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Combinatorics of Option Spreads: The Margining Aspect

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Abstract

In December 2005, the U.S. Securities and Exchange Commission approved margin rules for complex option spreads with 5, 6, 7, 8, 9, 10 and 12 legs. Only option spreads with 2, 3 or 4 legs were recognized before. Taking advantage of option spreads with a large number of legs substantially reduces margin requirements and, at the same time, adequately estimates risk for margin accounts with positions in options. In this paper we present combinatorial models for known and newly discovered option spreads with up to 134 legs. We propose their full characterization in terms of matchings, alternating cycles and chains in graphs with bicolored edges. We show that the combinatorial analysis of option spreads reveals powerful hedging mechanisms in the structure of margin accounts, and that the problem of minimizing the margin requirement for a portfolio of option spreads can be solved in polynomial time using network flow algorithms. We also give recommendations on how to create more efficient margin rules for options.

1 Introduction

The margining of accounts, i.e., the calculation of minimum regulatory margin requirements for *margin accounts*, is a critical intra-day and end-of-day risk management operation for any brokerage firm. There exist two approaches to margining portfolios, strategy-based and risk-based. Recently, Coffman et al. [2010] have published an experimental analysis of the two approaches to margining. Their results suggest that the risk-based approach, recently adopted in the US, has serious shortcomings. Specifically, it significantly undermargins stock option portfolios and does not provide any exit strategy. Coffman et al. conclude that strategy-based approach to margining is more appropriate for portfolios of stock options, although it is computationally and analytically more challenging and lacks in depth academic research. These conclusions call for a more detailed and comprehensive analysis of the strategy-based approach which we offer in this paper.

The strategy-based margining of an account without positions in options or other derivatives is simply the calculation of the total margin requirement for all positions in the account. Options, however, bring a nontrivial combinatorial component to the calculation because margin regulations for positions in options permit the use of different *offsets* for margin reductions that usually imitate trading strategies. Offsets involving only options are based on *option spreads*.

By the end of the nineties, it was commonly recognized that margin regulations impose excessively high minimum margin requirements. This can be partially explained by the fact that margin rules by that time permitted the use of option spreads with only two, three or four *legs*. The brokerage term "leg" stands for a single position in one *option series*. All options in an option series have the same type, i.e., *call* or *put*, the same *underlying instrument*, the same *expiry date* and the same *exercise price*.

On the other hand, the calculation of the minimum margin by using offsets with more than two legs is a computationally complex combinatorial problem that is not well understood. Despite the fact that margin regulations have a 75-year history dating from Regulation T in the Securities Act of 1934, the literature on margin calculations is surprisingly small. We can point to only two books [Geelan and Rittereiser, 1998; Curley, 2008], three papers [Rudd and Schroeder, 1982; Fiterman and Timkovsky, 2001; Coffman et al., 2010] devoted to margining algorithms and two papers [Fortune, 2000, 2003] devoted to margining practice. Literature on studying the influence of margin requirements on the market, such as for example [Moore, 1966; Luckett, 1982], is more representative; see the related survey in [Kupiec, 1998]. The vast majority of publications on margining consists primarily of regulatory circulares and manuals written by security market lawyers.

Consequently, margin calculation systems, developed and used in the brokerage industry up to 2005, ignore highly effective and broadly applicable combinatorial optimization methods. In particular, the reduction of the margin-minimization-by-pairing problem to the minimum-cost network-flow problem [Rudd and Schroeder, 1982] was seemingly forgotten for more than 20 years. As a result, existing margin calculation technology, faced with the combinatorial complexity of margin calculations, failed to take advantage of multi-leg option spreads. The vast majority of margin calculation systems used in the brokerage industry, as our study shows, uses offsets with two legs only; and they are based on outdated heuristics proposed by brokers in the mid seventies [Cox and Rubinsein, 1985; Geelan and Rittereiser, 1998]. The most advanced margining systems recognize offsets with up to four legs by using heuristics that cannot guarantee the minimum margin. However, as we show, the failure to use offsets with more that two legs can increase the margin requirement by several thousands of dollars.

The more legs an option spread has the more margin reduction it gives. Thus, the reduction of minimum margin requirements can be achieved by designing new option spreads with a larger number of legs. Option spreads with two, three and four legs, such as bull and bear spreads, butterfly spreads, condor spreads and box spreads, were known and permitted for margining since the mid seventies. Option spreads with more than four legs, a very efficient means of achieving adequate margin reductions, did not appear until 30 years later. Specifically, in August 2003 the CBOE (Chicago Board Options Exchange) proposed new margin rules for option spreads with up to 12 legs that were called *complex option spreads* [CBOE, 2003]. After two revisions of this proposal [CBOE, 2004, 2005], the SEC (U.S. Securities and Exchange Commission) approved these rules [SEC, 2005] and added them to NYSE Rule 431 in December 2005. In August 2007, these rules were also recognized in Canada [IDA, 2007].

The regulatory move of 2005 was a very important step in the development of margin regulations. Now it is important to understand how multi-leg spreads can be used in margin calculations and design efficient margin calculation algorithms that take advantage of multi-leg spreads. At the best of our knowledge, this kind of research has never been attempted. As we show in this paper, 12 legs is not the final step. We discover new multi-leg option spreads that have the same hedging mechanism as that of complex option spreads and propose a full characterization of multi-leg option spreads with up to 134 legs. We also show that the number of such spreads reaches several thousands. Therefore, any algorithm identifying all of them in a given margin account would be impractical. Note that the existing margining algorithms are based on spreads identification; see a discussion in [Rudd and Schroeder, 1982; Fiterman and Timkovsky, 2001]. We discover, however, that the problem of minimizing the margin requirement where multi-leg spreads are used for offsetting under the maximum loss margin rules, can be solved without identification of option spreads. Moreover, we show that it can be solved in polynomial time by network flow algorithms.

The remainder of the paper is organized as follows. Section 2 explains the advantage of using option spreads with a larger number of legs. Section 3 discusses types of margin requirements and explains that only the market risk component of the margin requirement for a portfolio of option spreads can be minimized. A model and a characterization of option spreads with two, three and four legs are proposed in Section 4. Section 6 is devoted to margining a portfolio of basic spreads. Sections 7 and 8 present complex spreads and their generalizations. Sections 9 is devoted to graph characterization and counting multi-leg spreads. Sections 10 contains a portfolio decomposition theorem and the margin minimization algorithm using offsets based on multi-leg spreads. In conclusion, we outline possible directions of further research and give recommendations on how to create more efficient margin rules for options.

2 Why Counting Legs Matters

As a matter of fact, every leg saves money. In this section, we show that the advantage of using an additional leg in margining equals the product of the difference between exercise prices and the contract size of the options.

Let us consider a margin account which consists of a long position in one call option A, a long position in one call option B and a short position in two call options C. The options' exercise prices and market prices are, respectively,

$$\begin{aligned} \mathsf{Ae} &= \$70.00, & \mathsf{Ap} &= \$55.90, \\ \mathsf{Be} &= \$90.00, & \mathsf{Bp} &= \$40.90, \\ \mathsf{Ce} &= \$80.00, & \mathsf{Cp} &= \$50.60. \end{aligned}$$

Each of the options expires by the end of day, January 15, 2010, and has the contract size of 100 shares. The market price of the underlying stock[1] is $\mathsf{Up} = \$123.62$.

In what follows, it will be convenient to denote a long or short position in an option, say, C, as $+\mathsf{C}$ or $-\mathsf{C}$, respectively; so, a set of positions then can be written as a formal sum of the positions. Following the definitions form NYSE Rule 431(f)(2)(C), we can conclude that, since $\mathsf{Ae} < \mathsf{Ce} < \mathsf{Be}$ and $\mathsf{Ce} - \mathsf{Ae} = \mathsf{Be} - \mathsf{Ce}$, the account represents a long butterfly spread $\mathsf{A} + \mathsf{B} - 2\mathsf{C}$ whose components are spreads $\mathsf{A} - \mathsf{C}$ and $\mathsf{B} - \mathsf{C}$.

Next, we show that the *regulatory minimum initial* and *maintenance* margin requirements for this account are $\$1000$ less if considered as the long butterfly spread, which has three legs, in comparison with the case where it is considered as a consolidation of the two two-leg spreads, which are its components.

Indeed, in accordance with NYSE Rule 431(f)(2)(G)(v)(1), the initial margin requirement for $\mathsf{A} + \mathsf{B} - 2\mathsf{C}$ is the total market price of A and B, i.e.,

$$100 \cdot (\mathsf{Ap} + \mathsf{Bp}) = 100 \cdot (\$55.90 + \$40.90) = \$9680.$$

NYSE Rule 431(f)(2)(G)(i) states that the initial margin requirement for a two-leg spread is the market value of the option carried long plus the lesser of the initial margin requirement for the option carried short and the spread exercise loss.

Since $\mathsf{Ae} < \mathsf{Ce}$, the spread $\mathsf{A} - \mathsf{C}$ exercise loss, i.e., $100 \cdot \max\{\mathsf{Ae} - \mathsf{Ce}, 0\}$, is zero, and hence the initial margin requirement for $\mathsf{A} - \mathsf{C}$ is $100 \cdot \mathsf{Ap} = \$5590$.

Since $\mathsf{Be} > \mathsf{Ce}$, the spread $\mathsf{B} - \mathsf{C}$ exercise loss, i.e., $100 \cdot \max\{\mathsf{Be} - \mathsf{Ce}, 0\}$, is $\$1000$. In accordance with NYSE Rule 431(f)(2)(D)(i), the initial margin requirement for $-\mathsf{C}$ is $100 \cdot (\mathsf{Cp} + \mathsf{Cm}) = 100 \cdot (\$50.60 + \$24.724)$, where

$$\begin{aligned}
\mathsf{Cm} &= \max\{0.2 \cdot \mathsf{Up} - \mathsf{Co}, 0.1 \cdot \mathsf{Up}\} \\
&= \max\{0.2 \cdot \$123.62 - \$0.00, 0.1 \cdot \$123.62\} = \$24.724, \\
\mathsf{Co} &= \max\{\mathsf{Ce} - \mathsf{Up}, 0\} \\
&= \max\{\$80.00 - \$123.62, 0\} = \$0.00.
\end{aligned}$$

Note that Co here is the out-of-the-money amount of the call option C. Thus, the initial margin requirement for $\mathsf{B} - \mathsf{C}$ is $100 \cdot \mathsf{Bp} + \$1000 = \$5090$. Therefore, if the account is considered as a consolidation of the two two-leg spreads, the initial margin requirement for it is $\$5590 + \$5090 = \$10680$, which is $\$1000$ more.

Deducting the total market value of the options in the long positions A and B in both cases, we obtain the maintenance margin requirements, i.e., $\$0$ for $\mathsf{A} + \mathsf{B} - 2\mathsf{C}$ and $\$1000$ for the consolidation of $\mathsf{A} - \mathsf{C}$ and $\mathsf{B} - \mathsf{C}$. Thus, we have the advantage of $\$1000$ in the maintenance margin requirement as well.

We have demonstrated the advantage of three legs over two legs. However, the more legs an option spread has the more advantageous it is. As we will see, the advantage of using multi-leg spreads is a multiple of the product of the difference between exercise prices and the contract size of the options involved in the spreads.

[1] The data are taken from `http://finance.yahoo.com` as of the end of day, May 21, 2008, at NYSE for the symbol IBM.

3 Market Risk Margin Requirement

In the above example, we calculated a regulatory minimum *initial* and *maintenance margin* requirements for option spreads, which are based on the estimation of the *current loss* in accordance with the current market prices of the options and the underlying stock. Any margin charge below this minimum is illegal. However, brokers/brokerage houses are allowed to use more stringent margin requirements in accordance with their *house margin rules*. Although these rules may vary, more stringent margin requirements for option spreads are usually based on the estimation of the *maximum loss*.

To explain the difference between *current loss* and *maximum loss* margin requirements for option spreads, let us recall that the current loss initial margin requirement for $\mathsf{B} - \mathsf{C}$, see Section 2, is $100 \cdot (\mathsf{Bp} + \mathsf{Rm})$, where

$$\mathsf{Rm} = \min\{\mathsf{Cm}, \max\{\mathsf{Be} - \mathsf{Ce}, 0\}.$$

This margin consists of the following two components: $100 \cdot \mathsf{Bp}$, the *premium margin requirement*, and $100 \cdot \mathsf{Rm}$, the *market risk margin requirement*. Note that the latter remains the same in the calculation of maintenance margin requirements.

It is clear that we obtain a more stringent margin requirement if we replace the market risk component by the spread exercise loss

$$\mathsf{Rmax} = \max\{\mathsf{Be} - \mathsf{Ce}, 0\}$$

since $\mathsf{Rmax} \geq \mathsf{Rm}$. The new market risk component, i.e., $100 \cdot \mathsf{Rmax}$, ignores the fact that, if the current market price of the underlying stock falls, the current loss on the option in the short position can be less than the spread exercise loss and associates the market risk only with the worst case scenario in which both options of the spread are exercised. Note that the current loss on the spread $\mathsf{B} - \mathsf{C}$ in Section 2 is also the maximum loss because $\mathsf{Rm} < \mathsf{Cm}$, and hence $\mathsf{Rm} = \mathsf{Rmax}$.

The formula $100 \cdot \mathsf{Rmax}$ representing the maximum loss market risk margin requirement should be recognized as commonly used and more preferable for margining stock option spreads in practice; see for example [CBOE, 2000]. This preference can be explained by the weighty argument that stocks are the most volatile securities on the market, and therefore the current loss calculated at the present moment can easily become the maximum loss in a few seconds.

On the other hand, the maximum loss margining is much more tractable than the current loss margining because, as can be seen from the formula for Rmax, the calculation of the initial or maintenance margin requirements for options in the short position can be avoided, and therefore the current market price of the underlying stock is not needed for the calculation. This paper takes advantage of this fact and considers only maximum loss margin requirements.

Regardless of what margin requirement we calculate, initial or maintenance, current loss of maximum loss, for a portfolio of option spreads, its premium component remains invariant to offsetting spreads in the portfolio. It is either the total market value of the options in long positions if we calculate the initial margin requirement, or zero if we calculate the maintenance margin requirement. Only the market risk component can be reduced by offsetting.

In what follows, we will be dealing with only the *maximum loss market risk margin requirements*. So, any margin formula we will obtain for margining option spreads will represent either the maintenance margin requirement or the market risk component of the initial margin requirement. The initial margin requirement can be easily obtained from its market risk component by adding the total market value of all options in the long positions of the portfolio. We will also call a maximum loss market risk margin requirement simply *market risk* to be short.

4 Main Option Spreads

The model presented in this section follows the regulatory definitions related to option spreads from NYSE Rule 431(f)(2) that can be found at http://rules.nyse.com/nyse/.

Option spreads of dimension h can be formally defined as integer vectors

$$\mathbf{v} = (\ c_1 \quad c_2 \quad \dots \quad c_h \ \ : \ \ p_1 \quad p_2 \quad \dots \quad p_h\ )$$

whose components c_j/p_j, $1 \le j \le h$, are said to be on the *call/put side* of the spread and associated with the number of option contracts in the jth position in a call/put option with the exercise price e_j on the same underlying instrument.

A positive/negative component of $\mathbf{v}$ is called a *long/short leg* of the spread. A *zero spread* that we denote by $\mathbf{0}$ has no legs. Let $l_1, l_2, \dots, l_k$ be the sequence of legs of a spread when scanning its components from left to right or right to left. If $l_i = l_{k-i+1}$ or $l_i = -l_{k-i+1}$ for all $i = 1, 2, \dots, \lfloor k/2 \rfloor$, then the spread is called *symmetric* or *antisymmetric*, respectively. *Single-side spreads*, i.e., *call-side/put-side spreads*, have legs only on the call/put side. *Two-side spreads* have legs on both sides.

The exercise prices are assumed to be all different and placed in the increasing order, i.e., $e_1 < e_2 < \dots < e_h$. The set $\{e_1, e_2, \dots, e_h\}$ is called an *exercise domain*.

Treating spreads as vectors we can add them, multiply by an integer scalar, cyclicly shift their components and take their *transpositions*, i.e., create the spreads $\bar{\mathbf{v}}$, where the components c_i and p_i are transposed for all $i = 1, 2, \dots, h$.

Let a be a positive integer and $a > 1$. Then $a\mathbf{v}$ is a *multiple* of $\mathbf{v}$ and a is a *divisor* of $a\mathbf{v}$. A spread without divisors is *prime*. We call two spreads *isomorphic* if one can be cyclicly shifted into the other such that the legs do not change their sides.

Antisymmetric two-leg spreads and simplest symmetric three-leg or four-leg spreads are well known and widely used as trading strategies and offsets in margining practice. Prime spreads among them can be defined as follows.

Definition 1 *Let 0^k denote k successive zero components of a vector. Then the fol-*

lowing antisymmetric prime two-leg spreads are basic spreads:

$$\begin{array}{lcllllllll} \textit{the } i\textit{th bull call spread} & = & & & (& 0^{i-1} & 1 & -1 & 0^{h-i-1} & : & 0^h \ ) \\ \textit{the } i\textit{th bear call spread} & = & & & (& 0^{i-1} & -1 & 1 & 0^{h-i-1} & : & 0^h \ ) \\ \textit{the } i\textit{th bull put spread} & = & (\ 0^h & : & & 0^{i-1} & 1 & -1 & 0^{h-i-1} &) & \\ \textit{the } i\textit{th bear put spread} & = & (\ 0^h & : & & 0^{i-1} & -1 & 1 & 0^{h-i-1} &) & \end{array}$$

where $i = 1, 2, \ldots, h-1$. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be a bull spread and a bear spread, respectively. Then* $\mathbf{x}+\mathbf{y}$ *is a symmetric prime three-leg or four-leg spread.*

If the exercise prices are separated by the same price interval, then its length is

$$e_2 - e_1 = e_3 - e_2 = \ldots = e_h - e_{h-1} = \mathsf{D} = \textit{an exercise differential,}$$

and the exercise domain and spreads on this domain are called *uniform*. Note that only uniform spreads are permitted for margining purposes [SEC, 2005]. Although, in what follows, our attention will be focused on the case of dimension four, all further results except counting, are valid for any dimension higher than four. Note that four is the minimum dimension that takes into consideration four-leg spreads.

Table 1 presents all 12 call-side spreads, where isomorphic spreads are numbered. For example, there are only two isomorphic long call butterfly spreads. The abbreviations "dr" and "cr" mark *debit spreads* and *credit spreads*.[2] Transposing spreads in Table 1 and changing the word "call" into "put" in the second column we can get all 12 put-side spreads. The last 9 rows of Table 2 present all 9 *debit two-side spreads*. Negating these spreads, transposing the words "long" and "short" in the second column and replacing "dr" by "cr" in the last column, we can get all 9 *credit two-side spreads*. We do not show the three bull put spreads, the 12 put-side spreads and the 9 credit two-side spreads because their structure is clear. Thus, the number of all prime spreads of dimension four that meet Definition 1 is 42. We call them further *main spreads*.

The set of basic spreads, which we refer to as $A \cup B$, generates the *crown graph* on 12 vertices. It is a bipartite graph with the vertex set $A \cup B$, where the parts

$$A = \{-\mathbf{a}, -\mathbf{b}, -\mathbf{c}, -\mathbf{e}, -\mathbf{f}, -\mathbf{g}\} \quad \text{and} \quad B = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}, \mathbf{f}, \mathbf{g}\}$$

represent bear spreads and bull spreads, respectively, and the set of edges

$$A + B = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in A, \ \mathbf{v} \in B\}$$

represents symmetric spreads. Since $|A| = |B|$, the crown graph is a *balanced* bipartite graph. It can be converted into a complete balanced bipartite graph by adding *horizontal edges* that connect six bull/bear spreads with their negations and therefore represent zero spreads. We denote the crown graph on $2n$ vertices by C_n; see Fig. 1.

It is not hard to verify that with the exception of the box spreads, symmetric main spreads are long/short depending only on whether their leg with the lowest exercise

[2]They are called so because the net positions in these spreads are the results of net debit/credit trades, i.e. where the cost of the long options is more/less than the cost of the short options.

Table 1: Call-Side Spreads

spread	spread name	calls				puts				net
a	1st bull call	1	-1							dr
b	2nd bull call		1	-1						dr
c	3rd bull call			1	-1					dr
-a	1st bear call	-1	1							cr
-b	2nd bear call		-1	1						cr
-c	3rd bear call			-1	1					cr
$\mathbf{a}-\mathbf{b}$	1st long call butterfly	1	-2	1						dr
$\mathbf{b}-\mathbf{c}$	2nd long call butterfly		1	-2	1					dr
$\mathbf{b}-\mathbf{a}$	1st short call butterfly	-1	2	-1						cr
$\mathbf{c}-\mathbf{b}$	2nd short call butterfly		-1	2	-1					cr
$\mathbf{a}-\mathbf{c}$	long call condor	1	-1	-1	1					dr
$\mathbf{c}-\mathbf{a}$	short call condor	-1	1	1	-1					cr

Table 2: Bull Call Spreads, Bear Put Spreads and Debit Two-Side Spreads

spread	spread name	calls				puts				net
a	1st bull call	1	-1							dr
b	2nd bull call		1	-1						dr
c	3rd bull call			1	-1					dr
$-\mathbf{e}$	1st bear put					-1	1			dr
$-\mathbf{f}$	2nd bear put						-1	1		dr
$-\mathbf{g}$	3rd bear put							-1	1	dr
$\mathbf{a}-\mathbf{e}$	1st long box	1	-1			-1	1			dr
$\mathbf{b}-\mathbf{f}$	2nd long box		1	-1			-1	1		dr
$\mathbf{c}-\mathbf{g}$	3rd long box			1	-1			-1	1	dr
$\mathbf{a}-\mathbf{f}$	1st long call iron butterfly	1	-1				-1	1		dr
$\mathbf{b}-\mathbf{g}$	2nd long call iron butterfly		1	-1				-1	1	dr
$\mathbf{b}-\mathbf{e}$	1st short put iron butterfly		1	-1		-1	1			dr
$\mathbf{c}-\mathbf{f}$	2nd short put iron butterfly			1	-1		-1	1		dr
$\mathbf{a}-\mathbf{g}$	long call iron condor	1	-1					-1	1	dr
$\mathbf{c}-\mathbf{e}$	short put iron condor			1	-1	-1	1			dr

price is long/short, respectively. The box spreads have both a long and a short leg with the lowest exercise price, therefore they cannot be classified in this way. The box spread is long/short if its call leg with the lowest exercise price is long/short.

Analogously, with the exception of the box spreads, symmetric main spreads are call/put spreads depending only on whether their leg with the lowest exercise price is a call/put leg, respectively. The box spreads have both a call and a put leg with the lowest exercise price, therefore they are call spreads and also put spreads.

One can also observe that symmetric debit/credit spreads correspond to the edges of the crown graph C_6 with a positive/negative slope. This property, of course, holds only if the vertices of the crown graph are positioned on the plan as shown in Fig. 1.

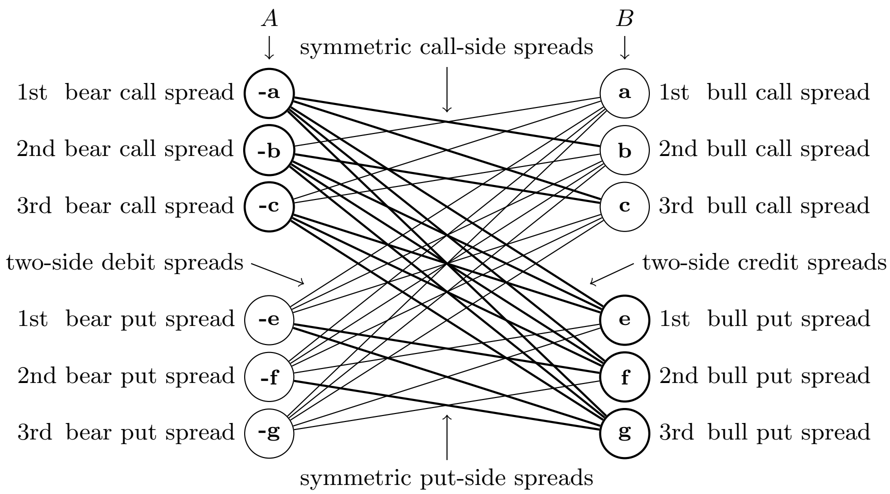

Figure 1: The crown graph C_6 of main spreads: thick lines depict credit spreads.

Other properties of main spreads can be expressed by the following equations:

$$\begin{array}{rcl}
-\text{short call [name] spread} & = & \overline{\text{long put [name] spread}} \\
-\text{long call [name] spread} & = & \overline{\text{short put [name] spread}} \\
-\text{short put [name] spread} & = & \overline{\text{long call [name] spread}} \\
-\text{long put [name] spread} & = & \overline{\text{short call [name] spread}} \\
\text{debit/credit spread} & = & \overline{\text{debit/credit spread}} \\
-\text{debit/credit spread} & = & \text{credit/debit spread}
\end{array}$$

It is well known, see [McMillan, 2002] for example, that debit spreads are free of market risk, i.e., they have no loss associated with underlying security price changes. Credit spreads, in contrast, are not free of market risk. The maximum loss on a credit spread $\mathbf{x}$ associated with underlying security price changes is D or $2\mathsf{D}$ if it is a short call iron butterfly or condor spread. Thus, the market risk $m(\mathbf{x})$ of an option spread $\mathbf{x}$ is

$$m(\mathbf{x}) = \begin{cases} \mathsf{0} & \text{if } \mathbf{x} \text{ is a debit spread,} \\ \mathsf{D} & \text{if } \mathbf{x} \text{ is a credit spread that is not} \\ & \qquad\quad \text{a short call iron butterfly or condor spread,} \\ 2\mathsf{D} & \text{if } \mathbf{x} \text{ is a short call iron butterfly or condor spread.} \end{cases} \tag{1}$$

Among symmetric credit spreads, only a *short call iron butterfly* spread and a *short call iron condor* spread do not give an advantage in the market risk in comparison with the pairs of their basic components, therefore they are not used as trading strategies. Their transpositions, i.e., a *short put iron butterfly* spread and a *short put iron condor* spread, however, are commonly used as trading strategies. The word “put” in their names is usually omitted, so they are called simply a *short iron butterfly* spread and a *short iron condor* spread; see, e.g., [Cohen, 2005]. As we will see, there exist only 12 among 30 symmetric spreads that give an advantage in the market risk.

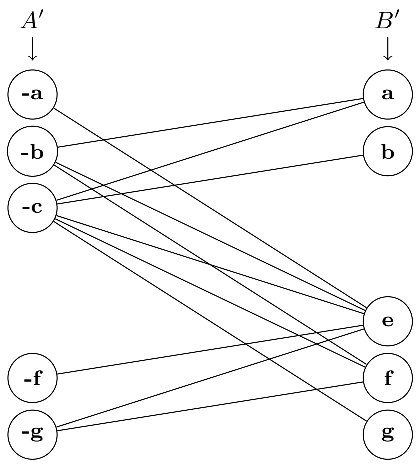

Figure 2: The offset graph.

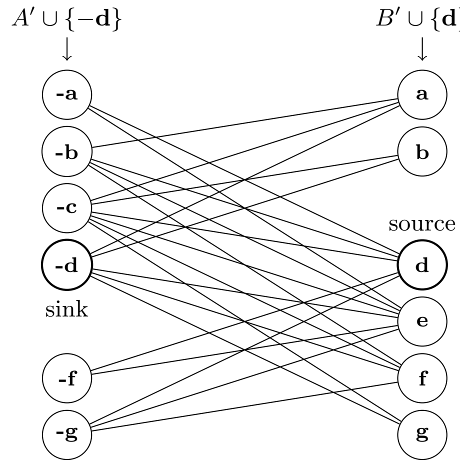

Figure 3: The offset network.

5 Offsets

An option spread is an *offset* if its market risk is lower than the total market risk of its components. Thus, a four-leg main spread $\mathbf{u}+\mathbf{v}$ is an offset if $m(\mathbf{u})+m(\mathbf{v}) > m(\mathbf{u}+\mathbf{v})$. Among four-leg main spreads only offsets are advantageous for margin reductions.

As we mentioned in Section 4, a short call iron butterfly and a short call iron condor are not offsets because $m(\mathbf{u}+\mathbf{v}) = 2\mathsf{D}$ while $m(\mathbf{u}) = \mathsf{D}$ and $m(\mathbf{v}) = \mathsf{D}$. The following lemma generalizes this result.

Lemma 1 *Symmetric single-side credit spreads, two-side debit spreads, short call iron butterfly and short call iron condor spreads are not offsets, while symmetric single-side debit spreads and two-side credit spreads which are not short call iron butterfly or short call iron condor spreads are offsets that reduce the market risk by* D.

Proof Formula (1) and Tables 1 and 2 imply that $m(\mathbf{u})+m(\mathbf{v})-m(\mathbf{u}+\mathbf{v}) = 0$ if $\mathbf{u}+\mathbf{v}$ is a symmetric single-side credit spread, two-side debit spread, short call iron butterfly or short call iron condor spread, and D if $\mathbf{u}+\mathbf{v}$ is another symmetric spread. $\square$

Figure 2 depicts the *offset graph*, i.e., the subgraph of C_6 whose edges represent offsets. Note that it does not contain the 3rd bull call spread $\mathbf{c}$ and the 1st bear put spread $-\mathbf{e}$ because they are not components of offsets, therefore they can be margined separately. Let us set $A' = A \setminus \{-\mathbf{e}\}$, $B' = B \setminus \{\mathbf{c}\}$ and denote the set of all 12 offsets by O.

6 Four-Leg Margining of Portfolios of Basic Spreads

A *position in a basic spread* $\mathbf{x}$ is the pair $[\mathbf{x}, q(\mathbf{x})]$, where $q(\mathbf{x})$ is a nonnegative integer indicating how many basic spreads $\mathbf{x}$ are involved in the position. A *portfolio of basic* *spreads* is a set of positions in basic spreads with the same underlying instrument and exercise differential. Without loss of generality, we can assume that such a portfolio contains all 12 basic spreads, where some of them have zero quantities.

To define a *portfolio market risk* associated with a portfolio of basic spreads, let us first calculate the total market risk for all basic spreads $\mathbf{x} \in A \cup B$, i.e.,

$$\mathsf{M} = \sum_{\mathbf{x} \in A \cup B} q(\mathbf{x}) m(\mathbf{x}).$$

It is clear that only credit spreads $\mathbf{x}$ with positive quantities $q(\mathbf{x})$ contribute to this sum because market risk of debit spreads is zero. The total market risk M overestimates the portfolio market risk if the portfolio has a bear spread $\mathbf{u}$ and a bull spread $\mathbf{v}$ such that $\mathbf{u} + \mathbf{v}$ is an offset. By Lemma 1, an offset reduces market risk by D, therefore the portfolio market risk is at most $\mathsf{M} - \mathsf{D}$.

It is important to notice that the offsets with the maximum of four legs are used here as a simplest mechanism of hedging market risk associated with basic credit spreads. So, the problem considered in this section is the simplest *portfolio margin minimization* (PMM) problem with this hedging mechanism. Consequently, the margin requirement found as a solution to this problem we call the minimum *four-leg margin* requirement. In Section 10 we consider the PMM problem with much more powerful hedging mechanisms that yield substantially lower margin requirements.

Decreasing $q(\mathbf{u})$ and $q(\mathbf{v})$ by one we can apply the described above offsetting operation to the residual portfolio and choose the next offset and prove that the portfolio market risk is at most $\mathsf{M} - 2\mathsf{D}$, etc. It is clear that on a certain step the residual portfolio will not contain offsets because the *uncovered quantities* of the positions, i.e., not covered by chosen offsets, remain on only one side, bullish or bearish. Let us define the *total offset quantity* to be the number of offsets created during this procedure.

Theorem 1 *Let x be the total offset quantity. Then the portfolio market risk is at most $\mathsf{M} - x\mathsf{D}$, and hence this difference is a four-leg margin requirement for the portfolio.*

Proof Directly follows from Lemma 1. □

Thus, the problem of minimizing the four-leg margin requirement for a portfolio of basic spreads is equivalent to the problem of maximizing the total offset quantity.

Let us show a reduction of this problem to the *maximum flow problem* [Ford and Fukerson, 1962]: Given a network with the set of nodes N, a *source* $s \in N$, a *sink* $t \in N$, $s \neq t$, and the set of edges E with *capacities* $c(e)$ of running a flow through $e \in E$, find a maximum flow from s to t.

Let us introduce a dummy bull spread $\mathbf{d}$, a dummy bear spread $-\mathbf{d}$ and create an *offset network* setting $N = A' \cup B' \cup \{\mathbf{d}, -\mathbf{d}\}$, $s = \mathbf{d}$, $t = -\mathbf{d}$,

$$E = \{(\mathbf{u}, \mathbf{d}) : \mathbf{u} \in A'\} \cup \{(-\mathbf{d}, \mathbf{v}) : \mathbf{v} \in B'\} \cup \{(\mathbf{u}, \mathbf{v}) : \mathbf{u} + \mathbf{v} \in O\},$$

$c(\mathbf{u}, \mathbf{d}) = q(\mathbf{u})$ for all $\mathbf{u} \in A'$, $c(-\mathbf{d}, \mathbf{v}) = q(\mathbf{v})$ for all $\mathbf{v} \in B'$ and make the capacities $c(\mathbf{u}, \mathbf{v})$ unrestricted for all $\mathbf{u} + \mathbf{v} \in O$; see Fig. 3.

Feasible integer flows in this network can be interpreted as follows: Let $\mathbf{u}+\mathbf{v} \in O$. Then the amount of flow through the edges $(\mathbf{u}, \mathbf{v})$ represents the quantities $x(\mathbf{u}+\mathbf{v})$ of the offsets $\mathbf{u}+\mathbf{v}$; the differences $q(\mathbf{u}) - \sum_{\mathbf{u}+\mathbf{v}\in O} x(\mathbf{u}+\mathbf{v})$ and $q(\mathbf{v}) - \sum_{\mathbf{u}+\mathbf{v}\in O} x(\mathbf{u}+\mathbf{v})$ represent uncovered position quantities of the bear spread $\mathbf{u}$ and the bull spread $\mathbf{v}$, respectively; the flow from $\mathbf{d}$ to $-\mathbf{d}$ represents the total offset quantity. If x^* is the maximal offset quantity, then $\mathsf{M} - x^*\mathsf{D}$ defines the portfolio *four-leg market risk.*

7 Complex Option Spreads

The regulatory amendment of December 14, 2005, initiated by the CBOE, was motivated by the observation that some combinations of main spreads have the same risk profile as single main spreads such as bull and bear spreads, condor spreads, iron butterfly and iron condor spreads. This phenomenon is explained by the fact that the summation of main spreads in such a combination turns out to be also a main spread. These combinations were named *complex spreads.*

Ten of the complex spreads are presented in Table 3. The other ten are their transpositions, where the names of the components have the words “call” and “put” interchanged. Negations of these 20 give 20 additional complex spreads, where the names of the components have the words “long” and “short” interchanged. Thus, Table 3 defines a total of 40 complex spreads. Since complex spreads 1 and 2, 4 and 5, 7 and 8 are isomorphic, there exist only seven types of the complex spreads.[3]

Margin calculations for complex spreads follow the ways of margin calculations for their resulting spreads; and a complex spread is an offset if its margin requirement is less than the total margin requirement for its components. Hence, not all complex spreads are offsets.

For example, the complex spread 6 in Table 3 has three components: the 1st long call butterfly spreads $\mathbf{b}-\mathbf{c}$, the 2nd long call butterfly spread $\mathbf{a}-\mathbf{b}$ and the 3rd bull call spread $\mathbf{c}$. All the three are debit spreads. By formula (1), the margin requirements for these spreads are zeros. The resulting spread is the 1st bull call spread $\mathbf{a}$, which is also a debit spread. Therefore, the margin requirement for the complex spread 6 is also zero. Thus, the complex spread 6 is not an offset, and there is no advantage of using it for margin reductions. It is not hard to verify that all complex spreads in Table 3 are not offsets. However, their negations are offsets.

For example, since the bear call spread $-\mathbf{a}$ is a credit spread, the margin requirement for the negation of the complex spread 6 is D, while the total margin requirement for $\mathbf{b}-\mathbf{a}$, $\mathbf{c}-\mathbf{b}$ and $-\mathbf{c}$, which are all credit spreads, is $3\mathsf{D}$. Thus, the complex spread 6 is an offset that gives the advantage of two exercise differentials.

In general, if a complex spread with the resulting debit/credit spread is an offset, it reduces the total margin requirement for basic spreads by $k\mathsf{D}/(k-1)\mathsf{D}$, respectively,

[3] The regulatory definition in SEC Release 34-52738, the CBOE Regulatory Circular and NYSE Rule 431 contains only descriptions of these seven types. The CBOE gave some of them the same names as those of their resulting main spreads. To avoid confusions, we do not use these names.

Table 3: Complex Spreads, Their Components and Resulting Spreads

cmplx sprd: cmpnt sum = result sprd	calls				puts				net
1. $\mathbf{b}+(\mathbf{a}-\mathbf{b})=\mathbf{a}:$				5 legs					
2nd bull call		1	-1						dr
+ 1st long call butterfly	1	-2	1						dr
= 1st bull call	1	-1							dr
2. $\mathbf{c}+(\mathbf{b}-\mathbf{c})=\mathbf{b}:$				5 legs					
3nd bull call			1	-1					dr
+ 2st long call butterfly		1	-2	1					dr
= 2nd bull call		1	-1						dr
3. $(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})=\mathbf{a}-\mathbf{c}:$				6 legs					
2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
= long call condor	1	-1	-1	1					dr
4. $(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}-\mathbf{b}:$				7 legs					
1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= 1st long put iron butterfly		-1	1		1	-1			cr
5. $(\mathbf{b}-\mathbf{c})+(\mathbf{f}-\mathbf{b})=\mathbf{f}-\mathbf{c}:$				7 legs					
2nd long call butterfly		1	-2	1					dr
+ 2nd short box		-1	1			1	-1		cr
= 2nd long put iron butterfly			-1	1		1	-1		cr
6. $\mathbf{c}+(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})=\mathbf{a}:$				8 legs					
3rd bull call			1	-1					dr
+ 2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
= 1st bull call	1	-1							dr
7. $\mathbf{b}+(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}:$				9 legs					
2nd bull call		1	-1						dr
+ 1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= 1st bull put					1	-1			cr
8. $\mathbf{c}+(\mathbf{b}-\mathbf{c})+(\mathbf{f}-\mathbf{b})=\mathbf{f}:$				9 legs					
3rd bull call			1	-1					dr
+ 2nd long call butterfly		1	-2	1					dr
+ 2nd short box		-1	1			1	-1		cr
= 2nd bull put						1	-1		cr
9. $(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}-\mathbf{c}:$				10 legs					
2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= long put iron condor			-1	1	1	-1			cr
10. $\mathbf{c}+(\mathbf{b}-\mathbf{c})+(\mathbf{a}-\mathbf{b})+(\mathbf{e}-\mathbf{a})=\mathbf{e}:$				12 legs					
3rd bull call			1	-1					dr
+ 2nd long call butterfly		1	-2	1					dr
+ 1st long call butterfly	1	-2	1						dr
+ 1st short box	-1	1			1	-1			cr
= 1st bull put					1	-1			cr

where k is the number of credit components. Thus, the negations of complex spreads 1 through 5, 6 through 9 and 10 reduce margin requirement by D, $2\mathsf{D}$ and $3\mathsf{D}$, respectively.

8 Generalizations of Complex Option Spreads

Generalizing the concept of complex spreads, we define a *centipede* to be a combination of main spreads such that their summation, i.e., the resulting spread, is also main. The margin rule for complex spreads we formulated in Section 7 depends only on whether the resulting spread is debit or credit. Therefore, it naturally applies to centipedes. Besides, centipedes provide the same margin reduction as complex spreads.

The goal of this section is to characterize centipedes and define centipedes with extreme properties. Although centipedes are low-risk option combinations for the same reason as complex spreads, they are not permitted for margining purposes.

Let $\mathbf{A}$ be the 8×42 matrix whose columns are all 42 main spreads. Then the centipedes with the resulting spread $\mathbf{b}$ can be found as 0–1 solutions to the equation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x}$ is a 0–1 column vector of size 42.

To measure the efficiency of centipedes for the purpose of margin reductions, we introduce the following two criteria. Let $\mathbf{l}$ and $\mathbf{d}$ be the row vectors of size 42 whose components are the numbers of legs and the exercise differential components in margin requirements for the main spreads, respectively. Thus, the ith component of $\mathbf{d}$ is 0 or D depending on whether the ith main spread is debit or credit.

Then $\mathbf{lx}$ and $\mathbf{dx}$ are the total number of legs and the margin advantage of the centipede $\mathbf{x}$, respectively. The larger $\mathbf{lx}$ the more legs are covered by the margin rule for $\mathbf{x}$. The larger $\mathbf{dx}$ the more margin reduction can be achieved using $\mathbf{x}$. Centipedes that maximize $\mathbf{lx}$ and $\mathbf{dx}$ are solutions to the corresponding 0–1 programs with the constraint $\mathbf{Ax} = \mathbf{b}$. They can be found as follows.

Let b be the number of legs of $\mathbf{b}$. All 42 main spreads have 136 legs in total and the resulting zero spread. Hence, the set of 41 main spreads without $-\mathbf{b}$ constitutes the centipede with $136 - b$ legs and the resulting spread $\mathbf{b}$. It is clear that centipedes with the resulting spread $\mathbf{b}$ and the number of legs more than $136 - b$ do not exist. Since the minimum value of b is 2, the maximum number of legs a centipede can have is 134.

To find a centipede with the maximum margin advantage we observe that the margin advantage a centipede gives is D times the number of its credit components minus $0/\mathsf{D}$ if the resulting spread is debit/credit. Hence, a centipede with 134 legs whose resulting spread is debit has the maximum margin advantage of $21\mathsf{D}$ because there exist exactly 21 credit main spreads.

Now we consider spread combinations which, in a sense, are even better than centipedes. They are based on the concept of a *horizontal option spread*, i.e., a long option combined with a short option on the same underlying security of the same type and exercise price. A horizontal option spread is invariant to underlying security market price changes and therefore market risk-free.

To illustrate, let us consider a *horizontal call spread* where the long call option lC and the short call option sC have the same exercise price e. Each option contracts, say, 100 underlying units. If sC is exercised, then the spread holder is obliged to sell 100 underlying units to the holder of sC at the price e. In this case, the spread holder can exercise lC, i.e., buy 100 underlying units at the same price, and deliver them to the holder of sC with no loss. If sC is not exercised and lC is out-of-the-money, then lC can be kept unexercised. A *horizontal put spread* has the same hedging mechanism except that exercising put options triggers the sell of underlying units.

We define a *millipede* to be a nonempty set of main spreads whose resulting spread is zero. Using induction on the number of components, it is easy to verify that the set of legs of a millipede can be partitioned into pairs such that each pair is a horizontal call or put spread. Therefore a millipede is a market risk-free option combination that should be margined in the same way as centipedes with the resulting debit spread. Thus, the margin requirement for a millipede is zero. Millipedes have the same margin advantage as centipedes with the resulting debit spread, i.e., $k\mathsf{D}$, where k is the number of credit components of the millipede.

There is a simple relationship between centipedes and millipedes. Indeed, any centipede with the resulting spread $\mathbf{b}$ being complemented by the spread $-\mathbf{b}$ is a millipede since $\mathbf{b} - \mathbf{b} = \mathbf{0}$. On the other hand, any component $\mathbf{b}$ of a millipede generates a centipede with the resulting spread $-\mathbf{b}$.

As well as centipedes, millipedes can be associated with 0–1 column vectors $\mathbf{x}$ of size 42 for which $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{A}$, as before, is the 8×42 matrix of main spreads. However, we can avoid using the equation $\mathbf{Ax} = \mathbf{0}$ for solving optimization problems related to millipedes and reduce them to classical problems on bipartite graphs if we take into account the structure of main spreads considered in Section 9.

We call a millipede/centipede *symmetric*, if all its components are symmetric spreads, and *asymmetric* otherwise. For example, complex spreads 3, 4, 5 and 9 in Table 3 are symmetric centipedes; the other six complex spreads in Table 3 are asymmetric.

9 Graph Characterization and Counting

The graph characterization of millipedes and centipedes will allow us to establish their key properties that will be used in the proof of the portfolio decomposition theorem in Section 10. This theorem will show that, if only maximum loss margin rules are used for margining main spreads, then the multi-leg margining problem for a portfolio of basic spreads, where millipedes and centipedes are used for offsetting, can be solved in polynomial time without their identification. Thus, the algorithm we propose in Section 10 is free of any enumeration of millipedes and centipedes.

Although it is not clear how the portfolio decomposition theorem can be extended to the current loss margin rules, it is obvious that the related margining algorithms are impossible without identification and hence at least partial enumeration of millipedes and centipedes. In this section, we estimate the complexity of their full enumeration providing only the numbers of all possible millipedes and centipedes of dimension four. Since these numbers reach several thousands, such a full enumeration for a portfolio with even a dozen of basic spreads is practically impossible.

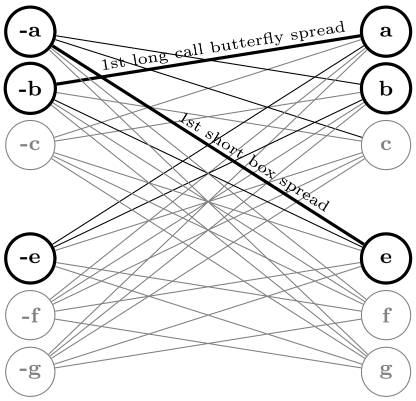

Figure 4: A balanced matching subgraph of C_6 with the set of vertices $\{\mathbf{a}, -\mathbf{a}, \mathbf{b}, -\mathbf{b}, \mathbf{e}, -\mathbf{e}\}$ and the set of edges $\{\mathbf{a} - \mathbf{b}, \mathbf{e} - \mathbf{a}\}$ that corresponds to the millipede $\mathbf{b} + (\mathbf{a} - \mathbf{b}) + (\mathbf{e} - \mathbf{a}) - \mathbf{e}$.

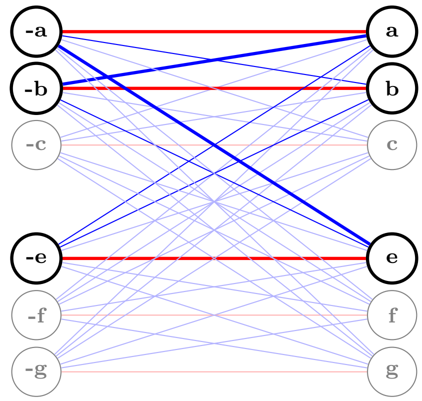

Figure 5: A red-blue-red alternating chain $(\mathbf{b}, -\mathbf{b}, \mathbf{a}, -\mathbf{a}, \mathbf{e}, -\mathbf{e})$ in C'_6 that corresponds to the independent asymmetric millipede $\mathbf{b} + (\mathbf{a} - \mathbf{b}) + (\mathbf{e} - \mathbf{a}) - \mathbf{e}$.

In addition, our counts permit to estimate the size of the margin rule book for option spreads if the CBOE or another option exchange decides to include in there margin rules for all possible millipedes and centipedes. If these rules are described in the traditional text form, as those in NYSE Rule 431, the work on such a book would take several years. Thus, a compact form of presenting the margin rules is necessary.

A subgraph with the set of vertices V and the set of edges E of a graph is called *vertex-induced/edge-induced* and denoted by $[V]/[E]$ if it is induced by V/E. Let us consider the crown graph C_6 defined in Section 4. We call a subgraph of C_6 *balanced/quasi-balanced* if the sum of its vertices is a zero/main spread. A balanced vertex-induced subgraph of C_n is also crown, so it is called a *balanced crown subgraph*.

A *matching* in a graph is a set of edges without common vertices. A matching is *perfect* if it covers all vertices of the graph. Let M be a matching in a vertex-induced subgraph $[V]$ of C_n. Then the subgraph of $[V]$ with the set of vertices V and the set of edges M is a *matching subgraph* (V, M) of C_n. The difference $V - M$ will denote the set of isolated vertices in (V, M), i.e., that are not covered by M.

Lemma 2 *Millipedes/centipedes* $\mathbf{x}$ *are in a one-to-one correspondence with balanced/quasi-balanced matching subgraphs* (V, M) *of* C_6*. A vertex in* $V - M$ *represents a component of* $\mathbf{x}$ *which is a bull or bear spread. An edge in* M *represents a component of* $\mathbf{x}$ *which is a symmetric spread.*

Proof Trivially follows from the definition of C_n and the definitions of balanced, quasi-balanced and matching subgraphs of C_n; see Fig. 4. □

Thus, the numbers of all millipedes and centipedes equal to the numbers of all balanced and quasi-balanced, respectively, matching subgraphs of C_6. We will be able to find this number after establishing other properties of crown graphs. At this point, however, we can count only symmetric millipedes and centipedes.

Lemma 3 *Symmetric millipedes/centipedes are in a one-to-one correspondence with perfect/maximal matchings in balanced/quasi-balanced vertex-induced subgraphs of* C_6.

Proof Lemma 2 implies that a millipede or centipede is symmetric if and only if the corresponding matching subgraph does not have isolated vertices or has only one isolated vertex, respectively. Hence, the related matchings of the matching subgraphs representing symmetric millipedes/centipedes are perfect/maximal. □

Lemma 4 *The number of perfect matchings in* C_n *is*

$$p_n = \sum_{k=0}^{n-2} (-1)^k \binom{n}{k} (n-k)^k (n-k-1)^{n-k}.$$

Proof Follows from an application to C_n Ryser's formula [Ryser, 1963]

$$\sum_{X \subseteq R} (-1)^{|X|} \prod_{u \in L} \sum_{v \notin X} 1_{uv \in E}$$

for counting perfect matchings in a bipartite graph with the left/right part L/R and the set of edges E. □

Theorem 2 *There exist exactly* 719/3600 *symmetric millipedes/centipedes.*

Proof By Lemmas 3 and 4, the number of symmetric millipedes with n components, i.e., the total number of perfect matchings in all balanced crown subgraphs C_n of C_6 is $P_n = \binom{6}{n} p_n$, where $n \geq 2$. Hence the number of all symmetric millipedes, i.e., the total number of perfect matchings in all balanced crown subgraphs of C_6 is

$$\sum_{n=2}^{6} P_n = 15 \cdot 1 + 20 \cdot 2 + 15 \cdot 9 + 6 \cdot 44 + 1 \cdot 265 = 719.$$

Since every component $\mathbf{v}$ of a millipede with n components generates a centipede with $n-1$ components and the resulting spread $-\mathbf{v}$, the number of symmetric centipedes with $n-1$ components is nP_n. Hence the number of symmetric centipedes is

$$\sum_{n=2}^{6} nP_n = 2 \cdot 15 + 3 \cdot 40 + 4 \cdot 135 + 5 \cdot 264 + 6 \cdot 265 = 3600. \quad \square$$

We call a millipede/centipede *independent* if it does not contain another/a millipede. Hence, a millipede is a disjoint union of independent millipedes; and a centipede is a disjoint union of a millipede and an independent centipede.

For example, main spreads are trivial independent centipedes with a single component; complex spreads defined in Section 7 are also independent centipedes. Independent millipedes and centipedes can be naturally characterized by graphs whose edges are colored in two colors.

A cycle/chain in a graph with red and blue edges is *alternating* if it alternates red and blue edges. A cycle/chain is *even/odd* if it is of even/odd length. Obviously, an alternating cycle is even, and the colors of the end edges of an even alternating chain are different. If an even alternating chain is scanned starting from the red/blue end edge, then we call it *red-blue/blue-red* alternating.

An odd alternating chain is *red-blue-red/blue-red-blue alternating* if it has red/blue end edges. In particular, a single red/blue edge is a red-blue-red/blue-red-blue alternating chain. A cycle/chain in a graph is *Hamiltonian* if it covers all its vertices.

Let $'$ denote the operator coloring all edges of a subgraph of C_n in blue and adding all incident horizontal edges colored in red. Thus, if H is a balanced vertex-induced subgraph of C_n, then H' is a balanced complete bipartite subgraph of C'_n, where all horizontal edges are red and all the other edges are blue. We will also apply this operator to matchings M assuming that $M' = [M]'$.

Thus, if M is a matching in C_n, then M' is a collection of alternating cycles and red-blue-red alternating chains in C'_n. It is also clear that any alternating cycle/chain in C'_n that covers the set of vertices V is Hamiltonian in $[V]$.

A cycle/chain in the graph C'_n is *balanced* if together with a vertex it also contains its negation, and *unbalanced* otherwise. Thus, an alternating cycle and a red-blue-red alternating chain are balanced; an even alternating chain is unbalanced; and a blue-red-blue alternating chain is balanced if and only if its end vertices negate each other.

Lemma 5 *Independent millipedes are in the following one-to-one correspondence with alternating cycles and red-blue-red alternating chains in* C'_6 *:*

$$\begin{array}{r} (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_2 - \mathbf{v}_3) + \ldots + (\mathbf{v}_{n-1} - \mathbf{v}_n) + (\mathbf{v}_n - \mathbf{v}_1) \quad \textit{and} \\ -\mathbf{v}_1 + (\mathbf{v}_1 - \mathbf{v}_2) + (\mathbf{v}_2 - \mathbf{v}_3) + \ldots + (\mathbf{v}_{n-1} - \mathbf{v}_n) + \mathbf{v}_n \qquad\quad \end{array}$$

are an independent symmetric millipede with n *components and an asymmetric millipede with* $n+1$ *components if and only if*

$$\begin{array}{r} (\mathbf{v}_1, -\mathbf{v}_2, \mathbf{v}_2, -\mathbf{v}_3, \ldots, \mathbf{v}_{n-1}, -\mathbf{v}_n, \mathbf{v}_n, -\mathbf{v}_1) \quad \textit{and} \\ (-\mathbf{v}_1, \mathbf{v}_1, -\mathbf{v}_2, \mathbf{v}_2, -\mathbf{v}_3, \ldots, \mathbf{v}_{n-1}, -\mathbf{v}_n, \mathbf{v}_n) \qquad\quad \end{array}$$

are an alternating cycle of length $2n$*, where* $2 \le n \le 6$*, and a red-blue-red alternating chain of length* $2n-1$*, where* $1 \le n \le 6$*, respectively.*

Proof Let (V, M) be a matching subgraph, where $|M| = n$, that represents a millipede $\mathbf{x}$, and let V_M be the set of vertices covered by M. Then M' is not an alternating cycle of length $2n$ or red-blue-red alternating chain of length $2n-1$ in $(V, M)'$ if and only if M' contains a proper subset N' such that N' is an alternating cycle or red-blue-red alternating chain. Then Lemma 2 implies that (V_N, N) is a proper subgraph of (V, M)

that represents a proper millipede in $\mathbf{x}$. Lemma 3 implies that M' is an alternating cycle in C_6' if and only if $\mathbf{x}$ is an independent symmetric millipede; see Fig. 5 $\Box$

Note that balanced blue-red-blue alternating chains also represent independent symmetric millipedes. However, there is no one-to-one correspondence between them because the deletion of any red edge from an alternating cycle transforms it into a balanced blue-red-blue alternating chain.

Lemma 6 *Independent centipedes are in the following one-to-one correspondence with unbalanced alternating chains in C_6':*

$$\begin{aligned} (\mathbf{v}_1-\mathbf{v}_2)+(\mathbf{v}_2-\mathbf{v}_3)+\ldots+(\mathbf{v}_{n-1}-\mathbf{v}_n)&,\\ (\mathbf{v}_1-\mathbf{v}_2)+(\mathbf{v}_2-\mathbf{v}_3)+\ldots+(\mathbf{v}_{n-1}-\mathbf{v}_n)+\mathbf{v}_n &\quad \textit{and}\\ -\mathbf{v}_1+(\mathbf{v}_1-\mathbf{v}_2)+(\mathbf{v}_2-\mathbf{v}_3)+\ldots+(\mathbf{v}_{n-1}-\mathbf{v}_n)& \end{aligned}$$

are an independent symmetric centipede with the resulting spread $\mathbf{v}_1-\mathbf{v}_n$ and $n-1$ components and independent asymmetric centipedes with the resulting spreads $\mathbf{v}_1$ and $\mathbf{v}_n$ and n components if and only if

$$\begin{aligned} (\mathbf{v}_1,-\mathbf{v}_2,\mathbf{v}_2,-\mathbf{v}_3,\ldots,\mathbf{v}_{n-1},-\mathbf{v}_n)&,\\ (\mathbf{v}_1,-\mathbf{v}_2,\mathbf{v}_2,-\mathbf{v}_3,\ldots,\mathbf{v}_{n-1},-\mathbf{v}_n,\mathbf{v}_n)&\quad \textit{and}\\ (-\mathbf{v}_1,\mathbf{v}_1,-\mathbf{v}_2,\mathbf{v}_2,-\mathbf{v}_3,\ldots,\mathbf{v}_{n-1},-\mathbf{v}_n)& \end{aligned}$$

are an unbalanced blue-red-blue alternating chain of length $2n-3$, where $2\le n\le 6$, a blue-red alternating chain of length $2n-2$ and a red-blue alternating chain of length $2n-2$, where $1\le n\le 6$, respectively.

Proof Follows from Lemma 5 because the deletion of a red-blue-red alternating chain of length three together with the end vertices of the blue edge from an alternating cycle transforms it into an unbalanced blue-red-blue alternating chain; and the deletion of an end edge together with one of its end vertices from a red-blue-red alternating chain transforms it into an even alternating chain;[4] see Figs. 6 and 7. These operations correspond to the deletion of only one component of the millipede represented by an alternating cycle or a red-blue-red alternating chain. $\Box$

Theorem 3 *An independent symmetric millipede/centipede has at most six/five components, while an independent asymmetric millipede/centipede has at most seven/six components.*

Proof Follows from Lemmas 5 and 6 because an alternating chain/unbalanced blue-red-blue alternating chain on 12/10 vertices has six/five blue edges; a red-blue-red alternating chain on 12 vertices has five blue edges and two end red edges; an even alternating chain on 12 vertices has five blue edges and one end red edge. $\Box$

[4] A trivial red-blue-red alternating chain has only one red edge with two end vertices.

Lemma 7 *Independent centipedes and millipedes are in the following one-to-one correspondences:*

$$(\mathbf{v}_0 - \mathbf{v}_1) + (\mathbf{v}_1 - \mathbf{v}_2) + \ldots + (\mathbf{v}_{n-1} - \mathbf{v}_n) + (\mathbf{v}_n - \mathbf{v}_{n+1})$$

is an independent symmetric centipede with the resulting spread $\mathbf{v}_0 - \mathbf{v}_{n+1}$ *and* $n+1$ *components if and only if* (a)

$$\begin{array}{c} (\mathbf{v}_0 - \mathbf{v}_1) + (\mathbf{v}_1 - \mathbf{v}_2) + \ldots + (\mathbf{v}_{n-1} - \mathbf{v}_n) + \mathbf{v}_n \quad \textit{and} \\ -\mathbf{v}_1 + (\mathbf{v}_1 - \mathbf{v}_2) + \ldots + (\mathbf{v}_{n-1} - \mathbf{v}_n) \end{array}$$

are independent asymmetric centipedes with the resulting spreads $\mathbf{v}_0$ *and* $-\mathbf{v}_n$*, respectively, and* $n+1$ *components, and* (b)

$$-\mathbf{v}_1 + (\mathbf{v}_1 - \mathbf{v}_2) + \ldots + (\mathbf{v}_{n-1} - \mathbf{v}_n) + \mathbf{v}_n \ \textit{and} \ \ \mathbf{v}_1 - \mathbf{v}_n$$

are an independent asymmetric millipede with $n+1$ *components and a symmetric main spread, respectively, where* $1 \leq n \leq 5$*.*

Proof Follows from Lemmas 5, 6 and a one-to-one correspondence between unbalanced blue-red-blue alternating chains C and (a) pairs of even alternating chains produced from C by adding single red edges to the ends of C or (b) balanced blue-red-blue alternating chains produced from C by deleting end blue edges and blue edges connecting the end vertices of C. $\Box$

Note that alternating cycles/chains in C'_n form axially symmetric pairs: every alternating cycle/chain C' has its negation $-C'$ where the signs of the vertices are interchanged.[5] Figures 8 and 9 give an example of an axially symmetric pair.

Let /2 denote the operator applicable to graphs with red and blue edges that contracts all red edges into single vertices and removes blue loops and parallel blue edges. Thus, $\mathsf{C}'_n/2$ is a complete graph. If C' is an alternating cycle/chain in C'_n, then $C'/2$ is a blue cycle/chain in $\mathsf{C}'_n/2$. Obviously, $C'/2$ is a Hamiltonian cycle/chain in $[C'/2]$.

Lemma 8 *Pairs of axially symmetric alternating cycles/chains* C' *and* $-C'$ *in* C'_n *are in a one-to-one correspondence with cycles/chains* $C'/2 = -C'/2$ *in* $\mathsf{C}'_n/2$*.*

Proof Trivially follows from the definition of the operator /2. $\Box$

Lemma 9 *The numbers of Hamiltonian alternating cycles and red-blue-red alternating chains, maximal unbalanced blue-red-blue alternating chains and even alternating chains in* C'_n *are* $a_n = (n-1)!$ *and* $r_n = n!$*,* $u_n = n!$ *and* $e_n = 2n!$*, respectively; where* $2 \leq n \leq 6$*;* $a_1 = 0$ *and* $r_1 = 1$*,* $u_1 = 0$ *and* $e_1 = 0$*.*

[5] Axially symmetric alternating cycles/chains are not isomorphic, otherwise they would not be cycles/chains but collections of at least two cycles/chains.

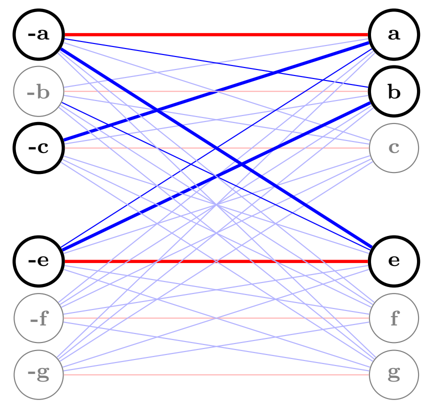

Figure 6: The unbalanced blue-red-blue alternating chain $(-\mathbf{c}, \mathbf{a}, -\mathbf{a}, \mathbf{e}, -\mathbf{e}, \mathbf{b})$ in C_6' that corresponds to the independent symmetric centipede $(\mathbf{a}-\mathbf{c})+(\mathbf{e}-\mathbf{a})+(\mathbf{b}-\mathbf{e})$ with the resulting spread $\mathbf{b}-\mathbf{c}$.

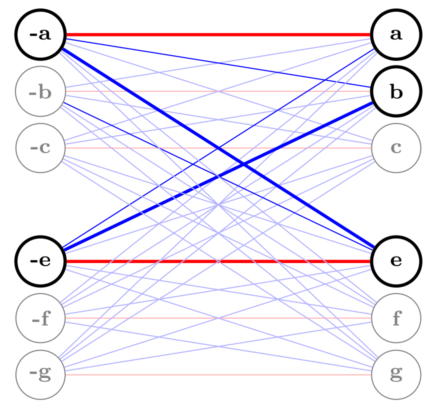

Figure 7: The red-blue alternating chain $(\mathbf{a}, -\mathbf{a}, \mathbf{e}, -\mathbf{e}, \mathbf{b})$ in C_6' that corresponds to the independent asymmetric centipede $\mathbf{a}+(\mathbf{e}-\mathbf{a})+(\mathbf{b}-\mathbf{e})$ with the resulting spread $\mathbf{b}$.

Proof Follows from Lemma 8 and that fact that the number of Hamiltonian cycles $C'/2$ in the complete graph $\mathsf{C}_n'/2$ is $(n-1)!/2$; the number of Hamiltonian red-blue-red alternating chains R' that can be produced from C' by deleting one blue edge is n; the number of unbalanced blue-red-blue alternating chains that can be produced from C' by deleting one blue edge with incident vertices and two red edges is also n; and the number of even alternating chains that can be produced from R' by deleting an end red edge and an incident end vertex is two. □

Theorem 4 *The numbers of independent symmetric millipedes with n components, asymmetric millipedes with $n+1$ components, symmetric centipedes with $n-1$ components and asymmetric centipedes with n components are*

$$A_n = \binom{6}{n}(n-1)!, \quad R_n = nA_n, \quad U_n = R_n \quad \textit{and} \quad E_n = 2U_n \quad \textit{if} \quad 2 \le n \le 6;$$

and $A_1 = 0$, $R_1 = 6$, $U_1 = 0$ and $E_1 = 0$, respectively. Thus, there exist exactly 409/1950 *independent symmetric millipedes/centipedes and* 1956/3900 *independent asymmetric millipedes/centipedes.*

Proof Every alternating cycle of length $2n$ or red-blue-red alternating chain of length $2n-1$ is Hamiltonian, and every unbalanced blue-red-blue alternating chain of length $2n-3$ or even alternating chain of length $2n-2$ is maximal in a balanced complete bipartite subgraph C_n' of C_6'. The number of such subgraphs is $\binom{6}{n}$. Hence, the theorem

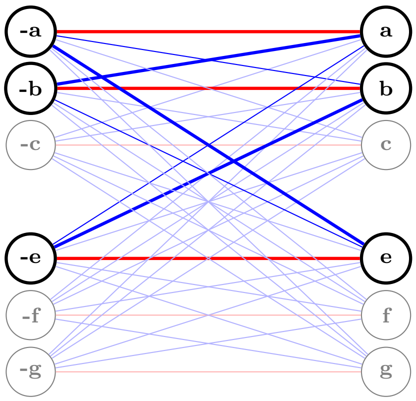

Figure 8: The alternating cycle $(\mathbf{a}, -\mathbf{a}, \mathbf{e}, -\mathbf{e}, \mathbf{b}, -\mathbf{b}, \mathbf{a})$ in C'_6 that corresponds to the independent millipede $(\mathbf{a}-\mathbf{b})+(\mathbf{b}-\mathbf{e})+(\mathbf{e}-\mathbf{a})$ and that is axially symmetric to the alternating cycle in Fig 9.

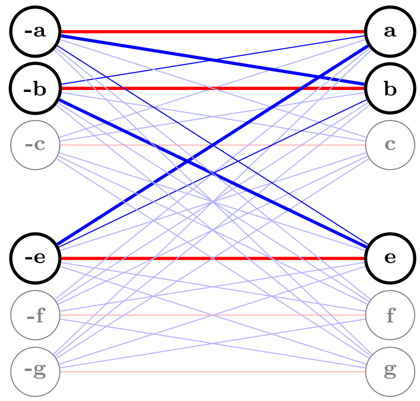

Figure 9: The alternating cycle $(-\mathbf{a}, \mathbf{a}, -\mathbf{e}, \mathbf{e}, -\mathbf{b}, \mathbf{b}, -\mathbf{a})$ in C'_6 that corresponds to the independent millipede $(\mathbf{b}-\mathbf{a})+(\mathbf{e}-\mathbf{b})+(\mathbf{a}-\mathbf{e})$ and that is axially symmetric to the alternating cycle in Fig 8.

follows from Lemmas 5, 6, 7(a) and 9 and the following count:

$$\begin{array}{llllll}
A_1=0, & A_2=15, & A_3=40, & A_4=90, & A_5=144, & A_6=120, \\
R_1=6, & R_2=30, & R_3=120, & R_4=360, & R_5=720, & R_6=720, \\
U_1=0, & U_2=30, & U_3=120, & U_4=360, & U_5=720, & U_6=720, \\
E_1=0, & E_2=60, & E_3=240, & E_4=720, & E_5=1440, & E_6=1440. \quad \Box
\end{array}$$

To characterize and count all millipedes we need the following definitions of graph covers. Let ϕ be a family of subgraphs of a graph C'_n. We say that a subfamily $\psi \subseteq \phi$ is a *cover* of C'_n *by subgraphs* from ϕ if subgraphs in ψ cover all vertices in C'_n. If no two subgraphs in ψ have common vertices, then ψ is an *exact cover* of C'_n. Associating cycles and chains with the subgraphs induced by their edges, we can define *covers* of C'_n *by cycles and chains* in C'_n in the same way.

Lemma 10 *Let ϕ be the family of alternating cycles and red-blue-red alternating chains in C'_6. Then millipedes are in a one-to-one correspondence with exact covers $\psi \subseteq \phi$ of balanced complete subgraphs of C'_6.*

Proof By Lemma 2, it will suffice to show a one-to-one correspondence between balanced matching subgraphs (V, M) of C'_6 and exact covers ψ of the balanced complete subgraph $[V]'$ of C'_6. The operator $'$ establishes such a correspondence: if (V, M) is a balanced matching subgraph, then $(V, M)'$ is a disjoint union ψ of alternating cycles and red-blue-red alternating chains covering all vertices of $[V]'$. $\Box$

Theorem 5 *There exist exactly* 17591 *millipedes and* 16872 *asymmetric millipedes.*

Proof By Lemma 10, we can count millipedes as exact covers of balanced complete bipartite subgraphs of C_6' by alternating cycles and red-blue-red alternating chains.

Let $m_1, m_2, \ldots, m_l, n$ be positive integers, and let $N = \{n_1 < n_2 < \ldots < n_l\}$ be a set of positive integers such that $\sum_{i=1}^{l} m_i n_i = n$. Then $P_n = \sum_{i=1}^{l} m_i \cdot n_i$ will denote a partition of n over N. For example, $2 \cdot 1 + 2 \cdot 2$ is a partition of 6 over $\{1 < 2\}$.

Let Φ be the family of all balanced complete subgraphs of C_n'. A single partition P_n defines a collection of sizes $2k$, $k \in N$, of balanced complete subgraphs of C_n' such that they form an exact cover $\Psi \subseteq \Phi$ of C_n'. By Lemma 9, the total number of Hamiltonian alternating cycles and red-blue-red alternating chains in a balanced complete subgraphs of size $2k$ is $h_k = a_k + r_k$. Thus, $h_1 = 1$ and $h_k = (k+1)(k-1)!$ if $k > 1$.

Hence, the number of exact covers ψ of C_n' by these cycles and chains per single exact cover Ψ is $h_{n_1}^{m_1} h_{n_2}^{m_2} \ldots h_{n_l}^{m_l}$.

Let $s_0 = 0$, and let $s_{j-1} = \sum_{i=1}^{j-1} m_i n_i$ if $j > 1$. Then the number of all exact covers Ψ is $g_1 g_2 \ldots g_l$, where $g_j = 1$ if $n_j = 1$ and

$$g_j = \prod_{k=0}^{m_j - 1} \binom{n - s_{j-1} - k n_j}{n_j} \text{ if } n_j > 1,$$

where $j = 1, 2, \ldots l$. Hence, the number of all exact covers ψ for all partitions P_n is

$$D_n = \sum_{P_n} \prod_{j=1}^{l} g_j h_{n_j}^{m_j}.$$

Table 4 shows that there exist 11 partitions P_6 and that $D_6 = 9650$. We leave for the reader to verify that $D_5 = 915$, $D_4 = 135$, $D_3 = 18$, $D_2 = 4$, $D_1 = 1$.

There exist $\binom{6}{n}$ complete balanced bipartite subgraphs of size $2n$ of the graph C_6', therefore the number of all millipedes equals

$$\sum_{n=1}^{6} \binom{6}{n} D_n = 6 \cdot 1 + 15 \cdot 4 + 20 \cdot 18 + 15 \cdot 135 + 6 \cdot 915 + 1 \cdot 9650 = 17591.$$

By Theorem 2, the number of all symmetric millipedes is 719, therefore the number of all asymmetric millipedes is $17591 - 719 = 16872$. □

Corollary 1 *There exist exactly* 47520 *centipedes and* 43920 *asymmetric centipedes.*

Proof A dependent centipede is a disjoint union of a millipede and an independent centipede. Therefore, by Lemmas 6 and 10, we can represent a centipede as a combination of an exact cover of a balanced complete bipartite subgraph of C_6' (by alternating cycles and red-blue-red alternating chains) and an unbalanced blue-red-blue or even alternating chain in its complement. Thus, by Theorems 4 and 5, the number of dependent centipedes equals

$$\sum_{n=1}^{4} \binom{6}{n} D_n \cdot \sum_{k=1}^{6-n} \binom{6-n}{k} (u_k + e_k) = 41670,$$

Table 4: Counting the number $D_6 = 9650$ of exact covers of the graph C_6', where $h_1 = 1$ and $h_k = (k+1)(k-1)!$ for $k > 1$, i.e., $h_2 = 3$, $h_3 = 8$, $h_4 = 30$, $h_5 = 144$, $h_6 = 820$.

N	partitions P_6	exact covers Ψ		exact covers ψ		product
{1}	$6 \cdot 1$	1	1	h_1^6	1^6	1
{1,2}	$4 \cdot 1 + 1 \cdot 2$	$\binom{6}{2}$	15	$h_1^4 h_2^1$	$1^4 \cdot 3^1$	45
{1,2}	$2 \cdot 1 + 2 \cdot 2$	$\binom{6}{2}\binom{4}{2}$	$15 \cdot 6$	$h_1^2 h_2^2$	$1^2 \cdot 3^2$	810
{1,3}	$3 \cdot 1 + 1 \cdot 3$	$\binom{6}{3}$	20	$h_1^3 h_3^1$	$1^3 \cdot 8^1$	160
{1,4}	$2 \cdot 1 + 1 \cdot 4$	$\binom{6}{4}$	15	$h_1^2 h_4^1$	$1^2 \cdot 30^1$	450
{1,5}	$1 \cdot 1 + 1 \cdot 5$	$\binom{6}{5}$	6	$h_1^1 h_5^1$	$1^1 \cdot 144^1$	864
{1,2,3}	$1 \cdot 1 + 1 \cdot 2 + 1 \cdot 3$	$\binom{6}{3}\binom{3}{2}$	$20 \cdot 3$	$h_1^1 h_2^1 h_3^1$	$1^1 \cdot 3^1 \cdot 8^1$	1440
{2}	$3 \cdot 2$	$\binom{6}{2}\binom{4}{2}\binom{2}{2}$	$15 \cdot 6 \cdot 1$	h_2^3	3^3	2430
{2,4}	$1 \cdot 2 + 1 \cdot 4$	$\binom{6}{4}\binom{2}{2}$	$15 \cdot 1$	$h_2^1 h_4^1$	$3^1 \cdot 30^1$	1350
{3}	$2 \cdot 3$	$\binom{6}{3}\binom{3}{3}$	$20 \cdot 1$	h_3^2	8^2	1280
{6}	$1 \cdot 6$	$\binom{6}{6}$	1	h_6^1	820^1	820
					total	9650

where $u_k + e_k = 3k!$. By Theorem 4, the number of independent centipedes is 5850, therefore the number of all centipedes is $5850 + 41670 = 47520$. By Theorem 2, the number of symmetric centipedes is 3600, therefore the number of asymmetric centipedes is $47520 - 3600 = 43920$. $\square$

10 Portfolio Decomposition

In the proof of the portfolio decomposition theorem we use the following definitions. Two collections of millipedes and centipedes are *equivalent* if they cover the same set of basic spreads. Lemma 7(b) implies that an independent centipede $\mathbf{z}$ with at least two components is equivalent to a pair that contains an independent millipede $\mathbf{x}$ and a main spread $\mathbf{y}$ that is the resulting spread of $\mathbf{z}$. Thus, $\mathbf{x}$ and $\mathbf{y}$ is an *equivalent pair* of $\mathbf{z}$. Lemma 7(b) also implies that an equivalent pair is unique.

Lemma 11 *Let an independent millipede* $\mathbf{x}$ *and a main spread* $\mathbf{y}$ *generate the equivalent pair of an independent centipede* $\mathbf{z}$*. Then* $m(\mathbf{y}) = m(\mathbf{z})$*.*

Proof Since $\mathbf{y}$ is the resulting spread of $\mathbf{z}$, we have $m(\mathbf{y}) = m(\mathbf{z})$. $\square$

As defined in Section 6, a portfolio P of basic spreads $\mathbf{x} \in A \cup B$ can be represented by their quantities $q(\mathbf{x})$. A portfolio P' with quantities $q'(\mathbf{x}) \leq q(\mathbf{x})$ is a *subportfolio* of P. Let $P - P'$ denote a portfolio with quantities $q(\mathbf{x}) - q'(\mathbf{x})$. Then P' and $P - P'$ constitute a *decomposition* of P. We also say that P is a *consolidation* of P' and $P - P'$. A portfolio P is *symmetric* if $q(\mathbf{x}) = q(-\mathbf{x})$ for all $\mathbf{x} \in A \cup B$ and *antisymmetric* if $q(\mathbf{x}) > 0$ implies $q(-\mathbf{x}) = 0$ for all $\mathbf{x} \in A \cup B$.

Now let us return to the PMM problem posed in Section 6 and consider its extension to the case where millipedes and centipedes are used as offsets. The minimum margin requirement found by using these offsets we call a *multi-leg margin requirement* meaning that the number of legs in offsets based on millipedes and centipedes is not restricted. Thus, the minimum multi-leg margin requirement is always not higher than the four-leg margin requirement; see Section 6.

Lemma 12 *Let P be a portfolio of basic spreads. If P is symmetric, then the minimum multi-leg margin requirement for P is zero. If P is antisymmetric, then the minimum multi-leg margin requirement for P is the minimum four-leg margin requirement.*

Proof In the symmetric case, consider an arbitrary millipede $\mathbf{x}$ in P. Regardless of its structure, the margin requirement for $\mathbf{x}$ is zero. Then we can take the next millipede in the residual portfolio, etc. Since the portfolio is symmetric, the position quantities will be exhausted on a certain step. This means that a symmetric portfolio is a consolidation of millipedes with zero total margin requirement.

In the antisymmetric case, P has no millipedes. Besides, P has no centipedes other than main spreads, otherwise a centipede has at least two components and hence contains a millipede, which is a contradiction. Therefore, the minimum four-leg margin requirement for P cannot be reduced by using millipedes and centipedes other than main spreads. $\square$

Theorem 6 *A portfolio of basic spreads P has a unique decomposition into a symmetric portfolio P' and an antisymmetric portfolio $P - P'$ such that the minimum multi-leg margin requirement for P is the minimum four-leg margin requirement for $P - P'$.*

Proof Let us set $q'(\mathbf{x}) = \min\{q(\mathbf{x}), q(-\mathbf{x})\}$. Then P' is a symmetric portfolio and $P - P'$ is an antisymmetric portfolio, which obviously constitute a unique decomposition of P. To show that this decomposition provides a minimum multi-leg margin requirement, we assume the opposite: There exists a nonempty set σ of other decompositions of P with lower multi-leg margin requirements. In general, they must involve the following three subportfolios: a symmetric portfolio P' that is a consolidation of millipedes, an antisymmetric portfolio A that is a consolidation of main spreads and the portfolio $P - P' - A$ that is a consolidation of centipedes with at least two components. Note that $P - P' - A$ is neither symmetric nor antisymmetric.

Let us consider a decomposition in σ where the portfolio P' is maximal and take a centipede $\mathbf{z}$ from $P - P' - A$. It can be only an independent centipede, otherwise it contains a millipede which can be separated from $\mathbf{z}$ and added to P' while the residual centipede would be left in $P - P' - A$. Since the margin requirement is not changed after this rearrangement, we have a contradiction because then P' is not maximal.

If $\mathbf{z}$ is an independent centipede, then $\mathbf{z}$ can be replaced by the equivalent pair which, by Lemma 11, has the same margin requirement. This pair contains a millipede that could be added to P' in the same way, which is a contradiction again. $\square$

Theorem 6 states that the minimum multi-leg margin requirement for a portfolio of basic spreads with quantities $q(\mathbf{x})$ is the minimum four-leg margin requirement for the subportfolio with quantities $q(\mathbf{x}) - q'(\mathbf{x})$, where

$$q'(\mathbf{x}) = \min\{q(\mathbf{x}), q(-\mathbf{x})\}.$$

Hence, it can be calculated by a maximum flow algorithm as described in Section 6. Thus, the complexity of the problem of margining a portfolio of basic spreads is primarily the complexity of finding a maximum flow in a network.

Since this network has only 12 vertices in the four-dimensional case, see Fig. 3, for each exercise differential, the complexity of margining a collection of portfolios of basic spreads is proportional to the number of these portfolios.

11 Concluding Remarks

We believe that we have taken only the first step in combinatorial modeling of derivative instruments. Being successful on further steps, this kind of modeling will lead to important inferences for margin regulators and eventually margin calculation practice. Our research literature review shows that nothing similar has been done before with the exception of only one paper devoted to combinatorics of stock index baskets offset by index options [Fiterman and Timkovsky, 2001].

Some open theoretical questions and recommendations for margin regulators are collected in the following remarks:

- All results in this paper related to margining are based on the concept of a uniform exercise domain that simplifies the margin rules for option spreads. They are also based on the margin rules following the concept of a maximum loss that disregards the current market prices of the underlying instrument and assigns more stringent margin requirements. The next step in the study of margining aspects of combinatorics of option spreads we believe must be devoted to the case where the exercise domain is not uniform or margin rules for options spreads are based on the concept of a current loss.

- We consider in this paper only the case where the spreads with number of legs more than two are simplest symmetrical spreads that are commonly used as trading strategies. Introducing new combinations of two-leg spreads will allow to find more sophisticated option spreads which represent more efficient hedging mechanisms. They can be used for margining purposes as well as for the design of new option trading strategies. We should mention, however, that even four-dimensional complex spreads whose components are antisymmetrical two-leg spreads and symmetrical three- or four-leg spreads are not being used by the existing margining systems; thus, they cannot take advantage of multi-leg margining even in the simplest form.

- Our model shows that together with 40 complex spreads of dimension four there exist many more spreads with the same hedging mechanism that can be margined, therefore, in the same way. If all four-dimensional centipedes and millipedes were recognized by margin regulators, then margin requirements for margin accounts with options could have been substantially decreased. Such a regulatory movement would release substantial free equity capital bursting investment activity in the options markets.

- As we have shown in Section 10, the problem of minimizing the multi-leg margin requirement for a portfolio of basic spreads can be solved in polynomial time. We can pose the following question: how to construct a portfolio of basic spreads from a portfolio of individual options such that the former has the minimum multi-leg margin requirement? This problem is equivalent to the problem of minimizing the multi-leg margin requirement for a portfolio of individual options whose complexity status remains unknown. A polynomial solution to this problem is known only in the case of finding the minimum two-leg margin requirement [Rudd and Schroeder, 1982].

Complex option spreads are the most efficient regulatory products of these days, which lead to the new era of lowered margin requirements in the stock market; without increasing risk but more careful study of hedging mechanisms hidden in option portfolios.

Acknowledgements

In conclusion, we would like to thank all people who were involved in the design of the complex option spreads, which inspired us to write this paper. We are also indebted to Eddie Anderson for valuable comments.

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