 2$. Then by Lemma 2.1.14,

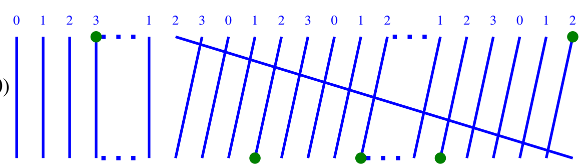
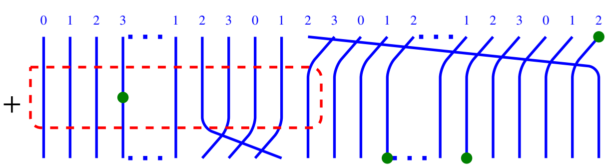
$e_\lambda y_\lambda \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}$

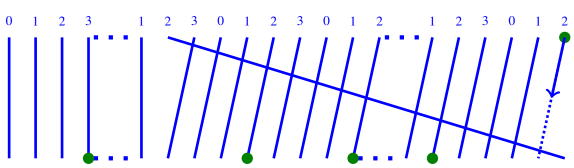
$=$ 

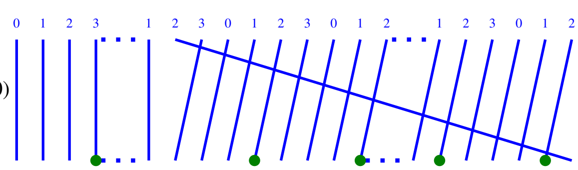
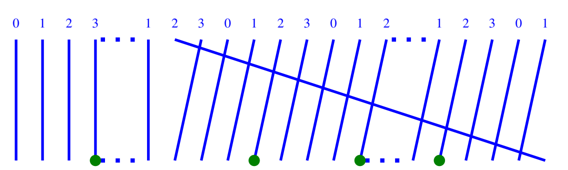
$\overset{(1.1.10)}{=}$  + 

$=_\lambda$  $=$ ....

$=_\lambda$ 

$\overset{(1.1.10)}{=_\lambda}$  + 

$=_\lambda$ 

$\overset{(1.1.10)}{=}$  + 

$= \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}e(\mathbf{i}_{\dot\lambda} \vee i)y_{\dot\lambda} + \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}e(\mathbf{i}_{\dot\mu} \vee i)y_{\dot\mu},$

which completes the proof. □

3.4.6. Remark. If $\lambda_1 > \lambda_2$ and t is the last Garnir tableau of λ, by Lemma 3.4.2 we have

$$\psi_{d(\mathsf{t})}\psi_r = \psi_{a_n,n}\psi_{a_{n-1},n-1}\dots\psi_{a_{r+2},r+2}\psi_{a_{r+1},r+1}.$$

Define w to be the last Garnir tableau of shape $\dot{\lambda}$, we can see that $\psi_{d(\mathsf{w})} = \psi_{a_{n-1},n-1}\dots\psi_{a_{r+2},r+2}\psi_{a_{r+1},r+1}$. Hence $e(\mathbf{i}_{\dot{\lambda}} \vee i)y_{\dot{\lambda}}\psi_{d(\mathsf{w})}\psi_r = \theta_i(\psi_{\mathsf{t}^{\dot{\lambda}}\mathsf{w}}\psi_r)$.

3.4.7. Lemma. *Suppose* t *and* $\dot{\mathsf{t}}$ *are the last Garnir tableau of shape* λ *and* $\dot{\lambda}$ *respectively with last Garnir entry r. Set*

$$\psi = \begin{cases} \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}y_n - \psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}, & \text{if } i = e-1,\ j \neq e-1. \\ \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}y_n - \psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1} - \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\lambda_{n-2}, & \text{if } i = j = e-1. \\ \psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}, & \text{otherwise.} \end{cases}$$

For any standard $\dot{\lambda}$*-tableau* $\dot{\mathsf{v}}$*, if* $d(\mathsf{t}) \le m_\lambda$ *and* $\dot{\mathsf{v}} \rhd \dot{\mathsf{t}}$*, then*

$$\begin{cases} \psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot{\lambda}}\dot{\mathsf{v}}}) \in R_n^{\ge\mu}, & \text{if } i = j = e-2, \\ \psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot{\lambda}}\dot{\mathsf{v}}}) \in R_n^{\ge\lambda}, & \text{otherwise.} \end{cases}$$

Proof. If it is not the case that $i = j = e-2$. By Lemma 3.4.5 we have

$$\psi\cdot e(\mathbf{i}_{\dot{\lambda}} \vee i)y_{\dot{\lambda}} =_\lambda e(\mathbf{i}_\lambda)y_\lambda\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}.$$

Then we have

$$\psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot{\lambda}}\dot{\mathsf{v}}}) = \psi\cdot e(\mathbf{i}_{\dot{\lambda}} \vee i)y_{\dot{\lambda}}\psi_{d(\dot{\mathsf{v}})} =_\lambda e_\lambda y_\lambda\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{v}})},$$

where as $\dot{\mathsf{v}} \rhd \dot{\mathsf{t}}$, then $d(\dot{\mathsf{v}}) < d(\dot{\mathsf{t}})$ and

$$l(\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{v}})}) < l(\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}) + l(d(\dot{\mathsf{t}})) = l(d(t)) \le m_\lambda.$$

Then by Lemma 3.2.1 we have $\psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot{\lambda}}\dot{\mathsf{v}}}) \in R_n^{\ge\lambda}$.

For $i = j = e-2$, set $\dot{\mu} = (\lambda_1 - 1, \lambda_2 - 1, 1)$,$\gamma = \lambda|_{n-1} = (\lambda_1, \lambda_2 - 1)$ and $\dot{\gamma} = (\lambda_1 - 1, \lambda_2 - 1)$. Because $y_{\dot{\gamma}} = y_{\dot{\mu}}$. By Lemma 3.4.5,

$$\begin{aligned} \psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot{\lambda}}\dot{\mathsf{v}}}) &= \psi_{\lambda_1,n}e(\mathbf{i}_{\dot{\lambda}} \vee i)y_{\dot{\lambda}}\psi_{d(\dot{\mathsf{v}})} =_\lambda e_\lambda y_\lambda\psi_{\lambda_1,n}\psi_{d(\dot{\mathsf{v}})} - \psi_{\lambda_1,n-1}e(\mathbf{i}_{\dot{\mu}} \vee i)y_{\dot{\mu}}\psi_{d(\dot{\mathsf{v}})} \\ &= e_\lambda y_\lambda\psi_{\lambda_1,n}\psi_{d(\dot{\mathsf{v}})} - \theta_i(\psi_{\lambda_1,n-1}e(\mathbf{i}_{\dot{\gamma}} \vee i)y_{\dot{\mu}}\psi_{d(\dot{\mathsf{v}})}) \\ &= e_\lambda y_\lambda\psi_{\lambda_1,n}\psi_{d(\dot{\mathsf{v}})} - \theta_i(\psi_{\lambda_1,n-1}e(\mathbf{i}_{\dot{\gamma}} \vee i)y_{\dot{\gamma}}\psi_{d(\dot{\mathsf{v}})}). \end{aligned}$$

Again by Lemma 3.4.5, $\psi_{\lambda_1,n-1}e(\mathbf{i}_{\dot{\gamma}} \vee i)y_{\dot{\gamma}}\psi_{d(\dot{\mathsf{v}})} =_\gamma e_\gamma y_\gamma\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{v}})}$. Since $\gamma = \lambda|_{n-1}$, by Lemma 2.1.14,

$$\theta_i(\psi_{\lambda_1,n-1}e(\mathbf{i}_{\dot{\gamma}} \vee i)y_{\dot{\gamma}}\psi_{d(\dot{\mathsf{v}})}) =_\lambda \theta_i(e_\gamma y_\gamma\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{v}})}).$$

Therefore

$$\psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot{\lambda}}\dot{\mathsf{v}}}) =_\lambda e_\lambda y_\lambda\psi_{\lambda_1,n}\psi_{d(\dot{\mathsf{v}})} - \theta_i(e_\gamma y_\gamma\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{v}})}).$$

As $\lambda \in \mathscr{S}_n^\Lambda$ and $|\gamma| = n-1 < |\lambda|$

$$e_\gamma y_\gamma\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{v}})} = \sum_{\mathsf{y}\in\mathrm{Shape}(\gamma)} c_{\mathsf{t}^\gamma\mathsf{y}}\psi_{\mathsf{t}^\gamma y} + \sum_{\mathsf{x},\mathsf{y}\in\mathrm{Std}(>\gamma)} c_{\mathsf{xy}}\psi_{\mathsf{xy}}. \tag{3.4.8}$$

For the first term of the left hand side of (3.4.8), because $\gamma = \lambda|_{n-1}$ and $j = i = e-2$, we have $b_i^\gamma = 2$. By Lemma 2.1.3 and the definition of γ, $\theta_i(\psi_{\mathsf{t}^\gamma\mathsf{y}}) \in R_n^{\ge\mu}$. For the second term of the left hand side of (3.4.8), as $\mathsf{x}, \mathsf{y} \in \mathrm{Std}(>\gamma) = \mathrm{Std}(>\lambda|_{n-1})$, $\psi_{\mathsf{xy}} \in R_n^{>\lambda|_{n-1}}$. By Lemma 2.1.14, $\theta_i(\psi_{\mathsf{xy}}) \in R_n^{>\lambda} \subseteq R_n^{\ge\mu}$. Therefore,

$$\theta_i(e_\gamma y_\gamma\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{v}})}) \in R_n^{\ge\mu}.$$

Finally, as

$$l(\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{v}})}) < l(\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}) + l(d(\dot{\mathsf{t}})) = l(d(\mathsf{t})) \le m_\lambda,$$

by Lemma 3.2.1 we have $e_\lambda y_\lambda \psi_{\lambda_1,n}\psi_{d(\dot{\mathsf{v}})} \in R_n^{>\lambda} \subseteq R_n^{\ge\mu}$. Hence $\psi{\cdot}\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}) \in R_n^{\ge\mu}$. This completes the proof. □

3.4.9. Lemma. *Suppose $\lambda_1 - \lambda_2 \equiv e-1 \pmod e$, i.e. $i = j$, and $(\lambda_1-\lambda_2+1)\lambda_2 - 1 \le m_\lambda$. Let $\dot{\mathsf{u}}$ and $\dot{\mathsf{v}}$ be standard $\lambda|_{n-1}$-tableaux with $\dot{\mathsf{u}} \rhd \mathsf{t}^{\dot\lambda}$. Assume $i = j \ne e-2$. Then set*

$$\psi = \begin{cases} \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}y_n - \psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1} - \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}, & \text{if } i=j=e-1,\\ \psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}, & \text{if } i=j\ne e-1,e-2. \end{cases}$$

and we have

$$\psi{\cdot}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{\ge\lambda}.$$

Proof. We assume that $i = j \ne e-2$. First we need to introduce some properties of $\dot{\mathsf{u}}$. Because $\dot{\mathsf{u}}$ is a standard $\lambda|_{n-1}$-tableau and $\dot{\mathsf{u}} \rhd \mathsf{t}^{\dot\lambda}$, the only possible choice of $\dot{\mathsf{u}}$ is the one such that $\dot{\mathsf{u}}|_{n-2} = \mathsf{t}^{(\lambda_1-1,\lambda_2-1)}$. Then define u and v to be the unique standard λ-tableau with $\mathsf{u}|_{n-1} = \dot{\mathsf{u}}$ and $\mathsf{v}|_{n-1} = \dot{\mathsf{v}}$, respectively. For example, when $\lambda = (7,4)$ and $e = 3$, then $\dot{\mathsf{u}} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&10\\\hline 7&8&9\\\cline{1-3}\end{array}$ and $\mathsf{u} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&10\\\hline 7&8&9&11\\\cline{1-4}\end{array}$. From the definitions of $\dot{\mathsf{u}}$, $\dot{\mathsf{v}}$ and u, v we can see that $d(\mathsf{v}) = d(\dot{\mathsf{v}})$ and $l(d(\mathsf{u})) = l(d(\dot{\mathsf{u}})) = \lambda_2 - 1$. Also notice that if $i = j \ne e-2$, then $\mathbf{i}_\lambda = \mathbf{i}_{\lambda|_{n-1}} \vee i$ and $y_{\lambda|_{n-1}} = y_\lambda$.

Now we consider different cases of i, j. Suppose $i = j \ne e-1, e-2$, then

$$\begin{aligned}\psi{\cdot}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) &= \psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{u}})}e(\mathbf{i}_{\lambda|_{n-1}}\vee i)y_{\lambda|_{n-1}}\psi_{d(\dot{\mathsf{v}})}\\ &= \psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{u}})}e_\lambda y_\lambda \psi_{d(\dot{\mathsf{v}})}.\end{aligned}$$

Recall that $e \ge 3$. As $\lambda_1 - \lambda_2 \equiv e-1 \pmod e$, we must have $\lambda_1 - \lambda_2 \ge e-1 \ge 2$. Also because of $\lambda_2 \ge 1$,

$$m_\lambda \ge (\lambda_1-\lambda_2+1)\lambda_2 - 1 \ge 3\lambda_2 - 1 \ge 2\lambda_2 > l(\psi_{\lambda_1,n}\psi_{d(\dot{\mathsf{u}})}) = 2\lambda_2 - 1.$$

Hence by Lemma 3.2.1

$$\begin{aligned}\psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{u}})}e_\lambda y_\lambda\psi_{d(\dot{\mathsf{v}})} &= \psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{u}})}e_\lambda y_\lambda\psi_{d(\mathsf{v})}\\ &= \psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}\psi_{d(\dot{\mathsf{u}})}\psi_{\mathsf{t}^\lambda\mathsf{v}} \in R_n^{\ge\lambda}.\end{aligned}$$

Suppose $i = j = e-1$, then

$$\begin{aligned}&\psi{\cdot}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}})\\ =\ & (\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}y_n - \psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1} - \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\lambda_{n-2})\psi_{d(\dot{\mathsf{u}})}e(\mathbf{i}_{\lambda|_{n-1}}\vee i)y_{\lambda|_{n-1}}\psi_{d(\dot{\mathsf{v}})}\\ =\ & \psi_{\lambda_1,n}y_n\psi_{d(\dot{\mathsf{u}})}\psi_{\mathsf{t}^\lambda\mathsf{v}} - \psi_{\lambda_1+1,n}\psi_{d(\dot{\mathsf{u}})}\psi_{\mathsf{t}^\lambda\mathsf{v}} - \psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{u}})}\psi_{\mathsf{t}^\lambda\mathsf{v}}.\end{aligned}$$

As $\psi_{d(\dot{\mathsf{u}})}$ doesn't involve ψ_{n-1}, by Proposition 3.2.5 and Lemma 2.1.12,

$$\psi_{\lambda_1,n}y_n\psi_{d(\dot{\mathsf{u}})}\psi_{\mathsf{t}^\lambda\mathsf{v}} = \psi_{\lambda_1,n}\psi_{d(\dot{\mathsf{u}})}y_n\psi_{\mathsf{t}^\lambda\mathsf{v}} \in R_n^{>\lambda},$$

and because $l(\psi_{\lambda_1+1,n}\psi_{d(\dot{\mathsf{u}})}) = l(\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{u}})}) = \lambda_2 - 1 + \lambda_2 - 1 = 2\lambda_2 - 2 < m_\lambda$, by Lemma 3.2.1, $\psi_{\lambda_1+1,n}\psi_{d(\dot{\mathsf{u}})}\psi_{\mathsf{t}^\lambda\mathsf{v}}$ and $\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathsf{u}})}\psi_{\mathsf{t}^\lambda\mathsf{v}}$ are both in $R_n^{\ge\lambda}$. Hence $\psi{\cdot}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{\ge\lambda}$. □

Now we are ready to prove that $\psi_{\mathsf{st}}\psi_r \in R_n^{\ge\lambda}$ when Shape(t) has only two rows.

3.4.10. Proposition. *Suppose $\lambda \in \mathscr{P}_n^\Lambda$ and $\mathfrak{t}$ is the last Garnir tableau of shape λ with r to be the last Garnir entry and $l(d(\mathfrak{t})) \le m_\lambda$, we have*

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \sum_{(\mathfrak{u},\mathfrak{v})\rhd(\mathfrak{t}^\lambda,\mathfrak{t})} c_{\mathfrak{u}\mathfrak{v}}\psi_{\mathfrak{u}\mathfrak{v}}.$$

Proof. By Lemma 3.4.2, as $\mathfrak{t}$ is the last Garnir tableau of shape λ, we have

$$\psi_{d(\mathfrak{t})}\psi_r = \psi_{\lambda_1,n}\psi_{\lambda_1-1,n-1}\dots\psi_{\lambda_2,r+1},$$

where $l(\psi_{\lambda_1,n}) = l(\psi_{\lambda_1-1,n-1}) = \dots = l(\psi_{\lambda_2,r+1}) = \lambda_2$. We prove the Proposition by induction on λ_1. Recall that we write $\dot\lambda = (\lambda_1-1,\lambda_2)$, $\mu = (\lambda_1,\lambda_2-1,1)$ and $\dot\mu = (\lambda_1-1,\lambda_2-1,1)$.

When $\lambda_1 = 1$, by definition of Garnir tableau, $\lambda_1 = \lambda_2$. Without loss of generality, we set $\Lambda = \Lambda_0$. In this case $i = 0$ and $j = e-1$. Hence

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \psi_{\mathfrak{t}^\lambda\mathfrak{t}^\lambda}\psi_r = e(0,e-1)\psi_1 = \psi_1 e(e-1,0) = 0 \in R_n^{\ge\lambda}.$$

So, when $\lambda_1 = 1$ the Proposition is true.

Assume for any partition of two rows with the length of its first row less than λ_1 the Proposition holds. By Lemma 3.4.5, we have

$$\begin{aligned}\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r &= e_\lambda y_\lambda \psi_{d(\mathfrak{t})}\psi_r = e_\lambda y_\lambda \psi_{\lambda_1,n}\psi_{d(\dot{\mathfrak{t}})}\psi_r \\ &= \begin{cases}\psi_{\lambda_1,n}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda}\psi_{d(\dot{\mathfrak{t}})}\psi_r + \psi_{\lambda_1,n-1}e(\mathbf{i}_{\dot\mu}\vee i)y_{\dot\mu}\psi_{d(\dot{\mathfrak{t}})}\psi_r & \\ \qquad = \psi_{\lambda_1,n}\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r) + \psi_{\lambda_1,n-1}\theta_i(e_{\dot\mu}y_{\dot\mu}\psi_{d(\dot{\mathfrak{t}})}\psi_r), & \text{if } i=j=e-2,\\ \psi\cdot\theta_i(e_{\dot\lambda}y_{\dot\lambda}\psi_{\lambda_1-1,n-1}\dots\psi_{\lambda_2+1,r+2}\psi_{\lambda_2,r+1}) = \psi\cdot\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r), & \text{otherwise.}\end{cases}\end{aligned} \tag{3.4.11}$$

where $\dot{\mathfrak{t}}$ is the last Garnir tableau with shape $\dot\lambda = (\lambda_1-1,\lambda_2)$, and

$$\psi = \begin{cases}\psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}y_n - \psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}, & \text{if } i=e-1,\ j\ne e-1,\\ \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}y_n - \psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1} - \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}, & \text{if } i=j=e-1,\\ \psi_{\lambda_1}\psi_{\lambda_2}\dots\psi_{n-2}\psi_{n-1}, & \text{otherwise.}\end{cases}$$

Now we separate the question into different cases.

Case 3.4.10a: $i \ne j$. By (3.4.11) we have

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \psi\cdot\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r).$$

By induction, $\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r = \sum_{\substack{\dot{\mathfrak{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathfrak{v}}\rhd\dot{\mathfrak{t}}}} c_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}} + \sum_{\dot{\mathfrak{u}},\dot{\mathfrak{v}}\in\mathrm{Std}(>\dot\lambda)} c_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}$. Therefore

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \sum_{\substack{\dot{\mathfrak{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathfrak{v}}\rhd\dot{\mathfrak{t}}}} c_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}\psi\cdot\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}) + \sum_{\dot{\mathfrak{u}},\dot{\mathfrak{v}}\in\mathrm{Std}(>\dot\lambda)} c_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}\psi\cdot\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}).$$

For $\dot{\mathfrak{u}},\dot{\mathfrak{v}} \in \mathrm{Std}(>\dot\lambda)$, by Lemma 1.4.4, $\mathrm{res}(\dot{\mathfrak{u}}) = \mathrm{res}(\mathfrak{t}^{\dot\lambda})$. Because $i \ne j$, we always have $\mathrm{Shape}(\dot{\mathfrak{u}}) > \lambda|_{n-1}$. Hence $\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}} \in R_n^{>\lambda|_{n-1}}$. Therefore by Lemma 2.1.14 and Lemma 2.1.12, $\psi\cdot\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) \in R_n^{>\lambda}$. So $\sum_{\dot{\mathfrak{u}},\dot{\mathfrak{v}}\in\mathrm{Std}(>\dot\lambda)} c_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}\psi\cdot\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) \in R_n^{>\lambda}$.

For $\dot{\mathfrak{v}} \in \mathrm{Std}(\dot\lambda)$ with $\dot{\mathfrak{v}}\trianglerighteq\dot{\mathfrak{t}}$, by Lemma 3.4.7, $\psi\cdot\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}) \in R_n^{\ge\lambda}$. Therefore $\sum_{\substack{\dot{\mathfrak{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathfrak{v}}\rhd\dot{\mathfrak{t}}}} c_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}\psi\cdot\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}) \in R_n^{\ge\lambda}$. These yield $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\ge\lambda}$.

Case 3.4.10b: $i = j \ne e-2$. By (3.4.11) we have

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \psi\cdot\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r).$$

By induction, $\psi_{\mathsf{t}^\lambda\mathsf{t}}\psi_r = \sum_{\substack{\dot{\mathsf{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathsf{v}}\rhd\dot{\mathsf{t}}}} c_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}} + \sum_{\substack{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(\lambda|_{n-1})\\ (\dot{\mathsf{u}},\dot{\mathsf{v}})\rhd(\mathsf{t}^{\dot\lambda},\dot{\mathsf{t}})}} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}} + \sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Shape}(>\lambda|_{n-1})} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}$. Therefore

$$\psi_{\mathsf{t}^\lambda\mathsf{t}}\psi_r = \sum_{\substack{\dot{\mathsf{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathsf{v}}\rhd\dot{\mathsf{t}}}} c_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}\psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}) + \sum_{\substack{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(\lambda|_{n-1})\\ (\dot{\mathsf{u}},\dot{\mathsf{v}})\rhd(\mathsf{t}^{\dot\lambda},\dot{\mathsf{t}})}} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi\cdot\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) + \sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Shape}(>\lambda|_{n-1})} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi\cdot\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}).$$

For $\dot{\mathsf{u}},\dot{\mathsf{v}} \in \mathrm{Std}(>\lambda|_{n-1})$, $\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}} \in R_n^{>\lambda|_{n-1}}$. As $\lambda \in \mathscr{S}_n^\Lambda$, by Lemma 2.1.14 we have $\psi\cdot\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{>\lambda}$. Hence $\sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(>\lambda|_{n-1})} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi\cdot\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{>\lambda}$.

For $\dot{\mathsf{u}},\dot{\mathsf{v}} \in \mathrm{Std}(\lambda|_{n-1})$ with $\dot{\mathsf{u}} \rhd \mathsf{t}^{\dot\lambda}$, because $m_\lambda \geq d(\mathsf{t}) = (\lambda_1-\lambda_2+1)\lambda_2 - 1$, by Lemma 3.4.9 we have $\psi\cdot\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{\geq\lambda}$. So $\sum_{\substack{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(\lambda|_{n-1})\\ (\dot{\mathsf{u}},\dot{\mathsf{v}})\rhd(\mathsf{t}^{\dot\lambda},\dot{\mathsf{t}})}} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi\cdot\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{\geq\lambda}$.

For $\dot{\mathsf{v}} \in \mathrm{Std}(\dot\lambda)$ with $\dot{\mathsf{v}}\rhd\dot{\mathsf{t}}$, by Lemma 3.4.7, $\psi\cdot\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}) \in R_n^{\geq\lambda}$. So $\sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Shape}(>\lambda|_{n-1})} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi\cdot\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{\geq\lambda}$.

Therefore we have $\psi_{\mathsf{t}^\lambda\mathsf{t}}\psi_r \in R_n^{\geq\lambda}$.

Case 3.4.10c: $i = j = e-2$. By (3.4.11) we have

$$\psi_{\mathsf{t}^\lambda\mathsf{t}}\psi_r = \psi_{\lambda_1,n}\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{t}}}\psi_r) + \psi_{\lambda_1,n-1}\theta_i(e_{\dot\mu}y_{\dot\mu}\psi_{d(\dot{\mathsf{t}})}\psi_r). \tag{3.4.12}$$

For the first term of (3.4.12), by induction,

$$\begin{aligned}\psi_{\lambda_1,n}\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{t}}}\psi_r) &= \sum_{\substack{\dot{\mathsf{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathsf{v}}\rhd\dot{\mathsf{t}}}} c_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}\psi_{\lambda_1,n}\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}) + \sum_{\substack{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(\lambda|_{n-1})\\ (\dot{\mathsf{u}},\dot{\mathsf{v}})\rhd(\mathsf{t}^{\dot\lambda},\dot{\mathsf{t}})}} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\lambda_1,n}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}})\\ &\quad+\sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(>\lambda|_{n-1})} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\lambda_1,n}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}).\end{aligned} \tag{3.4.13}$$

For $\dot{\mathsf{v}} \in \mathrm{Std}(\dot\lambda)$ with $\dot{\mathsf{v}} \rhd \dot{\mathsf{t}}$, by Lemma 3.4.7, we have $\psi_{\lambda_1,n}\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}) \in R_n^{\geq\mu}$. Therefore

$$\sum_{\substack{\dot{\mathsf{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathsf{v}}\rhd\dot{\mathsf{t}}}} c_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}\psi_{\lambda_1,n}\theta_i(\psi_{\mathsf{t}^{\dot\lambda}\dot{\mathsf{v}}}) \in R_n^{\geq\mu}. \tag{3.4.14}$$

For $\dot{\mathsf{u}},\dot{\mathsf{v}} \in \mathrm{Std}(\lambda|_{n-1})$ with $(\dot{\mathsf{u}},\dot{\mathsf{v}}) \rhd (\mathsf{t}^{\dot\lambda},\dot{\mathsf{t}})$, by Lemma 1.4.4, we have $\mathrm{res}(\dot{\mathsf{u}}) = \mathbf{i}_{\dot\lambda}$. So the choice of $\dot{\mathsf{u}}$ is unique, where $d(\dot{\mathsf{u}}) = \psi_{\lambda_1,n-1}$. Hence as $\mathbf{i}_\mu = \mathbf{i}_{\lambda|_{n-1}} \vee i$ and $y_{\lambda|_{n-1}} = y_\mu$

$$\psi_{\lambda_1,n}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) = \psi_{\lambda_1,n}\psi_{n-1,\lambda_1}e(\mathbf{i}_{\lambda|_{n-1}}\vee i)y_{\lambda|_{n-1}}\psi_{d(\dot{\mathsf{v}})} = \psi_{\lambda_1,n}\psi_{n-1,\lambda_1}e_\mu y_\mu\psi_{d(\dot{\mathsf{v}})}. \tag{3.4.15}$$

We work with $\psi_{\lambda_1,n}\psi_{n-1,\lambda_1}e_\mu y_\mu = \psi_{\lambda_1}\psi_{\lambda_1+1}\dots\psi_{n-2}\psi_{n-1}\psi_{n-2}\dots\psi_{\lambda_1+1}\psi_{\lambda_1}e_\mu y_\mu$ first. We define a partition $\sigma = (\lambda_1, \lambda_2-2, 1)$. Then

$$\begin{aligned}&\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-2}\psi_{n-1}\psi_{n-2}\psi_{n-3}\dots\psi_{\lambda_1}e_\mu y_\mu\\ =\;&\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-1}\psi_{n-2}\psi_{n-1}\psi_{n-3}\dots\psi_{\lambda_1}e_\mu y_\mu - \psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-3}\dots\psi_{\lambda_1}e_\mu y_\mu\\ =\;&\psi_{n-1}\theta_{i-1}(\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-2}\psi_{n-3}\dots\psi_{\lambda_1}e_\sigma y_\sigma)\psi_{n-1} - \psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-3}\dots\psi_{\lambda_1}e_\mu y_\mu.\end{aligned} \tag{3.4.16}$$

Consider the lefthand term in (3.4.16). As $\lambda \in \mathscr{S}_n^\Lambda$ and $|\sigma| = n-1 < |\lambda|$, we have

$$\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-2}\psi_{n-3}\dots\psi_{\lambda_1}e_\sigma y_\sigma = \sum_{\mathsf{u}\in\mathrm{Std}(\sigma)} c_{\mathsf{u}\mathsf{t}^\sigma}\psi_{\mathsf{u}\mathsf{t}^\sigma} + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\sigma)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}},$$

where $\mathrm{res}(\mathsf{u}) = \mathbf{i}_\sigma\cdot s_{\lambda_1}s_{\lambda_1+1}\dots s_{n-3}s_{n-2}s_{n-3}\dots s_{\lambda_1+1}s_{\lambda_1} = \mathbf{i}_\sigma$ by Lemma 1.4.4, and $\mathrm{res}(\mathsf{v}) = \mathbf{i}_\sigma$. Since $\min\{\lambda_1,\dots,n-2\} = \lambda_1$, by Lemma 3.3.1, $c_{\mathsf{u}\mathsf{t}^\sigma} \neq 0$ implies $\mathsf{u}|_{\lambda_1-1} \trianglerighteq \mathsf{t}^{\sigma|_{\lambda_1-1}}$. Then the unique choice for u is $\mathsf{u} = \mathsf{t}^\sigma$. Hence

$$\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-2}\psi_{n-3}\dots\psi_{\lambda_1}e_\sigma y_\sigma = c\cdot e_\sigma y_\sigma + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\sigma)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}}.$$

Further more if u is a standard tableau with Shape(u) $> \sigma$ and res(u) $= \mathbf{i}_\sigma$, we must have Shape(u) $> \lambda|_{n-1}$. Hence by Lemma 2.1.14,

$$\psi_{n-1}\theta_{i-1}(\sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\sigma)} c_{\mathsf{uv}}\psi_{\mathsf{uv}})\psi_{n-1} \in R_n^{>\lambda}.$$

Therefore

$$\begin{aligned}\psi_{n-1}\theta_{i-1}(\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-2}\psi_{n-3}\dots\psi_{\lambda_1}e_\sigma y_\sigma)\psi_{n-1} &=_\lambda c\cdot\psi_{n-1}\theta_{i-1}(e_\sigma y_\sigma)\psi_{n-1}\\ &= c\cdot\psi_{n-1}^2 e_\mu y_\mu = c\cdot(e_\lambda y_\lambda - e_\mu y_\mu y_{n-1}).\end{aligned}$$

By Proposition 3.2.5 we have $e_\mu y_\mu y_{n-1} \in R_n^{>\lambda}$, we have

$$\psi_{n-1}\theta_{i-1}(\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-2}\psi_{n-3}\dots\psi_{\lambda_1}e_\sigma y_\sigma)\psi_{n-1} =_\lambda c\cdot e_\lambda y_\lambda. \tag{3.4.17}$$

For the righthand term in (3.4.16), as $\lambda \in \mathscr{S}_n^\Lambda$, $\lambda|_{n-1} \in \mathscr{S}_{n-1}^\Lambda \cap (\mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda)$. By Lemma 3.3.2,

$$\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-3}\dots\psi_{\lambda_1}e_{\lambda|_{n-1}}y_{\lambda|_{n-1}} =_{\lambda|_{n-1}} \sum_{\dot{\mathsf{u}}\in\mathrm{Std}(\lambda|_{n-1})} c_{\dot{\mathsf{u}}\mathsf{t}^{\lambda|_{n-1}}}\psi_{\dot{\mathsf{u}}\mathsf{t}^{\lambda|_{n-1}}}.$$

Then by Lemma 2.1.14,

$$\begin{aligned}\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-3}\dots\psi_{\lambda_1}e_\mu y_\mu &= \theta_i(\psi_{\lambda_1}\dots\psi_{n-3}\psi_{n-3}\dots\psi_{\lambda_1}e_{\lambda|_{n-1}}y_{\lambda|_{n-1}})\\ &=_\lambda \sum_{\dot{\mathsf{u}}\in\mathrm{Std}(\lambda|_{n-1})} c_{\dot{\mathsf{u}}\mathsf{t}^{\lambda|_{n-1}}}\theta_i(\psi_{\dot{\mathsf{u}}\mathsf{t}^{\lambda|_{n-1}}})\\ &= \sum_{\mathsf{u}\in\mathrm{Std}(\mu)} c_{\dot{\mathsf{u}}\mathsf{t}^{\lambda|_{n-1}}}\psi_{\mathsf{u}\mathsf{t}^\mu},\end{aligned} \tag{3.4.18}$$

where u is the unique μ-tableau such that $\mathsf{u}|_{n-1} = \dot{\mathsf{u}}$.

So substitute (3.4.17) and (3.4.18) to (3.4.16), we have

$$\psi_{\lambda_1,n}\psi_{n-1,\lambda_1}e_\mu y_\mu =_\lambda \sum_{\mathsf{u}\in\mathrm{Std}(\mu)} c_{\mathsf{u}\mathsf{t}^\mu}\psi_{\mathsf{u}\mathsf{t}^\mu} \pm c\cdot\psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda}.$$

As $\dot{\mathsf{v}}$ is a standard tableau of shape $\lambda|_{n-1} = \mu|_{n-1}$, we can define v_1 and v_2 to be a standard μ-tableau and λ-tableau where $\mathsf{v}_1|_{n-1} = \mathsf{v}_2|_{n-1} = \dot{\mathsf{v}}$, respectively. Henceforth $d(\mathsf{v}_1) = d(\mathsf{v}_2) = d(\dot{\mathsf{v}})$ and by (3.4.15),

$$\begin{aligned}\psi_{\lambda_1,n}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) &= \psi_{\lambda_1,n}\psi_{n-1,\lambda_1}e_\mu y_\mu\psi_{d(\dot{\mathsf{v}})} =_\lambda \sum_{\mathsf{u}\in\mathrm{Std}(\mu)} c_{\mathsf{u}\mathsf{t}^\mu}\psi_{\mathsf{u}\mathsf{t}^\mu}\psi_{d(\dot{\mathsf{v}})} \pm c\cdot\psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda}\psi_{d(\dot{\mathsf{v}})}\\ &= \sum_{\mathsf{u}\in\mathrm{Std}(\mu)} c_{\mathsf{u}\mathsf{t}^\mu}\psi_{\mathsf{u}\mathsf{t}^\mu}\psi_{d(\mathsf{v}_1)} \pm c\cdot\psi_{\mathsf{t}^\lambda\mathsf{t}^\lambda}\psi_{d(\mathsf{v}_2)}\\ &= \sum_{\mathsf{u}\in\mathrm{Std}(\mu)} c_{\mathsf{u}\mathsf{t}^\mu}\psi_{\mathsf{u}\mathsf{v}_1} \pm c\cdot\psi_{\mathsf{t}^\lambda\mathsf{v}_2} \in R_n^{\geq\mu}.\end{aligned}$$

Therefore,

$$\sum_{\substack{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(\lambda|_{n-1})\\ (\dot{\mathsf{u}},\dot{\mathsf{v}})\rhd(\mathsf{t}^\lambda,\dot{\mathsf{t}})}} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\lambda_1,n}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{\geq\mu}. \tag{3.4.19}$$

Finally, suppose $\dot{\mathsf{u}},\dot{\mathsf{v}} \in$ Shape($> \lambda|_{n-1}$), by Lemma 2.1.14, we have $\psi_{\lambda_1,n}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{>\lambda}$. Therefore

$$\sum_{\dot{\mathsf{u}},\dot{\mathsf{v}}\in\mathrm{Std}(>\lambda|_{n-1})} c_{\dot{\mathsf{u}}\dot{\mathsf{v}}}\psi_{\lambda_1,n}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) \in R_n^{>\lambda}. \tag{3.4.20}$$

Substitute (3.4.14), (3.4.19) and (3.4.20) to (3.4.13), we have

$$\psi_{\lambda_1,n}\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r) \in R_n^{\geq\mu}. \tag{3.4.21}$$

For the second term of (3.4.12), by Lemma 3.4.5

$$\begin{aligned}\psi_{\lambda_1,n-1}\theta_i(e_{\dot\mu}y_{\dot\mu}\psi_{d(\dot{\mathfrak{t}})}\psi_r) &= \theta_i(\psi_{\lambda_1,n-1}e_{\dot\mu}y_{\dot\mu}\psi_{d(\dot{\mathfrak{t}})}\psi_r) = \theta_i(e_{\lambda|_{n-1}}y_{\lambda|_{n-1}}\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathfrak{t}})}\psi_r)\\ &= e_\mu y_\mu\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathfrak{t}})}\psi_r,\end{aligned}$$

where by Lemma 3.3.6, because $\psi_{\lambda_1,n_1}\psi_{d(\dot{\mathfrak{t}})}\psi_r$ doesn't involve ψ_{n-1},

$$e_\mu y_\mu\psi_{\lambda_1,n-1}\psi_{d(\dot{\mathfrak{t}})}\psi_r \in R_n^{\geq\mu}. \tag{3.4.22}$$

Therefore substitute (3.4.21) and (3.4.22) to (3.4.12), we have

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\mu}.$$

Then by Proposition 1.4.9 the proof is completed. □

3.4.23. **Example** We give an example of Case 3.4.10c. Suppose $\lambda = (7,4)$, $e = 4$ and $\Lambda = \Lambda_0$. Therefore $i = j = 3$ and

$$\mathfrak{t} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&7&9&10&11\\\hline 4&5&6&8\\\cline{1-4}\end{array} \qquad \mathfrak{t}^\lambda = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7\\\hline 8&9&10&11\\\cline{1-4}\end{array},$$

with $d(\mathfrak{t}) = s_7s_8s_9s_{10}s_6s_7s_8s_9s_5s_6s_7s_8s_4s_5s_6$ and $r = 7$.

By Lemma 3.4.5 we have

$$\begin{aligned}e_\lambda y_\lambda\psi_7\psi_8\psi_9\psi_{10} &= e(01230123012)y_4y_{11}\psi_7\psi_8\psi_9\psi_{10}\\ &= \psi_7\psi_8\psi_9\psi_{10}e(01230130122)y_4y_{11} + \psi_7\psi_8\psi_9e(01230130122)y_4\\ &= \psi_7\psi_8\psi_9\psi_{10}e(\mathbf{i}_{\dot\lambda}\vee i)y_{\dot\lambda} + \psi_7\psi_8\psi_9e(\mathbf{i}_{\dot\mu}\vee i)y_{\dot\mu}\\ &= \psi_7\psi_8\psi_9\psi_{10}\theta_i(e(\mathbf{i}_{\dot\lambda})y_{\dot\lambda}) + \psi_7\psi_8\psi_9\theta_i(e(\mathbf{i}_{\dot\mu})y_{\dot\mu}),\end{aligned}$$

where $\dot\lambda = (6,4)$ and $\dot\mu = (6,3,1)$. Therefore

$$\dot{\mathfrak{t}} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&7&9&10\\\hline 4&5&6&8\\\cline{1-4}\end{array}$$

and $d(\dot{\mathfrak{t}}) = s_6s_7s_8s_9s_5s_6s_7s_8s_4s_5s_6$, which indicates

$$\begin{aligned}\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r &= e_\lambda y_\lambda\psi_7\psi_8\psi_9\psi_{10}\psi_6\psi_7\psi_8\psi_9\psi_5\psi_6\psi_7\psi_8\psi_4\psi_5\psi_6\psi_7\\ &= \psi_7\psi_8\psi_9\psi_{10}\theta_i(e(\mathbf{i}_{\dot\lambda})y_{\dot\lambda})\psi_6\psi_7\psi_8\psi_9\psi_5\psi_6\psi_7\psi_8\psi_4\psi_5\psi_6\psi_7\\ &\quad +\psi_7\psi_8\psi_9\theta_i(e(\mathbf{i}_{\dot\mu})y_{\dot\mu})\psi_6\psi_7\psi_8\psi_9\psi_5\psi_6\psi_7\psi_8\psi_4\psi_5\psi_6\psi_7\\ &= \psi_7\psi_8\psi_9\psi_{10}\theta_i(e(\mathbf{i}_{\dot\lambda})y_{\dot\lambda})\psi_{d(\dot{\mathfrak{t}})}\psi_7 + \psi_7\psi_8\psi_9\theta_i(e(\mathbf{i}_{\dot\mu})y_{\dot\mu})\psi_{d(\dot{\mathfrak{t}})}\psi_7\\ &= \psi_{7,11}\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_7) + \psi_{7,10}\theta_i(e(\mathbf{i}_{\dot\mu})y_{\dot\mu}\psi_{d(\dot{\mathfrak{t}})}\psi_7).\end{aligned} \tag{3.4.24}$$

For the first term of (3.4.24),

$$\begin{aligned}\psi_{7,11}\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r) &= \sum_{\substack{\dot{\mathfrak{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathfrak{v}}\rhd\dot{\mathfrak{t}}}} c_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}\psi_{7,11}\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}) + \sum_{\substack{\dot{\mathfrak{u}},\dot{\mathfrak{v}}\in\mathrm{Std}(\lambda|_{n-1})\\ (\dot{\mathfrak{u}},\dot{\mathfrak{v}})\rhd(\mathfrak{t}^{\dot\lambda},\dot{\mathfrak{t}})}} c_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}\psi_{7,11}\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}})\\ &\quad + \sum_{\dot{\mathfrak{u}},\dot{\mathfrak{v}}\in\mathrm{Std}(>\lambda|_{n-1})} c_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}\psi_{7,11}\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}).\end{aligned} \tag{3.4.25}$$

For $\dot{\mathfrak{v}} \in \mathrm{Std}(\dot\lambda)$ with $\dot{\mathfrak{v}} \rhd \dot{\mathfrak{t}}$, by Lemma 3.4.7 we have

$$\sum_{\substack{\dot{\mathfrak{v}}\in\mathrm{Std}(\dot\lambda)\\ \dot{\mathfrak{v}}\rhd\dot{\mathfrak{t}}}} c_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}\psi_{7,11}\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{v}}}) \in R_n^{\geq\mu}. \tag{3.4.26}$$

For $\dot{\mathsf{u}}, \dot{\mathsf{v}} \in \mathrm{Std}(\lambda|_{n-1})$ with $(\dot{\mathsf{u}}, \dot{\mathsf{v}}) \rhd (\mathsf{t}^{\dot\lambda}, \mathsf{t})$, then $\mathrm{res}(\dot{\mathsf{u}}) = \mathbf{i}_{\dot\lambda} = 0123013012$, and because $\mathrm{Shape}(\dot{\mathsf{u}}) = \lambda|_{n-1} = (7,3)$ with residues

$$\begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&2&3&0&1&2\\ \hline 3&0&1\\ \cline{1-3}\end{array}$$

and

$$\dot{\mathsf{u}} \rhd \mathsf{t}^{\dot\lambda} = \begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\ \hline 7&8&9&10\\ \cline{1-4}\end{array}.$$

The only possible choice of $\dot{\mathsf{u}}$ is

$$\dot{\mathsf{u}} = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&10\\ \hline 7&8&9\\ \cline{1-3}\end{array},$$

with $d(\dot{\mathsf{u}}) = s_7 s_8 s_9 = \psi_{\lambda_1, n-1}$. Hence

$$\psi_{7,11}\theta_i(\psi_{\dot{\mathsf{u}}\dot{\mathsf{v}}}) = \psi_7\psi_8\psi_9\psi_{10}\psi_9\psi_8\psi_7 e(0123012301) y_4 \psi_{d(\dot{\mathsf{v}})}. \tag{3.4.27}$$

Notice we have

$$\begin{aligned}
&\psi_7\psi_8\psi_9\psi_{10}\psi_9\psi_8\psi_7 e(01230123012) y_4\\
=\;& \psi_7\psi_8\psi_9\psi_{10}\psi_9 e(01230130212)\psi_8\psi_7 y_4\\
=\;& \psi_7\psi_8\psi_{10}\psi_9\psi_{10} e(01230130212)\psi_8\psi_7 y_4 - \psi_7\psi_8 e(01230130212)\psi_8\psi_7 y_4\\
=\;& \psi_7\psi_8\psi_{10}\psi_9\psi_{10}\psi_8\psi_7 e(01230123012) y_4 - \psi_7\psi_8\psi_8\psi_7 e(01230123012) y_4\\
=\;& \psi_{10}\psi_7\psi_8\psi_9\psi_8\psi_7 e(01230123021) y_4 \psi_{10} - \psi_7\psi_8\psi_8\psi_7 e(01230123012) y_4\\
=\;& \psi_{10}\theta_1(\psi_7\psi_8\psi_9\psi_8\psi_7 e(0123012302) y_4)\psi_{10} - \psi_7\psi_8\psi_8\psi_7 e(01230123012) y_4\\
=\;& \psi_{10}\theta_1(\psi_7\psi_8\psi_9\psi_8\psi_7 e_\sigma y_\sigma)\psi_{10} - \psi_7\psi_8\psi_8\psi_7 e_\mu y_\mu, \qquad (3.4.28)
\end{aligned}$$

where $\sigma = (7,2,1)$. Consider the left term of (3.4.28), because $|\sigma| < |\lambda|$ and $\lambda \in \mathscr{S}_n^\Lambda$, we have

$$\psi_7\psi_8\psi_9\psi_8\psi_7 e_\sigma y_\sigma = \sum_{\mathsf{u}\in\mathrm{Std}(\sigma)} c_{\mathsf{u}\mathsf{t}^\sigma}\psi_{\mathsf{u}\mathsf{t}^\sigma} + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\sigma)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}}.$$

For $\mathsf{u} \in \mathrm{Std}(\sigma)$, by Lemma 3.3.1 and $\psi_7\psi_8\psi_9\psi_8\psi_7$ doesn't involve ψ_s with $s \le 6$, we have $u|_6 \rhd \mathsf{t}^\sigma|_6$. Then because $\mathrm{res}(\mathsf{u}) = \mathbf{i}_\sigma \cdot s_7 s_8 s_9 s_8 s_7 = 0123012302$, by the definition of σ

$$[\sigma] = \begin{array}{|c|c|c|c|c|c|c|}\hline \ &\ &\ &\ &\ &\ &\ \\ \hline \ &\ \\ \cline{1-2} \ \\ \cline{1-1}\end{array} \quad \text{with residues} \quad \begin{array}{|c|c|c|c|c|c|c|}\hline 0&1&2&3&0&1&2\\ \hline 3&0\\ \cline{1-2} 2\\ \cline{1-1}\end{array}.$$

Then the only possible choice of u is $\mathsf{t}^\sigma = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&5&6&7\\ \hline 8&9\\ \cline{1-2} 10\\ \cline{1-1}\end{array}$. Hence

$$\psi_7\psi_8\psi_9\psi_8\psi_7 e_\sigma y_\sigma = c\cdot\psi_{\mathsf{t}^\sigma\mathsf{t}^\sigma} + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\sigma)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}}.$$

For $\mathsf{u},\mathsf{v} \in \mathrm{Std}(>\sigma)$, we have $\mathrm{res}(\mathsf{u}) = \mathbf{i}_\sigma = 0123012302$. It is impossible that $\mathrm{Shape}(\mathsf{u}) = \lambda|_{n-1}$ because $\mathbf{i}_{\lambda|_{n-1}} = 0123012301$. Hence $\sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\sigma)} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}} = \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\lambda|_{n-1})} c_{\mathsf{u}\mathsf{v}}\psi_{\mathsf{u}\mathsf{v}} \in R_n^{>\lambda|_{n-1}}$. So

$\psi_7\psi_8\psi_9\psi_8\psi_7 e_\sigma y_\sigma = c\cdot\psi_{\mathfrak{t}^\sigma\mathfrak{t}^\sigma} + R_n^{>\lambda|_{n-1}}$ and hence by Lemma 2.1.14,

$$\begin{aligned}\psi_{10}\theta_1(\psi_7\psi_8\psi_9\psi_8\psi_7 e_\sigma y_\sigma)\psi_{10} &= c\cdot\psi_{10}\theta_1(\psi_{\mathfrak{t}^\sigma\mathfrak{t}^\sigma})\psi_{10} + \psi_{10}\theta_1(R_n^{>\lambda|_{n-1}})\psi_{10}\\ &= c\cdot\psi_{10}e(01230123021)y_4\psi_{10} + R_n^{>\lambda}\\ &=_\lambda c\cdot e(01230123012)y_4\psi_{10}^2\\ &= c\cdot e(01230123012)y_4y_{10} - c\cdot e(01230123012)y_4y_9\\ &=_\lambda c\cdot e(01230123012)y_4y_{10} = c\cdot e_\lambda y_\lambda. \end{aligned} \tag{3.4.29}$$

For the right term of (3.4.28), because $\lambda|_{n-1} \in \mathscr{S}_{n-1}^\Lambda \cap (\mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda)$, by Lemma 3.3.2 we have

$$\psi_7\psi_8\psi_8\psi_7 e_{\lambda|_{n-1}} y_{\lambda|_{n-1}} = \sum_{\dot{\mathfrak{u}}\in\mathrm{Std}(\lambda|_{n-1})} c_{\dot{\mathfrak{u}}\mathfrak{t}^{\lambda|_{n-1}}}\psi_{\dot{\mathfrak{u}}\mathfrak{t}^{\lambda|_{n-1}}} + R_n^{>\lambda|_{n-1}}.$$

Then by Lemma 2.1.14,

$$\begin{aligned}\psi_7\psi_8\psi_8\psi_7 e_\mu y_\mu &= \theta_2(\psi_7\psi_8\psi_8\psi_7 e_{\lambda|_{n-1}} y_{\lambda|_{n-1}})\\ &= \sum_{\dot{\mathfrak{u}}\in\mathrm{Std}(\lambda|_{n-1})} c_{\dot{\mathfrak{u}}\mathfrak{t}^{\lambda|_{n-1}}}\theta_2(\psi_{\dot{\mathfrak{u}}\mathfrak{t}^{\lambda|_{n-1}}}) + \theta_2(R_n^{>\lambda|_{n-1}})\\ &= \sum_{\mathfrak{u}\in\mathrm{Std}(\mu)} c_{\dot{\mathfrak{u}}\mathfrak{t}^{\lambda|_{n-1}}}\psi_{\mathfrak{u}\mathfrak{t}^\mu} + R_n^{>\lambda}. \end{aligned} \tag{3.4.30}$$

Substitute (3.4.29) and (3.4.30) back to (3.4.28), we have

$$\psi_7\psi_8\psi_9\psi_{10}\psi_9\psi_8\psi_7 e(01230123012)y_4 = \sum_{\mathfrak{u}\in\mathrm{Std}(\mu)} c_{\mathfrak{u}\mathfrak{t}^\mu}\psi_{\mathfrak{u}\mathfrak{t}^\mu} + c\cdot e_\lambda y_\lambda + R_n^{>\lambda}.$$

Recall $\dot{\mathfrak{v}}$ is a standard tableau of shape $\lambda|_{n-1} = \mu|_{n-1}$, we can define $\mathfrak{v}_1 \in \mathrm{Std}(\mu)$ and $\mathfrak{v}_2 \in \mathrm{Std}(\lambda)$ such that $d(\mathfrak{v}_1) = d(\mathfrak{v}_2) = d(\dot{\mathfrak{v}})$. Hence by (3.4.27),

$$\begin{aligned}\psi_{7,11}\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) &= \psi_7\psi_8\psi_9\psi_{10}\psi_9\psi_8\psi_7 e(0123012301)y_4\psi_{d(\dot{\mathfrak{v}})}\\ &= \sum_{\mathfrak{u}\in\mathrm{Std}(\mu)} c_{\mathfrak{u}\mathfrak{v}_1}\psi_{\mathfrak{u}\mathfrak{v}_1} + c\cdot\psi_{\mathfrak{t}^\lambda\mathfrak{v}_2} + R_n^{>\lambda} \in R_n^{\geq\mu},\end{aligned}$$

which yields

$$\sum_{\substack{\dot{\mathfrak{u}},\dot{\mathfrak{v}}\in\mathrm{Std}(\lambda|_{n-1})\\ (\dot{\mathfrak{u}},\dot{\mathfrak{v}})\rhd(\mathfrak{t}^{\dot\lambda},\dot{\mathfrak{t}})}} c_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}\psi_{7,11}\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) \in R_n^{\geq\mu}. \tag{3.4.31}$$

Finally, suppose $\dot{\mathfrak{u}}, \dot{\mathfrak{v}} \in \mathrm{Shape}(> \lambda|_{n-1})$, by Lemma 2.1.14 we have $\psi_{7,11}\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) \in R_n^{>\lambda}$. Therefore

$$\sum_{\dot{\mathfrak{u}},\dot{\mathfrak{v}}\in\mathrm{Std}(>\lambda|_{n-1})} c_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}\psi_{7,11}\theta_i(\psi_{\dot{\mathfrak{u}}\dot{\mathfrak{v}}}) \in R_n^{>\lambda}. \tag{3.4.32}$$

Substitute (3.4.26), (3.4.31) and (3.4.32) to (3.4.25), we have

$$\psi_{7,11}\theta_i(\psi_{\mathfrak{t}^{\dot\lambda}\dot{\mathfrak{t}}}\psi_r) \in R_n^{\geq\mu}. \tag{3.4.33}$$

For the second term of (3.4.24), by Lemma 3.4.5

$$\begin{aligned}\psi_{7,10}\theta_i(e(\mathbf{i}_{\dot\mu})y_{\dot\mu}\psi_{d(\dot{\mathfrak{t}})}\psi_7) &= \theta_2(\psi_7\psi_8\psi_9 e(0123013012)y_4\psi_{d(\dot{\mathfrak{t}})}\psi_7)\\ &= \theta_2(e(0123012301)y_4\psi_7\psi_8\psi_9\psi_{d(\dot{\mathfrak{t}})}\psi_7)\\ &= e(01230123012)y_4\psi_7\psi_8\psi_9\psi_{d(\dot{\mathfrak{t}})}\psi_7 = e_\mu y_\mu\psi_{7,10}\psi_{d(\dot{\mathfrak{t}})}\psi_r.\end{aligned}$$

Then by Lemma 3.3.6, because $\psi_{7,10}\psi_{d(\mathfrak{t})}\psi_r$ doesn't involve ψ_{10}, we have $e_\mu y_\mu \psi_{7,10}\psi_{d(\mathfrak{t})}\psi_r \in R_n^{\geq\mu}$. Therefore

$$\psi_{7,10}\theta_i(e(\mathbf{i}_{\dot{\mu}})y_{\dot{\mu}}\psi_{d(\mathfrak{t})}\psi_7) \in R_n^{\geq\mu}. \tag{3.4.34}$$

Substitute (3.4.33) and (3.4.34) to (3.4.24), we have $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\mu}$. Finally by Proposition 1.4.9, we have

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathfrak{t}^\lambda,\mathfrak{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}.$$

◇

Finally, we can extend the above Proposition to arbitrary multipartition using similar method we used in the last chapter.

3.4.35. Corollary. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$ and $\mathfrak{t}$ is the last Garnir tableau of shape λ with r the last Garnir entry and $l(d(\mathfrak{t})) \leq m_\lambda$. Therefore for any standard λ-tableau s, $\psi_{\mathsf{st}}\psi_r = \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathfrak{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}$.*

Proof. Write $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \ldots, \lambda_k^{(\ell)})$. If $\lambda^{(\ell)} = \emptyset$, then define $\bar{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(\ell-1)})$. As $l(\bar{\lambda}) = \ell - 1 < l(\lambda)$, we have $\bar{\lambda} \in \mathscr{P}_I^{\bar{\Lambda}} \cap \mathscr{P}_y^{\bar{\Lambda}} \cap \mathscr{P}_\psi^{\bar{\Lambda}}$. By Proposition 2.3.14, $\lambda \in \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$, and the Corollary follows.

Now suppose $\lambda^{(\ell)} \neq \emptyset$, First we assume $\mathsf{s} = \mathfrak{t}^\lambda$. As $\mathfrak{t}$ is the last Garnir tableau of shape λ, $k \geq 2$. Setting $m = \lambda_{k-1}^{(\ell)} + \lambda_k^{(\ell)}$. As $\mathfrak{t}$ is the last Garnir tableau, by the definition we can see that $\mathfrak{t}|_{n-m} = \mathfrak{t}^\lambda|_{n-m}$ and $k \geq 2$. Define i to be the residue of the node $(k-1, 1, \ell)$, $\Lambda' = \Lambda_i$, and $\tilde{\mathfrak{t}}$ to be the last Garnir tableau of shape $(\lambda_{k-1}^{(\ell)}, \lambda_k^{(\ell)})$. If we write $\mu = (\lambda^{(1)}, \ldots, \lambda^{(\ell-1)}, \mu^{(\ell)})$ with $\mu^{(\ell)} = (\lambda_1^{(\ell)}, \ldots, \lambda_{k-2}^{(\ell)})$ and $\gamma = (\lambda_{k-1}^{(\ell)}, \lambda_k^{(\ell)})$, then

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathfrak{t}^\gamma\tilde{\mathfrak{t}}}\hat{\psi}_{r-(n-m)})y_\mu.$$

Recall that $\hat{\psi}_{\mathfrak{t}^\gamma\tilde{\mathfrak{t}}}$ and $\hat{\psi}_{r-(n-m)}$ are elements of $\mathscr{R}_m$ and $\psi_{\mathfrak{t}^\gamma\tilde{\mathfrak{t}}}$ and $\psi_{r-(n-m)}$ are elements of $\mathscr{R}_m^{\Lambda'}$. Then by Proposition 3.4.10, we have $\psi_{\mathfrak{t}^\gamma\tilde{\mathfrak{t}}}\psi_{r-(n-m)} \in R_n^{\geq\gamma}$. Therefore we can write $\psi_{\mathfrak{t}^\gamma\tilde{\mathfrak{t}}}\psi_{r-(n-m)} = \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(\gamma)} c_{\mathsf{uv}}\psi_{\mathsf{uv}} + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\gamma)} c_{\mathsf{uv}}\psi_{\mathsf{uv}}$ and hence $\hat{\psi}_{\mathfrak{t}^\gamma\tilde{\mathfrak{t}}}\hat{\psi}_{r-(n-m)} = \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(\gamma)} c_{\mathsf{uv}}\hat{\psi}_{\mathsf{uv}} + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\gamma)} c_{\mathsf{uv}}\hat{\psi}_{\mathsf{uv}} + r$ where $r \in N_m^{\Lambda'}$. Therefore

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(\gamma)} c_{\mathsf{uv}}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}})y_\mu + \sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\gamma)} c_{\mathsf{uv}}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}})y_\mu + \hat{\theta}_{\mathbf{i}_\mu}(r)y_\mu.$$

For $\mathsf{u},\mathsf{v} \in \mathrm{Std}(\gamma)$, by Corollary 2.3.8 we have $\hat{\theta}_{\mathbf{i}_\mu}(\psi_{\mathsf{uv}}) \in R_n^{\geq\lambda}$. Hence $\sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(\gamma)} c_{\mathsf{uv}}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}})y_\mu \in R_n^{\geq\lambda}$.

For $\mathsf{u},\mathsf{v} \in \mathrm{Std}(>\gamma)$, write $\mathrm{Shape}(\mathsf{u}) = \mathrm{Shape}(\mathsf{v}) = \sigma$ and $\nu = \mu \vee \sigma$. By Corollary 2.3.6 we have $\nu > \lambda = \mu \vee \gamma$. Then by Corollary 2.3.15 and Lemma 2.1.14, $\hat{\theta}_{\mathbf{i}_\mu}(\psi_{\mathsf{uv}}) \in R_n^{>\lambda}$. Hence $\sum_{\mathsf{u},\mathsf{v}\in\mathrm{Std}(>\gamma)} c_{\mathsf{uv}}\hat{\theta}_{\mathbf{i}_\mu}(\hat{\psi}_{\mathsf{uv}})y_\mu \in R_n^{>\lambda}$.

Finally by Lemma 2.3.4, $\hat{\theta}_{\mathbf{i}_\mu}(r)y_\mu \in R_n^{>\lambda}$. These yield that

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r = \hat{\theta}_{\mathbf{i}_\mu}(\psi_{\mathfrak{t}^\gamma\tilde{\mathfrak{t}}}\psi_{r-(n-m)})y_\mu \in R_n^{\geq\lambda}.$$

Now choose any $\mathsf{s} \in \mathrm{Std}(\lambda)$. Because $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$, we have

$$\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r =_\lambda \sum_{\mathsf{v}\in\mathrm{Std}(\lambda)} c_{\mathfrak{t}^\lambda\mathsf{v}}\psi_{\mathfrak{t}^\lambda\mathsf{v}}.$$

Hence

$$\psi_{\mathsf{st}}\psi_r = \psi_{d(\mathsf{s})}^*\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r =_\lambda \sum_{\mathsf{v}\in\mathrm{Std}(\lambda)} c_{\mathfrak{t}^\lambda\mathsf{v}}\psi_{d(\mathsf{s})}^*\psi_{\mathfrak{t}^\lambda\mathsf{v}} = \sum_{\mathsf{v}\in\mathrm{Std}(\lambda)} c_{\mathfrak{t}^\lambda\mathsf{v}}\psi_{\mathsf{sv}}.$$

Therefore, $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. By Proposition 1.4.9 we completes the proof. □

3.4.36. Remark. Generally it is not easy to find c_{uv}. Kleshchev-Mathas-Ram [**14**] explicitly describes how to compute c_{uv} where $\mathrm{Shape}(\mathsf{u}) = \mathrm{Shape}(\mathsf{v}) = \lambda$. This thesis also gives an implicit method to compute these coefficients.

3.5. Completion of the ψ-problem

In this section we are going to prove that $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. We have claimed in Corollary 3.2.3 that if $\mathsf{t}\cdot s_r$ is standard and $d(\mathsf{t})\cdot s_r$ is reduced then the argument is true. So we will mainly consider that $\mathsf{t}\cdot s_r$ is not standard or $d(\mathsf{t})\cdot s_r$ is not reduced.

In the last section we have proved that when t is a Garnir-tableau then $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. In the beginning of this section we are going to consider some other cases of t. Then we will give some information about using $d(\mathsf{t})$ to determine the type of t, and use them to prove that $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$ for any $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$.

First we introduce two more types of t and r.

3.5.1. Lemma. *Suppose* t *and* s *are two standard* λ*-tableaux with* $d(\mathsf{t}) = d(\mathsf{s})\cdot s_k$ *for some* k *and* $l(d(\mathsf{t})) = l(d(\mathsf{s})) + 1$. *If for some* $r \notin \{k-1, k, k+1\}$, $\mathsf{t}\cdot s_r$ *is not standard or* $d(\mathsf{t})\cdot s_r$ *is not reduced, then* $\mathsf{s}\cdot s_r$ *is not standard or* $d(\mathsf{s})\cdot s_r$ *is not standard, respectively.*

Proof. When $\mathsf{t}\cdot s_r$ is not standard, we can see that r and $r+1$ in t are adjacent, either in the same row or in the same column. Since $d(\mathsf{t}) = d(\mathsf{s})\cdot s_k$, we have $\mathsf{s} = \mathsf{t}\cdot s_k$. As $r \notin \{k-1, k, k+1\}$, r and $r+1$ are in the same nodes in t as in s. Hence $\mathsf{s}\cdot s_r$ is not standard as well.

When $d(\mathsf{t})\cdot s_r$ is not reduced, we could see that $d(\mathsf{t})(r) > d(\mathsf{t})(r+1)$. As $d(\mathsf{t}) = d(\mathsf{s})\cdot s_k$ we have $d(\mathsf{t})(r) = d(\mathsf{s})\cdot s_k(r) = d(\mathsf{s})(r)$ and $d(\mathsf{t})(r+1) = d(\mathsf{s})\cdot s_k(r+1) = d(\mathsf{s})(r+1)$ as $r \notin \{k-1, k, k+1\}$. Hence $d(\mathsf{s})(r) > d(\mathsf{s})(r+1)$. Therefore $d(\mathsf{s})\cdot s_r$ is not reduced. This completes the proof. □

3.5.2. **Example** Suppose $\mathsf{t} =$

1	2	7	9
3	5	8	
4	6	10	

and $\mathsf{s} =$

1	2	7	8
3	4	9	
5	6	10	

. We have

$$\begin{aligned} d(\mathsf{t}) &= s_4s_5s_6s_7s_8s_6s_7s_3s_4s_5s_6s_4, \\ d(\mathsf{s}) &= s_7s_8s_4s_5s_6s_7s_3s_4s_5s_6s_4. \end{aligned}$$

Set $k = 8$, we have $d(\mathsf{t}) = d(\mathsf{s})\cdot s_8$ and $l(d(\mathsf{t})) = l(d(\mathsf{s})) + 1$. Let $r = 3$, i.e. $r \notin \{k-1, k, k+1\}$ and $\mathsf{t}\cdot s_r$ is not standard. We can see that $\mathsf{s}\cdot s_r$ is not standard either. Similarly, let $r = 6$, i.e. $r \notin \{k-1, k, k+1\}$ and $d(\mathsf{t})\cdot s_r$ is not reduced. We can see that $d(\mathsf{s})\cdot s_r$ is not reduced either.

◇

3.5.3. Definition. *Suppose* t *is a standard* λ*-tableau. If we can find a reduced expression* $s_{r_1}s_{r_2}\dots s_{r_l}$ *of* $d(\mathsf{t})$ *and* $1 \leq r \leq n-1$ *such that* $|r - r_l| > 1$, *we say* t *is* ***unlocked by*** s_r.

3.5.4. Lemma. *Suppose* $\lambda \in \mathscr{P}_n^{\Lambda}$ *and* t *is a standard* λ*-tableau with* $l(d(\mathsf{t})) \leq m_\lambda$. *If* t *is unlocked by* s_r, *then* $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$ *for any standard* λ*-tableau* s.

Proof. Suppose $\mathsf{t}\cdot s_r$ is standard and $d(\mathsf{t})\cdot s_r$ is reduced, by Corollary 3.2.3, $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$.

Suppose $\mathsf{t}\cdot s_r$ is not standard or $d(\mathsf{t})\cdot s_r$ is not reduced. Since t is a standard λ-tableau unlocked by s_r, by Definition 3.5.3, we can find a reduced expression $s_{r_1}s_{r_2}\dots s_{r_l}$ of $d(\mathsf{t})$ such that $|r - r_l| > 1$. Define $\mathsf{w} = \mathsf{t}^\lambda\cdot s_{r_1}s_{r_2}\dots s_{r_{l-1}}$. By Lemma 1.3.4, w is a standard λ-tableau. It is easy to see that $d(\mathsf{t}) = d(\mathsf{w})\cdot s_{r_l}$ and $l(d(\mathsf{t})) = l(d(\mathsf{w})) + 1$. Hence by Lemma 3.5.1, $\mathsf{w}\cdot s_r$ is not standard or $d(\mathsf{w})\cdot s_r$ is not reduced. So $\psi_{\mathsf{sw}}\psi_r =_\lambda \sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda) \\ \mathsf{v}\rhd\mathsf{w}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}$ because $l(d(\mathsf{w})) = l(d(\mathsf{t})) - 1 < w_\lambda$.

Because it is obvious that $d(\mathsf{w})\cdot s_{r_l}$ is a reduced expression of $d(\mathsf{t})$, say $d'(\mathsf{t}) = d(\mathsf{w})\cdot s_{r_l}$, by Lemma 3.2.2, we have

$$\sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda)\\ \mathsf{v}\rhd\mathsf{t}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}\psi_r =_\lambda \psi_{\mathsf{st}}\psi_r - \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{w})}\psi_{r_l}\psi_r = \psi_{\mathsf{st}}\psi_r - \psi_{\mathsf{sw}}\psi_r\psi_{r_l}.$$

Because $\mathsf{v} \rhd \mathsf{t}$, $l(d(\mathsf{v})) < l(d(\mathsf{t})) \leq m_\lambda$ and hence we have $\sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda)\\ \mathsf{v}\rhd\mathsf{t}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}\psi_r \in R_n^{\geq\lambda}$. For $\psi_{\mathsf{sw}}\psi_r\psi_{r_l} =_\lambda \sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda)\\ \mathsf{v}\rhd\mathsf{w}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}\psi_{r_l}$, because $\mathsf{v} \rhd \mathsf{w}$, we have $l(d(\mathsf{v})) < l(d(\mathsf{w})) < m_\lambda$, which yields that $\psi_{\mathsf{sw}}\psi_r\psi_{r_l} \in R_n^{\geq\lambda}$. Therefore we have $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. □

3.5.5. Lemma. *Suppose* t *is a standard* λ*-tableau and that we have a standard* λ*-tableau* w *such that* $d(\mathsf{t}) = d(\mathsf{w})\cdot s_r s_{r+1}$ *for some* r *and* $l(d(\mathsf{t})) = l(d(\mathsf{w})) + 2$*. If* $\mathsf{t}\cdot s_r$ *is not standard or* $d(\mathsf{t})\cdot s_r$ *is not reduced, then* $\mathsf{w}\cdot s_{r+1}$ *is not standard or* $d(\mathsf{w})\cdot s_{r+1}$ *is not reduced, respectively.*

Similarly suppose $d(\mathsf{t}) = d(\mathsf{w})\cdot s_r s_{r-1}$ *for some* r *and* $l(d(\mathsf{t})) = l(d(\mathsf{w})) + 2$*. If* $\mathsf{t}\cdot s_r$ *is not standard or* $d(\mathsf{t})\cdot s_r$ *is not reduced, then* $\mathsf{w}\cdot s_{r-1}$ *is not standard or* $d(\mathsf{w})\cdot s_{r-1}$ *is not reduced, respectively.*

Proof. Suppose $d(\mathsf{t}) = d(\mathsf{w})\cdot s_r s_{r+1}$. If $\mathsf{t}\cdot s_r$ is not standard, r and $r+1$ are adjacent in t. But r and $r+1$ occupy the same positions as $r+1$ and $r+2$, respectively in w. So $\mathsf{w}\cdot s_{r+1}$ is not standard. If $d(\mathsf{t})\cdot s_r$ is not reduced, as $d(\mathsf{w})^{-1}(r+1) = d(\mathsf{t})^{-1}(r)$ and $d(\mathsf{w})^{-1}(r+2) = d(\mathsf{t})^{-1}(r+1)$, by Proposition 1.2.9, $d(\mathsf{t})\cdot s_r$ is not reduced implies $d(\mathsf{w})\cdot s_{r+1}$ is not reduced. The other case is similar.

3.5.6. Remark. In Lemma 3.5.1 and Lemma 3.5.5, when we say $d(\mathsf{t}) = d(\mathsf{s})\cdot s_r$ or $d(\mathsf{t}) = d(\mathsf{s})\cdot s_r s_{r+1}$, it means $d(\mathsf{t})$ and $d(\mathsf{s})\cdot s_r$ or $d(\mathsf{t})$ and $d(\mathsf{s})\cdot s_r s_{r+1}$ are the same as permutations.

3.5.7. **Example** Let $\mathsf{t} = \begin{array}{|c|c|c|c|}\hline 1&2&3&12\\\hline 4&5&6&13\\\hline 7&8&11\\\cline{1-3} 9&10&14\\\cline{1-3}\end{array}$. Suppose $\mathsf{s} = \begin{array}{|c|c|c|c|}\hline 1&2&3&12\\\hline 4&5&6&13\\\hline 7&8&9\\\cline{1-3} 10&11&14\\\cline{1-3}\end{array}$, we have

$$\begin{aligned} d(\mathsf{t}) &= s_8s_9s_{10}s_{11}s_{12}s_4s_5s_6s_7s_8s_9s_{10}s_{11}s_9s_{10},\\ d(\mathsf{s}) &= s_8s_9s_{10}s_{11}s_{12}s_4s_5s_6s_7s_8s_9s_{10}s_{11}.\end{aligned}$$

So we have $d(\mathsf{t}) = d(\mathsf{s})s_9s_{10}$ and therefore $r = 9$. We can see that $\mathsf{t}\cdot s_r$ and $\mathsf{s}\cdot s_{r+1}$ are both non-standard.

Suppose $\mathsf{s} = \begin{array}{|c|c|c|c|}\hline 1&2&3&11\\\hline 4&5&6&13\\\hline 7&8&10\\\cline{1-3} 9&12&14\\\cline{1-3}\end{array}$, we have

$$\begin{aligned} d(\mathsf{t}) &= s_8s_9s_{10}s_{11}s_{12}s_4s_5s_6s_7s_8s_9s_{10}s_{11}s_9s_{10},\\ d(\mathsf{s}) &= s_8s_9s_{10}s_{11}s_{12}s_4s_5s_6s_7s_8s_9s_{10}s_9.\end{aligned}$$

So we have $d(\mathsf{t}) = d(\mathsf{s})s_{11}s_{10}$ and therefore $r = 11$. We can see that $d(\mathsf{t})\cdot s_r$ and $d(\mathsf{s})\cdot s_{r-1}$ are both non-reduced because in t, r is below $r+1$ and in s, $r-1$ is below r. ◇

3.5.8. Definition. *Suppose* t *is a standard* λ*-tableau. If we can find a reduced expression* $s_{r_1}s_{r_2}\dots s_{r_{l-1}}s_{r_l}$ *of* $d(\mathsf{t})$ *such that* $r_{l-1} = r$ *and* $r = r_l \pm 1$*, we say* t *is* ***unlocked by*** s_r ***on tails****.*

3.5.9. Lemma. *Suppose* $\lambda \in \mathscr{S}_n^\Lambda$ *and* t *is a standard* λ*-tableau with* $l(d(\mathsf{t})) \leq m_\lambda$*. If* t *is unlocked by* s_r *on tails, then* $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$ *for any standard* λ*-tableau* s*.*

Proof. Suppose $\mathsf{t}\cdot s_r$ is standard and $d(\mathsf{t})\cdot s_r$ is reduced. By Corollary 3.2.3, $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$.

Suppose $\mathsf{t}\cdot s_r$ is not standard or $d(\mathsf{t})\cdot s_r$ is not reduced. By Definition 3.5.8, we can find a reduced expression $s_{r_1}s_{r_2}\ldots s_{r_{l-1}}s_{r_l}$ of $d(\mathsf{t})$ such that $r_{l-1} = r$ and $r = r_l \pm 1$. Without loss of generality we set $r = r_l - 1$. Define $\mathsf{w} = \mathsf{t}^\lambda s_{r_1}s_{r_2}\ldots s_{r_{l-2}}$. By Lemma 1.3.4, w is a standard λ-tableau. It is easy to see that $d(\mathsf{t}) = d(\mathsf{w})\cdot s_r s_{r+1}$ and $l(d(\mathsf{t})) = l(d(\mathsf{w})) + 2$. Hence by Lemma 3.5.5, $\mathsf{w}\cdot s_{r+1}$ is not standard or $d(\mathsf{w})\cdot s_{r+1}$ is not reduced. So $\psi_{\mathsf{sw}}\psi_{r+1} =_\lambda \sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda)\\ \mathsf{v}\rhd\mathsf{w}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}$ because $l(d(\mathsf{w})) = l(d(\mathsf{t})) - 2 < w_\lambda$.

Because it is obvious that $d(\mathsf{w})\cdot s_r s_{r+1}$ is a reduced expression of $d(\mathsf{t})$, say $d'(\mathsf{t}) = d(\mathsf{w})\cdot s_r s_{r+1}$, by Lemma 3.2.2, we have

$$\sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda)\\ \mathsf{v}\rhd\mathsf{t}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}\psi_r =_\lambda \psi_{\mathsf{st}}\psi_r - \psi^*_{d(\mathsf{s})}e_\lambda y_\lambda \psi_{d(\mathsf{w})}\psi_r\psi_{r+1}\psi_r = \psi_{\mathsf{st}}\psi_r - \psi_{\mathsf{sw}}\psi_r\psi_{r+1}\psi_r. \tag{3.5.10}$$

Because $\mathsf{v} \rhd \mathsf{t}$, $l(d(\mathsf{v})) < l(d(\mathsf{t})) \leq m_\lambda$ and hence we have

$$\sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda)\\ \mathsf{v}\rhd\mathsf{t}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}\psi_r \in R_n^{\geq\lambda}. \tag{3.5.11}$$

For $\psi_{\mathsf{sw}}\psi_r\psi_{r+1}\psi_r$, write $\mathrm{res}(\mathsf{w}) = i_1 i_2 \ldots i_n$, the residue sequence of w. We have

$$\psi_{\mathsf{sw}}\psi_r\psi_{r+1}\psi_r = \begin{cases} \psi_{\mathsf{sw}}\psi_{r+1}\psi_r\psi_{r+1} \pm \psi_{\mathsf{sw}}, & \text{if } i_r = i_{r+2} = i_{r+1} \pm 1,\\ \psi_{\mathsf{sw}}\psi_{r+1}\psi_r\psi_{r+1}, & \text{otherwise.}\end{cases}$$

Because $\psi_{\mathsf{sw}}\psi_{r+1} =_\lambda \sum_{\substack{\mathsf{v}\in\mathrm{Std}(\lambda)\\ \mathsf{v}\rhd\mathsf{w}}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}$,

$$\psi_{\mathsf{sw}}\psi_{r+1}\psi_r\psi_{r+1} =_\lambda \sum_{\mathsf{v}\rhd\mathsf{w}} c_{\mathsf{sv}}\psi_{\mathsf{sv}}\psi_r\psi_{r+1}.$$

Since $\mathsf{v}\rhd\mathsf{w}$, $l(d(\mathsf{v})) < l(d(\mathsf{w})) = l(d(\mathsf{t}))-2 \leq m_\lambda - 2$. Hence $l(\psi_{d(\mathsf{v})}\psi_r\psi_{r+1}) = l(d(\mathsf{v}))+2 < m_\lambda$. By Lemma 3.2.1 we have $\psi_{\mathsf{sv}}\psi_r\psi_{r+1} \in R_n^{\geq\lambda}$ if $\mathsf{v} \rhd \mathsf{w}$. Therefore we always have

$$\psi_{\mathsf{sw}}\psi_{r+1}\psi_r\psi_{r+1} \in R_n^{\geq\lambda} \tag{3.5.12}$$

in both cases. Substitute (3.5.11) and (3.5.12) into (3.5.10), we have $\psi_{\mathsf{st}}\psi_r \in R_n^{\geq\lambda}$. □

Now we will introduce some information about using $d(\mathsf{t})$ to determine the type of t.

3.5.13. Lemma. *Suppose* $\mathsf{t} \in \mathrm{Std}(\lambda)$ *with* $d(\mathsf{t}) = s_{n-1}s_{n-2}\ldots s_{r+1}$*, and* $\mathsf{t}\cdot s_r$ *is not standard. Then* t *is the last Garnir tableau with shape* λ*.*

Proof. As $d(\mathsf{t})$ is the standard expression, we have $w_n = s_{n-1}, w_{n-1} = s_{n-2}, \ldots, w_{r+2} = s_{r+1}$ and $\mathsf{t} = \mathsf{t}^{(1)} = \mathsf{t}^{(2)} = \ldots = \mathsf{t}^{(r+1)}$. Write $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda^{(\ell)}_1, \ldots, \lambda^{(\ell)}_k)$, as $\mathsf{t}^{(n)} = \mathsf{t}^\lambda\cdot w_n = \mathsf{t}^\lambda\cdot s_{n-1}$ is standard, $n-1$ and n are not adjacent in t^λ. This forces $\lambda^{(\ell)}_k = 1$.

By Remark 1.3.2 we have

$$\mathsf{t}^{-1}(k) = \begin{cases} (\mathsf{t}^\lambda)^{-1}(k-1), & \text{if } r+2 \leq k \leq n,\\ (\mathsf{t}^\lambda)^{-1}(n), & \text{if } k = r+1,\\ (\mathsf{t}^\lambda)^{-1}(k), & \text{otherwise.}\end{cases}$$

and since $\mathsf{t}\cdot s_r$ is not standard, r and $r+1$ are adjacent in t. As $\mathsf{t}^{-1}(r+1) = (\mathsf{t}^\lambda)^{-1}(n) = (k, \lambda^{(\ell)}_k, \ell) = (k, 1, \ell)$, we must have $\mathsf{t}^{-1}(r) = (k-1, 1, \ell)$. This shows that t is the last Garnir tableau with shape λ. □

3.5.14. **Example** Suppose $\lambda = (4,4,1)$ and $\mathsf{t} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&7&8&9\\ \hline 6 \\ \cline{1-1}\end{array}$. Therefore $d(\mathsf{t}) = s_8 s_7 s_6$ and $\mathsf{t}{\cdot}s_5$ is not standard. Notice that t is the last Garnir tableau of shape λ. ◇

3.5.15. Lemma. *Suppose* $\mathsf{t} \in \mathrm{Std}(\lambda)$ *with* $d(\mathsf{t}) = s_r s_{r+1} \dots s_{n-2}$, *and* $\mathsf{t}{\cdot}s_{n-1}$ *is not standard. Then* t *is the last Garnir tableau with shape* λ.

Proof. As $d(\mathsf{t})$ is the standard expression, we have $w_n = s_r s_{r+1} \dots s_{n-2}$ and $w_{n-1} = \dots w_1 = 1$. Write $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda^{(\ell)}_1, \dots, \lambda^{(\ell)}_k)$, as $\mathsf{t} = \mathsf{t}^\lambda{\cdot}w_n$, we have $\mathsf{t}^{-1}(n) = (k, \lambda^{(\ell)}_k, \ell)$. By Remark 1.3.2 we have

$$\mathsf{t}^{-1}(k) = \begin{cases} (\mathsf{t}^\lambda)^{-1}(k+1), & \text{if } r \le k \le n-2,\\ (\mathsf{t}^\lambda)^{-1}(r), & \text{if } k = n-1,\\ (\mathsf{t}^\lambda)^{-1}(k), & \text{otherwise.} \end{cases}$$

As $\mathsf{t}{\cdot}s_{n-1}$ is not standard, $n-1$ and n are adjacent in t. So in r and n are adjacent in t^λ. But $r \le n-2$. Hence r has to be on the above of n in t^λ. i.e. $(\mathsf{t}^\lambda)^{-1}(r) = \mathsf{t}^{-1}(n-1) = (k-1, \lambda^{(\ell)}_k, \ell)$. This shows that $\lambda^{(\ell)}_{k-1} = \lambda^{(\ell)}_k$ and t is the last Garnir tableau of shape λ. □

3.5.16. **Example** Suppose $\lambda = (4,3,3)$ and $\mathsf{t} = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6&9 \\ \cline{1-3} 7&8&10 \\ \cline{1-3}\end{array}$. Therefore $d(\mathsf{t}) = s_7 s_8$ and $\mathsf{t}{\cdot}s_9$ is not standard. Notice that t is the last Garnir tableau of shape λ. ◇

3.5.17. Lemma. *Suppose* $\mathsf{t} \in \mathrm{Std}(\lambda)$ *and* $d(\mathsf{t}) = w_n w_{n-1} \dots w_1$ *with* $w_i \ne 1$ *if* $i \ge r+2$ *or* $i = r$ *and* $w_i = 1$ *if* $i < r$ *or* $i = r+1$, *i.e.* $d(\mathsf{t}) = w_n w_{n-1} \dots w_{r+2} w_r$. *If* $\mathsf{t}{\cdot}s_r$ *is not standard, then* $l(w_i) \ge l(w_r) + 1$ *for* $i \ge r+2$.

Proof. Write $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda^{(\ell)}_1, \dots, \lambda^{(\ell)}_k)$. Because $\mathsf{t}{\cdot}s_r$ is not standard, r and $r+1$ are adjacent in t. By Remark 1.3.2, as $w_i \ne 1$ for $i \ge r+2$

$$(k, \lambda^{(\ell)}_k, \ell) = (\mathsf{t}^{(n+1)})^{-1}(n) = (\mathsf{t}^{(n)})^{-1}(n-1) = \dots = (\mathsf{t}^{(r+3)})^{-1}(r+2) = (\mathsf{t}^{(r+2)})^{-1}(r+1).$$

Notice that $w_{r+1} = 1$ and w_r doesn't involve s_r or s_{r+1}, we have

$$(k, \lambda^{(\ell)}_k, \ell) = (\mathsf{t}^{(r+2)})^{-1}(r+1) = (\mathsf{t}^{(r+1)})^{-1}(r+1) = (\mathsf{t}^{(r)})^{-1}(r+1).$$

Since $w_r \ne 1$ and $w_{r+1} = 1$, recall $w_r = s_{a_r} s_{a_r+1} \dots s_{r-2} s_{r-1}$, by Remark 1.3.2 we have

$$(\mathsf{t}^{(r+2)})^{-1}(a_r) = (\mathsf{t}^{(r+1)})^{-1}(a_r) = (\mathsf{t}^{(r)})^{-1}(r).$$

Since $w_i = 1$ for $i < r$, we have $\mathsf{t}^{(r)} = \mathsf{t}$. Then $\mathsf{t}^{-1}(r+1) = (k, \lambda^{(\ell)}_k, \ell)$. Because $a_r \le r-1 < r+1$, by Remark 1.3.2, a_r is not on the left of $r+1$ in $\mathsf{t}^{(r+2)}$ because $\mathsf{t}^{(r+2)}|_{r+1} = \mathsf{t}^\mu$ with $\mu = \mathrm{Shape}(\mathsf{t}^{(r+2)}|_{r+1})$. As r and $r+1$ are adjacent in t and $(\mathsf{t}^{(r+2)})^{-1}(a_r) = (\mathsf{t}^{(r)})^{-1}(r) = \mathsf{t}^{-1}(r)$, we must have $\mathsf{t}^{-1}(r) = (k-1, \lambda^{(\ell)}_k, \ell)$. Therefore by the definition of the standard expression, we have $l(w_r) = \lambda^{(\ell)}_k - 1$.

Since $(k, \lambda^{(\ell)}_k, \ell) = (\mathsf{t}^{(n+1)})^{-1}(n) = (\mathsf{t}^{(n)})^{-1}(n-1) = \dots = (\mathsf{t}^{(r+2)})^{-1}(r+1)$ and Remark 1.3.2, we have $l(w_i) \ge \lambda^{(\ell)}_k = l(w_r) + 1$ for all $i \ge r+2$. □

3.5.18. Lemma. *Suppose* $\mathsf{t} \in \mathrm{Std}(\lambda)$ *and* $d(\mathsf{t}) = w_n w_{n-1} \dots w_1$ *with* $w_i \ne 1$ *if* $i > r+2$ *or* $i = r$ *and* $w_i = 1$ *if* $i < r$ *or* $i = r+1$. *If* $l(w_i) = l(w_r) + 1$ *for all* $i \ge r+2$, *i.e.* $d(\mathsf{t}) = w_n w_{n-1} \dots w_{r+2} w_r$, *and* $\mathsf{t}{\cdot}s_r$ *is not standard, then* t *is the last Garnir tableau of shape* λ.

Proof. Write $\lambda = (\lambda^{(1)}, \dots, \lambda^{(\ell)})$ and $\lambda^{(\ell)} = (\lambda_1^{(\ell)}, \dots, \lambda_k^{(\ell)})$. From the proof of Lemma 3.5.17 we have seen that $l(w_i) = \lambda_k^{(\ell)}$ for $i \geq r+2$ and $l(w_r) = \lambda_k^{(\ell)} - 1$. Therefore if $\mathsf{t}^\lambda(k-1, \lambda_{k-1}^{(\ell)}, \ell) = t$,

$$\begin{cases} w_n = s_t s_{t+1} \dots s_{n-1}, \\ w_{n-1} = s_{t-1} s_t \dots s_{n-2}, \\ \dots\dots\dots \\ w_{r+2} = s_{t-n+r+2} s_{t-n+r+3} \dots s_{r+1}, \\ w_r = s_{t-n+r+1} s_{t-n+r+2} \dots s_{r-1}, \end{cases}$$

and by direct calculation we can see that such $d(\mathsf{t})$ is the last Garnir tableau of shape λ. □

3.5.19. **Example** Suppose $\lambda = (7, 5, 3)$ and $\mathsf{t} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 8 & 9 & 12 & 14 & 15 \\ \cline{1-5} 10 & 11 & 13 \\ \cline{1-3} \end{array}$. Then

$$d(\mathsf{t}) = s_{12} s_{13} s_{14} \cdot s_{11} s_{12} s_{13} \cdot s_{10} s_{11}.$$

So we can write $d(\mathsf{t}) = w_{15} w_{14} w_{13} w_{12}$ where $w_{15} = s_{12} s_{13} s_{14}$, $w_{14} = s_{11} s_{12} s_{13}$, $w_{13} = 1$ and $w_{12} = s_{10} s_{11}$. Notice $l(w_{15}) = l(w_{14}) = l(w_{12}) + 1$ and $\mathsf{t} \cdot s_{12}$ is not standard, and furthermore, t is the last Garnir tableau of shape λ. ◇

Finally we are ready to prove the most important result of this section.

3.5.20. Proposition. *Suppose $\lambda \in \mathscr{P}_n^\Lambda$. For any standard λ-tableau t with $l(d(\mathsf{t})) \leq m_\lambda$, if $d(\mathsf{t}) \cdot s_r$ is not reduced or $\mathsf{t} \cdot s_r$ is not standard for some r,*

$$\psi_{\mathsf{st}} \psi_r = \sum_{(\mathsf{u},\mathsf{v}) \triangleright (\mathsf{s},\mathsf{t})} c_{\mathsf{uv}} \psi_{\mathsf{uv}}.$$

for any standard λ-tableau s.

Proof. First we set $\mathsf{s} = \mathsf{t}^\lambda$. Recall that the standard expression of $\psi_{d(\mathsf{t})}$ has the form

$$\psi_{w_n} \psi_{w_{n-1}} \dots \psi_{w_2},$$

with $\psi_{w_i} = \psi_{a_i} \psi_{a_i+1} \psi_{a_i+2} \dots \psi_{i-1}$ for some $a_i \leq i-1$ or $\psi_{w_i} = 1$. Let k be the integer such that $\psi_{w_k} \neq 1$ but $\psi_{w_i} = 1$ for all $i < k$. So $\psi_{d(\mathsf{t})} = \psi_{w_n} \psi_{w_{n-1}} \dots \psi_{w_k}$.

Recall that by Lemma 3.5.4 and 3.5.9, if $d(\mathsf{t})$ is unlocked by s_r or unlocked in tails by s_r, we have $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r \in R_n^{\geq \lambda}$.

We separate the problem into several cases:

Case 3.5.20a: $k - 1 \notin \{r-1, r, r+1\}$. Then

$$\psi_{d(\mathsf{t})} = \psi_{w_n} \dots \psi_{w_k} = \psi_{w_n} \dots \psi_{w_{k+1}} \psi_{a_k} \psi_{a_k+1} \dots \psi_{k-2} \psi_{k-1}.$$

In this case t is unlocked by s_r. Therefore by Lemma 3.5.4, $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r \in R_n^{\geq \lambda}$.

Case 3.5.20b: $k - 1 = r$. Define $\mathsf{w} = \mathsf{t} \cdot s_r$. Hence $d(\mathsf{t}) = d(\mathsf{w}) \cdot s_r$. Write $\mathbf{i}_\mathsf{w} = (i_1 i_2 \dots i_n)$.

$$e_\lambda y_\lambda \psi_{d(\mathsf{t})} \psi_r = y_\lambda \psi_{d(\mathsf{w})} e(\mathbf{i}_\mathsf{w}) \psi_r^2 = \begin{cases} 0, & \text{if } i_r = i_{r+1}, \\ y_\lambda \psi_{d(\mathsf{w})} e(\mathbf{i}_\mathsf{w}) = \psi_{\mathsf{t}^\lambda \mathsf{w}}, & \text{if } |i_r - i_{r+1}| > 1, \\ \pm\, y_\lambda \psi_{d(\mathsf{w})} e(\mathbf{i}_\mathsf{w})(y_r - y_{r+1}) & \\ \qquad = \pm\, \psi_{\mathsf{t}^\lambda \mathsf{w}}(y_r - y_{r+1}), & \text{if } i_r = i_{r+1} \pm 1. \end{cases}$$

By Proposition 3.2.5 we have $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r \in R_n^{\geq \lambda}$.

Case 3.5.20c: $k - 1 = r + 1$.

3.5.20c.1: $\psi_{w_i} = 1$ for some $i > k$ and $i \neq n$. Then we have

$$\begin{aligned}\psi_{d(\mathfrak{t})}\psi_r &= \psi_{w_n}\psi_{w_{n-1}}\ldots\psi_{w_{i+2}}\psi_{w_{i+1}}\psi_{w_{i-1}}\ldots\psi_{w_k}\psi_r\\ &= \psi_{w_n}\psi_{w_{n-1}}\ldots\psi_{w_{i+2}}(\psi_{a_{i+1}}\psi_{a_{i+1}+1}\ldots\psi_{i-1}\psi_i)\psi_{w_{i-1}}\ldots\psi_{w_k}\psi_r.\end{aligned}$$

As $i > k = r + 2 > r + 1$, we have

$$\psi_{d(\mathfrak{t})}\psi_r = (\psi_{w_n}\psi_{w_{n-1}}\ldots\psi_{w_{i+2}}\psi_{a_{i+1}}\psi_{a_{i+1}+1}\ldots\psi_{i-1}\psi_{w_{i-1}}\ldots\psi_{w_k}\psi_i)\psi_r,$$

which shows that $\mathfrak{t}$ is unlocked by s_r. By Lemma 3.5.4 we have $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$.

3.5.20c.2: $\psi_{w_n} = 1$. In this case ψ_{n-1} is not involved in $\psi_{d(\mathfrak{t})}\psi_r$. By Lemma 3.3.6 we have $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$.

3.5.20c.3: $\psi_{w_i} \neq 1$ for $i > k$ and $l(\psi_{w_k}) > 1$. Then we can see that $\mathfrak{t}$ is unlocked on tails by s_r. By Lemma 3.5.9 we have $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$.

3.5.20c.4: $\psi_{w_i} \neq 1$ for $i > k$, and we can find $k < j < n$ such that $l(\psi_{w_k}) = l(\psi_{w_{k+1}}) = \ldots = l(\psi_{w_{j-1}}) = 1$ and $l(\psi_{w_j}) > 1$. Then we have

$$w_j{\cdot}w_{j-1} = s_{a_j}s_{a_j+1}\ldots s_{j-3}s_{j-2}s_{j-1}{\cdot}s_{j-2} = s_{a_j}s_{a_j+1}\ldots s_{j-3}{\cdot}s_{j-1}s_{j-2}s_{j-1}.$$

Therefore

$$\begin{aligned}d(\mathfrak{t}) &= w_nw_{n-1}\ldots w_{j+1}{\cdot}s_{a_j}s_{a_j+1}\ldots s_{j-3}{\cdot}s_{j-1}s_{j-2}s_{j-1}{\cdot}w_{j-2}\ldots w_k\\ &= w_nw_{n-1}\ldots w_{j+1}s_{a_j}s_{a_j+1}\ldots s_{j-3}{\cdot}s_{j-1}s_{j-2}w_{j-2}\ldots w_k{\cdot}s_{j-1},\end{aligned}$$

and $j - 1 \geq k = r + 2 > r + 1$, s_{j-1} and s_r commute, which shows that $\mathfrak{t}$ is unlocked by s_r. By Lemma 3.5.4, we have $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$.

3.5.20c.5: $l(\psi_{w_k}) = l(\psi_{w_{k+1}}) = \ldots = l(\psi_{w_n}) = 1$. Then by Lemma 3.5.13, $\mathfrak{t}$ is the last Garnir tableau of shape λ. Hence by Proposition 3.4.10, $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$.

Case 3.5.20d: $k - 1 = r - 1$.
3.5.20d.1: $\psi_{w_{k+1}} \neq 1$. Then

$$\begin{aligned}d(\mathfrak{t}) &= w_nw_{n-1}\ldots w_{k+2}{\cdot}w_{k+1}w_k\\ &= w_nw_{n-1}\ldots w_{k+2}{\cdot}s_{a_{k+1}}s_{a_{k+1}+1}\ldots s_{k-1}s_k{\cdot}s_{a_k}s_{a_k+1}\ldots s_{k-2}s_{k-1}\\ &= w_nw_{n-1}\ldots w_{k+2}s_{a_{k+1}}s_{a_{k+1}+1}\ldots s_{k-1}s_{a_k}s_{a_k+1}\ldots s_{k-2}{\cdot}s_ks_{k-1},\end{aligned}$$

and as $r = k$, we can see that $\mathfrak{t}$ is unlocked on tails by s_r. Therefore $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$ by Lemma 3.5.9.

3.5.20d.2: $k = n-1$ and $\psi_{w_{k+1}} = \psi_{w_n} = 1$. The $\psi_{d(\mathfrak{t})}\psi_r = \psi_{w_{n-1}}\psi_{n-1} = \psi_{a_{n-1}}\psi_{a_{n-1}+1}\ldots\psi_{n-2}\psi_{n-1}$. Then by Lemma 3.5.15, $\mathfrak{t}$ is the last Garnir tableau of shape λ. Hence by Proposition 3.4.10, $\psi^\lambda_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$.

3.5.20d.3: $k < n - 1$, $\psi_{w_{k+1}} = 1$ and $\psi_{w_n} = 1$. Then $n - 1 > k = r$. So $\psi_{d(\mathfrak{t})}\psi_r$ doesn't involve ψ_{n-1}. By Lemma 3.3.6 we have $\psi_{\mathfrak{t}^\lambda\mathfrak{t}}\psi_r \in R_n^{\geq\lambda}$.

3.5.20d.4: $k < n-1$, $\psi_{w_{k+1}} = 1$ and we can find $k+1 < j < n$ such that $\psi_{w_j} = 1$ and $\psi_{w_{j+1}} \neq 1$. In this case we have

$$\begin{aligned} d(\mathsf{t}) &= w_n w_{n-1} \dots w_{j+2} w_{j+1} w_j w_{j-1} \dots w_k \\ &= w_n w_{n-1} \dots w_{j+2} \cdot s_{a_{j+1}} s_{a_{j+1}+1} \dots s_{j-1} s_j \cdot w_{j-1} \dots w_k \\ &= (w_n w_{n-1} \dots w_{j+2} \cdot s_{a_{j+1}} s_{a_{j+1}+1} \dots s_{j-1} \cdot w_{j-1} \dots w_k) \cdot s_j. \end{aligned}$$

As $j > k+1 = r+1$, ψ_j and ψ_r commute. Therefore t is unlocked by s_r. By Lemma 3.5.4, we have $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r \in R_n^{\geq \lambda}$.

3.5.20d.5: $k < n-1$, $\psi_{w_{k+1}} = 1$ and for any $j > k+1$, $\psi_{w_j} \neq 1$. Then by Lemma 3.5.17, we have $l(\psi_{w_j}) \geq l(\psi_{w_k}) + 1$ for all $j \geq k+2$.

3.5.20d.5.1: Suppose $l(\psi_{w_{k+2}}) > l(\psi_{w_k}) + 1$. So we have $a_{k+2} \leq a_k$, and therefore

$$\begin{aligned} w_{k+2} w_k &= s_{a_{k+2}} s_{a_{k+2}+1} \dots s_k s_{k+1} \cdot s_{a_k} s_{a_k+1} \dots s_{k-2} s_{k-1} \\ &= s_{a_k+1} \dots s_{k-1} s_k \cdot s_{a_{k+2}} s_{a_{k+2}+1} \dots s_k s_{k+1}. \end{aligned}$$

Therefore

$$\begin{aligned} d(\mathsf{t}) &= w_n w_{n-1} \dots w_{k+3} \cdot w_{k+2} w_k \\ &= w_n w_{n-1} w_{k+3} \cdot s_{a_k+1} \dots s_{k-1} s_k \cdot s_{a_{k+2}} s_{a_{k+2}+1} \dots s_k s_{k+1}. \end{aligned}$$

Then because $k = r$, t is unlocked by s_r on tails. Therefore, by Lemma 3.5.9, $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r \in R_n^{\geq \lambda}$.

3.5.20d.5.2: There exists $j > k+2$ such that $l(\psi_{w_{k+2}}) = l(\psi_{w_{k+3}}) = \dots = l(\psi_{w_{j-1}}) = l(\psi_{w_k}) + 1$ and $l(\psi_{w_j}) > l(\psi_{w_k}) + 1$. So we have $l(\psi_{w_j}) > l(\psi_{w_{j-1}})$ and $a_j \leq a_{j-1}$, and therefore

$$\begin{aligned} w_j w_{j-1} &= s_{a_j} s_{a_j+1} \dots s_{j-2} s_{j-1} \cdot s_{a_{j-1}} s_{a_{j-1}+1} \dots s_{j-3} s_{j-2} \\ &= s_{a_{j-1}+1} \dots s_{j-2} s_{j-1} \cdot s_{a_j} s_{a_j+1} \dots s_{j-2} s_{j-1}. \end{aligned}$$

Therefore

$$\begin{aligned} d(\mathsf{t}) &= w_n w_{n-1} \dots w_{j+1} w_j w_{j-1} w_{j-2} \dots w_k \\ &= w_n w_{n-1} \dots w_{j+1} \cdot s_{a_{j-1}+1} \dots s_{j-1} \cdot s_{a_j} \dots s_{j-2} s_{j-1} \cdot w_{j-2} \dots w_k \\ &= (w_n w_{n-1} \dots w_{j+1} \cdot s_{a_{j-1}+1} \dots s_{j-1} \cdot s_{a_j} \dots s_{j-2} \cdot w_{j-2} \dots w_k) \cdot s_{j-1}. \end{aligned}$$

Then because $j-1 > k+1 = r+1$, s_{j-1} and s_r commutes. Hence t is unlocked by s_r and therefore, by Lemma 3.5.4, $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r \in R_n^{\geq \lambda}$.

3.5.20d.5.3: $l(\psi_{w_{k+2}}) = l(\psi_{w_{k+3}}) = \dots = l(\psi_{w_{n-1}}) = l(\psi_{w_k}) + 1$. By Lemma 3.5.18, t is the last Garnir tableau of shape λ. By Proposition 3.4.10, we have $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r \in R_n^{\geq \lambda}$.

By the above cases, $\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r$ is always in $R_n^{\geq \lambda}$. Therefore by Proposition 1.4.9, we have

$$\psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r = \sum_{(\mathsf{u},\mathsf{v}) \rhd (\mathsf{t}^\lambda, \mathsf{t})} c_{\mathsf{uv}} \psi_{\mathsf{uv}} = \sum_{\mathsf{v} \rhd t} c_{\mathsf{t}^\lambda \mathsf{v}} \psi_{\mathsf{t}^\lambda \mathsf{v}} + \sum_{\mathsf{u},\mathsf{v} \in \mathrm{Std}(>\lambda)} c_{\mathsf{uv}} \psi_{\mathsf{uv}}.$$

Giving any standard λ-tableau s, we have $\psi_{\mathsf{st}} \psi_r = \psi^*_{d(\mathsf{s})} \psi_{\mathsf{t}^\lambda \mathsf{t}} \psi_r$. Notice $\psi^*_{d(\mathsf{s})} \psi_{\mathsf{t}^\lambda \mathsf{v}} = \psi_{\mathsf{sv}}$. For any $\mathsf{u}, \mathsf{v} \in \mathrm{Std}(> \lambda)$, $\psi_{\mathsf{uv}} \in R_n^{>\lambda}$. As $\lambda \in \mathscr{P}_n^\Lambda$, by Lemma 2.1.12, $R_n^{>\lambda}$ is an ideal. Therefore $\psi^*_{d(\mathsf{s})} \psi_{\mathsf{uv}} \in R_n^{>\lambda}$. These arguments yield that $\psi_{\mathsf{st}} \psi_r \in R_n^{\geq \lambda}$. By Proposition 1.4.9 we completes the proof. □

The following Corollary is straightforward by Corollary 3.2.3 and Proposition 3.5.20.

3.5.21. Corollary. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$, for any standard λ-tableau t with $l(d(\mathsf{t})) \le m_\lambda$, then*

$$\psi_{\mathsf{st}}\psi_r = \begin{cases} \psi_{\mathsf{t}^\lambda \mathsf{w}} + \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{w} = \mathsf{u}\cdot s_r \text{ is standard and } d(\mathsf{u})\cdot s_r \text{ is reduced,} \\ \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{u}\cdot s_r \text{ is not standard or } d(\mathsf{u})\cdot s_r \text{ is not reduced.} \end{cases}$$

for any standard λ-tableau s.

3.6. Integral basis Theorem

In this section we will complete the main Theorem of this thesis.

3.6.1. Theorem. *Suppose $\lambda \in \mathscr{S}_n^\Lambda$, we have $\lambda \in \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$.*

Proof. By Theorem 2.1.8 we have when $\lambda \in \mathscr{S}_n^\Lambda$ then $\lambda \in \mathscr{P}_I^\Lambda$. By Corollary 3.1.4, we have $0 < m_\lambda$, i.e. $1 \le m_\lambda$. Assume $l = l(d(\mathsf{u})$ for some $\mathsf{u} \in \operatorname{Std}(\lambda)$, by Proposition 3.2.5 and Corollary 3.5.21, for any $\mathsf{t} \in \operatorname{Std}(\lambda)$ with $l(d(\mathsf{t})) = l$, we have

$$\psi_{\mathsf{st}} y_r = \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}},$$

$$\psi_{\mathsf{st}}\psi_r = \begin{cases} \psi_{\mathsf{sw}} + \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{w} = \mathsf{t}\cdot s_r \text{ is standard and } d(\mathsf{u})\cdot s_r \text{ is reduced,} \\ \sum_{(\mathsf{u},\mathsf{v})\rhd(\mathsf{s},\mathsf{t})} c_{\mathsf{uv}}\psi_{\mathsf{uv}}, & \text{if } \mathsf{u}\cdot s_r \text{ is not standard or } d(\mathsf{u})\cdot s_r \text{ is not reduced.} \end{cases}$$

which yields that $l < m_\lambda$, i.e. $l + 1 \le m_\lambda$. So by induction, for any $\mathsf{t} \in \operatorname{Std}(\lambda)$, we have $l(d(\mathsf{t})) < m_\lambda$. Therefore $\lambda \in \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$. This completes the proof. □

3.6.2. Theorem. *The set $\{\psi_{st}^{\mathbb{Z}} \mid \mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda)$ for $\lambda \in \mathscr{P}_n^\Lambda\}$ is a graded cellular basis of $\mathscr{R}_n^\Lambda(\mathbb{Z})$.*

Proof. It's trivial that when $n = 1$ the Theorem holds. Assume for any $n' < n$ the Theorem follows. Suppose we can write all multipartitions of n as $\lambda_{[1]}, \lambda_{[2]}, \ldots, \lambda_{[k]}$ where $\lambda_{[1]} > \lambda_{[2]} > \ldots > \lambda_{[k]}$. As $\lambda_{[1]} = ((n), \emptyset, \ldots, \emptyset)$, by Lemma 2.1.9, Corollary 2.1.11 and 2.1.10, we have $\lambda_{[1]} \in \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$. Hence $\lambda_{[2]} \in \mathscr{S}_n^\Lambda$. Now assume $\lambda_{[i]} \in \mathscr{S}_n^\Lambda$, by Theorem 3.6.1, $\lambda_{[i]} \in \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$. Hence $\lambda_{[i+1]} \in \mathscr{S}_n^\Lambda$. Therefore for any i, $\lambda_{[i]} \in \mathscr{S}_n^\Lambda$. Hence for any $\lambda \in \mathscr{P}_n^\Lambda$, $\lambda \in \mathscr{P}_I^\Lambda \cap \mathscr{P}_y^\Lambda \cap \mathscr{P}_\psi^\Lambda$. Recall that

$$R_n^\Lambda = \{r \in \mathscr{R}_n^\Lambda(\mathbb{Z}) \mid r = \sum_{\substack{\mathsf{s},\mathsf{t}\in\operatorname{Std}(\mu)\\ \mu\in\mathscr{P}_n^\Lambda}} c_{\mathsf{st}}\psi_{\mathsf{st}}, c_{\mathsf{st}} \in \mathbb{Z}\}.$$

So R_n^Λ is an ideal.

Now for any $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$, set $\mathbf{j} = (i_1, i_2, \ldots, i_{n-1}) \in I^{n-1}$. Because $e(\mathbf{j}) \in \mathscr{R}_{n-1}^\Lambda$, by assumption we have $e(\mathbf{j}) = \sum_{\substack{\mu\in\mathscr{P}_{n-1}^\Lambda\\ \mathsf{u},\mathsf{v}\in\operatorname{Std}(\mu)}} c_{\mathsf{uv}}\psi_{\mathsf{uv}} \in R_{n-1}^\Lambda$ and hence $e(\mathbf{i}) = \theta_{i_n}(e(\mathbf{j})) = \sum_{\substack{\mu\in\mathscr{P}_{n-1}^\Lambda\\ \mathsf{u},\mathsf{v}\in\operatorname{Std}(\mu)}} c_{\mathsf{uv}}\theta_{i_n}(\psi_{\mathsf{uv}})$.

For any $\mu \in \mathscr{P}_{n-1}^\Lambda$ and $\mathsf{u},\mathsf{v} \in \operatorname{Std}(\mu)$, we have

$$\theta_{i_n}(\psi_{\mathsf{uv}}) = \psi_{d(\mathsf{u})}^* e(\mathbf{i}_\mu \vee i_n) y_\mu \psi_{d(\mathsf{v})}.$$

By Lemma 2.1.3 and Theorem 2.1.8, we have $e(\mathbf{i}_\mu \vee i_n) y_\mu y_n^0 \in R_n^\Lambda$. Then because R_n^Λ is an ideal,

$$e(\mathbf{i}) = \sum_{\substack{\mu\in\mathscr{P}_{n-1}^\Lambda\\ \mathsf{u},\mathsf{v}\in\operatorname{Std}(\mu)}} c_{\mathsf{uv}}\theta_{i_n}(\psi_{\mathsf{uv}}) \in R_n^\Lambda.$$

Then we have $R_n^\Lambda = \mathscr{R}_n^\Lambda(\mathbb{Z})$. By Corollary 1.4.10, the set $\{\psi_{st}^{\mathbb{Z}} \mid \mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda)$ for $\lambda \in \mathscr{P}_n^\Lambda\}$ is linearly independent. This yields that $\{\psi_{st}^{\mathbb{Z}} \mid \mathsf{s},\mathsf{t} \in \operatorname{Std}(\lambda)$ for $\lambda \in \mathscr{P}_n^\Lambda\}$ is a basis of $\mathscr{R}_n^\Lambda(\mathbb{Z})$.

By the definition of $\{\psi^{\mathbb{Z}}_{st} \mid \mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda) \text{ for } \lambda \in \mathscr{P}^{\Lambda}_n\}$ the elements in the set are homogeneous. The cellularity is trivial by Theorem 1.4.5 and Proposition 1.4.9. This completes the proof. □

The next Corollary is straightforward by Theorem 3.6.2.

3.6.3. Corollary. *For any* $\mathbf{i} \in I^n$, $e(\mathbf{i}) \neq 0$ *if and only if* $\mathbf{i}$ *is the residue sequence of a standard tableau* t.

Proof. Suppose $\mathbf{i}$ is the residue sequence of a standard tableau t. By Theorem 3.6.2 we have $\psi^{\mathbb{Z}}_{\mathsf{tt}} \neq 0$. Because $\psi^{\mathbb{Z}}_{\mathsf{tt}} = \psi^{\mathbb{Z}}_{\mathsf{tt}} e(\mathbf{i})$, we must have $e(\mathbf{i}) \neq 0$.

Suppose $\mathbf{i}$ is not the residue sequence of any standard tableau. By Theorem 3.6.2 we can write

$$1 = \sum_{\mathsf{s},\mathsf{t}} c_{\mathsf{st}} \psi^{\mathbb{Z}}_{\mathsf{st}},$$

and hence

$$e(\mathbf{i}) = 1 \cdot e(\mathbf{i}) = \sum_{\mathsf{s},\mathsf{t}} c_{\mathsf{st}} \psi^{\mathbb{Z}}_{\mathsf{st}} e(\mathbf{i}) = 0,$$

which completes the proof. □

CHAPTER 4

Basis of Affine KLR Algebras

In Theorem 3.6.2 we have shown that $\mathscr{R}_n^\Lambda(\mathbb{Z})$ is a $\mathbb{Z}$-free algebra with basis $\{\psi_{\mathsf{st}} \mid \mathsf{s},\mathsf{t} \in \mathrm{Std}(\lambda), \lambda \in \mathscr{P}_n^\Lambda\}$. In this chapter we will extend this result and find a graded cellular basis for the $\mathscr{R}_n(\mathbb{Z})$. Moreover, for any weight Λ we can delete a few elements from the basis of $\mathscr{R}_n(\mathbb{Z})$ and form a graded cellular basis of N_n^Λ. We then give a complete set of simple $\mathscr{R}_n$-modules by using the graded cellular basis of $\mathscr{R}_n$. Furthermore, in the previous chapters we set $e \neq 2$ during the proof. In this chapter we allow $e = 2$.

In this chapter, we will define a sequence of weights $(\Lambda^{(k)})$ with specific property and use such sequence to extend the graded cellular basis of $\mathscr{R}_n^\Lambda$ to $\mathscr{R}_n$.

4.1. Infinite sequence of weights and basis of $\mathscr{R}_\alpha^\Lambda$

In this section first we will introduce a special kind of sequence of weights $(\Lambda^{(k)})$. Then for $\mathbf{i} \in I^n$ we give a graded cellular basis for $\mathscr{R}_\alpha^\Lambda$, which will be extended to a graded cellular basis for $\mathscr{R}_\alpha$.

We fix a $e \geq 0$ and $e \neq 1$, and the ring of $\mathscr{R}_n$ to be $\mathbb{Z}$ and will write $\mathscr{R}_n$ instead of $\mathscr{R}_n(\mathbb{Z})$. Suppose $\Lambda = \sum_{i\in I} a_i\Lambda_i$ and $\Lambda' = \sum_{i\in I} a'_i\Lambda_i$ are two weights in P_+. We define a partial ordering on weights and write $\Lambda \leq \Lambda'$ if $a_i \leq a'_i$ for any $i \in I$, and $\Lambda < \Lambda'$ if $\Lambda \leq \Lambda'$ and $\Lambda \neq \Lambda'$.

4.1.1. Definition. *An **increasing sequence** of weights is a sequence $(\Lambda^{(k)})$ of weights in P^+ for $k \geq 1$ such that $\Lambda^{(k)} < \Lambda^{(k+1)}$ for all $k \geq 1$. The sequence $(\Lambda^{(k)})$ is **standard** if $\lim_{k\to\infty} a_i^{(k)} = \infty$, for all $i \in I$.*

4.1.2. **Example** Suppose $e > 0$. We define a sequence $(\Lambda^{(k)})$ where $\Lambda^{(1)} = \Lambda_1$ and $\Lambda^{(k)}-\Lambda^{(k-1)} = \Lambda_i$ with $k \equiv i \pmod e$. For example, when $e = 3$, we have

$$\begin{array}{lcl}
\Lambda^{(1)} & = & \Lambda_1, \\
\Lambda^{(2)} & = & \Lambda_1 + \Lambda_2, \\
\Lambda^{(3)} & = & \Lambda_1 + \Lambda_2 + \Lambda_0, \\
\Lambda^{(4)} & = & 2\Lambda_1 + \Lambda_2 + \Lambda_0, \\
\Lambda^{(5)} & = & 2\Lambda_1 + 2\Lambda_2 + \Lambda_0, \\
\Lambda^{(6)} & = & 2\Lambda_1 + 2\Lambda_2 + 2\Lambda_0, \\
\Lambda^{(7)} & = & 3\Lambda_1 + 2\Lambda_2 + 2\Lambda_0, \\
\cdots\cdots & \cdots & \cdots\cdots\cdots
\end{array}$$

So in this case we have $\lim_{k\to\infty} a_i^{(k)} = \infty$ for any $i \in I$ and $(\Lambda^{(k)})$ is a standard sequence. $\diamond$

4.1.3. **Example** Suppose $e = 0$. Define $(\Lambda^{(k)})$ where $\Lambda^{(1)} = \Lambda_0$ and $\Lambda^{(k)} - \Lambda^{(k-1)} = \Lambda_i$ with $i = (k-1) - (n-1)^2 - (n-1) = k - n^2 + n - 1$ if $(n-1)^2 < k \le n^2$. In more details,

$$\begin{array}{rcl}
\Lambda^{(1)} &=& \Lambda_0, \\
\Lambda^{(2)} &=& \Lambda_{-1} + \Lambda_0, \\
\Lambda^{(3)} &=& \Lambda_{-1} + 2\Lambda_0, \\
\Lambda^{(4)} &=& \Lambda_{-1} + 2\Lambda_0 + \Lambda_1, \\
\Lambda^{(5)} &=& \Lambda_{-2} + \Lambda_{-1} + 2\Lambda_0 + \Lambda_1, \\
\Lambda^{(6)} &=& \Lambda_{-2} + 2\Lambda_{-1} + 2\Lambda_0 + \Lambda_1, \\
\Lambda^{(7)} &=& \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + \Lambda_1, \\
\Lambda^{(8)} &=& \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1, \\
\Lambda^{(9)} &=& \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1 + \Lambda_2, \\
\Lambda^{(10)} &=& \Lambda_{-3} + \Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1 + \Lambda_2, \\
\Lambda^{(11)} &=& \Lambda_{-3} + 2\Lambda_{-2} + 2\Lambda_{-1} + 3\Lambda_0 + 2\Lambda_1 + \Lambda_2, \\
\cdots\cdots & \cdots & \cdots\cdots\cdots
\end{array}$$

So in this case we have $\lim_{k\to\infty} a_i^{(k)} = \infty$ for any $i \in I$ and $(\Lambda^{(k)})$ is a standard sequence. ◇

Recall that for any weight $\Lambda = \sum_{i\in I} a_i\Lambda_i$, we can define an two-sided ideal in $\mathscr{R}_n$, N_n^Λ, which is generated by elements $e(\mathbf{i})y_1^{a_{i_1}}$ for all $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$. Then the cyclotomic KLR algebra $\mathscr{R}_n^\Lambda \cong \mathscr{R}_n/N_n^\Lambda$. In this case we can also write $\mathscr{R}_n \cong \mathscr{R}_n^\Lambda \oplus N_n^\Lambda$ as $\mathbb{Z}$-modules.

Recall $Q_+ = \sum_{i\in I} \mathbb{N}\alpha_i$ is defined in Section 1.1. For $\alpha = \sum_{i\in I} a_i\alpha_i \in Q_+$, define $|\alpha| = \sum_{ß\in I} a_i$. Then for any $\alpha \in Q_+$ with $|\alpha| = n$, define I^α to be the set of all $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ such that $a_i = |\{\, 1 \le r \le n \mid i_r = i \,\}|$. By the definition if $\mathbf{i}, \mathbf{j} \in I^\alpha$ then we can find $v \in \mathfrak{S}_n$ such that $\mathbf{i} = \mathbf{j}{\cdot}v$. Define $\hat{e}_\alpha = \sum_{\mathbf{i}\in I^\alpha} \hat{e}(\mathbf{i}) \in \mathscr{R}_n$ and $e_\alpha = \sum_{\mathbf{i}\in I^\alpha} e(\mathbf{i}) \in \mathscr{R}_n^\Lambda$.

The following result is trivial by the relations of $\mathscr{R}_n$.

4.1.4. Lemma. *Suppose $\alpha, \beta \in Q_+$. Then $\mathscr{R}_n\hat{e}_\alpha \neq 0$ and $\hat{e}_\beta\mathscr{R}_n\hat{e}_\alpha = \delta_{\alpha\beta}\mathscr{R}_n e_\alpha = \delta_{\alpha\beta}\hat{e}_\beta\mathscr{R}_n$.*

We then define $\mathscr{R}_\alpha = \mathscr{R}_n\hat{e}_\alpha$, $\mathscr{R}_\alpha^\Lambda = \mathscr{R}_n^\Lambda e_\alpha$ and $N_\alpha^\Lambda = N_n^\Lambda \hat{e}_\alpha$. We can see that $\mathscr{R}_\alpha\hat{e}(\mathbf{j}) = 0$ if $\mathbf{j} \notin I^\alpha$. Finally, because

$$\mathscr{R}_n = \bigoplus_{\alpha\in Q_+} \mathscr{R}_\alpha \quad \text{and} \quad \mathscr{R}_n^\Lambda = \bigoplus_{\alpha\in Q_+} \mathscr{R}_\alpha^\Lambda,$$

and by the relations $\mathscr{R}_\alpha$ and $\mathscr{R}_\alpha^\Lambda$'s are subalgebras of $\mathscr{R}_n$ and $\mathscr{R}_n^\Lambda$, respectively. Hence we will mainly work in $\mathscr{R}_\alpha$, $\mathscr{R}_\alpha^\Lambda$ and N_α^Λ and extend the basis of $\mathscr{R}_\alpha^\Lambda$ to $\mathscr{R}_\alpha$ and hence generate a graded cellular basis of $\mathscr{R}_n$.

By Theorem 3.6.2 and the orthogonality of $e(\mathbf{i})$'s we can give a basis for $\mathscr{R}_\alpha^\Lambda$.

4.1.5. Proposition. *Suppose $\mathbf{i} \in I^n$ and $\Lambda \in P_+$. The set*

$$\{\, \psi_{\mathsf{st}} \mid \lambda \in \mathscr{P}_n^\Lambda, \mathsf{s}, \mathsf{t} \in \operatorname{Std}(\lambda), \operatorname{res}(\mathsf{t}) \in I^\alpha \,\}$$

is a graded cellular basis of $\mathscr{R}_\alpha^\Lambda$.

4.2. Minimum degree of N_α^Λ

Fix $\alpha \in Q_+$. In the last section we introduced a standard sequence of $(\Lambda^{(k)})$. For each k and $\mathbf{i} \in I^n$, we define the minimum degree of $N_\alpha^{\Lambda^{(k)}}$, $m_\alpha^{\Lambda^{(k)}} = \min\{\deg(r) \mid r \text{ is a homogeneous element in } N_\alpha^\Lambda\}$ and will prove that $m_\alpha^{\Lambda^{(k)}} \to \infty$ with $k \to \infty$. This result is quite important in the next section while extending the basis of $\mathscr{R}_\alpha^\Lambda$ to $\mathscr{R}_\alpha$.

First we need to find a general form of the homogeneous elements of N_α^Λ.

4.2.1. Lemma. *For $\Lambda = \sum_{i\in I} a_i\Lambda_i \in P_+$ and $\alpha \in Q_+$, the ideal N_α^Λ is spanned by*

$$\{\,\psi_u e(\mathbf{i}) y_1^{a_{i_1}} f(y)\psi_v \mid u, v \in \mathfrak{S}_n, f(y) \in \mathbb{Z}[y_1, y_2, \ldots, y_n], \mathbf{i} \in I^\alpha\,\}.$$

Proof. It is obvious that any element of N_α^Λ can be written as linear combination of elements of the form

$$\psi_{u_k} f_k(y)\psi_{u_{k-1}} \ldots \psi_{u_2} f_2(y)\psi_{u_1} f_1(y) e(\mathbf{i}) y_1^{a_{i_1}} g_1(y)\psi_{v_1} g_2(y)\psi_{v_2} \ldots \psi_{v_{l-1}} g_l(y)\psi_{v_l}, \tag{4.2.2}$$

where $u_i, v_i \in \mathfrak{S}_n$, $\mathbf{i} \in I^\alpha$ and $f_i(y), g_i(y) \in \mathbb{Z}[y_1, \ldots, y_n]$. Then by the view of Lemma 3.2.2 and [**5**, Proposition 2.5] it is obvious that any element in the form of (4.2.2) can be written as linear combination of $\psi_u e(\mathbf{i}) y_1^{a_{i_1}} f(y)\psi_v$'s. Hence N_α^Λ is spanned by those elements. □

By Lemma 4.2.1,

$$m_\alpha^\Lambda = \min\{\,\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}} f(y)\psi_v) \mid u, v \in \mathfrak{S}_n, f(y) \in \mathbb{Z}[y_1, y_2, \ldots, y_n], \mathbf{i} \in I^\alpha\,\}.$$

Hence we have an expression for m_α^Λ and we are ready to prove the result of this section.

4.2.3. Proposition. *For any standard sequence $(\Lambda^{(k)})$ and $\mathbf{i} \in I^n$, $\lim_{k\to\infty} m_\alpha^{\Lambda^{(k)}} = \infty$.*

Proof. By Lemma 4.2.1, we only need to work with $\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y)\psi_v)$ for all $\mathbf{i} \in I^\alpha$. We can write $\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y)\psi_v) = \deg(\psi_u e(\mathbf{i})) + \deg(y_1^{a_{i_1}^{(k)}}) + \deg(f(y)) + \deg(\psi_v e(\mathbf{i}{\cdot}v))$. As u and v are reduced expressions of permutations in $\mathfrak{S}_n$, $l(u) \le \frac{(n-1)n}{2}$, and $\deg(\psi_r e(\mathbf{i})) \ge -2$ for any $\mathbf{i}$. Hence $\deg(\psi_u e(\mathbf{i})) \ge -(n-1)n$. For the same reason $\deg(\psi_v e(\mathbf{i}{\cdot}v)) \ge -(n-1)n$. Then as $\deg(f(y)) \ge 0$, we have $\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y)\psi_v) \ge -2(n-1)n + 2a_{i_1}^{(k)}$.

Define $a_\alpha^{(k)} = \min_{\mathbf{i}\in I^\alpha} a_{i_1}^{(k)}$. We have

$$\deg(\psi_u e(\mathbf{i}) y_1^{a_{i_1}^{(k)}} f(y)\psi_v) \ge -2(n-1)n + 2a_\alpha^{(k)},$$

for any u, v and f. Therefore $m_\alpha^{\Lambda^{(k)}} \ge 2a_\alpha^{(k)} - 2(n-1)n$.

Choose $\mathbf{j} \in I^\alpha$. It is obvious that $I^\alpha = \{\,\mathbf{i} \in I^n \mid \mathbf{i} = \mathbf{j}{\cdot}v \text{ with } v \in \mathfrak{S}_n\,\}$. Then $|I^\alpha| \le |\mathfrak{S}_n| < \infty$. Then $a_{i_1}^{(k)} \to \infty$ as $k \to \infty$ for any $\mathbf{i} \in I^\alpha$ implies $a_\alpha^{(k)} \to \infty$ as $k \to \infty$ because I^α is finite. Therefore $m_\alpha^{\Lambda^{(k)}} \to \infty$. □

4.2.4. Remark. The set I^α is finite is important in the proof of Proposition 4.2.3. If I^α is infinite, $a_{i_1}^{(k)} \to \infty$ as $k \to \infty$ for all $\mathbf{i} \in I^\alpha$ is not sufficient to imply that $a_\alpha^{(k)} \to \infty$ as $k \to \infty$.

4.3. A graded cellular basis of $\mathscr{R}_n$

In this section we will prove the main result of this chapter. First we will introduce a special kind of multicharge κ corresponding to a standard sequence $(\Lambda^{(k)})$ which contains information for the multicharges $\kappa_{\Lambda^{(k)}}$ corresponding to $\Lambda^{(k)}$. Then for any $\alpha \in Q_+$, we will find a graded cellular basis $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ of $\mathscr{R}_\alpha$ corresponds to κ.

4.3.1. Definition. *Suppose $(\Lambda^{(k)})$ is a standard sequence. An* ***inverse multicharge sequence*** *for $(\Lambda^{(k)})$ is a infinite sequence $\kappa = (\ldots, \kappa_3, \kappa_2, \kappa_1)$ such that for any $k \ge 1$, if $\ell_k = l(\Lambda^{(k)})$, then $\kappa_{\Lambda^{(k)}} = (\kappa_{\ell_k}, \kappa_{\ell_k - 1}, \ldots, \kappa_2, \kappa_1)$ is a multicharge corresponding to $\Lambda^{(k)}$.*

4.3.2. **Example** Suppose $e = 3$. Using the standard sequence $(\Lambda^{(k)})$ introduced in Example 4.1.2, we can define a multicharge $\kappa = (\ldots, \kappa_3, \kappa_2, \kappa_1)$ where $\kappa_k \equiv k \pmod e$ for $k \ge 1$.

Therefore we can write $\kappa = (\dots, 0, 2, 1, 0, 2, 1, 0, 2, 1, 0, 2, 1)$, we have

$$\begin{array}{rcl} \kappa_{\Lambda^{(1)}} & = & (1), \\ \kappa_{\Lambda^{(2)}} & = & (2, 1), \\ \kappa_{\Lambda^{(3)}} & = & (0, 2, 1), \\ \kappa_{\Lambda^{(4)}} & = & (1, 0, 2, 1), \\ \kappa_{\Lambda^{(5)}} & = & (2, 1, 0, 2, 1), \\ \dots\dots & \dots & \dots\dots\dots \end{array}$$

These are all multicharges corresponding to $\Lambda^{(k)}$. ◇

Fix a standard sequence $(\Lambda^{(k)})$ and an inverse multicharge sequence κ for $(\Lambda^{(k)})$. An **affine multipartition** of n is an ordered sequence $\hat{\lambda} = (\dots, \lambda^{(2)}, \lambda^{(1)})$ of partitions such that $\sum_{i=1}^{\infty} |\lambda^{(i)}| = n$. Let $\mathscr{P}_n^{\kappa}$ be the set of all affine multipartitions of n. We define **young diagram** $[\hat{\lambda}]$ and **standard affine tableau** $\hat{\mathsf{s}}$ for affine multipartitions in the same way as for multipartitions. Let $\mathrm{Std}(\hat{\lambda})$ be the set of all standard affine tableaux of shape $\hat{\lambda}$.

We define the **level** of $\hat{\lambda}$ to be $l(\hat{\lambda}) = \ell$ if $\lambda^{(\ell)} \neq \emptyset$ and $\lambda^{(i)} = \emptyset$ for $i > \ell$. For any ℓ define a mapping $p_\ell \colon \mathscr{P}_n^{\kappa} \longrightarrow \mathscr{P}_n^{\Lambda}$, where $(\kappa_\ell, \kappa_{\ell-1}, \dots, \kappa_1)$ is a multicharge for Λ, sending $\hat{\lambda} = (\dots, \lambda^{(2)}, \lambda^{(1)})$ to $\lambda = (\lambda^{(\ell)}, \lambda^{(\ell-1)}, \dots, \lambda^{(2)}, \lambda^{(1)})$. In order to simplify the notation, suppose $l(\hat{\lambda}) = \ell$, we write $\lambda = p_\ell(\hat{\lambda})$. Then we can define a mapping $t \colon \mathrm{Std}(\hat{\lambda}) \longrightarrow \mathrm{Std}(\lambda)$ in the obvious way. It is obvious that t is a bijection. Again we will write s instead of $t(\hat{\mathsf{s}})$ in order to simplify the notation. Define the degree of each standard affine tableau to be $\deg(\hat{\mathsf{s}}) = \deg(\mathsf{s})$ and the residue sequence of the affine tableau $\mathrm{res}(\hat{\mathsf{s}}) = \mathrm{res}(\mathsf{s})$.

We extend dominance ordering $\trianglerighteq$ and lexicographic ordering $\geq$ to $\mathscr{P}_n^{\kappa}$. By defining $\hat{\lambda} \trianglerighteq \hat{\mu}$ if $l(\hat{\lambda}) > l(\hat{\mu})$ or $l(\hat{\lambda}) = l(\hat{\mu})$ and $\lambda \trianglerighteq \mu$ and $\hat{\lambda} \triangleright \hat{\mu}$ if $\hat{\lambda} \trianglerighteq \hat{\mu}$ and $\hat{\lambda} \neq \hat{\mu}$ for $\hat{\lambda}, \hat{\mu} \in \mathscr{P}_n^{\kappa}$. We define $\geq$ and $>$ in a similar way.

4.3.3. **Example** Suppose $\hat{\lambda} = (\dots |0|0|0|4, 3, 1|2, 1|3, 3)$. Then $\lambda = (4, 3, 1|2, 1|3, 3)$ and

$$\hat{\mathsf{s}} = \left(\dots \middle| \emptyset \middle| \emptyset \middle| \begin{array}{|c|c|c|c|}\hline 1&8&13&16\\\hline 7&12&15\\\cline{1-3} 10\\\cline{1-1}\end{array} \middle| \begin{array}{|c|c|}\hline 2&6\\\hline 3\\\cline{1-1}\end{array} \middle| \begin{array}{|c|c|c|}\hline 4&5&11\\\hline 9&14&17\\\hline\end{array}\right) \in \mathrm{Std}(\hat{\lambda}),,$$

and

$$\mathsf{s} = t(\hat{\mathsf{s}}) = \left(\begin{array}{|c|c|c|c|}\hline 1&8&13&16\\\hline 7&12&15\\\cline{1-3} 10\\\cline{1-1}\end{array} \middle| \begin{array}{|c|c|}\hline 2&6\\\hline 3\\\cline{1-1}\end{array} \middle| \begin{array}{|c|c|c|}\hline 4&5&11\\\hline 9&14&17\\\hline\end{array}\right) \in \mathrm{Std}(\lambda).$$

◇

Suppose $\Lambda \in P_+$ and $\lambda = (\lambda^{(\ell)}, \dots, \lambda^{(1)}) \in \mathscr{P}_n^{\Lambda}$. Then for any $\mathsf{s}, \mathsf{t} \in \mathrm{Std}(\lambda)$, in Definition 1.4.1 we have defined $\hat{\psi}_{\mathsf{st}}$ and $\psi_{\mathsf{st}} = \hat{\psi}_{\mathsf{st}} + N_n^{\Lambda} \in \mathscr{R}_n^{\Lambda}$. For any standard affine tableau $\hat{\mathsf{s}}$, $\hat{\mathsf{t}}$ we define $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} = \hat{\psi}_{\mathsf{st}}$. Also we can define $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}^* = \psi_{\hat{\mathsf{t}}\hat{\mathsf{s}}}$.

The next Lemma is straightforward by the definition of $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}$ and $\deg(\hat{\mathsf{s}})$.

4.3.4. Lemma. *Suppose $\hat{\lambda} \in \mathscr{P}_n^{\kappa}$ and $\hat{\mathsf{s}}, \hat{\mathsf{t}} \in \mathrm{Std}(\hat{\lambda})$. Then $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}$ are homogeneous elements of $\mathscr{R}_n$ and* $\deg(\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}) = \deg(\hat{\mathsf{s}}) + \deg(\hat{\mathsf{t}})$.

4.3.5. **Example** Suppose $\kappa = (\dots, 0, 2, 1, 0, 2, 1, 0, 2, 1)$ as in Example 4.3.2. For

$$\hat{\mathsf{s}} = \left(\dots \middle| \emptyset \middle| \emptyset \middle| \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \middle| \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array} \middle| \begin{array}{|c|}\hline 6\\\hline\end{array}\right) \qquad \hat{\mathsf{t}} = \left(\dots \middle| \emptyset \middle| \emptyset \middle| \begin{array}{|c|c|c|}\hline 1&2&4\\\hline\end{array} \middle| \begin{array}{|c|}\hline 3\\\hline 6\\\hline\end{array} \middle| \begin{array}{|c|}\hline 5\\\hline\end{array}\right),$$

with

$$\mathsf{s} = t(\hat{\mathsf{s}}) = \left(\boxed{1\,|\,2\,|\,3} \;\middle|\; \begin{array}{|c|}\hline 4\\\hline 5\\\hline\end{array} \;\middle|\; \boxed{6} \right) \qquad \mathsf{t} = t(\hat{\mathsf{t}}) = \left(\boxed{1\,|\,2\,|\,4} \;\middle|\; \begin{array}{|c|}\hline 3\\\hline 6\\\hline\end{array} \;\middle|\; \boxed{5} \right).$$

Then $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} = \hat{\psi}_{\mathsf{st}} = e(012211)y_2y_3^2y_5\psi_5\psi_3 \in \mathscr{R}_n$. ◇

Fix $\alpha \in Q_+$, a standard sequence $(\Lambda^{(k)})$ and an inverse multicharge sequence κ corresponds to $(\Lambda^{(k)})$. We define a set of homogeneous elements of $\mathscr{R}_\alpha$,

$$\mathscr{B}_\alpha^{(\Lambda^{(k)})} = \{\, \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \mid \hat{\lambda} \in \mathscr{P}_n^\kappa, \hat{\mathsf{s}}, \hat{\mathsf{t}} \in \operatorname{Std}(\hat{\lambda}), \operatorname{res}(\hat{\mathsf{t}}) \in I^\alpha \,\}.$$

Note that by definition $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ depends on the choice of κ and hence $(\Lambda^{(k)})$. Remarkably, the main results of this chapter are true for any inverse multicharge sequence corresponds to $(\Lambda^{(k)})$.

4.3.6. Proposition. *The set $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is a homogeneous basis of $\mathscr{R}_\alpha$.*

Proof. By Lemma 4.3.4, all elements of $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ are homogeneous. So we only have to prove that $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is a basis of $\mathscr{R}_\alpha$. First of all we show that $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ spans $\mathscr{R}_\alpha$.

Given any $r \in \mathscr{R}_\alpha$, we can write r as a linear combination of homogeneous elements, i.e. $r = \sum_{i\in\mathbb{N}} c_i r_i$, where $c_i \in \mathbb{Z}$, $\deg(r_i) = i$ and there are only finite many $i \in \mathbb{N}$ with $c_i \neq 0$. It is enough to prove that any homogeneous element $r \in \mathscr{R}_\alpha$ is a linear combination of $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$.

For any $\Lambda < \Lambda'$, it is obvious that $N_\alpha^{\Lambda'} \subseteq N_\alpha^{\Lambda}$. Moreover, $N_\alpha^{\Lambda'}$ is a $\mathscr{R}_\alpha$-ideal of N_α^{Λ}. Hence we can define an infinite filtration

$$\mathscr{R}_\alpha > N_\alpha^{\Lambda^{(1)}} > N_\alpha^{\Lambda^{(2)}} > N_\alpha^{\Lambda^{(3)}} > \dots.$$

By Proposition 4.2.3, $\lim_{k\to\infty} m_\alpha^{\Lambda^{(k)}} = \infty$, so if $r \in \mathscr{R}_\alpha$ is homogeneous then there exists an integer $k(r)$ such that $m_\alpha^{\Lambda^{(k)}} > \deg(r)$ whenever $k > k(r)$. Fix $k > k(r)$ and hence $r \notin N_\alpha^{\Lambda^{(k)}}$.

By Proposition 4.1.5, choosing a multicharge κ corresponding to Λ, $\mathscr{R}_\alpha^\Lambda \cong \mathscr{R}_\alpha/N_\alpha^\Lambda$ has a homogeneous basis $\{\, \psi_{\mathsf{st}} \mid \lambda \in \mathscr{P}_n^\Lambda, \mathsf{s}, \mathsf{t} \in \operatorname{Std}(\lambda), \operatorname{res}(\mathsf{t}) \in I^\alpha \,\}$. Fix a multicharge $(\kappa_{\ell_k}, \kappa_{\ell_{k-1}}, \dots, \kappa_2, \kappa_1)$ corresponding $\Lambda^{(k)}$. For any homogeneous element $r \in \mathscr{R}_\alpha$, we can find $c_{\mathsf{st}} \in \mathbb{Z}$ with $\operatorname{res}(\mathsf{t}) \in I^\alpha$ such that

$$r + N_\alpha^{\Lambda^{(k)}} = \sum_{\mathsf{s},\mathsf{t}} c_{\mathsf{st}}\psi_{\mathsf{st}} = \sum_{\mathsf{s},\mathsf{t}} c_{\mathsf{st}}\hat{\psi}_{\mathsf{st}} + N_\alpha^{\Lambda^{(k)}} = \sum_{\hat{\mathsf{s}},\hat{\mathsf{t}}} c_{\mathsf{st}}\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} + N_\alpha^{\Lambda^{(k)}}$$

$$\Rightarrow \quad r - \sum_{\hat{\mathsf{s}},\hat{\mathsf{t}}} c_{\mathsf{st}}\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in N_\alpha^{\Lambda^{(k)}}.$$

But as r is a homogeneous element which is not in $N_\alpha^{\Lambda^{(k)}}$, we must have $r - \sum_{\hat{\mathsf{s}},\hat{\mathsf{t}}} c_{\mathsf{st}}\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} = 0$, i.e. $r = \sum_{\hat{\mathsf{s}},\hat{\mathsf{t}}} c_{\mathsf{st}}\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}$ with $\operatorname{res}(\hat{\mathsf{t}}) = \operatorname{res}(\mathsf{t}) \in I^\alpha$. This shows that r belongs to the span of $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$. Hence $\mathscr{R}_\alpha$ is spanned by $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$.

Next we will prove that $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is linearly independent. Suppose S_α is a finite subset of $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$. Write $m_{S_\alpha} = \max\{\deg(\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}) \mid \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in S_\alpha\}$. By Proposition 4.2.3 we can find some k such that $m_\alpha^{\Lambda^{(k)}} > m_{S_\alpha}$. Hence $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \notin N_\alpha^{\Lambda^{(k)}}$ for any $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in S_\alpha$. This means that for any $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in S_\alpha$, $\psi_{\mathsf{st}} \in \mathscr{R}_\alpha^{\Lambda^{(k)}}$ is nonzero. As by the definition, $\{\, \psi_{\mathsf{st}} \mid \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in S_\alpha \,\}$ is a subset of the basis of $\mathscr{R}_\alpha^{\Lambda^{(k)}}$. We have $\sum_{\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\in S_\alpha} c_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in N_\alpha^{\Lambda^{(k)}}$ if and only if $\sum_{\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\in S_\alpha} c_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\mathsf{st}} = 0$ if and only if all $c_{\hat{\mathsf{s}}\hat{\mathsf{t}}} = 0$. But $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \notin N_\alpha^{\Lambda^{(k)}}$ for any $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in S_\alpha$, the above result yields that $\sum_{\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\in S_\alpha} c_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} = 0$ if and only if $c_{\hat{\mathsf{s}}\hat{\mathsf{t}}} = 0$. This shows that $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is linearly independent. Hence $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is a basis of $\mathscr{R}_\alpha$. □

Notice that in the definition of $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$, it is well-defined for any inverse multicharge sequence κ corresponds to $(\Lambda^{(k)})$. Hence for any weight Λ with $\ell = l(\Lambda)$, by the definition of the standard sequence, we can set $\Lambda^{(1)} = \Lambda$. Therefore, we obtain a subset of $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$:

$$\mathscr{B}_\Lambda^{(\Lambda^{(k)})} = \{\, \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \mid \hat{\lambda} \in \mathscr{P}_n^\kappa \text{ with } l(\hat{\lambda}) \le \ell, \hat{\mathsf{s}}, \hat{\mathsf{t}} \in \operatorname{Std}(\hat{\lambda}), \operatorname{res}(\hat{\mathsf{t}}) \in I^\alpha \,\}.$$

4.3.7. Corollary. *Suppose Λ is a weight with level ℓ and $(\Lambda^{(k)})$ is a standard sequence with $\Lambda^{(1)} = \Lambda$. Then*

$$\mathscr{B}_\alpha^{(\Lambda^{(k)})} \backslash \mathscr{B}_\Lambda^{(\Lambda^{(k)})} = \{\, \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \mid \hat{\lambda} \in \mathscr{P}_n^\kappa \textit{ with } l(\hat{\lambda}) > \ell = l(\Lambda), \hat{\mathsf{s}}, \hat{\mathsf{t}} \in \operatorname{Std}(\hat{\lambda}), \operatorname{res}(\hat{\mathsf{t}}) \in I^\alpha \,\}$$

is a basis of N_α^Λ.

Proof. By Proposition 4.1.5, $\mathscr{R}_\alpha^\Lambda$ has a basis $\{\, \psi_{\mathsf{st}} \mid \lambda \in \mathscr{P}_n^\Lambda, \mathsf{s}, \mathsf{t} \in \operatorname{Std}(\lambda), \operatorname{res}(\mathsf{t}) \in I^\alpha \,\}$. It is easy to see that when $\Lambda^{(1)} = \Lambda$,

$$\{\, \psi_{\mathsf{st}} \mid \lambda \in \mathscr{P}_n^\Lambda, \mathsf{s}, \mathsf{t} \in \operatorname{Std}(\lambda), \operatorname{res}(\mathsf{t}) \in I^\alpha \,\} = \{\, \psi_{\mathsf{st}} = \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} + N_n^\Lambda \mid \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in \mathscr{B}_\Lambda^{(\Lambda^{(k)})} \,\}.$$

So for $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in \mathscr{B}_\Lambda^{(\Lambda^{(k)})}$, we must have $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \notin N_\alpha^\Lambda$.

Now suppose $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in \mathscr{B}_\alpha^{(\Lambda^{(k)})} \backslash \mathscr{B}_\Lambda^{(\Lambda^{(k)})}$. Then $\hat{\mathsf{s}}, \hat{\mathsf{t}} \in \operatorname{Std}(\hat{\lambda})$ with $l(\hat{\lambda}) > \ell$. By the definition it is obvious that $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in N_\alpha^\Lambda$ when $\Lambda^{(1)} = \Lambda$. Then N_α^Λ is spanned by $\mathscr{B}_n^{(\Lambda^{(k)})} \backslash \mathscr{B}_\Lambda^{(\Lambda^{(k)})}$. $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is a basis implies the linearly independence of $\mathscr{B}_\alpha^{(\Lambda^{(k)})} \backslash \mathscr{B}_\Lambda^{(\Lambda^{(k)})}$. So $\mathscr{B}_\alpha^{(\Lambda^{(k)})} \backslash \mathscr{B}_\Lambda^{(\Lambda^{(k)})}$ is a basis of N_α^Λ. □

Recall for any $\hat{\mathsf{s}}, \hat{\mathsf{t}} \in \operatorname{Std}(\hat{\lambda})$ with $\hat{\lambda} \in \mathscr{P}_n^\kappa$, we define $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}^* = \psi_{\hat{\mathsf{t}}\hat{\mathsf{s}}}$. By Proposition 4.3.6, $*$ can be defined to be a linear bijection from $\mathscr{R}_\alpha$ to $\mathscr{R}_\alpha$. The next Corollary is straightforward by Corollary 4.3.7.

4.3.8. Corollary. *Suppose $* : \mathscr{R}_\alpha \longrightarrow \mathscr{R}_\alpha$ is defined as above. Then it can be restricted to a linear bijection $* : N_\alpha^\Lambda \longrightarrow N_\alpha^\Lambda$.*

Now we can prove the main result of this chapter.

4.3.9. Proposition. *The set $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is a graded cellular basis of $\mathscr{R}_\alpha$.*

Proof. Recall Definition 1.2.1 gives the definition of graded cellular basis. Proposition 4.3.6 shows that $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is a homogeneous basis of $\mathscr{R}_\alpha$. To prove the Theorem we need to establish properties 1.2.1(b) and 1.2.1(c) of $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$.

Suppose a is an element of $\mathscr{R}_\alpha$ and $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \in \mathscr{B}_\alpha^{(\Lambda^{(k)})}$ with $\hat{\mathsf{s}}, \hat{\mathsf{t}} \in \operatorname{Std}(\hat{\lambda})$. We can write $a = \sum_{i \in \mathbb{N}} c_i a_i$ where $c_i \in \mathbb{Z}$ and a_i are homogeneous elements in $\mathscr{R}_\alpha$ with $\deg(a_i) = i$. Define $d_1 = \deg(\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}})$ and $d_2 = \max\{i \mid c_i \neq 0\}$. By Proposition 4.2.3 we can find some k such that $m_\alpha^{\Lambda^{(k)}} > \max\{d_1, d_2, d_1 + d_2\}$. This means that $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}$, a and $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} a$ are not elements of $N_\alpha^{\Lambda^{(k)}}$. This means that $\psi_{\mathsf{st}} = \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} + N_\alpha^{\Lambda^{(k)}}$, $a + N_\alpha^{\Lambda^{(k)}}$ and $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} a + N_\alpha^{\Lambda^{(k)}}$ are nonzero elements of $\mathscr{R}_\alpha^{\Lambda^{(k)}}$. By Proposition 4.1.5 and because t is a bijection,

$$\psi_{\mathsf{st}}(a + N_\alpha^{\Lambda^{(k)}}) = (\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} + N_\alpha^{\Lambda^{(k)}})(a + N_\alpha^{\Lambda^{(k)}}) = \sum_{\mathsf{v} \in \operatorname{Std}(\lambda)} c_{\mathsf{sv}} \psi_{\mathsf{sv}} + \sum_{\substack{\mathsf{u},\mathsf{v} \in \operatorname{Std}(\mu) \\ \mu \rhd \lambda}} c_{\mathsf{uv}} \psi_{\mathsf{uv}}$$

$$\Rightarrow \quad \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} a + N_\alpha^{\Lambda^{(k)}} = \sum_{\hat{\mathsf{v}} \in \operatorname{Std}(\hat{\lambda})} c_{\hat{\mathsf{s}}\hat{\mathsf{v}}} \psi_{\hat{\mathsf{s}}\hat{\mathsf{v}}} + \sum_{\substack{\hat{\mathsf{u}},\hat{\mathsf{v}} \in \operatorname{Std}(\hat{\mu}) \\ \hat{\mu} \rhd \hat{\lambda}}} c_{\hat{\mathsf{u}}\hat{\mathsf{v}}} \psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}} + N_\alpha^{\Lambda^{(k)}}$$

$$\Rightarrow \quad \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} a - \Big(\sum_{\hat{\mathsf{v}} \in \operatorname{Std}(\hat{\lambda})} c_{\hat{\mathsf{s}}\hat{\mathsf{v}}} \psi_{\hat{\mathsf{s}}\hat{\mathsf{v}}} + \sum_{\substack{\hat{\mathsf{u}},\hat{\mathsf{v}} \in \operatorname{Std}(\hat{\mu}) \\ \hat{\mu} \rhd \hat{\lambda}}} c_{\hat{\mathsf{u}}\hat{\mathsf{v}}} \psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}} \Big) \in N_\alpha^{\Lambda^{(k)}}.$$

Since the left hand side of the above equation is homogeneous to d_1+d_2 and $m_\alpha^{\Lambda^{(k)}} > d_1+d_2$, we can see that

$$\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}a = \sum_{\hat{\mathsf{v}}\in\mathrm{Std}(\hat\lambda)} c_{\hat{\mathsf{s}}\hat{\mathsf{v}}}\psi_{\hat{\mathsf{s}}\hat{\mathsf{v}}} + \sum_{\substack{\hat{\mathsf{u}},\hat{\mathsf{v}}\in\mathrm{Std}(\hat\mu)\\ \hat\mu\rhd\hat\lambda}} c_{\hat{\mathsf{u}}\hat{\mathsf{v}}}\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}}.$$

which shows that $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ satisfies 1.2.1(b).

For 1.2.1(c), choose arbitrary $\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}, \psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}} \in \mathscr{B}_\alpha^{(\Lambda^{(k)})}$. Suppose $\deg(\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}) = k_1$ and $\deg(\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}}) = d_2$. By Proposition 4.2.3 we can choose k such that $m_\alpha^{\Lambda^{(k)}} > \max\{k_1, k_2, k_1+k_2\}$. Then by Corollary 4.3.8,

$$\begin{aligned}(\psi_{\mathsf{st}}\psi_{\mathsf{uv}})^* &= ((\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} + N_\alpha^{\Lambda^{(k)}})(\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}} + N_\alpha^{\Lambda^{(k)}}))^* = (\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}} + N_\alpha^{\Lambda^{(k)}})^* = (\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}})^* + N_\alpha^{\Lambda^{(k)}},\\ \psi_{\mathsf{vu}}\psi_{\mathsf{ts}} &= (\psi_{\hat{\mathsf{v}}\hat{\mathsf{u}}} + N_\alpha^{\Lambda^{(k)}})(\psi_{\hat{\mathsf{t}}\hat{\mathsf{s}}} + N_\alpha^{\Lambda^{(k)}}) = \psi_{\hat{\mathsf{v}}\hat{\mathsf{u}}}\psi_{\hat{\mathsf{t}}\hat{\mathsf{s}}} + N_\alpha^{\Lambda^{(k)}},\end{aligned}$$

which implies that $(\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}})^* - \psi_{\hat{\mathsf{v}}\hat{\mathsf{u}}}\psi_{\hat{\mathsf{t}}\hat{\mathsf{s}}} = N_\alpha^{\Lambda^{(k)}}$. Then because $m_\alpha^{\Lambda^{(k)}} > k_1+k_2$, $(\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}})^* - \psi_{\hat{\mathsf{v}}\hat{\mathsf{u}}}\psi_{\hat{\mathsf{t}}\hat{\mathsf{s}}} = 0$, i.e. $(\psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}}\psi_{\hat{\mathsf{u}}\hat{\mathsf{v}}})^* = \psi^*_{\hat{\mathsf{u}}\hat{\mathsf{v}}}\psi^*_{\hat{\mathsf{t}}\hat{\mathsf{s}}}$. Because $*$ is a linear bijection and $\mathscr{B}_\alpha^{(\Lambda^{(k)})}$ is a basis of $\mathscr{R}_\alpha$, this shows that $*: \mathscr{R}_\alpha \longrightarrow \mathscr{R}_\alpha$ is an anti-isomorphism. Hence $*$ satisfies 1.2.1(c). This completes the proof. □

Combining the above two Propositions and Corollary 4.3.7 we can get the following results.

4.3.10. Theorem. *For any standard sequence* $(\Lambda^{(k)})$, *the set*

$$\mathscr{B}_n^{(\Lambda^{(k)})} = \{\, \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \mid \hat\lambda \in \mathscr{P}_n^\kappa, \hat{\mathsf{s}}, \hat{\mathsf{t}} \in \mathrm{Std}(\hat\lambda)\,\}$$

is a graded cellular basis of $\mathscr{R}_n$.

Proof. By definition we have $\mathscr{B}_n^{(\Lambda^{(k)})} = \bigoplus_{\alpha\in Q_+} \mathscr{B}_\alpha^{(\Lambda^{(k)})}$ and $\mathscr{R}_n = \bigoplus_{\alpha\in Q_+} \mathscr{R}_\alpha$. By the relations of $\mathscr{R}_n$ we can see that $\mathscr{R}_\alpha$ are subalgebras. The Theorem follows by Proposition 4.3.9 straightforward. □

4.3.11. Corollary. *Suppose* Λ *is a weight with level* ℓ *and* $(\Lambda^{(k)})$ *is a standard sequence with* $\Lambda^{(1)} = \Lambda$. *Then*

$$\{\, \psi_{\hat{\mathsf{s}}\hat{\mathsf{t}}} \mid \hat\lambda \in \mathscr{P}_n^\kappa \text{ with } l(\hat\lambda) > \ell = l(\Lambda), \hat{\mathsf{s}}, \hat{\mathsf{t}} \in \mathrm{Std}(\hat\lambda)\,\}$$

is a basis of N_n^Λ.

4.4. Graded simple $\mathscr{R}_n$-modules

Theorem 4.3.10 gives a graded cellular basis of $\mathscr{R}_n$. Graham and Lehrer [**7**] described a complete set of irreducible representations of finite dimensional cellular algebra, however, their results do not apply to $\mathscr{R}_n$ because it is an infinite dimensional algebra. In this section we use Graham and Lehrer's results to construct a complete set of graded simple $\mathscr{R}_n$-modules. The graded simple $\mathscr{R}_n$-modules have been desccribed by Brundan and Kleshchev [**4**, Theorem 5.19]. See Remark 4.4.15 for more detials.

First we need to state some properties of the simple $\mathscr{R}_n$-modules. We start by showing that the graded dimension of an simple $\mathscr{R}_n$-module is bounded below.

4.4.1. Lemma. *Suppose* $r \in \mathscr{R}_n$ *is a homogeneous element. Then* $\deg(r) \geq -n(n-1)$.

Proof. By (1.4.7) we have the following basis of $\mathscr{R}_n$:

$$\{\, \hat e(\mathbf{i})\hat y_1^{\ell_1}\hat y_2^{\ell_2}\dots\hat y_n^{\ell_n}\hat\psi_w \mid \mathbf{i}\in I^n, w\in\mathfrak{S}_n, \ell_1,\ell_2,\dots,\ell_n \geq 0\,\}.$$

If $\mathbf{i}\in I^n$, $w\in\mathfrak{S}_n$ and $\ell_1,\ell_2,\dots,\ell_n\geq 0$, then

$$\begin{aligned}\deg(\hat e(\mathbf{i})\hat y_1^{\ell_1}\hat y_2^{\ell_2}\dots\hat y_n^{\ell_n}\hat\psi_w) &\geq \deg(\hat e(\mathbf{i})\hat y_1^{\ell_1}\hat y_2^{\ell_2}\dots\hat y_n^{\ell_n}) + \deg(\hat e(\mathbf{i})\hat\psi_w)\\ &= 2(\ell_1+\ell_2+\ldots+\ell_n) + \deg(\hat e(\mathbf{i})\hat\psi_w)\\ &\geq \deg(\hat e(\mathbf{i})\hat\psi_w).\end{aligned}$$

As $w \in \mathfrak{S}_n$, we have $l(w) \le \frac{n(n-1)}{2}$ and by definition, $\hat{e}(\mathbf{i})\hat{\psi}_r \ge -2$ for any r and $\mathbf{i} \in I^n$. Therefore

$$\deg(\hat{e}(\mathbf{i})\hat{\psi}_w) \ge -2 \times \frac{n(n-1)}{2} = -n(n-1).$$

Hence $\deg(\hat{e}(\mathbf{i})\hat{y}_1^{\ell_1}\hat{y}_2^{\ell_2}\dots\hat{y}_n^{\ell_n}\hat{\psi}_w) \ge -n(n-1)$. This completes the proof. □

Recall that for $|\alpha| = \sum_{i\in I} a_i$ and $\hat{e}_\alpha = \sum_{\mathbf{i}\in I^\alpha} \hat{e}(\mathbf{i}) \in \mathscr{R}_n$.

4.4.2. Lemma. *Suppose S is a simple $\mathscr{R}_n$-module. Then there exists $\alpha \in Q_+$ with $|\alpha| = n$ such that for any $\beta \in Q_+$ with $|\beta| = n$, $\hat{e}_\beta S = \delta_{\alpha\beta} S$.*

Proof. Suppose S is a simple $\mathscr{R}_n$-module. Because $1 = \sum_{\mathbf{j}\in I^n} \hat{e}(\mathbf{j}) = \sum_{\substack{\beta\in Q_+\\|\beta|=n}} \hat{e}_\beta$, we can write $S = \bigoplus_{\substack{\beta\in Q_+\\|\beta|=n}} \hat{e}_\beta S$. Suppose $\hat{e}_\alpha S \ne 0$ for some $\alpha \in Q_+$. Choose any nonzero element $s \in S$ and $\beta \in Q_+$ with $\beta \ne \alpha$. By Lemma 4.1.4, $\hat{e}_\alpha \mathscr{R}_n \hat{e}_\beta = 0$. So we must have $\hat{e}_\beta \cdot s = 0$. Hence $\hat{e}_\beta S = 0$. Therefore $S = \bigoplus_{\substack{\beta\in Q_+\\|\beta|=n}} \hat{e}_\beta S = \hat{e}_\alpha S$. This completes the proof. □

It is well-known that the irreducible representations of the affine Hecke algebra are finite dimensional as, by Bernstein, the affine Hecke algebra is finite dimensional over its centre. See for example, Proposition 4.1 and Corollary 4.2 of Grojnowski [**8**], or Proposition 2.12 of Khovanov-Lauda [**13**]. The next Proposition gives a different approach.

4.4.3. Proposition. *Suppose S is a graded simple $\mathscr{R}_n$-module and $\alpha \in Q_+$ is such that $\hat{e}_\beta S = \delta_{\alpha\beta} S$ for $\beta \in Q_+$. If $\Lambda \in P_+$ with $m_\alpha^\Lambda > n(n-1)$, then S is isomorphic to a graded simple $\mathscr{R}_n^\Lambda$-module.*

Proof. By Lemma 4.4.2 we can find $\alpha \in Q_+$ such that $\hat{e}_\beta S = \delta_{\alpha\beta} S$ for $\beta \in Q_+$. Then we choose an arbitrary nonzero homogeneous element $s \in S$ and suppose $\deg(s) = d$. Now for any nonzero homogeneous element $t \in S$, because S is simple, we can find a homogeneous element $a \in \mathscr{R}_\alpha$ such that $t = a\cdot s$. Therefore

$$\deg(t) = \deg(a\cdot s) = \deg(a) + \deg(s) \ge d - n(n-1)$$

where by Lemma 4.4.1 we have $\deg(a) \ge -n(n-1)$. So for any homogeneous nonzero element $t \in S$, we have

$$\deg(t) \ge d - n(n-1). \tag{4.4.4}$$

Similarly, since for any nonzero homogeneous element $t \in \mathscr{R}_n$ we can find homogeneous element $a \in \mathscr{R}_\alpha$ such that $s = a\cdot t$, we have

$$\deg(t) \le d + n(n-1). \tag{4.4.5}$$

Combining (4.4.4) and (4.4.5), we have $|\deg(s) - \deg(t)| \le n(n-1)$ for any nonzero homogeneous element $t \in S$. Because s is chosen arbitrarily, we have

$$|\deg(s) - \deg(t)| \le n(n-1) \tag{4.4.6}$$

for any nonzero homogeneous elements $s, t \in S$.

Suppose $\Lambda \in P_+$ with $m_\alpha^\Lambda > n(n-1)$. For any homogeneous element $a \in N_\alpha^\Lambda$ and $t \in S$, we have $a\cdot t = 0$ because $\deg(a\cdot t) - \deg(t) = \deg(a) > n(n-1)$ and (4.4.6).

For any $s \in S$, we can define a map $f : \mathscr{R}_n \longrightarrow S$ by sending a to $a\cdot s$. It is a homomorphism and it is obvious that $N_\alpha^\Lambda \subseteq \ker f$. If $\beta \in Q_+$ and $\beta \ne \alpha$, then by Lemma 4.4.2 we have $\hat{e}_\beta \cdot s = 0$. Therefore $N_\beta^\Lambda \subseteq \ker f$. Hence $N_n^\Lambda \subseteq \ker f$. Therefore we can consider S as a simple $\mathscr{R}_n/N_n^\Lambda$-module, i.e. $\mathscr{R}_n^\Lambda$-module. This completes the proof. □

4.4.7. Corollary. *Suppose S is a simple graded $\mathscr{R}_n$-module. Then S is finite-dimensional.*

Building on Ariki's [**1**] work in the ungraded case, Hu and Mathas [**9**] constructed all graded simple $\mathscr{R}_n^\Lambda$-modules in the sense of Graham-Lehrer [**7**]. They proved that, up to shift, graded simple $\mathscr{R}_n^\Lambda$-modules are labeled by the **Kleshchev multipartitions** of n, which were introduced by Ariki and Mathas [**2**]. Readers may also refer to Brundan and Kleshchev [**4**, (3.27)] (where they are called **restricted multipartitions**).

Suppose $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(\ell)}) \in \mathscr{P}_n^\Lambda$ and we consider the Young diagram $[\lambda]$. Let $\gamma = (r, c, l)$ be a node in the Young diagram with residue i, i.e. $i \equiv r - c + \kappa_l \pmod e$. Then γ is an addable i-node if $\gamma \notin [\lambda]$ and $[\lambda] \cup \{\gamma\}$ is the Young diagram of a multipartition, and γ is a removable i-node if $\gamma \in [\lambda]$ and $[\lambda] \backslash \{\gamma\}$ is the Young diagram of a multipartition.

For each $\lambda \in \mathscr{P}_n^\Lambda$, we read all addable and removable i-nodes in the following order: we start with the first row of $\lambda^{(1)}$, and then read rows in $\lambda^{(1)}$ downward. We then read the first row of $\lambda^{(2)}$, and repeat the same procedure, until we finish reading all rows of λ. We write A for an addable i-node, and R for a removable i-node. Hence we get a sequence of A and R. We then delete RA as many as possible. For example, if we have a sequence $RARARRAAARRAR$, the resulting sequence will be $---------AR--R$. The node corresponding to the leftmost R is the **good i-node**.

The Kleshchev multipartition can then be defined recursively as follows.

4.4.8. Definition. [**1**, Definition 2.3] *We declare that $\emptyset$ is Kelshchev. Assume that we have already defined the set of Kleshchev multipartitions up to size $n-1$. Let λ be a multipartition of n. We say that λ is a Kleshchev multipartition if there is a good node γ in $[\lambda]$ such that if $[\mu] = [\lambda] \backslash \{\gamma\}$ and μ is a Kleshchev multipartition.*

Let $\mathscr{P}_0^\Lambda$ be the set of Kleshchev multipartitions in $\mathscr{P}_n^\Lambda$. Let S^λ be the cell module of $\mathscr{R}_n^\Lambda$(it is called the **Specht module** in $\mathscr{R}_n^\Lambda$), which was introduced in Section 1.2, and $D^\lambda = S^\lambda/\text{rad}\, S^\lambda$. By Hu-Mathas [**9**, Corollary 5.11] we can give a set of complete non-isomorphic graded simple $\mathscr{R}_n^\Lambda$-modules. Brundan and Kleshchev [**4**, Theorem 4.11] gives the same classification.

4.4.9. Theorem. *The set*

$$\{\, D^\lambda\langle k\rangle \mid \lambda \in \mathscr{P}_0^\Lambda, k \in \mathbb{Z}\,\}$$

is a complete set of pairwise non-isomorphic graded simple $\mathscr{R}_n^\Lambda$-modules.

We can consider S^λ and $D^\lambda\langle k\rangle$ as $\mathscr{R}_n$-modules. The actions of $\hat{e}(\mathbf{i})$, $\hat{y}_r$ and $\hat{\psi}_s$ on S^λ and $D^\lambda\langle k\rangle$ are the same as the actions of $e(\mathbf{i})$, y_r and ψ_s. Therefore $D^\lambda\langle k\rangle$ is also a simple $\mathscr{R}_n$-module. Hence we can define a set of graded simple $\mathscr{R}_n$-modules similar as in Theorem 4.4.9.

The next Lemma is straightforward by the definition of D^λ.

4.4.10. Lemma. *Suppose $\lambda, \mu \in \mathscr{P}_n^\Lambda$. Then $D^\lambda \cong D^\mu$ as $\mathscr{R}_n^\Lambda$-modules if and only if $D^\lambda \cong D^\mu$ as $\mathscr{R}_n$-modules.*

Now we can classify all graded simple $\mathscr{R}_n$-modules. Following the process in Section 1.2, for each $\hat{\lambda} \in \mathscr{P}_n^\kappa$ we can define the cell module $S^{\hat{\lambda}}$ of $\mathscr{R}_n$(which is called the **Specht module** as well), associated with a bilinear form $\langle\cdot,\cdot\rangle$. Then we can define rad $S^{\hat{\lambda}}$ and hence a graded simple module $D^{\hat{\lambda}} = S^{\hat{\lambda}}/\text{rad}\, S^{\hat{\lambda}}$.

4.4.11. Lemma. *Suppose $\hat{\lambda} \in \mathscr{P}_n^\kappa$ and $\mu = p_k(\hat{\lambda})$ for some $k \geq l(\hat{\lambda})$. Then $S^\mu \cong S^{\hat{\lambda}}$ as $\mathscr{R}_n$-modules.*

Proof. It is trivial by the definition of Specht modules in $\mathscr{R}_n$ and $\mathscr{R}_n^\Lambda$. □

The next Corollary is straightforward by Lemma 4.4.11.

4.4.12. Corollary. *Suppose $\hat{\lambda} \in \mathscr{P}_n^\kappa$ and $\mu = p_k(\hat{\lambda})$ for some $k \geq l(\hat{\lambda})$. Then $D^\mu \cong D^{\hat{\lambda}}$ as $\mathscr{R}_n$-modules.*

Hence we can prove the following Lemma.

4.4.13. Lemma. *Suppose* $\hat{\lambda}, \hat{\mu} \in \mathscr{P}_n^\kappa$*. Then* $D^{\hat{\lambda}} \cong D^{\hat{\mu}}$ *if and only if* $\hat{\lambda} = \hat{\mu}$*.*

Proof. The if part is trivial. Now suppose $D^{\hat{\lambda}} \cong D^{\hat{\mu}}$. Choose $k > \max\{l(\hat{\lambda}), l(\hat{\mu})\}$ and set $\nu = p_k(\hat{\lambda})$ and $\sigma = p_k(\hat{\mu})$. Then by Corollary 4.4.12 we have $D^\nu \cong D^\sigma$ as $\mathscr{R}_n$-modules. Then Theorem 4.4.9 and Lemma 4.4.10 implies $\nu = \sigma$. Therefore by the definition of k we have $\hat{\lambda} = \hat{\mu}$. This completes the proof. □

Now we extend Kleshchev multipartitions to affine multipartitions. Define $\hat{\lambda} \in \mathscr{P}_n^\kappa$ to be an **affine Kleshchev multipartition** if λ is a Kleshchev multipartition and $\mathscr{P}_0^\kappa$ as the set of all affine Kleshchev multipartitions in $\mathscr{P}_n^\kappa$. Hence we can give a complete set of pairwise non-isomorphic graded simple $\mathscr{R}_n$-modules.

4.4.14. Theorem. *The set*

$$\{ D^{\hat{\lambda}}\langle k\rangle \mid \hat{\lambda} \in \mathscr{P}_0^\kappa, k \in \mathbb{Z} \}$$

is a complete set of pairwise non-isomorphic graded simple $\mathscr{R}_n$*-modules.*

Proof. By the definition of (affine) Kleshchev multipartitions, [**9**, Corollary 5.11] and Corollary 4.4.12, $D^{\hat{\lambda}}\langle k\rangle \cong D^\lambda\langle k\rangle \neq 0$ if and only if $\hat{\lambda} \in \mathscr{P}_0^\kappa$.

Suppose S is a graded simple $\mathscr{R}_n$-module. By Lemma 4.4.2 we can find $\alpha \in Q_+$ such that $\hat{e}_\beta S = \delta_{\alpha\beta} S$ for $\beta \in Q_+$. Then by Proposition 4.2.3 we can choose i such that $m_\alpha^{\Lambda^{(i)}} > n(n-1)$ and hence by Proposition 4.4.3, S is isomorphic to a graded simple $\mathscr{R}_n^{\Lambda^{(i)}}$-module. Therefore by Theorem 4.4.9 we can find some $\mu \in \mathscr{P}_n^{\Lambda^{(i)}}$ and $k \in \mathbb{Z}$ such that $S \cong D^\mu\langle k\rangle$ as $\mathscr{R}_n^{\Lambda^{(i)}}$-modules, and hence as $\mathscr{R}_n$-modules. Suppose $l(\mu) = \ell$. We can choose $\hat{\lambda} \in \mathscr{P}_n^\kappa$ such that $p_\ell(\hat{\lambda}) = \mu$ with $l(\hat{\lambda}) \leq \ell$. By Corollary 4.4.12 we have $D^{\hat{\lambda}} \cong D^\mu$ as $\mathscr{R}_n$-modules. Therefore $S \cong D^{\hat{\lambda}}\langle k\rangle$. So

$$\{ D^{\hat{\lambda}}\langle k\rangle \mid \hat{\lambda} \in \mathscr{P}_0^\kappa, k \in \mathbb{Z} \}$$

is a complete set of graded simple $\mathscr{R}_n$-modules.

By Lemma 4.4.13, the set $\{ D^{\hat{\lambda}}\langle k\rangle \mid \hat{\lambda} \in \mathscr{P}_0^\kappa, k \in \mathbb{Z} \}$ is a set of pairwise non-isomorphic graded $\mathscr{R}_n$-modules. This completes the proof. □

4.4.15. Remark. Ariki-Mathas [**2**] showed that the simple H_n-modules are indexed by aperiodic multisegments. Khovanov and Lauda [**13, 12**] also gives a classification of all graded simple $\mathscr{R}_n$-modules of arbitrary type. Interested readers may also refer to [**15**], [**4**], [**16**], [**19**], [**27**], [**11**] and [**22**]. As far as we are aware the construction and classification in Theorem 4.4.14 is new.

CHAPTER 5

Idempotents and Jucys-Murphy Elements

In this chapter we will give an explicit expression for the KLR idempotent $e(\mathbf{i})$ using the generators of cyclotomic Hecke algebras H_n^Λ when $e > 0$ and $p > 0$, and show the periodic properties of Jucys-Murphy elements x_r and X_r in H_n^Λ. The main idea is using the nilpotency of y_r's in cyclotomic KLR algebras. Recall that H_n^Λ can either be a degenerate or non-degenerate Hecke algebra. We will work with these cases separately in this chapter.

5.1. Explicit expression of $e(\mathbf{i})$

Recall the cyclotomic Hecke algebras H_n^Λ introduced in (1.2.7). In this section we will introduce the detailed definition of $e(\mathbf{i})$ in H_n^Λ.

We can define a set of pairwise orthogonal idempotents $\{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$ for both degenerate and non-degenerate cases. Brundan and Kleshchev [**3**] defined $e(\mathbf{i})$ of $\mathscr{R}_n^\Lambda$ in H_n^Λ in the sense of $\mathscr{R}_n^\Lambda \cong H_n^\Lambda$. Suppose M is a finite dimensional H_n^Λ-module. By Kleshchev [**17**, Lemma 7.1], the eigenvalues of each x_r or X_r on M belongs to I. So $M = \bigoplus_{\mathbf{i} \in I^d} M_{\mathbf{i}}$ of its weight space

$$M_{\mathbf{i}} = \{v \in M \mid (x_r - q_{i_r})^N v = 0 \text{ for all } r = 1, \dots, d \text{ and } N \gg 0\},$$

where q_{i_r} is introduced in (1.2.6). Then we deduce that there is a system $\{\, e(\mathbf{i}) \mid \mathbf{i} \in I^n \,\}$ such that $e(\mathbf{i})M = M_{\mathbf{i}}$.

Murphy [**23**, (1,2)] defined a complete set of primitive orthogonal idempotents in symmetric groups and Mahtas [**21**, Definition 3.1] generalized this result to the degenerate and non-degenerate cyclotomic Hecke algebras.

Hu and Mathas [**9**, Lemma 4.1] proves that the idempotents $e(\mathbf{i})$ in H_n^Λ is equivalent to the primitive idempotents defined by Mathas in H_n^Λ. Murphy's approach gives an explicit formula for the idempotents $e(\mathbf{i})$. Unfortunatesly, it is not very efficient for actual calculations. Recall e and p are parameters of H_n^Λ defined in Section 1.2. We call p the **characteristic** of H_n^Λ and e the **quantum characteristic** of H_n^Λ. In this section we will give a more explicit expression of $e(\mathbf{i})$ in degenerate and non-degenerate cyclotomic Hecke algebras using x_r or X_r when $p > 0$ and $e > 0$.

In the rest of this chapter we fix $p > 0$ and $e > 0$. We have the following well-known facts.

5.1.1. Lemma. *Suppose $\mathbb{F}_p$ is a field with char $\mathbb{F}_p = p > 0$ and $r_1, r_2 \in H_n^\Lambda$. For any non-negative integer k we always have $(r_1 - r_2)^{p^k} = r_1^{p^k} - r_2^{p^k}$.*

5.1.2. Remark. Notice the above Lemma is a well-known result and will be applied without mention in this chapter.

5.1.3. Lemma. *Suppose p and e are characteristic and quantum characteristic of non-degenerate H_n^Λ with $e, p > 0$. Then gcd$(e, p) = 1$. Moreover, we can find l such that $p^l \equiv 1 \pmod{e}$.*

Proof. In non-degenerate case, $\gcd(e, p) = 1$ is well-known. So by Chinese Remainder Theorem we can find $a, b \in \mathbb{Z}$ such that $ap + be = 1$. Now consider the sequence $p, p^2, p^3, p^4 \dots$. We can find k_1, k_2 such that $p^{k_1} \equiv p^{k_2} \pmod{e}$. Choose k_2 such that $k_2 - k_1 > k_1$. Hence write

$l = k_2 - k_1$ and $p^l \equiv s \pmod e$ where $0 \le s \le e-1$. So we have $p^{2l} \equiv p^l \pmod e$ which implies $s^2 \equiv s \pmod e$. So we can write $s^2 - s = ke$ for some $k \in \mathbb{Z}$. So

$$s^2 - s = ke \quad \Rightarrow \quad as(s-1) = ake \quad \Rightarrow \quad (1-be)(s-1) = ake \quad \Rightarrow \quad s-1 = (b(s-1)+ak)e$$

which implies $e \mid s-1$. But because $0 \le s \le e-1$, we have $s = 1$. Therefore $p^l \equiv 1 \pmod e$. □

In the degenerate case we have $e = p$ and in non-degenerate case we have $\gcd(e,p) = 1$. Fix a residue sequence $\mathbf{i} = (i_1, i_2, \ldots, i_n)$. For any $1 \le r \le n$ and any $j \in I$ with $j \ne i_r$, choose $N \gg 0$ and define $L_{i_r,j} = 1 - (\frac{q_{i_r}-x_r}{q_{i_r}-q_j})^N$ in both degenerate and non-degenerate cases.

Notice that by the definition of $e(\mathbf{i})$ given by Brundan and Kleshchev [**3**], for any $\mathbf{j} \in I$ and $1 \le r \le n$, we have

$$(x_r - q_{j_r})^N e(\mathbf{j}) = 0$$

for $N \gg 0$.

5.1.4. Lemma. *Suppose* $1 \le r \le n$ *and* $\mathbf{j} = (j_1, j_2, \ldots, j_n) \in I^n$*, for* $j \in I$ *and* $N_j \gg 0$ *we have*

$$L_{i_r,j}^{N_j} e(\mathbf{j}) = \begin{cases} e(\mathbf{j}), & \text{if } j_r = i_r, \\ 0, & \text{if } j_r = j. \end{cases}$$

Proof. Suppose $j_r = i_r$. Because $(x_r - q_{i_r})^N e(\mathbf{j}) = (x_r - q_{j_r})^N e(\mathbf{j}) = 0$ for $N \gg 0$, we have

$$L_{i_r,j} e(\mathbf{j}) = (1 - (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^N) e(\mathbf{j}) = e(\mathbf{j}) - \frac{1}{(q_{i_r} - q_j)^N}(q_{i_r} - x_r)^N e(\mathbf{j}) = e(\mathbf{j}).$$

Therefore $L_{i_r,j}^{N_j} e(\mathbf{j}) = L_{i_r,j}^{N_j-1} e(\mathbf{j}) = \ldots = L_{i_r,j} e(\mathbf{j}) = e(\mathbf{j})$.

Suppose $j_r = j$. We have

$$\begin{aligned} L_{i_r,j} &= 1 - (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^N \\ &= -\sum_{k=1}^{N} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k + \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k \\ &= \frac{x_r - q_{i_r}}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k + \frac{q_{i_r} - q_j}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k \\ &= (x_r - q_{i_r} + q_{i_r} - q_j) \frac{1}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k \\ &= (\frac{1}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k)(x_r - q_j). \end{aligned}$$

Therefore for $N_j \gg 0$,

$$L_{i_r,j}^{N_j} e(\mathbf{j}) = (\frac{1}{q_{i_r} - q_j} \sum_{k=0}^{N-1} (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^k)^{N_j} (x_r - q_j)^{N_j} e(\mathbf{j}) = 0$$

because when $j_r = j$ we have $(x_r - q_j)^{N_j} e(\mathbf{j}) = 0$, which completes the proof. □

Now we define $L_r(\mathbf{i}) = \prod_{\substack{j \in I \\ j \ne i_r}} L_{i_r,j}$. In the product, $j \in I \backslash \{i_r\}$, which is a finite product since $e > 0$. So $L_r(\mathbf{i})$ is well defined. We have the following Lemma.

5.1.5. Lemma. *Suppose* $1 \le r \le n$. *We can choose* $N_r(\mathbf{i}) \gg 0$ *such that*

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = \sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} e(\mathbf{j}).$$

Proof. By Lemma 5.1.4, for any $j \in I$ with $j \neq i_r$ we can find N_j large enough such that

$$L_{i_r,j}^{N_j} e(\mathbf{j}) = \begin{cases} e(\mathbf{j}), & \text{if } j_r = i_r, \\ 0, & \text{if } j_r = j. \end{cases}$$

Now choose $N_r(\mathbf{i}) \geq \max\{N_j \mid j \in I, j \neq i_r\}$, which is finite since $e > 0$. Therefore $L_r(\mathbf{i})^{N_r(\mathbf{i})} = \prod_{\substack{j\in I \\ j\neq i_r}} L_{i_r,j}^{N_r(\mathbf{i})}$. Hence for any $e(\mathbf{j})$, if $j_r \neq i_r$,

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = \prod_{\substack{j\in I \\ j\neq i_r}} L_{i_r,j}^{N_r(\mathbf{i})} e(\mathbf{j}) = (\prod_{\substack{j\in I \\ j\neq i_r, j_r}} L_{i_r,j}^{N_r(\mathbf{i})}) L_{r,j_r}^{N_r(\mathbf{i})} e(\mathbf{j}) = 0, \tag{5.1.6}$$

and if $j_r = i_r$,

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = e(\mathbf{j}), \tag{5.1.7}$$

because for any j, $L_{i_r,j}^{N_j} e(\mathbf{j}) = e(\mathbf{j})$.

Therefore, because $\sum_{\mathbf{j}\in I^n} e(\mathbf{j}) = 1$, by (5.1.6) and (5.1.7),

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = L_r(\mathbf{i})^{N_r(\mathbf{i})}(\sum_{\mathbf{j}\in I^n} e(\mathbf{j})) = \sum_{\mathbf{j}\in I^n} L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = \sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} L_r(\mathbf{i})^{N_r(\mathbf{i})} e(\mathbf{j}) = \sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} e(\mathbf{j})$$

which completes the proof. □

As the idempotents $e(\mathbf{j})$'s are pairwise orthogonal, Lemma 5.1.5 immediately implies the following.

5.1.8. Corollary. *For any* $\mathbf{i} \in I^n$, *we have*

$$e(\mathbf{i}) = \prod_{r=1}^{n} L_r(\mathbf{i})^{N_r(\mathbf{i})}.$$

The previous results are true in both the degenerate and non-degenerate cases. Notice that when we define $L_{i_r,j} = 1 - (\frac{q_{i_r} - x_r}{q_{i_r} - q_j})^{N_j}$ and $L_r(\mathbf{i})^{N_r(\mathbf{i})}$, the only restriction is that N_j and $N_r(\mathbf{i})$ are large enough. As we now show, by choosing specific values for N_j and $N_r(\mathbf{i})$, it is possible to simplify the expression of $L_r(\mathbf{i})^{N_r(\mathbf{i})}$ even further and give a more explicit expression of $e(\mathbf{i})$. We emphasize the simplified expressions of $L_r(\mathbf{i})^{N_r(\mathbf{i})}$ are different for degenerate and non-degenerate H_n^Λ.

We start with the degenerate cyclotomic Hecke algebras. Recall that in this case $e = p$.

5.1.9. Proposition. *Suppose* $q = 1$. *For any* $i_r \in I$ *there exists* $s \gg 0$ *such that*

$$\sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} e(\mathbf{j}) = \begin{cases} 1 - x_r^{p^s(1-p)}, & \textit{when } i_r = 0, \\ -\sum_{k=1}^{p-1} \frac{x_r^{kp^s}}{i_r^k}, & \textit{when } i_r \neq 0. \end{cases}$$

Proof. By Lemma 5.1.5 the Proposition is equivalent to claim that

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = \begin{cases} 1 - x_r^{p^s(1-p)}, & \text{when } i_r = 0, \\ -\sum_{k=1}^{p-1} \frac{x_r^{kp^s}}{i_r^k}, & \text{when } i_r \neq 0. \end{cases}$$

for $s \gg 0$.

By the definition of $L_r(\mathbf{i})$, because $I = \mathbb{Z}/p\mathbb{Z}$ we have

$$L_r(\mathbf{i}) = \prod_{\substack{j\in I\\ j\neq i_r}} L_{i_r,j} = \prod_{\substack{j\in I\\ j\neq i_r}}(1 - (\frac{i_r - x_r}{i_r - j})^{N_j}) = \prod_{j=1}^{p-1}(1 - (\frac{i_r - x_r}{j})^{N_j}).$$

Take $k \gg 0$ and $N_j = p^k$. Hence because H_n^Λ is defined over a field $\mathbb{F}_p$ of characteristic p, we have $j^{N_j} = j$. And because p is a prime, we have

$$L_r(\mathbf{i}) = \prod_{j=1}^{p-1}(1 - (\frac{i_r - x_r}{j})^{N_j}) = \prod_{j=1}^{p-1}(1 - \frac{(i_r - x_r)^{N_j}}{j}) = \prod_{j=1}^{p-1}(1 - j\cdot(i_r - x_r)^{N_j}) = 1 - (i_r - x_r)^{(p-1)N_j}.$$

Without loss of generality, choose $N_r(\mathbf{i}) = p^l$ with $l \gg 0$. We have

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = (1 - (i_r - x_r)^{(p-1)N_j})^{N_r(\mathbf{i})} = 1 - (i_r - x_r)^{(p-1)N_jN_r(\mathbf{i})}.$$

Setting $s = k+l$, we have $N_jN_r(\mathbf{i}) = p^{k+l} = p^s$. Now we consider two cases, which are $i_r = 0$ and $i_r \neq 0$.

Suppose first $i_r = 0$. We have

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = 1 - (i_r - x_r)^{(p-1)N_jN_r(\mathbf{i})} = 1 - (-x_r)^{(p-1)p^s} = 1 - x_r^{(p-1)p^s}. \tag{5.1.10}$$

Suppose $i_r \neq 0$. We have

$$\begin{aligned}(i_r - x_r)^{(p-1)N_jN_r(\mathbf{i})} &= (i_r - x_r)^{p^{s+1}-p^s} = \frac{(i_r - x_r)^{p^{s+1}}}{(i_r - x_r)^{p^s}}\\ &= \frac{i_r - x_r^{p^{s+1}}}{i_r - x_r^{p^s}} = \frac{1 - (\frac{x_r}{i_r})^{p^{s+1}}}{1 - (\frac{x_r}{i_r})^{p^s}}\\ &= 1 + (\frac{x_r}{i_r})^{p^s} + (\frac{x_r}{i_r})^{2p^s} + \ldots + (\frac{x_r}{i_r})^{(p-1)p^s}\\ &= 1 + \frac{x_r^{p^s}}{i_r} + \frac{x_r^{2p^s}}{i_r^2} + \ldots + \frac{x_r^{(p-1)p^s}}{i_r^{p-1}} = \sum_{k=0}^{p-1}\frac{x^{kp^s}}{i_r^k}.\end{aligned}$$

Hence,

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = 1 - (i_r - x_r)^{(p-1)N_jN_r(\mathbf{i})} = 1 - \sum_{k=0}^{p-1}\frac{x^{kp^s}}{i_r^k} = -\sum_{k=1}^{p-1}\frac{x^{kp^s}}{i_r^k}. \tag{5.1.11}$$

By combining (5.1.10) and (5.1.11), we complete the proof. □

Finally, by combining Corollary 5.1.8 and Proposition 5.1.9, we have an explicit expression of $e(\mathbf{i})$ for the degenerate cyclotomic Hecke algebras.

5.1.12. Theorem. *Suppose* $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ *and* $q = 1$, *then*

$$e(\mathbf{i}) = \prod_{r=1}^{n} L_r(\mathbf{i})^{N_r(\mathbf{i})}$$

where

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = \begin{cases} 1 - x_r^{p^s(1-p)}, & \text{when } i_r = 0,\\ -\sum_{k=1}^{p-1}\frac{x_r^{kp^s}}{i_r^k}, & \text{when } i_r \neq 0.\end{cases}$$

for $s \gg 0$.

We now give a similar expression for the non-degenerate cyclotomic Hecke algebras. First we give two Lemmas which will be used later.

5.1.13. Lemma. *For any $k \in \mathbb{Z}$ with $k \not\equiv 0 \pmod e$, we have*

$$1 + q^k + q^{2k} + \ldots + q^{(e-1)k} = 0.$$

Proof. By the definition, we have

$$\begin{aligned} & 1 + q + q^2 + \ldots + q^{e-1} = 0 \\ \Rightarrow\ & (1 + q + q^2 + \ldots + q^{e-1})(1-q) = 0 \\ \Rightarrow\ & 1 - q^e = 0 \\ \Rightarrow\ & q^e = 1 \\ \Rightarrow\ & (q^e)^k = q^{ke} = (q^k)^e = 1 \\ \Rightarrow\ & (q^k)^e - 1 = (1 + q^k + q^{2k} + \ldots + q^{(e-1)k})(q^k - 1) = 0. \end{aligned}$$

Because $k \in \mathbb{Z}$ and $k \not\equiv 0 \pmod e$, we have $q^k - 1 \neq 0$. Therefore we must have $1 + q^k + q^{2k} + \ldots + q^{(e-1)k} = 0$. □

5.1.14. Lemma. *Suppose $i_r \in I$ and $f(x) = \prod_{j \neq i_r}(1 - \frac{r^{i_r} - x}{r^{i_r} - r^j}) \in \mathbb{F}_p[x]$ with $r = q^s$ for some positive integer $s \not\equiv 0 \pmod e$ and $q \in \mathbb{F}_p^\times$. Then $e^{-1} \in \mathbb{F}_p$ and*

$$f(x) = e^{-1}(1 + \frac{x}{r^{i_r}} + (\frac{x}{r^{i_r}})^2 + \ldots + (\frac{x}{r^{i_r}})^{e-1}).$$

Proof. By Lemma 5.1.3 we have $\gcd(e,p) = 1$ and hence $e^{-1} \in \mathbb{F}_p$. Define $g(x) = e^{-1}(1 + \frac{x}{r^{i_r}} + (\frac{x}{r^{i_r}})^2 + \ldots + (\frac{x}{r^{i_r}})^{e-1})$. We prove that $f(x) = g(x)$ by first comparing their roots. It is obvious that the roots of $f(x)$ are all of the form r^j with $j \in I$ and $j \neq i_r$. Then for any such r^j,

$$g(r^j) = e^{-1}(1 + r^{j-i_r} + r^{2(j-i_r)} + \ldots + r^{(e-1)(j-i_r)}) = e^{-1}(1 + r^k + r^{2k} + \ldots + r^{(e-1)k})$$

for $k \equiv j - i_r \pmod e$ and $k \neq 0$. Because $r = q^s$ and $s \not\equiv 0 \pmod e$, we must have $sk \not\equiv 0 \pmod e$. Therefore by Lemma 5.1.13 we have $g(r^j) = 0$. Because $f(x)$ and $g(x)$ are both polynomials of degree $e-1$, they have $e-1$ roots, which means that $g(x)$ and $f(x)$ have the same roots. This yields that $f(x) = kg(x)$ for some $k \in \mathbb{F}_p$.

Now because $f(r^{i_r}) = 1 = g(r^{i_r})$, we have $k = 1$. Therefore $f(x) = g(x)$, which completes the proof. □

5.1.15. Proposition. *Suppose $q \neq 1$. For any $i_r \in I$, there exists $s \gg 0$ such that*

$$\sum_{\substack{\mathbf{j} \in I^n \\ j_r = i_r}} e(\mathbf{j}) = e^{-1}(1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1}).$$

Proof. By Lemma 5.1.5 the Proposition is equivalent to prove that

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = e^{-1}(1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1}).$$

By the definition of $L_r(\mathbf{i})$, because $I = \mathbb{Z}/e\mathbb{Z}$, if $N_j, N_r(\mathbf{i}) \gg 0$ then we have

$$L_r(\mathbf{i})^{N_r(\mathbf{i})} = \prod_{\substack{j \in I \\ j \neq i_r}} (1 - (\frac{q^{i_r} - X_r}{q^{i_r} - q^j})^{N_j})^{N_r(\mathbf{i})}.$$

Suppose $N_j = p^k$ and $N_r(\mathbf{i}) = p^l$ with $k, l \gg 0$. We have

$$\begin{aligned} L_r(\mathbf{i})^{N_r(\mathbf{i})} &= \prod_{j \neq i_r} (1 - (\frac{q^{i_r} - X_r}{q^{i_r} - q^j})^{p^k})^{p^l} \\ &= \prod_{j \neq i_r} (1 - (\frac{q^{i_r} - X_r}{q^{i_r} - q^j})^{p^{k+l}}) \\ &= \prod_{j \neq i_r} (1 - \frac{q^{p^{k+l} \cdot i_r} - X_r^{p^{k+l}}}{q^{p^{k+l} \cdot i_r} - q^{p^{k+l} \cdot j}}) \\ &= \prod_{j \neq i_r} (1 - \frac{r^{i_r} - X_r^{p^s}}{r^{i_r} - r^j}), \end{aligned}$$

where $s = k + l$ and $r = q^{p^s} \in \mathbb{F}_p$. Notice that by Lemma 5.1.3, we have $p^s \not\equiv 0 \pmod{e}$.

Now we set $f(x) = \prod_{j \neq i_r} (1 - \frac{r^{i_r} - x}{r^{i_r} - r^j}) \in \mathbb{F}_p[x]$. By Lemma 5.1.14 we have

$$f(x) = e^{-1}(1 + \frac{x}{r^{i_r}} + (\frac{x}{r^{i_r}})^2 + \ldots + (\frac{x}{r^{i_r}})^{e-1}).$$

Therefore

$$\begin{aligned} L_r(\mathbf{i})^{N'} &= f(X_r^{p^s}) = e^{-1}(1 + \frac{X_r^{p^s}}{r^{i_r}} + (\frac{X_r^{p^s}}{r^{i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{r^{i_r}})^{e-1}) \\ &= e^{-1}(1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1}) \end{aligned}$$

which completes the proof. □

Finally we can get an explicit expression of $e(\mathbf{i})$ for the non-degenerate H_n^Λ using Proposition 5.1.15 and the orthogonality of $e(\mathbf{i})$'s.

5.1.16. Theorem. *Suppose* $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in I^n$ *and* $q \neq 1$, *we have*

$$e(\mathbf{i}) = e^{-n} \prod_{r=1}^{n} (1 + \frac{X_r^{p^s}}{q^{p^s \cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s \cdot i_r}})^{e-1})$$

for $s \gg 0$.

5.2. Periodic property of x_r in degenerate case

In the degenerate cyclotomic Hecke algebra, when $e = p > 0$ the algebra is finite. We know that $\dim H_n^\Lambda = \ell^n n!$. Hence over $\mathbb{F}_p$ the algebra has $p\ell^n n!$ elements. Therefore, by choosing $k > p\ell^n n!$, for any r we must be able to find k_1, k_2 with $1 \leq k_1 < k_2 \leq k$ such that $x_r^{k_1} = x_r^{k_2}$. Therefore for any r we can find integers d_r and N such that for any $N' \geq N$, $x_r^{N'} = x_r^{N'+d_r}$. We define the **period** of x_r to be the smallest positive integer d_r such that $x_r^N = x_r^{N+d_r}$ for some N. In this section we will give information on the period d_r and the minimal N such that $x_r^N = x_r^{N+d_r}$.

Recall that y_r is the generator of $\mathscr{R}_n^\Lambda$. By Brundan and Kleshchev [**3**, (3.21)], $y_r = \sum_{\mathbf{i} \in I^n} (x_r - i_r) e(\mathbf{i})$ and by [**3**, Lemma 2.1], $y_r^s = 0$ for $s \gg 0$.

5.2.1. Lemma. *Suppose s is an integer. For any r,* $x_r^{p^{s+1}} = x_r^{p^s}$ *if and only if* $y_r^{p^s} = 0$.

Proof. For any $i \in I$, we have

$$x_r^{p^{s+1}} - x_r^{p^s} = (x_r^{p^{s+1}} - i) - (x_r^{p^s} - i) = (x_r - i)^{p^{s+1}} - (x_r - i)^{p^s}.$$

Suppose $y_r^{p^s} = \sum_{\mathbf{i}\in I^n}(x_r - i_r)^{p^s} e(\mathbf{i}) = 0$. Therefore for any $\mathbf{i}$, $(x_r - i_r)^{p^s} e(\mathbf{i}) = 0$. Then for any $\mathbf{i} \in I^n$ with $i_r = i$ we have

$$(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = (x_r - i)^{p^{s+1}} e(\mathbf{i}) - (x_r - i)^{p^s} e(\mathbf{i}) = 0.$$

Then as $\sum_{\mathbf{i}\in I^n} e(\mathbf{i}) = 1$, we have

$$(x_r^{p^{s+1}} - x_r^{p^s}) = \sum_{\mathbf{i}\in I^n}(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = 0,$$

which shows that $x_r^{p^{s+1}} = x_r^{p^s}$.

Suppose $y_r^{p^s} = \sum_{\mathbf{i}\in I^n}(x_r - i_r)^{p^s} e(\mathbf{i}) \neq 0$, we must be able to find a $\mathbf{i} \in I^n$ such that $(x_r - i_r)^{p^s} e(\mathbf{i}) \neq 0$. Assume

$$(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = (x_r - i)^{p^{s+1}} e(\mathbf{i}) - (x_r - i)^{p^s} e(\mathbf{i}) = 0,$$

which means that $y_r^{p^{s+1}} e(\mathbf{i}) = y_r^{p^s} e(\mathbf{i}) \neq 0$. Because $p^{s+1} > p^s$ and $y_r^{p^s} e(\mathbf{i}) \neq 0$, we can find k where $y_r^{p^{s+1}+k} e(\mathbf{i}) = 0$ and $y_r^{p^s+k} e(\mathbf{i}) \neq 0$. But $y_r^{p^{s+1}+k} e(\mathbf{i}) = y_r^k y_r^{p^{s+1}} e(\mathbf{i}) = y_r^k y_r^{p^s} e(\mathbf{i}) = y_r^{p^s+k} e(\mathbf{i}) \neq 0$, which leads to contradiction. Therefore we must have $(x_r^{p^{s+1}} - x_r^{p^s})e(\mathbf{i}) = (x_r - i)^{p^{s+1}} e(\mathbf{i}) - (x_r - i)^{p^s} e(\mathbf{i}) \neq 0$, which yields that $x_r^{p^{s+1}} \neq x_r^{p^s}$. □

Choose $s \gg 0$ such that $y_r^s = 0$. By Lemma 5.2.1 we have $x_r^{p^s} = x_r^{p^{s+1}} = x_r^{p^s+(p-1)p^s}$. So the period d_r divides $p^s(p-1)$. Then $d_r = p^m$ or $p^m(p-1)$ with $m \geq 0$.

5.2.2. Lemma. *Suppose d_r is the period of x_r. Then $(p-1) \mid d_r$.*

Proof. When $p = 2$ there is nothing to prove. Hence we set $p > 2$ so that p is odd. Assume that $d_r = p^m$ for some m. Consider $\lambda = (r-1, 1^{n-r+1})$ and $\mathfrak{t} = \mathfrak{t}^\lambda$. Let $\mathbf{j} = (j_1, j_2, \ldots, j_n) = \operatorname{res}(\mathfrak{t})$, it is easy to see that $j_r = e - 1 = p - 1$. Now $\mathbf{j}$ is a residue sequence so that $e(\mathbf{j}) \neq 0$ by Corollary 3.6.3. So we must have $\sum_{\substack{\mathbf{i}\in I^n \\ i_r = p-1}} e(\mathbf{i}) \neq 0$. Choose $s \gg m$. By Proposition 5.1.9,

$$\begin{aligned} L_r(\mathbf{j})^{N_r(\mathbf{j})} &= -\frac{x_r^{p^s}}{p-1} - \frac{x_r^{2p^s}}{(p-1)^2} - \ldots - \frac{x_r^{(p-1)p^s}}{(p-1)^{p-1}} \\ &= x_r^{p^s} - x_r^{2p^s} + x_r^{3p^s} - \ldots - x_r^{(p-1)p^s}. \end{aligned}$$

By assumption, because $s \gg m$, we have $x_r^{p^s} = x_r^{2p^s} = \ldots = x_r^{(p-1)p^s}$. Therefore

$$L_r(\mathbf{j})^{N_r(\mathbf{j})} = x_r^{p^s} - x_r^{2p^s} + x_r^{3p^s} - \ldots - x_r^{(p-1)p^s} = (1 - 1 + 1 - \ldots - 1)x_r^{p^s} = 0.$$

But by Lemma 5.1.5 we have $L_r(\mathbf{j})^{N_r(\mathbf{j})} = \sum_{i_r = p-1} e(\mathbf{i}) \neq 0$, which leads to contradiction. Therefore $d_r = p^m(p-1)$ and hence $(p-1)|d_r$. □

Now we know that $d_r = p^m(p-1)$ for some m. We can give a more specific value of m. Define l to be the integer such that $y_r^{p^l} = 0$ and $y_r^{p^{l-1}} \neq 0$. First we introduce two Lemmas.

5.2.3. Lemma. *Suppose $f(x) \in \mathbb{F}_p[x]$, $h \in H_n^\Lambda$ and $e(\mathbf{i})h \neq 0$. Then $f(x_r)e(\mathbf{i})h = 0$ only if $f(i_r) = 0$.*

Proof. We prove this Lemma by contradiction. Because $\mathbb{F}_p$ is a field, $f(i_r) = 0$ only if $(x - i_r) \mid f(x)$. Assume $f(x_r)e(\mathbf{i})h = 0$. Suppose $f(i_r) \neq 0$, we can write $f(x) = (x - i_r)g(x) + j$ with $j \neq 0$. Set $s \gg 0$ such that $(x_r - i_r)^{p^s} e(\mathbf{i}) = 0$. Because $f(x_r)e(\mathbf{i})h = 0$, we have

$$f^{p^s}(x_r)e(\mathbf{i})h = ((x_r - i_r)g(x_r) + j)^{p^s} e(\mathbf{i})h = g^{p^s}(x_r)(x_r - i_r)^{p^s} e(\mathbf{i})h + j \cdot e(\mathbf{i})h = j \cdot e(\mathbf{i})h \neq 0$$

because $j \neq 0$ and $e(\mathbf{i})h \neq 0$, which leads to contradiction. Therefore $f(x_r)e(\mathbf{i})h \neq 0$ when $f(i_r) \neq 0$. This completes the proof. □

5.2.4. Lemma. *Suppose $k \in \mathbb{Z}$ and $t \in \mathbb{Z}$. For any $i \in I$ with $i \neq 0$, we have*

$$x^k - x^{k+p^t(p-1)} = f(x)(i-x)^{p^t}$$

with $f(x) = \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})$.

Proof. Suppose $i \in I$ and $i \neq 0$. We have

$$\begin{aligned} x^k - x^{k+p^t(p-1)} &= x^k(1 - x^{p^t(p-1)}) = x^k(1 - (\frac{x^{p^t}}{i})^{p-1}) \\ &= x^k(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})(1 - \frac{x^{p^t}}{i}) \\ &= \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})(i^{p^t} - x^{p^t}) \\ &= \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})(i - x)^{p^t} \\ &= f(x)(i-x)^{p^t} \end{aligned}$$

with $f(x) = \frac{x^k}{i}(1 + \frac{x^{p^t}}{i} + (\frac{x^{p^t}}{i})^2 + \ldots + (\frac{x^{p^t}}{i})^{p-2})$. This completes the proof. □

5.2.5. Proposition. *Suppse l is the smallest non-negative integer such that $y_r^{p^l} = 0$. Then the period of x_r is $d_r = p^l(p-1)$.*

Proof. Suppose $d_r = p^m(p-1)$. By Lemma 5.2.1 we have $x_r^{p^{l+1}} = x_r^{p^l + p^l(p-1)} = x_r^{p^l}$. Therefore $d_r \mid p^l(p-1)$ which indicates that $m \leq l$. Now take $s \gg 0$, by Lemma 5.2.4 we have

$$(x_r^{p^s} - x_r^{p^s + p^{l-1}(p-1)})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^{l-1}}$$

where $f(x) = \frac{x^{p^s}}{i_r}(1 + \frac{x^{p^{l-1}}}{i_r} + (\frac{x^{p^{l-1}}}{i_r})^2 + \ldots + (\frac{x^{p^{l-1}}}{i_r})^{p-2}) \in \mathbb{F}_p[x]$. It is easy to see that $f(i_r) = p-1 \neq 0$. By the definition of l, $e(\mathbf{i})(i_r - x_r)^{p^{l-1}} \neq 0$. Then by Lemma 5.2.3 we have

$$(x_r^{p^s} - x_r^{p^s + p^{l-1}(p-1)})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^{l-1}} \neq 0.$$

Therefore $x_r^{p^s} - x_r^{p^s + p^{l-1}(p-1)} \neq 0$, i.e. $x_r^{p^s} \neq x_r^{p^s + p^{l-1}(p-1)}$, which yields $m \geq l$. This shows that $m = l$ and $d_r = p^l(p-1)$. □

Now we know that the period of x_r is $d_r = p^l(p-1)$, and we still need to find the smallest non-negative integer N such that $x_r^N = x_r^{N+d_r}$.

5.2.6. Proposition. *Suppose $1 \leq r \leq n$ and we can find a residue sequence $\mathbf{i}$ such that $i_r = 0$. If N is the smallest non-negative integer such that $x_r^N \sum_{i_r=0} e(\mathbf{i}) = 0$, then $x_r^N = x_r^{N+d_r}$ and $x_r^{N-1} \neq x_r^{N-1+d_r}$.*

Proof. By the definition of N, we can find $\mathbf{i}$ with $i_r = 0$ such that $x_r^{N-1}e(\mathbf{i}) \neq 0$ and $x_r^N e(\mathbf{i}) = 0$. Suppose $s \gg 0$. Because $d_r \geq 1$ we have

$$(x_r^{N-1} - x_r^{N-1+d_r})e(\mathbf{i}) = x_r^{N-1}e(\mathbf{i}) - x_r^{N-1+d_r}e(\mathbf{i}) = x_r^{N-1}e(\mathbf{i}) \neq 0$$

which indicates that $x_r^{N-1} \neq x_r^{N-1+d_r}$.

Next we will prove that $x_r^N = x_r^{N+d_r}$. Suppose $\mathbf{i} \in I^n$ with $i_r = 0$, then

$$(x_r^N - x_r^{N+d_r})e(\mathbf{i}) = (1 - x_r^{d_r})x_r^N e(\mathbf{i}) = 0$$

by the definition of N. Now suppose $\mathbf{i} \in I^n$ with $i_r \neq 0$. By Proposition 5.2.5, $d_r = p^l(p-1)$ where $y_r^{p^l} = 0$. So by Lemma 5.2.4,

$$(x_r^N - x_r^{N+d_r})e(\mathbf{i}) = (x_r^N - x_r^{N+(p-1)p^l})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^l} = f(x_r)e(\mathbf{i})(-y_r)^{p^l} = 0$$

with $f(x) \in \mathbb{F}_p[x]$. Therefore we have $(x_r^N - x_r^{N+d_r})e(\mathbf{i}) = 0$ for any $\mathbf{i} \in I^n$ and hence $x_r^N = x_r^{N+d_r}$. This completes the proof. □

Notice that in Proposition 5.2.6 we require $1 \le r \le n$ such that we can find a residue sequence $\mathbf{i}$ with $i_r = 0$. If no such residue sequence exists we obtain a different result.

5.2.7. Proposition. *Suppose* $1 \le r \le n$ *and for any residue sequence* $\mathbf{i}$ *we always have* $i_r \neq 0$. *Then* $x_r^{d_r} = 1$.

Proof. By Proposition 5.2.5, $d_r = p^l(p-1)$ where $y_r^{p^l} = 0$. And for any $\mathbf{i} \in I^n$, we have $i_r \neq 0$. Then by Lemma 5.2.4,

$$(1 - x_r^{d_r})e(\mathbf{i}) = (1 - x_r^{p^l(p-1)})e(\mathbf{i}) = f(x_r)e(\mathbf{i})(i_r - x_r)^{p^l} = f(x_r)e(\mathbf{i})(-y_r)^{p^l} = 0,$$

which shows that $x_r^{d_r}e(\mathbf{i}) = e(\mathbf{i})$ for any $\mathbf{i} \in I^n$. Hence $x_r^{d_r} = 1$. □

Finally we give the main Theorem of this section by combining Proposition 5.2.5, Proposition 5.2.6 and Proposition 5.2.7.

5.2.8. Theorem. *In the degenerate cyclotomic Hecke algebras, suppose* l *is the smallest nonnegative integer such that* $y_r^{p^l} = 0$ *and* N *is the smallest nonnegative integer such that* $x_r^N \sum_{i_r=0} e(\mathbf{i}) = 0$. *Then* $x_r^k = x_r^{k+p^m(p-1)}$ *if and only if* $m \ge l$ *and* $k \ge N$.

5.3. Periodic property of X_r in non-degenerate case

In non-degenerate cyclotomic Hecke algebras, when $p > 0$ the algebra is finite. So by the same reason as degenerate case, X_r must have a periodic property. We define the period d_r of X_r similarly as in degenerate cases. In this section we will give an analogues result for the non-degenerate case when $e > 0$.

Recall by Brundan and Kleshchev [**3**, (3.21)], $y_r = \sum_{\mathbf{i}\in I^n}(1 - q^{-i_r}X_r)e(\mathbf{i})$ and by [**3**, Lemma 2.1], $y_r^s = 0$ for $s \gg 0$. It is easy to imply that $(X_r - q^{i_r})^s e(\mathbf{i}) = 0$ for $s \gg 0$. We will use this fact without mention.

5.3.1. Lemma. *Suppose* $s \gg 0$ *and* $1 \le r \le n$. *We have* $X_r^{ep^s} = 1$.

Proof. By Proposition 5.1.15, for any $i_r \in I$, we have

$$\begin{aligned}
(X_r - q^{i_r})^{p^s} \sum_{\substack{\mathbf{j}\in I^n \\ j_r=i_r}} e(\mathbf{j}) &= (X_r^{p^s} - q^{p^s\cdot i_r}) \sum_{\substack{\mathbf{j}\in I^n \\ j_r=i_r}} e(\mathbf{j}) \\
&= e^{-1}(X_r^{p^s} - q^{p^s\cdot i_r})(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + (\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^2 + \ldots + (\frac{X_r^{p^s}}{q^{p^s\cdot i_r}})^{e-1}) \\
&= e^{-1}(\frac{X_r^{ep^s}}{q^{(e-1)p^s\cdot i_r}} - q^{p^s\cdot i_r}) = 0,
\end{aligned}$$

which leads to

$$\frac{X_r^{ep^s}}{q^{(e-1)p^s\cdot i_r}} = q^{p^s\cdot i_r} \quad \Rightarrow \quad X_r^{ep^s} = q^{ep^s\cdot i_r} = 1$$

because $q^e = 1$. □

Define d_r to be the period of X_r. By Lemma 5.3.1 we have $d_r \mid ep^s$. Therefore $d_r = ep^m$ with $m \ge 0$ or $d_r = p^m$ with $m \ge 1$. In the following Lemma we are going to give more information about the form of d_r.

5.3.2. Lemma. *Suppose* d_1 *is the period of* X_1. *We have* $d_1 = p^m$ *if* $\Lambda = \ell\Lambda_0$.

Proof. By (1.2.7) we have $(X_1 - q^0)^\ell = (X_1 - 1)^\ell = 0$. Choose s such that $p^s \geq \ell$, we have $(X_1 - 1)^{p^s} = X_1^{p^s} - 1 = 0$, which means $X_1^{p^s} = 1$. Hence $d_1 \mid p^s$ and therefore $d_1 = p^m$. □

5.3.3. Remark. When we set $r = 1$ and $\Lambda = \ell\Lambda_0$, it means that $e(\mathbf{i}) = 0$ if $i_r = i_1 \neq 0$. So Lemma 5.3.2 is actually:

Suppose $1 \leq r \leq n$ *and for any* $\mathbf{i} \in I^n$, $e(\mathbf{i}) = 0$ *if* $i_r \neq 0$. *Then* $d_r = p^m$.

because the only possible r and Λ for such condition is giving in Lemma 5.3.2.

5.3.4. Lemma. *Suppose d_r is the period of X_r. We have $e \mid d_r$ if $r > 1$ or $r = 1$ and $\Lambda \neq \ell\Lambda_0$.*

Proof. We prove the Lemma by contradiction. Assume that $d_r = p^m$. Choose $i_r \in I$ with $i_r \neq 0$. Because $r > 1$ or $r = 1$ and $\Lambda \neq \ell\Lambda_0$ we must can find $\mathbf{j} \in I^n$ with $j_r = i_r$ with $e(\mathbf{j}) \neq 0$. Then $\sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} e(\mathbf{j}) \neq 0$. Choose $s \gg m$. By Lemma 5.1.3, $\gcd(e, p) = 1$. Then because $i_r \neq 0$, $p^s \cdot i_r \not\equiv 0 \pmod{e}$. Then by Lemma 5.1.13 and Proposition 5.1.15, we have

$$\begin{aligned}
\sum_{\substack{\mathbf{j}\in I^n \\ j_r = i_r}} e(\mathbf{j}) &= e^{-1}\Big(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^{e-1}\Big) \\
&= e^{-1}\Big(1 + \frac{1}{q^{p^s\cdot i_r}} + \frac{1}{q^{2p^s\cdot i_r}} + \ldots + \frac{1}{q^{(e-1)p^s\cdot i_r}}\Big)X_r^{p^s} \\
&= \frac{e^{-1}}{q^{(e-1)p^s\cdot i_r}}(1 + q^{p^s\cdot i_r} + (q^{p^s\cdot i_r})^2 + \ldots + (q^{p^s\cdot i_r})^{e-1})X_r^{p^s} = 0,
\end{aligned}$$

which leads to contradiction. Hence $d_r \neq p^m$ and therefore $e \mid d_r$. □

Now we know that $d_r = p^m$ when $r = 1$ and $\Lambda = \ell\Lambda_0$ and $d_r = ep^m$ otherwise. In the rest of the section we will find the value of m. First we give the simpler case.

5.3.5. Lemma. *Suppose $s \geq 0$ and $\Lambda = \ell\Lambda_0$. We have $X_1^{p^s} = 1$ if and only if $y_1^{p^s} = 0$.*

Proof. Suppose $y_1^{p^s} = 0$. For any $\mathbf{i} \in I^n$,

$$y_1^{p^s} e(\mathbf{i}) = (1 - X_1)^{p^s} e(\mathbf{i}) = e(\mathbf{i}) - X_1^{p^s} e(\mathbf{i}) = 0 \quad\Rightarrow\quad X_1^{p^s} e(\mathbf{i}) = e(\mathbf{i}).$$

Therefore $X_1^{p^s} = 1$.

Suppose $y_1^{p^s} \neq 0$. Then we can find $\mathbf{i} \in I^n$ with $y_1^{p^s} e(\mathbf{i}) \neq 0$. So

$$y_1^{p^s} e(\mathbf{i}) = (1 - X_1)^{p^s} e(\mathbf{i}) = e(\mathbf{i}) - X_1^{p^s} e(\mathbf{i}) \neq 0 \quad\Rightarrow\quad X_1^{p^s} e(\mathbf{i}) \neq e(\mathbf{i}).$$

Therefore $X_1^{p^s} \neq 1$. □

Now we consider the case when $r \neq 1$ or $r = 1$ and $\Lambda \neq \ell\Lambda_0$.

5.3.6. Lemma. *For any non-negative integer s, we can find $k \gg s$ such that $q^{p^k} = q^{p^s}$ and $p^{k-s} \equiv 1 \pmod{e}$.*

Proof. By Lemma 5.1.3, we can find l such that $p^l \equiv 1 \pmod{e}$. Because $q^e = 1$, choose $t \gg 0$ and set $k = s + tl$, we have $q^{p^k} = q^{p^{s+tl}} = q^{p^s p^{tl}} = (q^{p^{tl}})^{p^s} = q^{p^s}$. Moreover, $p^{k-s} = p^{tl} \equiv 1^t \pmod{e} \equiv 1 \pmod{e}$. This completes the proof. □

Now we are ready to give more information of d_r.

5.3.7. Lemma. *We have $X_r^{ep^s} = 1$ for some s only if $y_r^{p^s} = 0$.*

Proof. Fix s such that $X_r^{ep^s} = 1$. By Lemma 5.3.6, we can find $k \gg 0$ such that $q^{p^{s+k}} = q^{p^s}$ and $p^k \equiv 1 \pmod{e}$. Therefore $p^{s+k} - p^s = p^s(p^k - 1)$ and hence $ep^s \mid p^s(p^k - 1)$. So $X_r^{p^{s+k}} = X_r^{p^s}$.

Then for any $i_r \in I$, by Proposition 5.1.15, we have

$$\begin{aligned}\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) &= e^{-1}\Big(1 + \frac{X_r^{p^{s+k}}}{q^{p^{s+k}\cdot i_r}} + \Big(\frac{X_r^{p^{s+k}}}{q^{p^{s+k}\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^{s+k}}}{q^{p^{s+k}\cdot i_r}}\Big)^{e-1}\Big)\\ &= e^{-1}\Big(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^{e-1}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}(X_r - q^{i_r})^{p^s}\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) &= e^{-1}(X_r - q^{i_r})^{p^s}\Big(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^{e-1}\Big)\\ &= e^{-1}(X_r^{p^s} - q^{p^s\cdot i_r})\Big(1 + \frac{X_r^{p^s}}{q^{p^s\cdot i_r}} + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^2 + \ldots + \Big(\frac{X_r^{p^s}}{q^{p^s\cdot i_r}}\Big)^{e-1}\Big)\\ &= e^{-1}\Big(\frac{X_r^{ep^s}}{q^{(e-1)p^s\cdot i_r}} - q^{p^s\cdot i_r}\Big) = e^{-1}\Big(\frac{1}{q^{(e-1)p^s\cdot i_r}} - q^{p^s\cdot i_r}\Big) = 0\end{aligned}$$

because $\frac{1}{q^{(e-1)p^s\cdot i_r}} = q^{p^s\cdot i_r}$. This means that $y_r^{p^s}\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) = (1 - q^{-i_r}X_r)^{p^s}\sum_{\substack{\mathbf{j}\in I^n\\ j_r=i_r}} e(\mathbf{j}) = 0$ for any $i_r \in I$. Hence $y_r^{p^s} = 0$. □

5.3.8. Lemma. *Suppose $y_r^{p^s} = 0$ for some s. Then we have $X_r^{ep^s} = 1$.*

Proof. Fix s such that $y_r^{p^s} = 0$. For any $\mathbf{i} \in I^n$, by Lemma 5.3.6 we can choose $k \gg s$ such that $q^{p^k} = q^{p^s}$. Then

$$\begin{aligned}(X_r^{p^k} - X_r^{p^s})e(\mathbf{i}) &= (X_r^{p^k} - q^{p^s\cdot i_r} - X_r^{p^s} + q^{p^s\cdot i_r})e(\mathbf{i})\\ &= (X_r^{p^k} - q^{p^k\cdot i_r})e(\mathbf{i}) - (X_r^{p^s} - q^{p^s\cdot i_r})e(\mathbf{i})\\ &= (X_r - q^{i_r})^{p^k}e(\mathbf{i}) - (X_r - q^{i_r})^{p^s}e(\mathbf{i}) = q^{p^k\cdot i_r}(-y_r)^{p^k}e(\mathbf{i}) - q^{p^s\cdot i_r}(-y_r)^{p^s}e(\mathbf{i}) = 0.\end{aligned}$$

So $(X_r^{p^k} - X_r^{p^s})e(\mathbf{i}) = 0$ for any $\mathbf{i} \in I^m$. Therefore we must have $X_r^{p^k} - X_r^{p^s} = 0$ for some $k \gg 0$. Hence

$$X_r^{p^k} - X_r^{p^s} = X_r^{p^s}(X_r^{p^k - p^s} - 1) = 0 \quad\Rightarrow\quad X_r^{p^k-p^s} = 1,$$

which implies that $d_r \mid p^k - p^s$. We know that $d_r = ep^m$ for some m and $p^k - p^s = p^s(p^{k-s} - 1)$. It is obvious that $m \le s$. Hence $X_r^{ep^s} = 1$. □

The next Corollary follows straightforward by combining Lemma 5.3.7 and Lemma 5.3.8.

5.3.9. Corollary. *Suppose $s \ge 0$, $r = 1$ and $\Lambda \neq \ell\Lambda_0$. We have $X_r^{ep^s} = 1$ if and only if $y_r^{p^s} = 0$.*

Finally, combining all the results above, we have the final Theorem.

5.3.10. Theorem. *In non-degenerate H_n^Λ, we have $X_r^{d_r} = 1$ with*

$$d_r = \begin{cases} p^m, & \text{if } r = 1 \text{ and } \Lambda = \ell\Lambda_0,\\ ep^m, & \text{otherwise.}\end{cases}$$

if and only if $y_r^{p^m} = 0$.

Proof. The Theorem follows straightforward by Lemma 5.3.2, Lemma 5.3.4, Lemma 5.3.5 and Corollary 5.3.9. □

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