Inverse Scattering Transform Method for Lattice Equations

Samuel Butler

A thesis submitted for the degree of
Doctor of Philosophy

Applied Mathematics
University of Sydney

September 2012

Abstract

The main original contribution of this thesis is the development of a fully discrete inverse scattering transform (IST) method for nonlinear partial difference equations. The equations we solve are nonlinear partial difference equations on a quad-graph, also called lattice equations, which are known to be multidimensionally consistent in N dimensions for arbitrary N. Such equations were discovered by Nijhoff, Quispel and Capel and Adler and later classified by Adler, Bobenko and Suris.

The main equation solved by our IST framework is the $Q3_\delta$ lattice equation. Our approach also solves all of its limiting cases, including H1, known as the lattice potential KdV equation. Our results provide the discrete analogue of the solution of the initial value problem on the real line. We provide a rigorous justification that solves the problem for wide classes of initial data given along initial paths in a multi-dimensional lattice.

Moreover, we show how soliton solutions arise from the IST method and also utilise asymptotics of the eigenfunctions to construct infinitely many conservation laws.

Acknowledgements

I most sincerely thank my supervisor Nalini Joshi for her skilled guidance and unwavering support throughout this project. It has been a privilege to be part of the excellent and extensive research group that Nalini has created. I am most grateful to Frank Nijhoff, Claude Viallet and Kenji Kajiwara for their kind hospitality and inspiring discussions, and am much obliged to James Atkinson for the comments and perspectives he has offered.

Finally I would like to thank Philip Howes, Dean Thomson, Ben Attfield, John Shaw, Tom Walker, Sarah Wood, Sean Randal, Megan Lovesey, Michael Hay, Yang Shi, Robert Marangell, Martin Wechselberger, Lewis Mitchell, Karen Leung, Matt Gibson, Nobutaka Nakazono, the helpful staff at the School of Mathematics and Statistics, and of course my wonderful family.

CONTENTS

Abstract **3**

Acknowledgements **4**

Chapter 1. Introduction **7**

Chapter 2. The Continuous Inverse Scattering Transform **16**

2.1 Linear Example of the Inverse Scattering Transform 17

2.2 The Inverse Scattering Transform for the Korteweg-de Vries Equation 20

2.3 One-soliton Solution for the KdV Equation 26

2.4 Arbitrary Reflectionless Potential 26

Chapter 3. Lattice Equations **32**

3.1 Motivation for Discretisation 32

3.2 Bäcklund Transformations of the KdV 34

3.3 The Discrete KdV Equation 37

3.4 Integrability for Lattice Equations 49

3.5 Multidimensional Soliton Solutions of ABS Equations 58

Chapter 4. The Discrete Inverse Scattering Transform **62**

4.1 Linear Problem for $Q3_\delta$ 63

4.2 Motivation for the Forward Scattering Problem 65

4.3 Forward Scattering of ϕ_1 68
4.4 "Time" Evolution of the Scattering Data 86
4.5 Inverse Problem for ϕ_1 90
4.6 Inverse Problem for ϕ_2 92
4.7 Reconstruction of the Solution of $Q3_\delta$ 97
4.8 Discrete Inverse Scattering Transform for the Remaining ABS Equations 104
4.9 One-Soliton Solutions 121
4.10 Other Lattice Equations 123
4.11 Chapter Summary 127

Chapter 5. Conservation Laws 128
5.1 Conservation Laws for the KdV Equation 128
5.2 Conservation Laws for Lattice Equations 132
5.3 Conservation Laws for Lattice Equations from the Discrete Inverse Scattering Transform 135

Chapter 6. Conclusions 142

Chapter 7. Appendix 146
7.1 Proof of Theorem 4.3.6 146
7.2 Proof of Theorem 4.3.7 151

Notation 155

References 157

1 Introduction

The field of nonlinear waves and integrable systems has a long and colourful history. It began in the nineteenth century with the pioneering work of Stokes [**83**], Boussinesq [**26**] and Korteweg and de Vries [**54**], all of whom studied the dynamics of fluids. Many of the models that were derived were nonlinear partial differential equations, and without computational assistance very little could be said at the time about their solutions. In the second half of the twentieth century some of these models were then rediscovered by researchers such as Kruskal and Zabusky in 1965 [**92**], who used a combination of mathematical analysis and computational power to explain the Fermi-Pasta-Ulam (FPU) paradox. This was an observation of recurring states of energy (rather than the expected dissipation) within a one-dimensional string of connected masses with nonlinear spring interactions. The equation that Kruskal and Zabusky found as a model for a continuum limit of the FPU system was

$$u_t + uu_x + \delta u_{xxx} = 0, \tag{1.1}$$

where δ is a parameter, which is a nonlinear partial differential equation in two independent variables and is known as the KdV equation. This is in fact the equation found by Korteweg and de Vries while studying shallow water waves, which they showed admits periodic "cnoidal" solutions, and

was also found earlier by Boussinesq. One of the most important observations that Kruskal and Zabusky made was the existence of solitary waves in the solutions of (1.1). These are localised waves which, unlike linear waves, interact elastically with neighbouring waves, and have a direct relationship between amplitude and speed. This particle-like nature led the authors to label them as *solitons*. An example of a two-soliton solution is

$$u(x,t) = -12\left(\frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{\left[3\cosh(x-28t)+\cosh(3x-36t)\right]^2}\right), \tag{1.2}$$

whose graph as a function of x and t is shown in Figure 1.

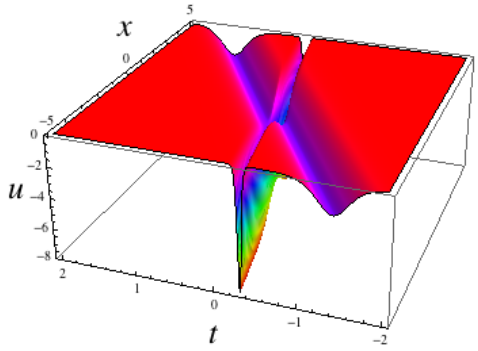

FIGURE 1. Two-soliton solution (1.2) of the KdV equation

The next great advancement in the field was the famous 1967 publication by the Princeton group of Gardner, Greene, Kruskal and Miura [**43**], who gave a new method of finding solutions to (1.1) with decaying boundary conditions. The method involved forward scattering (Sturm-Louiville) theory, where it was shown how the solitons were related to reflectionless potentials and the time-independent discrete eigenvalues, as well as inverse scattering theory which had been known to quantum mechanical physicists such as Jost and Kohn [**52**] and Gel'fand and Levitan [**45**] since the 1950s. The Princeton group showed how the KdV equation could be viewed as the compatibility condition for a linear system of equations, and

that the solution u of (1.1) could be obtained by solving a linear Volterra-type integral equation. This effectively was a way of linearising the KdV equation. In 1968 they also gave a transformation [**61**] between (1.1) and the modified KdV equation (mKdV)

$$v_t - 6v^2 v_x + v_{xxx} = 0 \tag{1.3}$$

and showed how to construct an infinite number of nontrivial conservation laws [**62**] to the KdV equation.

In the years that followed there were a great number of advancements as researchers found ways of applying this new method of solution to a number of physically important systems. One of the first applications was in 1971 from Zakharov and Shabat [**93**], who used ideas of Lax [**56**] to solve the initial-value problem for the nonlinear Schrödinger equation

$$iu_t = u_{xx} + ku^2 u^* \quad k > 0 \tag{1.4}$$

for solutions with decaying boundary conditions. Like the case of the KdV equation the authors found soliton solutions and an infinite number of conservation laws. In 1972 Wadati [**89**] then solved the mKdV (1.3), and in 1973 Ablowitz, Kaup, Newell and Segur (AKNS) [**3**] applied this method to solve the sine-Gordon equation

$$u_{xt} = \sin(u) \tag{1.5}$$

for which they found soliton solutions, breather solutions and an infinite number of conservation laws. The wide applicability of this method then led AKNS [**4**] [**5**] to show that equations (1.1), (1.3), (1.4) and (1.5) are in fact all related to a single matrix eigenvalue problem, from which many physically important systems are obtainable. Noting the similarity between this method of solving partial differential equations and the method of Fourier transform, they also labelled it the *Inverse Scattering Transform* (IST).

Another difficult problem was the periodic boundary value problem for integrable systems. This was solved for the KdV equation in the 1970s by the Russian school of Dubrovin, Novikov and Matveev [**34**] [**35**] [**36**]. This thesis however does not consider such an initial-value problem but instead focuses on an initial-value problem given on a discrete version of the real line.

Since these pioneering applications the IST has been used in a vast array of applications in mathematical physics. It not only gives a method of finding solutions to a number of nonlinear equations, but can also be used as a tool to obtain conservation laws, recursion operators and hierarchies of higher compatible flows (this was done for the KdV equation in [**44**]), as well as knowledge of the asymptotics of solutions [**9**] and many other properties of these systems. It has also been adapted to the case of the half-line [**41**]. References for the applications of the IST include the books by Ablowitz and Segur [**10**] and Ablowitz and Clarkson [**2**]. Importantly the advent of increased computational power has also allowed mathematical physicists to obtain numerical approximations to the solutions given by the IST. This has had much success, for example, in the areas of optics, electromagnetism and quantum mechanics.

There were however many physically important nonlinear systems for which the IST was not directly applicable, due to the fact that one or more of the variables appearing in the equation were discrete. Examples of this

include the mass and spring problem considered by FPU and the Toda lattice [**84**]. These systems are comprised of isolated point masses, which naturally give rise to discrete terms within the system, and their time dependence, which was assumed to be continuous. As such the nonlinear equations governing the motion of these systems contains both discrete and continuous differential operators (called differential-difference equations), for which application of the IST required amendment.

The first studies into the possibility of applying the IST to discrete equations date back to 1973 with Case and Kac [**30**] and Case [**29**]. These authors looked at differential-difference equations, and considered a direct discretisation of the time-independent Schrödinger equation which appears in the IST for the KdV equation. They found that for the intial-value problem posed along the half-line, the solution to their original nonlinear equation was expresible in terms of the solution to a linear discrete Volterra-type integral equation. In 1974 Flaschka [**38**] then showed how this procedure could be applied to solutions of the Toda lattice, where he considered linear difference equations (coupled with continuous time evolution) in which the coefficients depended on the Hamiltonian of the lattice. Ablowitz and Ladik [**6**] [**7**] [**8**] then derived a new discrete scattering problem, which was in fact a discrete analogue of that given by Zhakarov and Shabat, and found this to be applicable to a number of important systems. They also showed how to extend these idea to partial difference equations. More recent applications of the IST to differential difference equations can be seen in [**72**], [**75**], [**23**], [**25**], [**79**] and [**1**].

Parallel to these advancements in the application of the IST to continuous and semi-continuous systems, over the past forty years there has emerged a new area of mathematical physics which is the study of *discrete integrable systems*. These are systems for which all independent variables

are discrete, and as such are modelled by ordinary difference equations or partial difference equations. Our interest here however is solely on integrable nonlinear partial difference equations in two discrete independent variables.

Some of the first such equations to be identified as integrable were particular discretisations of KdV-type systems, such as the lattice KdV equation

$$u_{n+1,m+1} - u_{n,m} = \lambda \left(\frac{1}{u_{n,m+1}} - \frac{1}{u_{n+1,m}} \right), \tag{1.6}$$

which is attributed to Hirota [**50**]. This was found by discretising the bilinear form of (1.1). A different method known as "direct linearization" was advanced by the Dutch group of Capel et al. [**67**] [**74**] in the 1980s. Here the authors started with variations of the singular integral equation obtained in the IST for the KdV and through the application of Bäcklund transformations showed that certain quantities obeyed nonlinear partial difference equations. An example of this is the NQC equation

$$\frac{1-(p+\beta)s_{n+1,m}+(p-\alpha)s_{n,m}}{1-(q+\beta)s_{n,m+1}+(q-\alpha)s_{n,m}} = \frac{1-(q+\alpha)s_{n+1,m+1}+(q-\beta)s_{n+1,m}}{1-(p+\alpha)s_{n+1,m+1}+(p-\beta)s_{n,m+1}}, \tag{1.7}$$

which is in fact a particular discretisation of the Krichever-Novikov equation [**55**] in the case of a degenerate curve. There were also several other known partial difference equations of physical interest, including discrete versions of the mKdV equation (1.3) and the sine-Gordon equation (1.5).

Further to the knowledge of the equations themselves, there also followed significant research into the various integrability properties of these systems. Since these were discrete versions of known continuous integrable systems it was expected that similar integrability results should hold in the discrete case, and this was indeed found to be true. For example, conservation laws were found by Orfanidis [**70**] in 1978 for the lattice sine-Gordon equation, and by Wiersma and Capel [**91**] in 1987 for the lattice potential

KdV equation

$$(p+q+w_{n,m}-w_{n+1,m+1})(p-q+w_{n,m+1}-w_{n+1,m})=p^2-q^2. \quad (1.8)$$

Lax pairs for these and other discrete systems were then found (see e.g. [**66**] or [**63**]), and a number of integrability tests for discrete equations were developed. These include the obeying of singularity confinement [**46**], the possession of an infinite number of symmetries [**58**], the vanishing of algebraic entropy [**21**] and the notion of multidimensional consistency [**69**] [**24**]. This last property then formed the basis of a classification for scalar affine-linear partial difference equations of two independent variables, which was completed in 2003 [**12**] by Adler, Bobenko and Suris (ABS), and generalised [**13**] in 2009.

With this early work as a foundation, in the last decade or so there has been a great amount of research into the integrable partial difference equations characterised by ABS, and associated systems. Researchers have found N-soliton solutions [**65**] [**49**] [**20**] [**64**], infinite numbers of conservation laws [**51**] [**76**] [**77**] [**78**], Bäcklund transformations [**80**] [**16**] and many other important properties such as Hamiltonian structures and Calogero-Moser systems. It thus appears that these nonlinear discrete systems possess at least as much interesting and physically relevant structure as their continuous counterparts, yet are by their very nature more general, since one integrable discrete equation has an infinite number of continuum limits. One could argue that these equations lie at the very core of mathematical physics, and will in due course play a large role in a number of different fields.

An obvious gap in the overall understanding of discrete integrable systems was the existence and application of a fully discrete IST, as a method

of finding solutions to these nonlinear equations and as a means of obtaining other properties of the systems, such as conservation laws, asymptotics and characterising special solutions. The IST for the lattice potential KdV equation (1.8) was first considered by Levi and Petrera [**57**], and then in 2010 Butler and Joshi [**28**] put this on a rigorous footing, where they showed how real-valued solutions of (1.8) could be obtained through the solution of a discrete Volterra-type integral equation. In 2012 Butler [**27**] then showed how this could be generalised. The IST was developed in an N-dimensional setting for arbitrary N, and complex-valued solutions of the majority of the ABS equations were shown to be obtainable through solving a singular integral equation. It was also shown that this integral equation is related to the linearisation of the KdV and Painlevé II equations, found by Fokas and Ablowitz [**40**] in 1981.

The aim of this thesis is to combine the ideas in [**28**] and [**27**] in order to rigorously derive a fully discrete IST in a multidimensional setting as a means of solving a large class of nonlinear integrable partial difference equations. The solutions obtained will be real-valued, depend on an arbitrary number of independent variables and parameters, must obey a summability condition and will be shown to be obtainable through solving a singular integral equation. While the complex case was considered in [**27**], we will see that restricting to real-valued solutions significantly relaxes the summability restriction placed on solutions. We also give an example of one of the utilities of the discrete IST, which is its ability to be used to generate and infinite number of conservation laws for these discrete systems.

The outline of this thesis is as follows: In Chapter 2 we give an introduction to the theory behind the IST for continuous equations, where we consider both the linear heat equation and the nonlinear KdV equation. We

state the main results of the procedure and give examples of soliton solutions. In Chapter 3 we then move to lattice equations, where we look at a variety of lattice equations including those classified by ABS. We consider solutions, symmetries, initial-value problems and various integrability properties of these systems. These two chapters then provide a basis for the development of the discrete IST carried out in Chapter 4. The results of this chapter are new and are the main results of the thesis. In Chapter 5 we then look at conservation laws for lattice equations and show how these are obtainable from the new discrete IST, and finally in the Appendix we give some rigorous proofs of theorems stated in Chapter 4.

2 The Continuous Inverse Scattering Transform

The Inverse Scattering Transform (IST) is a method of finding solutions to linear and integrable nonlinear partial differential equations. In this chapter we look at the mathematical structure of the IST in its application to solve both the heat equation

$$u_t = u_{xx} \tag{2.1}$$

and the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.2}$$

This will provide an underlying framework for the development of the *discrete* IST, used to solve partial *difference* equations, which will be given in Chapter 4.

The heat equation and the KdV equation are both partial differential equations in one spatial and one temporal dimension, however one fundamental difference between these two equations is that (2.1) is *linear* in u, while (2.2) is not. Despite this difference they are both able to be represented as the compatibility condition for an auxiliary *linear* system, which provides the basis for the IST. The IST for the heat equation is relatively

simple, owing to the fact that the equation itself is linear. In Section 2.1 we provide a complete description of the IST for this equation, and compare the solution with that obtained by separation of variables or Fourier transform, which are standard methods for solving linear partial differential equations. In fact the IST essentially reduces to the Fourier transform in the linear case, and can thus be thought of as a nonlinear generalization of this technique. In Section 2.2 we then look at the more complicated mathematical structure of the IST for the KdV equation, and we then give examples of a one-soliton solution, and a solution obtained from an arbitrary reflectionless potential.

2.1 Linear Example of the Inverse Scattering Transform

Consider the heat equation (2.1) with an initial condition $u(x,0)$ satisfying

$$\int_{-\infty}^{+\infty} |u(x,0)|\,dx < \infty. \tag{2.3}$$

This can be solved by separation of variables or Fourier transform to give the general solution as

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{u}\,(k)\,e^{-ikx-k^2t}\,dk \tag{2.4}$$

where $\widehat{u}\,(k)$ is the Fourier transform of the initial condition $u(x,0)$. Alternatively one can generate [**39**] the Lax pair

$$\phi_x - i\zeta\phi = u \tag{2.5a}$$

$$\phi_t + \zeta^2\phi = u_x + i\zeta u, \tag{2.5b}$$

whose consistency gives $\phi_{xt} - \phi_{tx} = u_t - u_{xx}$. Equation (2.5a) defines the forward scattering problem, where we solve it for $\phi(x, 0; \zeta)$ along the initial-value space $t = 0$. The second Lax equation (2.5b) is then used to determine $\phi(x, t; \zeta)$, from which $u(x, t)$ may then be constructed from (2.5a).

To solve (2.5a) for $\phi(x, 0; \zeta)$ we first consider the limit $|x| \to \infty$, where

$$\phi_x(x, 0; \zeta) - i\zeta\phi(x, 0; \zeta) = 0 \;\Rightarrow\; \phi(x, 0; \zeta) \sim Ae^{i\zeta x}$$

for some constant A. We then define the unique Jost solutions φ and ψ of (2.5a) by the boundary conditions

$$\lim_{x\to-\infty} \left| \varphi(x, 0; \zeta)e^{-i\zeta x} - 0 \right| = 0$$
$$\lim_{x\to+\infty} \left| \psi(x, 0; \zeta)e^{-i\zeta x} - 0 \right| = 0.$$

In terms of the initial condition $u(x, 0)$ we have

$$\varphi(x, 0; \zeta) := \int_{-\infty}^{x} u(s, 0)\, e^{i\zeta(x-s)}\, ds \tag{2.6a}$$

$$\psi(x, 0; \zeta) := -\int_{x}^{+\infty} u(s, 0)\, e^{i\zeta(x-s)}\, ds. \tag{2.6b}$$

By virtue of (2.3) these integrals will exist in the half-planes $\mathrm{Im}(\zeta) \geq 0$ and $\mathrm{Im}(\zeta) \leq 0$ respectively. Furthermore since these are both solutions of (2.5a) it follows that

$$\varphi(x, 0; \zeta) - \psi(x, 0; \zeta) = \mathtt{B}(0; \zeta)\, e^{i\zeta x} \tag{2.7}$$

for some function $\mathtt{B}$ which is independent of x, and by (2.6) is expressible as

$$\mathtt{B}(0; \zeta) = \int_{-\infty}^{+\infty} u(s, 0)\, e^{-i\zeta s}\, ds. \tag{2.8}$$

We have now completed the direct scattering procedure, that is we have constructed solutions to the first Lax equation (2.5a) along the initial-value space $t = 0$, which are given by the integrals (2.6a) and (2.6b). These expressions however are inadequate for determining the time dependent Jost solutions $\varphi(x, t; \zeta)$ and $\psi(x, t; \zeta)$ since this would involve the knowledge of

$u(x,t)$, which is the very thing we are aiming to find. A more fruitful endeavour is to consider the time dependence of the function $\mathtt{B}(t;\zeta)$, defined by

$$\varphi(x,t;\zeta) - \psi(x,t;\zeta) = \mathtt{B}(t;\zeta)\, e^{i\zeta x}. \tag{2.9}$$

By substituting this equation into the second Lax equation (2.5b) and taking the limit $x \to +\infty$ we have

$$\mathtt{B}_t + \zeta^2 \mathtt{B} = 0 \;\Rightarrow\; \mathtt{B}(t;\zeta) = \mathtt{B}(0;\zeta)\, e^{-\zeta^2 t}, \tag{2.10}$$

where we have assumed that $u \to 0$ and $u_x \to 0$ as $x \to +\infty$ for all $t \geq 0$. Now consider a summary of the mathematical objects defined thus far: The Jost solutions φ and ψ are two sectionally holomorphic functions (assuming that the integrals (2.6a) and (2.6b) may be differentiated) defined in the half-planes $\mathrm{Im}(\zeta) \geq 0$ and $\mathrm{Im}(\zeta) \leq 0$ respectively, and are related on the boundary $\mathrm{Im}(\zeta) = 0$ by the jump condition (2.9). Furthermore by integrating by parts we have

$$\begin{aligned}
\varphi(x,t;\zeta) &= -\frac{u(x,t)}{i\zeta} + \frac{1}{i\zeta}\int_{-\infty}^{x} u_s(s,t)\, e^{i\zeta(x-s)}\, ds \\
\psi(x,t;\zeta) &= -\frac{u(x,t)}{i\zeta} - \frac{1}{i\zeta}\int_{x}^{\infty} u_s(s,t)\, e^{i\zeta(x-s)}\, ds,
\end{aligned}$$

and so providing that u is bounded and that

$$\int_{-\infty}^{+\infty} |u_s(s,t)|\, ds < \infty \tag{2.11}$$

it follows that

$$\varphi(x,t;\zeta) = \mathcal{O}\left(\frac{1}{\zeta}\right) \text{ as } |\zeta| \to \infty \text{ in } \mathrm{Im}(\zeta) \geq 0 \tag{2.12a}$$

$$\psi(x,t;\zeta) = \mathcal{O}\left(\frac{1}{\zeta}\right) \text{ as } |\zeta| \to \infty \text{ in } \mathrm{Im}(\zeta) \leq 0. \tag{2.12b}$$

Note that (2.11) is a sufficient condition for the Jost solutions to be holomorphic. We thus have all the ingredients for a Riemann-Hilbert problem, that is the determination of two sectionally holomorphic functions φ and

ψ, with given boundary behaviour (2.12), which satisfy the jump condition (2.9) on the common boundary of their regions of analyticity. The solution of this problem in $\mathrm{Im}(\zeta) \geq 0$ (see e.g. Gakhov [**42**] Section 4) is

$$\varphi(x,t;\zeta) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{\mathtt{B}(0;k)}{k-\zeta}\right)e^{ikx-k^2t}\,dk. \tag{2.13}$$

Through the solution of the Riemann-Hilbert problem we have obtained a singular integral equation for $\varphi(x,t;\zeta)$, where importantly the time dependence now enters only through the known time dependence of $\mathtt{B}(t;\zeta)$. From the first Lax equation the solution $u(x,t)$ of (2.1) is therefore

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{i\,(k-\zeta)\,\mathtt{B}(0;k)}{k-\zeta}\right)e^{ikx-k^2t}\,dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathtt{B}(0;k)\,e^{ikx-k^2t}\,dk. \end{aligned} \tag{2.14}$$

This is identical to the result (2.4), where we recognise $\mathtt{B}(0;k)$ as the Fourier transform of the initial condition $u(x,0)$.

2.2 The Inverse Scattering Transform for the Korteweg-de Vries Equation

The procedure for solving the KdV (2.2) via the IST is considerably more complicated than that for the heat equation (2.1). This method was first discovered by Gardner, Greene, Kruskal and Miura [**43**] [**44**] in the 1960s. Here we give an outline of the important mathematical features of this procedure, and for a detailed analysis on the forward scattering problem considered here see [**33**], while the inverse problem is treated in [**45**].

A Lax pair for equation (2.2) is given by

$$\phi_{xx} + (u+\zeta^2)\,\phi = 0 \tag{2.15}$$

$$\phi_t = (u_x + c)\,\phi + (4\zeta^2 - 2u)\,\phi_x, \tag{2.16}$$

where, provided that $c_t = \zeta_t = 0$, the consistency of the system forces u to satisfy the KdV equation:

$$\phi_{txx} - \phi_{xxt} = \Big(u_t + 6uu_x + u_{xxx}\Big)\,\phi.$$

The first of these equations (2.15) defines the forward scattering problem, while the second (2.16) defines the time evolution of the system. Assume that there exists some real-valued initial condition $u(x,0)$ which satisfies

$$\int_{-\infty}^{+\infty} |u(x,0)|(1+|x|)\,dx < \infty, \tag{2.17}$$

which as an acceptable integrability condition (i.e. one for which the inverse problem is able to be solved uniquely) on the initial condition was first proposed by Faddeev **[37]**. This was later proved in **[33]** where detailed estimates for the Schrödinger scattering problem were obtained. The forward scattering problem is then the determining of the solution $\phi(x,0;\zeta)$ to

$$\phi_{xx} + (u(x,0) + \zeta^2)\,\phi = 0. \tag{2.18}$$

This is a linear second-order equation for ϕ, which is in fact a Sturm-Louiville equation, and as such can be solved (at worst numerically) using standard methods. We first consider the limit $|x| \to \infty$, in which equation (2.15) becomes

$$\phi_{xx} + \zeta^2\phi = 0 \;\Rightarrow\; \phi \sim Ae^{i\zeta x} + Be^{-i\zeta x},$$

for some constant A and B. We then define two unique pairs of linearly independent Jost solutions by the boundary conditions

$$\left.\begin{aligned} \varphi(x,0;\zeta) &\sim e^{-i\zeta x} \\ \mathring{\varphi}(x,0;\zeta) &\sim e^{i\zeta x} \end{aligned}\right\} \text{ as } x \to -\infty \tag{2.19}$$

$$\left.\begin{aligned} \psi(x,0;\zeta) &\sim e^{i\zeta x} \\ \mathring{\psi}(x,0;\zeta) &\sim e^{-i\zeta x} \end{aligned}\right\} \text{ as } x \to +\infty. \tag{2.20}$$

Since (2.18) is invariant under the transformation $\zeta \to -\zeta$, by uniqueness of the boundary value problem it follows that $\varphi(x,0;\zeta) = \mathring{\varphi}(x,0;-\zeta)$ and $\psi(x,0;\zeta) = \mathring{\psi}(x,0;-\zeta)$. Furthermore since the solution space of equation (2.18) is two-dimensional we may write

$$\varphi(x,0;\zeta) = \mathtt{A}(0;\zeta)\,\mathring{\psi}(x,0;\zeta) + \mathtt{B}(0;\zeta)\,\psi(x,0;\zeta), \tag{2.21}$$

where the two functions A and B are independent of x, and by considering the Wronskian

$$W(\varphi,\mathring{\varphi}) := \varphi\,\mathring{\varphi}_x - \varphi_x\,\mathring{\varphi}$$

one can also show that these functions satisfy

$$|\mathtt{A}(0;\zeta)|^2 - |\mathtt{B}(0;\zeta)|^2 = 1 \tag{2.22}$$

on $\mathrm{Im}(\zeta) = 0$. Note that this implies that A does not vanish on $\mathrm{Im}(\zeta) = 0$. We now state several results about the Jost solutions and the functions A and B. Derivations can be seen in **[33] [10] [2]**, and are obtained by integrating (2.18) and obtaining a series representations for the Jost solutions. We also mention that similar results hold for the discrete case considered in Chapter 4.

Proposition 2.2.1. *The Jost solutions and the spectral functions* A *and* B *have the following analyticity properties:*

- $\varphi(x,0;\zeta)$ *and* $\psi(x,0;\zeta)$ *exist and are continuous in* ζ *in the closed half-plane* $\mathrm{Im}(\zeta) \geq 0$*, and are analytic in* ζ *in the open half-plane* $\mathrm{Im}(\zeta) > 0$
- $\mathring{\varphi}(x,0;\zeta)$ *and* $\mathring{\psi}(x,0;\zeta)$ *exist and are continuous in* ζ *in the closed half-plane* $\mathrm{Im}(\zeta) \leq 0$*, and are analytic in* ζ *in the open half-plane* $\mathrm{Im}(\zeta) < 0$
- $\mathtt{A}(0;\zeta)$ *exists and is continuous in* ζ *in the closed half-plane* $\mathrm{Im}(\zeta) \geq 0$*, and is analytic in* ζ *in the open half-plane* $\mathrm{Im}(\zeta) > 0$*.*
- $\mathtt{B}(0;\zeta)$ *exists and is continuous in* ζ *on* $\mathrm{Im}(\zeta) = 0$*.*

Proposition 2.2.2. *The Jost solutions and the spectral functions* $\mathtt{A}$ *and* $\mathtt{B}$ *have the following asymptotic properties:*

$$\left.\begin{aligned}\varphi(x,0;\zeta) &= e^{-i\zeta x} + \mathcal{O}\left(\tfrac{1}{\zeta}\right)\\ \psi(x,0;\zeta) &= e^{ikx} + \mathcal{O}\left(\tfrac{1}{\zeta}\right)\end{aligned}\right\} \text{ as } |\zeta| \to \infty \text{ in } \mathrm{Im}(\zeta) \geq 0 \tag{2.23}$$

$$\left.\begin{aligned}\mathring{\varphi}(x,0;\zeta) &= e^{i\zeta x} + \mathcal{O}\left(\tfrac{1}{\zeta}\right)\\ \mathring{\psi}(x,0;\zeta) &= e^{-ikx} + \mathcal{O}\left(\tfrac{1}{\zeta}\right)\end{aligned}\right\} \text{ as } |\zeta| \to \infty \text{ in } \mathrm{Im}(\zeta) \leq 0 \tag{2.24}$$

$$\mathtt{A}(0;\zeta) = 1 + \mathcal{O}\left(\frac{1}{\zeta}\right) \text{ as } |\zeta| \to \infty \text{ in } \mathrm{Im}(\zeta) \geq 0 \tag{2.25}$$

$$\mathtt{B}(0;\zeta) = \mathcal{O}\left(\frac{1}{\zeta}\right) \text{ as } |\zeta| \to \infty \text{ on } \mathrm{Im}(\zeta) = 0. \tag{2.26}$$

Theorem 2.2.3. *The function* $\mathtt{A}(0;\zeta)$ *has a finite number of zeroes* $\zeta_1, ..., \zeta_N$ *in the open half-plane* $\mathrm{Im}(\zeta) > 0$*, and does not vanish on* $\mathrm{Im}(\zeta) = 0$*. Moreover all of these zeroes are simple and lie on the imaginary axis* $\mathrm{Re}(\zeta_k) = 0$*. At each* ζ_k *we have* $\varphi(x,0;\zeta_k) = C_k(0)\,\psi(x,0;\zeta_k)$ *for some constant* $C_k(0)$*.*

The N zeroes of $\mathtt{A}$ form a set of discrete eigenvalues, which we will show are in fact associated with the N solitons which exist within the solution $u(x,t)$ of the KdV equation.

This completes the forward scattering procedure, in which we have constructed the four Jost solutions φ, $\mathring{\varphi}$, ψ and $\mathring{\psi}$ as well as the spectral functions $\mathtt{A}$ and $\mathtt{B}$. As was done in the case of the IST for the heat equation in Section 2.1, we now determine the time-dependent spectral functions $\mathtt{A}(t;\zeta)$ and $\mathtt{B}(t;\zeta)$. Since these functions are independent of x we calculate this in the limit $|x| \to \infty$, where we assume that $u \to 0$ and $u_x \to 0$ for all $t > 0$. Inserting the relation (2.21) in the second Lax equation (2.16) and

taking this limit yields

$$\frac{\partial \mathtt{A}}{\partial t} = 0 \;\Rightarrow\; \mathtt{A}(t;\zeta) = \mathtt{A}(0;\zeta) \tag{2.27}$$

$$\frac{\partial \mathtt{B}}{\partial t} = 8i\zeta^3 \mathtt{B} \;\Rightarrow\; \mathtt{B}(t;\zeta) = \mathtt{B}(0;\zeta)\, e^{8i\zeta^3 t} \tag{2.28}$$

$$\frac{\partial C_k}{\partial t} = 8i\zeta^3 C_k \;\Rightarrow\; C_k(t) = C_k(0)\, e^{8i\zeta_k^3 t} \tag{2.29}$$

The inverse transform involves using $\mathtt{A}$ and $\mathtt{B}$ to reconstruct the time-dependent solution $u(x,t)$. As for the heat equation this is done from the setting of a Riemann-Hilbert problem. Importantly, since the boundary conditions for u are independent of time, the analyticity and asymptotic results of Propositions 2.2.1 and 2.2.2 and Theorem 2.2.3 continue to hold for all $t > 0$. By these results the relation

$$\frac{\varphi(x,t;\zeta)}{\mathtt{A}(0;\zeta)} - \mathring{\psi}(x,t;\zeta) = \left(\frac{\mathtt{B}(t;\zeta)}{\mathtt{A}(0;\zeta)}\right) \psi(x,t;\zeta)$$

defines a jump condition between the two sectionally meromorphic functions $\dfrac{\varphi}{\mathtt{A}}$ and $\mathring{\psi}$ along the real ζ-axis, with known boundary behaviour. For $\mathrm{Im}(\zeta) > 0$ the solution of this is given by the singular integral

$$\begin{aligned}\psi(x,t;\zeta)e^{-i\zeta x} = 1 &- \sum_{k=1}^{N} \left(\frac{C_k(0)\,\psi(x,t;\zeta_k)}{\mathtt{A}_\zeta(0;\zeta_k)\,(\zeta+\zeta_k)}\right) e^{i\zeta_k x + 8i\zeta_k^3 t} \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \left(\frac{\mathtt{B}(0;\sigma)\,\psi(x,t;\sigma)}{\mathtt{A}(0;\sigma)\,(\sigma+\zeta)}\right) e^{i\sigma x + 8i\sigma^3 t}\, d\sigma \end{aligned} \tag{2.30}$$

where the sum is over the N simple zeroes of $\mathtt{A}$ in $\mathrm{Im}(\zeta) > 0$. This is a closed-form singular integral equation for $\psi(x,t;\zeta)$, where all time dependence is known from $\mathtt{B}(t;\zeta)$.

One can also isolate the dependence of ψ on the spectral parameter ζ by expressing it in the form

$$\psi(x,t;\zeta) = e^{i\zeta x} + \int_x^{+\infty} K(x,y,t)\, e^{i\zeta y}\, dy. \tag{2.31}$$

That such a K exists follows from inserting this expression into (2.15), thereby obtaining a Goursat problem for K. It can be shown that the solution of this

Goursat problem exists and is unique. The motivation for this choice comes from the fact that one of the boundary conditions for K gives a simple relation between it and the solution $u(x,t)$ to the KdV equation:

$$u(x,t) = -2\frac{\partial}{\partial x}\Big[K(x,x,t)\Big]. \tag{2.32}$$

In order to obtain a linear equation for $K(x,y,t)$ we substitute equation (2.31) into the singular integral equation for ψ. Since K is related to the Fourier transform of ψ, by taking the inverse Fourier transform we obtain the following Gel'fand-Levitan integral equation, valid for $y \geq x$:

$$K(x,y,t) + L(x+y,t) + \int_x^\infty K(x,s,t)L(s+y,t)\, ds = 0, \tag{2.33}$$

where the quantity L is given by

$$L(x,t) = -i\sum_{k=1}^{N}\left(\frac{C_k(0)}{\mathtt{A}_\zeta(0;\zeta_k)}\right)e^{i\zeta_k x + 8i\zeta_k^3 t} + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left(\frac{\mathtt{B}(0;\sigma)}{\mathtt{A}(0;\sigma)}\right)e^{i\sigma x + 8i\sigma^3 t}\, d\sigma. \tag{2.34}$$

Furthermore by considering the Wronskian $W(\psi,\psi_\zeta)$ one can also show that

$$\frac{-i\,C_k(0)}{\mathtt{A}_\zeta(0;\zeta_k)} = \left(\int_{-\infty}^{+\infty}\psi(x,0;\zeta_k)^2\, dx\right)^{-1} > 0, \tag{2.35}$$

which follows from the fact that the Jost solutions are real whenever $\mathrm{Re}(\zeta_k) = 0$. Thus from the knowledge of

$$\Big\{\mathtt{A}(0;\zeta),\ \mathtt{B}(0;\zeta),\ C_k(0),\ \{\zeta_k : k = 1,...,N\}\Big\}$$

one can construct the full time-dependent solution $u(x,t)$ through the linear Volterra-type integral equation (2.33). The quantity $T := \dfrac{1}{\mathtt{A}}$ is known as the transmission coefficient and $R := \dfrac{\mathtt{B}}{\mathtt{A}}$ is known as the reflection coefficient. Initial conditions for which the relfection coefficient is identically zero on $\mathrm{Im}(\zeta) = 0$ are known as *reflectionless potentials*.

2.3 One-soliton Solution for the KdV Equation

Consider an arbitrary initial condition $u(x,0)$ which satisfies (2.17) and gives rise to a spectral function $\mathtt{A}$ which has the following properties:

- $\mathtt{A}(0;\zeta)$ has exactly one zero ζ_1 in $\mathrm{Im}(\zeta) > 0$, which we denote by $\zeta_1 = ik$, where $k > 0$,
- $\mathtt{B}(0;\zeta) = 0$ for all ζ on $\mathrm{Im}(\zeta) = 0$.

The function L appearing in the Gel'fand-Levitan equation is

$$L(x,t) = -i\left(\frac{C_1(0)}{\mathtt{A}_\zeta(0;ik)}\right)e^{-kx+8k^3t} =: 2kA\,e^{-kx+8k^3t},$$

for some constant A. Assuming the natural form

$$K(x,y,t) = -\Psi(x)\,e^{-ky+8k^3t}, \tag{2.36}$$

equation (2.33) becomes

$$\Psi(x) - 2kAe^{-kx} + 2kA\Psi(x)\int_x^{+\infty} e^{-2ks+8k^3t}\,ds = 0,$$

from which we obtain the unique solution

$$K(x,y,t) = \frac{-2kA\,e^{-k(x+y)+8k^3t}}{1+A\,e^{-2kx+8k^3t}}. \tag{2.37}$$

By defining $A =: e^{2kx_o}$, equation (2.32) gives the solution to the KdV as

$$u(x,t) = 2k^2\,\mathrm{sech}^2\Big[k\left(x-4k^2t-x_o\right)\Big], \tag{2.38}$$

which is precisely the one-soliton solution. This is shown in Figure 1.

2.4 Arbitrary Reflectionless Potential

We now consider the case of an arbitrary reflectionless potential, that is an initial condition $u(x,0)$ for which

- $\mathtt{A}(0;\zeta)$ has N simple zeroes $\{\zeta_1,...,\zeta_N\}$ in $\mathrm{Im}(\zeta) > 0$, which we denote by $\{ik_1,...,ik_N\}$, where $k_j > 0$ for each j,

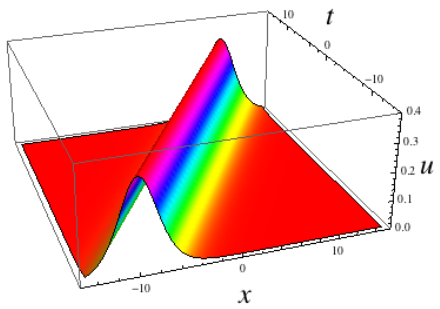

FIGURE 1. One-soliton solutions of the KdV equation.
$k = 0.40$, $x_o = -0.30$

- $\mathtt{B}(0;\zeta) = 0$ for all ζ on $\mathrm{Im}(\zeta) = 0$.

Following the analysis given in **[44]**, due to (2.35) we express the function L in the form

$$L(x,t) = -i\sum_{k=1}^{N}\left(\frac{C_k(0)}{\mathtt{A}_\zeta(0;\zeta_k)}\right)e^{i\zeta_k x + 8i\zeta_k^3 t} = \sum_{j=1}^{N} c_j(t)^2 e^{-k_j x} \tag{2.39}$$

where we have defined the positive time-dependent *normalisation constants* $c_j(t)$ by

$$c_j(t) := \left(\frac{-iC_j(0)}{\mathtt{A}_\zeta(0;\zeta_j)}\right)^{\frac{1}{2}} e^{4k_j^3 t} > 0. \tag{2.40}$$

That these normalisation constants must be positive follows from the details of the forward scattering problem **[44]**. The Gel'fand-Levitan equation (2.33) becomes

$$K(x,y,t) + \sum_{j=1}^{N} c_j(t)^2 e^{-k_j(x+y)} + \sum_{j=1}^{N} c_j(t)^2 e^{-k_j y}\int_x^\infty K(x,s,t)\,e^{-k_j s}\,ds = 0. \tag{2.41}$$

To find the solution $K(x,y,t)$ we assume the form

$$K(x,y,t) = -\sum_{r=1}^{N} c_r(t)\Psi_r(x)\,e^{-k_r y}, \tag{2.42}$$

and by insering this into the Gel'fand-Levitan equation and demanding that the coefficients of all the exponentials $e^{-k_j y}$ vanish we obtain the N

equations

$$\Psi_j(x) + \sum_{r=1}^{N} \left(\frac{c_j(t)\, c_r(t)}{k_j + k_r} \right) \Psi_r(x)\, e^{-(k_j+k_r)x} = c_j(t)\, e^{-k_j x}. \tag{2.43}$$

Define the matrix C_{mn} by

$$\begin{aligned} C_{mn}(x,t) &:= \left[\left(\frac{c_m(t)\, c_n(t)}{k_m + k_n} \right) e^{-(k_m+k_n)x} \right] \\ &= \left[\left(\frac{c_m(0)\, c_n(0)}{k_m + k_n} \right) e^{-(k_m+k_n)x + 4(k_m^3+k_n^3)t} \right], \end{aligned} \tag{2.44}$$

and let Ψ and E denote the column vectors

$$\Psi(x) := \begin{bmatrix} \psi_1(x) \\ \cdot \\ \cdot \\ \psi_N(x) \end{bmatrix}, \qquad E(x,t) := \begin{bmatrix} c_1(t) e^{-k_1 x} \\ \cdot \\ \cdot \\ c_N(t) e^{-k_N x} \end{bmatrix}. \tag{2.45}$$

Equation (2.43) may then be expressed in matrix form as

$$\Big[I + C(x,t) \Big]\, \Psi(x) = E(x,t). \tag{2.46}$$

To show that $I + C$ is invertible, it suffices to show that C (and thus $I + C$) is positive definite, since then all eigenvalues are positive. Given a real nonzero dummy vector $\mathbf{z}$ we have

$$\begin{aligned} \mathbf{z}^T C \mathbf{z} &= \sum_{m=1}^{N} \sum_{n=1}^{N} \left(\frac{z_m\, z_n\, c_m(t)\, c_n(t)}{k_m + k_n} \right) e^{-(k_m+k_n)x} \\ &= \int_x^{+\infty} \left[\sum_{n=1}^{N} z_n\, c_n(t)\, e^{-k_n s} \right]^2 ds \\ &> 0 \end{aligned}$$

as required. Letting $\Delta(x,t) := \det \Big(I + C(x,t) \Big)$ and $Q_{mn}(x,t)$ denote the matrix of cofactors of $I + C$, Cramer's rule gives

$$\Psi_n(x) = \frac{1}{\Delta(x,t)} \sum_{m=1}^{N} c_m(t)\, Q_{mn}(x,t)\, e^{-k_n x}, \tag{2.47}$$

and hence

$$\begin{aligned} K(x,x,t) &= -\sum_{n=1}^{N} c_n(t)\,\Psi_n(x)\,e^{-k_n x} \\ &= -\frac{1}{\Delta(x,t)}\sum_{m=1}^{N}\sum_{n=1}^{N} c_m(t)\,c_n(t)\,Q_{mn}(x,t)\,e^{-(k_m+k_n)x} \\ &= \frac{1}{\Delta(x,t)}\frac{\partial}{\partial x}\Big(\Delta(x,t)\Big) \end{aligned} \tag{2.48}$$

where we have used the formula for the derivative of a determinant:

$$\frac{\partial}{\partial x}\Big(\det(M)\Big) = \det(M)\,\mathrm{Tr}\left(M^{-1}\frac{\partial M}{\partial x}\right). \tag{2.49}$$

Thus finally we arrive at

Theorem 2.4.1. *Every reflectionless potential can be expressed as*

$$u(x,t) = -2\frac{\partial^2}{\partial x^2}\Big[\log\big(\det\left(I+C\right)\big)\Big], \tag{2.50}$$

where the matrix $C_{mn}(x,t)$ is defined by (2.44)*, with positive normalisation constants $c_n(t)$ given by* (2.40) *and distinct positive constants k_n.*

Every solution of the KdV therefore, which is of the form (2.50), will give rise to a reflectionless potential. Furthermore in [**44**] the authors show that these solutions are in fact pure soliton solutions (i.e. solutions comprised *only* of solitons, with no radiation present). This gives a new characterisation of soliton solutions - those which give rise to reflectionless potentials. A three-soliton solution is shown in Figure 2. It was also shown in [**44**] that pure soliton solutions can be represented in terms of the square eigenfunctions as

$$u(x,t) = -4\sum_{n=1}^{N} k_n\,\psi^2(x,t;ik_n), \tag{2.51}$$

where each term in the sum is a single soliton with discrete eigenvalue ik_n.

In general however solutions to (2.2) are comprised not just of solitons, but also of background radiation, which is expressed by transcendental

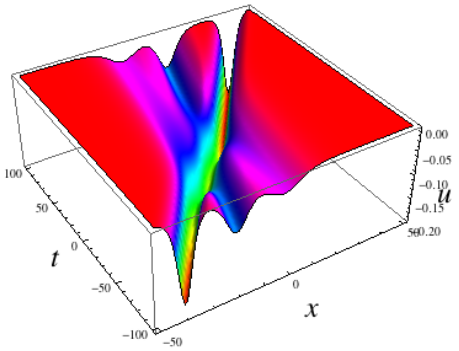

FIGURE 2. Three-soliton solutions of the KdV equation. $c_1 = e^{-0.13}$, $c_2 = e^{0.20}$, $c_3 = e^{1.2}$, $k_1 = 0.13$, $k_2 = 0.20$, $k_3 = 0.30$

functions. We cannot write down such a solution explicitly, but through the forward scattering procedure of the IST one can determine exactly how many solitons are present within a given solution (by counting the number of discrete eigenvalues ζ_k) and give their amplitudes and wave speeds, *without knowing the solution itself*. We also mention that given a solution to the Schrödinger equation (2.15), it is possible **[81]** to define new solutions (through a recursion relation) which solve (2.15) for a different potential u. This new potential will either gain a discrete eigenvalue and hence gain a soliton (the "dressing" procedure), or lose a discrete eigenvalue and hence lose a soliton (the "undressing" procedure). Thus given a potential containing N solitons, the undressing procedure can be applied N times to obtain a solution which is comprised solely of background radiation.

An arbitrary transcendental solution to the KdV is expressed implicitly through the integral equation (2.33), which is derived from a singular integral equation for ϕ. As a final note we mention that this idea was generalised in 1981 by Fokas and Ablowitz **[40]**, who showed that if $\phi(x, t; k)$

satisfies

$$\phi(x,t;k) + ie^{ikx+ik^3t} \int_L \frac{\phi(x,t;l)}{l+k}\, d\lambda(l) = e^{ikx+ik^3t}, \tag{2.52}$$

for an approriate contour L and measure λ, then a solution of (2.2) is given by

$$u(x,t) = -\frac{\partial}{\partial x} \int_L \phi(x,t;k)\, d\lambda(k). \tag{2.53}$$

The integral equation (2.52) was also proposed to study the Painlevé II reduction of the KdV equation and its corresponding transcendental solutions. This direct method of starting with linear integral equations in order to obtain formal solutions of nonlinear evolution equations has also been applied to other nonlinear partial differential equations (see e.g. [**68**]).

This concludes our study of the continuous IST. We have looked at its application in solving the linear heat equation and the nonlinear KdV equation, both of which are important prevalent equations in mathematical physics. The main features of the IST discussed in this chapter will also provide a framework for the analysis carried out in Chapter 4, where we will rigorously develop a discrete version of the IST as a means of solving nonlinear partial difference equations. These equations, some of which are discrete versions of the KdV equation (2.2), will be first derived and studied in Chapter 3, before being solved using the discrete IST in Chapter 4.

3 Lattice Equations

This chapter is dedicated to the study of discretisations of such integrable partial differential equations as the KdV equation, which appeared in Chapter 2. We consider the problem of determining the "correct" discretisation for a given equation, the sorts of solutions which exist, and the notion of an initial-value problem in the discrete setting. In Section 3.4 we then move away from partial differential equations altogether, and look at a class of integrable nonlinear partial *difference* equations, which are nonlinear evolution equations depending on two discrete independent variables. We look at various properties of these equations, such as their known solutions, symmetries, initial-value problems, and how to interpret the notion of integrability in the discrete setting. This will provide a natural setting for the development of the discrete IST in Chapter 4.

3.1 Motivation for Discretisation

A fundamental property of integrable partial *differential* equations is that the solution u is dependent on *continuous* independent variables x and t (which may be vectors). The Korteweg-de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{3.1}$$

is such an equation. As was shown in chapter 2 this equation is solvable with the Inverse Scattering Transform (IST), it possesses a class of special

soliton solutions and has an infinite number of conservation laws. As such it is known as a completely integrable system. The IST solution for (3.1) is found by solving the Volterra-type integral equation

$$K(x,y,t) + L(x+y,t) + \int_x^\infty K(x,s,t)L(s+y,t)\,ds = 0, \qquad (3.2)$$

which is equation (2.33), for the function $K(x,y,t)$. The solution of (3.1) is then given by

$$u(x,t) = -2\frac{\partial}{\partial x}\Big[K(x,x,t)\Big].$$

The function L is a known function of the scattering data (including the reflection coefficient) and is given by equation (2.34). In Chapter 2 it was shown that if the reflection coefficient R is identically zero then the solution is comprised solely of N solitons. If $R \neq 0$ however the solution of (3.2) will be composed of solitons as well as transcendental background radiation. At present the only means of solving this equation is with some numerical scheme, which necessarily involves discretisation of the integration variable. There are many ways of choosing this discretisation, depending on whether one is interested in rapid convergence of the solution or the conservation of some quantity such as energy. The point is however that to date there is no non-numerical method for solving the KdV when radiation is present in the solution. Thus even for the most elementary non-special solutions of the KdV, the solution obtained from the numerical method (used on the exact integral representation of the solution) will be *discrete* in the spatial variable, that is its dependence on x will only be taken at countably infinite isolated points.

Since the solution is discrete in x, one is tempted to ask whether there exist versions of the KdV equation itself which depend continuously on time, but in which the spatial variable has been discretised. Furthermore, what if one were to demand that time were also discretised? Of course each

time one discretises there are an infinite number of possibilities, such as

$$u_x \to \frac{u(x+h,t)-u(x,t)}{h}, \qquad u_x \to \frac{u(x+h,t)-u(x-h,t)}{2h},$$

and thus an infinite number of possible “discrete versions" of the KdV. Yet not all of these remain integrable (in some sense) in the discrete setting, for example the equation may not define a well-posed initial-value problem, it may not retain the same number of conservation laws, or (see e.g. [**47**]) certain discretisations may induce chaotic behaviour.

3.2 Bäcklund Transformations of the KdV

In order to determine what could be considered a “natural" discretisation of the KdV, we first consider whether or not there already exists any sort of natural discrete property of this equation, from which we could base our discretisation. Consider the Miura transformation

$$u = -v^2 - v_x + \frac{1}{4}p^2, \tag{3.3}$$

where the added parameter p (satisfying $p_x = p_t = 0$) comes from the fact that the KdV equation is Galilean invariant, that is it remains unchanged under the change of variables

$$t' = t, \qquad x' = x - \frac{3}{2}p^2 t, \qquad u' = u - \frac{1}{4}p^2.$$

This maps solutions of

$$v_t - 6v^2 v_x + v_{xxx} = -\frac{3}{2}p^2 v_x \tag{3.4}$$

to those of the KdV equation (3.1):

$$u_t + 6uu_x + u_{xxx} = -\left[2v + \frac{\partial}{\partial x}\right]\left[v_t - 6v^2 v_x + v_{xxx} + \frac{3}{2}p^2 v_x\right].$$

Note that if $p = 0$ then equation (3.4) becomes the Modified KdV equation. A trivial symmetry of equation (3.4) is its invariance under the map $v \to$

$-v$, however this generates a highly nontrivial transformation in the action of the Miura transformation **[31]**. If u denotes the solution obtained from (3.3), then let $\widetilde{u}$ denote the solution obtained from the same transformation, but after the map $v \to -v$, that is

$$u = -v^2 - v_x + \frac{1}{4}p^2 \tag{3.5a}$$

$$\widetilde{u} = -v^2 + v_x + \frac{1}{4}p^2. \tag{3.5b}$$

We emphasise that u and $\widetilde{u}$ are *both* solutions of the KdV equation (3.1). Adding and subtracting these yields

$$u + \widetilde{u} = -2v^2 + \frac{1}{2}p^2 \tag{3.6a}$$

$$u - \widetilde{u} = -2v_x. \tag{3.6b}$$

We now define the quantity

$$w(x,t) := \int u(x,t)\,dx$$

from which it follows that $w(x,t)$ satisfies the potential KdV equation

$$w_t + 3w_x^2 + w_{xxx} = 0. \qquad (\mathrm{pKdV}) \tag{3.7}$$

Note that in the definition of w we have chosen to set the arbitrary constant to be zero. By integrating (3.6b) and again setting the arbitrary constant to be zero one can combine this with (3.6a) to eliminate v and obtain

$$2(w + \widetilde{w})_x = p^2 - (w - \widetilde{w})^2. \tag{3.8}$$

This relation describes the x-dependence of a transformation between two *different* solutions w and $\widetilde{w}$ of the *same* equation (3.7), which (now indirectly) corresponds to the change $v \to -v$ in (3.4). This transformation involves a free parameter p and is known as a *Bäcklund transformation*. Given one solution w, it provides a means of iterating through the solution space of (3.7) to obtain a new solution $\widetilde{w}$ of the same equation.

As an example, consider the trivial solution $w \equiv 0$ of (3.7), which also corresponds to the trivial solution $u \equiv 0$ of the KdV equation. The Bäcklund transfomation (3.8) becomes

$$2\widetilde{w}_x + \widetilde{w}^{\,2} = p^2$$

which is a Riccati equation, and is thus linearisable by the substitution $w = 2\log(\psi)_x$. Solving yields

$$\widetilde{w}(x,t) = p\tanh\left(\frac{p}{2}x + \alpha(t)\right),$$

where the time dependence $\alpha(t)$ can be found by using (3.7):

$$\alpha(t) = -\frac{p}{2}(p^2 t + x_o)$$

for some constant x_o. Thus the new solution of the pKdV obtained via Bäcklund transformation is

$$\widetilde{w}(x,t) = p\tanh\left(\frac{p}{2}(x - p^2 t - x_o)\right), \tag{3.9}$$

which is in fact a one-soliton solution for this equation. Differentiating in x then gives the new solution of the KdV equation

$$\widetilde{u}(x,t) = \frac{p^2}{2}\mathrm{sech}^2\left(\frac{p}{2}(x - p^2 t - x_o)\right),$$

which is also the one-soliton solution for the KdV. For completeness we also note that the corresponding solution of equation (3.4) is

$$v(x,t) = \frac{p}{2}\tanh\left(\frac{p}{2}(x - p^2 t - x_o)\right).$$

Thus from the Bäcklund transformation we have transformed the zero solution of the KdV equation into the one-soliton solution. This process can be repeated [**90**] to obtain further new solutions, with each transformation

corresponding to a mapping within the solution space of the KdV. For example starting with the solution (3.9) and performing the Bäcklund transformation (3.8) a second time yields a second new solution

$$\widetilde{\widetilde{w}}(x,t) = \frac{4p}{p\big(x-3p^2t-x_1\big)\mathrm{sech}^2\left(\frac{p}{2}\big(x-p^2t-x_o\big)\right)+2\tanh\left(\frac{p}{2}\big(x-p^2t-x_o\big)\right)}$$

where x_1 is a second arbitrary constant. This therefore naturally defines a *discrete* process associated with this equation, which is the mapping between the solutions $w \to \widetilde{w} \to \widetilde{\widetilde{w}} \to ...$, and with the correct insight, will serve as a basis for obtaining a discretisation of the KdV equation itself.

3.3 The Discrete KdV Equation

Consider the Bäcklund transformation (3.8) given in Section 3.2

$$2(w+\widetilde{w})_x = p^2 - (w-\widetilde{w})^2, \tag{3.10}$$

which maps one solution w of the pKdV equation (3.7) to another solution $\widetilde{w}$ of the same equation. Given w one can solve this equation for $\widetilde{w} = \widetilde{w}(x,t;p)$, which will depend on the free parameter p. We denote the action of this transformation by

$$\mathrm{BT}_p\big(w\big) = \widetilde{w}.$$

Suppose then that we define another Bäcklund transformation

$$2(w+\widehat{w})_x = q^2 - (w-\widehat{w})^2, \tag{3.11}$$

denoted by BT_q, which maps the solution w to a different solution $\widehat{w} = \widehat{w}(x,t;q)$ of the same equation, but which now depends on the parameter q rather than p. From these two Bäcklund transformations (3.10) and (3.11) it is possible therefore to define an infinite number of new solutions

to the pKdV through the various permutations of superpositions of Bäcklund transformations, such as

$$\widehat{\widetilde{w}} = \mathrm{BT}_q\left(\mathrm{BT}_p\left(\mathrm{BT}_p\left(w\right)\right)\right), \qquad \widetilde{\widehat{w}} = \mathrm{BT}_p\left(\mathrm{BT}_q\left(\mathrm{BT}_q\left(w\right)\right)\right).$$

With such compositions it is natural to ask whether there exists any sort of permutability between the various superpositions, in particular whether the diagram in Figure 1 commutes. This in fact is the case[1], that is for any

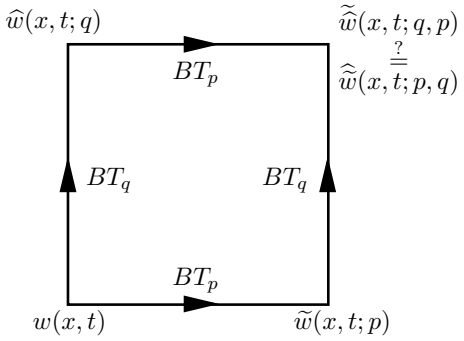

FIGURE 1. Permutability of the Bäcklund Transformation for the pKdV

solution w of the pKdV, one may choose the integration constants resulting from the Bäcklund transformations in such a way that

$$\widehat{\widetilde{w}}\,(x,t;p,q) = \mathrm{BT}_q\left(\mathrm{BT}_p\left(w\right)\right) = \mathrm{BT}_p\left(\mathrm{BT}_q\left(w\right)\right) = \widetilde{\widehat{w}}\,(x,t;q,p).$$

Given w, there are two ways of calculating the quantity $\widehat{\widetilde{w}}$. Firstly one can use (3.10) to calculate $\widetilde{w}$, then by applying BT_q to (3.10) use this to determine $\widehat{\widetilde{w}}$:

$$2(\,\widetilde{w} + \widehat{\widetilde{w}}\,)_x = q^2 - (\,\widetilde{w} - \widehat{\widetilde{w}}\,)^2. \tag{3.12}$$

[1]A similar permutability property of the sine-Gordon equation was found in 1899 by Bianchi [**22**]

Alternatively one can use (3.11) to find $\widehat{w}$ and then apply BT_q to (3.10) to again determine $\widehat{\widetilde{w}}$:

$$2(\widehat{w}+\widehat{\widetilde{w}})_x = p^2 - (\widehat{w}-\widehat{\widetilde{w}})^2. \tag{3.13}$$

Moreover by taking the difference of equations (3.12) and (3.13) we obtain

$$2(\widehat{w}-\widetilde{w})_x = p^2 - q^2 + 2\widehat{\widetilde{w}}(\widehat{w}-\widetilde{w}) + (\widetilde{w}^2 - \widehat{w}^2),$$

and by combining this with the difference of equations (3.10) and (3.11) it is possible to eliminate all derivatives and obtain

$$(w-\widehat{\widetilde{w}})(\widehat{w}-\widetilde{w}) = p^2 - q^2. \tag{3.14}$$

This equation defines a nonlinear relationship between the four solutions w, $\widetilde{w}$, $\widehat{w}$ and $\widehat{\widetilde{w}}$ of the pKdV. The equation is affine-linear, so that given any three of these quantities one can solve uniquely for the fourth. For example it was shown that the Bäcklund transformation acting on the seed solution $w \equiv 0$ yielded the one-soliton solution (3.9), and thus by (3.14) a new solution $\widehat{\widetilde{w}}$ of the pKdV depending on the two distinct paramters p and q is

$$\widehat{\widetilde{w}} = \frac{p^2-q^2}{p\tanh\left(\frac{p}{2}\left(x-p^2t-x_o\right)\right) - q\tanh\left(\frac{q}{2}\left(x-q^2t-x_1\right)\right)}.$$

If we set $p > q > 0$ and $-px_o \to -px_o + \dfrac{i\pi}{2}$ (so that the denominator does not vanish for real x and t) with x_o and x_1 real, then this defines a two-soliton solution to the pKdV:

$$\widehat{\widetilde{w}}(x,t;p,q) = \frac{p^2-q^2}{p\coth\left(\frac{p}{2}\left(x-p^2t-x_o\right)\right) - q\tanh\left(\frac{q}{2}\left(x-q^2t-x_1\right)\right)}, \tag{3.15}$$

which is shown in Figure 2. By differentiating this solution respect to x we obtain a two-soliton solution to the KdV equation, shown in Figure 3.

An important feature of equation (3.14) is that it is does not explicitly depend on x, t or any derivatives. The procedure of mapping between solutions within the solution space of the pKdV can therefore be carried

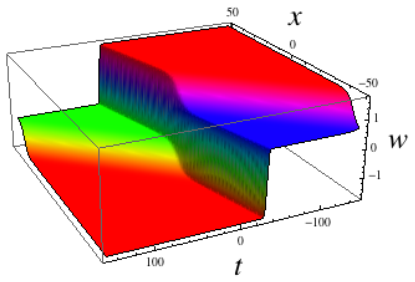

FIGURE 2. Two-soliton solution of the pKdV equation.
$p = 1.16$, $q = 0.55$, $x_o = -2.70$, $x_1 = -1.90$

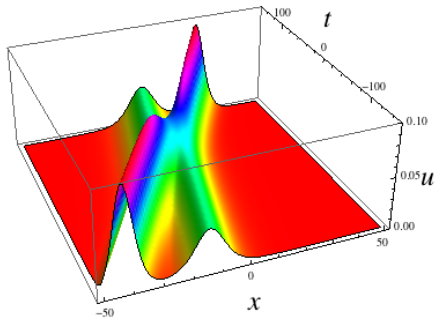

FIGURE 3. Two-soliton solutions of the KdV equation.
$p = 0.44$, $q = 0.28$, $x_o = -10.0$, $x_1 = -7.25$

out purely algebraically in terms of w, $\widetilde{w}$, $\widehat{w}$, etc., *independently of* x *and* t. Thus for equation (3.14) the quantities x and t no longer play the role of independent variables, but rather exist as *parameters* within the solutions space. The independent variables for this equation are in fact represented by the shifts $\widetilde{\ }$ and $\widehat{\ }$. To make this more explicit we redefine this notation:

$$w = w_{n,m}, \qquad \widetilde{w} = w_{n+1,m}, \qquad \widehat{w} = w_{n,m+1}, \qquad \widehat{\widetilde{w}} = w_{n+1,m+1}.$$

Each shift $\widetilde{\ }$ with parameter p corresponds to a shift in the independent variable n such that $n \to n+1$, while each shift $\widehat{\ }$ with parameter q corresponds to $m \to m+1$. In this light equation (3.14) defines a nonlinear

discrete evolution equation for w, and since it depends on two discrete independent variables n and m we call this equation a *partial difference equation* (PΔE). This equation exists within an infinite two-dimensional lattice $\mathbb{Z} \times \mathbb{Z}$, where the dependent variable w depends explicitly on the parameters p and q, and implicitly on the parameters x and t.

In order to obtain a solution equation (3.14) we start with known solutions of the pKdV equation and postulate the dependence on n and m. Given that $c_o x - 3c_o^2 t + c_1$ solves the pKdV for any constants c_o and c_1, trying a solution with linear dependence on n and m yields

$$w_{n,m} = pn + qm + \frac{1}{4}(p^2 + k^2)x - \frac{3}{16}(p^2 + k^2)^2 t + C, \tag{3.16}$$

where we have chosen $4c_o = p^2 + k^2$, $c_1 = C$. Performing the Bäcklund transformation (3.10) (with new solution denoted by $\overline{w}$) then gives

$$\overline{w} = w - k \tanh\left(-\frac{k}{2}\Big(x - k(k+6)t\Big) + f_{n,m}\right) \tag{3.17}$$

and by again seeking a linear function $f_{n,m}$ we obtain

$$\begin{aligned}\overline{w}_{n,m} = {} & pn + qm + \frac{1}{4}(p^2 + k^2)x - \frac{3}{16}(p^2 + k^2)^2 t + C \\ & - k \tanh\left(\mu n + \nu m - \frac{k}{2}\Big(x - k(k+6)t\Big) + D\right)\end{aligned} \tag{3.18}$$

where $p\tanh(\mu) = q\tanh(\nu) = k$ and D is constant. Let us emphasise the dual nature of this solution. By the definition of the Bäcklund transformation, for fixed n and m this function solves the pKdV partial differential equation (3.7) as a function of the continuous variables x and t. We have now shown however that if instead we hold x and t fixed, this function also solves the partial difference equation (3.14) as a function of the discrete variables n and m. The PΔE (3.14) was derived as a mapping within the solution space of the pKdV, however this duality result suggests that perhaps there is a stronger connection between the two equations. It turns out in fact that equation (3.14) is a particular discretisation of equation (3.7),

and as such is called the *lattice potential KdV equation* (lpKdV). The pKdV is obtained from the lpKdV by taking the following two-step limit process **[50] [91] [66]**: To begin we set

$$q = p - \delta, \qquad \delta m = \tau, \qquad m \to \infty, \qquad \delta \to 0, \tag{3.19}$$

and redefine the independent variable n to be $n' := n + m$, which we keep finite. By writing $w = w_{n'}(m)$ we then have

$$\widetilde{w} = w_{n'+1}(m), \qquad \widehat{w} = w_{n'+1}(m+1),$$

and by expanding $\widehat{w}$ as

$$\widehat{w} = w_{n'+1}(m) + \delta \frac{\partial}{\partial(\delta m)} w_{n'+1}(m) + ... = w_{n+1} + \delta \frac{\partial w_{n+1}}{\partial \tau} + ...$$

(where we have dropped the m-dependence and the primes), the lpKdV (with w replacing u) becomes

$$1 + \frac{\partial w_n}{\partial \tau} = \frac{2p}{2p + w_{n-1} - w_{n+1}}. \tag{3.20}$$

This is a differential-difference equation for the quantity $w = w(n, \tau)$, which is related to the Kac-van Moerbeke-Volterra equation **[53]**. The second continuum limit, which gives the potential KdV equation (3.7), is then performed by letting

$$p \to \infty, \qquad n \to \infty, \qquad \tau \to \infty, \tag{3.21}$$

while labelling

$$x = x(n, \tau) = \frac{2n}{p} + \frac{2\tau}{p^2}, \qquad t = t(n; \tau) = \frac{2n}{3p^3} + \frac{2\tau}{p^4},$$

and keeping both of these quantities finite.

Let us now move away from the view of the lpKdV (3.14) existing as a map within the solution space of the pKdV (3.7), and rather consider its properties as a discrete version of this equation, with n and m representing

discrete space and time. In this light the solution (3.18) is more conveniently expressed as

$$w_{n,m} = pn + qm + C - \frac{2k}{1+\rho(k)} \tag{3.22}$$

where the plane-wave factor is defined by

$$\rho(k) := \rho_o \left(\frac{p+k}{p-k}\right)^n \left(\frac{q+k}{q-k}\right)^m, \tag{3.23}$$

and the constants C and ρ_o (into which we have absorbed the x and t dependence) have been redefined appropriately. The solution (3.22) is in fact a one-soliton solution to (3.14). A smooth plot of the soliton kink is shown in Figure 4, however it is understood that the solution exists only at the vertices of the overlaid lattice.

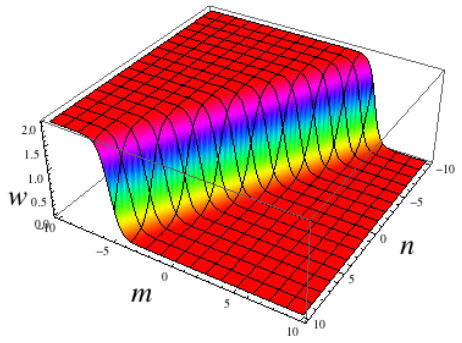

FIGURE 4. Plot of $w = \frac{2k}{1+\rho(k)}$. $p = 1.5$, $q = 1.1$, $\rho_o = k = 1$

One can also construct the two-soliton solution

$$w = pn + qm + C - \frac{2(k_1+k_2) + 2k_2\,\rho(k_1) + 2k_1\,\rho(k_2)}{1 + \rho(k_1) + \rho(k_2) + \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 \rho(k_1)\,\rho(k_2)} \tag{3.24}$$

where

$$\rho(k_i) = \rho_i \left(\frac{p+k_i}{p-k_i}\right)^n \left(\frac{q+k_i}{q-k_i}\right)^m$$

with ρ_i constant. This double kink is shown in Figure 5.

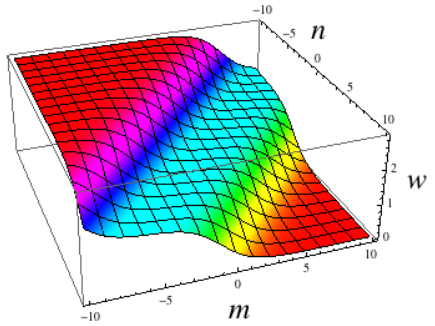

FIGURE 5. Plot of $w_{n,m} - pn - qm - C$ with $w_{n,m}$ given by (3.24). $p = 1.9$, $q = 1.7$, $\rho_1 = 5$, $\rho_2 = 1$, $k_1 = 0.75$, $k_2 = 0.70$

It is also possible to define a well-posed initial-value problem for equation (3.14) **[71] [73]**. Since the equation is linear in each of the quantities w, $\widetilde{w}$, $\widehat{w}$ and $\widehat{\widetilde{w}}$, given any three of these values one can solve uniquely for the fourth. Thus the initial-value space shown in Figure 6 defines a well-posed initial-value problem, as each remaining point in the lattice can be found uniquely using equation (3.14) on each individual lattice plaquette.

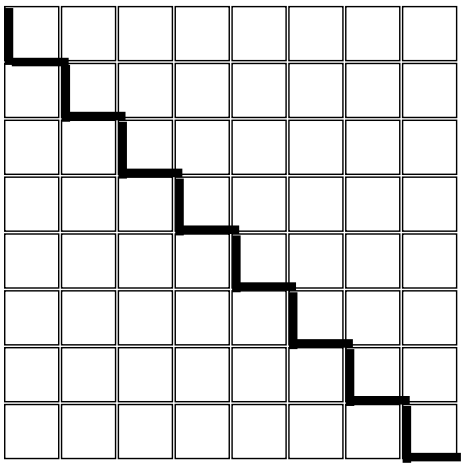

FIGURE 6. Well-posed Initial-value Problem for the lpKdV

Let us now compare equation (3.14) with a more naive discretisation of the pKdV (3.7) obtained by setting

$$w_t = \widehat{w} - w, \qquad w_x = \widetilde{w} - w, \qquad w_{xxx} = \widetilde{\widetilde{w}} - 3\widetilde{\widetilde{w}} + 3\widetilde{w} - w.$$

The resulting PΔE is

$$\widehat{w} + \widetilde{\widetilde{w}} - 3\widetilde{\widetilde{w}} + 3\widetilde{w}^2 - 6w\widetilde{w} + 3\widetilde{w} + 3w^2 - 2w = 0 \tag{3.25}$$

which exists on five points of the lattice. The asymmetry of this equation affects its ability to evolve within the lattice. For example given an initial condition along some line of points in the $\widetilde{\ }$ direction, the evolution of this equation is unique in the positive $\widehat{\ }$ direction, but grossly underdetermined in the negative $\widehat{\ }$ direction. Furthermore given any four of the five shifts of w appearing in this equation, due to its quadratic nature it is not always possible to solve uniquely for the remaining term. It is for these reasons that equation (3.14) is a "better", or more natural discretisation of the pKdV than (3.25).

It is however possible to define an initial-value space for which equation (3.14) defines either an over-determined or under-determined initial-value problem (see e.g. [**15**] or [**87**]). This is shown in Figure 7. The points with filled circles are over-determined, that is based on the initial data there are two ways of calculating these values which in general will not coninncide. The points with open circles are under-determined, that is there is not sufficient initial data to determine their values uniquely.

One remaining question is whether we can use the lattice potential KdV to obtain a discrete version of the KdV equation itself. It turns out that the most natural way to do this is to define the variable $u := \widetilde{w} - \widehat{w}$, where by equation (3.14) the quantity u satisfies

$$u - \widehat{\widetilde{u}} = (p^2 - q^2)\left(\frac{1}{\widehat{u}} - \frac{1}{\widetilde{u}}\right). \tag{3.26}$$

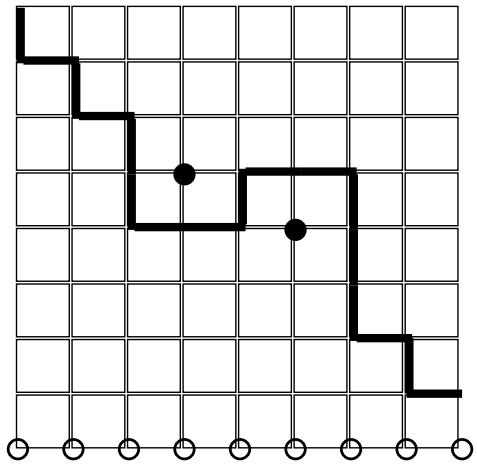

FIGURE 7. Ill-posed Initial-value Problem for the lpKdV

This is in fact a discretisation of the KdV equation **[50]** and is known as the *lattice KdV equation* (lKdV). From (3.22) the one-soliton solution to this equation is given by

$$u = (p-q)\left(\frac{\left(1+\rho(k)\right)\left(1+\widehat{\widetilde{\rho}}(k)\right)}{\left(1+\widetilde{\rho}(k)\right)\left(1+\widehat{\rho}(k)\right)}\right), \tag{3.27}$$

and is shown in Figure 8. Equation (3.24) then gives the two-soliton solu-

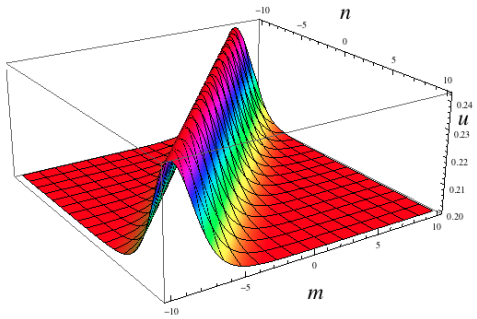

FIGURE 8. One-soliton solution of the lKdV. $p = 1.9$, $q = 1.7$, $\rho_o = 1$, $k = 0.75$

tion

$$u = (p - q) \left(\frac{F \, \widehat{\widetilde{F}}}{\widetilde{F} \, \widehat{F}} \right), \tag{3.28}$$

where the quantity F is given by

$$F = 1 + \rho(k_1) + \rho(k_2) + \left(\frac{k_1 - k_2}{k_1 + k_2} \right)^2 \rho(k_1)\, \rho(k_2).$$

The solution (3.28) is shown in Figure 9.

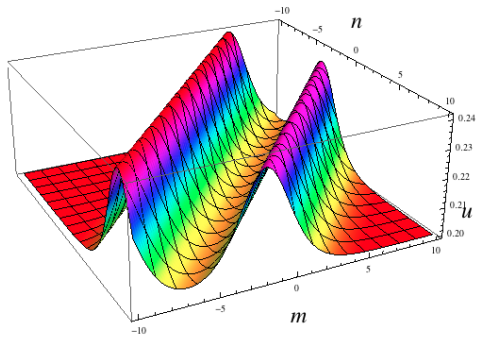

FIGURE 9. Two-soliton solution of the lKdV. $p = 1.9$, $q = 1.7$, $\rho_o = 5$, $\rho_2 = 1$, $k_1 = 0.75$, $k_2 = 0.70$

Many other fully discrete counterparts to famous partial differential equations were obtained in the 80's by the Dutch group of Capel et al. who advanced the so-called direct linearization method [**67**]. The group started with variations of the singular integral equation (2.52) which linearises the KdV and Painlevé II equations [**40**], and through the application of Bäcklund transformations showed that certain terms within the integral equation obeyed nonlinear partial difference equations [**74**]. These equations were then labeled as integrable since their solutions (including classes of special solutions such as solitons) could be found through the singular integral equations themselves. Examples of such partial difference equations

include the NQC equation **[67]**

$$\frac{(p+\alpha)\,\widehat{\widetilde{s}}-(p-\beta)\,\widetilde{s}-1}{(q+\alpha)\,\widehat{\widetilde{s}}-(q-\beta)\,\widetilde{s}-1}=\frac{(q+\beta)\,\widehat{s}-(q-\alpha)s-1}{(p+\beta)\,\widetilde{s}-(p-\alpha)s-1}, \tag{3.29}$$

the lattice sine-Gordon equation **[50]** **[70]** **[74]**

$$\sin\left(w+\widetilde{w}+\widehat{w}+\widehat{\widetilde{w}}\,\right)=pq\sin\left(w-\widetilde{w}-\widehat{w}+\widehat{\widetilde{w}}\,\right), \tag{3.30}$$

as well as a a range of KdV-type lattice equations including the lattice modified KdV equation **[67]** **[74]** **[66]**

$$p\left(V\,\widehat{V}-\widetilde{V}\,\widehat{\widetilde{V}}\,\right)=q\left(V\,\widetilde{V}-\widehat{V}\,\widehat{\widetilde{V}}\,\right), \tag{3.31}$$

the lattice potential modified KdV equation **[65]**

$$\frac{\widehat{\widetilde{W}}}{W}=\frac{\left(p\widehat{W}-q\right)\left(p-q\widetilde{W}\right)}{\left(p-q\widehat{W}\right)\left(p\widetilde{W}-q\right)} \tag{3.32}$$

and the lattice Schwarzian KdV equation **[66]**

$$\frac{\left(Z-\widetilde{Z}\right)\left(\widehat{Z}-\widehat{\widetilde{Z}}\right)}{\left(Z-\widehat{Z}\right)\left(\widetilde{Z}-\widehat{\widetilde{Z}}\right)}=\frac{q^2}{p^2}. \tag{3.33}$$

As a final remark we mention an alternative approach to finding discretisations of famous partial differential equations pioneered by Hirota, which is the finding of discretisations of their bilinear forms. By redefining the independent variable in the KdV equation (3.1) according to $u=2\partial_x^2\log(f)$ then the equation for f becomes

$$ff_{xxxx}-4f_xf_{xxx}+3f_{xx}^2+ff_{xt}-f_xf_t=0,$$

which is expressed most conveniently as

$$D_x\left(D_t+D_x^3\right)f\cdot f=0, \tag{3.34}$$

where the Hirota derivatives are defined by

$$D_x^n D_t^m f(x,t) \cdot g(x',t') \equiv$$
$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m f(x,t)\, g(x',t')\Big|_{x'=x,t'=t}.$$

The exponentiated Hirota operator is then given by

$$e^{\alpha D_x} f \cdot g = f(x+\alpha)g(x-\alpha).$$

The lattice KdV equation (3.26) was in fact discovered by Hirota [**50**] through finding corresponding differential-difference and difference-difference bilinear forms of (3.34).

3.4 Integrability for Lattice Equations

The lpKdV (3.14) is a nonlinear partial difference equation which is a fully discrete counterpart to the continuous pKdV equation (3.7). Based on the various definitions of integrability for continuous equations (Liouville integrability, Painlevé property, infinite number of conservation laws and higher symmetries, etc.), equation (3.7) is a completely integrable system, and a natural question which arises is whether the lpKdV retains any of these integrability characteristics in the discrete setting. Integrability for partial difference equations however is not something that has been to date defined precisely, and there are various inherent properties of systems which indicate some notion of integrability as it is understood for continuous equations. The properties that we consider are three-dimensional (3D) consistency (which has no analogue in the continuous case), the existence of a Lax pair, and the existence of an infinite number of conservation laws, though this final property will not be considered until Chapter 5. Other present notions of integrability for partial difference equations are the vanishing of algebraic entropy [**21**], the existence of an infinite number

of symmetries [**58**] and the obeying of singularity confinement [**46**]. The 3D consistent equations presented below, which are all scalar affine-linear partial difference equations in two independent variables, do indeed have zero algebraic entropy and pass the singularity confinement test.

The idea of 3D consistency, also known as consistency around a cube, goes back to [**69**] [**24**]. Consider the lpKdV equation, which we now write as

$$Q_{pq}\big(w,\widetilde{w},\widehat{w},\widehat{\widetilde{w}}\,\big) := (w-\widehat{\widetilde{w}}\,)(\,\widetilde{w}-\widehat{w}\,)+p^2-q^2=0. \tag{3.35}$$

This equation naturally lives on an elementary lattice plaquette shown in Figure 10.

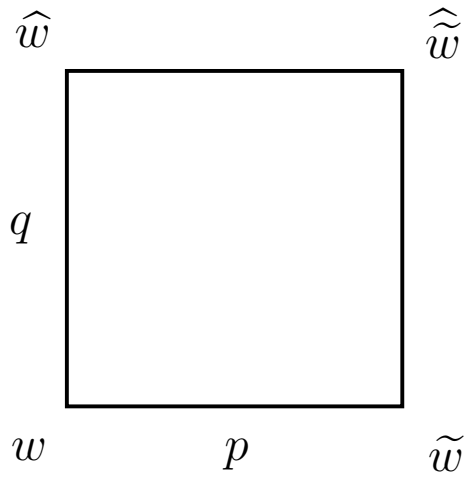

FIGURE 10. Elementary lattice plaquette

By considering the effect of the square symmetries of this lattice plaquette on the functional Q defined in (3.35) we see that

$$Q_{pq}\big(w,\widetilde{w},\widehat{w},\widehat{\widetilde{w}}\,\big) = Q_{pq}\big(\,\widetilde{w},w,\widehat{\widetilde{w}},\widehat{w}\,\big) = -Q_{qp}\big(w,\widehat{w},\widetilde{w},\widehat{\widetilde{w}}\,\big),$$

and since the lpKdV is the equation $Q=0$ it follows that this equation is invariant under each of these symmetries. Suppose now that we consider

the square in Figure 10 as the base of a cube, where the third orthogonal direction is denoted by $\overline{w}$ (and a new independent variable l), with associated parameter r, as shown in Figure 11.

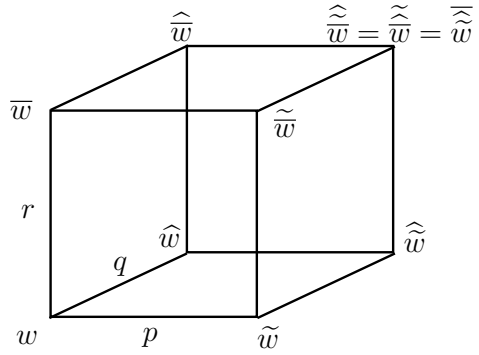

FIGURE 11. 3D Consistency

Furthermore we assume that equation (3.7) is imposed on every face of the cube, thus from the three faces touching the vertex denoted by w we have the three equations

$$Q_{pq}\big(w, \widetilde{w}, \widehat{w}, \widehat{\widetilde{w}}\big) := (w - \widehat{\widetilde{w}})(\widetilde{w} - \widehat{w}) + p^2 - q^2 = 0$$

$$Q_{qr}\big(w, \widehat{w}, \overline{w}, \widehat{\overline{w}}\big) := (w - \widehat{\overline{w}})(\widehat{w} - \overline{w}) + q^2 - r^2 = 0$$

$$Q_{rp}\big(w, \overline{w}, \widetilde{w}, \widetilde{\overline{w}}\big) := (w - \widetilde{\overline{w}})(\overline{w} - \widetilde{w}) + r^2 - p^2 = 0.$$

If we assume that the initial conditions w, $\widetilde{w}$, $\widehat{w}$ and $\overline{w}$ are given, then these three equations can be used to determine uniquely the values $\widehat{\widetilde{w}}$, $\widetilde{\overline{w}}$ and $\widehat{\overline{w}}$. This however leaves three means of determining the remaining point of the cube (antipodal to w), as there are three faces which touch it. 3D consistency is the property of an equation whereby given the initial conditions described above, the three means of calculating this final vertex of the cube give the same result, that is

$$\widehat{\widetilde{\overline{w}}} = \widetilde{\widehat{\overline{w}}} = \widetilde{\overline{\widehat{w}}}.$$

This can be seen for the lpKdV algebraically [**90**] by the symmetry of this value on the parameters p, q and r, and their respective initial conditions $\widetilde{w}$, $\widehat{w}$ and $\overline{w}$:

$$\widehat{\widetilde{w}} = \frac{p^2\,\widetilde{w}\,(\widehat{w}-\overline{w}) + q^2\,\widehat{w}\,(\overline{w}-\widetilde{w}) + r^2\,\overline{w}\,(\widetilde{w}-\widehat{w})}{p^2(\widehat{w}-\overline{w}) + q^2(\overline{w}-\widetilde{w}) + r^2(\widetilde{w}-\widehat{w})}.$$

Interestingly for the lpKdV this quantity does not depend on the initial condition w, a phenomenon known as the tetrahedron property.

This 3D consistency property is one of the present definitions of integrability for partial difference equations. This property naturally allows the equation to be embedded in multiple dimensions in a consistent manner, which we will see is reflected in the covariance of the dependence of their solutions on the multiple lattice variables. In terms of an initial-value problem it also allows one to generalise the regular square lattice to an arbitrary quad-graph [**15**]. Many examples of equations exhibiting this property were previously known, e.g. [**50**] [**67**] [**74**] [**32**], and in 2003 Adler, Bobenko and Suris (ABS) [**12**] (see also [**13**] where this was generalised) classified all such affine-linear scalar partial difference equations of two independent variables of the form

$$Q_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = 0$$

which are 3D consistent, are invariant under the group of square symmetries

$$Q_{pq}(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}) = \epsilon Q_{pq}(\widetilde{u},u,\widehat{\widetilde{u}},\widehat{u}) = \sigma Q_{qp}(u,\widehat{u},\widetilde{u},\widehat{\widetilde{u}}), \qquad \sigma,\epsilon = \pm 1, \quad (3.36)$$

and which satisfy the tetrahedron property. This led to a list of nine equations, separated into three classes (named type-Q, type-A and type-H), which are unique up to constant (i.e. independent of the lattice variables) Möbius transformations of the dependent variables and point transformations of the parameters. Within these nine equations there exists a natural

hierarchy defined by a degeneration scheme. The degeneration diagram between the type-Q equations was first given in [**14**] and is shown in Figure 12.

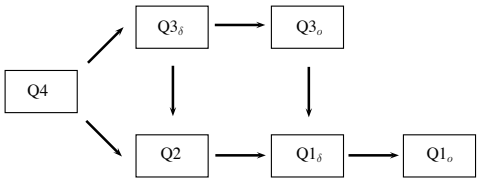

FIGURE 12. Degeneration scheme between the type-Q equations in the ABS list

The equation Q4 which lies at the top of this hierarchy is given by

$$\begin{aligned}\mathrm{Q4}: \quad & \dot{p}(u\,\widetilde{u}+\widehat{u}\,\widehat{\widetilde{u}})-\dot{q}(u\,\widehat{u}+\widetilde{u}\,\widehat{\widetilde{u}}) \\ & -\left(\frac{\dot{p}\dot{Q}-\dot{q}\dot{P}}{1-\dot{p}^2\dot{q}^2}\right)\Big((\widetilde{u}\,\widehat{u}+u\,\widehat{\widetilde{u}})-\dot{p}\dot{q}(1+u\,\widetilde{u}\,\widehat{u}\,\widehat{\widetilde{u}})\Big)=0, \qquad (3.37)\end{aligned}$$

where $\dot{P}^2=\dot{p}^4-\gamma\dot{p}^2+1$ and $\dot{Q}^2=\dot{q}^4-\gamma\dot{q}^2+1$, where γ is some constant. This particular parametrisation was first given in [**48**]. From this equation one can perform various degeneration schemes (by taking limits of fixed parameters and dependent variables) and obtain every other equation in the ABS list. Q4 was first written down by Adler [**11**] who discovered it as the permutability condition for Bäcklund transformations of the Krichever-Novikov equation. The remaining equations of type-Q are

$$\mathrm{Q1}_\delta: \quad \mathring{p}(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-\mathring{q}(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})+\delta^2\mathring{p}\mathring{q}(\mathring{p}-\mathring{q})=0 \qquad (3.38a)$$

$$\begin{aligned}\mathrm{Q2}: \quad & \mathring{p}(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-\mathring{q}(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) \\ & +\mathring{p}\mathring{q}(\mathring{p}-\mathring{q})(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})-\mathring{p}\mathring{q}(\mathring{p}-\mathring{q})(\mathring{p}^2-\mathring{p}\mathring{q}+\mathring{q}^2)=0 \qquad (3.38b)\end{aligned}$$

$$\begin{aligned}\mathrm{Q3}_\delta: \quad & P(u\,\widehat{u}+\widetilde{u}\,\widehat{\widetilde{u}})-Q(u\,\widetilde{u}+\widehat{u}\,\widehat{\widetilde{u}}) \\ & -(p^2-q^2)\left((\widetilde{u}\,\widehat{u}+u\,\widehat{\widetilde{u}})+\frac{\delta^2}{4PQ}\right)=0, \qquad (3.38c)\end{aligned}$$

where $\mathring{p} = \frac{a^2}{p^2-a^2}$, $\mathring{q} = \frac{a^2}{q^2-a^2}$ and the parameters $\mathfrak{p} := (p, P)$, $\mathfrak{q} := (q, Q)$ lie on the Jacobi elliptic curve $\{(x, X) : X^2 = (x^2 - a^2)(x^2 - b^2)\}$. This form of $\mathrm{Q3}_\delta$ is due to **[18]** and **[65]**. The equation $\mathrm{Q1}_o$ is in fact the Schwarzian KdV or cross-ratio equation (3.33), while $\mathrm{Q3}_o$ is the NQC equation (3.29). Both of these had been known to be integrable (in the sense of having N-soliton solutions and Lax pairs) partial difference equations well before the ABS classification. The next family of equations is the type-H equations, given by

$$\mathrm{H1}: \quad (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+p^2-q^2=0 \tag{3.39a}$$

$$\mathrm{H2}: \quad (u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+(p^2-q^2)(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})-p^4+q^4=0 \tag{3.39b}$$

$$\mathrm{H3}_\delta: \quad Q(u\,\widetilde{u}+\widehat{u}\,\widehat{\widetilde{u}})-P(u\,\widehat{u}+\widetilde{u}\,\widehat{\widetilde{u}})+\delta\left(\frac{p^2-q^2}{PQ}\right)=0 \tag{3.39c}$$

where $P^2 = a^2 - p^2,\ Q^2 = a^2 - q^2$.

These three equations can all be obtained through degneration of type-Q equations, with H1 lying at the bottom of this degeneration hierarchy. Here H1 is the lpKdV equation, while $\mathrm{H3}_o$ is the lattice modified KdV equation (3.31) or Hirota equation, both of which were previously-known discretisations of KdV-type integrable systems **[66]**. Importantly the equations $\mathrm{Q3}_\delta \to \mathrm{Q1}_\delta$ as well as all type-H equations are parametrised in such a way that the common lattice parameters p and q which appear in all equations are unaffected by the degeneration scheme **[65]** between these equations. The final family is the type-A equations

$$\mathrm{A1}_\delta: \quad \mathrm{p}(u+\widehat{u})(\widetilde{u}+\widetilde{u})-\mathrm{q}(u+\widetilde{u})(\widehat{u}+\widehat{\widetilde{u}})-\delta^2\mathrm{pq}(\mathrm{p}-\mathrm{q})=0 \tag{3.40a}$$

$$\begin{aligned}\mathrm{A2}: \quad & \mathrm{p}(1-\mathrm{q}^2)(u\,\widehat{u}+\widetilde{u}\,\widehat{\widetilde{u}})-\mathrm{q}(1-\mathrm{p}^2)(u\,\widetilde{u}+\widehat{u}\,\widehat{\widetilde{u}}) \\ & +(\mathrm{p}^2-\mathrm{q}^2)\left(1+u\,\widetilde{u}\,\widehat{u}\,\widehat{\widetilde{u}}\right)=0,\end{aligned} \tag{3.40b}$$

however these are in fact related by straightforward gauge transformations to $\mathrm{Q1}_\delta$ and $\mathrm{Q3}_o$ and as such are not treated here as separate cases.

An important properties of the equations classified by ABS is that they all possess a Lax pair, which is naturally inherited by the affine-linearity and 3D consistency of the equations. The construction of the such a Lax pair was first shown in **[63]** **[24]** and is as follows: Suppose we are given an equation $Q_{pq}\big(u,\widetilde{u},\widehat{u},\widehat{\widetilde{u}}\big)=0$ which exists on the bottom face of the cube in Figure 11. The equation on the front face can then be expressed as

$$Q_{pr}\big(u,\widetilde{u},\overline{u},\widetilde{\overline{u}}\big)=a_o+a_1\overline{u}+a_2\widetilde{\overline{u}}+a_3\overline{u}\,\widetilde{\overline{u}}=0$$

where each of the coefficients a_i may depend on u, $\widetilde{u}$, p and r. As an equation in the new dependent variable $\overline{u}$ this is a discrete Riccati equation and as such we introduce the natural decomposition $\overline{u}=:f/g$ which leads to

$$\frac{\widetilde{g}}{\widetilde{f}}=\frac{a_2g+a_3f}{-a_0g-a_1f}.$$

Defining $\boldsymbol{\phi}:=(g,f)^T$ this can be rewritten in matrix form as

$$\widetilde{\boldsymbol{\phi}}=\kappa_1\mathcal{L}\,\boldsymbol{\phi}:=\kappa_1\begin{pmatrix}a_2 & a_3\\ -a_0 & -a_1\end{pmatrix}\boldsymbol{\phi} \tag{3.41}$$

where κ_1 is an as yet unspecified separation function. This is the first Lax equation, where the third lattice parameter r plays the role of the spectral parameter, and in terms of the partial derivatives of Q with respect to the dependent variables the elements of this matrix are

$$a_o=Q-\overline{u}\,Q_{\overline{u}}-\widetilde{\overline{u}}\,Q_{\widetilde{\overline{u}}}+\overline{u}\,\widetilde{\overline{u}}\,Q_{\overline{u}\,\widetilde{\overline{u}}}$$

$$a_1=Q_{\overline{u}}-\widetilde{\overline{u}}\,Q_{\overline{u}\,\widetilde{\overline{u}}}\quad a_2=Q_{\widetilde{\overline{u}}}-\overline{u}\,Q_{\overline{u}\,\widetilde{\overline{u}}}\quad a_3=Q_{\overline{u}\,\widetilde{\overline{u}}}.$$

The second is obtained in a similar manner by considering

$$Q_{qr}\big(u,\widehat{u},\overline{u},\widehat{\overline{u}}\big)=b_o+b_1\overline{u}+b_2\widehat{\overline{u}}+b_3\overline{u}\,\widehat{\overline{u}}=0$$

which gives

$$\widehat{\boldsymbol{\phi}} = \kappa_2 \mathcal{M}\, \boldsymbol{\phi} := \kappa_2 \begin{pmatrix} b_2 & b_3 \\ -b_0 & -b_1 \end{pmatrix} \boldsymbol{\phi}$$

for some other separation function κ_2. The consistency of this Lax pair is the vanishing of the difference $\widehat{\widetilde{\boldsymbol{\phi}}} - \widetilde{\widehat{\boldsymbol{\phi}}}$, which is equivalent to demaning that $\widehat{\widetilde{u}} = \widetilde{\widehat{u}}$, which will only hold if u satisfies the original 3D consistent equation. The matrix equation which arises from this consistency condition is

$$\widehat{\kappa}_1 \kappa_2 \widehat{\mathcal{L}} \mathcal{M} = \kappa_1 \widetilde{\kappa}_2 \widetilde{\mathcal{M}} \mathcal{L}, \tag{3.42}$$

and by taking the determinant of this one obtains a restriction on the functions κ_1 and κ_2. Using the above values for the a_i and the machinery developed in [**12**] it turns out that the determinant of the Lax matrix $\mathcal{L}$ is

$$\det(\mathcal{L}) = Q\, Q_{\overline{u}\,\widetilde{u}} - Q_{\overline{u}}\, Q_{\widetilde{u}} = \mathtt{k}(p,r)\mathcal{H}(u,\widetilde{u}\,;p) \tag{3.43}$$

where the function $\mathtt{k}(p,r)$ is antisymmetric and the biquadratic $\mathcal{H}$ does not depend on r. Due to the symmetry of the equation we also have

$$\det(\mathcal{M}) = Q\, Q_{\overline{u}\,\widehat{u}} - Q_{\overline{u}}\, Q_{\widehat{u}} = \mathtt{k}(q,r)\mathcal{H}(u,\widehat{u}\,;q).$$

If we introduce the function $\mathcal{U}$ by the first-order relations

$$\mathcal{H}(u,\widetilde{u}\,;p) = \mathcal{U}\widetilde{\mathcal{U}}, \qquad \mathcal{H}(u,\widehat{u}\,;q) = \mathcal{U}\widehat{\mathcal{U}},$$

then by the consistency relations of the biquadratics $\mathcal{H}$ shown in [**65**], this new object $\mathcal{U}$ is covariant in all lattice directions. $\mathcal{U}$ is defined uniquely up to a factor of c^σ for some constant c, where σ alternates between ± 1 with every lattice shift, and the determinant of the consistency condition (3.42) becomes

$$\widehat{\kappa}_1 \kappa_2 \widehat{\mathcal{U}} = \pm \kappa_1 \widetilde{\kappa}_2 \widetilde{\mathcal{U}}.$$

A solution of this is $\kappa_1 = \kappa_2 = \mathcal{U}^{-1}$, where we have taken the arbitrary constant factors to be unity without loss of generality. As an example the Lax pair for H2 is given by

$$\widetilde{\boldsymbol{\phi}} = \frac{1}{\mathcal{U}} \begin{pmatrix} -\widetilde{u} + p^2 - r^2 & 1 \\ (p^2 - r^2)(r^2 - \mathcal{U}\widetilde{\mathcal{U}}) - u\,\widetilde{u} & u - (p^2 - r^2) \end{pmatrix} \boldsymbol{\phi}$$

$$\widehat{\boldsymbol{\phi}} = \frac{1}{\mathcal{U}} \begin{pmatrix} -\widehat{u} + q^2 - r^2 & 1 \\ (q^2 - r^2)(r^2 - \mathcal{U}\widehat{\mathcal{U}}) - u\,\widehat{u} & u - (q^2 - r^2) \end{pmatrix} \boldsymbol{\phi}$$

where

$$\mathcal{U}\widetilde{\mathcal{U}} = u + \widetilde{u} - p^2, \qquad \mathcal{U}\widehat{\mathcal{U}} = u + \widehat{u} - q^2, \qquad \mathtt{k}(p, r) = 2(p^2 - r^2).$$

These relations between u and $\mathcal{U}$ in fact constitute Miura-type relations between H2 for u and H1 for $\mathcal{U}$ **[16]**. The consistency condition gives

$$\widetilde{\widehat{\boldsymbol{\phi}}} - \widehat{\widetilde{\boldsymbol{\phi}}} = Q_{pq}(u, \widetilde{u}, \widehat{u}, \widehat{\widetilde{u}}) \times \begin{pmatrix} u - r^2 & 1 \\ K & -u + p^2 + q^2 - r^2 \end{pmatrix} \boldsymbol{\phi},$$

where $K = u(\widetilde{u} + \widehat{u}) + (\widetilde{u} - p^2)(\widehat{u} - q^2) - r^2(p^2 + q^2) + r^4$. As a final note on the Lax pairs of the ABS equations, for equations $\mathrm{Q1}_\delta \to \mathrm{Q3}_\delta$ as well as all type-H equations, the coefficients a_3 and b_3 are constant. The second-order scalar equation for the first component of $\boldsymbol{\phi}$ which exists on the points $\boldsymbol{\phi} \to \widetilde{\boldsymbol{\phi}} \to \widetilde{\widehat{\boldsymbol{\phi}}}$ and $\boldsymbol{\phi} \to \widehat{\boldsymbol{\phi}} \to \widehat{\widetilde{\boldsymbol{\phi}}}$ are then

$$\widetilde{\widehat{\phi}}_1 + \left(\frac{a_1 - \widetilde{b}_2}{\widetilde{\mathcal{U}}} \right) \widetilde{\phi}_1 + \mathtt{k}(p, r)\, \phi_1 = 0$$

$$\widehat{\widetilde{\phi}}_1 + \left(\frac{b_1 - \widehat{a}_2}{\widehat{\mathcal{U}}} \right) \widehat{\phi}_1 + \mathtt{k}(q, r)\, \phi_1 = 0$$

respectively. We will see in Chapter 4 that these equations play a major role in the direct scattering problem for these particular ABS equations.

3.5 Multidimensional Soliton Solutions of ABS Equations

Solutions to KdV-type integrable lattice equations in the ABS hierarchy (i.e. $Q3_o$, $Q1_o$, $H3_o$ and H1) have been known since the invention of the direct linearization approach **[67]** **[74]**. The linear singular integral equations used in this technique can be used to construct soliton solutions to these equations, and also in principle contain solutions with background radiation. Solutions to equations Q4, Q2, H2 and the δ-deformations of the KdV-type equations have been found more recently. With the exception of Q4 it was shown in **[16]** that there exists Bäcklund transformations between these new equations and older ones, which led to the application of direct linearization to the ABS equations. N-soliton solutions to 3D consistent equations were first explored in **[19]** **[17]** **[18]**, and in **[65]** N-soliton solutions to all ABS equations below and including $Q3_\delta$ were given, where they were found from a Cauchy matrix approach. The Q3 solutions was found first, and then by taking limits on fixed parameters the remaining solutions of all other equations were found through the degeneration scheme depicted in Figure 13.

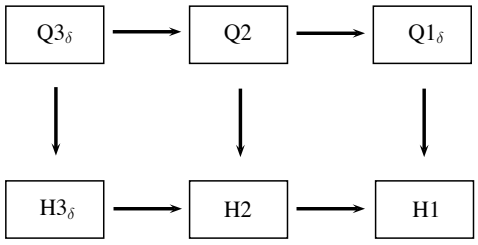

FIGURE 13. Degeneration scheme between the type-Q and type-H equations used in **[65]**

These same solutions were also found in **[49]** using Casorati determinants and bilinear forms. The authors of **[65]** also found a number of Miura-type relations between the equations which highlights an underlying relation between these equations and the various components which comprise their solutions, which is much deeper than simply their relation via the degeneration scheme. Elliptic N-soliton solutions, which are the result of N Bäcklund transformations on elliptic seed solutions to the equations, were then presented in **[64]**.

As an example of the emergence of the Cauchy matrix structure in the soliton solutions, the N-soliton solution to H1 (lpKdV) is given by

$$u_{n,m} = pn + qm + A - \mathbf{c}^T \big(I + \mathcal{M}\big)^{-1} \mathbf{r}$$

where A is constant, $\mathbf{c} = c_i$ is a vector of nonzero constants, $\mathbf{r} = \rho_i$ is a vector of plane wave factors

$$\rho_i := \rho_i^o \left(\frac{p + k_i}{p - k_i}\right)^n \left(\frac{q + k_i}{q - k_i}\right)^m \tag{3.44}$$

and the Cauchy matrix $\mathcal{M}$ is defined by

$$\mathcal{M}_{ij} = \frac{\rho_i c_j}{k_i + k_j}. \tag{3.45}$$

This solution can be easily extended to an M-dimensional lattice with independent variables n_i and associated parameters p_i by extending the linear seed solution and plane wave factors covariantly:

$$\begin{gathered} pn + qn \ \to p_1 n_1 + p_2 n_2 + ... + p_M n_M \\ \left(\frac{p + k_i}{p - k_i}\right)^n \left(\frac{q + k_i}{q - k_i}\right)^m \to \left(\frac{p_1 + k_i}{p_1 - k_i}\right)^{n_1} \left(\frac{p_2 + k_i}{p_2 - k_i}\right)^{n_2} ... \left(\frac{p_M + k_i}{p_M - k_i}\right)^{n_M}. \end{gathered}$$

The search for soliton solutions of Q4 has been a more recent venture, and these have been found in **[20]**. The solutions are obtained through a discrete Riccati-type linearisation arising from Bäcklund transformations, which we explain using H1 as an example. Due to their 3D consistency,

all ABS equations define their own auto-Bäcklund transformations. Thus given one solution, one can use the same equation with a new parameter to generate a new solution. For the case of H1, given the linear seed solution

$$u_o = pn + qm + c,$$

if we consider the equations $Q_{pk}\big(u, \widetilde{u}, \overline{u}, \widetilde{\overline{u}}\big)$ then this can be viewed as a discrete Riccati equation for the new dependent variable $v := \overline{u}$

$$(u_o - \widetilde{v})(\widetilde{u}_o - v) + p^2 - k^2 = 0.$$

This is linearisable by making the substitution $v = u_o + p + \widetilde{h}/h$, giving

$$\widetilde{\overline{h}} + 2p\widetilde{h} + (p^2 - k^2)h = 0 \quad \Rightarrow \quad v = u_o + k\left(\frac{1 - A\left(\frac{p+k}{p-k}\right)^n}{1 + A\left(\frac{p+k}{p-k}\right)^n}\right).$$

The m-dependence can then be found in a similar fashion using $Q_{qk}\big(u, \widehat{u}, \overline{u}, \widehat{\overline{u}}\big)$, which gives

$$v = pn + qm + c + k\left(\frac{1 - A\left(\frac{p+k}{p-k}\right)^n\left(\frac{q+k}{q-k}\right)^m}{1 + A\left(\frac{p+k}{p-k}\right)^n\left(\frac{q+k}{q-k}\right)^m}\right).$$

This solution, which is in fact a one-soliton solution, is the result of applying one Bäcklund transformation to the linear seed solution u_o. In principle this process can be repeated an arbitrary number of times with distinct parameters to obtain an N-soliton solution, however this is in general an unwieldy task. A linearisation of this method was discovered in [**20**], in which new solutions are constructed from Bäcklund transformations, yet one is only required to solve first-order linear homogeneous equations. Suppose in the above calculations we chose the new solution $\overline{u}$ to be simply $\overline{u} = u_o + k$. We can then likewise define another new solution $\underline{u}$ by $\underline{u} = u_o - k$. We now construct a new solution w to H1 by defining

$$w = \frac{\underline{u} - \eta\,\overline{u}}{1 - \eta}.$$

It then follows that the quantity η satisfies the first-order equations

$$\widetilde{\eta} = -\left(\frac{Q_{pk}\big(u_o, \widetilde{u}_o, \overline{u}, \widetilde{\underline{u}}\big)}{Q_{pk}\big(u_o, \widetilde{u}_o, \underline{u}, \widetilde{\overline{u}}\big)}\right)\eta, \qquad \widehat{\eta} = -\left(\frac{Q_{qk}\big(u_o, \widehat{u}_o, \overline{u}, \widehat{\underline{u}}\big)}{Q_{qk}\big(u_o, \widehat{u}_o, \underline{u}, \widehat{\overline{u}}\big)}\right)\eta.$$

For H1 this reduces to

$$\widetilde{\eta} = \left(\frac{p-k}{p+k}\right)\eta, \qquad \widehat{\eta} = \left(\frac{q-k}{q+k}\right)\eta$$

and thus we obtain

$$w = \frac{\underline{u} - \eta\,\overline{u}}{1-\eta} = \frac{u_o - k - \eta\,(u_o + k)}{1-\eta} = u_o + k\left(\frac{1+\eta^{-1}}{1-\eta^{-1}}\right)$$

which is again the one-soliton solution. This linearisation procedure was used in [**20**] to give an N-soliton solution to Q4.

We now have successfully determined a class of integrable partial difference equations, including discretisations of such famous partial differential equations as the KdV equation (3.1). We have looked at some of their known solutions, the notion of an initial-value problem and different ways of interpreting the notion of integrability in the discrete setting. In the following chapter we then extend these ideas into formulating a discrete IST as a means of obtaining solutions to these partial difference equations.

4 The Discrete Inverse Scattering Transform

In this Chapter we derive a discrete inverse scattering transform as a method of finding solutions to the Q3$_\delta$ equation (3.38c), which is

$$P(u\,\widehat{u}+\widetilde{u}\,\widehat{\widetilde{u}})-Q(u\,\widetilde{u}+\widehat{u}\,\widehat{\widetilde{u}})-(p^2-q^2)\left((\widetilde{u}\,\widehat{u}+u\,\widehat{\widetilde{u}})+\frac{\delta^2}{4PQ}\right)=0, \quad (4.1)$$

where $P^2=(p^2-a^2)(p^2-b^2)$ and $Q^2=(q^2-a^2)(q^2-b^2)$, and then show how this can be used as a tool for obtaining solutions to a number of nonlinear lattice equations. As was discussed in Chapter 3 it is natural to consider equation (4.1) existing in an N-dimensional lattice, with independent variables $n_1,...,n_{\rm N}$ (all elements of $\mathbb{Z}$), and parameters $p_1,...,p_{\rm N}$. The evolution of u in each pair of lattice directions is found by imposing copies of (4.1) on each elementary quadrilateral within the lattice. The solution that we will obtain from the discrete IST will depend on all N lattice variables n_k, all N lattice parameters p_k, as well as the additional parameters a and b. We do however make the following assumptions:

- The solution u and all parameters $a,b,p_1,...,p_{\rm N}$ are **real**.
- Since the equation only depends on the squares $a^2,b^2,p_1^2,...,p_{\rm N}^2$, we choose to set $a>b>0$ and $p_k>0$ for all $k=1,...,{\rm N}$.

Unlike in the continuous case where the initial-value space is usually given along the line $t = 0$, in the discrete setting we have the freedom to generalise this to a multidimensional *staircase* within the N-dimensional lattice. There are some restrictions on how one can choose this staircase, which are discussed in Section 4.2, but essentially the IST allows for this generalisation without causing serious complications. The boundary conditions that we assume for the solution are those exhibited by the known soliton solutions, and are given in Section 4.3, where the discrete IST is rigorously developed over the following four Sections. In Section 4.8 we then show to obtain solutions for all lower ABS equations, derive their one-soliton solutions in Section 4.9, and in Section 4.10 show how these methods apply to some previously-known lattice equations such as the lattice KdV equation. For convenience of the reader we have collected the notation we employ in this chapter on pages 155-156.

4.1 Linear Problem for Q3$_\delta$

A Lax pair for equation (4.1), first obtained in [**64**] by the direct linearization approach, is

$$(p^2-\zeta^2)^{\frac{1}{2}}\widetilde{\boldsymbol{\phi}} = \frac{1}{\mathcal{U}}\begin{pmatrix} P\widetilde{u}-(p^2-b^2)u & \zeta^2-b^2 \\ \mathcal{U}\widetilde{\mathcal{U}}-\frac{\delta^2(p^2-b^2)}{4P(\zeta^2-b^2)} & (p^2-b^2)\widetilde{u}-Pu \end{pmatrix}\boldsymbol{\phi} \tag{4.2}$$

$$(q^2-\zeta^2)^{\frac{1}{2}}\widehat{\boldsymbol{\phi}} = \frac{1}{\mathcal{U}}\begin{pmatrix} Q\widehat{u}-(q^2-b^2)u & \zeta^2-b^2 \\ \mathcal{U}\widehat{\mathcal{U}}-\frac{\delta^2(q^2-b^2)}{4Q(\zeta^2-b^2)} & (q^2-b^2)\widehat{u}-Qu \end{pmatrix}\boldsymbol{\phi}, \tag{4.3}$$

where this system is consistent (i.e. $\widehat{\widetilde{\boldsymbol{\phi}}} = \widetilde{\widehat{\boldsymbol{\phi}}}$) if and only if u solves (4.1). The dual function $\mathcal{U}$ is determined by solving the first-order equations

$$\mathcal{U}\widetilde{\mathcal{U}} = P(u^2 + \widetilde{u}^2) - (2p^2 - a^2 - b^2)u\,\widetilde{u} + \frac{\delta^2}{4P} \tag{4.4}$$

$$\mathcal{U}\widehat{\mathcal{U}} = Q(u^2 + \widehat{u}^2) - (2q^2 - a^2 - b^2)u\,\widehat{u} + \frac{\delta^2}{4Q}. \tag{4.5}$$

The integration constants are determined by the boundary conditions that will be imposed on u. This is the setup for solving Q3$_\delta$ in the (n, m)-plane. The equation however is multidimensionally consistent, and thus if we choose an arbitrary direction with variable n_k and parameter p_k, then let $\widehat{u}$ denote a shift of u in this direction, that is

$$u = u(n_1, n_2, ..., n_k, ..., n_N; p_1, p_2, ..., p_k, ...p_{\mathrm{N}})$$

$$\widehat{u} = u(n_1, n_2, ..., n_k + 1, ..., n_N; p_1, p_2, ..., p_k, ...p_{\mathrm{N}}).$$

Due to the symmetry of the equation and its multidimensional consistency, we then have the Lax equation

$$(p_k^2 - \zeta^2)^{\frac{1}{2}}\widehat{\boldsymbol{\phi}} = \frac{1}{\mathcal{U}} \begin{pmatrix} P_k\widehat{u} - (p_k^2 - b^2)u & \zeta^2 - b^2 \\ \mathcal{U}\widehat{\mathcal{U}} - \frac{\delta^2(p_k^2 - b^2)}{4P_k(\zeta^2 - b^2)} & (p_k^2 - b^2)\widehat{u} - P_k u \end{pmatrix} \boldsymbol{\phi}, \tag{4.6}$$

and by combining this with any of the remaining N-1 similar Lax equations, say in the n_l-direction, this will form a Lax pair for Q3$_\delta$ as an equation to be solved in the (k, l)-plane. We therefore have $\frac{1}{2}$N(N-1) Lax pairs for Q3$_\delta$, depending on in which two lattice directions we choose to solve the equation. The dependence of the function $\mathcal{U}$ on each of the N variables is determined by solving

$$\mathcal{U}\widehat{\mathcal{U}} = P_k(u^2 + \widehat{u}^2) - (2p_k^2 - a^2 - b^2)u\widehat{u} + \frac{\delta^2}{4P_k},$$

and again using the boundary conditions of u to determine the integration constants.

4.2 Motivation for the Forward Scattering Problem

The Lax equations (4.6) are the basis of the forward scattering problem. Depending on how one chooses to define the initial data, the forward scattering problem is the determination of the eigenfunction $\boldsymbol{\phi}$ as a function of all N independent variables, and the spectral parameter ζ.

Suppose first that we choose to give the initial condition on a line Γ_o spanned by the original variable n (and parameter p), which corresponds to one of the N variables n_k (and parameter p_k). This is the usual setup for the continuous IST for equations such as the KdV equation. Along this line all other N-1 variables are held constant. Then from the Lax equation (4.6), the first component of $\boldsymbol{\phi}$ is found by solving the second-order equation

$$(p^2-\zeta^2)^{\frac{1}{2}}\,\widetilde{\widetilde{\phi}}_1-\left(\frac{P(\widetilde{\widetilde{u}}-u)}{\widetilde{\mathcal{U}}}\right)\widetilde{\phi}_1+(p^2-\zeta^2)^{\frac{1}{2}}\,\phi_1=0,$$

and the second component in then constructed from

$$(\zeta^2-b^2)\,\phi_2=(p^2-\zeta^2)^{\frac{1}{2}}\,\mathcal{U}\,\widetilde{\phi}_1-(P\,\widetilde{u}-(p^2-b^2)u)\,\phi_1.$$

This gives $\boldsymbol{\phi}$ as a function of n along the line Γ_o, which is the direct scattering problem. The dependence of $\boldsymbol{\phi}$ on the remaining N-1 lattice variables can then be determined in the inverse problem by consider the remaining N-1 Lax equations. These remaining lattice variables play the role of N-1 discrete "time" variables, and in the language of the continuous theory this discrete IST can be considered as a 1+(N-1)-type scattering problem.

Contrary to the continuous case, one of the benefits of the discrete setup is that we can easily change the one-dimensional manifold along which we specify the initial conditions. Perhaps the most natural such manifold is a (1,1)-staircase, that is the staircase of points encountered by successive alternating iterations in two lattice directions. Let us consider this as an initial-value space for the scattering problem, where this staircase Γ_1 lies

in the original (n, m)-plane with parameters p, q. For each 3-point segment of the staircase which iterates first in the n-direction and then in the m-direction we have

$$(q^2-\zeta^2)^{\frac{1}{2}}\,\widehat{\widetilde{\phi}}_1 - \left(\frac{Q\,\widehat{\widetilde{u}} - (q^2-p^2)\,\widetilde{u} - Pu}{\widetilde{\mathcal{U}}}\right)\widetilde{\phi}_1 + (p^2-\zeta^2)^{\frac{1}{2}}\,\phi_1 = 0,$$

while for each 3-point segment which iterates first in the m-direction and then in the n-direction

$$(p^2-\zeta^2)^{\frac{1}{2}}\,\widehat{\widetilde{\phi}}_1 - \left(\frac{P\,\widehat{\widetilde{u}} - (p^2-q^2)\,\widehat{u} - Qu}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_1 + (q^2-\zeta^2)^{\frac{1}{2}}\,\phi_1 = 0.$$

Solving these equations gives ϕ_1, and then ϕ_2 may be constructed from similarly considering the first component of either (4.2) or (4.3). Since this first Lax equation contains additional terms involving the functions u and $\mathcal{U}$, the second component ϕ_2 contains the structure of the Q3$_\delta$ solution, whereas ϕ_1 contains only the structure of the simpler H1 solution. This property of the eigenfunctions is also clear from the analysis given in **[64]**.

The result of the forward scattering problem is that we know $\boldsymbol{\phi}$ as a function along Γ_1, that is in terms of some independent staircase variable which depends on n and m. If we let this new variable be denoted by i, and let i_o correspond to the point (n_o, m_o) on Γ_1, then assuming that we iterate first in the n-direction, the change of variables from i to n, m is given by

$$n - n_o = \left\lfloor \frac{1}{2}(i+1-i_o) \right\rfloor, \qquad m - m_o = \left\lfloor \frac{1}{2}(i-i_o) \right\rfloor. \tag{4.7}$$

Here the brackets $\lfloor\ \rfloor$ denote the floor function. Hence even though the staircase exists in two lattice directions, it is still one-dimensional and as such the result is that we only know $\boldsymbol{\phi}$ as a function of this one staircase variable i. We do not know $\boldsymbol{\phi}$ as a function explicitly of n and m. We will see however that by imposing sufficient asymptotic dependence of the

eigenfunctions on the variables n, m and the remaining N-2 lattice variables, that one can construct $\boldsymbol{\phi}$ as a function of all N lattice variables. Thus once again we have a 1+(N-1)-type scattering problem.

Since the initial condition may be given on a (1,1)-staircase, it is natural to consider how this may be generalised. To do so we consider an arbitrary staircase Γ which has the following properties:

- Γ is an infinite staircase which exists in I of the N lattice directions, where $1 \leq \mathrm{I} \leq \mathrm{N}$
- Every iteration along Γ corresponds to a positive iteration in one of the lattice variables n_k.
- Γ may be written as a $(a_1, a_2, ..., a_{\mathrm{I}})$-staircase for some positive integers $a_1, a_2, ...a_{\mathrm{I}}$, i.e. it is defined through some stepping algorithm

As an upshot Γ must span each of the I lattice directions in which it exists. We now consider Γ as an initial-value space for $\mathrm{Q3}_\delta$. To do so it is convenient to introduce the staircase variable i, which will be related to the I lattice variables by a relation similar to (4.7). To take care of the fact that the parameters p_k will also change along the staircase we introduce the staircase parameter $\mathtt{p} = \mathtt{p}(i)$ (and $\mathtt{P}^2 = (\mathtt{p}^2 - a^2)(\mathtt{p}^2 - b^2)$), which will cycle through the parameters p_k encountered along the staircase. This new variable i and parameter $\mathtt{p}(i)$ allow the scattering problem for ϕ_1 along Γ to be conveniently expressed as

$$(\overline{\mathtt{p}}^2 - \zeta^2)^{\frac{1}{2}} \overline{\overline{\phi}}_1 - \left(\frac{\overline{\mathtt{P}}\,\overline{\overline{u}} - (\overline{\mathtt{p}}^2 - \mathtt{p}^2)\,\overline{u} - \mathtt{P}\,u}{\overline{\mathcal{U}}} \right) \overline{\phi}_1 + (\mathtt{p}^2 - \zeta^2)^{\frac{1}{2}}\, \phi_1 = 0, \quad (4.8)$$

where

$$\phi_1 = \phi_1(i;\zeta), \quad \overline{\phi}_1 = \phi_1(i+1;\zeta), \quad \mathtt{p} = \mathtt{p}(i), \quad \overline{\mathtt{p}} = \mathtt{p}(i+1).$$

The second component of the eigenfunction is then constructed from

$$(\zeta^2 - b^2)\,\phi_2 = (\mathtt{p}^2 - \zeta^2)^{\frac{1}{2}}\,\mathcal{U}\,\overline{\phi}_1 - (\mathtt{P}\,\overline{u} - (\mathtt{p}^2 - b^2)u)\,\phi_1. \quad (4.9)$$

Equations (4.8) and (4.9) define the forward scattering problem, which is summarised as follows: Given an initial condition u as a function of i, firstly the function $\mathcal{U}$ is determined by solving

$$\mathcal{U}\overline{\mathcal{U}} = \mathtt{P}(u^2 + \overline{u}^2) - (2\mathtt{p}^2 - a^2 - b^2)u\overline{u} + \frac{\delta^2}{4\mathtt{P}}, \tag{4.10}$$

and setting the arbitrary constant to be unity, and then ϕ_1 is found by solving (4.8). Given ϕ_1, one then uses (4.9) to determine the corresponding ϕ_2, and thus $\boldsymbol{\phi}$ is known as a function of the staircase variable i.

4.3 Forward Scattering of ϕ_1

The forward scattering of ϕ_1 is the solving of equation (4.8). We will see that in fact this is the same scattering problem as for H1, and represents the fact that the soliton solutions of Q3$_\delta$ are comprised of elements of the soliton solutions of H1. Let us consider how the function

$$\Omega := \frac{\overline{\mathtt{P}}\,\overline{u} - (\overline{\mathtt{p}}^2 - \mathtt{p}^2)\,\overline{u} - \mathtt{P}\,u}{\overline{\mathcal{U}}} \tag{4.11}$$

behaves for the known soliton solutions of Q3$_\delta$ **[18]** **[65]**. If we define

$$\rho(k) := \prod_{r=1}^{N} \left(\frac{p_r + k}{p_r - k}\right)^{n_r}, \quad \mathcal{F}(a,b) := \prod_{r=1}^{N} \left(\frac{(p_r + a)(p_r + b)}{(p_r - a)(p_r - b)}\right)^{\frac{1}{2}n_r} \tag{4.12}$$

$$S(a,b) := \frac{1 + \left(\frac{(a-k)(b-k)}{(a+k)(b+k)}\right)\rho(k)}{1 + \rho(k)}, \quad V(a) := \frac{1 + \left(\frac{(a-k)}{(a+k)}\right)\rho(k)}{1 + \rho(k)} \tag{4.13}$$

then a one-soliton solution to Q3 (which depends on all N lattice variables) is given by

$$\begin{aligned} u = {} & \mathcal{A}\,\mathcal{F}(a,b)\,S(a,b) + \mathcal{B}\,\mathcal{F}(a,-b)\,S(a,-b) \\ & + \mathcal{C}\,\mathcal{F}(-a,b)\,S(-a,b) + \mathcal{D}\,\mathcal{F}(-a,-b)\,S(-a,-b), \end{aligned} \tag{4.14}$$

where the four constants are restrained by

$$\mathcal{A}\mathcal{D}(a+b)^2 - \mathcal{B}\mathcal{C}(a-b)^2 = -\frac{\delta^2}{16ab}. \tag{4.15}$$

The corresponding dual function $\mathcal{U}$ is

$$\begin{aligned}\mathcal{U} = (a+b)\mathcal{A}\,\mathcal{F}(a,b)\,V(a)V(b) + (a-b)\mathcal{B}\,\mathcal{F}(a,-b)\,V(a)V(-b)\\ -(a-b)\mathcal{C}\,\mathcal{F}(-a,b)\,V(-a)V(b) - (a+b)\mathcal{D}\,\mathcal{F}(-a,-b)\,V(-a)V(-b).\end{aligned} \tag{4.16}$$

Using this as a guide, and bearing in mind the choice $a > b > 0$, the boundary conditions that we assume on the solution u are

$$u \sim \mathcal{C}\,\mathcal{F}(-a,b) + \mathcal{D}\,\mathcal{F}(-a,-b) \quad \text{as } i \to -\infty \tag{4.17a}$$

$$u \sim \mathcal{K}_o\mathcal{A}\,\mathcal{F}(a,b) + \mathcal{K}_1\mathcal{B}\,\mathcal{F}(a,-b) \quad \text{as } i \to +\infty, \tag{4.17b}$$

where $\mathcal{K}_o$ and $\mathcal{K}_1$ are constants and the plane-wave factors $\mathcal{F}$ satisfy

$$\overline{\mathcal{F}}(a,b) = \left(\frac{(\mathtt{p}(i)+a)(\mathtt{p}(i)+b)}{(\mathtt{p}(i)-a)(\mathtt{p}(i)-b)}\right)^{\frac{1}{2}}\mathcal{F}(a,b). \tag{4.18}$$

The corresponding boundary conditions for $\mathcal{U}$ are

$$\mathcal{U} \sim -(a-b)\mathcal{C}\,\mathcal{F}(-a,b) - (a+b)\mathcal{D}\,\mathcal{F}(-a,-b) \quad \text{as } i \to -\infty \tag{4.19a}$$

$$\mathcal{U} \sim \mathcal{K}_o(a+b)\mathcal{A}\,\mathcal{F}(a,b) + \mathcal{K}_1(a-b)\mathcal{B}\,\mathcal{F}(a,-b) \quad \text{as } i \to +\infty. \tag{4.19b}$$

Then at either end of the staircase (i.e. as $i \to \pm\infty$) we can explicitly calculate that

$$\Omega \sim \mathtt{p} + \overline{\mathtt{p}} \quad \text{as } \; i \to \pm\infty. \tag{4.20}$$

For soliton solutions all other terms decay exponentially, that is they decay like $\lambda^{-|i|}$ for some constant $\lambda > 1$. This asymptotic result shows that the object Ω behaves like a difference of H1-type soliton solutions as $i \to \pm\infty$, as shown in **[28]**. This is perhaps not surprising as in **[65]** and **[64]** Miura-type relations between soliton solutions of H1 and Q3$_\delta$ were found, which take precisely the form of the quantity Ω. The potential term appearing in the forward scattering problem for ϕ_1 can therefore essentially be thought of as this difference of H1 solutions.

Definition 4.3.1. *Given an initial condition $u = u(i)$ along the staircase Γ, the potential $v = v(i)$ is defined to be*

$$v(i+1) \equiv \overline{v} := \Omega - \mathtt{p} - \overline{\mathtt{p}} = \left(\frac{\overline{\mathtt{P}}\,\overline{\overline{u}} - (\overline{\mathtt{p}}^2 - \mathtt{p}^2)\,\overline{u} - \mathtt{P}\,u}{\overline{\mathcal{U}}} \right) - \mathtt{p} - \overline{\mathtt{p}}, \tag{4.21}$$

where $\mathcal{U}$ is determined by (4.10)*.*

Due to the boundary conditions on u the potential vanishes at either end of the staircase Γ. Thus we rewrite the scattering problem for ϕ_1 as

$$(\overline{\mathtt{p}}^2 - \zeta^2)^{\frac{1}{2}}\,\overline{\overline{\phi}}_1 - \left(\mathtt{p} + \overline{\mathtt{p}} + \overline{v}\right)\overline{\phi}_1 + (\mathtt{p}^2 - \zeta^2)^{\frac{1}{2}}\,\phi_1 = 0. \tag{4.22}$$

We now construct solutions to this equation. In doing so we assume that all solutions and parameters are real, and that $\mathtt{p}(r) > 0$ for all $r \in \mathrm{I}$.

Definition 4.3.2. *The Jost solutions $\varphi, \mathring{\varphi}$ are defined by the boundary conditions*

$$\varphi(i;\zeta) \sim \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r) + \zeta}{\mathtt{p}(r) - \zeta} \right)^{\frac{1}{2}} \quad \text{as} \;\; i \to -\infty \tag{4.23a}$$

$$\mathring{\varphi}(i;\zeta) \sim \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta} \right)^{\frac{1}{2}} \quad \text{as} \;\; i \to -\infty, \tag{4.23b}$$

and the Jost solutions $\psi, \mathring{\psi}$ to equation (4.22) *are defined by the boundary conditions*

$$\psi(i;\zeta) \sim \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta} \right)^{\frac{1}{2}} \quad \text{as} \;\; i \to +\infty \tag{4.23c}$$

$$\mathring{\psi}(i;\zeta) \sim \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r) + \zeta}{\mathtt{p}(r) - \zeta} \right)^{\frac{1}{2}} \quad \text{as} \;\; i \to +\infty. \tag{4.23d}$$

Since equation (4.22) is invariant under the map $\zeta \to -\zeta$, it follows by the definition of the boundary conditions for the Jost solutions and uniqueness of the boundary value problem [**59**], that

$$\mathring{\varphi}(i;\zeta) = \varphi(i;-\zeta), \quad \mathring{\psi}(i;\zeta) = \psi(i;-\zeta). \tag{4.24}$$

Since the general solution to (4.22) involves two linearly independent solutions we may write

$$\psi = \mathtt{A}\,\mathring{\varphi} + \mathtt{B}\,\varphi, \quad \mathring{\psi} = \mathring{\mathtt{A}}\,\varphi + \mathring{\mathtt{B}}\,\mathring{\varphi} \tag{4.25}$$

where $\mathtt{A}$ and $\mathtt{B}$ are independent of i and $\mathring{\mathtt{A}}(\zeta) = \mathtt{A}(-\zeta)$ and $\mathring{\mathtt{B}}(\zeta) = \mathtt{B}(-\zeta)$.

Proposition 4.3.1. *If ζ is purely imaginary then*

$$|\mathtt{A}(\zeta)|^2 = 1 + |\mathtt{B}(\zeta)|^2. \tag{4.26}$$

Proof. Firstly given any two solutions $x(i)$ and $y(i)$ of (4.22), by eliminating the potential term one can show that the discrete Wronskian (or Casoratian)

$$W(x, y) := (\mathtt{p}^2 - \zeta^2)^{\frac{1}{2}}(x\,\overline{y} - \overline{x}\,y) \tag{4.27}$$

is independent of i. Furthermore if ζ is purely imaginary then equation (4.22) is purely real and thus φ^* and ψ^* (the complex conjugates of φ and ψ) are also solutions of this equation. By comparing the boundary conditions for $\varphi^*, \mathring{\varphi}$ and $\psi^*, \mathring{\psi}$, by the uniqueness of the boundary value problem we have

$$\varphi^*(i;\zeta^*) \equiv \mathring{\varphi}(i;\zeta), \qquad \psi^*(i;\zeta^*) \equiv \mathring{\psi}(i;\zeta).$$

By taking the complex conjugate of (4.25) we then have $\mathring{\mathtt{A}}(\zeta) \equiv \mathtt{A}^*(\zeta^*)$ and $\mathring{\mathtt{B}}(\zeta) \equiv \mathtt{B}^*(\zeta^*)$. Now due to the linearity and anti-symmetry of the Wronskian we have

$$W(\psi, \psi^*) = W(\mathtt{A}\varphi^* + \mathtt{B}\varphi, \mathtt{A}^*\varphi + \mathtt{B}^*\varphi^*) = \big(|\mathtt{A}|^2 - |\mathtt{B}|^2\big)W(\varphi^*, \varphi),$$

and since the Wronskian is independent of i, these may be evaluated at the relevant boundaries which gives $W(\psi, \psi^*) = W(\varphi^*, \varphi) = 2\zeta$. This proves (4.26). □

4.3.1 Analyticity and Asymptoticity Properties of the Jost Solutions

We now determine asymptoticity properties of the Jost solutions as functions of the discrete independent variabe i, and analyticity and asymptoticity properties of the Jost solutions as functions of the spectral parameter ζ.

Definition 4.3.3. *The functions $\Lambda, \mathring{\Lambda}$ and $\Upsilon, \mathring{\Upsilon}$ are defined by*

$$\varphi(i;\zeta) = \Lambda(i;\zeta)\prod_{r=0}^{i-1}\left(\frac{\mathtt{p}(r)+\zeta}{\mathtt{p}(r)-\zeta}\right)^{\frac{1}{2}}, \quad \mathring{\varphi}(i;\zeta) = \mathring{\Lambda}(i;\zeta)\prod_{r=0}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)^{\frac{1}{2}} \tag{4.28a}$$

$$\psi(i;\zeta) = \Upsilon(i;\zeta)\prod_{r=0}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)^{\frac{1}{2}}, \quad \mathring{\psi}(i;\zeta) = \mathring{\Upsilon}(i;\zeta)\prod_{r=0}^{i-1}\left(\frac{\mathtt{p}(r)+\zeta}{\mathtt{p}(r)-\zeta}\right)^{\frac{1}{2}}. \tag{4.28b}$$

Proposition 4.3.2. *For $\zeta \neq 0$ the functions Λ and Υ satisfy the following summation equations:*

$$\Lambda(i;\zeta) = 1 + \frac{1}{2\zeta}\sum_{l=-\infty}^{i-1}\left[1 - \prod_{r=l}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)\right]\upsilon(l)\Lambda(l;\zeta) \tag{4.29}$$

$$\Upsilon(i;\zeta) = 1 + \frac{1}{2\zeta}\sum_{l=i+1}^{+\infty}\left[1 - \prod_{r=i}^{l-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)\right]\upsilon(l)\Upsilon(l;\zeta). \tag{4.30}$$

Proof. Equation (4.22) for Λ becomes

$$(\overline{\mathtt{p}}+\zeta)\overline{\overline{\Lambda}} - (\mathtt{p}+\overline{\mathtt{p}})\overline{\Lambda} + (\mathtt{p}-\zeta)\Lambda = \overline{\upsilon}\overline{\Lambda}$$

which in terms of i may be written as

$$\begin{aligned}\mathtt{p}(i+1)\Big[\Lambda(i+2;\zeta) - \Lambda(i+1;\zeta)\Big] - \mathtt{p}(i)\Big[\Lambda(i+1;\zeta) - \Lambda(i;\zeta)\Big] \\ + \zeta\Big[\Lambda(i+2;\zeta) - \Lambda(i;\zeta)\Big] = \upsilon(i+1)\Lambda(i+1;\zeta),\end{aligned}$$

and after summing from $l = -\infty$ to $l = i - 1$, and using $\Lambda \to 1$ as $i \to -\infty$, we have

$$\Lambda(i+1;\zeta)\Big[\mathtt{p}(i)+\zeta\Big] - \Lambda(i;\zeta)\Big[\mathtt{p}(i)-\zeta\Big] = 2\zeta + \sum_{l=-\infty}^{i} \upsilon(l)\Lambda(l;\zeta).$$

We now multiply this equation by the summing factor $s(i) := \prod_{r=0}^{i-1}\left(\frac{\mathtt{p}(r)+\zeta}{\mathtt{p}(r)-\zeta}\right)$, which gives

$$\begin{aligned}
&\Lambda(i+1;\zeta)s(i+1) - \Lambda(i;\zeta)s(i) \\
&\quad = \Big[s(i+1)-s(i)\Big] + \frac{1}{2\zeta}\Big[s(i+1)-s(i)\Big]\sum_{l=-\infty}^{i}\upsilon(l)\Lambda(l;\zeta)
\end{aligned}$$

and then we sum from $j = i_o \le i - 1$ to $j = i - 1$, obtaining

$$\begin{aligned}
\Lambda(i;\zeta)s(i) - \Lambda(i_o;\zeta)s(i_o) &= \Big[s(i)-s(i_o)\Big] \\
&+ \frac{1}{2\zeta}\sum_{j=i_o}^{i-1}\Big[s(j+1)-s(j)\Big]\sum_{l=-\infty}^{j}\upsilon(l)\Lambda(l;\zeta).
\end{aligned}$$

We now let $i_0 \to -\infty$ and assume that $s(i) \to 0$ as $i \to -\infty$. For negative values of i we define $s(-1) = 1$, $s(-a) = s(a-2)^{-1}$ for $a \ge 2$. By changing the order of summation the double sum can be rewritten as

$$\begin{aligned}
&\sum_{j=-\infty}^{i-1}\sum_{l=-\infty}^{j}\Big[s(j+1)-s(j)\Big]\upsilon(l)\Lambda(l;\zeta) \\
&\qquad = \sum_{l=-\infty}^{i-1}\upsilon(l)\Lambda(l;\zeta)\sum_{j=l}^{i-1}\Big[s(j+1)-s(j)\Big] \\
&\qquad = \sum_{l=-\infty}^{i-1}\Big[s(i)-s(l)\Big]\upsilon(l)\Lambda(l;\zeta)
\end{aligned}$$

and so the summation equation becomes

$$\begin{aligned}
&\Lambda(i;\zeta)s(i) = s(i) + \frac{1}{2\zeta}\sum_{l=-\infty}^{i-1}\Big[s(i)-s(l)\Big]\upsilon(l)\Lambda(l;\zeta) \\
&\Rightarrow \quad \Lambda(i;\zeta) = 1 + \frac{1}{2\zeta}\sum_{l=-\infty}^{i-1}\left[1 - \prod_{r=l}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)\right]\upsilon(l)\Lambda(l;\zeta),
\end{aligned}$$

which is equation (4.29). Equation (4.30) follows in a similar manner by summing to $i = +\infty$ and using $\Upsilon \to 1$ as $i \to +\infty$. □

Proposition 4.3.3. *At $\zeta = 0$ the functions Λ and Υ satisfy the following summation equations:*

$$\Lambda(i;0) = 1 + \sum_{l=-\infty}^{i-1} \left[\sum_{j=l}^{i-1} \frac{1}{\mathtt{p}(j)} \right] \upsilon(l)\Lambda(l;0) \tag{4.31}$$

$$\Upsilon(i;0) = 1 + \sum_{l=i+1}^{+\infty} \left[\sum_{j=i}^{l-1} \frac{1}{\mathtt{p}(j)} \right] \upsilon(l)\Upsilon(l;0). \tag{4.32}$$

Proof. At $\zeta = 0$, equation (4.22) for Λ becomes

$$\begin{aligned} &\mathtt{p}(i+1)\Big[\Lambda(i+2;0) - \Lambda(i+1;0)\Big] - \mathtt{p}(i)\Big[\Lambda(i+1;0) - \Lambda(i;0)\Big] \\ &\qquad = \upsilon(i+1)\Lambda(i+1;0), \end{aligned}$$

which after summing from $l = -\infty$ to $l = i - 1$ gives

$$\mathtt{p}(i)\Big[\Lambda(i+1;0) - \Lambda(i;0)\Big] = \sum_{l=-\infty}^{i} \upsilon(l)\Lambda(l;0).$$

After dividing through by $\mathtt{p}(i)$ and summing again from $j = -\infty$ to $j = i - 1$ we obtain

$$\Lambda(i;0) = 1 + \sum_{j=-\infty}^{i-1} \sum_{l=-\infty}^{j} \left[\frac{1}{\mathtt{p}(j)} \right] \upsilon(l)\Lambda(l;0),$$

and by changing the order of summation we have

$$\Lambda(i;0) = 1 + \sum_{l=-\infty}^{i-1} \left[\sum_{j=l}^{i-1} \frac{1}{\mathtt{p}(j)} \right] \upsilon(l)\Lambda(l;0)$$

which is equation (4.29). The result (4.30) follows in a similar manner. □

Proposition 4.3.4. *For $\zeta \neq 0$ the summation equations* (4.29) *and* (4.30) *have the Neumann series solutions*

$$\Lambda(i;\zeta) = \sum_{k=0}^{+\infty} \frac{H_k(i;\zeta)}{\zeta^k}, \qquad \Upsilon(i;\zeta) = \sum_{k=0}^{+\infty} \frac{J_k(i;\zeta)}{\zeta^k} \tag{4.33}$$

where

$$H_0 = 1, \qquad H_{k+1}(i;\zeta) = \frac{1}{2}\sum_{l=-\infty}^{i-1}\left[1 - \prod_{r=l}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)\right]\upsilon(l)H_k(l;\zeta), \quad (4.34)$$

$$J_0 = 1, \qquad J_{k+1}(i;\zeta) = \frac{1}{2}\sum_{l=i+1}^{+\infty}\left[1 - \prod_{r=i}^{l-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)\right]\upsilon(l)J_k(l;\zeta). \quad (4.35)$$

Proof. Inserting this series expression for Λ into the summation equation (4.29) gives

$$\begin{aligned}
\Lambda(i;\zeta) &= 1 + \frac{1}{2\zeta}\sum_{l=-\infty}^{i-1}\left[1 - \prod_{r=l}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)\right]\upsilon(l)\left(\sum_{k=0}^{+\infty}\frac{H_k(l;\zeta)}{\zeta^k}\right) \\
&= 1 + \sum_{k=0}^{+\infty}\frac{1}{\zeta^{k+1}}\left(\frac{1}{2}\sum_{l=-\infty}^{i-1}\left[1 - \prod_{r=l}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)\right]\upsilon(l)H_k(l;\zeta)\right) \\
&= 1 + \sum_{k=0}^{+\infty}\frac{H_{k+1}(i;\zeta)}{\zeta^{k+1}} \\
&= \sum_{k=0}^{+\infty}\frac{H_k(i;\zeta)}{\zeta^k}
\end{aligned}$$

as required. The proof for Υ follows in a similar fashion. □

Proposition 4.3.5. *At $\zeta = 0$ the summation equations* (4.31) *and* (4.32) *have the Neumann series solutions*

$$\Lambda(i;0) = \sum_{k=0}^{+\infty}H_k^o(i), \qquad \Upsilon(i;0) = \sum_{k=0}^{+\infty}J_k^o(i) \quad (4.36)$$

where

$$H_0^o = 1, \qquad H_{k+1}^o(i) = \sum_{l=-\infty}^{i-1}\left[\sum_{j=l}^{i-1}\frac{1}{\mathtt{p}(j)}\right]\upsilon(l)H_k^o(l), \quad (4.37)$$

$$J_0^o = 1, \qquad J_{k+1}^o(i) = \sum_{l=i+1}^{+\infty}\left[\sum_{j=i}^{l-1}\frac{1}{\mathtt{p}(j)}\right]\upsilon(l)J_k^o(l). \quad (4.38)$$

Proof. Inserting this series expression for Λ into the summation equation (4.31) gives

$$\begin{aligned}
\Lambda(i;0) &= 1 + \sum_{l=-\infty}^{i-1} \left[\sum_{j=l}^{i-1} \frac{1}{\mathtt{p}(j)} \right] \upsilon(l) \left(\sum_{k=0}^{+\infty} H_k^o(l) \right) \\
&= 1 + \sum_{k=0}^{+\infty} \left(\sum_{l=-\infty}^{i-1} \left[\sum_{j=l}^{i-1} \frac{1}{\mathtt{p}(j)} \right] \upsilon(l) H_k^o(l) \right) \\
&= 1 + \sum_{k=0}^{+\infty} H_{k+1}^o(i) \\
&= \sum_{k=0}^{+\infty} H_k^o(i)
\end{aligned}$$

as required. The proof for Υ is similar. □

Theorem 4.3.6. *Assume that*

$$\sum_{i=-\infty}^{+\infty} |\upsilon(i)|(1+|i|) < \infty, \tag{4.39}$$

and that $\mathtt{p}(r) > 0$ *for all* $r \in \mathrm{I}$. *Let* $\mathcal{R}^+$ *denote the half-plane*

$$\mathcal{R}^+ := \big\{ \zeta \ : \ \mathrm{Re}(\zeta) \geq 0 \big\}. \tag{4.40}$$

Then for $\zeta \in \mathcal{R}^+$,

$$|\Lambda(i;\zeta) - 1| \leq C_1 \quad \text{for } \ \zeta \neq 0 \tag{4.41a}$$

$$|\Lambda(i;\zeta) - 1| \leq C_2(1 + \max\{0, i\}) \tag{4.41b}$$

$$|\Upsilon(i;\zeta) - 1| \leq C_3 \quad \text{for } \ \zeta \neq 0 \tag{4.41c}$$

$$|\Upsilon(i;\zeta) - 1| \leq C_4(1 + \max\{0, -i\}) \tag{4.41d}$$

where $C_1 \to C_4$ *are constants. For all* $\zeta \in \mathcal{R}^+$ *the series solutions for* Λ *and* Υ *converge absolutely in* i, *and uniformly if* $\zeta \neq 0$. *For each* i, Λ *and* Υ *are continuous functions of* ζ *in* $\mathcal{R}^+$, *and analytic functions of* ζ *in the interior of this half-plane.*

Proof. The proof of this Theorem is obtained by showing absolute and uniform convergence of the Neumann series representation of the Jost solutions. The complete details, reminiscent of the analysis given in **[33]** for the continuous case, are given in Section 7.1 of the Appendix. The estimates obtained agree with those obtained in **[25]** and **[82]** for the same spectral problem, however those stated here are more precise. □

Theorem 4.3.7. *Assume that*

$$\sum_{i=-\infty}^{+\infty} (1+i^2)|\upsilon(i)| < \infty, \tag{4.42}$$

and that $\mathrm{p}(r) > 0$ *for all* $r \in \mathrm{I}$. *Then for* $\zeta \in \mathcal{R}^+$ *the* ζ*-derivatives of the functions* Λ *and* Υ *satisfy*

$$|\Lambda'(i;\zeta)| \leq C_5(1+|i|\max\{1,i\}) \tag{4.43}$$

$$|\Upsilon'(i;\zeta)| \leq C_6(1+|i|\max\{1,-i\}) \tag{4.44}$$

for some constants C_5 *and* C_6. *For each* i, Λ' *and* Υ' *are continuous in* ζ *for all* $\zeta \in \mathcal{R}^+$.

Proof. This proof is obtained by first differentiating the Neumann series representations of the Jost solutions, and then showing that these converge absolutely and uniformly. Rigorous details are given in Section 7.2 of the Appendix. □

Corollary 4.3.8. *For* $\zeta \in \mathcal{R}^+$ *the functions* Λ *and* Υ *have the following asymptotic behaviour:*

$$\Lambda(i;\zeta) = 1 + \mathcal{O}\left(\frac{1}{\zeta}\right) \quad \text{as } |\zeta| \to \infty \tag{4.45a}$$

$$\Upsilon(i;\zeta) = 1 + \mathcal{O}\left(\frac{1}{\zeta}\right) \quad \text{as } |\zeta| \to \infty. \tag{4.45b}$$

Proof. From the series solution of Λ we have

$$\Lambda(i;\zeta) = 1 + \sum_{k=1}^{+\infty} \frac{H_k(i;\zeta)}{\zeta^k}.$$

For $\zeta \in \mathcal{R}^+$, $\zeta \neq 0$ however, from Section 7.1 of the Appendix we have that $|H_k| \leq K$ for some constant K, and thus $H_k = \mathcal{O}(1)$ as $|\zeta| \to \infty$, for all $k \geq 1$. This proves the result, and a similar argument works for the series solution for Υ. □

Corollary 4.3.9. *Theorems 4.3.6 and 4.3.7 and Corollary 4.3.8 hold for the functions $\mathring{\Lambda}$ and $\mathring{\Upsilon}$ for ζ in the half-plane*

$$\mathcal{R}^- := \big\{\zeta : \mathrm{Re}(\zeta) \leq 0\big\}.$$

Proof. This follows from the fact that $\mathring{\Lambda}(i;\zeta) = \Lambda(i;-\zeta)$ and $\mathring{\Upsilon}(i;\zeta) = \Upsilon(i;-\zeta)$. □

4.3.2 Analyticity and Asymptoticity Properties of A and B

We now look at analyticity and asymptoticity properties of $\mathtt{A} = \mathtt{A}(\zeta)$ and $\mathtt{B} = \mathtt{B}(\zeta)$, which are defined by equation (4.25). These are related to the reflection coefficient R and transmission coefficient T by

$$R = \frac{\mathtt{B}}{\mathtt{A}}, \qquad T = \frac{1}{\mathtt{A}}.$$

Proposition 4.3.10. A *and* B *have the following properties:*

- A *is analytic in the interior of $\mathcal{R}^+$ and continuous in $\mathcal{R}^+$, except possibly at $\zeta = 0$*
- B *is continuous on the imaginary ζ-axis, except possibly at $\zeta = 0$.*

Proof. Taking the Wronskian of $\psi = \mathtt{A}\mathring{\varphi} + \mathtt{B}\varphi$ we have

$$\begin{aligned}
\mathtt{A}(\zeta) &= \frac{1}{2\zeta} W(\psi, \varphi) \\
&= \frac{1}{2\zeta}\Big((\mathtt{p}(i) + \zeta)\Lambda(i+i;\zeta)\Upsilon(i;\zeta) - (\mathtt{p}(i) - \zeta)\Lambda(i;\zeta)\Upsilon(i+1;\zeta) \Big) \\
\mathtt{B}(\zeta) &= \frac{1}{2\zeta} W(\mathring{\varphi}, \varphi) = \left(\frac{\mathtt{p}(i) - \zeta}{2\zeta}\right) \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta}\right) \\
&\qquad \times \Big(\mathring{\Lambda}(i;\zeta)\Upsilon(i+1;\zeta) - \mathring{\Lambda}(i+1;\zeta)\Upsilon(i;\zeta) \Big)
\end{aligned}$$

Since Λ and Υ are continuous in $\mathcal{R}^+$ and analytic in the interior of this region, $\mathtt{A}$ also has this property, except possibly at $\zeta = 0$. The expression for $\mathtt{B}$ however is only valid on the intersection of $\mathcal{R}^+$ and $\mathcal{R}^-$, i.e. the imaginary ζ-axis. Since $\mathring{\Lambda}$ and Υ are continuous here, $\mathtt{B}$ also has this property, except possibly at $\zeta = 0$. □

Proposition 4.3.11. *For $\zeta \neq 0$ the functions $\mathtt{A}$ and $\mathtt{B}$ can be expressed as*

$$\mathtt{A}(\zeta) = 1 + \frac{1}{2\zeta} \sum_{l=-\infty}^{+\infty} \upsilon(l)\, \Upsilon(l;\zeta) \tag{4.46}$$

$$\mathtt{B}(\zeta) = \frac{1}{2\zeta} \sum_{l=-\infty}^{+\infty} \left[\prod_{r=0}^{l-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta}\right) \right] \upsilon(l)\, \Upsilon(l;\zeta). \tag{4.47}$$

Proof. The summation equation (4.30) for Υ may be written as

$$\begin{aligned}
\Upsilon(i;\zeta) = &\left(1 + \frac{1}{2\zeta} \sum_{l=i+1}^{+\infty} \upsilon(l)\Upsilon(l;\zeta) \right) \\
&+ \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta}\right) \left(\frac{1}{2\zeta} \sum_{l=i+1}^{+\infty} \left[\prod_{r=0}^{l-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta}\right) \right] \upsilon(l)\, \Upsilon(l;\zeta) \right).
\end{aligned} \tag{4.48}$$

Taking the limit $i \to -\infty$ and comparing this with

$$\Upsilon(i;\zeta) \sim \mathtt{A}(\zeta) + \mathtt{B}(\zeta) \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta}\right) \quad \text{as} \quad i \to -\infty$$

gives the desired result. □

Proposition 4.3.12. *For $\zeta \in \mathcal{R}^+$ we have*

$$\mathtt{A}(\zeta) = 1 + \mathcal{O}\left(\frac{1}{\zeta}\right) \quad \text{as} \quad |\zeta| \to \infty \tag{4.49}$$

and for ζ on the imaginary axis we have

$$\mathtt{B}(\zeta) = \mathcal{O}\left(\frac{1}{\zeta}\right) \quad \text{as} \quad |\zeta| \to \infty. \tag{4.50}$$

Proof. Inserting the asymptotic behaviour (4.45b) of Υ into the expression (4.46) for $\mathtt{A}$ gives

$$\mathtt{A}(\zeta) = 1 + \frac{1}{2\zeta} \sum_{l=-\infty}^{+\infty} \upsilon(l) \left[1 + \mathcal{O}\left(\frac{1}{\zeta}\right)\right]$$

and since

$$\left| \sum_{l=-\infty}^{+\infty} \upsilon(l) \right| \leq \sum_{l=-\infty}^{+\infty} |\upsilon(l)| < \infty$$

this proves (4.49). Performing the same task for the expression (4.47) for $\mathtt{B}$ gives

$$\mathtt{B}(\zeta) = \frac{1}{2\zeta} \sum_{l=-\infty}^{+\infty} \left[\prod_{r=0}^{l-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta}\right)\right] \upsilon(l) \left[1 + \mathcal{O}\left(\frac{1}{\zeta}\right)\right],$$

and since for purely imaginary ζ we have

$$\left| \sum_{l=-\infty}^{+\infty} \left[\prod_{r=0}^{l-1} \left(\frac{\mathtt{p}(r) - \zeta}{\mathtt{p}(r) + \zeta}\right)\right] \upsilon(l) \right| \leq \sum_{l=-\infty}^{+\infty} |\upsilon(l)| < \infty,$$

this proves (4.50). □

Theorem 4.3.13. *The function $\mathtt{A}$ has a finite number of bounded isolated zeroes $\{\zeta_k,\, k = 1, ..., M\}$ in the interior of $\mathcal{R}^+$, and moreover every ζ_k is purely real and satisfies $\zeta_k \leq p_r$ for all parameters p_r existing along Γ. At each zero of $\mathtt{A}$ we have $\psi(i;\zeta_k) = \mathtt{B}(\zeta_k)\varphi(i;\zeta_k)$, and*

$$\sum_{i=-\infty}^{+\infty} \left(\frac{\varphi(i;\zeta_k)\varphi(i+1;\zeta_k)}{(\mathtt{p}(i)^2 - \zeta_k^2)^{\frac{1}{2}}}\right) = \frac{\mathtt{A}'(\zeta_k)}{\mathtt{B}(\zeta_k)}, \tag{4.51}$$

where $\mathtt{A}'(\zeta)$ denotes differentiation with respect to the variable ζ.

Proof. Since $\mathtt{A} \sim 1$ as $|\zeta| \to \infty$ it follows that there exists some constant C_o such that $|\zeta_k| < C_o$ for every k. Since $2\zeta\mathtt{A}(\zeta) = W(\psi, \varphi)$ it follows that for every k, $\varphi(i;\zeta_k)$ and $\psi(i;\zeta_k)$ are linearly dependent, so we may write $\psi(i;\zeta_k) = b_k\varphi(i;\zeta_k)$ for some constant b_k. This implies that

$$\Upsilon(i;\zeta_k) \sim b_k \prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r)-\zeta_k}{\mathtt{p}(r)+\zeta_k}\right) \quad \text{as} \quad i \to -\infty,$$

and so by equation (4.48) we have

$$\begin{aligned}
\zeta_k &= -\frac{1}{2}\sum_{l=-\infty}^{+\infty} \upsilon(l)\,\Upsilon(l;\zeta_k) \\
b_k &= \frac{1}{2\zeta_k}\sum_{l=-\infty}^{+\infty} \left[\prod_{r=0}^{l-1}\left(\frac{\mathtt{p}(r)-\zeta_k}{\mathtt{p}(r)+\zeta_k}\right)\right] \upsilon(l)\,\Upsilon(l;\zeta_k).
\end{aligned}$$

Thus $b_k = \mathtt{B}(\zeta_k)$ for every k. Now consider the scattering problem (4.22) for φ at $\zeta = \zeta_k$:

$$(\overline{\mathtt{p}}^2 - \zeta_k^2)^{\frac{1}{2}}\,\varphi(i+2;\zeta_k) - \big(\mathtt{p} + \overline{\mathtt{p}} + \overline{\upsilon}\big)\,\varphi(i+1;\zeta_k) + (\mathtt{p}^2 - \zeta_k^2)^{\frac{1}{2}}\,\varphi(i;\zeta_k) = 0.$$

For every ζ_k in the interior of $\mathcal{R}^+$ we have

$$\begin{aligned}
\varphi(i;\zeta_k) &\sim \prod_{r=0}^{i-1}\left(\frac{\mathtt{p}(r)+\zeta_k}{\mathtt{p}(r)-\zeta_k}\right)^{\frac{1}{2}} \to 0 \ \text{ as } \ i \to -\infty, \\
\varphi(i;\zeta_k) &\sim \mathtt{B}(\zeta_k)\prod_{r=0}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta_k}{\mathtt{p}(r)+\zeta_k}\right)^{\frac{1}{2}} \to 0 \ \text{ as } \ i \to +\infty,
\end{aligned}$$

and thus φ is summable over all i. If we multiply the scattering problem for φ by $\varphi^*(i+1;\zeta_k^*)$, sum over all i and define

$$s(i;\zeta_k) := \varphi(i+1;\zeta_k)\varphi^*(i;\zeta_k^*) + \varphi^*(i+1;\zeta_k^*)\varphi(i;\zeta_k) \in \mathbb{R}$$

then we have

$$\sum_{i=-\infty}^{+\infty} (\mathtt{p}(i)^2 - \zeta_k^2)^{\frac{1}{2}} s(i;\zeta_k) = \sum_{i=-\infty}^{+\infty} \Big[\mathtt{p}(i) + \mathtt{p}(i+1) + \upsilon(i+1)\Big]|\varphi(i+1;\zeta_k)|^2,$$

which implies that ζ_k must be real and that $0 < \zeta_k \leq \mathtt{p}(i)$ for every i. Thus ζ_k must be less than every parameter p_r through which $\mathtt{p}(i)$ cycles, which proves the given statement.

Now $\mathtt{A}$ has isolated zeroes along the positive real ζ axis which are all bounded. The only way that there could be an infinite number of these zeroes is if they formed a limiting sequence which accumulated at $\zeta = 0$. We will show that this is not possible. Suppose that such a sequence $\{\zeta_k\}$ of zeroes exists: $\lim_{k\to\infty} \zeta_k = 0$. Then at each ζ_k we have

$$\mathtt{B}(\zeta_k) = \frac{\psi(i;\zeta_k)}{\varphi(i;\zeta_k)},$$

and so

$$\lim_{k\to\infty} |\mathtt{B}(\zeta_k) - \mathtt{B}(0)| = \lim_{k\to\infty} \left| \frac{\psi(i;\zeta_k)}{\varphi(i;\zeta_k)} - \frac{\psi(i;0)}{\varphi(i;0)} \right| = 0$$

since the Jost solutions are continuous at $\zeta = 0$. At $\zeta = 0$ however we have $\varphi(i;0) = \mathring{\varphi}(i;0)$ and so

$$\mathtt{A}(0) + \mathtt{B}(0) = \left(\frac{\psi(i;0)}{\varphi(i;0)} \right) = \lim_{k\to\infty} \left(\frac{\psi(i;\zeta_k)}{\varphi(i;\zeta_k)} \right) = \lim_{k\to\infty} \mathtt{B}(\zeta_k) = \mathtt{B}(0)$$

which implies that $\mathtt{A}(0) = 0$, which in turn contradicts Proposition 4.3.1. Thus $\mathtt{A}$ has only a finite number of zeroes in $\mathcal{R}^+$.

Finally to prove (4.51) we define the following two useful functions:

$$W_\varphi(i;\zeta) := W(\varphi, \varphi'), \qquad W_\psi(i;\zeta) := W(\psi, \psi').$$

We differentiate $2\zeta\mathtt{A} = W(\psi, \varphi)$ to obtain

$$\begin{aligned} 2\zeta_k \mathtt{A}'(\zeta_k) &= W(\psi(i;\zeta_k), \varphi'(i;\zeta_k)) + W(\psi(i;\zeta_k), \varphi'(i;\zeta_k)) \\ &= \mathtt{B}(\zeta_k) W_\varphi(i;\zeta_k) - \frac{1}{\mathtt{B}(\zeta_k)} W_\psi(i;\zeta_k). \end{aligned} \tag{4.52}$$

Now consider the difference of two equations: firstly the derivative of the scattering problem (4.22) for φ multiplied by $\varphi(i+1;\zeta)$, and secondly the

(un-differentiated) scattering problem for φ multiplied by $\varphi'(i+1;\zeta)$. This gives

$$W_\varphi(i+1;\zeta) - W_\varphi(i;\zeta) = \zeta\left[\frac{\varphi(i;\zeta)\varphi(i+1;\zeta)}{(\mathtt{p}(i)^2-\zeta^2)^{\frac{1}{2}}} + \frac{\varphi(i+1;\zeta)\varphi(i+2;\zeta)}{(\mathtt{p}(i+1)^2-\zeta^2)^{\frac{1}{2}}}\right],$$

which may be summed to give

$$W_\varphi(i;\zeta) = \zeta\sum_{l=-\infty}^{i-1}\left[\frac{\varphi(l;\zeta)\varphi(l+1;\zeta)}{(\mathtt{p}(l)^2-\zeta^2)^{\frac{1}{2}}} + \frac{\varphi(l+1;\zeta)\varphi(l+2;\zeta)}{(\mathtt{p}(l+1)-\zeta^2)^{\frac{1}{2}}}\right].$$

One can then perform the same task with ψ, only instead this time summing from i to $+\infty$, to obtain

$$W_\psi(i;\zeta) = -\zeta\sum_{l=i}^{+\infty}\left[\frac{\psi(l;\zeta)\psi(l+1;\zeta)}{(\mathtt{p}(l)^2-\zeta^2)^{\frac{1}{2}}} + \frac{\psi(l+1;\zeta)\psi(l+2;\zeta)}{(\mathtt{p}(l+1)-\zeta^2)^{\frac{1}{2}}}\right].$$

Now set $\zeta=\zeta_k$ and rewrite $\psi(i;\zeta_k)=\mathtt{B}(\zeta_k)\varphi(i;\zeta_k)$. The expression (4.52) then becomes

$$\begin{aligned}2\zeta_k\mathtt{A}'(\zeta_k) &= \zeta_k\mathtt{B}(\zeta_k)\sum_{l=-\infty}^{+\infty}\left[\frac{\varphi(l;\zeta_k)\varphi(l+1;\zeta_k)}{(\mathtt{p}(l)^2-\zeta_k^2)^{\frac{1}{2}}} + \frac{\varphi(l+1;\zeta_k)\varphi(l+2;\zeta_k)}{(\mathtt{p}(l+1)-\zeta_k^2)^{\frac{1}{2}}}\right]\\ &= 2\zeta_k\mathtt{B}(\zeta_k)\sum_{l=-\infty}^{+\infty}\left[\frac{\varphi(l;\zeta_k)\varphi(l+1;\zeta_k)}{(\mathtt{p}(l)^2-\zeta_k^2)^{\frac{1}{2}}}\right],\end{aligned}$$

which gives (4.51). □

The sum in equation (4.51) is of fundamental importance to the scattering problem, and as such we make the following definition.

Definition 4.3.4. *The* **square eigenfunction** Φ *is defined to be*

$$\Phi(i;\zeta) := \frac{\varphi(i;\zeta)\varphi(i+1;\zeta)}{(\mathtt{p}(i)^2-\zeta^2)^{\frac{1}{2}}}, \tag{4.53}$$

and we define the **normalisation constants** $\{\mathtt{c}_k,\, k=1,...,M\}$ *to be*

$$\mathtt{c}_k := \frac{\mathtt{B}(\zeta_k)}{\mathtt{A}'(\zeta_k)} = \left[\sum_{i=-\infty}^{+\infty}\Phi(i;\zeta_k)\right]^{-1}. \tag{4.54}$$

From this definition it is clear that we have the following result

Theorem 4.3.14. *If the normalisation constants* $\mathtt{c}_k$ *are all finite, then every zero* $\mathtt{A}$ *in* $\mathcal{R}^+$ *is simple.*

There are a number of ways of ensuring that all the normalisation constants are finite. For example since

$$\begin{aligned}\frac{1}{\mathtt{c}_k} &= \sum_{i=-\infty}^{+\infty}\left(\frac{\varphi(i;\zeta_k)\varphi(i+1;\zeta_k)}{(\mathtt{p}(i)^2-\zeta_k^2)^{\frac{1}{2}}}\right)\\ &= \sum_{i=-\infty}^{+\infty}\big[\mathtt{p}(i)+\mathtt{p}(i+1)+\upsilon(i+1)\big]|\varphi(i;\zeta_k)|^2,\end{aligned}$$

if one imposes (for example) that $\big[\mathtt{p}(i)+\mathtt{p}(i+1)+\upsilon(i+1)\big] > 0$ for all i, then every $\mathtt{c}_k$ will be finite. This was assumed in **[28]**, however is a rather strong restriction. A more useful endeavour, which was shown in **[27]**, is to consider the perturbed scattering problem obtained by setting $\upsilon \to \lambda\upsilon$ for some real parameter λ. One can then show that the Jost solutions and $\mathtt{A}$ are all differentiable functions of λ in some open interval about $\lambda = 1$. Thus if we expand $\mathtt{A}$ as a Taylor series about any zero ζ_k:

$$\mathtt{A}(\zeta,\lambda) = \sum_{n=1}^{+\infty} a_n(\lambda)(\zeta-\zeta_k(\lambda))$$

then the coefficient a_1 will be a non-constant differentiable function of λ, and thus if $a_1(1) = 0$ then for the perturbed scattering problem (i.e. setting λ close to but not equal to 1), $a_1(\lambda) \neq 0$, i.e. the normalisation constants become finite. We illustrate this with an example.

Example 4.3.15. *Take the N=2 case with the initial staircase* Γ *being a line in the* n*-direction with parameter* p*. Consider the potential*

$$\upsilon(n) = \lambda(\delta_0(n)+\delta_1(n)),$$

where $\delta_m(n)$ *denotes the delta function (or Kronecker delta-symbol), which is defined by* $\delta_m(n) = 0$ *if* $n \neq m$*, and* $\delta_m(m) = 1$*. Here* λ *is the perturbation*

parameter mentioned above. Note that for the case of H1 the potential is simply

$$\upsilon(n) = w(n+1,0) - w(n,0),$$

where we have rewritten the H1 dependent variable as $u(n,m) = pn + qm + w(n,m)$*. Thus for this example we have chosen to set* $w(n,0) = \lambda H_1(n)$ *along the initial-value space, where the discrete Heaviside step function is defined by* $H_o(n) = 1$ *if* $n \geq 0$ *and* $H_o(n) = 0$ *if* $n < 0$*. By summing the linear equation* (4.22) *for* Υ *along the* n*-axis one finds firstly that* $\Upsilon(n;\zeta) = 1$ *for* $n \geq 1$*, and then*

$$\Upsilon(0;\zeta) = 1 + \frac{\lambda}{p+\zeta}, \qquad \Upsilon(-1;\zeta) = 1 + \frac{\lambda}{p+\zeta} + \frac{\lambda(2p+\lambda)}{(p+\zeta)^2}.$$

We now use equation (4.25) *to find* A *and* B*. Since* $\Lambda(n;\zeta) = 1$ *for all* $n \leq 0$ *we therefore have*

$$\mathtt{A}(\zeta) + \mathtt{B}(\zeta) = 1 + \frac{\lambda}{p+\zeta}$$
$$\mathtt{A}(\zeta) + \mathtt{B}(\zeta)\left(\frac{p-\zeta}{p+\zeta}\right) = 1 + \frac{\lambda}{p+\zeta} + \frac{\lambda(2p+\lambda)}{(p+\zeta)^2}$$

from which we obtain

$$\mathtt{A}(\zeta) = \frac{\zeta^2 + \zeta(p+\lambda) + \lambda\left(p + \frac{\lambda}{2}\right)}{\zeta(p+\zeta)}.$$

The numerator of A *is a quadratic and if we look at the discriminant* Δ *of this we have*

$$\Delta = -\lambda^2 - 2p\lambda + p^2.$$

In particular if $\lambda = -p(1+\sqrt{2})$ *then the discriminant vanishes and moreover*

$$\mathtt{A}(\zeta) = \frac{\left(\zeta - \frac{1}{\sqrt{2}}p\right)^2}{\zeta(p+\zeta)}.$$

Thus for this particular choice of λ*,* A *has a double zero at* $\zeta_k = \frac{1}{\sqrt{2}}p$*. For all other values of* λ *however,* A *has only simple zeroes in* $\mathcal{R}^+$*.*

As Example 4.3.15 illustrates, since a_1 is a non-constant differentiable function of λ, in the event that a multiple zero of $\mathtt{A}$ does occur, one can simply perturb the system by setting $\upsilon \to \lambda\upsilon$, and then for λ in some small interval around 1 all zeroes of $\mathtt{A}$ in $\mathcal{R}^+$ will become simple. We do not dwell on this further, but assume henceforth that all normalisation constants are finite. This concludes the forward scattering of ϕ_1.

4.4 "Time" Evolution of the Scattering Data

We now consider how the spectral functions $\mathtt{A}$, $\mathtt{B}$ and the normalisation constants $\mathtt{c}_k$ depend on the N lattice variables. As an analogy to the continuous theory, this is the calculation of the "time" dependence of these functions, with respect to the arbitrary number of discrete "time" variables.

From the Lax pair the equation governing the evolution of the Jost solutions in any one particular lattice direction with variable n_k and parameter p_k is given by

$$(p_k^2-\zeta^2)^{\frac{1}{2}}\widehat{\varphi}-\left(\frac{P_k(\widehat{u}-u)}{\widehat{\mathcal{U}}}\right)\widehat{\varphi}+(p_k^2-\zeta^2)^{\frac{1}{2}}\varphi=0 \tag{4.55}$$

where $\widehat{u}$ denotes an iteration of u in the n_k-direction. From the boundary conditions (4.17) for u, for all $k=1,...,\mathrm{N}$ we have

$$\left(\frac{P_k(\widehat{u}-u)}{\widehat{\mathcal{U}}}\right)\sim 2p_k \quad \text{as} \quad i\to\pm\infty. \tag{4.56}$$

In other words at both ends of the staircase the dependence of the Jost solutions on the lattice variable n_k is governed by

$$(p_k^2-\zeta^2)^{\frac{1}{2}}\widehat{\varphi}-2p_k\widehat{\varphi}+(p_k^2-\zeta^2)^{\frac{1}{2}}\varphi=0. \tag{4.57}$$

The boundary conditions (4.17) are assumed to hold *independently* of the remaining "non-staircase" lattice variables n_s, $s\notin \mathrm{I}$. We also assume that the boundary conditions (4.23a) (4.23c) for the Jost solutions hold *independently*

of n_s, $s \notin \mathrm{I}$. In other words these hold for all values of the discrete "time" variables, which is the assumption made in the continuous IST theory for the KdV discussed in Chapter 2.

Let us first consider the case where the n_k-direction is one of the staircase directions, i.e. $k \in \mathrm{I}$. Here the boundary conditions for the Jost solutions are given by (4.23a) (4.23c), yet these are inconsistent[1] with equation (4.57). We illustrate this with an example. Consider a $(1,1)$-staircase in the (n,m)-plane, which was discussed in Section 4.2. Assuming that we iterate first in the n-direction, the change of variables from i to n, m is given by

$$n = \left\lfloor \frac{1}{2}(i+1) \right\rfloor, \qquad m = \left\lfloor \frac{1}{2} i \right\rfloor .$$

Thus the plane-wave factors appearing in the boundary conditions for the Jost solutions are proportional to

$$\prod_{r=0}^{i-1} \left(\frac{\mathtt{p}(r)+\zeta}{\mathtt{p}(r)-\zeta} \right)^{\frac{1}{2}} \propto \left(\frac{p+\zeta}{p-\zeta} \right)^{\frac{n}{4}} \left(\frac{q+\zeta}{q-\zeta} \right)^{\frac{m}{4}} .$$

As functions of n and m these are inconsistent with equation (4.57) in the n-direction or m-direction respectively. To deal with this we redefine the boundary conditions for the Jost solutions *as functions of* $n_k, k \in \mathrm{I}$, to be

$$\varphi \sim \prod_{r\in\mathrm{I}} \left(\frac{p_r+\zeta}{p_r-\zeta} \right)^{\frac{1}{2}n_r}, \qquad \mathring{\varphi} \sim \prod_{r\in\mathrm{I}} \left(\frac{p_r-\zeta}{p_r+\zeta} \right)^{\frac{1}{2}n_r} \quad \text{as } i \to -\infty \tag{4.58a}$$

$$\psi \sim \prod_{r\in\mathrm{I}} \left(\frac{p_r-\zeta}{p_r+\zeta} \right)^{\frac{1}{2}n_r}, \qquad \mathring{\psi} \sim \prod_{r\in\mathrm{I}} \left(\frac{p_r+\zeta}{p_r-\zeta} \right)^{\frac{1}{2}n_r} \quad \text{as } i \to +\infty. \tag{4.58b}$$

When one restricts these boundary conditions to Γ these agree with the previous boundary conditions (4.23a) (4.23c). Now however the Jost functions are consistent with equation (4.57) for $k \in \mathrm{I}$. Furthermore φ and $\mathring{\varphi}$ are now linearly independent functions of equation (4.55), so the functions $\mathtt{A}$ and $\mathtt{B}$, and the normalisation constants $\mathtt{c}$, are independent of n_k for all $k \in \mathrm{I}$.

[1] Unless Γ is a line, in which case (4.55) is the forward scattering problem itself

If we now consider one of the discrete "time" directions with variable n_s-direction, $s \notin \mathrm{I}$, then by (4.58a) (4.58b), as $n_s \to \pm\infty$ the Jost solutions do *not* depend on n_s, and are thus inconsistent with (4.57). To circumvent this we make the following definition.

Definition 4.4.1. *Let* J *denote the collection of lattice directions in which* Γ *does not iterate, so that the union of the sets* I *and* J *gives all N lattice directions. The N-dimensional Jost solutions* $\varphi^{(N)}, \mathring{\varphi}^{(N)}$ *and* $\psi^{(N)}, \mathring{\psi}^{(N)}$*, which are solutions to all N equations* (4.55)*, are defined to be*

$$\varphi^{(N)} := \varphi \prod_{s\in \mathrm{J}} \left(\frac{p_s+\zeta}{p_s-\zeta}\right)^{\frac{1}{2}n_s}, \qquad \mathring{\varphi}^{(N)} := \mathring{\varphi} \prod_{s\in \mathrm{J}} \left(\frac{p_s-\zeta}{p_s+\zeta}\right)^{\frac{1}{2}n_s} \tag{4.59a}$$

$$\psi^{(N)} := \psi \prod_{s\in \mathrm{J}} \left(\frac{p_s-\zeta}{p_s+\zeta}\right)^{\frac{1}{2}n_s}, \qquad \mathring{\psi}^{(N)} := \mathring{\psi} \prod_{s\in \mathrm{J}} \left(\frac{p_s+\zeta}{p_s-\zeta}\right)^{\frac{1}{2}n_s}. \tag{4.59b}$$

We then have the following result about the "time" dependence of the spectral functions $\mathtt{A}$ and $\mathtt{B}$ and the normalisation constants $\mathtt{c}$. Note that these functions were originally defined along Γ, which spans the collection I of lattice directions. The span of these lattice directions forms a subspace of the N-dimensional lattice, which is defined by setting $n_s = const.$ for all $s \in \mathrm{J}$. Thus the notation $\mathtt{A}(\zeta)$ in fact means $\mathtt{A}$ evaluated at $n_s = const.$ for all $s \in \mathrm{J}$. We then write its "time-dependent" counterpart as $\mathtt{A}(n_s;\zeta)$, which may depend on all discrete "times" $n_s, s \in \mathrm{J}$. Similar notation is used for $\mathtt{B}$ and $\mathtt{c}$.

Theorem 4.4.1.

- *The function* $\mathtt{A}$ *is independent of all lattice variables* n_k*,* $k = 1, ..., N$*:*

$$\mathtt{A}(n_s;\zeta) = \mathtt{A}(\zeta) \tag{4.60}$$

- *The dependence of the functions* $\mathtt{B}$ *and* $\mathtt{c}$ *on* $n_s, s \in \mathrm{J}$ *is given by*

$$\mathtt{B}(n_s;\zeta) = \mathtt{B}(\zeta)\prod_{s\in\mathrm{J}}\left(\frac{p_s+\zeta}{p_s-\zeta}\right)^{n_s} \tag{4.61}$$

$$\mathtt{c}(n_s;\zeta) = \mathtt{c}(\zeta)\prod_{s\in\mathrm{J}}\left(\frac{p_s+\zeta}{p_s-\zeta}\right)^{n_s}. \tag{4.62}$$

Proof. Consider the n_s-direction, where $s \in \mathrm{J}$. The evolution equation for the N-dimensional Jost solutions in this direction is

$$(p_s^2-\zeta^2)^{\frac{1}{2}}\widehat{\varphi}^{(N)} - \left(\frac{P_s(\widehat{u}-u)}{\widehat{u}}\right)\widehat{\varphi}^{(N)} + (p_s^2-\zeta^2)^{\frac{1}{2}}\varphi^{(N)} = 0,$$

and by their boundary conditions the functions $\varphi^{(N)}$ and $\mathring{\varphi}^{(N)}$ are linearly independent solutions of this equation. We may therefore write

$$\psi^{(N)} = C_1\,\mathring{\varphi}^{(N)} + C_2\,\varphi^{(N)},$$

where C_1 and C_2 are independent of n_s, but may depend on all other lattice variables. This is equivalent to

$$\psi = C_1\,\mathring{\varphi} + C_2\,\varphi\prod_{s\in\mathrm{J}}\left(\frac{p_s+\zeta}{p_s-\zeta}\right)^{n_s},$$

and by comparing this with

$$\psi = \mathtt{A}\,\mathring{\varphi} + \mathtt{B}\,\varphi\prod_{r\in\mathrm{I}}\left(\frac{p_r+\zeta}{p_r-\zeta}\right)^{n_r},$$

it follows that $\mathtt{A}$ is independent of n_s, while the dependence of $\mathtt{B}$ on n_s is through the plane-wave factor $\left(\frac{p_s+\zeta}{p_s-\zeta}\right)^{n_s}$. Repeating this argument for all $s \in \mathrm{J}$ gives the desired results for $\mathtt{A}$ and $\mathtt{B}$. The result for $\mathtt{c}$ follows from its definition (4.54). □

Example 4.4.2. *Take for example the case of N=4, with* Γ *being a* $(2,3)$*-staircase in the* (n,m)*-plane with respective parameters* p *and* q*, and let the remaining two lattice directions be enumerated by the independent variables* τ_1 *and* τ_2 *with respective parameters* σ_1 *and* σ_2*. Then the set* I *is the union of the* n*-lattice direction and* m*-lattice direction (i.e. the* (n,m)*-plane), while the set* J *is the union of the*

τ_1-lattice direction and τ_2-lattice direction (another plane in the four-dimensional space). By the results of the above Theorem, $\mathtt{A}$ *does not depend on any of the four lattice variables, while the quantities* $\mathtt{B}(\tau_1,\tau_2;\zeta)$ *and* $\mathtt{B}(\tau_1,\tau_2;\zeta)$ *are given by*

$$\mathtt{B}(\tau_1,\tau_2;\zeta)=\mathtt{B}(\zeta)\left(\frac{\sigma_1+\zeta}{\sigma_1-\zeta}\right)^{\tau_1}\left(\frac{\sigma_2+\zeta}{\sigma_2-\zeta}\right)^{\tau_2}$$
$$\mathtt{c}(\tau_1,\tau_2;\zeta)=\mathtt{c}(\zeta)\left(\frac{\sigma_1+\zeta}{\sigma_1-\zeta}\right)^{\tau_1}\left(\frac{\sigma_2+\zeta}{\sigma_2-\zeta}\right)^{\tau_2}.$$

4.5 Inverse Problem for ϕ_1

We now consider the inverse problem for ϕ_1, that is the construction of the Jost solutions as functions of all N lattice variables. In doing so we alter the notation of all eigenfunctions by writing

$$\varphi(n_1,...,n_N;\zeta)\to\varphi(\zeta),$$

and similarly for the other functions. Here $\varphi(\zeta)$ is understood to depend on *all* lattice variables $n_1,...,n_N$, however for the inverse problem it is convenient to suppress this dependence in the notation.

Consider the equation (4.25), which we rewrite as

$$\frac{\Upsilon(\zeta)}{\mathtt{A}(\zeta)}-\mathring{\Lambda}(\zeta)=R(\zeta)\Lambda(\zeta)\rho(\zeta), \tag{4.63}$$

where the reflection coefficient R is given by

$$R(\zeta)=\frac{\mathtt{B}(\zeta)}{\mathtt{A}(\zeta)}$$

and the plane-wave factors ρ are defined by

$$\rho(\zeta):=\prod_{r\in\mathrm{I}}\left(\frac{p_r+\zeta}{p_r-\zeta}\right)^{n_r}\prod_{s\in\mathrm{J}}\left(\frac{p_s+\zeta}{p_s-\zeta}\right)^{n_s}=\prod_{r=1}^{N}\left(\frac{p_r+\zeta}{p_r-\zeta}\right)^{n_r}.$$

This defines a jump condition between two sectionally meromorphic functions along the contour $\mathrm{Re}(\zeta)=0$. The functions $\frac{\Upsilon}{\mathtt{A}}$ and $\mathring{\Lambda}$ are analytic in the interior of the regions $\mathcal{R}^+$ and $\mathcal{R}^-$ respectively, and both are continuous along $\mathrm{Re}(\zeta)=0$. Given the jump condition and their boundary conditions,

the question of determining a function which is equal to these in their respective half-planes is a Riemann-Hilbert problem. The method of solving such a problem is well-known (see e.g. [**42**]): Consider the singular integral

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{R(\sigma)\Lambda(\sigma)}{\sigma+\zeta}\rho(\sigma)\,d\sigma\\ &=\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Upsilon(\sigma)}{\mathtt{A}(\sigma)(\sigma+\zeta)}\,d\sigma-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\mathring{\Lambda}(\sigma)}{\sigma+\zeta}\,d\sigma\end{aligned}\tag{4.64}$$

where $\zeta\in\mathcal{R}^+$. Here the contour of integration is the imaginary σ-axis. Since the function $\frac{\Upsilon}{\mathtt{A}}$ has M simple poles in $\mathcal{R}^+$ and has the boundary behaviour $\frac{\Upsilon}{\mathtt{A}}\sim 1$ as $|\zeta|\to\infty$, one may use the residue theorem to calculate

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Upsilon(\sigma)}{\mathtt{A}(\sigma)(\sigma+\zeta)}\,d\sigma=\frac{1}{2}-\sum_{k=1}^{M}\frac{\Upsilon(\zeta_k)}{\mathtt{A}'(\zeta_k)(\zeta+\zeta_k)}.$$

By then using the fact that

$$\Upsilon(\zeta_k)=\mathtt{B}(\zeta_k)\Lambda(\zeta_k)\rho(\zeta_k)$$

this can written as

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Upsilon(\sigma)}{\mathtt{A}(\sigma)(\sigma+\zeta)}\,d\sigma=\frac{1}{2}-\sum_{k=1}^{M}\frac{\mathtt{c}_k\Lambda(\zeta_k)}{(\zeta+\zeta_k)}\rho(\zeta_k).$$

Now since $\mathring{\Lambda}$ is analytic in $\mathcal{R}^-$ one can determine that

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\mathring{\Lambda}(\sigma)}{\sigma+\zeta}\,d\sigma=-\frac{1}{2}+\mathring{\Lambda}(-\zeta)=-\frac{1}{2}+\Lambda(\zeta),$$

and thus the singular integral equation becomes

$$\Lambda(\zeta)=1-\sum_{k=1}^{M}\frac{\mathtt{c}_k\Lambda(\zeta_k)}{(\zeta+\zeta_k)}\rho(\zeta_k)-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{R(\sigma)\Lambda(\sigma)}{\sigma+\zeta}\rho(\sigma)\,d\sigma.\tag{4.65}$$

This is the discrete version of equation (2.30) obtained in the continuous IST for the KdV equation[2]. Given the ζ_k, the normalisation constants $\mathtt{c}_k$ and the reflection coefficient R, one can use equation (4.67) to determine

[2]If one allows the quantities R and ρ to depend on the KdV independent variables in the correct way these two integral equations are in fact identical.

the Jost solution Λ *as a function of* ζ *and all* N *lattice variables*. Remarkably all of the dependence on the lattice variables is contained in the plane-wave factors ρ. If instead we started with the relation

$$\Lambda(\zeta) = \mathtt{A}(\zeta)\mathring{\Upsilon}(\zeta) - \mathtt{B}(-\zeta)\Upsilon(\zeta)\rho(-\zeta), \tag{4.66}$$

which is consistent with (4.25), then by following the same procedure as above one finds that $\Upsilon(\zeta)$ is determined by solving the singular integral equation

$$\Upsilon(\zeta) = 1 - \sum_{k=1}^{M} \frac{\mathtt{d}_k \Upsilon(\zeta_k)}{(\zeta+\zeta_k)}\rho(-\zeta_k) + \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{S(\sigma)\Upsilon(\sigma)}{\sigma+\zeta}\rho(-\sigma)\,d\sigma, \tag{4.67}$$

where

$$\mathtt{d}_k := \frac{1}{\mathtt{A}'(\zeta_k)\mathtt{B}(\zeta_k)}, \qquad S(\zeta) := \frac{\mathtt{B}(-\zeta)}{\mathtt{A}(\zeta)}. \tag{4.68}$$

Note that one could, in the spirit of the continuous IST, isolate the dependence of the Jost solution Λ on the spectral parameter ζ and thereby obtain a discrete Gel'fand-Levitan integral (summation) equation. The details of this can be seen in [**28**]. For the discrete setting however it is more convenient to use (4.67) to obtain Λ as a function of ζ, as we will see that the solution of Q3$_\delta$ and all of the other equations considered in this chapter are simply expressible in terms of the Jost solutions evaluated at specific values of ζ.

4.6 Inverse Problem for ϕ_2

Now that we have constructed the first component of the eigenfunction $\boldsymbol{\phi}$ of (4.6), we use the first component of the Lax equations to determine ϕ_2:

$$(\zeta^2 - b^2)\,\phi_2(\zeta) = (p_k^2 - \zeta^2)^{\frac{1}{2}}\mathcal{U}\,\widehat{\phi}_1(\zeta) - (P_k\widehat{u} - (p_k^2 - b^2)\,u\,)\,\phi_1(\zeta). \tag{4.69}$$

Let $\varphi_2^{(N)}$ and $\psi_2^{(N)}$ be the corresponding second components for the Jost solutions $\varphi^{(N)}$ and $\psi^{(N)}$ respectively, and define the functions Λ_2 and Υ_2

by

$$\varphi_2^{(N)} = \Lambda_2 \prod_{r=1}^{N} \left(\frac{p_r + \zeta}{p_r - \zeta} \right)^{n_r}, \qquad \psi_2^{(N)} = \Upsilon_2 \prod_{r=1}^{N} \left(\frac{p_r - \zeta}{p_r + \zeta} \right)^{n_r}.$$

Then Λ_2 is given by

$$(\zeta^2 - b^2)\,\Lambda_2(\zeta) = (p_k + \zeta)\,\mathcal{U}\,\widehat{\Lambda}(\zeta) - (P_k\widehat{u} - (p_k^2 - b^2)\,u\,)\,\Lambda(\zeta), \tag{4.70}$$

and Υ_2 is given by

$$(\zeta^2 - b^2)\,\Upsilon_2(\zeta) = (p_k - \zeta)\,\mathcal{U}\,\widehat{\Upsilon}(\zeta) - (P_k\widehat{u} - (p_k^2 - b^2)\,u\,)\,\Upsilon(\zeta). \tag{4.71}$$

We see that $\Lambda_2(\zeta)$ and $\Upsilon_2(\zeta)$ are analytic in $\mathcal{R}^+$, except for a simple pole at $\zeta = b$. Thus $\mathring{\Lambda}_2(\zeta) = \Lambda_2(-\zeta)$ and $\mathring{\Upsilon}_2(\zeta) = \Upsilon_2(-\zeta)$, which are the second components for the Jost solutions $\mathring{\Lambda}$ and $\mathring{\Upsilon}$ respectively, are analytic in $\mathcal{R}^-$ except for a simple pole at $\zeta = -b$. Furthermore all of these functions are continuous on the imaginary ζ-axis. Now since the two eigenfunctions

$$\boldsymbol{\varphi}^{(N)} := \begin{pmatrix} \varphi^{(N)} \\ \varphi_2^{(N)} \end{pmatrix}, \qquad \mathring{\boldsymbol{\varphi}}^{(N)} := \begin{pmatrix} \mathring{\varphi}^{(N)} \\ \mathring{\varphi}_2^{(N)} \end{pmatrix},$$

are linearly independent solutions of the Lax equations (4.6), we may write

$$\boldsymbol{\psi}^{(N)} := \begin{pmatrix} \psi^{(N)} \\ \psi_2^{(N)} \end{pmatrix} = \mathtt{A}(\zeta)\,\mathring{\boldsymbol{\varphi}}^{(N)} + \mathtt{B}(\zeta)\,\boldsymbol{\varphi}^{(N)},$$

whose second component may be written in terms of Λ_2 and Υ_2 as

$$\frac{\Upsilon_2(\zeta)}{\mathtt{A}(\zeta)} - \mathring{\Lambda}_2(\zeta) = R(\zeta)\,\Lambda_2(\zeta)\,\rho(\zeta). \tag{4.72}$$

Note that these are the same functions $\mathtt{A}$ and $\mathtt{B}$ as those that appear in the integral equation (4.67) for $\Lambda(\zeta)$. As in the inverse problem for ϕ_1, equation (4.72) becomes the jump condition for a Riemann-Hilbert problem, and as

such we look at the singular integral

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{R(\sigma)\Lambda_2(\sigma)}{\sigma+\zeta}\rho(\sigma)\,d\sigma\\ =&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Upsilon_2(\sigma)}{\mathtt{A}(\sigma)(\sigma+\zeta)}\,d\sigma-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\mathring{\Lambda}_2(\sigma)}{\sigma+\zeta}\,d\sigma.\end{aligned}\tag{4.73}$$

The difference in this case is that Υ_2 and $\mathring{\Lambda}_2$ have simple poles at $+b$ and $-b$ respectively, and both of these function are $\mathcal{O}(\,\zeta^{-1}\,)$ as $|\zeta|\to\infty$. Let us first consider the integral involving Υ_2. By using the residue theorem one has

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\Upsilon_2(\sigma)}{\mathtt{A}(\sigma)(\sigma+\zeta)}\,d\sigma=-\frac{\mathrm{Res}_{\zeta=b}\big[\Upsilon_2\big]}{\mathtt{A}(b)(\zeta+b)}-\sum_{k=1}^{M}\frac{\mathtt{c}_k\Lambda_2(\zeta_k)}{(\zeta+\zeta_k)}\rho(\zeta_k).\tag{4.74}$$

From the Lax equations however, by eliminating the first component ϕ_1 one can show that at $\zeta=b$ the second-order linear equation for ϕ_2 in the n_k-direction drastically simplifies to

$$(p_k^2-a^2)^{\frac{1}{2}}\,\widehat{\widehat{\phi}}_2-\left(\frac{P_k(\widehat{\widehat{u}}-u)}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_2+(p_k^2-a^2)^{\frac{1}{2}}\,\phi_2=0,\tag{4.75}$$

which we identify as equation (4.55) at $\zeta=a$. Since this holds for every lattice direction we may write

$$(\zeta-b)\psi_2^{(N)}(\zeta)\big|_{\zeta=b}=\alpha\,\varphi^{(N)}(a)+\beta\,\psi^{(N)}(a)$$

for some constants α and β. In terms of Υ and Λ this implies that

$$\begin{aligned}(\zeta-b)\Upsilon_2(\zeta)\big|_{\zeta=b}=&\alpha\,\Lambda(a)\prod_{r=1}^{N}\left(\frac{(p_r+a)(p_r+b)}{(p_r-a)(p_r-b)}\right)^{\frac{1}{2}n_r}\\ &+\beta\,\Upsilon(a)\prod_{r=1}^{N}\left(\frac{(p_r-a)(p_r+b)}{(p_r+a)(p_r-b)}\right)^{\frac{1}{2}n_r}\\ =&\alpha\,\Lambda(a)\,\mathcal{F}(a,b)+\beta\,\Upsilon(a)\,\mathcal{F}(-a,b),\end{aligned}\tag{4.76}$$

where the plane-wave factors $\mathcal{F}$ are defined by (4.18). In order to determine α and β we consider equation (4.76) in the limit $i\to\pm\infty$. Firstly we have

the asymptotic behaviour

$$\Lambda(a) \sim \mathtt{A}(a) + c_o\rho(-a) \ \text{ as } \ i \to +\infty$$
$$\Upsilon(a) \sim \mathtt{A}(a) + c_1\rho(a) \ \text{ as } \ i \to -\infty,$$

for some constants c_o and c_1. Since $a > 0$ both of these plane-wave factors are exponentially small. Thus using the boundary conditions of u and $\mathcal{U}$ in (4.71), as $i \to -\infty$ we have

$$\begin{aligned}(\zeta - b)\Upsilon_2(\zeta)\big|_{\zeta=b} &\sim \frac{\mathtt{A}(b)}{2b}\Big[(p_k - b)\,\mathcal{U} - (P_k\widehat{u} - (p_k^2 - b^2)\,u)\Big] \\ &\sim (a-b)\mathtt{A}(b)\mathcal{CF}(-a,b),\end{aligned}$$

which implies that $\beta\mathtt{A}(a) = (a-b)\,\mathcal{C}\mathtt{A}(b)$. By then taking $i \to +\infty$ we have

$$\begin{aligned}(\zeta - b)\Upsilon_2(\zeta)\big|_{\zeta=b} &\sim \frac{1}{2b}\Big[(p_k - b)\,\mathcal{U} - (P_k\widehat{u} - (p_k^2 - b^2)\,u)\Big] \\ &\sim -(a+b)\,\mathcal{K}_o\mathcal{AF}(a,b),\end{aligned}$$

which gives $\alpha\mathtt{A}(a) = -(a+b)\,\mathcal{K}_o\mathcal{A}$. Therefore the integral in equation (4.74) becomes

$$\begin{aligned}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{\Upsilon_2(\sigma)}{\mathtt{A}(\sigma)(\sigma+\zeta)}\,d\sigma &= (a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\left(\frac{\Lambda(a)}{(\zeta+b)}\right) \\ &\quad - (a-b)\mathcal{CF}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)(\zeta+b)}\right) - \sum_{k=1}^{M}\frac{\mathtt{c}_k\Lambda_2(\zeta_k)}{(\zeta+\zeta_k)}\rho(\zeta_k).\end{aligned}$$

The integral involving $\mathring{\Lambda}$ in (4.73) can be evaluated to be

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\mathring{\Lambda}_2(\sigma)}{\sigma+\zeta}\,d\sigma = \Lambda_2(\zeta) + \frac{1}{\zeta - b}\lim_{\sigma\to -b}\Big[\mathring{\Lambda}_2(\sigma)(\sigma+b)\Big], \tag{4.77}$$

and by similar reasoning and using the fact that $\mathring{\Lambda}(-a) = \Lambda(a)$ etc., we find

$$\begin{aligned}\lim_{\sigma\to -b}\Big[\mathring{\Lambda}_2(\sigma)(\sigma+b)\Big] &= -\,(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\Lambda(a) \\ &\quad + (a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right).\end{aligned}$$

Thus the integral (4.77) becomes

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{\mathring{\Lambda}_2(\sigma)}{\sigma+\zeta}\,d\sigma = \Lambda_2(\zeta) - (a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\left(\frac{\Lambda(a)}{\zeta-b}\right) + (a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)(\zeta-b)}\right).$$

Using these in (4.73) then gives the following closed-form singular integral equation for Λ_2:

$$\begin{aligned}\Lambda_2(\zeta) =& (a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\left(\frac{\Lambda(a)}{\zeta+b}\right)\\ &+(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\left(\frac{\Lambda(a)}{\zeta-b}\right)\\ &-(a-b)\mathcal{CF}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)(\zeta+b)}\right)\\ &-(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)(\zeta-b)}\right)\\ &-\sum_{k=1}^{M}\left(\frac{\mathtt{c}_k\Lambda_2(\zeta_k)}{(\zeta+\zeta_k)}\right)\rho(\zeta_k) - \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left(\frac{R(\sigma)\Lambda_2(\sigma)}{\sigma+\zeta}\right)\rho(\sigma)\,d\sigma. \quad (4.78)\end{aligned}$$

A natural ansatz for this is

$$\begin{aligned}\Lambda_2(\zeta) =& (a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\Lambda(a)\xi_b(\zeta)\\ &+(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\Lambda(a)\xi_{-b}(\zeta)\\ &-(a-b)\mathcal{CF}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)\xi_b(\zeta)\\ &-(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)\xi_{-b}(\zeta), \quad (4.79)\end{aligned}$$

where by equation (4.78) the functions $\xi_{\pm b}(\zeta)$ are calculated by solving

$$\xi_{\pm b}(\zeta) = \frac{1}{\zeta\pm b} - \sum_{k=1}^{M}\left(\frac{\mathtt{c}_k\,\xi_{\pm b}(\zeta_k)}{\zeta+\zeta_k}\right)\rho(\zeta_k) - \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left(\frac{R(\sigma)\,\xi_{\pm b}(\sigma)}{\sigma+\zeta}\right)\rho(\sigma)\,d\sigma. \quad (4.80)$$

Note that the ingredients in these equations are the scattering data from the forward scattering of ϕ_1, and (4.80) differs from (4.67) only in the source term.

Remark 4.6.1.

If we define the quantity

$$S(a,b) := \frac{1}{a+b} - \xi_b(a), \tag{4.81}$$

then by comparing (4.80) *with the integral equation considered in the direct linearization approach* **[67]** *we see that this object is in fact a solution (containing solitons and radiation) of the NQC equation* (1.7) *with* $\alpha \to a$, $\beta \to b$.

4.7 Reconstruction of the Solution of Q3$_\delta$

We now show how one can recontruct the solution u as a function of all N lattice variables.

4.7.1 Solution in terms of ϕ_2

We consider how to construct u and $\mathcal{U}$ from the knowledge of the second Lax component $\Lambda_2(\zeta)$, obtained by solving the integral equation (4.80). Take equation (4.70), which holds for any $n_k,\ k = 1, ..., N$. By dividing through by $(p_k + \zeta)$ and taking the limit $|\zeta| \to \infty$ we have

$$\mathcal{U} = \lim_{|\zeta|\to\infty} \big[\zeta \Lambda_2(\zeta)\big]. \tag{4.82}$$

Using equations (4.79), (4.80) and (4.82) this may be expressed as

$$\begin{aligned}
\mathcal{U} =& (a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{A}\mathcal{F}(a,b)\Lambda(a)V(b) \\
&+(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{B}\mathcal{F}(a,-b)\Lambda(a)V(-b) \\
&-(a-b)\mathcal{C}\mathcal{F}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)V(b) \\
&-(a+b)\mathcal{D}\mathcal{F}(-a,-b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)V(-b),
\end{aligned} \tag{4.83}$$

where $V(\pm b)$ is given by

$$V(\pm b) = 1 - \sum_{k=1}^{M} \mathsf{c}_k \, \xi_{\pm b}(\zeta_k)\rho(\zeta_k) - \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} R(\sigma)\, \xi_{\pm b}(\sigma)\rho(\sigma)\, d\sigma. \quad (4.84)$$

Equation (4.70) could then in principle be summed to find u. There is a way however to obtain a closed-form expression for u rather than its derivative. To do this we first note that in the Lax equations (4.6), one is free to interchange the roles of the parameters a and b. In other words the N equations

$$(p_k^2 - \zeta^2)^{\frac{1}{2}}\widehat{\boldsymbol{\phi}} = \frac{1}{\mathcal{U}} \begin{pmatrix} P_k\widehat{u} - (p_k^2 - a^2)u & \zeta^2 - a^2 \\ \mathcal{U}\widehat{\mathcal{U}} - \frac{\delta^2(p_k^2 - a^2)}{4P_k(\zeta^2 - a^2)} & (p_k^2 - a^2)\widehat{u} - P_k u \end{pmatrix} \boldsymbol{\phi} \quad (4.85)$$

are also N Lax equations for Q3$_\delta$. Note however that we are *not* swapping a and b in the functions u and $\mathcal{U}$, nor in our initial condition nor boundary conditions. We are simply repeating the entire IST with the new Lax equations (4.85) in place of (4.6), which is permissible due to the symmetric dependence of Q3$_\delta$ on a and b. There are however some important remarks to be made. Firstly the forward scattering problem (4.22) for ϕ_1 is independent of a and b, and thus *interchanging a and b in the Lax equations will not change the Jost solutions Λ and Υ, nor the scattering data*. The second component ϕ_2 however will now be calculated by swapping a and b in the Lax equations (4.70) and (4.71), and re-deriving the integral equation (4.80). To make a clear distinction between the original quantities Λ_2, Υ_2 and their new counterparts, we rewrite the original functions as

$$\Lambda_2 \to \Lambda_2^{(b)}, \qquad \Upsilon_2 \to \Upsilon_2^{(b)},$$

and then denote the new functions, which are obtained by swapping a and b in (4.70) and (4.71), by $\Lambda_2^{(a)}$ and $\Upsilon_2^{(a)}$. By repeating the analysis of the

previous section we find that

$$\begin{aligned}\Lambda_2^{(a)}(\zeta) =& (a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\Lambda(b)\xi_a(\zeta)\\ &+(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right)\xi_a(\zeta)\\ &-(a-b)\mathcal{CF}(-a,b)\Lambda(b)\xi_{-a}(\zeta)\\ &-(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right)\xi_{-a}(\zeta). \end{aligned} \tag{4.86}$$

Then finally given the quantities $\Lambda_2^{(b)}$ and $\Lambda_2^{(a)}$ one can combine the two versions of (4.70), namely

$$\begin{aligned}(\zeta^2-b^2)\,\Lambda_2^{(b)}(\zeta) &= (p_k+\zeta)\,\mathcal{U}\,\widehat{\Lambda}(\zeta) - (P_k\widehat{u} - (p_k^2-b^2)\,u\,)\,\Lambda(\zeta)\\ (\zeta^2-a^2)\,\Lambda_2^{(a)}(\zeta) &= (p_k+\zeta)\,\mathcal{U}\,\widehat{\Lambda}(\zeta) - (P_k\widehat{u} - (p_k^2-a^2)\,u\,)\,\Lambda(\zeta)\end{aligned}$$

to express the N-dimensional solution of Q3$_\delta$ as

$$u = \frac{(\zeta^2-b^2)\,\Lambda_2^{(b)}(\zeta) - (\zeta^2-a^2)\,\Lambda_2^{(a)}(\zeta)}{(a^2-b^2)\,\Lambda(\zeta)}. \tag{4.87}$$

Since however the solution is independent of ζ, we may take it to be large, in which case the solution can be expressed as

$$u = \lim_{|\zeta|\to\infty}\left(\frac{\zeta^2\big[\Lambda_2^{(b)}(\zeta) - \Lambda_2^{(a)}(\zeta)\big]}{a^2-b^2}\right). \tag{4.88}$$

Remark 4.7.1.

For the reflectionless case $\mathtt{B}\equiv 0$ *the Jost solutions exist at the points* $\zeta=-a$ *and* $\zeta=-b$*, and we have* $\mathcal{K}_o=\mathtt{A}(a)\mathtt{A}(b)$ *and* $\mathcal{K}_1=\mathtt{A}(a)/\mathtt{A}(b)$*. Comparing the expressions for* u *obtained by setting* $\zeta=\pm a$ *and* $\pm b$ *in* (4.87)*, and using the fact that* $\Upsilon(\zeta)=\mathtt{A}(\zeta)\Lambda(-\zeta)$ *we find*

$$u = (a+b)\mathcal{AF}(a,b)\xi_a(b) + (a-b)\mathcal{BF}(a,-b)\xi_a(-b) \tag{4.89}$$

$$-(a-b)\mathcal{CF}(-a,b)\xi_{-a}(b) - (a+b)\mathcal{DF}(-a,-b)\xi_{-a}(-b). \tag{4.90}$$

Using the NQC variable $S(a,b)$ defined by (4.81) *this becomes*

$$\begin{aligned} u =& \mathcal{A}\mathcal{F}(a,b)\big[1-(a+b)S(a,b)\big] + \mathcal{B}\mathcal{F}(a,-b)\big[1-(a-b)S(a,-b)\big] \\ &+ \mathcal{C}\mathcal{F}(-a,b)\big[1+(a-b)S(-a,b)\big] + \mathcal{D}\mathcal{F}(-a,-b)\big[1+(a+b)S(-a,-b)\big]. \end{aligned} \tag{4.91}$$

This is precisely the form of the N-soliton solution of Q3$_\delta$ obtained in **[65]**. *The corresponding dual function $\mathcal{U}$ is given by*

$$\begin{aligned} \mathcal{U} =& (a+b)\mathcal{A}\mathcal{F}(a,b)\Lambda(a)\Lambda(b) + (a-b)\mathcal{B}\mathcal{F}(a,-b)\Lambda(a)\Lambda(-b) \\ &- (a-b)\mathcal{C}\mathcal{F}(-a,b)\Lambda(-a)\Lambda(b) - (a+b)\mathcal{D}\mathcal{F}(-a,-b)\Lambda(-a)\Lambda(-b), \end{aligned} \tag{4.92}$$

which was also obtained in **[65]**.

4.7.2 Solution in terms of ϕ_1

Remarkaby there is another way of expressing the quantities u and $\mathcal{U}$ in terms of *only* the quantities $\Lambda(a), \Lambda(b), \Upsilon(a)$ and $\Upsilon(b)$, i.e. just in terms of the first component of the Lax equations. This involves considering the second-order linear equation for ϕ_2 at specific values of ζ. From Section 4.6, by considering the second-order equation for ϕ_2 at the point $\zeta = b$ and using the boundary conditions for u and $\mathcal{U}$ we found that

$$\begin{aligned} (\zeta-b)\Upsilon_2^{(b)}(\zeta)\big|_{\zeta=b} =& -(a+b)\mathtt{A}(b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{A}\mathcal{F}(a,b)\Lambda(a) \\ &+ (a-b)\mathtt{A}(b)\mathcal{C}\mathcal{F}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right). \end{aligned}$$

Thus from the Lax equation (4.71) we have

$$\begin{aligned} &(p_k-b)\,\mathcal{U}\left(\frac{\widehat{\Upsilon}(b)}{\mathtt{A}(b)}\right) - (P_k\widehat{u} - (p_k^2-b^2)\,u\,)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right) \\ =& -2b(a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{A}\mathcal{F}(a,b)\Lambda(a) + 2b(a-b)\mathcal{C}\mathcal{F}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right). \end{aligned} \tag{4.93}$$

In a similar manner one can show that

$$\begin{aligned}(\zeta-b)\Lambda_2^{(b)}(\zeta)\big|_{\zeta=b} =&(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\Lambda(a)\\ &-(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right),\end{aligned}$$

and thus we also have

$$\begin{aligned}&(p_k+b)\mathcal{U}\widehat{\Lambda}(b)-(P_k\widehat{u}-(p_k^2-b^2)\,u\,)\Lambda(b)\\ =&2b(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\Lambda(a)-2b(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right).\end{aligned} \tag{4.94}$$

By combining equations (4.93) and (4.94) we may eliminate u, and using the Wronksian identity for $\mathtt{A}$ we have

$$\begin{aligned}\mathcal{U}=&(a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\Lambda(a)\Lambda(b)\\ &+(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\Lambda(a)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right)\\ &-(a-b)\mathcal{CF}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)\Lambda(b)\\ &-(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)\Upsilon(b)}{\mathtt{A}(a)\mathtt{A}(b)}\right).\end{aligned} \tag{4.95}$$

Remark 4.7.2.

By comparing (4.95) *with* (4.83) *we find that*

$$V(b)=\Lambda(b),\qquad V(-b)=\frac{\Upsilon(b)}{\mathtt{A}(b)}. \tag{4.96}$$

As will be shown by equation (4.174) *the function $V(b)$ therefore satisfies the lattice potential modified KdV equation* (3.31) *with slightly generalised coefficients,*

$$(p-b)V\widehat{V}-(p+b)\widetilde{V}\widehat{\widetilde{V}}=(q-b)V\widetilde{V}-(q+b)\widehat{V}\widehat{\widetilde{V}}. \tag{4.97}$$

The solution to this equation including radiation is given by (4.84).

If instead we use equations (4.93) and (4.94) to eliminate $\mathcal{U}$ we have

$$\begin{aligned} P_k\widehat{u} - (p_k^2 - b^2)u &= (p+b)(a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\Lambda(a)\widehat{\Lambda}(b) \\ &\quad + (p-b)(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\Lambda(a)\left(\frac{\widehat{\Upsilon}(b)}{\mathtt{A}(b)}\right) \\ &\quad - (p+b)(a-b)\mathcal{CF}(-a,b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)\widehat{\Lambda}(b) \\ &\quad - (p-b)(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(a)\widehat{\Upsilon}(b)}{\mathtt{A}(a)\mathtt{A}(b)}\right). \end{aligned} \tag{4.98}$$

Now we repeat this process with $\Lambda_2^{(a)}$ and $\Upsilon_2^{(a)}$ rather than $\Lambda_2^{(b)}$ and $\Upsilon_2^{(b)}$. This gives the additional two equations

$$\begin{aligned} &(p_k - a)\,\mathcal{U}\left(\frac{\widehat{\Upsilon}(a)}{\mathtt{A}(a)}\right) - (P_k\widehat{u} - (p_k^2 - a^2)\,u\,)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right) \\ &\qquad = -2a(a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\Lambda(b) \\ &\qquad\quad - 2b(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right) \end{aligned} \tag{4.99}$$

$$\begin{aligned} &(p_k + a)\mathcal{U}\widehat{\Lambda}(a) - (P_k\widehat{u} - (p_k^2 - a^2)\,u\,)\Lambda(a) \\ &= -2a(a-b)\mathcal{CF}(-a,b)\Lambda(b) - 2a(a+b)\mathcal{DF}(-a,-b)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right), \end{aligned} \tag{4.100}$$

which can be combined to give

$$\begin{aligned} P_k\widehat{u} - (p_k^2 - a^2)u &= (p+a)(a+b)\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\widehat{\Lambda}(a)\Lambda(b) \\ &\quad + (p+a)(a-b)\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\widehat{\Lambda}(a)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right) \\ &\quad - (p-a)(a-b)\mathcal{CF}(-a,b)\left(\frac{\widehat{\Upsilon}(a)}{\mathtt{A}(a)}\right)\Lambda(b) \\ &\quad - (p-a)(a+b)\mathcal{DF}(-a,-b)\left(\frac{\widehat{\Upsilon}(a)\Upsilon(b)}{\mathtt{A}(a)\mathtt{A}(b)}\right). \end{aligned} \tag{4.101}$$

Equations (4.98) and (4.101) can then be combined to give the solution u of Q3$_\delta$ as

$$
\begin{aligned}
u = &\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)\mathtt{A}(b)}\right)\mathcal{AF}(a,b)\left(\frac{(p_k+a)\widehat{\Lambda}(a)\Lambda(b)-(p_k+b)\Lambda(a)\widehat{\Lambda}(b)}{(a-b)}\right)\\
&+\left(\frac{\mathcal{K}_1\mathtt{A}(b)}{\mathtt{A}(a)}\right)\mathcal{BF}(a,-b)\left(\frac{(p_k+a)\widehat{\Lambda}(a)\left(\frac{\Upsilon(b)}{\mathtt{A}(b)}\right)-(p_k-b)\Lambda(a)\left(\frac{\widehat{\Upsilon}(b)}{\mathtt{A}(b)}\right)}{(a+b)}\right)\\
&+\mathcal{CF}(-a,b)\left(\frac{(p_k+b)\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)\widehat{\Lambda}(b)-(p_k-a)\left(\frac{\widehat{\Upsilon}(a)}{\mathtt{A}(a)}\right)\Lambda(b)}{(a+b)}\right)\\
&+\mathcal{DF}(-a,-b)\left(\frac{(p_k-b)\left(\frac{\Upsilon(a)\widehat{\Upsilon}(b)}{\mathtt{A}(a)\mathtt{A}(b)}\right)-(p_k-a)\left(\frac{\widehat{\Upsilon}(a)\Upsilon(b)}{\mathtt{A}(a)\mathtt{A}(b)}\right)}{(a-b)}\right). \qquad (4.102)
\end{aligned}
$$

Equation (4.102) is an explicit respresentation of the solution u of Q3$_\delta$ purely in terms of ϕ_1. The shift $\frown$ involved (with corresponding parameter p_k) can be made in any one of the N lattice directions.

Remark 4.7.3.

The fact that equation (4.102) *holds for shifts in any one of the N lattice directions implies that the quantities* $\Lambda(a)$ *and* $\Lambda(b)$ *satisfy the nonlinear relation*

$$(p+a)\widetilde{\Lambda}(a)\Lambda(b)-(p+b)\Lambda(a)\widetilde{\Lambda}(b)=(q+a)\widehat{\Lambda}(a)\Lambda(b)-(q+b)\Lambda(a)\widehat{\Lambda}(b) \quad (4.103)$$

in the (n,m)*-plane. Of course this may be extended to any pair of lattice directions within the N-dimensional lattice.*

4.8 Discrete Inverse Scattering Transform for the Remaining ABS Equations

We now give an IST for all lower[3] ABS equations, namely Q2, Q1$_\delta$, H3$_\delta$, H2 and H1. To do so we use the Lax pair for each equation obtained from the multidimensional consistency, which was discussed in Chapter 3, and impose the boundary conditions exhibited by the soliton solutions, which are given in [**65**]. By considering the two eigenfunctions separately we will derive an explicit expression for the solution u (or its derivative) of each equation explicitly.

The method for obtaining a Lax pair for 3D consistent equations was explained in Section 3.4 of Chapter 3. We now revisit this concept. If we consider any of the aforementioned equations on an elementary lattice quadrilateral in a plane spanned by the n_k-direction (with parameter p_k) with iterations denoted by $u \to \widehat{u}$, and some other lattice direction with parameter ζ and iterations denoted by $u \to \dot{u}$, then we may express the equation as

$$a_o + a_1\dot{u} + a_2\widehat{\dot{u}} + a_3\dot{u}\widehat{\dot{u}} = 0, \tag{4.104}$$

where the four coefficients $a_o \to a_3$ depend on p_k, ζ, u and $\widehat{u}$ only. A Lax equation for equation (4.104) is then

$$\alpha\widehat{\boldsymbol{\phi}} = \frac{1}{\mathcal{U}}\begin{pmatrix} -a_2 & -a_3 \\ a_0 & a_1 \end{pmatrix}\boldsymbol{\phi}, \tag{4.105}$$

where for convenience we have redefined $\boldsymbol{\phi} \to \boldsymbol{\phi}(-\alpha)^{n_k}$ from equation (3.41). Here the function $\mathcal{U}$ is defined through the first-order equation

$$\mathcal{U}\widehat{\mathcal{U}} = \mathcal{H}(u, \widehat{u}; p_k)$$

[3]The two A-type equations are related by straightforward gauge transformations to Q1$_\delta$ and Q3$_o$ and are therefore not treated as distinct examples

where we choose to set the arbitrary constant appearing in the solution to be unity, and α is chosen such that

$$\det\left[\frac{1}{\alpha\mathcal{U}}\begin{pmatrix} -a_2 & -a_3 \\ a_0 & a_1 \end{pmatrix}\right] = \frac{\widehat{\mathcal{U}}}{\mathcal{U}}.$$

Since however by equation (3.43) we have

$$\det\begin{pmatrix} -a_2 & -a_3 \\ a_0 & a_1 \end{pmatrix} = \mathtt{K}(p_k,\zeta)\mathcal{H}(u,\widehat{u};p_k)$$

where $\mathtt{K}$ is antisymmetric, it follows that $\alpha = \sqrt{\mathtt{K}(p_k,\zeta)}$. Using the fact that for all equations considered here we have $a_3 = const.$, the linear equation which governs the evolution of the first component ϕ_1 in the n_k-direction is

$$\sqrt{\mathtt{K}(p_k,\zeta)}\,\widehat{\widehat{\phi}}_1 - \left(\frac{a_1 - \widehat{a}_2}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_1 + \sqrt{\mathtt{K}(p_k,\zeta)}\,\phi_1 = 0. \tag{4.106}$$

In comparison to the evolution of ϕ_1 for Q3$_\delta$, this equation can easily be generalised to define evolution along the staircase Γ, and importantly, for each of the equations Q2→H1, if we impose the same boundary conditions (at either end of the staircase, i.e. $i \to \pm\infty$) as exhibited by the known soliton solutions, equation (4.106) gives rise to *exactly the same forward scattering problem as for* Q3$_\delta$. In other words the forward scattering problem for ϕ_1 for every equation Q3$_\delta$ →H1 is *identical*. This is an extremely useful result, as all of the results from the forward scattering of Q3$_\delta$ hold for these remaining equations.

The differences in the IST for these equations however arises when we consider the second eigenfunction ϕ_2. By the Lax equation (4.105) the evolution of ϕ_2 in the n_k-direction is

$$\sqrt{\mathtt{K}(p_k,\zeta)}\,\widehat{\widehat{\phi}}_2 - \left(\frac{\widehat{a}_1 - \left(\frac{\widehat{a}_o}{a_o}\right)a_2}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_1 + \sqrt{\mathtt{K}(p_k,\zeta)}\left(\frac{\widehat{a}_o}{a_o}\right)\phi_1 = 0, \tag{4.107}$$

which due to the fact that a_o depends on n_k is a much more unwieldy equation to deal with. It does however contain much more fine structure than equation (4.106), and it is therefore worthy of attention. To do this we consider the *singular points* of the equation, which we define to be the values of ζ on the real positive axis for which $|a_o| \to \infty$. This sends the ratio $\frac{\widehat{a_o}}{a_o} \to 1$ which dramatically reduces the complexity of equation (4.107), for which the solution can be expressed in terms of known functions Λ or Υ, or perhaps even simpler exponential functions. Crucially however, at these singular points we retain just enough information to be used in conjuction with ϕ_1 such that u may be determined from the Lax equation

$$a_3\phi_2 = -a_2\phi_1 - \sqrt{\mathtt{K}(p_k,\zeta)}\,\mathcal{U}\,\widehat{\phi}_1. \tag{4.108}$$

This equation holds for every lattice direction, which enables one to determine u and $\mathcal{U}$ as functions of all N lattice variables and parameters.

We also make use of the Wronskian identity for $\mathtt{A}(a)$:

$$(p_k+\zeta)\widehat{\Lambda}(\zeta)\Upsilon(\zeta) - (p_k-\zeta)\Lambda(\zeta)\widehat{\Upsilon}(\zeta) = 2\zeta\mathtt{A}(\zeta), \tag{4.109}$$

which holds whenever $\zeta \in \mathcal{R}^+$. This implies that

$$\Lambda(\zeta) \sim \mathtt{A}(\zeta) + c_o\rho(-\zeta) \ \text{ as } \ i \to +\infty$$
$$\Upsilon(\zeta) \sim \mathtt{A}(\zeta) + c_1\rho(\zeta) \ \text{ as } \ i \to -\infty,$$

for some constants c_o and c_1. Thus for example at $\zeta = a > 0$ we have

$$\Lambda(a) \sim \mathtt{A}(a) \ \text{ as } \ i \to +\infty, \qquad \Upsilon(a) \sim \mathtt{A}(a) \ \text{ as } \ i \to -\infty.$$

Finally we make use of the Wronskian identities resulting from eliminating the potential terms in the second-order equations for the pairs (Λ, Λ'), (Λ', Υ), (Λ, Υ') and (Υ, Υ'). For example eliminating the potential term in

the equations for $\Lambda(a)$ and $\Lambda'(a)$ yields the identity

$$(p_k+a)(\widehat{\widetilde{\Lambda}}'(a)\widehat{\Lambda}(a)-\widehat{\widetilde{\Lambda}}(a)\widehat{\Lambda}'(a))-(p_k-a)(\widehat{\Lambda}'(a)\Lambda(a)-\widehat{\Lambda}(a)\Lambda'(a))$$
$$+\widehat{\Lambda}(a)(\widehat{\widetilde{\Lambda}}(a)-\Lambda(a))=0.$$

This is the entirety of the machinery used to obtain solutions to each of the equations Q2→H1 through a discrete IST. For each equation we give the elements of the Lax matrix $a_0 \to a_3$, we specify the biquadratic $\mathcal{H}$ and the function $\mathtt{K}(p_k;\zeta)$, and state the boundary conditions that we assume on the solution u (from which one obtains the boundary conditions for $\mathcal{U}$) at either end of the staircase Γ. As we will see, some of the constants specified in the boundary conditions (though which do not appear in the IST for ϕ_1) are free, while others must be set to specific values determined by the IST. By following the above procedure we give an explicit expression for u and $\mathcal{U}$ for each equation.

4.8.1 Q2

The Q2 equation is

$$\mathring{p}(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}})-\mathring{q}(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}})+\mathring{p}\mathring{q}(\mathring{p}-\mathring{q})(u+\widetilde{u}+\widehat{u}+\widehat{\widetilde{u}})$$
$$-\mathring{p}\mathring{q}(\mathring{p}-\mathring{q})(\mathring{p}^2-\mathring{p}\mathring{q}+\mathring{q}^2)=0, \tag{4.110}$$

where $\mathring{p}=\frac{a^2}{p^2-a^2}$ and $\mathring{q}=\frac{a^2}{q^2-a^2}$, and by letting $\mathring{\zeta}=\frac{a^2}{\zeta^2-a^2}$ we have

$$\mathcal{H}(u,\widehat{u};p_k)=\frac{1}{\mathring{p}_k}\Big((u-\widehat{u})^2-2\mathring{p}_k^2(u+\widehat{u})+\mathring{p}_k^4\Big)$$
$$\sqrt{\mathtt{K}(p_k,\zeta)}=\left[\frac{\mathring{p}_k\mathring{\zeta}}{a}\right](p_k^2-\zeta^2)^{\frac{1}{2}}.$$

The elements of the Lax matrix are

$$a_0 = \mathring{p}_k u\widehat{u} + \mathring{p}_k\mathring{\zeta}(\mathring{p}_k - \mathring{\zeta})(u + \widehat{u}) - \mathring{p}_k\mathring{\zeta}(\mathring{p}_k - \mathring{\zeta})(\mathring{p}_k^2 - \mathring{p}_k\mathring{\zeta} + \mathring{\zeta}^2)$$

$$a_1 = -\mathring{p}_k\widehat{u} + \mathring{\zeta}\big(\widehat{u} - u + \mathring{p}_k^2\big) - \mathring{p}_k\mathring{\zeta}^2$$

$$a_2 = -\mathring{p}_k u + \mathring{\zeta}\big(u - \widehat{u} + \mathring{p}_k^2\big) - \mathring{p}_k\mathring{\zeta}^2$$

$$a_3 = \mathring{p}_k$$

and thus the Lax equations for ϕ_1, Λ_2 and Υ_2 are

$$(p_k^2 - \zeta^2)^{\frac{1}{2}}\widehat{\widehat{\phi}}_1(\zeta) - \frac{a}{\mathring{p}_k}\left(\frac{\widehat{\widehat{u}} - u}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_1(\zeta) + (p_k^2 - \zeta^2)^{\frac{1}{2}}\phi_1(\zeta) = 0 \tag{4.111}$$

$$\begin{aligned}\Lambda_2(\zeta) = &\left[u + \left(\frac{p_k^2 - a^2}{\zeta^2 - a^2}\right)(\widehat{u} - u - \mathring{p}_k^2) + \left(\frac{a^2}{\zeta^2 - a^2}\right)^2\right]\Lambda(\zeta) \\ &- a\left(\frac{p_k + \zeta}{\zeta^2 - a^2}\right)\mathcal{U}\widehat{\Lambda}(\zeta)\end{aligned} \tag{4.112}$$

$$\begin{aligned}\Upsilon_2(\zeta) = &\left[u + \left(\frac{p_k^2 - a^2}{\zeta^2 - a^2}\right)(\widehat{u} - u - \mathring{p}_k^2) + \left(\frac{a^2}{\zeta^2 - a^2}\right)^2\right]\Upsilon(\zeta) \\ &- a\left(\frac{p_k - \zeta}{\zeta^2 - a^2}\right)\mathcal{U}\widehat{\Upsilon}(\zeta),\end{aligned} \tag{4.113}$$

and the boundary conditions that we impose on u and $\mathcal{U}$ at either end of the staircase Γ are

$$u \sim (\mu + \mu_o)^2 + \frac{1}{4} + \mathcal{A}\mathcal{D} + \frac{1}{2}\mathcal{D}\rho(-a) \text{ as } i \to -\infty \tag{4.114}$$

$$u \sim (\mu + \mu_o)^2 - \mathcal{K}_1(\mu + \mu_o) + \mathcal{K}_2 + \frac{1}{4} + \mathcal{A}\mathcal{D} + \frac{1}{2}\mathcal{K}_o\mathcal{A}\rho(a) \text{ as } i \to +\infty, \tag{4.115}$$

which imply that $\mathcal{U}$ has the boundary behaviour

$$\mathcal{U} \sim -2(\mu + \mu_o) - \mathcal{D}\rho(-a) \text{ as } i \to -\infty$$

$$\mathcal{U} \sim -2(\mu + \mu_o) + \mathcal{K}_1 + \mathcal{K}_o\mathcal{A}\rho(a) \text{ as } i \to +\infty.$$

Here the function μ is defined by

$$\mu := \sum_{r=1}^{N} \left(\frac{ap_r}{a^2 - p_r^2} \right) n_r, \tag{4.116}$$

$\mathcal{K}_o, \mathcal{A}$ and $\mathcal{D}$ are free constants, while $\mathcal{K}_1$ and $\mathcal{K}_2$ are fixed. The singular point for this equation is $\zeta = a$, and in this limit $\zeta = a + \epsilon$ with $\epsilon \ll 1$ we have

$$\frac{\widehat{a}_o}{a_o} = 1 + \frac{4(u - \widehat{\widehat{u}})}{a^2}\epsilon^2 + O\big(\epsilon^3\big),$$

and thus

$$\frac{a}{\mathring{p}_k \mathring{\zeta}} \left[\widehat{a}_1 - \left(\frac{\widehat{a}_o}{a_o} \right) a_2 \right] = \frac{a}{\mathring{p}_k}(\widehat{\widehat{u}} - u) + 4(u - \widehat{\widehat{u}})\epsilon + O(\epsilon^2).$$

By writing

$$\phi_2 = \frac{\phi_2^{\epsilon^2}}{\epsilon^2} + \frac{\phi_2^{\epsilon}}{\epsilon} + O(1), \tag{4.117}$$

from equations (4.112) and (4.113) we firstly have

$$\Lambda_2^{\epsilon^2} = \frac{a^2}{4}\Lambda(a), \qquad \Upsilon_2^{\epsilon^2} = \frac{a^2}{4}\Upsilon(a). \tag{4.118}$$

If we now look at equation (4.107) for Λ_2, by taking the $O(\epsilon)$ terms we have

$$\begin{aligned}(p_k + a)\widehat{\widehat{\Lambda}}_2^{\epsilon^2}(a) - \frac{a}{\mathring{p}_k}\left(\frac{\widehat{\widehat{u}} - u}{\widehat{\mathfrak{U}}} \right) \widehat{\Lambda}_2^{\epsilon}(a) + (p_k - a)\Lambda_2^{\epsilon}(a) \\ = \frac{a^2}{4}\Lambda(a) + a^2 \left(\frac{u - \widehat{\widehat{u}}}{\widehat{\mathfrak{U}}} \right) \widehat{\Lambda}(a) - \frac{a^2}{4}\widehat{\widehat{\Lambda}}(a),\end{aligned}$$

but using the equation (4.111) for $\Lambda(a)$ this may be rewritten as

$$\begin{aligned}(p_k + a)\widehat{\widehat{\Lambda}}_2^{\epsilon^2}(a) - \frac{a}{\mathring{p}_k}\left(\frac{\widehat{\widehat{u}} - u}{\widehat{\mathfrak{U}}} \right) \widehat{\Lambda}_2^{\epsilon}(a) + (p_k - a)\Lambda_2^{\epsilon}(a) \\ = \frac{a^2}{4}\Lambda(a) - a\mathring{p}_k\Big((p_k + a)\widehat{\widehat{\Lambda}}(a) + (p_k - a)\Lambda(a) \Big) - \frac{a^2}{4}\widehat{\widehat{\Lambda}}(a).\end{aligned}$$

Since this is an inhomogeneous form of equation (4.111) for Λ we can write down the general solution

$$\Lambda_2^{\epsilon} = \alpha\Lambda(a) + \beta\Upsilon(a) \left(\frac{p_k - a}{p_k + a} \right)^{n_k} + \left(\frac{a^2 p_k n_k}{a^2 - p_k^2} \right) \Lambda(a) - \left(\frac{3a^2}{4} \right) \Lambda'(a), \tag{4.119}$$

where α and β are constants, and since this process may be carried out in every lattice direction we have

$$\Lambda_2^{\epsilon} = \alpha\Lambda(a) + \beta\Upsilon(a)\rho(-a) + a\mu\Lambda(a) - \left(\frac{3a^2}{4}\right)\Lambda'(a). \tag{4.120}$$

From equation (4.112) however we also have

$$\begin{aligned}\Lambda_2^{\epsilon} = &\left[\left(\frac{p_k^2 - a^2}{2a}\right)(\widehat{u} - u - \mathring{p}_k^2) - \frac{a}{4}\right]\Lambda(a) \\ &+ \left(\frac{a^2}{4}\right)\Lambda'(a) - \left(\frac{p_k + a}{2}\right)\mathcal{U}\widehat{\Lambda}(a),\end{aligned} \tag{4.121}$$

which will enable us to determine the constants α and β through using the known boundary conditions for u and $\mathcal{U}$. Firstly equating (4.120) and (4.121) as $i \to -\infty$ gives

$$\alpha + \beta c_1 + \beta\mathtt{A}(a)\rho(-a) + a\mu \sim a(\mu + \mu_o) + \frac{a}{4} + a\mathcal{D}\rho(-a)$$

which implies $\beta\mathtt{A}(a) = a\mathcal{D}$. Then by letting $i \to +\infty$ we have

$$\begin{aligned}&\alpha\mathtt{A}(a) + a\mu\mathtt{A}(a) - \frac{3a^2}{4}\mathtt{A}'(a) \\ &\sim \frac{a\mathtt{A}(a)}{4} + \frac{a^2\mathtt{A}'(a)}{4} + a(\mu + \mu_o)\mathtt{A}(a) - \frac{a\mathcal{K}_1\mathtt{A}(a)}{2}\end{aligned}$$

which gives

$$\alpha = \frac{a}{4} + \frac{a^2\mathtt{A}'(a)}{\mathtt{A}(a)} + a\mu_o - \frac{a\mathcal{K}_1}{2}. \tag{4.122}$$

Now combining (4.120) and (4.121) gives

$$\begin{aligned}&\left(\frac{p_k^2 - a^2}{2a}\right)(\widehat{u} - u - \mathring{p}_k^2)\Lambda(a) - \left(\frac{p_k + a}{2}\right)\mathcal{U}\widehat{\Lambda}(a) \\ &= \left(\frac{a}{2} + \frac{a^2\mathtt{A}'(a)}{\mathtt{A}(a)} + a(\mu + \mu_o) - \frac{a\mathcal{K}_1}{2}\right)\Lambda(a) - a^2\Lambda'(a) + a\mathcal{D}\Psi(a)\rho(-a).\end{aligned} \tag{4.123}$$

where for convenience we have defined

$$\Psi(\zeta) := \frac{\Upsilon(\zeta)}{\mathtt{A}(\zeta)}. \tag{4.124}$$

We now use equation (4.113) to perform the same task with Υ_2^{ς}. In terms of Ψ this yields the second equation

$$\begin{aligned}
&\left(\frac{p_k^2-a^2}{2a}\right)(\widehat{u}-u-\mathring{p}_k^2)\Psi(a)-\left(\frac{p_k-a}{2}\right)\mathcal{U}\widehat{\Psi}(a)\\
&=\left(\frac{a}{2}-a(\mu+\mu_o)\right)\Psi(a)-a^2\Psi'(a)\\
&\qquad+a\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\mathcal{A}\Lambda(a)\rho(a).
\end{aligned}\tag{4.125}$$

Equations (4.123) and (4.125) can then be combined to give an explicit expression for $\mathcal{U}$:

$$\begin{aligned}
\mathcal{U}=&\left(-2(\mu+\mu_o)+\frac{\mathcal{K}_1}{2}-\frac{a\mathtt{A}'(a)}{\mathtt{A}(a)}\right)\Lambda(a)\Psi(a)+a\big(\Lambda'(a)\Psi(a)-\Lambda(a)\Psi'(a)\big)\\
&+\mathcal{A}\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\Lambda(a)^2\rho(a)-\mathcal{D}\Psi(a)^2\rho(-a),
\end{aligned}\tag{4.126}$$

from which we obtain $\mathcal{K}_1=2a\left(\frac{\mathtt{A}'(a)}{\mathtt{A}(a)}\right)$ in order that the boundary conditions of $\mathcal{U}$ be satisfied. Inserting this gives

$$\begin{aligned}
\mathcal{U}=&-2(\mu+\mu_o)\Lambda(a)\Psi(a)+a\big(\Lambda'(a)\Psi(a)-\Lambda(a)\Psi'(a)\big)\\
&+\mathcal{A}\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\Lambda(a)^2\rho(a)-\mathcal{D}\Psi(a)^2\rho(-a).
\end{aligned}\tag{4.127}$$

Alternatively one can combine (4.123) and (4.125), as well as the value for $\mathcal{K}_1$, to give the explicit representation of the discrete derivative of the solution to Q2:

$$\begin{aligned}
\widehat{u}-u=&\frac{a^2p_k^2}{(p_k^2-a^2)^2}-\left(\frac{a(\mu+\mu_o)}{p_k^2-a^2}\right)\Big((p_k+a)\widehat{\Lambda}(a)\Psi(a)+(p_k-a)\Lambda(a)\widehat{\Psi}(a)\Big)\\
&-\left(\frac{a^2}{p_k^2-a^2}\right)\Big((p_k+a)\widehat{\Lambda}(a)\Psi'(a)-(p_k-a)\Lambda'(a)\widehat{\Psi}(a)\Big)\\
&+\frac{1}{2}\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\mathcal{A}\Lambda(a)\widehat{\Lambda}(a)\Big(\widehat{\rho}(a)-\rho(a)\Big)\\
&+\frac{1}{2}\mathcal{D}\Psi(a)\widehat{\Psi}(a)\Big(\widehat{\rho}(-a)-\rho(-a)\Big).
\end{aligned}\tag{4.128}$$

Using the various Wronskian identities mentioned at the start of this Section we can directly sum this equation to obtain

$$\begin{aligned} u =& (\mu+\mu_o)^2 - (\mu+\mu_o)\mathcal{G}(a) - \frac{a}{2}\mathcal{X}(a) + \frac{1}{4} + \mathcal{A}\mathcal{D} \\ &+ \frac{1}{2}\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\mathcal{A}\rho(a)\mathcal{Y}(a) + \frac{1}{2}\mathcal{D}\rho(-a)\mathcal{Z}(a), \end{aligned} \tag{4.129}$$

where

$$\mathcal{G}(a) := (p_k+a)\widehat{\Lambda}'(a)\Psi(a) - (p_k-a)\Lambda'(a)\widehat{\Psi}(a) + \widehat{\Lambda}(a)\Psi(a) - 1 \tag{4.130}$$

$$\mathcal{X}(a) := \Big((p_k+a)\widehat{\Lambda}'(a) + \widehat{\Lambda}(a)\Big)\Psi'(a) - \Big((p_k-a)\widehat{\Psi}'(a) - \widehat{\Psi}(a)\Big)\Lambda'(a) \tag{4.131}$$

$$\mathcal{Y}(a) := (p_k+a)\big[\widehat{\Lambda}'(a)\Lambda(a) - \widehat{\Lambda}(a)\Lambda'(a)\big] + \Lambda(a)\widehat{\Lambda}(a) \tag{4.132}$$

$$\mathcal{Z}(a) := (p_k-a)\big[\widehat{\Psi}'(a)\Psi(a) - \widehat{\Psi}(a)\Psi'(a)\big] + \Psi(a)\widehat{\Psi}(a). \tag{4.133}$$

From this we see that we must set $\mathcal{K}_2 = \left(\dfrac{a\mathtt{A}'(a)}{\mathtt{A}(a)}\right)^2$.

4.8.2 $\mathbf{Q1}_\delta$

The Q1 equation is

$$\mathring{p}(u-\widehat{u})(\widetilde{u}-\widehat{\widetilde{u}}) - \mathring{q}(u-\widetilde{u})(\widehat{u}-\widehat{\widetilde{u}}) + \delta^2\mathring{p}\mathring{q}(\mathring{p}-\mathring{q}) = 0, \tag{4.134}$$

where again $\mathring{p} = \frac{a^2}{p^2-a^2}$ and $\mathring{q} = \frac{a^2}{q^2-a^2}$, and

$$\mathcal{H}(u,\widehat{u};p_k) = \frac{1}{\mathring{p}_k}\Big((u-\widehat{u})^2 - \delta^2\mathring{p}_k^2\Big)$$

$$\sqrt{\mathtt{K}(p_k,\zeta)} = \left[\frac{\mathring{p}_k\mathring{\zeta}}{a}\right](p_k^2-\zeta^2)^{\frac{1}{2}}.$$

The elements of the Lax matrix are

$$a_0 = \mathring{p}_k u\widehat{u} + \delta^2\mathring{p}_k\mathring{\zeta}(\mathring{p}_k - \mathring{\zeta}) \qquad a_1 = -\mathring{p}_k\widehat{u} + \mathring{\zeta}(\widehat{u}-u)$$

$$a_2 = -\mathring{p}_k u + \mathring{\zeta}(u-\widehat{u}) \qquad a_3 = \mathring{p}_k,$$

and the Lax equations for ϕ_1, Λ_2 and Υ_2 are

$$(p_k^2-\zeta^2)^{\frac{1}{2}}\widehat{\widehat{\phi}}_1(\zeta)-\frac{a}{\mathring{p}_k}\left(\frac{\widehat{\widehat{u}}-u}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_1(\zeta)+(p_k^2-\zeta^2)^{\frac{1}{2}}\phi_1(\zeta)=0 \tag{4.135}$$

$$\Lambda_2(\zeta)=\left[u+\left(\frac{p_k^2-a^2}{\zeta^2-a^2}\right)(\widehat{u}-u)\right]\Lambda(\zeta)-a\left(\frac{p_k+\zeta}{\zeta^2-a^2}\right)\mathcal{U}\widehat{\Lambda}(\zeta) \tag{4.136}$$

$$\Upsilon_2(\zeta)=\left[u+\left(\frac{p_k^2-a^2}{\zeta^2-a^2}\right)(\widehat{u}-u)\right]\Upsilon(\zeta)-a\left(\frac{p_k-\zeta}{\zeta^2-a^2}\right)\mathcal{U}\widehat{\Upsilon}(\zeta). \tag{4.137}$$

The boundary conditions we assume for u are

$$u\sim\mathcal{B}(\mu+\mu_o)+\mathcal{D}\rho(-a)\text{ as }\ i\to-\infty, \tag{4.138}$$

$$u\sim\mathcal{K}_o\mathcal{A}\rho(a)+\mathcal{B}(\mu+\mu_o+\mathcal{K}_1)\text{ as }\ i\to+\infty, \tag{4.139}$$

which implies that

$$\mathcal{U}\sim-\mathcal{B}-2\mathcal{D}\rho(-a)\text{ as }\ i\to-\infty, \tag{4.140}$$

$$\mathcal{U}\sim 2\mathcal{K}_o\mathcal{A}\rho(a)-\mathcal{B}\text{ as }\ i\to+\infty. \tag{4.141}$$

Here $\mathcal{K}_o$ and μ_o are free constants, while $\mathcal{K}_1$ is fixed. The remaining constants are restrained by

$$16\mathcal{A}\mathcal{D}+\mathcal{B}^2=\delta^2.$$

The singular point for this equation is $\zeta=a$, and in this limit $\zeta=a+\epsilon$ with $\epsilon\ll 1$ we have

$$\frac{\widehat{a}_o}{a_o}=1+O\big(\epsilon^2\big)$$

and so by writing

$$\phi_2=\frac{\phi_2^\epsilon}{\epsilon}+O(1),$$

equation (4.107) for Λ_2^ϵ becomes

$$(p_k+a)\widehat{\widehat{\Lambda}}_2^\epsilon(a)-\frac{a}{\mathring{p}_k}\left(\frac{\widehat{\widehat{u}}-u}{\widehat{\mathcal{U}}}\right)\widehat{\Lambda}_2^\epsilon(a)+(p_k-a)\Lambda_2^\epsilon(a)=0.$$

This however is just equation (4.135) for Λ, and using the fact that this holds for every lattice direction, there exists constants α and β such that

$$\Lambda_2^\epsilon = \alpha\Lambda(a) + \beta\Upsilon(a)\rho(-a).$$

Equating this with (4.136) then gives

$$(p_k^2-a^2)(\widehat{u}-u)\Lambda(a)-a(p_k+a)\mathcal{U}\widehat{\Lambda}(a) = 2a\alpha\Lambda(a)+2a\beta\Upsilon(a)\rho(-a). \quad (4.142)$$

By using the boundary conditions for u and $\mathcal{U}$ we can then determine α and β, and thus

$$(p_k^2 - a^2)(\widehat{u} - u)\Lambda(a) - a(p_k + a)\mathcal{U}\widehat{\Lambda}(a) = a^2\mathcal{B}\Lambda(a) + 4a^2\mathcal{D}\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)\rho(-a). \quad (4.143)$$

We may now repeat this process using equation (4.137) for Υ_2. This yelds the second equation

$$(p_k^2-a^2)(\widehat{u}-u)\Upsilon(a)-a(p_k-a)\mathcal{U}\widehat{\Upsilon}(a) = 4a^2\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)}\right)\mathcal{A}\Lambda(a)\rho(a)-a^2\mathcal{B}\Upsilon(a). \quad (4.144)$$

Equations (4.143) and (4.144) can then be combined to firstly give

$$\mathcal{U} = 2\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\mathcal{A}\Lambda(a)^2\rho(a)-\mathcal{B}\left(\frac{\Lambda(a)\Upsilon(a)}{\mathtt{A}(a)}\right)-2\mathcal{D}\left(\frac{\Upsilon(a)}{\mathtt{A}(a)}\right)^2\rho(-a), \quad (4.145)$$

and also

$$\begin{aligned}\widehat{u}-u = &\left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\mathcal{A}\Lambda(a)\widehat{\Lambda}(a)\Big(\widehat{\rho}(a)-\rho(a)\Big)\\ &+\frac{2ap_k\mathcal{B}}{p_k^2-a^2}+a\mathcal{B}\left(\frac{(p+a)\widehat{\Lambda}(a)\Upsilon(a)+(p-a)\Lambda(a)\widehat{\Upsilon}(a)}{2\mathtt{A}(a)(p_k^2-a^2)}\right)\\ &+\mathcal{D}\left(\frac{\Upsilon(a)\widehat{\Upsilon}(a)}{\mathtt{A}(a)^2}\right)\Big(\widehat{\rho}(-a)-\rho(-a)\Big). \quad (4.146)\end{aligned}$$

By now using the Wronskian identities mentioned at the start of this section we may sum this to obtain the solution of $\mathrm{Q1}_\delta$

$$u = \left(\frac{\mathcal{K}_o}{\mathtt{A}(a)^2}\right)\mathcal{A}\rho(a)\mathcal{Y}(a)+\mathcal{B}\left\{\mu+\mu_o-\frac{1}{2}\mathcal{G}(a)\right\}+\mathcal{D}\rho(-a)\mathcal{Z}(a), \quad (4.147)$$

where $\Psi(\zeta) = \dfrac{\Upsilon(\zeta)}{\mathtt{A}(\zeta)}$ and $\mathcal{G}, \mathcal{Y}$ and $\mathcal{Z}$ are given by (4.130), (4.132) and (4.133) respectively. Here the boundary condition as $i \to -\infty$ are satisfied, however to satisfy the boundary condition as $i \to +\infty$ we must set $\mathcal{K}_1 = -\dfrac{a\mathtt{A}'(a)}{\mathtt{A}(a)}$.

4.8.3 H3$_\delta$

The H3$_\delta$ equation is

$$Q(u\,\widetilde{u} + \widehat{u}\,\widehat{\widetilde{u}}) - P(u\,\widehat{u} + \widetilde{u}\,\widehat{\widetilde{u}}) + \delta\left(\frac{p^2 - q^2}{PQ}\right) = 0 \tag{4.148}$$

where $P^2 = a^2 - p^2$ and $Q^2 = a^2 - q^2$, and

$$\mathcal{H}(u, \widehat{u}; p_k) = u\widehat{u} + \frac{\delta}{P_k}$$
$$\sqrt{\mathtt{K}(p_k, \zeta)} = (p^2 - \zeta^2)^{\frac{1}{2}}.$$

The elements of the Lax matrix elements are

$$a_0 = (a^2 - \zeta^2)^{\frac{1}{2}} u\widehat{u} + \delta\left(\frac{p_k^2 - \zeta^2}{P_k(a^2 - \zeta^2)^{\frac{1}{2}}}\right) \qquad a_1 = -P_k u$$
$$a_2 = -P_k\widehat{u} \qquad a_3 = (a^2 - \zeta^2)^{\frac{1}{2}},$$

and the Lax equations for ϕ_1, Λ_2 and Υ_2 are

$$(p_k^2 - \zeta)^{\frac{1}{2}}\widehat{\widetilde{\phi}}_1(\zeta) - \left(\frac{P_k(\widehat{\widetilde{u}} - u)}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_1(\zeta) + (p_k^2 - \zeta^2)^{\frac{1}{2}}\phi_1(\zeta) = 0 \tag{4.149}$$

$$(a^2 - \zeta^2)^{\frac{1}{2}}\Lambda_2(\zeta) = P_k\widehat{u}\Lambda(\zeta) - (p_k + \zeta)\mathcal{U}\widehat{\Lambda}(\zeta) \tag{4.150}$$

$$(a^2 - \zeta^2)^{\frac{1}{2}}\Upsilon_2(\zeta) = P_k\widehat{u}\Upsilon(\zeta) - (p_k - \zeta)\mathcal{U}\widehat{\Upsilon}(\zeta). \tag{4.151}$$

The boundary conditions for u are

$$u \sim \big(\mathcal{C}\sigma + \mathcal{D}\big)\Theta^{-1} \ \text{ as } i \to -\infty, \tag{4.152}$$

$$u \sim \mathcal{K}\big(\mathcal{A} + \sigma\mathcal{B}\big)\Theta \ \text{ as } i \to +\infty \tag{4.153}$$

which imply that

$$\mathcal{U} \sim \big(\mathcal{C}\sigma - \mathcal{D}\big)\Theta^{-1} \ \text{ as } i \to -\infty, \tag{4.154}$$

$$\mathcal{U} \sim \mathcal{K}\big(\mathcal{A} - \sigma\mathcal{B}\big)\Theta \ \text{ as } i \to +\infty. \tag{4.155}$$

The new functions Θ and σ are defined by

$$\Theta := \prod_{r=1}^{N} \left(\frac{a+p_r}{a-p_r}\right)^{\frac{1}{2}n_r}, \qquad \sigma = (-1)^{n_1+\ldots+n_{\mathrm{N}}},$$

while $\mathcal{K}$ is a free constant, and the remaining constants are restrained by

$$\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C} = -\frac{\delta}{4a}.$$

The singular point for the equation is $\zeta = a$, and in this limit $\zeta = a + \epsilon$ with $\epsilon \ll 1$, by writing

$$\Lambda_2(a) = \frac{\Lambda_2^{\epsilon}}{\epsilon^{\frac{1}{2}}} + O\left(\epsilon^{\frac{1}{2}}\right),$$

equation (4.107) for Λ_2 reduces dramatically to

$$(p_k + a)\widehat{\widehat{\Lambda}}_2^{\epsilon}(a) + (p_k - a)\Lambda_2^{\epsilon}(a) = 0,$$

and since this holds for all lattice directions we have

$$\Lambda_2^{\epsilon} = (\alpha + \beta\sigma)\Theta^{-1} \tag{4.156}$$

for some constants α and β. Equating this with equation (4.150) and using the boundary conditions for u and $\mathcal{U}$ then gives the equation

$$P_k\widehat{u}\Lambda(a) - (p_k + a)\mathcal{U}\widehat{\Lambda}(a) = 2a(\mathcal{D} - \sigma\mathcal{C})\Theta^{-1}. \tag{4.157}$$

If we now repeat this process using equation (4.151) for Υ_2 we find

$$P_k\widehat{u}\Upsilon(a) - (p_k - a)\mathcal{U}\widehat{\Upsilon}(a) = 2a\mathcal{K}(\mathcal{A} - \sigma\mathcal{B})\Theta. \tag{4.158}$$

Combining equations (4.157) and (4.158) then gives

$$\mathcal{U} = \left(\frac{\mathcal{K}}{\mathtt{A}(a)}\right)\big(\mathcal{A} - \sigma\mathcal{B}\big)\Lambda(a)\Theta + \big(\mathcal{C}\sigma - \mathcal{D}\big)\Psi(a)\Theta^{-1} \tag{4.159}$$

where $\Psi(\zeta) = \dfrac{\Upsilon(\zeta)}{\mathtt{A}(\zeta)}$. The solution of H3$_\delta$ is

$$u = \left(\frac{\mathcal{K}}{\mathtt{A}(a)}\right)\big(\mathcal{A} + \sigma\mathcal{B}\big)\Lambda(a)\Theta + \big(\mathcal{C}\sigma + \mathcal{D}\big)\Psi(a)\Theta^{-1}. \tag{4.160}$$

4.8.4 H2

The H2 equation is

$$(u - \widehat{\widetilde{u}})(\widetilde{u} - \widehat{u}) + (p^2 - q^2)(u + \widetilde{u} + \widehat{u} + \widehat{\widetilde{u}}) - p^4 + q^4 = 0, \tag{4.161}$$

and

$$\mathcal{H}(u, \widehat{u}; p_k) = 2(u + \widehat{u} - p_k^2)$$
$$\sqrt{\mathtt{K}(p_k, \zeta)} = (p^2 - \zeta^2)^{\frac{1}{2}}.$$

The elements of the Lax matrix are

$$a_0 = u\widehat{u} + (p_k^2 - \zeta^2)(u + \widehat{u}) - p_k^4 + \zeta^4 \qquad a_1 = -u + p_k^2 - \zeta^2$$
$$a_2 = -\widehat{u} + p_k^2 - \zeta^2 \qquad a_3 = 1,$$

and the Lax equations for ϕ_1 and Λ_2 are

$$(p_k^2 - \zeta^2)^{\frac{1}{2}}\widehat{\phi}_1(\zeta) - \left(\frac{\widehat{\widetilde{u}} - u}{\widehat{\mathcal{U}}}\right)\widehat{\phi}_1(\zeta) + (p_k^2 - \zeta^2)^{\frac{1}{2}}\phi_1(\zeta) = 0 \tag{4.162}$$

$$\Lambda_2(\zeta) = (\widehat{u} - p_k^2 + \zeta^2)\Lambda(\zeta) - (p_k + \zeta)\mathcal{U}\widehat{\Lambda}(\zeta). \tag{4.163}$$

The boundary conditions for u are

$$u \sim (\tau + \tau_o)^2 + 2\mathcal{A}\sigma(\tau + \tau_1) - \mathcal{A}^2 \text{ as } i \to -\infty$$
$$u \sim \big(\tau + \tau_o\big)^2 + 2\mathcal{K}_o(\tau + \tau_o)$$
$$+ 2\mathcal{A}\sigma(\tau + \tau_1 + \mathcal{K}_o) + \mathcal{K}_1^2 - \mathcal{A}^2 \text{ as } i \to +\infty,$$

which imply that

$$\mathcal{U} \sim 2\big[\tau + \tau_o - \sigma\mathcal{A}\big] \text{ as } i \to -\infty$$
$$\mathcal{U} \sim 2\big[\tau + \tau_o + \mathcal{K}_o - \sigma\mathcal{A}\big] \text{ as } i \to -\infty.$$

Here the function τ is defined by

$$\tau = \sum_{r=1}^{N} p_r n_r,$$

and $\mathcal{A}, \tau_o$ and τ_1 are all unrestrained constants while $\mathcal{K}_o$ and $\mathcal{K}_1$ are fixed. The singular point of the equation is $|\zeta| \to \infty$, and in the limit $\zeta = \frac{1}{\epsilon}$ with $\epsilon \ll 1$ we have

$$\frac{\widehat{a}_o}{a_o} = 1 + \epsilon^2(\widehat{\widehat{u}} - u) + O(\epsilon^4) \quad \Rightarrow \quad \widehat{a}_1 - \left(\frac{\widehat{a}_o}{a_o}\right) a_2 = u - \widehat{\widehat{u}} + O(\epsilon^2)$$

and so by writing

$$\Lambda = 1 + \epsilon\Lambda^{\epsilon} + \epsilon^2\Lambda^{\epsilon^2} + O(\epsilon^3) \text{ as } |\zeta| \to \infty$$
$$\Lambda_2 = \frac{\Lambda_2^{-\epsilon^2}}{\epsilon^2} + \frac{\Lambda_2^{-\epsilon}}{\epsilon} + O(1) \text{ as } |\zeta| \to \infty$$

then by (4.163) we have $\Lambda_2^{-\epsilon^2} = 1$. Equation (4.107) for Λ_2 then becomes

$$\widehat{\widehat{\Lambda}}_2^{-\epsilon} - \Lambda_2^{-\epsilon} = -2p_k - \frac{\widehat{\widehat{u}} - u}{\widehat{\mathcal{U}}},$$

but by taking this limit in equation (4.162) for Λ we find that

$$\left(\frac{\widehat{\widehat{u}} - u}{\widehat{\mathcal{U}}}\right) = 2p_k + \widehat{\widehat{\Lambda}}^{\epsilon} - \Lambda^{\epsilon}$$

and so

$$\Lambda_2^{-\epsilon} = \alpha + \beta(-1)^{n_k} - 2p_k n_k - \Lambda^{\epsilon}$$

for some constants α and β. Since this holds for all lattice directions and using the Lax equation (4.163) we then have

$$\Lambda^{\epsilon} - \mathcal{U} = \alpha + \beta\sigma - 2\tau - \Lambda^{\epsilon},$$

and then using the boundary conditions for $\mathcal{U}$ we have

$$\mathcal{U} = 2\Big(\tau + \tau_o - \mathcal{A}\sigma + \Lambda^\epsilon\Big). \tag{4.164}$$

From the biquadratic $\mathcal{H}$ and the definition of $\mathcal{U}$ we can then write

$$\begin{aligned} u + \widehat{u} =& p_k^2 + \frac{1}{2}\mathcal{U}\widehat{\mathcal{U}} \\ =& (\tau + \tau_o)^2 + (\tau + \tau_o + p_k)^2 + 2(\tau + \tau_o)\Lambda^\epsilon \\ &+ 2(\tau + \tau_o + p_k)\widehat{\Lambda}^\epsilon - 2\mathcal{A}^2 - 2\mathcal{A}\sigma(p_k + \widehat{\Lambda}^\epsilon - \Lambda^\epsilon) \\ &+ 2p(\Lambda^\epsilon - \widehat{\Lambda}^\epsilon) + 2\Lambda^\epsilon\widehat{\Lambda}^\epsilon. \end{aligned} \tag{4.165}$$

We can however go further: For purely imaginary ζ, by equating the equations (4.162) for $\Lambda(\zeta)$ and $\Lambda(-\zeta)$ we have

$$(p_k + \zeta)\widehat{\Lambda}(\zeta)\Lambda(-\zeta) - (p_k - \zeta)\Lambda(\zeta)\widehat{\Lambda}(-\zeta) = 2\zeta,$$

and by taking the limit $\zeta = \frac{1}{\epsilon}$ in this equation we obtain the identity

$$p_k(\Lambda^\epsilon - \widetilde{\Lambda}^\epsilon) + \Lambda^\epsilon\widetilde{\Lambda}^\epsilon = \Lambda^{\epsilon^2} + \widetilde{\Lambda}^{\epsilon^2}.$$

We can therefore directly sum equation (4.165) and use the boundary conditions for u to give the solution of H2 as

$$u = (\tau + \tau_o)^2 + 2(\tau + \tau_o)\Lambda^\epsilon + 2\Lambda^{\epsilon^2} - \mathcal{A}^2 + 2\sigma\mathcal{A}(\tau + \tau_1 + \Lambda^\epsilon). \tag{4.166}$$

Writing

$$\mathtt{A}(\zeta) \sim 1 + \frac{\mathtt{A}^\epsilon}{\zeta} + \frac{\mathtt{A}^{\epsilon^2}}{\zeta^2} + \ldots \text{ as } |\zeta| \to \infty,$$

this result shows that we must set

$$\mathcal{K}_o = \lim_{i\to+\infty} \Lambda^\epsilon = \mathtt{A}^\epsilon, \qquad \mathcal{K}_1^2 = \lim_{i\to+\infty} 2\Lambda^{\epsilon^2} = 2\mathtt{A}^{\epsilon^2}.$$

4.8.5 H1

The H1 equation is

$$(u-\widehat{\widetilde{u}})(\widetilde{u}-\widehat{u})+(p^2-q^2)=0, \tag{4.167}$$

and

$$\mathcal{H}(u,\overline{u};p_k)=1 \;\;\Rightarrow\;\; \mathcal{U}=1$$
$$\sqrt{\mathtt{K}(p_k,\zeta)}=(p_k^2-\zeta^2)^{\frac{1}{2}}.$$

The elements of the Lax matrix are

$$a_0=u\widehat{u}+p_k^2-\zeta^2 \qquad a_1=-u \qquad a_2=-\widehat{u} \qquad a_3=1.$$

The boundary conditions we assume on u are

$$u\sim\tau+\tau_o \;\text{ as } i\to-\infty, \tag{4.168}$$

$$u\sim\tau+\tau_o+\mathcal{K} \;\text{ as } i\to+\infty \tag{4.169}$$

where τ_o is a free constant and $\mathcal{K}$ is fixed. The Lax equation for Λ is

$$(p_k+\zeta)\widehat{\widetilde{\Lambda}}(\zeta)-\left(\widehat{\widetilde{u}}-u\right)\widehat{\Lambda}(\zeta)+(p_k-\zeta)\Lambda(\zeta)=0$$

and if we let $|\zeta|\to\infty$ and use the fact that this holds for every lattice direction we obtain

$$\begin{aligned}\widehat{\widetilde{u}}-u &= 2p_k+\widehat{\widetilde{\Lambda}}^{\epsilon}-\Lambda^{\epsilon}\\ \Rightarrow u &= \tau+\tau_o+\Lambda^{\epsilon}.\end{aligned} \tag{4.170}$$

From this solution we see that we must set $\mathcal{K}=\mathtt{A}^{\epsilon}$. Note that if u_o solves H1 then so does $\mathcal{A}(u_o)+\sigma\mathcal{B}(u_o+const.)$, provided that $\mathcal{A}^2-\mathcal{B}^2=1$, so this can be generalised to

$$u=\mathcal{A}(\tau+\tau_o+\Lambda^{\epsilon})+\sigma\mathcal{B}(\tau+\tau_1+\Lambda^{\epsilon}). \tag{4.171}$$

4.9 One-Soliton Solutions

We now construct explicit solutions to all of the above equations for the special case where the reflection coefficient R is identically zero and the function $\mathtt{A}$ has exactly one zero in $\mathcal{R}^+$ at $\zeta_1 = k > 0$. Since the solution of each equation is essentially comprised of the same ingredients, the only equation we are required to solve is the singular integral equation for $\Lambda(\zeta)$. By setting $\mathtt{c}_1 = 2kc$ where c is a constant, this integral equation (4.67) is

$$\Lambda(\zeta) = 1 - \left(\frac{2kc\Lambda(k)}{\zeta + k}\right)\rho(k),$$

which has the solution

$$\Lambda(\zeta) = \frac{1 + \mathtt{A}(\zeta)\rho(k)}{1 + \rho(k)} \Rightarrow \Upsilon(\zeta) = \frac{\mathtt{A}(\zeta) + \rho(k)}{1 + \rho(k)},$$

where we have absorbed the constant c into the plane-wave factor ρ, and the function $\mathtt{A}$ is given by

$$\mathtt{A}(\zeta) = \left(\frac{\zeta - k}{\zeta + k}\right).$$

By choosing boundary conditions exhibited by the known one-soliton solutions [**65**], it is then straightforward to use the above formulae to compute the following:

a) Q3$_\delta$: Set $\mathcal{K}_o = \left(\frac{(a-k)(b-k)}{(a+k)(b+k)}\right)$ and $\mathcal{K}_1 = \left(\frac{(a-k)(b+k)}{(a+k)(b-k)}\right)$. Then

$$\begin{aligned} u = &\mathcal{A}\mathcal{F}(a,b)\left(\frac{1+\left(\frac{(a-k)(b-k)}{(a+k)(b+k)}\right)\rho(k)}{1+\rho(k)}\right) \\ &+\mathcal{B}\,\mathcal{F}(a,-b)\left(\frac{1+\left(\frac{(a-k)(b+k)}{(a+k)(b-k)}\right)\rho(k)}{1+\rho(k)}\right) \\ &+\mathcal{C}\,\mathcal{F}(-a,b)\left(\frac{1+\left(\frac{(a+k)(b-k)}{(a-k)(b+k)}\right)\rho(k)}{1+\rho(k)}\right) \\ &+\mathcal{D}\,\mathcal{F}(-a,-b)\left(\frac{1+\left(\frac{(a+k)(b+k)}{(a-k)(b-k)}\right)\rho(k)}{1+\rho(k)}\right) \end{aligned}$$

where $\mathcal{A}\mathcal{D}(a+b)^2 - \mathcal{B}\mathcal{C}(a-b)^2 = -\frac{\delta^2}{16ab}$.

b) Q2: Set $\mathcal{K}_o = \left(\frac{a-k}{a+k}\right)^2$, $\mathcal{K}_1 = \frac{4ak}{a^2-k^2}$ and $\mathcal{K}_2 = \frac{4a^2k^2}{(a^2-k^2)^2}$. Then

$$\begin{aligned} u =&(\mu+\mu_o)^2 - \left(\frac{4ak}{a^2-k^2}\right)(\mu+\mu_o)\left(\frac{\rho(k)}{1+\rho(k)}\right) \\ &+\left(\frac{2ak}{a^2-k^2}\right)^2\left(\frac{\rho(k)}{1+\rho(k)}\right)+\frac{1}{4}+\mathcal{A}\mathcal{D} \\ &+\frac{1}{2}\mathcal{A}\rho(a)\left(\frac{1+\left(\frac{a-k}{a+k}\right)^2\rho(k)}{1+\rho(k)}\right)+\frac{1}{2}\mathcal{D}\rho(-a)\left(\frac{1+\left(\frac{a+k}{a-k}\right)^2\rho(k)}{1+\rho(k)}\right). \end{aligned}$$

c) Q1$_\delta$: Set $\mathcal{K}_o = \left(\frac{a-k}{a+k}\right)^2$ and $\mathcal{K}_1 = \frac{2ak}{k^2-a^2}$. Then

$$\begin{aligned} u =&\mathcal{A}\rho(a)\left(\frac{1+\left(\frac{a-k}{a+k}\right)^2\rho(k)}{1+\rho(k)}\right)+\mathcal{D}\rho(-a)\left(\frac{1+\left(\frac{a+k}{a-k}\right)^2\rho(k)}{1+\rho(k)}\right) \\ &+\mathcal{B}\left(\mu+\mu_o+\left(\frac{2ak}{k^2-a^2}\right)\frac{\rho(k)}{1+\rho(k)}\right) \end{aligned}$$

where $16\mathcal{A}\mathcal{D}+\mathcal{B}^2 = \delta^2$.

d) H3$_\delta$: Set $\mathcal{K} = \left(\frac{a-k}{a+k}\right)$. Then

$$u = \left(\mathcal{A} + \sigma\mathcal{B}\right)\left(\frac{1+\left(\frac{a-k}{a+k}\right)\rho(k)}{1+\rho(k)}\right)\Theta + \left(\mathcal{C}\sigma + \mathcal{D}\right)\left(\frac{1+\left(\frac{a+k}{a-k}\right)\rho(k)}{1+\rho(k)}\right)\Theta^{-1}.$$

e) H2: Set $\mathcal{K}_o = -2k$ and $\mathcal{K}_1^2 = 4k^2$. Then

$$u = (\tau+\tau_o)^2 + 2(\tau+\tau_o)\left(\frac{-2k\rho(k)}{1+\rho(k)}\right) + \left(\frac{4k^2\rho(k)}{1+\rho(k)}\right) - \mathcal{A}^2 + 2\sigma\mathcal{A}\left[\tau+\tau_1+\left(\frac{-2k\rho(k)}{1+\rho(k)}\right)\right].$$

f) H1: Set $\mathcal{K} = -2k$. Then

$$u = \mathcal{A}\left[\tau+\tau_o+\left(\frac{-2k\rho(k)}{1+\rho(k)}\right)\right] + \sigma\mathcal{B}\left[\tau+\tau_1+\left(\frac{-2k\rho(k)}{1+\rho(k)}\right)\right] \tag{4.172}$$

For the case of an arbitrary reflectionless potential where A has exactly N zeroes in $\mathcal{R}^+$, one can show that the solution of (4.67) is

$$\Lambda(\zeta) = 1 - \mathbf{c}^T(\zeta I + K)^{-1}(I+\mathcal{M})^{-1}\mathbf{r}$$

where $\mathbf{c}, \mathbf{r}$ and $\mathcal{M}$ are defined in Section 3.5 of Chapter 3 by equations (3.44) and (3.45), and $K = K_{ij}$ is the diagonal matrix $k_i\delta_{ij}$. By using the various identities in **[65]** one can show that these solutions are exactly the N-soliton solutions given in this paper. This was shown explicitly in **[27]** by utilising the degeneration procedure between the ABS equations from **[65]**.

4.10 Other Lattice Equations

As a final section to this chapter we give a short discussion on how we can identify solutions of some well-known lattice equations within the machinery of the discrete IST. These are all obtainable by considering the first component ϕ_1 of the Lax equations, and thus to obtain solutions to the

following equations one must simply solve the singular integral equation (4.67). We begin by deriving a closed-form lattice equation for the quantity $\Lambda(\zeta)$. Consider the linear equation for $\Lambda(\zeta)$ taken along opposing sides of a lattice element in the n- and m-directions:

$$(q+\zeta)\frac{\widehat{\widetilde{\Lambda}}(\zeta)}{\widetilde{\Lambda}(\zeta)} + (p-\zeta)\frac{\Lambda(\zeta)}{\widetilde{\Lambda}(\zeta)} = \frac{Q\,\widehat{\widetilde{u}} - (q^2-p^2)\,\widetilde{u} - Pu}{\widetilde{\mathcal{U}}} \tag{4.173a}$$

$$(p+\zeta)\frac{\widehat{\widetilde{\Lambda}}(\zeta)}{\widehat{\Lambda}(\zeta)} + (q-\zeta)\frac{\Lambda(\zeta)}{\widehat{\Lambda}(\zeta)} = \frac{P\,\widehat{\widetilde{u}} - (p^2-q^2)\,\widehat{u} - Qu}{\widehat{\mathcal{U}}} \tag{4.173b}$$

One can then show that if u solves $\mathrm{Q3}_\delta$ then the quantities on the right-hand side of these equation are in fact equal[4], and thus equating these shows that $\Lambda(\zeta)$ satisfies the lattice equation

$$(p+\zeta)\frac{\widehat{\widetilde{\Lambda}}(\zeta)}{\widehat{\Lambda}(\zeta)} - (p-\zeta)\frac{\Lambda(\zeta)}{\widetilde{\Lambda}(\zeta)} = (q+\zeta)\frac{\widehat{\widetilde{\Lambda}}(\zeta)}{\widetilde{\Lambda}(\zeta)} - (q-\zeta)\frac{\Lambda(\zeta)}{\widehat{\Lambda}(\zeta)}. \tag{4.174}$$

Now let $\alpha_p := (p^2-\zeta^2)^{\frac{1}{2}}$, $\alpha_q := (p^2-\zeta^2)^{\frac{1}{2}}$, then in terms of the Jost function φ this becomes

$$\alpha_p\Big(\varphi(\zeta)\,\widehat{\varphi}(\zeta) - \widetilde{\varphi}(\zeta)\,\widehat{\widetilde{\varphi}}(\zeta)\Big) = \alpha_q\Big(\varphi(\zeta)\,\widetilde{\varphi}(\zeta) - \widehat{\varphi}(\zeta)\,\widehat{\widetilde{\varphi}}(\zeta)\Big), \tag{4.175}$$

that is $\varphi(\zeta)$ satisfies the lattice potential modified KdV equation **[67]** **[74]** with parameters α_p and α_q. By setting $W(\zeta) := \dfrac{\widehat{\varphi}(\zeta)}{\widetilde{\varphi}(\zeta)}$ we then see that W satisfies

$$\frac{\widehat{\widetilde{W}}(\zeta)}{W(\zeta)} = \frac{\big(\alpha_p\widehat{W}(\zeta)-\alpha_q\big)\big(\alpha_p-\alpha_q\widetilde{W}(\zeta)\big)}{\big(\alpha_p\widetilde{W}(\zeta)-\alpha_q\big)\big(\alpha_p-\alpha_q\widehat{W}(\zeta)\big)}, \tag{4.176}$$

which is the lattice modified KdV equation **[65]**. Of course equivalent equations to (4.174), (4.175) and (4.176) hold in any pair of lattice directions, and each of these equations is multidimensionally consistent. If we now break

[4] This is the (1,2) element of the consistency equation for the Lax pairs in the n- and m-directions

the covariance between the n- and m- lattice directions in (4.174) by setting $\zeta = p$, we have the following lattice equation for the quantity $\Lambda(p)$:

$$2p = (p+q)\frac{\widehat{\Lambda}(p)}{\widetilde{\Lambda}(p)} + (p-q)\frac{\Lambda(p)}{\widehat{\widetilde{\Lambda}}(p)}. \tag{4.177}$$

The (weaker) multidimensional consistency properties of this equation are discussed in [**65**]. By setting $\zeta = p$ in (4.173a) and $\zeta = q$ in (4.173b), then equating the right-hand sides we have $\widehat{\widetilde{\Lambda}}(p)\widehat{\Lambda}(q) = \widehat{\widetilde{\Lambda}}(q)\widetilde{\Lambda}(p)$ which shows that it is self-consistent to express $\Lambda(p)$ in terms of the τ-function:

$$\widetilde{\Lambda}(p) = \frac{F}{\widetilde{\widehat{F}}}\,, \qquad \widehat{\Lambda}(q) = \frac{F}{\widehat{\widetilde{F}}}\,.$$

Using this in (4.177) gives the 6-point equation

$$2pF\,\widehat{F} = (p+q)\underset{\sim}{\widehat{F}}\,\widetilde{F} + (p-q)\underset{\sim}{F}\,\widehat{\widetilde{F}}.$$

By supplementing this with the similar equation obtained by setting $\zeta = q$ in (4.174), one can obtain Hirota's discrete-time Toda equation [**50**]:

$$(p-q)^2\underset{\widehat{\sim}}{F}\,\widehat{\widetilde{F}} - (p+q)^2\underset{\sim}{\widehat{F}}\,\underset{\widehat{\ }}{\widetilde{F}} + 4pqF^2 = 0.$$

We now derive two final lattice equation of KdV type. From the $\mathrm{Q1}_\delta$ equation, if one makes the degeneration

$$a = \epsilon, \qquad u \to \epsilon u,$$

then the equation becomes

$$\frac{\big(u-\widetilde{u}\big)\big(\widehat{u}-\widehat{\widetilde{u}}\big)}{\big(u-\widehat{u}\big)\big(\widetilde{u}-\widehat{\widetilde{u}}\big)} = \frac{q^2}{p^2}, \tag{4.178}$$

which is the Schwarzian KdV equation [**66**], also known as the cross-ratio equation (or $\mathrm{Q1}_o$). By choosing the constants such that

$$\mathcal{A} \to \frac{\mathcal{B}}{4}, \qquad \mathcal{B} \to -\frac{\mathcal{B}}{2}, \qquad \mathcal{D} \to \frac{\mathcal{B}}{4}, \qquad \xi_o \to 1 - 2\epsilon\mu_o$$

with μ_o constant, the solution to (4.178) is then given by

$$u = \mathcal{B}\Big(\eta + \eta_o - \mathcal{T}\Big), \tag{4.179}$$

where

$$\mathcal{T} = \frac{p}{2}\Big(\widetilde{\Lambda}(0)\Lambda''(0) - \Lambda(0)\widetilde{\Lambda}''(0)\Big) - \Lambda(0)\widetilde{\Lambda}'(0), \qquad \eta = \sum_{r=1}^{M} \frac{n_r}{p_r}.$$

Alternatively from the solution (4.171) of H1, also known as the lattice potential KdV equation [**50**] [**67**] [**74**], if one defines $\omega := \widetilde{u} - \widehat{u}$ then the quantity ω satisfies the lattice KdV equation [**50**]

$$\omega - \widehat{\widetilde{\omega}} = (p^2 - q^2)\left(\frac{1}{\widehat{\omega}} - \frac{1}{\widetilde{\omega}}\right), \tag{4.180}$$

to which we have the solution

$$\omega = \Big(\mathcal{B} - (-1)^{n+m}\mathcal{A}\Big)\Big(p - q + \widetilde{\Lambda}^{\epsilon} - \widehat{\Lambda}^{\epsilon}\Big). \tag{4.181}$$

Note that equation (4.180) is not multidimensionally consistent with itself, and thus the solution depends only on the variables n and m, while all other lattice variables are held constant. Moreover from the integral equation (4.67), for an arbitrary reflectionless potential with M discrete eigenvalues we have

$$\Lambda^{\epsilon} = \sum_{k=1}^{M} \mathsf{c}_k \Lambda(\zeta_k)\rho(\zeta_k).$$

Thus the general solution (4.181) is comprised of the background solution plus the M distinct quantities

$$\mathsf{c}_k\Big(\widetilde{\Lambda}(\zeta_k)\,\widetilde{\rho}\,(\zeta_k) - \widehat{\Lambda}(\zeta_k)\,\widehat{\rho}\,(\zeta_k)\Big)$$

which are the individual solitons within the solution. This is the corresponding result to (2.51) for the KdV equation.

4.11 Chapter Summary

In this chapter we have rigorously derived a discrete IST for the $Q3_\delta$ lattice equation. The initial-value space was given on a multidimensional staircase and the solution obtained depends on N discrete independent variables and N distinct lattice parameters. The assumptions made on the solution were that it be real and that the initial profile satisfy the summability condition (4.39). The solution to $Q3_\delta$ is expressed explicitly in terms of the eigenfunctions $\Lambda(a), \Lambda(b), \Upsilon(a)$ and $\Upsilon(b)$, all of which are obtainable by solving the singular integral equation (4.67).

After giving the solution to $Q3_\delta$ we then used this machinery to give a discrete IST to all lower ABS equations. This was possible due to the fact that for every equation considered, the first component of the Lax equations was the same, and is essentially the H1 solution. This is reflective of the fact that the soliton solutions to all of these equations are comprised of the same ingredients **[65]**. In each case an explicit representation of the solution was given in terms of these same eigenfunctions.

As a final exercise we showed that the solutions obtained for reflectionless potentials having exactly one discrete eigenvalue are exactly the one-soliton solutions given in **[65]**. We also showed how to construct some previously known integrable lattice equations, some of which having been studied for over 40 years.

We have found therefore that the solutions of all these lattice equations are essentially goverened by the singular integral equation (4.67). The knowledge of solutions to this equation is equivalent to knowledge of solutions to the nonlinear lattice equations. In fact this integral equation is a discrete analogue to (2.30) of Chapter 2, which one obtains from the IST for the KdV equation.

5 Conservation Laws

In this final chapter we explore one of the many uses of the inverse scattering transform, which is the the ability to generate an infinite number of nontrivial conservation laws for nonlinear equations. We begin by looking at the KdV equation, where discuss different methods for obtaining an infinite number of conservation laws. In Section 5.2 we then review some methods of obtaining conservation laws for nonlinear partial difference equations, and in Section 5.3 we then show how these can be generated directly from the discrete IST developed in Chapter 4. We also mention possible generalisations of this method.

5.1 Conservation Laws for the KdV Equation

One of the key integrability properties of the KdV equation (2.2) is that it has an infinite number of nontrivial conservation laws **[62]** **[44]**. In general a (local) conservation law for partial differential equations is an equation of the form

$$\frac{\partial}{\partial t}\Big[T\Big] = \frac{\partial}{\partial x}\Big[X\Big], \tag{5.1}$$

where X and T are functionals of u and its various derivatives. For the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{5.2}$$

we may assume that both X and T depend only on $u, u_x, u_{xx}, ...$ since all t-derivatives may be replaced with x-derivatives using the KdV equation itself. In physics T is referred to as the *density* and X the *flux*. For example the KdV equation itself may be written in the form

$$\frac{\partial}{\partial t}\Big[u\Big] = \frac{\partial}{\partial x}\Big[-3u^2 - u_{xx}\Big],$$

and thus u is a conserved density and $-3u^2 - u_{xx}$ is the corresponding flux. The reason for such labelling is that if we assume decaying boundary conditions on u and its derivatives such that the integral

$$\int_{-\infty}^{+\infty} X\, dx$$

exists, then the quantity

$$I := \int_{-\infty}^{+\infty} T\, dx \tag{5.3}$$

is a constant of motion for solutions of the KdV equation, i.e. it satisfies $I_t = 0$. This can be seen by taking a partial time derivative of (5.3), passing the derivative inside the integral (which we assume is possible) and using the conservation law (5.1) and the boundary conditions of X.

We now look at one method of proving that the KdV has an infinite number of conservation laws, which was first given in **[62]**, and is based around the Miura transformation

$$u = -v_x - v^2$$

which is the special case of $p = 0$ for equation (3.3) in Chapter 3. This maps solutions of the modified KdV equation (mKdV)

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

to those of the KdV equation (5.2). Since the KdV is Galilean-invariant, it remains unchanged through the change of variables

$$t' = t, \qquad x' = x + \frac{6}{\epsilon^2}t, \qquad u'(x', t') = u(x, t) + \frac{1}{\epsilon^2},$$

however if we set

$$v = -\epsilon w + \frac{1}{\epsilon}$$

then this change of variables transforms the mKdV equation into

$$w_t + \Big(6w^2 - 2\epsilon^2 w^3 + w_{xx}\Big)_x = 0, \tag{5.4}$$

where we have dropped the primes for notational clarity. Thus we see that this particular choice for v has turned the mKdV equation into another local conservation law (note that the mKdV is itself already a conservation law with density v and flux $2v^3 - v_{xx}$), where $T = w$ is the local conserved density and $X = -6w^2 + 2\epsilon^2 w^3 - w_{xx}$ is the corresponding flux. In order to turn this into a conservation law for u', we observe that this change of variables maps the Miura transformation to

$$u' = 2w + \epsilon w_x - \epsilon^2 w^2,$$

which may be solved recursively for w as a formal series in ϵ:

$$w = \frac{u'}{2} - \epsilon\left(\frac{u'_x}{4}\right) + \frac{\epsilon^2}{8}\Big(u'^2 + u'_{xx}\Big) + \dots.$$

By then substituting this into (5.4) and imposing that this equation be satisfied at every power of ϵ, equation (5.4) yields an infinite number of conservation laws. The first two conserved densities and fluxes are (dropping the primes), up to a common multiplicative factor,

$$\begin{aligned}
&T_o = u, \qquad X_o = -3u^2 - u_{xx} \\
&T_1 = -u_x, \qquad X_1 = 6uu_x + u_{xxx} \\
&T_2 = u^2 + u_{xx}, \qquad X_2 = -4u^3 - 5u_x^2 - 8uu_{xx} - u_{xxxx}.
\end{aligned}$$

By inspection however the second conservation law reads

$$(-u_x)_t = (6uu_x + u_{xxx})_x$$

which after using the KdV equation to replace the right-hand side, simply states that the cross-derivatives of u are equal, which is of course true not only for solutions of the KdV, but for a much wider class of functions. For this reason we call such conservation laws *trivial*. In fact it turns out that all odd powers of ϵ give rise to trivial conservation laws, but the conservation laws for even powers of ϵ are all nontrivial [**62**].

A different means of obtaining an infinite number of conservation laws for the KdV however is by using the inverse scattering transform. This is just one of the many utilities of the IST as a means of obtaining information about the solutions of the KdV equation. Other uses include the ability to study the asymptotics of solutions [**9**] as well as to generate an infinite hierarchy of compatible flows (using the so-called Lenard scheme, given in [**44**]). By using a square eigenfunction expansion, the authors of [**44**] showed that if one defines the quantity A_n through the recursion relation

$$\frac{\partial}{\partial x}\Big(A_{n+1}\Big) = \frac{1}{4}(-A_{n_{xxx}} + 4uA_{n_x} + 2u_xA_n) \tag{5.5}$$

with $A_0 = 1$, then for every $n \geq 0$, A_n is a conserved density for the equation

$$u_t - uu_x + u_{xxx} = 0,$$

which is related to the KdV equation by the change of variables $u \to -6u$. By explicit calculation we have

$$A_1 = \frac{u}{2}, \qquad A_2 = \frac{1}{8}(3u^2 - u_{xx}),$$

where we have chosen the integration constants to be zero so that the densities have the required boundary conditions. The corresponding fluxes can then be calculated from the conservation law equation (5.1) by replacing all t-derivatives with $-6uu_x - u_{xxx}$, calculating the required antiderivative and setting the integration constant to be zero. For each n this yields a nontrivial conservation law for the KdV equation.

5.2 Conservation Laws for Lattice Equations

We now look at the existence of conservation laws for the partial difference equations considered in Chapter 4. In the discrete setting a conservation law for a partial difference equation of two variables n and m is an equation of the form

$$\Delta_m\Big[M\Big] = \Delta_n\Big[N\Big] \tag{5.6}$$

where Δ_n and Δ_m are discrete derivative operators

$$\begin{aligned}\Delta_n\Big[F(n,m)\Big] &= F(n+1,m) - F(n,m)\\ \Delta_m\Big[F(n,m)\Big] &= F(n,m+1) - F(n,m),\end{aligned}$$

and $M = M(n,m)$ and $N = N(n,m)$ are the discrete equivalents of the density and flux respectively (here we take m to be the discrete time direction). Again if we assume decaying boundary condition on the flux, then by defining $I := \sum_{n=-\infty}^{+\infty} M(n,m)$ we have

$$\Delta_m\Big[I\Big] = \sum_{n=-\infty}^{+\infty} \Delta_m\Big[M(n,m)\Big] = \sum_{n=-\infty}^{+\infty} \Delta_n\Big[N(n,m)\Big] = 0,$$

and thus I is a constant of motion, in the sense that it is independent of the discrete time variable. The question of the existence of conservation laws for the nonlinear difference equations considered in Chapter 4 goes back (to the best of the author's knowledge) to Orfanides **[70]** in 1978 who found conservations laws for the lattice sine-Gordon equation. In 1987 Wiersma and Capel **[91]** then looked at conservation laws for the lattice potential KdV equation, and its various continuum limits. More recently conservation laws for the ABS equation have been obtained in **[51]** **[76]** **[77]**. Here the authors start with a conservation law for an unknown density and flux, substitute the nonlinear partial difference equation into the conservation law, and by using certain differential operators reduce this to a system of

partial differential equations, which can in principle be solved to obtain the required density and flux as functions of the dependent variable. These have also been obtained from an algebraic approach in [**60**]. Conservation laws of reductions of integrable lattice equations have also been found, see e.g. [**88**] [**86**] [**87**].

Here we briefly describe the so-called "Gardner method" [**78**] for the lattice potential KdV equation (H1)

$$(u-\widehat{\widetilde{u}})(\widehat{u}-\widetilde{u})=p^2-q^2,$$

which is a method of obtaining an infinite number of nontrivial conservation laws for this equation, and is a discrete analogue of the above method for the KdV equation using the Miura transformation. The multidimensional consistency of the ABS equations implies that they define their own auto-Bäcklund transformations, and these have been studied in [**80**] [**16**]. Given one solution $u=u_{n,m}$ of H1, a new solution $v=v_{n,m}$ of H1 obtained through the transformation $u\to v$ is found by solving

$$(u-\widetilde{v})(\widetilde{u}-v)=r^2-p^2 \tag{5.7a}$$

$$(u-\widehat{v})(\widehat{u}-v)=r^2-q^2. \tag{5.7b}$$

These are discrete Riccati equations for the new quantity v. One can start by making the substitution $v=\widetilde{f}f^{-1}$ in (5.7a) so as to linearise it. The solution of this must then be used in (5.7b) to determine the m-dependence, where a similar substitution could be made on the first integration constant. There are however two degenerate values of the Bäcklund parameter r^2 for which one of the equations (5.7) factorise, namely $r^2=p^2$ or $r^2=q^2$. At $r^2=p^2$ we have the solution $v=\widetilde{u}$ or $\widetilde{v}=u$ and at r^2-q^2 we have $v=\widehat{u}$ or $\widehat{v}=u$. Given these explicit solutions we look for series solutions near these degenerate points. For example if we write $r^2=p^2+\epsilon$ and look for a

solution

$$v_{n,m} = u_{n+1,m} + \sum_{i=1}^{+\infty} v_{n,m}^{(i)} \epsilon^i, \tag{5.8}$$

then the quantities $v^{(i)}$ must satisfy

$$\epsilon = \left(\sum_{i=1}^{+\infty} v_{n,m}^{(i)} \epsilon^i \right) \left(u_{n,m} - u_{n+2,m} - \sum_{i=1}^{+\infty} v_{n+1,m}^{(i)} \epsilon^i \right). \tag{5.9}$$

To leading order we then have

$$v_{n,m}^{(1)} = \frac{1}{u_{n,m} - u_{n+2,m}},$$

and the higher-order terms are found by the relation

$$v_{n,m}^{(i)} = \frac{1}{u_{n,m} - u_{n+2,m}} \sum_{j=1}^{i-1} v_{n,m}^{(j)} v_{n+1,m}^{(i-j)}. \tag{5.10}$$

As in the continuous case the idea is to start with a single known conservation law and then expand in the parameter ϵ. By using equations (5.7a) and (5.7b) one can show that if we define the density and flux by

$$M(n,m) = \ln(v_{n,m} - u_{n+1,m}), \qquad N(n,m) = \ln(v_{n,m} - u_{n,m+1}) \tag{5.11}$$

then this indeed yields a conservation law for H1. Since the H1 equation is independent of ϵ, by using the ϵ-expansion of v in the components M and N of the conservation law we will obtain a new conservation law at each power of ϵ. If we define

$$\mathtt{V}_n := \frac{1}{u_{n+2,m} - u_{n,m}}, \qquad \mathtt{W} := \frac{1}{u_{n+1,m} - u_{n,m+1}} \tag{5.12}$$

then the $O(\epsilon)$ conserved density and flux are

$$M = \mathtt{V}_n \mathtt{V}_{n+1}, \qquad N = -\mathtt{W}\mathtt{V}_n, \tag{5.13}$$

and at $O(\epsilon^2)$ we have

$$M = \mathtt{V}_n \mathtt{V}_{n+1}^2 \mathtt{V}_{n+2} + \frac{1}{2} \mathtt{V}_n^2 \mathtt{V}_{n+1}^2, \qquad N = -\mathtt{W}\mathtt{V}_n^2 \mathtt{V}_{n+1} - \frac{1}{2} \mathtt{W}^2 \mathtt{V}_n^2.$$

The corresponding conserved quantities appear naturally along the line $m = const.$, however by instead expanding around $r^2 = q^2 + \epsilon$ the method yields an infinite number of conserved quantities along the line $n = const.$. Furthermore by taking a particular continuum limit it was shown in [78] that these are all indeed nontrivial.

5.3 Conservation Laws for Lattice Equations from the Discrete Inverse Scattering Transform

We now look at how one can obtain an infinite number of conservation laws for the partial difference equations considered in Chapter 4, all from the machinery of the discrete IST. This method is based on that for the continuous case, where we first derive a single conservation law for a modification of the square eigenfunction, and then expand in some small parameter ϵ. Here we consider only the first component of the Lax equations, which for *all* of the equations in Chapter 4 is the same. The conservation laws derived here therefore apply not only to H1, but also to H2$\to$Q3$_\delta$. It is unclear however what effect the various Miura transformations relating, for example, the potential v to the solution u of Q3$_\delta$, will have on the these conservation laws. As in Section 5.2 the conserved quantities appear naturally along the lines $m = const.$ or $n = const.$, however due to the multidimensional nature of the discrete IST we indicate how one could generalise this to a higher-dimensional staircase. Finally we note that since it is likely that similar calculations can also be performed for the second component of the Lax equations, we do not claim that these are exhaustive. In other words it is not clear whether the infinite sequence of conservation laws and conserved quantities generated here is sufficient to uniquely characterise solutions of the equations.

For each equation in Chapter 4, the evolution of the first component of the Lax equations along the line $m = const.$ is given by equation (4.55), which is

$$\alpha_p \phi_1(n+2;\zeta) - \mathtt{V}_{n+1}\phi_1(n+1;\zeta) + \alpha_p \phi_1(n;\zeta) = 0, \tag{5.14}$$

where $\alpha_p := (p^2 - \zeta^2)^{\frac{1}{2}}$ and we assume that $\mathtt{V}_n \to 2p$ as $n \to \pm\infty$. It was also shown in Section of Chapter 4 that ϕ_1 satisfies

$$\alpha_p\Big(\phi_1\widehat{\phi}_1 - \widetilde{\phi}_1\widehat{\widetilde{\phi}}_1\Big) = \alpha_q\Big(\phi_1\widetilde{\phi}_1 - \widehat{\phi}_1\widehat{\widetilde{\phi}}_1\Big) \tag{5.15}$$

which is the lattice modified KdV equation, where $\alpha_p = (p^2 - \zeta^2)^{\frac{1}{2}}$ and $\alpha_q = (q^2 - \zeta^2)^{\frac{1}{2}}$. In fact by rederiving this using the Jost functions φ and ψ one can obtain the more convenient form

$$\frac{\widetilde{\psi}\,\widehat{\widetilde{\varphi}} - \psi\,\widehat{\varphi}}{\alpha_q} = \frac{\widehat{\varphi}\,\widehat{\widetilde{\psi}} - \varphi\,\widetilde{\psi}}{\alpha_p}. \tag{5.16}$$

Equation (5.16) is a conservation law with density M and flux N given by

$$M = \frac{\varphi\,\widetilde{\psi}}{\alpha_p} - \frac{\mathtt{A}}{p+\zeta} = \frac{\Lambda\widetilde{\Upsilon} - \mathtt{A}}{p+\zeta} \tag{5.17}$$

$$N = \frac{\psi\,\widehat{\varphi}}{\alpha_q} - \frac{\mathtt{A}}{q-\zeta} = \frac{\Upsilon\widetilde{\Lambda} - \mathtt{A}}{q-\zeta}, \tag{5.18}$$

where the extra constant terms have been added so that M and N decay to zero at the relevant boundaries. Here $\mathtt{A} = \mathtt{A}(\zeta)$ is the spectral function from Chapter 4, i.e. the reciprocal of the transmission coefficient. Now in order to generate an infinite number of conservation laws from (5.16) we note that both M and N are analytic around the point $\zeta = p$, and we thus may expand each as a power series in $\epsilon := \zeta - p$. We now obtain a recursion relation for the quantity

$$\mathtt{F}_n := \frac{\varphi(n-1;\zeta)\psi(n;\zeta)}{\alpha_p}$$

as a function defined in the n-direction. Consider equation (5.14) for ψ:

$$\alpha_p\Big(\psi(n+2;\zeta)+\psi(n;\zeta)\Big)=\mathtt{V}_{n+1}\psi(n+1;\zeta). \tag{5.19}$$

If we multiply this by $\varphi(n+1)$ and use the identity

$$\alpha_p\psi(n;\zeta)\varphi(n+1;\zeta)=2\zeta\mathtt{A}(\zeta)+\alpha_p\varphi(n;\zeta)\psi(n+1;\zeta),$$

then in terms of $\mathtt{F}$ we have

$$\varphi(n+1;\zeta)\psi(n+1;\zeta)=\frac{\alpha_p^2(\mathtt{F}_{n+2}+\mathtt{F}_{n+1})-2\zeta\mathtt{A}(\zeta)}{\mathtt{V}_{n+1}}. \tag{5.20}$$

Now multiply equation (5.19) by φ to obtain

$$\varphi(n;\zeta)\psi(n+2;\zeta)=\mathtt{V}_{n+1}\mathtt{F}_{n+1}-\varphi(n;\zeta)\psi(n;\zeta), \tag{5.21}$$

and by then taking equation (5.19) for φ and multiplying it by $\psi(n+2;\zeta)$ and using (5.20) and (5.21), we obtain the closed-form linear expression for $\mathtt{F}$:

$$\begin{aligned}\mathtt{F}_{n+2}-\mathtt{F}_{n+1}=&\left[\frac{\alpha_p^2(\mathtt{F}_{n+3}+\mathtt{F}_{n+2})-2\zeta\mathtt{A}(\zeta)}{\mathtt{V}_{n+1}\mathtt{V}_{n+2}}\right]\\&-\left[\frac{\alpha_p^2(\mathtt{F}_{n+1}+\mathtt{F}_{n})-2\zeta\mathtt{A}(\zeta)}{\mathtt{V}_{n}\mathtt{V}_{n+1}}\right].\end{aligned} \tag{5.22}$$

The general solution is first found by solving the homogeneous equation

$$\mathtt{H}_{n+2}-\mathtt{H}_{n+1}=\alpha_p^2\left[\frac{\mathtt{H}_{n+3}+\mathtt{H}_{n+2}}{\mathtt{V}_{n+1}\mathtt{V}_{n+2}}\right]-\alpha_p^2\left[\frac{\mathtt{H}_{n+1}+\mathtt{H}_{n}}{\mathtt{V}_{n}\mathtt{V}_{n+1}}\right] \tag{5.23}$$

and then adding the particular solution

$$\mathtt{H}_n^p=\frac{-2\zeta\mathtt{A}(\zeta)}{\mathtt{V}_{n-1}\mathtt{V}_n}.$$

We now make a Taylor expansion around the point $\zeta=p$ by letting $\zeta=p+\epsilon$, which implies $\alpha_p^2=-2p\epsilon-\epsilon^2$. Firstly the particular solution $\mathtt{H}^p$ will become

$$\mathtt{H}_n^p=\frac{-2}{\mathtt{V}_{n-1}\mathtt{V}_n}\Big(p\mathtt{A}(p)+(p\mathtt{A}'(p)+\mathtt{A}(p))\epsilon+\ldots\Big)$$

and thus will only change by a scaling factor. By writing

$$\mathtt{H}_n = \mathtt{H}_n^{(o)} + \mathtt{H}_n^{(1)}\epsilon + \mathtt{H}_n^{(2)}\epsilon^2 + ...$$

the homogeneous equation (5.23) gives at $O(1)$

$$\mathtt{H}_{n+2}^{(o)} - \mathtt{H}_{n+1}^{(o)} = 0 \quad \Rightarrow \quad \mathtt{H}_n^{(o)} = C_o$$

for some constant C_o. Thus at $O(1)$ we have

$$\mathtt{H} + \mathtt{H}^p - \frac{\mathtt{A}(\zeta)}{p+\zeta} = C_o - \frac{2p\mathtt{A}(p)}{\mathtt{V}_{n-1}\mathtt{V}_n} - \frac{\mathtt{A}(p)}{2p},$$

and since the left-hand side decays to zero as $n \to \pm\infty$ we have $C_o = \frac{\mathtt{A}(p)}{p}$ and so the $O(\epsilon^o)$ conserved density is

$$M^{(o)} = \frac{\mathtt{A}(p)}{2p}\left(1 - \frac{4p^2}{\mathtt{V}_{n-1}\mathtt{V}_n}\right). \tag{5.24}$$

Now by considering the $O(\epsilon)$ terms in equation (5.23) we have

$$\begin{aligned}\mathtt{H}_{n+2}^{(1)} - \mathtt{H}_{n+1}^{(1)} &= \left[\frac{2\mathtt{A}(p)}{\mathtt{V}_n\mathtt{V}_{n+1}}\right] - \left[\frac{2\mathtt{A}(p)}{\mathtt{V}_{n+1}\mathtt{V}_{n+2}}\right] \\ \Rightarrow \mathtt{H}_n^{(1)} &= C_1 - \frac{2\mathtt{A}(p)}{\mathtt{V}_{n-1}\mathtt{V}_n}\end{aligned}$$

for some constant C_1, and so adding this to the relevant term in the expansion of $\mathtt{H}^p$ and using the boundary conditions of $\mathtt{F}$ we find firstly that $C_1 = \frac{\mathtt{A}'(p)}{p} + \frac{3\mathtt{A}(p)}{4p^2}$ and then that the $O(\epsilon)$ conserved density is

$$\begin{aligned}M^{(1)} &= \frac{\mathtt{A}(p)}{p^2} + \frac{\mathtt{A}'(p)}{2p} - \frac{4\mathtt{A}(p) + 2p\mathtt{A}'(p)}{\mathtt{V}_{n-1}\mathtt{V}_n} \\ &= \left[\frac{2}{p} + \frac{\mathtt{A}'(p)}{\mathtt{A}(p)}\right] M^{(o)}\end{aligned}$$

and thus we see that the $O(\epsilon)$ conserved density is simply a scalar multiple of $M^{(o)}$. If we now look at $O(\epsilon^r)$ we have

$$\begin{aligned}\mathtt{H}_{n+2}^{(r)} - \mathtt{H}_{n+1}^{(r)} =& 2p\left[\frac{\mathtt{H}_{n+1}^{(r-1)} + \mathtt{H}_n^{(r-1)}}{\mathtt{V}_n\mathtt{V}_{n+1}}\right] - 2p\left[\frac{\mathtt{H}_{n+3}^{(r-1)} + \mathtt{H}_{n+2}^{(r-1)}}{\mathtt{V}_{n+1}\mathtt{V}_{n+2}}\right] \\ &+ \left[\frac{\mathtt{H}_{n+1}^{(r-2)} + \mathtt{H}_n^{(r-2)}}{\mathtt{V}_n\mathtt{V}_{n+1}}\right] - \left[\frac{\mathtt{H}_{n+3}^{(r-2)} + \mathtt{H}_{n+2}^{(r-2)}}{\mathtt{V}_{n+1}\mathtt{V}_{n+2}}\right].\end{aligned} \tag{5.25}$$

Equation (5.25) defines a recursion operator which enables one to successively determine every $\mathtt{H}^{(r)}$ for $r \geq 0$, which can be added to the relevant term in the expansion of $\mathtt{H}^p$ to obtain the $O(\epsilon^r)$ conserved density $M^{(r)}$. The integration constant in each case is chosen such that $M^{(r)} \to 0$ as $n \to \pm\infty$. This bears a great resemblance to the recursion operator (5.5) for the KdV equation, and in both cases is a generator for conserved densities. Going to $O(\epsilon^2)$ we find

$$\begin{aligned}\mathtt{H}^{(2)}_{n+1} = \frac{5\mathtt{A}(p)}{8p^3} + \frac{5\mathtt{A}'(p)}{4p^2} + \frac{\mathtt{A}''(p)}{2p} - \left(\frac{2C_o + 4pC_1}{\mathtt{V}_n\mathtt{V}_{n+1}}\right) \\ + 4p\mathtt{A}(p)\left(\frac{1}{\mathtt{V}_{n-1}\mathtt{V}_n^2\mathtt{V}_{n+1}} + \frac{1}{\mathtt{V}_n^2\mathtt{V}_{n+1}^2} + \frac{1}{\mathtt{V}_n\mathtt{V}_{n+1}^2\mathtt{V}_{n+2}}\right),\end{aligned}$$

which implies that

$$\begin{aligned}M^{(2)} = \left[\frac{5}{2p^2} + \frac{3\mathtt{A}'(p)}{p\mathtt{A}(p)} + \frac{\mathtt{A}''(p)}{2\mathtt{A}(p)}\right] M^{(o)} \\ + \frac{\mathtt{A}(p)}{4p^3}\left(\frac{16p^4}{\mathtt{V}_{n-2}\mathtt{V}_{n-1}^2\mathtt{V}_n} + \frac{16p^4}{\mathtt{V}_{n-1}^2\mathtt{V}_n^2} + \frac{16p^4}{\mathtt{V}_{n-1}\mathtt{V}_n^2\mathtt{V}_{n+1}} - 3\right).\end{aligned} \tag{5.26}$$

Thus up to a scalar multiple the first two functionally independent conserved densities are

$$\begin{aligned}M^{(1)} &= \frac{1}{\mathtt{V}_n\mathtt{V}_{n+1}} - \frac{1}{4p^2} \\ M^{(2)} &= \frac{1}{\mathtt{V}_n\mathtt{V}_{n+1}^2\mathtt{V}_{n+2}} + \frac{1}{\mathtt{V}_{n+1}^2\mathtt{V}_{n+2}^2} + \frac{1}{\mathtt{V}_{n+1}\mathtt{V}_{n+2}^2\mathtt{V}_{n+3}} - \frac{3}{16p^4}.\end{aligned}$$

For the H1 equation we have

$$\mathtt{V}_n = u_{n+2,m} - u_{n,m},$$

and since

$$\sum_{n=-\infty}^{+\infty}\left(\frac{16p^4}{\mathtt{V}_{n-2}\mathtt{V}_{n-1}^2\mathtt{V}_n} - 1\right) = \sum_{n=-\infty}^{+\infty}\left(\frac{16p^4}{\mathtt{V}_{n-1}\mathtt{V}_n^2\mathtt{V}_{n+1}} - 1\right)$$

the densities obtained here agree with those obtained by the Gardner method in Section 5.2, where they were shown to be nontrivial. Of course we have the added constant terms, which allow these conservation laws to exist for

soliton solutions, and all solutions obtained through the discrete IST. The corresponding fluxes can then be obtained by making a similar expansion in N. In order to obtain conserved quantities along the line $n = const.$ one simply replaces equation (5.14) with

$$(q^2 - \zeta^2)^{\frac{1}{2}}\phi_1(m+2;\zeta) - \mathtt{V}_{m+1}\phi_1(m+1;\zeta) + (q^2 - \zeta^2)^{\frac{1}{2}}\phi_1(m;\zeta) = 0,$$

where the function $\mathtt{V}_m \to 2q$ as $m \to \pm\infty$, and then repeat same calculations in the m-direction.

A natural question that arises is whether one can then obtain conserved quantities along some staircase Γ. From Chapter 4 we found that the evolution of ϕ_1 along an arbitrary staircase Γ is given by equation (4.22), which is

$$(\mathtt{p}(i+1)^2-\zeta^2)^{\frac{1}{2}}\,\phi_1(i+2;\zeta)-\mathtt{V}_{i+1}\phi_1(i+1;\zeta)+(\mathtt{p}(i)^2-\zeta^2)^{\frac{1}{2}}\,\phi_1(i;\zeta) = 0, \quad (5.27)$$

where i is the independent variable along the Γ, the function $\mathtt{V}_{i+1} \to \mathtt{p}(i) + \mathtt{p}(i+1)$ as $i \to \pm\infty$, and $\mathtt{p}(i)$ are the staircase parameters, which cycle through the lattice parameters p_k following a specified stepping algorithm. For example if N=4 and Γ is a $(1,2,3)$-staircase in the (n,m,l)-directions with parameters p,q,r respectively, then $\mathtt{p}$ will cycle through
$p,q,q,r,r,r,p,q,q,r,r,r,....$ If we donte the remaining lattice variable and parameter by y and s respectively, then a conservation law involving a shift along Γ is an equation of the form

$$\Delta_y\Big[Y\Big] = \Delta_i\Big[I\Big], \quad (5.28)$$

where i may represent either n, m or l due to the multidimensional consistency. Using the above results we may set

$$Y = \frac{\varphi(i, y; \zeta)\psi(i+1, y; \zeta)}{(p - \zeta^2)^{\frac{1}{2}}} - \frac{\mathtt{A}}{p + \zeta} \tag{5.29}$$

$$I = \frac{\psi(i, y+1; \zeta)\varphi(i, y+1; \zeta)}{(s^2 - \zeta^2)^{\frac{1}{2}}} - \frac{\mathtt{A}}{s - \zeta}. \tag{5.30}$$

It then follows that

$$\Delta_y \left(\sum_{i=-\infty}^{+\infty} Y \right) = 0, \tag{5.31}$$

in other words the quantity in the brackets is unchanged when iterated in the transverse y-direction. If we now follow the above methodology by setting $\zeta = p_k + \epsilon$, for some staircase parameter p_k, then in principle one could use equation (5.27) to obtain a recursion relation for the density I at every power of ϵ, and thus obtain an infinite number of conservation laws, yielding an infinite number of conserved quantities along Γ. We do not go into the specific calculations here, but rather just mention this possible generalisation of this technique.

In this chapter we have looked at conservation laws for both continuous and discrete integrable nonlinear equations. In particular we have shown how these may be obtained directly from the machinery of the discrete IST. The method is based on the conservation law (5.16) obtained from the square eigenfunctions, which naturally gives conserved quantities along the line $m = const.$ or $n = const.$, and these were shown to agree with those obtained by previous methods. We also showed how one can generalise this to obtain conserved quantities along an arbitrary staircase.

6 Conclusions

The application of the inverse scattering transform (IST) as a method of solving physically relevant nonlinear partial differential equations has seen much growth and use since its discovery in the late 1960s. It provides a way of linearising these systems, that is a means of obtaining their solutions through the solving of *linear* equations. Some of the major successes of the IST in mathematical physics have been the solving of the Korteweg-de Vries equation and its variants, the nonlinear Schrödinger equation and the sine-Gordon equation, all of which have great importance in the field. The IST gives a wide class of solutions satisfying decaying boundary conditions, which are typically the most physically relevant scenarios. It also provides a means of obtaining other characteristic properties of these equations such as conservation laws and asymptotics of solutions.

Many physical systems however are naturally modelled by discrete or semi-discrete equations, and as such there was a great need for an adaptation of the IST to deal with this. In the 1970s several investigations were made, and authors such as Case, Kac, Flaschka, Ablowitz and Ladik had success in adapting the IST to the semi-discrete setting. The systems that were solved were often discrete versions of well-known physical systems, and as in the continuous case this gave a method of solving such systems

for solutions with decaying boundary conditions. Investigations into fully discrete applications of the IST were also made by Ablowitz and Ladik.

Over the last decade or so however the field of discrete integrable systems has blossomed, and knowledge of the variety of discrete systems, their solutions and their integrability properties has increased profoundly. An important part of this picture was the work of Adler, Bobenko and Suris (ABS) [**12**] who exhaustively classified a restricted class of multidimensionally consistent lattice equations. With this rapid and fruitful development there was a great need for a rigorous formulation of a discrete IST as a tool for solving this new class of equations, which took into account their defining inherent properties such as their multidimensional consistency. This has been the aim of our work.

In our study we have rigorously derived a discrete IST as a tool for solving a wide class of integrable nonlinear partial difference equations. Our method has combined the ideas of Butler and Joshi [**28**] and Butler [**27**] to give a new result, which is a method of solving the initial-value problem for the majority of the ABS equations, where the initial condition is posed on a multidimensional staircase within an N-dimensional lattice. This is one of the benefits of the discrete setting, as such generalisations are not so natural in the continuous setting. In each case the solution was assumed to be real, and boundary conditions were chosen such that the potential term appearing in the scattering problem decayed at the boundary.

The solution of each nonlinear equation was shown to be expressible in terms of the solution to a singular integral equation, which is in fact a discrete analogue of the linearisation of the KdV and Painlevé II equations found by Fokas and Ablowitz [**40**] in 1981. Due to the multidimensional

nature of the discrete IST, the solutions obtained depended on all N independent variables and lattice parameters. We also found that for each equation the soliton solutions corresponded exactly to the initial conditions for which the reflection coefficient was identically zero.

The class of solutions for which the discrete IST is applicable is determined by the reality condition and the summability condition assumed on the potential. The summability condition that we found was the same as that obtained in [**28**] for the lattice potential KdV equation, however in that paper the generalisation to an N-dimensional lattice was not made. It is satisfying therefore to see that this natural extension of the IST imposes no further restrictions on the class of solutions obtained. In contrast the summability condition assumed in [**27**] was significantly stronger, however this work allowed for both a multidimensional lattice and complex-valued solutions. It would be of great interest to determine whether the discrete IST presented here could be generalised to allow for complex-valued solutions without having to strengthen this summability condition.

Another useful application of the IST is its ability to generate an infinite number of conservation laws for these nonlinear equations. Here we have shown how to do this for partial difference equations. We began with a single conservation law obtained from the first component of the forward scattering problem, and by expanding in a small parameter we found a recursion relation which can be used to generate an infinite number of nontrivial conservation laws. We also indicated how this method could be generalised to give conservation laws along an arbitrary staircase. There is however the question of whether a similar technique can be applied to the second component of the forward scattering problem, and whether this will yield additional conservation laws which are independent to those found here. Furthermore as in the continuous case it is likely that the discrete IST

can be used as a tool for obtaining qualitative and quantitative information about other properties of these nonlinear systems, such as the long-time asymptotics of solutions. These are important questions, which should be answerable from the results given here.

Finally there is the question of how this discrete IST can be applied to more general systems. The Q4 equation, which lies at the top of the ABS hierarchy, does not fall within the scope of our work here. It is likely however that the methods developed here should be applicable to this equation, after generalisation to allow for a different type of scattering problem. There are also a great number of physically important systems which lie outside the ABS classification, such as the lattice Boussinesq systems [**85**]. These are multicomponent systems which retain many of the well-known properties of integrable systems. This suggests that an adaptation of our work in this direction is also a possibility, and will form the basis of future investigations flowing from this thesis.

7 Appendix

7.1 Proof of Theorem 4.3.6

We first prove (4.41a). For $\zeta \in \mathcal{R}^+,\ \zeta \neq 0$ the recursion relation (4.34) can be upper-bounded by

$$|H_{k+1}(i;\zeta)| \leq \sum_{l=-\infty}^{i-1} |\upsilon(l)||H_k(l;\zeta)|. \tag{7.1}$$

We then claim that

$$|H_k(i;\zeta)| \leq \frac{F(i)^k}{k!}, \tag{7.2}$$

where

$$F(i) = \sum_{r=-\infty}^{i-1} |\upsilon(r)|.$$

Clearly this holds for $k = 0$. To prove the inductive step we use (7.1), summation by parts and the fact that $F(i+1) \geq F(i)$:

$$\begin{aligned}
|H_{k+1}(i;\zeta)| &\leq \sum_{l=-\infty}^{i-1} |\upsilon(l)| \frac{F(l)^k}{k!} \\
&= \frac{1}{k!} \sum_{l=-\infty}^{i-1} \big[F(l+1) - F(l)\big] F(l)^k \\
&= \frac{F(i)^{k+1}}{k!} - \frac{1}{k!} \sum_{l=-\infty}^{i-1} \big[F(l+1)^k - F(l)^k\big] F(l+1)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{F(i)^{k+1}}{k!} - \frac{1}{k!}\sum_{l=-\infty}^{i-1}\big[F(l+1)-F(l)\big]F(l+1) \\
&\qquad \times \left(\sum_{r=0}^{k-1} F(l+1)^{k-1-r}F(l)^r\right) \\
&\leq \frac{F(i)^{k+1}}{k!} - \frac{k}{k!}\sum_{l=-\infty}^{i-1}\big[F(l+1)-F(l)\big]F(l)^k,
\end{aligned}
$$

and so by examining the second and last lines we have

$$
\frac{1}{k!}\sum_{l=-\infty}^{i-1}\big[F(l+1)-F(l)\big]F(l)^k \leq \frac{F(i)^{k+1}}{(k+1)!},
$$

which then shows that

$$
|H_{k+1}(i;\zeta)| \leq \frac{F(i)^{k+1}}{(k+1)!}.
$$

Thus the estimate (7.2) holds. The series solution for Λ can then be upper-bounded by

$$
\begin{aligned}
|\Lambda(i;\zeta)-1| &\leq \sum_{k=1}^{+\infty}\frac{|H_k(i;\zeta)|}{|\zeta|^k} \leq \sum_{k=1}^{+\infty}\frac{F(i)^k}{|\zeta|^k k!} \\
&\leq \left(\frac{F(+\infty)}{|\zeta|}\right)\exp\left[\frac{F(+\infty)}{|\zeta|}\right] \leq C_1
\end{aligned}
$$

since (4.39) holds. Thus (4.41a) is proved and so for any $\zeta \neq 0$ the series solution for Λ converges absolutely and uniformly in i. We now prove (4.41b). To allow for $\zeta = 0$ we give an alternative upper-bound for the

summation equation (4.29). One can easily verify that

$$1-\prod_{r=l}^{i-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right)=\sum_{j=l}^{i-1}\left[1-\left(\frac{\mathtt{p}(j)-\zeta}{\mathtt{p}(j)+\zeta}\right)\right]\prod_{r=l}^{j-1}\left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right) \quad (7.3)$$

$$\Rightarrow |\Lambda(i;\zeta)| \leq 1+\sum_{l=-\infty}^{i-1}\left(\sum_{j=l}^{i-1}\frac{1}{|\mathtt{p}(j)+\zeta|}\prod_{r=l}^{j-1}\left|\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta}\right|\right)|\upsilon(l)||\Lambda(l;\zeta)|$$

$$\leq 1+\sum_{l=-\infty}^{i-1}\left(\sum_{j=l}^{i-1}\frac{1}{|\mathtt{p}(j)+\zeta|}\right)|\upsilon(l)||\Lambda(l;\zeta)|.$$

For $\zeta \in \mathcal{R}^+$ however we have

$$|\mathtt{p}(j)+\zeta| \geq |\mathtt{p}(j)-\zeta| \geq \big|\,|\mathtt{p}(j)|-|\zeta|\,\big| \geq |\mathtt{p}(j)|-|\zeta| \geq |\mathtt{p}(j)|$$

and so we have

$$|\Lambda(i;\zeta)| \leq 1+\eta\sum_{l=-\infty}^{i-1}(i-l)\,|\upsilon(l)|\,|\Lambda(l;\zeta)|, \quad (7.4)$$

where $\eta = \max\{\,|\mathtt{p}(r)^{-1}| : r \in \mathrm{I}\,\}$. Equation (7.4) is a majorant for both the summation equations (4.29) and (4.31), and thus may be used to estimate $\Lambda(i;\zeta)$ for all $\zeta \in \mathcal{R}^+$. Thus we have

$$|\Lambda(i;\zeta)| \leq \sum_{k=0}^{+\infty}\eta^k H_k^*(i)$$

where

$$H_0^* = 1, \qquad H_{k+1}^*(i) = \sum_{l=-\infty}^{i-1}(i-l)\,|\upsilon(l)|\,H_k^*(l).$$

We claim that

$$|H_k^*(i)| \leq \frac{G(i,i)^k}{k!} \quad (7.5)$$

where

$$G(i,j) = \sum_{r=-\infty}^{j-1}(i-r)\,|\upsilon(r)|.$$

Clearly this holds for $k = 0$. To prove the inductive step we use the recursion relation for H_k^*, and again summation by parts and the properties of G:

$$\begin{aligned}
|H_{k+1}^*(i)| &\leq \sum_{l=-\infty}^{i-1} (i-l)\,|v(l)| \left(\frac{G(l,l)^k}{k!}\right) \\
&\leq \frac{1}{k!} \sum_{l=-\infty}^{i-1} \big[G(i,l+1) - G(i,l)\big] G(i,l)^k \\
&= \frac{1}{k!} G(i,i)^{k+1} - \frac{1}{k!} \sum_{l=-\infty}^{i-1} \big[G(i,l+1)^k - G(i,l)^k\big] G(i,l+1) \\
&= \frac{1}{k!} G(i,i)^{k+1} - \frac{1}{k!} \sum_{l=-\infty}^{i-1} \big[G(i,l+1) - G(i,l)\big] G(i,l+1) \\
&\qquad \times \left(\sum_{r=0}^{k-1} G(i,l+1)^{k-1-r} G(i,l)^r \right) \\
&\leq \frac{1}{k!} G(i,i)^{k+1} - \frac{k}{k!} \sum_{l=-\infty}^{i-1} \big[G(i,l+1) - G(i,l)\big] G(i,l)^k,
\end{aligned}$$

and again by examining the second and last lines we have

$$\frac{1}{k!} \sum_{l=-\infty}^{i-1} \big[G(i,l+1) - G(i,l)\big] G(i,l)^k \leq \frac{G(i,i)^{k+1}}{(k+1)!}$$

which implies

$$|H_{k+1}^*(i)| \leq \frac{G(i,i)^{k+1}}{(k+1)!}$$

and completes the inductive step. Thus for all $\zeta \in \mathcal{R}^+$

$$|\Lambda(i;\zeta) - 1| \leq \sum_{k=1}^{+\infty} \frac{\eta^k G(i,i)^k}{k!} \leq \eta\, G(i,i) \exp\Big[\eta\, G(i,i)\Big].$$

Let us first consider $i \leq 0$. In this case we have

$$\begin{aligned}
|\Lambda(i;\zeta) - 1| &\leq \eta \exp\Big[\eta\, G(i,i)\Big] \left[i \sum_{r=-\infty}^{i-1} |\upsilon(r)| + \sum_{r=-\infty}^{i-1} (-r)|\upsilon(r)| \right] \\
&\leq \eta \exp\Big[\eta\, G(0,0)\Big] \left[\sum_{r=-\infty}^{-1} (-r)|\upsilon(r)| \right] \\
&\leq D_1,
\end{aligned}$$

for some constant D_1, courtesy of (4.39). To examine the case $i > 0$ we consider the majorant (7.4)

$$\begin{aligned}
|\Lambda(i;\zeta)| &\leq 1 + \eta \sum_{l=-\infty}^{i-1} (-l)\, |\upsilon(l)|\, |\Lambda(l;\zeta)| + i\,\eta \sum_{l=-\infty}^{i-1} |\upsilon(l)|\, |\Lambda(l;\zeta)| \\
&\leq 1 + \eta \sum_{l=-\infty}^{-1} (-l)\, |\upsilon(l)|\, |\Lambda(l;\zeta)| + i\,\eta \sum_{l=-\infty}^{i-1} |\upsilon(l)|\, |\Lambda(l;\zeta)| \\
&\leq D_2 + i\,\eta \sum_{l=-\infty}^{i-1} |\upsilon(l)|\, |\Lambda(l;\zeta)|
\end{aligned}$$

for some constant D_2, where we have used (4.39) and the fact that $\Lambda(i;\zeta)$ can be upper-bounded by a constant for $i \leq 0$. Write $\Lambda(i;\zeta) = D_2(1 + i)\Xi(i;\zeta)$, then the upper-bound for Ξ becomes

$$\begin{aligned}
|\Xi(i;\zeta)| &\leq 1 + \eta \left(\frac{i}{1+i} \right) \sum_{l=-\infty}^{i-1} (1 + |l|)\, |\upsilon(l)|\, |\Xi(l;\zeta)| \\
&\leq 1 + \eta \sum_{l=-\infty}^{i-1} (1 + |l|)\, |\upsilon(l)|\, |\Xi(l;\zeta)|.
\end{aligned}$$

Making similar arguments to those presented above it follows that

$$\begin{aligned}
|\Xi(i;\zeta)| &\leq \exp\left(\eta \sum_{l=-\infty}^{i-1} (1 + |l|)\, |\upsilon(l)| \right) \\
&\leq \exp\left(\eta \sum_{l=-\infty}^{+\infty} (1 + |l|)\, |\upsilon(l)| \right)
\end{aligned}$$

which in turn implies that for $i > 0$,

$$|\Lambda(i;\zeta)| \le C_2(1+i)$$

for some constant C_2. Combining this with the result for $i \le 0$ proves (4.41b). Thus for each i the series solution for Λ converges absolutely and uniformly in ζ for $\zeta \in \mathcal{R}^+$. Since the iterates H_k are continuous in this region and analytic in its interior, Λ also has this property. The results for Υ follow in a similar fashion.

7.2 Proof of Theorem 4.3.7

To prove (4.43) we first use (7.3) to rewrite the summation equation (4.29) as

$$\Lambda(i;\zeta) = 1 + \sum_{l=-\infty}^{i-1} \left(\sum_{j=l}^{i-1} \frac{1}{(\mathtt{p}(j)+\zeta)} \prod_{r=l}^{j-1} \left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta} \right) \right) \upsilon(l)\Lambda(l;\zeta),$$

which agrees with (4.31) and is therefore valid for all $\zeta \in \mathcal{R}^+$. Taking a derivative of this equation gives

$$\begin{aligned}\Lambda'(i;\zeta) = &\sum_{l=-\infty}^{i-1} \left(\sum_{j=l}^{i-1} \frac{1}{(\mathtt{p}(j)+\zeta)} \prod_{r=l}^{j-1} \left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta} \right) \right) \upsilon(l)\Lambda'(l;\zeta) \\ &- \sum_{l=-\infty}^{i-1} \sum_{j=l}^{i-1} \left(\frac{1}{(\mathtt{p}(j)+\zeta)^2} \prod_{r=l}^{j-1} \left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta} \right) \right. \\ &\left. + \frac{1}{(\mathtt{p}(j)+\zeta)} \sum_{s=l}^{j-1} \frac{2\mathtt{p}(s)}{(\mathtt{p}(s)+\zeta)^2} \prod_{r=l, r\neq s}^{j-1} \left(\frac{\mathtt{p}(r)-\zeta}{\mathtt{p}(r)+\zeta} \right) \right) \upsilon(l)\Lambda(l;\zeta)\end{aligned}$$

which can be upper-bounded by

$$|\Lambda'(i;\zeta)| \le \eta \sum_{l=-\infty}^{i-1} (i-l)\,\upsilon(l)\,\Lambda'(l;\zeta) + \eta^2 \sum_{l=-\infty}^{i-1} (i-l)^2\,|\upsilon(l)|\,|\Lambda(l;\zeta)| \quad (7.6)$$

Firstly if $i \leq 0$ then since $|\Lambda(i;\zeta)|$ can be bounded by a constant, the summability condition (4.42) implies that

$$\eta^2 \sum_{l=-\infty}^{i-1} (i-l)^2 \, |\upsilon(l)| \, |\Lambda(l;\zeta)| \leq D_1$$

for some constant D_1. If on the other hand $i > 0$ then using $(i-l)^2 \leq 2(i^2+l^2)$ we have

$$\begin{aligned}
\eta^2 \sum_{l=-\infty}^{i-1} (i-l)^2 \, |\upsilon(l)| \, |\Lambda(l;\zeta)| &\leq 2\eta^2 \sum_{l=-\infty}^{i-1} l^2 \, |\upsilon(l)| \, |\Lambda(l;\zeta)| \\
&\quad + 2\eta^2 \sum_{l=-\infty}^{i-1} i^2 \, |\upsilon(l)| \, |\Lambda(l;\zeta)| \\
&\leq 2\eta^2 \sum_{l=-\infty}^{-1} l^2 \, |\upsilon(l)| \, |\Lambda(l;\zeta)| + 2\eta^2 \sum_{l=1}^{i-1} l^2 \, |\upsilon(l)| \, |\Lambda(l;\zeta)| \\
&\quad + 2i^2 \eta^2 \sum_{l=-\infty}^{i-1} |\upsilon(l)| \, |\Lambda(l;\zeta)| \\
&\leq D_1 + 2i^2 \eta^2 \sum_{l=1}^{i-1} |\upsilon(l)| \, |\Lambda(l;\zeta)| + 2i^2 \eta^2 \sum_{l=-\infty}^{i-1} |\upsilon(l)| \, |\Lambda(l;\zeta)| \\
&\leq D_1 + i^2 D_3 \sum_{l=-\infty}^{i-1} (1+|l|) \, |\upsilon(l)| \\
&\leq D_4(1+i^2),
\end{aligned}$$

for some constants $D_1 \to D_4$, where we have used (4.41b) to upper-bound $\Lambda(i;\zeta)$ for $i > 0$. Thus the upper-bound (7.6) becomes

$$|\Lambda'(i;\zeta)| \leq D_5(1 + i \max\{0,i\}) + \eta \sum_{l=-\infty}^{i-1} (i-l) \, |\upsilon(l)| \, |\Lambda'(l;\zeta)|. \tag{7.7}$$

Now consider the function θ defined by $\Lambda'(i;\zeta) = D_5\,(1+i\max\{0,i\})\,\theta(i;\zeta)$. If $i \leq 0$ then we have

$$|\theta(i;\zeta)| \leq 1 + \eta \sum_{l=-\infty}^{i-1} (i-l)\,|\upsilon(l)|\,|\theta(l;\zeta)|$$
$$\Rightarrow |\theta(i;\zeta)| \leq \exp\big[\eta\, G(i,i)\big] \leq \exp\big[\eta\, G(0,0)\big]$$

where $G(i,j)$ is defined in the proof of Theorem 4.3.6. If $i > 0$ then we have

$$\begin{aligned}
|\theta(i;\zeta)| &\leq 1 + \eta \sum_{l=-\infty}^{i-1} (i-l)\,|\upsilon(l)|\,|\theta(l;\zeta)| \left[\frac{1+j\max\{0,j\}}{1+i^2}\right] \\
&= 1 + \eta \sum_{l=-\infty}^{-1} (i-l)\,|\upsilon(l)|\,|\theta(l;\zeta)| \left[\frac{1}{1+i^2}\right] \\
&\qquad + \eta \sum_{l=0}^{i-1} (i-l)\,|\upsilon(l)|\,|\theta(l;\zeta)| \left[\frac{1+j^2}{1+i^2}\right] \\
&\leq 1 + \eta \sum_{l=-\infty}^{i-1} (i-l)\,|\upsilon(l)|\,|\theta(l;\zeta)| \\
&\Rightarrow\ |\theta(i;\zeta)| \leq \exp\big[\eta\, G(i,i)\big].
\end{aligned}$$

Thus for all i we have

$$|\Lambda'(i;\zeta)| \leq D_5\,(1 + i\max\{0,i\})\,\exp\big[\eta\, G(i,i)\big],$$

which allows us to rewrite (7.7) as

$$\begin{aligned}
|\Lambda'(i;\zeta)| \leq D_5(1 + i\max\{0,i\}) + \eta \sum_{l=-\infty}^{i-1} (-l)\,|\upsilon(l)|\,|\Lambda'(l;\zeta)| \\
+ i\eta \sum_{l=-\infty}^{i-1} |\upsilon(l)|\,|\Lambda'(l;\zeta)|
\end{aligned}$$

$$\leq D_5(1 + i\max\{0, i\}) + \eta \sum_{l=-\infty}^{-1} (-l)\,|\upsilon(l)|\,|\Lambda'(l;\zeta)|$$

$$+ i\,\eta \sum_{l=-\infty}^{i-1} |\upsilon(l)|\,|\Lambda'(l;\zeta)|$$

$$\leq D_6(1 + i\max\{0, i\}) + i\,\eta \sum_{l=-\infty}^{i-1} |\upsilon(l)|\,|\Lambda'(l;\zeta)|.$$

Now for $i \leq 0$ we have

$$|\Lambda'(i;\zeta)| \leq D_6 + i\,\eta \sum_{l=-\infty}^{i-1} |\upsilon(l)|\,|\Lambda'(l;\zeta)|,$$

so by defining $\Lambda'(i;\zeta) = D_6(1 + |i|)\Xi_1(i;\zeta)$ we have

$$|\Xi_1(i;\zeta)| \leq 1 + \eta \sum_{l=-\infty}^{i-1} (1 + |l|)\,|\upsilon(l)|\,|\Xi_1(l;\zeta)|$$

$$\Rightarrow |\Xi_1(i;\zeta)| \leq D_7 \;\Rightarrow\; |\Lambda'(i;\zeta)| \leq D_8(1 + |i|)$$

for new constants D_7 and D_8. If $i > 0$ then by defining $\Lambda'(i;\zeta) = D_6(1 + i^2)\Xi_2(i;\zeta)$ we have

$$|\Xi_2(i;\zeta)| \leq 1 + \eta \sum_{l=-\infty}^{i-1} (1 + l^2)\,|\upsilon(l)|\,|\Xi_2(l;\zeta)|$$

$$\Rightarrow |\Xi_2(i;\zeta)| \leq D_9 \;\Rightarrow |\Lambda'(i;\zeta)| \leq D_{10}(1 + i^2).$$

This proves (4.43), and (4.44) is proved in a similar manner. Thus one can iterate the derivatives of the summation equations to obtain series solutions for Λ' and Υ', which for any given i, converge absolutely and uniformly in ζ for all $\zeta \in \mathcal{R}^+$, and for a given $\zeta \in \mathcal{R}^+$, converge absolutely in i. These functions are therefore continuous functions of ζ in $\mathcal{R}^+$.

Notation

n_k	discrete independent variables
p_k	continuous lattice parameters
a, b	additional parameters independent of p_k
u	lattice dependent variable
$\widehat{u}$	u shifted one unit in the n_k-direction, for arbitrary k
ζ	continuous spectral parameter
Γ	staircase within the N-dimensional lattice
I	set of lattice directions in which Γ exists
J	set of lattice directions in which Γ does not exist
$\overline{u}$	u shifted one unit along Γ
i	discrete independent variable along Γ
$\mathtt{p}(i)$	parameters along Γ
φ, ψ	Jost solutions analytic in $\mathrm{Re}(\zeta) > 0$
$\mathring{\varphi}, \mathring{\psi}$	Jost solutions analytic in $\mathrm{Re}(\zeta) < 0$
Λ, Υ	normalised Jost solutions analytic in $\mathrm{Re}(\zeta) > 0$
$\mathring{\Lambda}, \mathring{\Upsilon}$	normalised Jost solutions analytic in $\mathrm{Re}(\zeta) < 0$
$W(x, y)$	Wronskian of x and y
$\mathtt{A}, \mathtt{B}$	spectral functions relating ψ to φ and $\mathring{\varphi}$
R	reflection coefficient
φ^*	complex conjugate of φ

Notation

$\mathcal{R}^+$	half-plane $\mathrm{Re}(\zeta) \geq 0$
$\mathcal{R}^-$	half-plane $\mathrm{Re}(\zeta) \leq 0$
Φ	square eigenfunction
$\mathtt{c}_k$	normalisation constants
$\varphi^{(N)}$	N-dimensional Jost solution
$\Lambda^{(a)}$	normalised Jost solution obtained by swapping a and b in the Lax equations
$\mathcal{H}$	biquadratic of a lattice equation
$\mathtt{K}$	antisymmetric function obtained from the determinant of the Lax matrix

References

[1] M. Ablowitz, G. Biondini, and B. Prinari. Inverse scattering transform for the integrable discrete nonlinear Schrödinger equation with non-vanishing boundary conditions. *Inv. Prob.*, 23(1711-1758), 2007.

[2] M. Ablowitz and P. Clarkson. *Solitons, nonlinear evolution equations and inverse scattering*. Cambridge University Press, 1991.

[3] M. Ablowitz, D. Kaup, A. Newell, and H. Segur. Method for solving the sine-Gordon equation. *Phys. Rev. Lett.*, 30:1262–1264, 1973.

[4] M. Ablowitz, D. Kaup, A. Newell, and H. Segur. Nonlinear evolution equations of physical significance. *Phys. Rev. Lett.*, 31:125–127, 1973.

[5] M. Ablowitz, D. Kaup, A. Newell, and H. Segur. The inverse scattering transform - Fourier analysis for nonlinear problems. *Stud. Appl. Math.*, 53:249–315, 1974.

[6] M. Ablowitz and J. Ladik. Nonlinear differential-difference equations. *J. Math. Phys.*, 16:598–603, 1975.

[7] M. Ablowitz and J. Ladik. Nonlinear differential-difference equations and Fourier analysis. *J. Math. Phys.*, 17(1011-1018), 1976.

[8] M. Ablowitz and J. Ladik. On the solutions of a class of nonlinear partial difference equations. *Stud. Appl. Math.*, 57:1–12, 1977.

[9] M. Ablowitz and H. Segur. Asymptotic solutions of the Korteweg-de Vries equation. *Stud. Appl. Math.*, 57:13–44, 1977.

[10] M. Ablowitz and H. Segur. *Solitons and the inverse scattering transform*. Society for Industrial and Applied Mathematics Philadelphia, 1981.

[11] V. Adler. Bäcklund transformation for the Krichever-Novikov equation. *Int. Math. Res. Not.*, 1:1–4, 1998.

[12] V. Adler, A. Bobenko, and Y. Suris. Classification of integrable equations on quad-graphs. the consistency approach. *Comm. Math. Phys.*, 233:513–543, 2003.

[13] V. Adler, A. Bobenko, and Y. Suris. Discrete nonlinear hyperbolic equations. classification of integrable cases. *Funct. Anal. Appl.*, 43(1):3–17, 2009.

[14] V. Adler and Y. Suris. Q4: Integrable master equation related to an elliptic curve. *Int. Math. Res. Not. IMRN*, 47:2523–2553, 2004.

[15] V. Adler and A. Veselov. Cauchy problem for integrable discrete equations on quad-graphs. *Acta Appl. Math.*, 84:237–262, 2004.

[16] J. Atkinson. Bäcklund transformations for integrable lattice equations. *J. Phys. A*, 41:135202, 2008.

[17] J. Atkinson, J. Hietarinta, and F. Nijhoff. Seed and soliton solutions of Adler's lattice equation. *J. Phys. A*, 40:F1–F8, 2007.

[18] J. Atkinson, J. Hietarinta, and F. Nijhoff. Soliton solutions for Q3. *J. Phys. A*, 41:142001, 2008.

[19] J. Atkinson and F. Nijhoff. Solutions of Adler's lattice equations associated with 2-cycles of the Bäcklund transformations. *J. Nonlinear Math. Phys.*, 15:34–42, 2007.

[20] J. Atkinson and F. Nijhoff. A constructive approach to the soliton solutions of integrable quadrilateral lattice equations. *Comm. Math. Phys.*, 299:283–304, 2010.

[21] M. Bellon and C. Viallet. Algebraic entropy. *Comm. Math. Phys.*, 204:425–437, 1999.

[22] L. Bianchi. *Vorlesungen über differenzialgeometrie*. Teubner Leipzig, 1899.

[23] H. Blohm. Solutions of a discrete inverse scattering problem and the Cauchy problem of a class of discrete evolution equations. *J. Math. Phys.*, 40:4374–4392, 1999.

[24] A. Bobenko and Y. Suris. Integrable systems on quad-graphs. *Int. Math. Res. Not. IMRN*, 11:573–611, 2002.

[25] M. Boiti, F. Pempinelli, B. Prinari, and A. Spire. An integrable discretization of KdV at large times. *Inv. Prob.*, 17:515–526, 2001.

[26] J. Boussinesq. *Essai sur la théorie des eaux courantes. Memoires présentés par divers savantes à l'Acad. Des Sci.* Inst. Nat. France XXIII, 1877.

[27] S. Butler. Multidimensional inverse scattering of integrable lattice equations. *Nonlinearity*, 25:1613–1634, 2012.

[28] S. Butler and N. Joshi. An inverse scattering transform for the lattice potential KdV equation. *Inv. Prob.*, 26:115012, 2010.

[29] K. Case. On discrete inverse scattering problems II*. *J. Math. Phys.*, 14(7):916–920, 1973.

[30] K. Case and M. Kac. A discrete version of the inverse scattering problem. *J. Math. Phys.*, 14(5):594–603, 1973.

[31] H.-H. Chen. General derivation of Bäcklund transformations from inverse scattering problems. *Phys. Rev. Lett.*, 33(15):925–928, 1974.

[32] E. Date, M. Jimbo, and T. Miwa. Method for generating discrete soliton equations. I-V. *J. Phys. Soc. Japan*, 51, 52:4116–4131, 761–771, 1982, 1983.

[33] P. Deift and E. Trubowitz. Inverse scattering on the line. *Comm. Pure Appl. Math.*, 32:121–251, 1979.

[34] B. Dubrovin. Periodic problem for the Korteweg-de Vries equation in the class of finite band potentials. *Funct. Anal. Appl.*, 9:215–223, 1975.

[35] B. Dubrovin, V. Matveev, and S. Novikov. Nonlinear equations of Korteweg-de Vries type, finite zoned linear operators, and Abelian varieties. *Russ. Math. Surv.*, 31:59–146, 1976.

[36] B. Dubrovin and S. Novikov. Periodic and conditionally periodic analogues of the many-soliton solutions of the Korteweg-de Vries equation. *Sov. Phys. JETP*, 40:1058–1063, 1974.

[37] L. Faddeev and (trans. from russian by B. Seckler). The inverse problem in the quantum theory of scattering. *J. Math. Phys.*, 4:72–104, 1963.

[38] H. Flaschka. On the Toda lattice II. *Progr. Theoret. Phys*, 51(3):703–716, 1974.

[39] A. Fokas. A unified transform method for solving linear and certain nonlinear PDEs. *Proc. R. Soc. Lond. A*, 453:1411–1443, 1997.

[40] A. Fokas and M. Ablowitz. Linearization of the Korteweg-de Vries and Painlevé ii equations. *Phys. Rev. Lett.*, 47:1096, 1981.

[41] A. Fokas, A. Its, and L.-Y. Sung. The nonlinear Schrödinger equation on the half-line. *Nonlinearity*, 18:1771–1822, 2005.

[42] F. Gakhov. *Boundary value problems*. Pergamon Press Oxford, 1966.

[43] C. Gardner, J. Greene, M. Kruskal, and R. Miura. Method for solving the Korteweg-de Vries equation. *Phys. Rev. Lett.*, 19:1095–1097, 1967.

[44] C. Gardner, J. Greene, M. Kruskal, and R. Miura. Korteweg-de Vries equation and generalizations. VI. Methods for exact solution. *Comm. Pure Appl. Math.*, 27:97–133, 1974.

[45] I. Gel'fand and B. Levitan. On the determination of a differential equation from its spectral function. *Proc. St. Pet. Math. Soc.*, 1:253–304, 1955.

[46] B. Grammaticos, A. Ramani, and V. Papageorgiou. Do integrable mappings have the Painlevé property? *Phys. Rev. Lett.*, 67:1825, 1991.

[47] B. Herbst and M. Ablowitz. Numerically induced chaos in the nonlinear Schrödinger equation. *Phys. Rev. Lett.*, 62(18):2065–2068, 1989.

[48] J. Hietarinta. Searching for CAC-maps. *J. Nonlinear Math. Phys.*, 12(2):233–230, 2005.

[49] J. Hietarinta and D. Zhang. Soliton solutions for ABS lattice equations: Casoratians and bilinearization. *J. Phys. A*, 42:404006, 2009.

[50] R. Hirota. Nonlinear partial difference equations I-III. *J. Phys. Soc. Japan*, 43(1424-1433, 2074-2089), 1977.

[51] P. Hydon. Conservation laws of partial difference equations with two independent variables. *J. Phys. A*, 34:10347–10355, 2001.

[52] R. Jost and W. Kohn. Construction of a potential from a phase shift. *Phys. Rev.*, 87(6):977–992, 1952.

[53] M. Kac and R. van Moerbeke. On an explicitly soluble system of nonlinear differential equations related to certain Toda lattices. *Adv. Math.*, 16:160–169, 1975.

[54] D. Korteweg and G. de Vries. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Phil. Mag.*, 39:422–443, 1895.

[55] I. Krichever and S. Novikov. Holomorphic bundles over algebraic curves and nonlinear equations. *Russ. Math. Surv.*, 35:53–79, 1980.

[56] P. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[57] D. Levi and M. Petrera. Continuous symmetries of the lattice potential KdV equation. *J. Phys. A*, 40:4141–4159, 2007.

[58] D. Levi and P. Winternitz. Continuous symmetries of difference equations. *J. Phys. A*, 39:R1–R63, 2006.

[59] R. Mickens. *Difference equations: Theory and applications (2nd Edition)*. Van Nostrand Reinhold, 1990.

[60] A. Mikhailov, J. Wang, and P. Xenitides. Recursion operators, conservation laws and integrability conditions for difference equations. *Theo.*

Math. Phys., 167(1):421–443, 2011.

[61] R. Miura. Korteweg-de vries equation and generalizations. I. a remarkable explicit nonlinear transformation. *J. Math. Phys.*, 9:1202–1204, 1968.

[62] R. Miura, C. Gardner, and M. Kruskal. Korteweg-de vries equation and generalizations. II. existence of conservation laws and constants of motion. *J. Math. Phys.*, 9:1204–1209, 1968.

[63] F. Nijhoff. Lax pair for the Adler (lattice Krichever-Novikov) system. *Phys. Lett. A*, 297(1-2):49–58, 2002.

[64] F. Nijhoff and J. Atkinson. Elliptic N-soliton solutions of ABS lattice equations. *Int. Math. Res. Not. IMRN*, 20:3837–3895, 2010.

[65] F. Nijhoff, J. Atkinson, and J. Hietarinta. Soliton solutions for ABS lattice equations: I. Cauchy matrix approach. *J. Phys. A*, 42:404005, 2009.

[66] F. Nijhoff and H. Capel. The discrete Korteweg-de Vries equation. *Acta Appl. Math.*, 39:133–158, 1995.

[67] F. Nijhoff, G. Quispel, and H. Capel. Direct linearization of nonlinear difference-difference equations. *Phys. Lett. A*, 97(125-128), 1983.

[68] F. Nijhoff, G. Quispel, J. van der Linden, and H. Capel. On some linear integral equations generating solutions of nonlinear partial differential equations. *Physica A*, 119(101-142), 1983.

[69] F. Nijhoff and A. Walker. The discrete and continuous Painlevé VI hierarchy and the Garnier systems. *Glas. Math. J.*, 43A:109–123, 2001.

[70] S. Orfanidis. Discrete sine-Gordon equation. *Phys. Rev. D*, 18(10):3822–3827, 1978.

[71] V. Papageorgiou, F. Nijhoff, and H. Capel. Integrable mappings and nonlinear integrable lattice equations. *Phys. Lett.*, 147A:106–114, 1990.

[72] L. Pilloni and D. Levi. The inverse scattering transform for solving the discrete sine-Gordon equation. *Phys. Lett.*, 92A(1):5–8, 1982.

[73] G. Quispel, H. Capel, V. Papageorgiou, and F. Nijhoff. Integrable mappings derived from soliton equations. *Physica A*, 173:243–266, 1991.

[74] G. Quispel, F. Nijhoff, H. Capel, and J. van der Linden. Linear integral equations and nonlinear difference-difference equations. *Physica A*, 125:344–380, 1984.

[75] O. Ragnisco, P. Santini, S. Chitlaru-Briggs, and M. Ablowitz. An example of $\overline{\partial}$ problem arising in a finite difference context: Direct and inverse problem for the discrete analog of the equation $\psi_{xx} + u\psi = \sigma\psi_y$. *J. Math. Phys.*, 28(4):777–780, 1987.

[76] O. Rasin and P. Hydon. Conservation laws for NQC-type difference equations. *J. Phys. A*, 39:14055–14066, 2006.

[77] O. Rasin and P. Hydon. Conservation laws for integrable difference equations. *J. Phys. A*, 40:12763–12773, 2007.

[78] O. Rasin and J. Schiff. Infinitely many conservation laws for the discrete KdV equation. *J. Phys. A*, 42:175205, 2009.

[79] S. Ruijsenaars. A new class of reflectionless second-order AΔOs and its relation to nonlocal solitons. *Regul. Chaotic Dyn.*, 7(4):351–391, 2002.

[80] J. Schiff. Loop groups and discrete KdV equations. *Nonlinearity*, 16:257–275, 2003.

[81] A. Shabat. The infinite-dimensional dressing dynamical system. *Inv. Prob.*, 8:303–308, 1992.

[82] A. Shabat. Discretization of the Schrödinger spectral problem. *Inv. Prob.*, 18:1003–1011, 2002.

[83] G. Stokes. On the theory of oscillatory waves. *Camb. Trans.*, 8:441–473, 1847.

[84] M. Toda. Waves in nonlinear lattice. *Progr. Theoret. Phys. Suppl.*, 45:174–200, 1970.

[85] A. Tongas and F. Nijhoff. The Boussinesq integrable system. Compatible lattice and continuum structures. *Glas. Math. J.*, 47A:205–219, 2005.

[86] D. Tran, P. van der Kamp, and G. Quispel. Closed-form expressions for integrals of traveling wave reductions of integrable lattice equations. *J. Phys. A*, 42:225201–225221, 2009.

[87] P. van der Kamp and G. Quispel. The staircase method: Integrals for periodic reductions of integrable lattice equations. *J. Phys. A*, 43:465207–465241, 2010.

[88] P. van der Kamp, O. Rojas, and G. Quispel. Closed-form expressions for integrals of mKdV and sine-Gordon maps. *J. Phys. A*, 40:12789–12798, 2007.

[89] M. Wadati. The exact solutions of the modified Korteweg-de Vries equation. *J. Phys. Soc. Japan*, 32:1681, 1972.

[90] H. Wahlquist and F. Estabrook. Bäcklund transformation for solutions of the Korteweg-de Vries equation. *Phys. Rev. Lett.*, 31(23):1386–1390, 1973.

[91] G. Wiersma and H. Capel. Lattice equations, hierarchies and hamiltonian structures. *Physica A*, 142:199–244, 1987.

[92] N. Zabusky and M. Kruskal. Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. *Phys. Rev. Lett.*, 15:240–243, 1965.

[93] V. Zakharov and A. Shabat. Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP*, 34:62–69, 1972.