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# When to quit when you're ahead

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# When to quit when you're ahead

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**Abstract**

It is common for rewards to be given on the basis of a rank ordering, so that relative performance amongst a cohort is the criterion. In this paper we formulate an equilibrium model in which an agent makes successive decisions on whether or not to gamble and is rewarded on the basis of a rank ordering of final wealth. This is a model of the behaviour of mutual fund managers who are paid depending on funds under management which in turn are largely determined by annual or quarterly rank orderings. In this model fund managers can elect either to pick stocks or to use a market tracking strategy. In equilibrium the final distribution of rewards will have a negative skew. We explore how this distribution depends on the number of players, the probability of success when gambling, the structure of the rewards, and on information regarding the other player's performance.

## 1   Introduction

There are many areas in life in which the aim is to come first, even by a small amount. More generally rewards are often based, not on the absolute values of some variables, but instead on the rank ordering between firms or individuals who compete against each other. Business School academics are used to a world in which significant rewards accrue to Schools who do well in various published rankings. Such an environment may well persuade firms or individuals to adopt some relatively high risk strategies to give a chance of breaking into the group with the highest rewards.

Even without any external ranking process, the relative wealth of individuals may be an important motivator for behavior. This reflects the importance of people's concept of

status and the patterns of conspicuous consumption which go along with this. Many authors have discussed the impact of status-seeking behavior (see Hopkins and Kornienko (8) and the references therein). It is clear that if individuals care about their relative wealth then there are implications for attitudes to risk as has been considered by Robson (12) and others.

In this paper we explore the way in which negative skew is introduced by players who are in part competing against other players for a high ranking in a tournament. We model an environment in which players derive benefit from doing better than other players, though their objective may also include some measure of absolute return. Our model is one in which each player's strategy has the effect of selecting a distribution of possible returns. Each player then receives a benefit that is a function of its own return and the returns of the other players. Players act in a way to maximize their expected utility.

This is a situation that can occur in a number of different environments: for example we could consider a group of managers who are seeking to maximize the relative profits of their part of the overall organization (perhaps competing store managers in a retail chain, or sales managers for different regions), or we could consider fund managers seeking a higher ranking for their funds. We will show that in these circumstances players can be expected to choose actions which lead to a small probability of large negative returns, while at the same time foregoing the balancing large positive returns in favour of a safer strategy giving a relatively high probability of a modest positive return. Later we will look in more detail at the shapes of the distributions that occur, but we can summarize by saying that distribution of returns has negative skew. The intuition for this is simple. If the return is already better than the other players then from a competitive standpoint nothing more is gained by improving it further, while if the return is worse than the return of the other players then from a competitive standpoint nothing is lost if the return is made even worse.

Where a principal wishes to motivate a group of agents, all subject to common exogenous factors, then it may well be appropriate to reward on the basis of a rank ordering of performance, rather than absolute performance values. Doing so eliminates the possibilities of rewarding players who simply benefit from good conditions and retains the incentives for effort even when overall conditions are bad. The same sort of argument applies in a fund management setting where the rankings provide a useful tool for investors who want to look beyond the absolute level of returns. In this context however the introduction of negative skew may well be contrary to the aims of the principal: neither shareholders nor investors are

likely to favour an environment in which a high probability of positive returns is balanced by a small probability of disastrous results.

The literature in this area has two main strands. There are a number of papers which analyze situations where rewards based on ranking are given in order to motivate effort applied by agents. Some of this literature is in labour economics, and job promotions comprise one obvious way in which the best relative performance generates a large prize. An early paper by Lazear and Rosen (11) formulates the problem with output related to investment (or effort) together with a random component. The reward structure is then related to rank and the optimum choice of effort will be related to the difference between rewards for coming first, second etc. Hvide (9) introduces the ability for the agent to take risks (increasing the variance of the random component) and shows that a result with very high variance and low effort occurs as an equilibrium. A recent study by Casas-Arce and Martinez-Jerez (4) considers the tournament game that arises when a manufacturer rewards retailers with prizes for good sales performance. Again the primary aim is to induce retailers to expend more sales effort.

A second strand of literature, which is more relevant to this paper, arises from research on fund performance. It has been recognized for some time that the rankings of mutual funds, which are regularly published, have a considerable impact on the flow of money into these funds. Since fund managers are often compensated through a flat fee plus a percentage of the funds under management it is reasonable to suppose that these managers will behave in ways that are similar to a tournament game in which their reward is directly related to their ranking amongst similar funds. Both Brown et al. (3) and Chevalier and Ellison (7) provide evidence that fund managers who do well in the first half of the year tend to use less risky strategies in the second half of the year and propose that this is a consequence of the fact that in a tournament a fund that starts out badly needs to achieve a higher volatility in order to have any chance of winning. There is now quite a substantial literature in this area. First the empirical evidence is more mixed than the early studies suggested and also the tournament model may produce counter-intuitive results when formulated in a way that reflects the choice between investing in a risky asset and investing in the index. Taylor (15) provides a two period model in which winning managers are more likely to choose a risky portfolio than losing managers. There are a number of more recent papers such as (1) that discuss these issues.

Our focus is on the dynamics of these competitive environments: this enables us to model

a situation with a rich structure for the distribution of final returns. We look at a multiple period model in which a simple ('gamble or not' ) decision is taken at each period. The net effect is to produce a distribution of returns with some negative skew in the equilibrium. It has been observed by a number of authors (10) (2) that returns from both mutual funds and hedge funds are far from normal, with evidence of negative skew in many cases.

In this paper we discuss the properties of the equilibrium solution. We are able to analyze small cases exactly and prove some results characterizing the equilibrium strategies, as well as providing a numerical method for larger problems. We investigate the impact of changes in the probability of winning a gamble, of increasing the number of players, and of changing from a winner takes all payoff to one in which say the first, second and third placed players all benefit. Finally we consider the impact of information about the performance of other players. Our basic model has the final returns of all the players revealed only at the end: What is the result of having information on the performance of other players available throughout the game?

## 2    The model

We suppose that there are $N$ players who compete in a tournament. The game ends at time $T$. At this point player $i$ has a total return $x_i(T)$. We will use the terminology 'return' or 'position' interchangeably to refer to the characteristic on which competing players are evaluated. In a fund management context this will be equal to wealth. The payoff or utility received by player $i$ has two components related to the absolute value of its return and the relative value with respect to the other players. Thus we can suppose that player $i$ seeks to maximize the expectation $E\left[H_i(x_i(T)) + \varphi_i(x_i(T), x_{-i}(T)]\right]$ where we write $x_{-i}(T)$ for $\{x_j(T) : j \neq i\}$ and the function $\varphi(a, \underline{b})$ is based entirely on the rank ordering of $a$ within the set of values $\underline{b}$. We look for equilibrium solutions to this game.

We begin by focussing on a simple class of these problems in which $\varphi_i(x_i(T), x_{-i}(T)) = 1/m$ if there are $m$ players including player $i$ in the set of winning players. Thus if we define the set of winning players as $W = \{j : x_j(T) = \max(x_1(T), ...x_n(T))\}$ then $\varphi_i(x_i(T), x_j(T)) = 1/|W|$ if $i \in W$ and $\varphi_i(x_i(T), x_{-i}(T)) = 0$ otherwise. Hence the component of the player's expected utility which is given by $E\left[\varphi_i(x_i(T), x_{-i}(T))\right]$ is simply the probability of player $i$ winning when the winner is selected at random from amongst the players achieving the best return at time $T$.

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We will suppose that there are $T$ periods and the firm's action at stage $t$ depends on its current position, $x(t)$. We shall assume that players have no information on the behavior of other players. There is a choice of doing nothing so $x(t+1) = x(t)$, or investing in a risky asset ('gambling') with a probability $p$ of $x(t+1) = x(t) + \delta$ and a probability $1 - p$ of $x(t+1) = x(t) - \delta$. In the special case that $p = 0.5$ and the component $H_i(x(T))$ is linear so that players are risk neutral with respect to final return then, since expected final return is fixed, we may ignore the $H_i$ term in the player payoff. We look for a Nash equilibrium in strategies.

Since one of our applications relates to the behavior of fund managers, we begin by showing how this simple "$\pm\delta$" random walk with drift can be translated into a framework more familiar from the perspective of financial investment. We can suppose that the players may choose either to invest in a market tracking instrument which follows some stochastic behavior $M(t)$ or to invest in a risky stock which in each period achieves a return of $e^{\delta}M(t)$ with probability $p$ and a return of $e^{-\delta}M(t)$ with probability $1 - p$. From a fund manager's perspective this is achieved by letting the fund become significantly overweight in a few stocks relative to their natural weighting in the asset class. Thus if wealth is $w(t)$ at time $t$ then investing in the market tracking instrument gives wealth $w(t)M(t)$ at time $t + 1$. and the log of wealth is either $\log(w(t)) + \log(M(t))$ or $\log(w(t)) + \log(M(t)) \pm \delta$ depending on whether or not the market tracking instrument is selected. The log of wealth at a time $t$ is given by $\log(w(0)) + \sum_{i=i}^{t} \log(M(i))$ plus a term that captures the random walk which takes place when there is an investment in a risky stock. Hence we can track the log market behavior and take this away from the log return to exactly reproduce our original problem. Notice however that we can no longer simply remove the $H_i$ term in the player payoff if it is linear. In this context 'gambling' will leave expected log wealth unaltered if $p = 0.5$, but will lead to a change in expected wealth. Note that it is not unreasonable in some circumstances to treat log wealth as a (risk averse) utility function.

We will assume that the first period has $t = 1$ so there are $T - 1$ opportunities to gamble (or not) at time periods $1, ...T - 1$. We use the notation that, for a policy $\pi$, $\pi(x, t)$ is the probability of gambling at time $t$ if the position at time $t$ is $x$. We will often refer to this as the state $(x, t)$. A pure strategy will have either $\pi(x, t) = 1$ or $\pi(x, t) = 0$. However since we are interested in a Nash equilibrium we may need to use mixed strategies in which $0 < \pi(x, t) < 1$ for some values of $x$ and $t$.

We consider an optimal strategy for player $i$ given that the strategy for each of the other players $j \neq i$, $1 \leq j \leq m$, is fixed and each of the other players is using the same strategy. Thus each of the other players has the same distribution over final positions $x(T)$: we let $G(z) = \Pr(x_j(T) \leq z)$ and $g(z) = \Pr(x_j(T) = z)$.

We write $V(x,t)$ for the expected value for player $i$ starting at state $(x,t)$ and behaving optimally. The dynamics of the process is that if we gamble in state $(x,t)$ then with probability $p$ we move to $x+1$ at time $t+1$ and with probability $1-p$ we move to $x-1$ at time $t+1$. The probability of winning if we end in state $(x,T)$ can be calculated by considering the probability that all $N-1$ other players end at positions less than $x$; plus the probability that $N-2$ players end at positions less than $x$, and one other player ends at $x$ (when we split the prize); plus the probability that $N-3$ players end at positions less than $x$, and two other players end at $x$; and so on. We obtain

$$
\begin{aligned}
V(x,T) &= G(x-1)^{N-1} + \frac{1}{2}(N-1)g(x)G(x-1)^{N-2} \\
&\quad + \frac{1}{3}\frac{(N-1)(N-2)}{1.2}g(x)^2 G(x-1)^{N-3} + ... + \frac{1}{N}g(x)^{N-1} \\
&= \frac{1}{Ng(x)}\left(\big(g(x)+G(x-1)\big)^N - G(x-1)^N\right) = \frac{1}{Ng(x)}\left(G(x)^N - G(x-1)^N\right).
\end{aligned}
$$

When $N = 2$ this becomes

$$
V(x,T) = G(x-1) + 0.5g(x) = 0.5G(x-1) + 0.5G(x). \tag{1}
$$

The dynamic programming recursions are as follows (where in each case the condition holds for all $x \in \mathcal{X} = \{-T, -T+1, ...T-1, T\}$ and $t \in \mathcal{T} = \{1, 2...T-1\}$ ):

$$
V(x,t) = \max\{pV(x+1,t+1) + (1-p)V(x-1,t+1), V(x,t+1)\}. \tag{2}
$$

Then the optimal policy is to gamble, i.e. $\pi(x,t) = 1$, if

$$
V(x,t+1) < pV(x+1,t+1) + (1-p)V(x-1,t+1), \tag{3}
$$

and the optimal policy is not to gamble, i.e. $\pi(x,t) = 0$, if

$$
V(x,t+1) > pV(x+1,t+1) + (1-p)V(x-1,t+1). \tag{4}
$$

**Lemma 1.** If for some given $w$ both

$$V(w,t) < pV(w+1,t) + (1-p)V(w-1,t) \qquad (5)$$

and

$$V(w,s) > pV(w+1,s) + (1-p)V(w-1,s), \qquad (6)$$

then $s > t$.

**Proof.** Suppose on the contrary that $s < t$. Choose the last such $s$, so

$$V(w,r) = pV(w+1,r) + (1-p)V(w-1,r),$$

for $s < r < t$. When $s < t-1$ we may use (2) and deduce that $V(w,s) = V(w,s+1) = ... = V(w,t-1)$. If $s = t-1$ then this is trivial. Now from (5) and (2)

$$V(w,t-1) = pV(w+1,t) + (1-p)V(w-1,t).$$

Hence

$$V(w,s) = pV(w+1,t) + (1-p)V(w-1,t). \qquad (7)$$

Now we can apply (2) repeatedly to show that

$$V(w-1,s) \geq V(w-1,s+1) \geq ... \geq V(w-1,t)$$

and that

$$V(w+1,s) \geq V(w+1,s+1) \geq ... \geq V(w+1,t).$$

Hence, from (6), $V(w,s) > pV(w+1,t) + (1-p)V(w-1,t)$ which contradicts (7) and establishes the result. ∎

The result of this lemma shows that once there is a definite advantage to not gambling, then gambling will not take place again. We can say that the player then 'quits'. In the fund management context once there is a definite advantage to switching out of the risky asset, then it is never optimal to switch back again. Thus there are functions $\alpha(x) < \beta(x)$ for $x \in \mathcal{X}$ taking values in $\mathcal{T} \cup \{0, T\}$ with the properties:

(a) $V(x,t+1) < pV(x+1,t+1) + (1-p)V(x-1,t+1)$ and $\pi(x,t) = 1$ for $t \leq \alpha(x)$ and

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$t \in \mathcal{T}$ ;

(b) $V(x, t+1) > pV(x+1, t+1) + (1-p)V(x-1, t+1)$ and $\pi(x, t) = 0$ for $t \geq \beta(x)$ and $t \in \mathcal{T}$ ;

(c) $V(x, t+1) = pV(x+1, t+1) + (1-p)V(x-1, t+1)$ for $\alpha(x) < t < \beta(x)$ and $t \in \mathcal{T}$ .

In the case that $\alpha(x) = 0$ then (a) is empty and when $\beta(x) = T$ then (b) is empty. If $\alpha(x) = \beta(x) - 1$ then (c) is empty.

Notice that for $\beta(x) < T$ then

$$V(x, \beta(x)) = V(x, \beta(x) + 1) = ... = V(x, T) = 0.5G(x-1) + 0.5G(x).$$

Values of $x$ for which $\alpha(x) = T - 1$ have the property that a player at position $x$ always gambles whatever the time period. We expect this set to include any value sufficiently small (i.e. large negative values). We can be more specific since these $x$ values are determined by the fact that

$$V(x, T) < pV(x+1, T) + (1-p)V(x-1, T).$$

When there are two players, substituting from (1) gives

$$G(x-1) + G(x) < pG(x) + pG(x+1) + (1-p)G(x-2) + (1-p)G(x-1)$$

which simplifies to

$$(1-p)(g(x) + g(x-1)) < p(g(x+1) + g(x)). \tag{8}$$

Under this condition we gamble at $(x, t)$ when $t = T - 1$ and hence (from the lemma) for any value of $t$. On the other hand (reversing all the inequalities) shows that if

$$(1-p)(g(x) + g(x-1)) > p(g(x+1) + g(x)),$$

then $\beta(x) < T$ and it is optimal not to gamble at $(x, T-1)$.

# 3   Equilibrium solutions for two examples

We begin by considering a two player game with $p = 0.5$. Small cases can be solved by hand and Figure 1 shows an equilibrium solution with $T = 4$. At the nodes marked A and C mixing
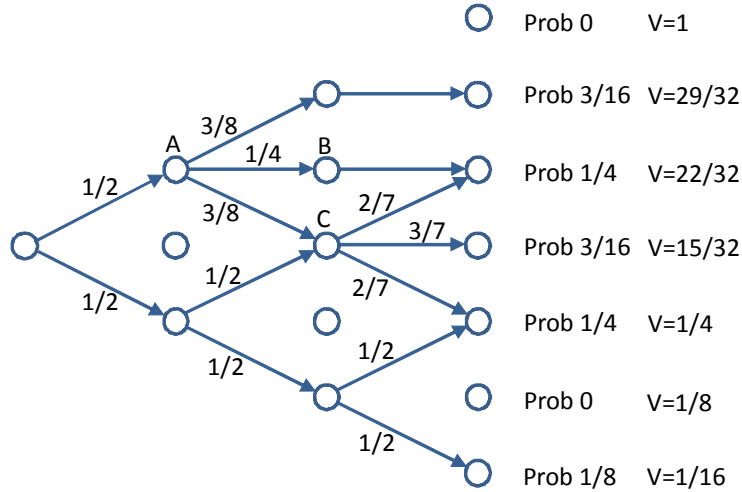
Figure 1: The equilibrium solution for $T = 4$

takes place: there is a 3/4 probability of gambling at A and a 4/7 probability of gambling at C. With the policy shown the probabilities of achieving each of the final wealth values are as shown on the right of the Figure. From these values the final values $V(x, T)$ are obtained (also shown on the right of the Figure) and then the dynamic programming recursions can be used to determine the values of $V$ throughout the tree. It is not hard to check that at each node the 'right' decisions are made and at the two nodes at which mixing takes place there are equal values obtained from the gambling or quitting options.

It turns out that there are many different equilibrium solutions. There will be a range of equilibrium solutions in which the basic structure is the same but the exact choice of gamble probabilities varies: all of these will share the same values of the probabilities at $T$ and hence of $V(x, T)$. Specifically if $\pi_A$, $\pi_B$, and $\pi_C$ are the probabilities of gambling at these three nodes, then we have

$$\pi_C = \frac{1}{1 + \pi_A}, \ \pi_B = \frac{0.75 - \pi_A}{1 - \pi_A}$$

and $\pi_A$ can take any value in the range $[0, 0.75]$.

We note that the condition for mixing at $(x, t)$ is

$$V(x, t + 1) = 0.5V(x + 1, t + 1) + 0.5V(x - 1, t + 1).$$

i.e. that $V(x, t+1) - V(x-1, t+1) = V(x+1, t+1) - V(x, t+1)$. Using the same approach as in (8) we can simplify this expression for $t = T - 1$ and we see that the condition for mixing
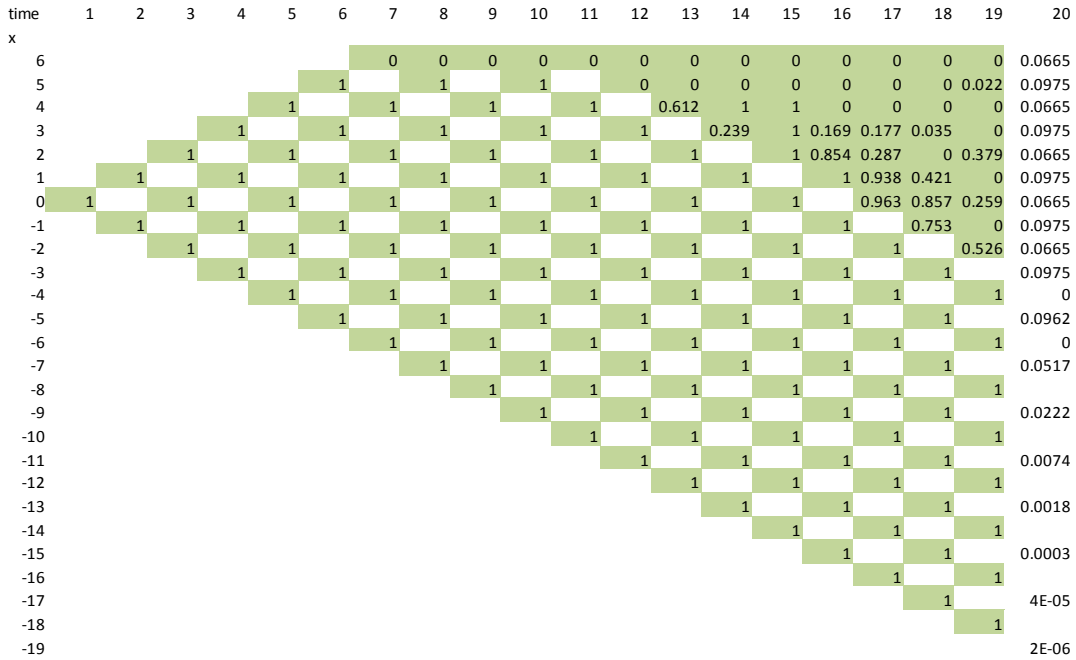
9

| x | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| 6 |  |  |  |  |  |  | 0 |  | 0 |  | 0 |  | 0 |  | 0 |  | 0 |  | 0 | 0.0665 |
| 5 |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 0 |  | 0 |  | 0 |  | 0.022 |  | 0.0975 |
| 4 |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.612 | 1 | 1 | 0 | 0 | 0 | 0 | 0.0665 |
| 3 |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.239 | 1 | 0.169 | 0.177 | 0.035 | 0 | 0.0975 |
| 2 |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 | 0.854 | 0.287 | 0 | 0.379 | 0.0665 |
| 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 | 0.938 | 0.421 | 0 | 0.0975 |
| 0 | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.963 | 0.857 | 0.259 | 0.0665 |
| -1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.753 | 0 | 0.0975 |
| -2 |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.526 | 0.0665 |
| -3 |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.0975 |
| -4 |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 | 0 |
| -5 |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.0962 |
| -6 |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 | 0 |
| -7 |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.0517 |
| -8 |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  |
| -9 |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.0222 |
| -10 |  |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 1 |  |
| -11 |  |  |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  | 0.0074 |
| -12 |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 1 |  |
| -13 |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  | 0.0018 |
| -14 |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 1 |  |
| -15 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  | 0.0003 |
| -16 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  | 1 |  |
| -17 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  | 4E-05 |
| -18 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 1 |  |
| -19 |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  | 2E-06 |

Figure 2: Pattern of equilibrium behaviour when $T = 20$

at $(x, T-1)$ is simply

$$g(x-1) = g(x+1). \tag{9}$$

We note the duplication of the $g$ values $3/16$ and $1/4$ in the $T = 4$ equilibrium solution, just as is required by this equation.

Our second example uses $T = 20$ with $N = 2$ and $p = 0.5$. This is straightforward to solve using the computational approach which will be discussed later and Figure 2 shows an equilibrium solution. Shading represents the points where there is a positive probability mass The final column gives the $g$ values and the other elements in the table are $\pi(x,t)$, the probabilities of gambling at each position.

The pattern of behavior here is quite clear. For low values of $t$ it is best to gamble but there is an absorbing barrier - so that if $x$ becomes large enough (equal to 6 with $T = 20$) then gambling stops. However at a certain point mixing starts to occur and this is carried out in a way that gives rise to alternating probability values at the final time for sufficiently large $x$ ( $x \geq -3$ in this case). We have a triangular region in which mixing takes place (with $T = 20$ the region is determined by $x + t \geq 17$).

# 4 Finding an equilibrium solution

Consider first the two-player game in which each player selects a value of $\pi \in \{0, 1\}$ at each state $(x, t)$. We take $\Pi$ to be the (finite) set of such $\pi$. This is the same problem except that we have added a restriction to rule out $0 < \pi(x, t) < 1$. Then this is a two player constant sum game with a finite action space (all the pure strategies with $\pi(x, t) \in \{0, 1\}$ for $x \in \mathcal{X}$ and $t \in \mathcal{T}$) and so has a minimax solution in mixed strategies. Now observe that playing a mixture $\sum \mu_i \pi_i$ with each $\pi_i(x, t) \in \{0, 1\}$ is equivalent to playing the strategy defined by $\pi(x, t) = \sum \mu_i \pi_i(x, t)$. Hence we have established that there is an equilibrium in our original game with two players.

The $N$ player case requires more care. We can use a general result which goes back to Nash (1951) but see also Cheng et al. (2004) for a discussion of this result. They show (Theorem 4) that a symmetric game of this form will have an equilibrium solution which is also symmetric, and hence each player gets a payoff of $1/N$. Thus the existence of an equilibrium is established. Notice, however, that the equilibrium may not be unique. Indeed our examples above indicate that there will not be a unique policy. The result of playing any policy is clearly determined by the final distribution of return values, and so strategies that share the same distribution, $g(x)$, will be equivalent. We will show that in many cases an equilibrium solution is unique up to this equivalence.

Our next task is to establish that a (symmetric) solution for the $N$ player case can be found using a method related to the linear programming solution for the two player case. We let $\phi_k$ be the prize awarded to a player in position $k$ so the winner gets $\phi_1$, the second place player $\phi_2$, etc. If there is a tie for places $j, j+1, ...j+h-1$ then the $h$ prizes are awarded randomly amongst these players with each prize going to a different player. The consequence is that each of these $h$ players has expected payoff $(\phi_j + \phi_{j+1} + ... + \phi_{j+h-1})/h$. We say that the prize structure determined by $\phi$ is convex if either $N = 2$, or if $N > 2$ and $\phi_i + \phi_{i+2} \geq 2\phi_{i+1}$, $i = 1, 2, ..N - 2$.

The difficulties in many of the calculations for the multi-player version of this game are associated with the possibility of a number of players ending at the same position. Since it is only rank ordering that matters, it will turn out to be convenient to define the perturbed game in which each player after ending at position $x$ then undergoes a further randomization giving a final distribution of positions which is uniform over an interval $[x - \varepsilon, x + \varepsilon]$. Thus all rank orderings amongst the players with a position $x$ are equally likely. So the expected

payoff in the perturbed game matches that in the original game and they are equivalent.

Now consider one player using a pure strategy $i$ and all the other players using mixed strategies. We let $I$ be the set of all pure strategies and $|I| = M$. If there are $N$ players, let $A^{(i)}(v^{(2)}, v^{(3)}, ..., v^{(N)})$ be the payoff to a player 1 using pure strategy $i$ when the $k$'th player mixes with probabilities $v^{(k)} \in \mathbb{R}^M$ over the set of pure strategies. Thus each $v^{(k)}$ is an $M$-vector. For an $M$-vector $v$ we define $A^{(i)}(v) = A^{(i)}(v, v, ..., v)$, to be the expected payoff to player 1 when each of the other players uses the same mixed strategy $v$.

**Lemma 2.** If the prize structure is convex then $A^{(i)}(v)$ is convex in $v$.

**Proof.** Let $G_v(z)$ be the probability of ending at a position less than $z$ in the perturbed game using strategy $v$. We write this as $G$. Then the probability of winning prize $\phi_k$ if $(x, T) = z$ is given by

$$h_k(G) := \frac{(N-1)!}{(k-1)!(N-k)!} G^{N-k}(1-G)^{k-1}.$$

Hence the expected payoff in this final position is

$$\rho(G) := \sum_{k=1}^{N} \phi_k h_k(G).$$

First suppose that $N > 2$. Then

$$\begin{aligned}
\rho'(G) &= \sum_{k=1}^{N} \phi_k \frac{(N-1)!}{(k-1)!(N-k)!}((N-k)G^{N-k-1}(1-G)^{k-1} - (k-1)G^{N-k}(1-G)^{k-2}) \\
&= (N-1) \sum_{k=1}^{N-1} \frac{(N-2)!}{(k-1)!(N-k-1)!}(\phi_k - \phi_{k+1})G^{N-k-1}(1-G)^{k-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\rho''(G) &= (N-1) \sum_{k=1}^{N-1}(\phi_k - \phi_{k+1})\frac{(N-2)!}{(k-1)!(N-k-1)!}[(N-k-1)G^{N-k-2}(1-G)^{k-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (k-1)G^{N-k-1}(1-G)^{k-2}] \\
&= (N-1)(N-2) \sum_{k=1}^{N-2}[(\phi_k - 2\phi_{k+1} + \phi_{k+2})\frac{(N-3)!}{(k-1)!(N-k-2)!}G^{N-k-2}(1-G)^{k-1}
\end{aligned}$$

which is non-negative by our assumption on $\phi$, demonstrating that $\rho$ is a convex function of $G$.

If $N = 2$ then $\rho'(G) = (\phi_1 - \phi_2)$ and $\rho$ is linear.

Notice that we can write

$$A^{(i)}(v) = \sum_x g_i(x) \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \rho(G_v(z)) dz.$$

But if $w = \mu v + (1-\mu)u$ is a convex combination of two mixed strategies $v$ and $u$ then $G_w(z) = \mu G_v(z) + (1-\mu)G_u(z)$. And so, from the convexity of $\rho$,

$$\rho(G_w(z)) \le \mu \rho(G_v(z)) + (1-\mu)\rho(G_u(z)).$$

Thus

$$\begin{aligned} A^{(i)}(w) &= \sum_x g_i(x) \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \rho(G_w(z)) dz \\ &\le \mu A^{(i)}(v) + (1-\mu) A^{(i)}(u) \end{aligned}$$

as required. ∎

We are now ready to characterize a symmetric Nash equilibrium for the $N$ player game. We define the problem

$$\begin{aligned} P_N(I): \quad &\text{minimize} \quad \lambda \\ &\text{subject to} \quad A^{(i)}(v) \le \lambda \text{ for all } i \in I \\ &\qquad\qquad \sum_{j=1}^M v_j = 1 \\ &\qquad\qquad v_j \ge 0, \ j = 1, 2, ..., M \end{aligned}$$

**Theorem 1.** If the prize structure is convex and $\lambda^*$, $v^*$ is an optimal solution to the problem $P_N(I)$, then $\lambda^* = 1/N$ and each player using $v^*$ is a symmetric Nash equilibrium for the $N$ player game

**Proof.** The problem $P_N(I)$ has a linear objective and from Lemma 2 the feasible region is convex. Moreover a constraint qualification is satisfied (simply take $\lambda$ large). Hence a Lagrangian duality result holds.

The Lagrangian for this problem is

$$L(v, z, \lambda, \mu, \rho) = \lambda + \sum_{i \in I} \mu_i(A^{(i)}(v) + z_i - \lambda) + \rho(1 - \sum_{j=1}^{M} v_j)$$

$$= \lambda(1 - \sum_{i \in I} \mu_i) + \sum_i \mu_i A^{(i)}(v) + \rho(1 - \sum_{j=1}^{M} v_j) + \sum_{i \in I} \mu_i z_i.$$

Lagrangian duality implies that there are values for $\mu, \rho$ such that

$$\text{minimize } L(v, z, \lambda, \mu, \rho) \text{ over } v_j \geq 0, \ z_i \geq 0$$

is achieved at the optimal solution to the original problem and moreover these values maximize the minimum of $L$. We deduce $\mu_i \geq 0$ and $\sum_i \mu_i = 1$ with complementary slackness between $\mu$ and $z$ .

Hence there exists a scalar $\rho$ and vector $\mu$ with $\mu_i \geq 0$ with $\sum_i \mu_i = 1$ such that

$$\min_{v_j \geq 0} \sum_i \mu_i A^{(i)}(v) + \rho(1 - \sum_{j=1}^{M} v_j) = \lambda^*.$$

Thus, since one possible value for $v$ is $v = \mu$ and noting that $\sum_{j=1}^{M} v_j = 1$, we have

$$\sum_i \mu_i A^{(i)}(\mu) \geq \lambda^*.$$

But the left hand side here is just the result of playing $\mu$ against itself and is therefore equal to $1/N$, and hence $\lambda^* \leq 1/N$.

Now consider the first player using the mixed strategy $v^*$ which solves $P_N$. Then all the strategies are the same and the player has a payoff of $1/N$. i.e. $\sum_{i \in I} v_i^* A^{(i)}(v^*) = 1/N$ . Hence at least one pure strategy $j \in I$ has $A^{(j)}(v^*) \geq 1/N$ and so $\lambda^* \geq 1/N$. Putting these results together we have $\lambda^* = 1/N$.

This completes the proof that each player using $v^*$ is a symmetric Nash equilibrium for this game. One player varying from this can achieve at most $\max_{i \in I} A^{(i)}(v^*) = 1/N$ so no player can improve its payoff by unilaterally changing its strategy. ∎

This result can be used to establish uniqueness of the solution up to the values of $g$ in the case that the prize structure us strictly convex (with the obvious interpretation of this) and $N > 2$. The problem can be reformulated with the $g$ values as the fundamental choice rather

than $\pi$ and then use the fact that an optimization problem with a strictly convex feasible set and linear objective has a unique optimal solution. A more specific approach will be needed to show this with $N = 2$

## A computational approach for large problems

The fact that an equilibrium solution can be obtained from solving $P_N(I)$ suggests a computational approach that can be used for larger examples. We adapt a method using a column generation approach to solve the convex problem $P_N(I)$. The set of all pure strategy solutions, $I$, is very large since each $\beta(x)$ can take values from $|x|$ to $T$ and to define a pure strategy solution involves picking a $\beta(x)$ for each $x \in \mathcal{X}$. This gives $(T!)^2(T+1)$ pure strategies. When $T = 10$ this is already more than $10^{14}$.

Our algorithm proceeds as follows. At step $k$ we use a set of pure strategies $I_k = \{\pi_1, \pi_2, ...\pi_k\}$, and solve the problem $P_N(I_k)$ as though $I_k$ was the complete set of pure strategies. Suppose this gives an optimal solution $v^{(k)}$ which gives weight $v_j^{(k)}$ to pure strategy $\pi_j$. Then we suppose that all the players $2, 3, ..., N$ use strategy $v^{(k)}$ and search for the best strategy for player 1. This is done simply by solving the dynamic programming recursions (1) and (2). This new strategy, call it $\pi_{k+1}$, is then added to the set $I_k$ to produce $I_{k+1} = I_k \cup \{\pi_{k+1}\}$. Then $k$ is incremented by 1 and the whole process is repeated (solving the problem $P_N(I_{k+1})$ ). We stop when the new optimal policy $\pi_{k+1}$ is already in the policy set, or when $\pi_{k+1}$ achieves a value of $1/N$ for player 1, indicating that the new policy achieves no improvement over the existing set of policies. This will guarantee an equilibrium since it shows that no player can improve their return by changing from $v^{(k)}$.

We implement this by starting with a set of $K$ solutions (all the same) and then adding one solution at a time by replacing a solution with zero weight in the current best mixture. In fact we continue for longer than $K$ steps, at each stage replacing the solution with the lowest average recent weight (we use an exponential smoothing approach to form a measure of the recent average probabilities applied to each solution). At each stage the payoff function $A^{(i)}$ needs to be found for the new pure strategy $\pi_i$ being introduced.

## 5   Patterns of behavior in larger problems

With the computational approach described above we can find equilibrium solutions for larger values of $T$ and we observe a consistent pattern of behavior. This is broadly similar to the

behavior with $T = 20$, having a triangular region where mixing takes place. But there may also be a region with higher values of $x$ where the behavior is not to gamble for $t$ large enough. In this region $\alpha(x)$ is strictly greater than $\alpha(x+1)-1$ whereas in the mixing region $\alpha(x)$ increases by 1 for each increment in $x$. In this section we explore how changes in the parameters of the problem effect the equilibrium solutions.

**Varying the drift**

The parameter $p$ gives the advantage or disadvantage of gambling. In an investment context it is natural to consider an upward drift corresponding to $p > 0.5$. We expect that as $p$ increases then gambling becomes more and more attractive. We begin by asking whether for sufficiently high $p$ value there is an equilibrium where players always gamble. Consider this 'always gamble' solution with $T$ even and let $S = T/2$. In this case the probabilities $g(x)$ are given by the binomial distribution with

$$g(x) = p^{S+x/2}(1-p)^{S-x/2}\frac{T!}{(S+x/2)!(S-x/2)!} \text{ if } x = -T, -T+2, ... -2, 0, 2, ...T-2, T$$

and $g(x) = 0$ when $x$ is odd. Now observe that we can evaluate the decision to gamble at $x + 1$ at time $T - 1$ where $x = 2y$ is even. As we saw before it is optimal to gamble if the inequality (8) holds, and since $g(x + 1) = 0$ we can rewrite this condition as

$$(1-p)(p^{S+y}(1-p)^{S-y}\frac{1}{(S+y)!(S-y)!}) < p(p^{S+y+1}(1-p)^{S-y-1}\frac{1}{(S+y+1)!(S-y-1)!}).$$

This simplifies to

$$p^2(S-y) > (1-p)^2(S+y+1).$$

This inequality is less likely to be satisfied as $y$ increases (making the positive term smaller and the negative term higher). The highest value of $y$ occurs when $x + 1 = T - 1$ and so $y = S - 1$. Hence an always-gamble solution will be an equilibrium if $p/(1 - p) > \sqrt{2S}$,i.e. $p > \sqrt{T}/(1 + \sqrt{T})$. Thus for high enough value of $p$ the always-gamble solution will be an equilibrium, but the switching point for $p$ approaches 1 as $T$ gets large. For $T = 80$ we need $p > 0.899$.

In the other direction we can consider values of $p$ less than 0.5 when gambling becomes less attractive. For small enough values of $p$ we expect that the solution of never gambling
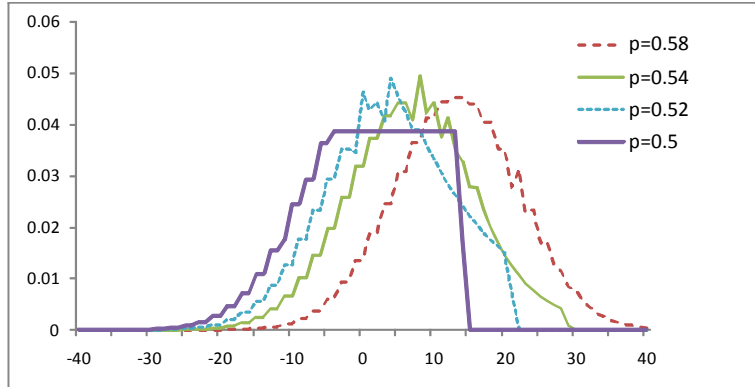
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Figure 3: Distribution of final position with upward drift

will be an equilibrium. Against another player using this strategy it is clearly best to gamble at any $x, t$ position with $x < 0$ (otherwise we never win) and not to gamble if $x > 0$ (since winning is assured). Hence we need to calculate the value of $V$ at $x = -1$, $t = 1$ to check that not gambling is optimal at the start. Unfortunately this is a very complex expression and does not give rise to a simple expression for the limiting value of $p$ at which gambling becomes never worthwhile. We can work out the values numerically and for $T = 80$ it turns out that for $p < 0.333344$ it is never worth gambling.

In the limit of large $T$ we have $V(-1, t) = 0.5 \Pr(x = 0 \text{ for some } s > t)$. Let $\gamma$ be the probability for $T$ infinite that we never reach the position one step larger in $x$. Then $\gamma = (1 - 2p)/(1 - p)$, which can be seen from a standard analysis of a gambler's ruin probability for an infinitely rich adversary - see e.g. Ross (2007). Thus the limiting value of $V(-1, 1)$ for large $T$ is $0.5(1 - \gamma)$ which can be simplified to $p/[2(1 - p)]$ . Hence we have established that the never gamble solution is an equilibrium when

$$0.5 > p + (1 - p) \frac{p}{2(1 - p)}$$

i.e. for $p < 1/3$.

Figures 3 and 4 show the behavior of $g$ (i.e. the distribution of final positions) for different values of $p$ near 0.5 and $T = 80$. Because over part of the range (at the left hand end) the $g$ values alternate with every other one being zero it makes better sense to plot the two point moving average: $(g(x) + g(x - 1))/2$. With this size of problem we begin to reach some computational difficulties. The solutions below were obtained using the method outlined above, using GAMS and the CONOPT solver, with a maximum number of 35 solutions in the
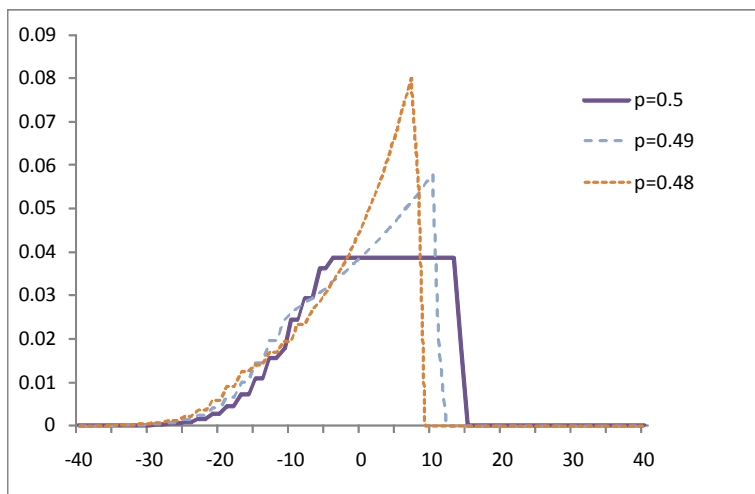
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Figure 4: Distribution of final position with downward drift

set $I_k$.

Notice that for $p = 0.5$ there is a region where the values of $g(x)$ alternate with $g(x) = g(x+2)$, which is the condition for mixing. On this segment of the curve the two point moving average plotted has a constant value. For values of $p \neq 0.5$ there is a segment of the curve where mixing takes place and (from (8)) the two point moving average is multiplied by a factor $(1 - p)/p$ at each increase in $x$ by 1. this is apparent in all the curves drawn except for the case $p = 0.58$ where the mixing region (starting at $x = 23$) is disguised by the shape of the bell curve.

## Varying the number of competitors

In order to investigate the effect of varying the number of competitors we carry out some experiments with $T = 80$. Figure 5 shows what happens for different values of $N$ and $p = 0.5$. Again we plot the two point moving average, but even so we observe some oscillation in values around $x = 0$ for $N = 3$ and $N = 4$. Each of the curves exhibits a sharp cutoff on the right-hand side - corresponding to a value of $x$ that is sufficiently large that gambling is no longer attractive, but the point at which we stop gambling is pushed higher as the number of players increases. This is intuitive - with a larger number of players the chance of winning is reduced and the winning player will be likely to have a larger value of $x$ at time $T$.
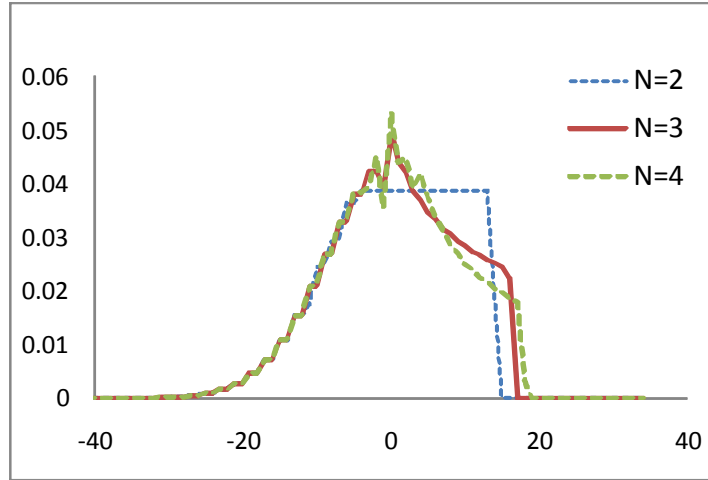
Figure 5: Distribution of final position for different numbers of players ($p = 0.5$ and $T = 80$)

Each of the curves has a region where

$$V(x,T) = pV(x+1,T) + (1-p)V(x-1,T)$$

so that players are indifferent whether to gamble or not at $T-1$. For $N = 2$ we have seen that this gives a flat section of the curve but for higher values of $N$ the condition is more complicated and implies a decreasing and convex segment in the $g$ function.

**Varying the reward structure**

Up to now our experiments have considered the case where the prize is divided amongst the winners. In this section we consider different prize structures. We have already observed that a convex prize structure has properties enabling a solution approach through a particulat convex optimization problem. Non-convex prize structures lead to problems which are computationally harder. Before discussing an example of a non-convex prize structure we give a result for the linear case.

**Lemma 4.** Suppose that the prize structure is linear with $\phi_i - \phi_{i+1} = k > 0$ for $i = 2, 3..., N$, then there is an equilibrium which exactly matches the equilibrium in the two player case.
**Proof.** We consider the perturbed game so that we do not need to consider two players ending at the same position. We may assume that in an equilibrium all other players use the

same strategy. The expected payoff from ending at $x$ in the $N$ player game is

$$\sum_{j=1}^{N} \phi_j \Pr(N - j \text{ other players below } x) = \phi_N + k \sum_{j=1}^{N} (N - j) \Pr(N - j \text{ other players below } x)$$
$$= \phi_N + kE(\text{number of other players below } x)$$
$$= \phi_N + k(N - 1)G(x)$$

where $G(x) = \Pr(\text{the player finishes at a position less than } x)$. Thus the payoff is simply an affine transformation of the payoff in a two player game (which is $\phi_2 + (\phi_1 - \phi_2)G(x)$ ) and this is enough to establish the result we require. ∎

Next we consider the possibility that the prize is divided amongst the top two or three players. In particular we investigate the case with $N = 3$, when the prize is shared between the top two players (a non-convex prize structure). We use the framework from section 4 with $\phi_1 = 0.5$ and $\phi_2 = 0.5$, and hence if one player is first and there is a tie between the second and third players, then the first placed player receives 0.5 and the other two players each receive 0.25. Then

$$V(x, T) = (1/2)G(x - 1)^2 + G(x - 1)(1 - G(x - 1)) + (1/2)g(x)(1 - G(x)) + (1/3)g(x)^2.$$

The first term relates to both the other players doing worse than $x$; the second term arises from just one of the other players doing worse than $x$; the third term comes from one of the other players also getting $x$ while the third player does better; and the final term comes from both the other players getting $x$.

Since this is not a convex prize structure the approach based on solving $P_N(I)$ may not work (and in fact does not work in practice). We have instead used a more direct approach of finding the solutions to a set of non linear equations where the probabilities of gambling $\pi(x, t)$ at each position lead to the final probabilities $g(x)$ and these are used as the basis for a dynamic programming recursion with the aim of achieving the appropriate complementarity conditions where $\pi(x, t) = 1$ if (3) holds and $\pi(x, t) = 0$ if (4) holds. We use CONOPT as a solver and the result for $T = 80$ is shown in Figure 6. The winner-takes-all prize structure for $N = 3$ is also shown as a comparison.

Spreading the prize in this way is equivalent to switching from a reward for coming first
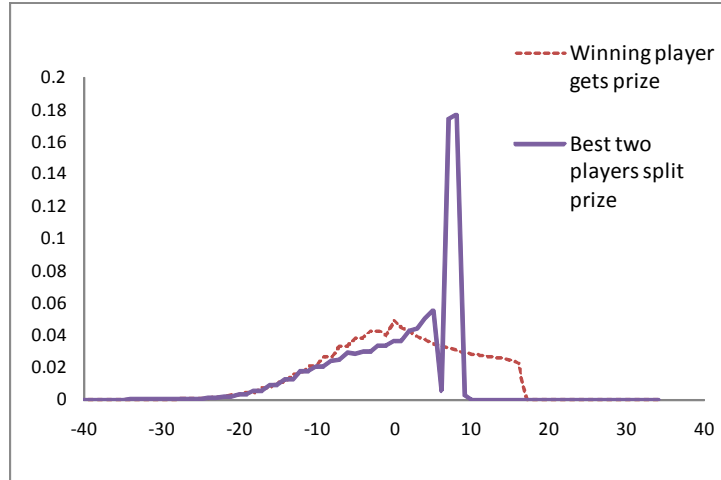
Figure 6: The effect of different prize structures for $N = 3$ and $p = 0.5$

to a punishment for coming last. This has a dramatic effect on the distribution of returns, giving a more strongly skewed shape

The case with a very large number of players is also of interest. Suppose that the reward a player receives is related to its percentile position. Thus there is some function $\theta(y)$ such that the reward given to player $i$ is $\theta$(proportion of players doing worse than $i$). We require $\theta$ increasing on $[0, 1]$ . Thus the expected reward with $x$ as $N$ goes to infinity is just $\theta(G(x))$. It is important to note that in this large $N$ case we no longer need to assume that players have no knowledge of other players' actions or results as decisions are taken. In equilibrium with a large number of other players we can expect the range and distribution of results simply to match the equilibrium distribution. As before with a linear prize structure where $\theta(x) = x$, so that the amount received is proportional to the percentile rank, then the reward is $G(x)$ which matches the two player case. Thus the equilibrium solution with a large number of players for this reward function is just the two player equilibrium discussed earlier.

## 6    Knowing the other player's position

Up to now we have assumed that there is no exchange of information until the end of the game when the winning player is revealed at time $T$. In many circumstances it makes more sense to assume that the positions of the other players are known at all times. It is easy to see that the decision on whether or not to gamble will depend only on the relative positions of the other players and the time till $T$. If there are just two players then the game can be

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analyzed easily since the state at time $t$ is then captured with a single number, while more generally the state with $N$ players requires an $N - 1$ vector to represent it. In this section we will concentrate on analyzing the two player case, indicated by $i$ and $j$. We suppose that $p$ is the probability of moving to a higher position and $(1 - p)$ is the probability of moving to a lower position.

We can use similar notation to before. The state variable $(x, t)$ represents the state where player $i$ has a value $x$ greater than the value of player $j$ at time $t$. For a policy $\pi$, $\pi(x, t)$ is the probability of gambling at state $(x, t)$. Hence $\pi(0, t)$ is the probability of player $i$ gambling if both players have the same value at time $t$, $\pi(1, t)$ is the probability of player $i$ gambling if player $i$ is one ahead player $j$, etc. A pure strategy for $i$ will have either $\pi(x, t) = 1$ or $\pi(x, t) = 0$. The value of the game is the expected payoff, and for finite $T$, this is a two player finite sum game and so has a Nash equilibrium. If the equilibrium involves mixed strategies, then this can be obtained with $0 < \pi(x, t) < 1$.

To find an optimal strategy for one player against a fixed strategy of the other we write $V_i(x, t)$ for the optimal expected payoff for player $i$ starting at the state $(x, t)$. Then the final conditions are $V_i(x, T) = 1$ if $x > 0$, $V_i(0, T) = 0.5$ and $V_i(x, T) = 0$ if $x < 0$.

If player $i$ is in the state $(x, t)$ then player $j$ is in the state $(-x, t)$. Writing $\pi_j$ for the strategy used by player $j$ we let $\rho = \pi_j(-x, t)$ be the probability that player $j$ gambles when player $i$ is in state $(x, t)$. Then we can write the following dynamic programming recursion for $V_i$:

$$V_i(x, t) \ \ = \ \ \max\{\rho(pV_i(x - 1, t + 1) + (1 - p)V_i(x + 1, t + 1)) + (1 - \rho)V_i(x, t + 1), \tag{10}$$
$$\rho((p^2 + (1 - p)^2)V_i(x, t + 1) + p(1 - p)V_i(x + 2, t + 1) + p(1 - p)V_i(x - 2, t + 1))$$
$$+ (1 - \rho)(pV_i(x + 1, t + 1) + (1 - p)V_i(x - 1, t + 1))\}$$

The first term tracks what happens if player $i$ does not gamble and the second term tracks the case where player $i$ does gamble. The optimal policy for player $i$ is determined by the larger term (with mixing only possible if they are equal).

Note that the expected payoffs to the two players sum to 1. In a symmetric equilibrium we can write $V = V_i = V_j$ and this translates into the requirement that

$$V(x, t) + V(-x, t) = 1. \tag{11}$$

which implies that $V(0, t) = 0.5$.

**Theorem 2.** When $0 < p < 0.5$ there is a symmetric equilibrium in which each player gambles if they are behind the other player, and does not gamble if they are either ahead or at the same position as the other player. In the case that $p = 0.5$ there is an equilibrium in which each player gambles if they are behind the other player, does not gamble if they are ahead and chooses an arbitrary probability of gambling if the two players are at the same position.

**Proof.** To prove this we only need to check the optimality conditions given by (10) i.e. that if player $j$ follows the policy stated then player $i$ will do so as well. At the same time as doing this we will establish inductively that for any equilibrium: (a) $V(x, t) \geq pV(x + 1, t) + (1 - p)V(x - 1, t)$ for $x > 0$ ; (b) $V(x, t) \leq pV(x + 1, t) + (1 - p)V(x - 1, t)$ for $x < 0$, and (c) $V(x, t)$ is non-decreasing in $x$. Notice that all these statements are true at $t = T$.

First consider the case $x = 0$. From (10) and letting $\rho = \pi_j(0, t)$ we have

$$
\begin{aligned}
V(x, t) \quad = \quad & \max\{\rho(pV(-1, t+1) + (1 - p)V(1, t+1)) + (1 - \rho)0.5, \\
& \rho(0.5p^2 + 0.5(1 - p)^2 + p(1 - p)V(2, t+1) + p(1 - p)V(-2, t+1)) \\
& + (1 - \rho)(pV(1, t+1) + (1 - p)V(-1, t+1))\}.
\end{aligned}
$$

We can use (11) to show that the first term is

$$
\rho(p + (1 - 2p)V(1, t+1)) + (1 - \rho)0.5,
$$

and the second is

$$
\rho(0.5p^2 + 0.5(1 - p)^2 + p(1 - p)) + (1 - \rho)((1 - p) - (1 - 2p)V(1, t+1))
$$

Hence the first term minus the second is $(V(1, t+1) - 0.5)(1 - 2p) \geq 0$ with equality when $p = 0.5$. Since this holds independent of $\rho$ we have established that when $p < 0.5$ we should never gamble but when $p = 0.5$ it does not matter what value of $\rho$ is chosen.

Now consider $x > 0$. We have $\rho = \pi_j(-x, t) = 1$ so

$$
\begin{aligned}
V(x, t) \quad = \quad & \max\{pV(x - 1, t+1) + (1 - p)V(x + 1, t+1)), \\
& (p^2 + (1 - p)^2)V(x, t+1) + p(1 - p)V(x + 2, t+1) + p(1 - p)V(x - 2, t+1)\}
\end{aligned}
$$

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From inductive assumption (a) we have

$$V(x-1, t+1) \geq pV(x, t+1) + (1-p)V(x-2, t+1),$$
$$V(x+1, t+1) \geq pV(x+2, t+1) + (1-p)V(x, t+1).$$

Note that in the case where $x = 1$ we need to deal with the first inequality differently. Using (11) and $V(0, t+1) = 0.5$, the inequality can be written

$$(1-2p)V(1, t+1) \geq 0.5 - p, \qquad (12)$$

which is satisfied because $V(1, t+1) \geq 0.5$ from inductive assumption (c) and $p \leq 0.5$. Hence the maximum in (??) occurs in the first term and it is optimal not to gamble. Moreover

$$
\begin{aligned}
&V(x, t) - pV(x+1, t) - (1-p)V(x-1, t) \\
&= p(V(x-1, t+1) - pV(x, t+1) - (1-p)V(x-2, t+1)) \\
&\quad + (1-p)(V(x+1, t+1) - pV(x+2, t+1) - (1-p)V(x, t+1)) \\
&\geq 0.
\end{aligned}
$$

Note that for $x = 1$ we have to use (12) in order to establish this inequality. In any case this establishes part (a) of the inductive assumptions. Finally observe that in this region

$$
\begin{aligned}
&V(x+1, t) - V(x, t) \\
&= p(V(x, t+1) - V(x-1, t+1)) + (1-p)(V(x+2, t+1) - V(x+1, t+1)) \geq 0
\end{aligned}
$$

which establishes part (c).

Next we consider $x < 0$. We have $\rho = \pi_j(-x, t) = 0$ so

$$V_i(x, t) = \max\{V_i(x, t+1)), pV_i(x+1, t+1) + (1-p)V_i(x-1, t+1)\}.$$

Now by our inductive assumption (b) the maximization occurs in the second term and it is

optimal to gamble. Moreover for $x < -1$

$$
\begin{aligned}
& V(x,t) - pV(x+1,t) - (1-p)V(x-1,t) \\
= \quad & p(V(x+1,t+1) - pV(x+2,t+1) - (1-p)V(x,t+1)) \\
& + (1-p)(V(x-1,t+1) - pV(x,t+1) - (1-p)V(x-2,t+1)) \\
\leq \quad & 0.
\end{aligned}
$$

For $x = -1$ we need a different expression for $V(0,t)$ and

$$
\begin{aligned}
& V(-1,t) - pV(0,t) - (1-p)V(-2,t) \\
= \quad & 0.5p + (1-p)(V(-2,t+1) - 0.5p - pV(-1,t+1) - (1-p)V(-3,t+1)) \\
\leq \quad & 0.
\end{aligned}
$$

by the inductive assumption (b). Thus we have established the inductive assumption (b). Finally observe that for $x < 0$

$$
\begin{aligned}
& V(x,t) - V(x-1,t) \\
= \quad & p(V(x+1,t+1) - V(x,t+1)) + (1-p)(V(x-1,t+1) - V(x-2,t+1)) \geq 0
\end{aligned}
$$

which establishes part (c).

It only remains to show that $V(-1,t) \leq V(0,t) = 0.5 \leq V(1,t)$ which has not been covered by our discussion above. Now

$$
V(1,t) = 0.5p + (1-p)V(2,t+1) \geq 0.5p + (1-p)V(1,t+1)
$$

and hence $V(1,t) \geq 0.5$. The other inequality $V(-1,t) \leq 0.5$ follows from (11). ■

Notice that there are other equilibrium solutions. Since the maximum change in $x$ at a single step is 2, whenever $|x| > 2T - 2t$ it no longer matters what policy is chosen. However our interest is in the distribution of final values, and this will be affected by the choice of whether or not to gamble when there is no longer any chance of reversing the rank ordering. The policy described in Theorem 2 is a natural one to consider and we show the distribution
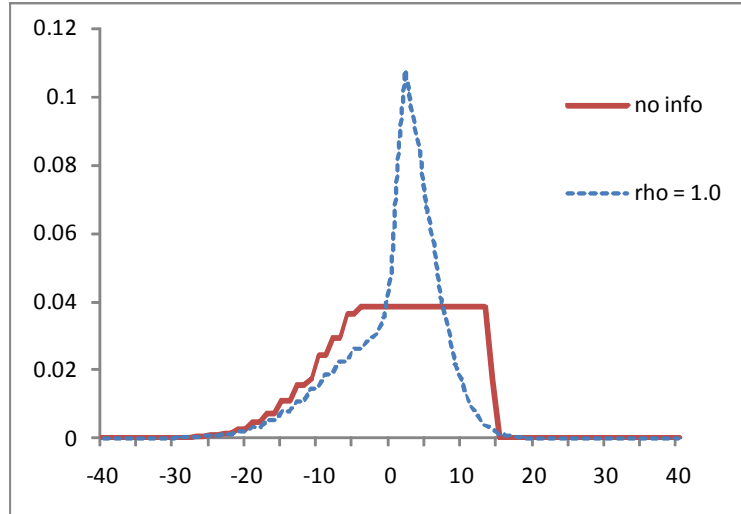
Figure 7: Distribution of final values with full information, $p = 0.5$

of final values under this policy when $p = 0.5$ in Figure 7. $\rho$ gives the probability of gambling when $x = 0$, which we take to be the same for both players. We have shown the solution when $\rho$ is 1, but different values for $\rho$ produce little difference in the final distribution The figure also shows the case where players have no information on the other player, as a comparison.

When $p > 0.5$ it is not possible to write down a simple equilibrium policy. Notice that as $p \rightarrow 1$ it will become best to gamble at almost all states. Numerical calculation of an equilibrium policy is simpler than for the problem without information on the other player's position. We can work out an equilibrium policy with one step to go, then with two steps to go and so on. As an example Figure 8 shows the structure of the equilibrium policy for $p = 0.54$ and $T = 5$. The nodes which are shaded are those at which the players gamble in equilibrium and those shown as empty are those at which the players do not gamble. Notice that the equilibrium has a complex structure and at $t = 3$ the players gamble when $x = 2$ but do not gamble when $x = 1$, which is somewhat counterintuitive.

# 7    Conclusions

We have explored the patterns of behavior that occur when individuals can control the riskiness of their actions and are rewarded only when they outperform their competitors. In our model individuals can choose to gamble or not at each time period. The typical equilibrium behaviour
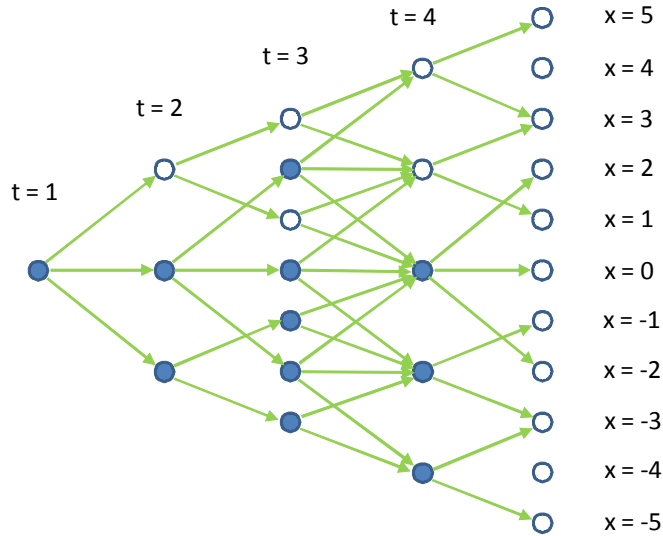
Figure 8: Possible states in equilibrium for $T = 5$ and $p = 0.54$

involves a time varying boundary: if at any time $t$ the current position is lower than the boundary it is best to continue gambling. However, equilibria in this game usually contain a range of values at which there is mixing with a positive probability both of gambling and not gambling. Thus, once a player hits the boundary, a possibility of not gambling is introduced, but the player may not quit gambling entirely. The shape of the boundary involves lower values of $x$ as the time horizon approaches, and for a fair gamble ($p = 0.5$) and a small number of players this will involve negative values of $x$ for $t$ close to $T$. Thus we might summarise by saying that we stop gambling (or reduce its probability) when we are sufficiently far ahead, but as the end approaches we may no longer need to be ahead of the starting position. In that sense we might choose to quit even though we are "behind".

As we would expect the equilibrium produces results that involve a negative skew, with small probabilities of bad outcomes being balanced by larger probabilities of better than average outcomes. In these models higher negative skew occurs at equilibria in which the players are more likely to quit early, rather than carrying on gambling. Our computations demonstrate that these higher skew distributions occur with fewer players rather than many players; with negative drift rather than positive drift; with punishment for poor relative performance rather than rewards for good relative performance; and with knowledge of the other player's ongoing performance rather than ignorance.

This work has involved two fundamental decisions on the type of model to employ. First

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we have assumed that the game is played over a finite time horizon, and second we have used a discrete time framework. Both of these choices present the opportunity for further work through exploring either an infinite time horizon or a continuous time model. In a fund management context it may be argued that either a quarterly or annual ranking exercise is important, making the finite time horizon a natural modelling choice. However there are many circumstances when competitors receive some benefit from their current relative position, which would lead naturally to a infinite time horizon model with discounting where rewards at each time period are given to the player who is ahead at that time.

In the case of the continuous time diffusion limit and a model without information on the performance of other players, we conjecture that there will be a monotonic switching curve at which gambling stops. We expect that a continuous time framework will eliminate the need to consider mixed strategies, since the desired final distribution of values $g(\cdot)$ can be obtained simply by using the right switching curve. This is closely related to the type of switching curve that arises in the exercise of an American option, but here occurs in an equilibrium context. Some related work is by Browne (6) who considers a stochastic differential game in which two players compete with a dynamic portfolio trading strategy. This varies from our model, not only through using a continuous time framework, but also through not having a finite time horizon. The game stops, and a winner is determined, when one player outperforms the other by a given percentage.

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