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Abstract

Supply function equilibria are used in the analysis of divisible good auctions with a large number of identical objects to be sold or bought. An important example occurs in wholesale electricity markets. Despite the substantial literature on supply function equilibria the existence of a pure strategy Nash equilibria for a uniform price auction in asymmetric cases has not been established in a general setting. In this paper we prove the existence of a supply function equilibrium for a duopoly with asymmetric firms having convex costs, with decreasing concave demand subject to an additive demand shock, provided the second derivative of the demand function is small enough. The proof is constructive and also gives insight into the structure of the equilibrium solutions.

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Supply Function Equilibria Always Exist

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Abstract Supply function equilibria are used in the analysis of divisible good auctions with a large number of identical objects to be sold or bought. An important example occurs in wholesale electricity markets. Despite the substantial literature on supply function equilibria the existence of a pure strategy Nash equilibria for a uniform price auction in asymmetric cases has not been established in a general setting. In this paper we prove the existence of a supply function equilibrium for a duopoly with asymmetric firms having convex costs, with decreasing concave demand subject to an additive demand shock, provided the second derivative of the demand function is small enough. The proof is constructive and also gives insight into the structure of the equilibrium solutions.

Keywords: Wholesale electricity markets; divisible good auctions; supply functions; existence of equilibria.
MSC Codes: 91A80, 91B26

1 Introduction

We consider a market in which firms offer to supply a homogeneous good at prices that depend on the quantity required. The function linking price and quantity is called a 'supply function'. Such markets can be difficult to analyze: not only are supply function equilibrium hard to find, but there may also be many different possible equilibria. Klemperer and Meyer[14] were the first to explore supply function equilibria in a general context. The supply function model can apply in a purchase context to the auctions of US treasury securities [17], but the most important examples of supply function bidding occur in electricity spot markets.

The usual pattern for wholesale electricity markets is a uniform price auction, in which the generation firms submit supply function bids every hour

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or even every half hour and each generation firm is paid the market clearing price for all the electricity it supplies. Since electricity cannot be effectively stored it is necessary to vary the amount dispatched on a short time interval: typically the market clears and a new price is established every 5 minutes. Electricity markets are also characterized by transmission networks with both the generation capacity and the demands distributed over different nodes of the network. In this paper we will ignore transmission issues and assume that all generation firms supply power at the same node.

Each electricity market will have its own rules of operation. Many markets only allow supply functions that are step functions with power offered at a limited number of different prices, while some operate with piecewise linear offer curves. Markets all have some minimal 'tick' size both for price increments and for supply amounts, and there will also be a price cap on the maximum price at which the market can clear. In the short-run electricity demand is very inelastic. Nevertheless a model that incorporates demand elasticity is useful: in practice this may arise not only from demand responsiveness, but also from imports or from non-strategic generators.

A number of authors have studied supply function equilibria as a model for generators' bidding behavior in spot markets for electricity. This work was initiated by Green and Newbery[8] and Bolle[5] who showed that the concept of supply function equilibria introduced by Klemperer and Meyer could be applied to electricity markets. In special cases supply function equilibria can be calculated analytically; for example if there are constant marginal costs [15][9] or linear marginal costs and linear demand [7][4]. It is important to understand equilibrium behavior in an oligopoly with a small number of players since many electricity markets involve only a handful of large generation firms. Supply function equilibria play an important role in understanding the market power possessed by dominant market participants. Recent investigation of the behavior of the electricity market in Texas (ERCOT) has shown that supply function models give a reasonable approximation of the behavior of the large firms in that market [13][16]. For a comprehensive review of the extensive literature on supply function equilibria in an electricity market context the reader is referred to Holmberg and Newbery[12].

A characteristic of many supply function equilibrium models is that there are a range of possible equilibria. Usually the wider the range of possible demand outcomes the more restricted the set of possible equilibria. In their original work Klemperer and Meyer[14] showed that, when demand is unbounded, then a single (linear) equilibrium solution can be identified. In an electricity market context it is important to correctly model capacity constraints: the interaction of extreme demand scenarios with the capacity constraints will often enable a unique equilibrium to be picked out [1][10].

This paper is closely connected to Anderson and Hu[1] (hereafter AH) who investigated the structure of supply function equilibria in general asymmetric cases. Alongside an improved understanding of the behavior of supply functions in equilibria with capacity constraints, there has also been progress on computational approaches for finding SFE [1][11]. However, though supply function equilibrium can be calculated for many examples, there are also many cases where no supply function equilibria of the form considered by AH can be found.

All previous research on SFE has concentrated on what AH call strong supply function equilibria. These have the property that each player's optimal solution, given the other player's supply function, is independent of the distribution of demand shocks (Holmberg and Newbery[12] use the terminology *ex-post optimality* for this property). AH go so far as to claim that in normal circumstances any (weaker) form of SFE in which the offer curves can vary with changes in demand distribution are unlikely to exist. We show that this conjecture is false.

We give an example of a duopoly in which both firms have fixed marginal costs, and with stochastic elastic demand having a range large enough to ensure that for low demand scenarios only the cheaper firm is used, but for high demand scenarios one firm reaches its capacity limit. In this situation the pair of ordinary differential equations describing the equilibrium behavior can be solved explicitly. This is helpful in showing that, given the right choice of parameters, there may be no strong supply function equilibrium. Moreover for this particular example we show how an equilibrium involving a vertical segment can be constructed: the equilibrium will then depend on the distribution of demand. An understanding of this form of SFE turns out to be critical in resolving the existence question. Our discussion will show that in quite general circumstances there will always exist an SFE in pure strategies for an asymmetric duopoly with capacities. We will also give conditions under which an SFE in pure strategies is unique.

In the next section we introduce the supply function model in detail and characterize a supply function equilibrium. In section 3, as motivation for our discussion, we give a particular duopoly example for which there is no strong supply function equilibrium, but there is nevertheless a supply function equilibrium in pure strategies which depends on the distribution of demand shocks. Then in section 4 we prove our main result giving conditions under which such an equilibrium always exists. In section 5 we show, under slightly stronger conditions, that the equilibrium is unique.

2 Supply function equilibria in a duopoly

We consider a duopoly in which both firms offer a supply function into the market. These non-decreasing supply functions indicate the amount that a firm is prepared to supply at any given price. The market operates as a uniform price auction, so that there is a single clearing price that applies to all firms. At the time when supply function offers are made demand is uncertain. After the two firms announce their supply functions, $s_1(p)$ and $s_2(p)$, demand occurs and the spot market price p and supply amount for each firm are determined from the intersection of the aggregate market supply function and the realized demand function. The demand $D(p, \varepsilon)$ is a function of the price p

and a random variable ε. So when demand occurs a particular value ε_0 of the demand uncertainty ε becomes known and the market clears at a price p such that $D(p, \varepsilon_0) = s_1(p) + s_2(p)$. If the market clears at price $\overline{p}$ then firm i supplies an amount $s_i(\overline{p})$ and is paid a total of $\overline{p}s_i(\overline{p})$ for this amount.

Each firm i has a cost function $C_i(q)$, for $q \geq 0$ giving the cost of supplying a quantity q, and each firm also has a maximum supply capacity $\bar{q}_i$, $i = 1, 2$. We will assume that firms have complete information in relation to the demand function, as well as the costs and capacity of the other firm.

We make the following set of assumptions in respect to the problem data and the supply functions offered:

Assumption 1

(a) $C_i(\cdot)$ is strictly increasing convex on $[0, \bar{q}_i]$ and twice continuously differentiable.

(b) $D(p, \varepsilon) = D(p) + \varepsilon$, where $D(\cdot)$ is strictly decreasing, concave, and smooth (i.e. continuously differentiable). The probability density of the demand shock has support $[\underline{\varepsilon}, \bar{\varepsilon}]$ and is well defined in that interval (i.e. no atoms in the shock distribution). There is some price p with $D(p) + \bar{\varepsilon} < \bar{q}_1 + \bar{q}_2$ (so there is always a potential clearing price).

(c) $s_i \in S$ where S is the set of functions $s : [0, \infty) \to [0, \bar{q}_i]$ which are non-decreasing, left-continuous, piecewise-smooth with a finite number of pieces and for which there is a uniform bound U on s' over each piece.

We will take the quantity q as the horizontal axis (so a vertical segment is an interval (p_1, p_2) on which a supply function $s(p)$ is constant and a horizontal segment corresponds to a discontinuity in the supply function). At points where the supply functions are discontinuous our assumption on left-continuity implies that, when there is a jump at price p_0, then we take the supply function value at p_0 as the lowest value possible. We write $s(p^+)$ for $\lim_{\delta \searrow 0} s(p + \delta)$ which exists from our assumption.

When there is a supply function that has a discontinuity with a jump at price p it indicates that the firm is willing to supply a range of quantities at that price. One consequence is that we need to revisit our definition of the clearing price and the way that the demand is allocated to different firms. We let the clearing price be the lowest price at which demand is met, so that when the demand shock takes the value ε_0 then the market clears at a price

$$p(\varepsilon_0) = \inf\{p : D(p) + \varepsilon_0 \leq s_1(p) + s_2(p)\}. \quad (1)$$

We define $p_{\min} = \inf\{p : D(p) + \underline{\varepsilon} \leq s_1(p) + s_2(p)\}$ and $p_{\max} = \inf\{p : D(p) + \bar{\varepsilon} \leq s_1(p) + s_2(p)\}$ as the lowest and highest clearing prices that may occur.

The existence of discontinuities means that firm i may be dispatched at any quantity between $s_i(p)$ and $s_i(p^+)$ at clearing price p. We need to decide what happens when both firms have supply functions that jump at the same price p. We shall assume that a sharing rule exists at a price p, so that excess demand is distributed amongst the firms in proportion to the size of the jump in quantity for each firm. In this case when the demand shock is ε_0, then firm i is dispatched an amount

$$q_i(s_1, s_2, \varepsilon_0) = s_i(p) + \frac{(D(p) + \varepsilon_0 - s_1(p) - s_2(p))\,(s_i(p^+) - s_i(p))}{s_1(p^+) + s_2(p^+) - s_1(p) - s_2(p)}. \tag{2}$$

where p is the clearing price determined by (1): this clearing price is a function of the supply functions offered as well as the demand shock and we can write it as $p(s_1, s_2, \varepsilon_0)$. Notice that the amount dispatched lies between $s_i(p(s_1, s_2, \varepsilon_0))$ and $s_i(p(s_1, s_2, \varepsilon_0)^+)$ and this definition collapses back to the simple form $q_i = s_i(p)$ when supply functions are continuous.

The profit for firm i from using a supply function s_i given the other supply function s_j and a particular demand shock ε_0, is

$$\pi_i(s_1, s_2, \varepsilon_0) = p(s_1, s_2, \varepsilon_0)q_i(s_1, s_2, \varepsilon_0) - C_i(q_i(s_1, s_2, \varepsilon_0)).$$

We look for a Nash equilibrium in supply functions. Using an expected profit framework, this is a pair of supply functions σ_1, σ_2 with the property that

$$E(\pi_1(\sigma_1, \sigma_2, \varepsilon)) = \max_{s_i \in S}\ E(\pi_1(s_i, \sigma_2, \varepsilon)), \tag{3}$$

$$E(\pi_2(\sigma_1, \sigma_2, \varepsilon)) = \max_{s_i \in S}\ E(\pi_2(\sigma_1, s_i, \varepsilon)). \tag{4}$$

The expectations here are taken with respect to the demand shock ε.

A fundamental observation is that the choice of an optimal supply function can often be made in a way that is independent of the demand distribution. Following AH we call this a strongly optimal supply function. We consider an optimal choice for firm i given a smooth supply function offer by firm j. For a fixed demand shock ε_0, firm i faces residual demand $D(p) + \varepsilon_0 - s_j(p)$. The market clears at the price where this decreasing function intersects the offer curve $s_i(p)$. Firm i will choose the intersection point to maximize its profit π_i. If we regard this as a problem of optimizing the price at which intersection occurs then we choose p to maximize

$$p(D(p) + \varepsilon_0 - s_j(p)) - C_i(D(p) + \varepsilon_0 - s_j(p)),$$

giving first order conditions

$$(D(p) + \varepsilon_0 - s_j(p)) + (p - C_i'(D(p) + \varepsilon_0 - s_j(p)))(D'(p) - s_j'(p)) = 0. \tag{5}$$

Choosing a supply function that has

$$s_i(p) = (p - C_i'(s_i(p)))(s_j'(p) - D'(p)) \tag{6}$$

will guarantee that (5) is satisfied at the intersection with the residual demand. This supply function is independent of the demand shock ε_0. We require that it is an increasing function for it to be a potential supply function, and we also need to check the second order conditions for a maximum. But if these requirements are satisfied it will be optimal for any choice of demand shock, and hence for any distribution of demand shocks. We call this a strongly optimal supply function.

Later we will need to make use of the market distribution function introduced by Anderson and Philpott[2]. We define this as follows for firm i:

$$\psi_i(q,p) = \Pr(D(p) + \varepsilon - s_j(p) < q).$$

Anderson and Philpott also define a function Z related to the Euler conditions for the calculus of variations problem implicit in choosing a supply function to maximize profit. We let $R_i(q,p) = qp - C_i(q)$ be the profit obtained from a dispatch of q at price p: then

$$\begin{aligned} Z_i(q,p) &= \frac{\partial R_i}{\partial q}\frac{\partial \psi_i}{\partial p} - \frac{\partial R_i}{\partial p}\frac{\partial \psi_i}{\partial q} \\ &= f(q + s_j(p) - D(p))[(p - C_i'(q))(s_j'(p) - D'(p)) - q]. \end{aligned} \tag{7}$$

In practice the solution of (6) is often monotonic and the second order conditions will also usually hold. We can therefore consider the existence of a Nash equilibrium in which each firm has a strongly optimal supply function given the offer of the other firm (we call this a strong SFE). Now we give a result describing the form of the supply function equilibria that can occur. This is simply an application of the necessary optimality conditions first given by Anderson and Philpott.

Lemma 1 *Suppose the pair $s_1(p)$, $s_2(p)$ is a pure strategy Nash supply function equilibrium in a duopoly, Then:*

(a) If $p \in (p_{\min}, p_{\max})$ with $s_i(p) \in (0, \bar{q}_i)$, $s_i'(p) > 0$ and both $s_i'(p)$ and $s_j'(p)$ are well-defined (so p is not at the boundary of one of the pieces for either s_i or s_j) then the first order condition (6) for $s_i(p)$ is satisfied;

(b) If $p_1, p_2 \in (p_{\min}, p_{\max})$ and $s_i(p) = q_0 \in (0, \bar{q}_i)$ for $p \in (p_1, p_2)$ with p_1 the start of this interval (i.e. if $s_i'(p) > 0$ as p approaches p_1 from below, or if $s_i(p_1^+) > s_i(p_1)$) then

$$\int_{p_1}^{p_2} f(q_0 + s_j(p) - D(p)) \left[(p - C_i'(q_0))(s_j'(p) - D'(p)) - q_0\right] dp \geq 0$$

with equality if p_2 is chosen as the end of the interval (i.e. if $s_i'(p) > 0$ as p approaches p_2 from above, or if $s_i(p_2^+) > s_i(p_2)$).

(c) If $p_0 \in (p_{\min}, p_{\max})$ and $s_i(p_0^+) > s_i(p_0)$ then for $q_0 \in [s_i(p_0), s_i(p_0^+)]$

$$\int_{s_i(p_0)}^{q_0} f(q + s_j(p_0) - D(p_0)) \left[(p_0 - C_i'(q))(s_j'(p_0) - D'(p_0)) - q\right] dq \leq 0$$

with equality if $q_0 = s_i(p_0^+)$.

Proof:

Part (a) follows directly from the optimality conditions given in Anderson and Philpott[2], who show (Lemma 4.2) that under these conditions, when the derivatives involved are well-defined and the supply function is strictly between its bounds, then an Euler equation holds and $Z(s_i(p), p) = 0$. As $p \in (p_{\min}, p_{\max})$ we have $f(s_i(p) + s_j(p) - D(p)) > 0$ and the result follows from (7).

Parts (b) and (c) also derive from optimality conditions given in [2], or can be obtained by rewriting a result of Anderson and Xu[3] (Theorem 3.1) taking account of the specific form of the Z function (7) and noting that the capacity constraints do not play any role here. ■

Now we derive a result establishing the key properties of a supply function equilibria. This is very similar to a result of AH for strong supply function equilibrium. But since we do not assume strong optimality for the solutions our result is more general and requires a different proof.

Theorem 1 *Suppose that there is a duopoly with asymmetric firms having $C_1'(0) \neq C_2'(0)$. If the pair $s_1(p)$, $s_2(p)$ is a pure strategy Nash equilibrium in supply functions, then:*

(a) $s_i(p) = 0$ if and only if $p \leq C_i'(0)$;

(b) $C_i'(s_i(p)) < p$ for all $p > C_i'(0)$;

(c) If $s_i(p)$ is discontinuous at p_0 in the range $(p_{\min}, p_{\max})$ then $p_0 = C_j'(0)$) (i.e. p_0 is the other firm's marginal cost function at zero)

Proof:

(a) We suppose that there is some p with $s_i(p) > 0$ for $p_{\min} < p < C_i'(0)$. Thus there is a non-zero probability of firm i being dispatched at a price less than $C_i'(0)$, and we may suppose this happens for $\varepsilon \in (\underline{\varepsilon}, \varepsilon_x)$. We replace $s_i(p)$ with a new supply function $\widetilde{s}_i(p) = 0$ for $p \leq C_i'(0)$ and $\widetilde{s}_i(p) = s_i(p)$ otherwise. Consider a fixed $\varepsilon_0 \in (\underline{\varepsilon}, \varepsilon_x)$: firm i is dispatched a quantity $\widetilde{q}_i = q_i(\widetilde{s}_i, s_j, \varepsilon_0)$ which is less than $q_i = q_i(s_i, s_j, \varepsilon_0)$ at a price $\widetilde{p} = p(\widetilde{s}_i, s_j, \varepsilon_0)$ which is higher than $p_0 = p(s_i, s_j, \varepsilon_0)$. Let $\delta = q_i - \widetilde{q}_i > 0$, then from the convexity of C_i,

$$C_i(q_i) > C_i(\widetilde{q}_i) + \delta C_i'(\widetilde{q}_i) \geq C_i(\widetilde{q}_i) + \delta C_i'(0).$$

So

$$\begin{aligned}\pi_i(\widetilde{s}_i, s_j, \varepsilon_0) &= \widetilde{q}_i\widetilde{p} - C_i(\widetilde{q}_i) > q_i p_0 - \delta p_0 - C_i(q_i) + \delta C_i'(0) \\ &> \pi_i(s_i, s_j, \varepsilon_0).\end{aligned}$$

Since this inequality holds for all $\varepsilon_0 \in (\underline{\varepsilon}, \varepsilon_x)$ this contradicts the optimality of s_i. Thus $s_i(p) = 0$ for $p_{\min} < p < C_i'(0)$ and since s_i is left continuous we also have $s_i(C_i'(0)) = 0$.

To establish the 'only if' part we suppose that $s_i(p_0) = 0$ with $p_0 > C_i'(0)$. We may choose $p_0 < p_1 := \sup(p : s_i(p) = 0)$. Consider p approaching p_1 from above. Then we can apply Lemma 1 part (a) to show

$$\begin{aligned} s_i(p) &= (p - C_i'(s_i(p)))(s_j'(p) - D'(p)) \\ &\geq (p - C_i'(s_i(p)))(-D'(0)) \end{aligned}$$

From this we deduce that $s_i(p)$ does not approach zero as p approaches p_1 (since if this were the case then the right hand side has a limit $(p_1 - C_i'(0))(-D'(0)) > 0$ giving a contradiction). We take $\widetilde{s}_i(p) = \delta$ for $p \in (p_0, p_1]$ and $\widetilde{s}_i(p) = s_i(p)$ otherwise. This will be monotonic for δ small enough. Then there is a fixed probability of ε falling into the region E_δ where the dispatch is increased from 0 to δ together with an $O(\delta^2)$ probability of the dispatch being increased, but by less than δ (which can happen when $p(s_i, s_j, \varepsilon) = p_1$). For ε in E_δ the new price remains above p_0, and we have

$$\begin{aligned} \pi_i(\widetilde{s}_i, s_j, \varepsilon_0) &= \delta\widetilde{p} - C_i(\delta) \\ &\geq \delta p_0 - C_i(0) - \delta C_i'(0) - O(\delta^2) \\ &= \pi_i(s_i, s_j, \varepsilon_0) + \delta(p_0 - C_i'(0)) - O(\delta^2) \end{aligned}$$

It is clear that this gives an improvement in the expected profit of order δ. The other changes have only an order δ^2 impact on expected profit and this contradicts the optimality of s_i for δ chosen small enough. Hence $s_i(p_0) > 0$ for $p_0 > C_i'(0)$.

(b) We begin by considering $p_z \in (p_{\min}, p_{\max})$ with $s_i(p_z) \in (0, \bar{q}_i)$ and $s_i'(p_z) > 0$ we note that for small enough $\delta > 0$ any choice of $p_w \in (p_z, p_z + \delta)$ has the property that both s_i and s_j are continuous at p_w and $s_i(p_w) \in (0, \bar{q}_i)$ and $s_i'(p_w) > 0$. This follows because s_i and s_j are discontinuous at only a finite number of points. Thus from Lemma 1 (a) we know that

$$s_i(p_w) = (p_w - C_i'(s_i(p_w)))(s_j'(p_w) - D'(p_w)).$$

Hence using the bound on s_j', the fact that s_i is non-decreasing, and the fact that $-D'$ is non-decreasing, we have

$$p_w - C_i'(s_i(p_w)) > \frac{s_i(p_z)}{(U - D'(p_{\max}))} > 0, \tag{8}$$

Since $p_w - C_i'(s_i(p_w))$ is bounded away from zero, we can deduce that the limit $p_z - C_i'(s_i(p_z)) > 0$ as required.

Now consider a p_z such that $s_i(p_z) = \bar{q}_i$. Then take $p_x = \inf(p : s_i(p) = \bar{q}_i)$ and apply the above argument to p approaching p_x from below. So $p_x - C_i'(s_i(p_x)) = p_x - C_i'(\bar{q}_i) > 0$. Thus $p_z - C_i'(\bar{q}_i) > 0$. In the rather pathological case that $s_i(p_x) < \bar{q}_i$, so that the supply function has a corner between a horizontal and a vertical segment, then we can apply Lemma 1 (c) which implies that the Z value at the end of a horizontal segment cannot be negative. This condition translates to $(p_x - C_i'(s_i(p_x^+)))(s_j'(p_x) - D'(p_x)) \geq s_i(p_x^+)$ in this case. Hence we have exactly the same inequality $p_x - C_i'(\bar{q}_i) > 0$.

Finally a similar argument applies when $s_i'(p_z) = 0$. In this case set $\widehat{q}_i = s_i(p_z)$ and define $p_y = \inf(p : s_i(p) = \widehat{q}_i)$. Letting p approach p_y from below gives $p_y - C_i'(s_i(p_y)) = p_y - C_i'(\widehat{q}_i) > 0$. Thus $p_z - C_i'(\widehat{q}_i) > 0$. Thus we have established the inequality for all p_z with $s_i(p_z) > 0$. i.e. (using part (a)) for all $p_z > C_i'(0)$.

(c) We suppose that $s_i(p_0)$ has a discontinuity at $p_0 > p_{\min}$, so $s_i(p_0) < s_i(p_0^+)$. Our approach is to show that in this case it is best for the other firm to offer a quantity slightly undercutting the price p_0. Since the amount by which the second firm undercuts the price of the first is arbitrary we see that no equilibrium is possible.

We need to first consider the case where firm j also has a jump at p_0, so $s_j(p_0) < s_j(p_0^+)$. Reversing the roles of i and j if necessary we may assume $s_i(p_0^+) - s_i(p_0) \geq s_j(p_0^+) - s_j(p_0))$. In this case consider a new solution $\widetilde{s}_j(p) = s_j(p_0^+)$ for $p \in (p_0 - \delta, p_0]$ and $\widetilde{s}_j(p) = s_j(p)$ otherwise. Let

$$E_0 = \{\varepsilon : s_i(p_0) + s_j(p_0) - D(p_0) \leq \varepsilon \leq s_i(p_0^+) + s_j(p_0^+) - D(p_0)\},$$

be the set of demand shocks such that the clearing price is p_0. We consider a subset $\widetilde{E}_0$ of E_0 to consist of the demand shocks in E_0 where

$$3(s_j(p_0^+) - s_j(p_0))/4 \leq D(p_0) - s_i(p_0) - s_j(p_0) + \varepsilon \leq (s_j(p_0^+) - s_j(p_0)).$$

From the definition of dispatch quantities (2), we can see that for ε in $\widetilde{E}_0$ firm j is dispatched a quantity at most

$$s_j(p_0) + \frac{(s_j(p_0^+) - s_j(p_0))^2}{(s_i(p_0^+) + s_j(p_0^+) - s_i(p_0) - s_j(p_0))} \leq s_j(p_0) + \frac{1}{2}(s_j(p_0^+) - s_j(p_0)).$$

Now consider what happens for $\varepsilon \in \widetilde{E}_0$ with the offer $\widetilde{s}_j$. If the price is still p_0 then the new dispatch is $s_j^+(p_0)$ and if the price is reduced to $p < p_0$ the dispatch quantity is

$$D(p) - s_i(p) + \varepsilon > D(p_0) - s_i(p_0) + \varepsilon > s_j(p_0) + \frac{3}{4}(s_j(p_0^+) - s_j(p_0)).$$

Thus for fixed ε in $\widetilde{E}_0$, there is an increase in dispatch of more than $\Delta = (s_j(p_0^+) - s_j(p_0))/4$ if $\widetilde{s}_j$ is used. Note from part (b) and (8) that $C_j'(s_j(p_0^+)) \leq p_0 - \delta_0$ where

$$\delta_0 = \frac{s_j(p_0)}{(U - D'(p_{\max}))}.$$

Observe from part (a) that since $s_j(p_0^+) > 0$ we must have $p_0 > C_j'(0)$, and hence again using part (a), $s_j(p_0) > 0$. We write q_j and $\widetilde{q}_j$ for the dispatch given ε under s_j and $\widetilde{s}_j$ respectively. Thus, using convexity, $C_j'(q) < p_0 - \delta_0$ for $q \in (q_j, \widetilde{q}_j)$. So $C_j(\widetilde{q}_j) < C_j(q_j) + (p_0 - \delta_0)(\widetilde{q}_j - q_j)$. Thus

$$\begin{aligned}
\pi_j(\widetilde{s}_j, s_i, \varepsilon) &= \widetilde{q}_j\widetilde{p} - C_j(\widetilde{q}_j) \\
&> \widetilde{q}_j\widetilde{p} - C_j(q_j) - (p_0 - \delta_0)(\widetilde{q}_j - q_j) \\
&= p_0 q_j - C_j(q_j) + \widetilde{q}_j(\widetilde{p} - p_0) + \delta_0(\widetilde{q}_j - q_j) \\
&> \pi_j(s_j, s_i, \varepsilon) + \widetilde{q}_j(\widetilde{p} - p_0) + \delta_0\Delta.
\end{aligned}$$

Since $\widetilde{p}$ approaches p_0 as δ approaches zero we see that using $\widetilde{s}_j$ increases profit for j by at least $\delta_0 \Delta/2$ for δ small enough. Since prices drop by at most δ and dispatch quantities can never drop, the reduction in profit that may occur with other demand shock values is of order δ. Since the improvement for ε in $\widetilde{E}_0$ is independent of δ, for δ chosen small enough we obtain an overall improvement in expected profit if $\widetilde{s}_j$ is used, which contradicts the optimality of s_j.

Now consider the other case when there is no jump in s_j at p_0 so $s_j(p_0) = s_j(p_0^+)$. Notice first that we exclude the case $p_0 = C_j'(0)$ since this is excluded in the theorem statement. We can also exclude the possibility that $p_0 < C_j'(0)$. For in this case $s_j(p) = 0$ in an interval around p_0. Since $s_i(p_0^+) > 0$ we can deduce from part (a) that $p_0 > C_i'(0)$. But then using part (a) $s_i(p) > 0$ for p approaching p_0 from below. Thus the equation (6) defines $s_i(p)$ both above and below p_0. But this implies continuity there and contradicts the fact that $s_i(p_0^+) > s_i(p_0)$

Hence we have established that if there is a jump in s_i at p_0, but s_j is continuous at this price, then $p_0 > C_j'(0)$ and so (from part (a)) $s_j(p_0) > 0$.

We set $\widetilde{s}_j(p) = s_j(p_0 + \delta_1)$ for $p \in (p_0 - \delta_2, p_0 + \delta_1)$ and $\widetilde{s}_j(p) = s_j(p)$ otherwise. We let E_0 be the set of demand shocks such that the clearing price is within δ_2 of p_0, i.e.

$$E_0 = [s_i(p_0-\delta_2)+s_j(p_0-\delta_2)-D(p_0-\delta_2), s_i(p_0+\delta_2)+s_j(p_0+\delta_2)-D(p_0+\delta_2)].$$

We consider a subset $\widetilde{E}_0$ of E_0, consisting of the demand shocks in E_0 such that

$$\varepsilon \geq s_i(p_0 - \delta_2) + s_j(p_0 - \delta_2) - D(p_0 - \delta_2) + (s_i(p_0 + \delta_2) - s_i(p_0 - \delta_2))/2.$$

Now consider what happens for $\varepsilon \in \widetilde{E}_0$ with the offer $\widetilde{s}_j$. If the price is greater than $p_0 - \delta_2$ then the new dispatch is $s_j(p_0 + \delta_1)$ and if the price is reduced to $p_0 - \delta_2$ the dispatch quantity is

$$D(p_0 - \delta_2) - s_i(p_0 - \delta_2) + \varepsilon > s_j(p_0 - \delta_2) + (s_i(p_0 + \delta_2) - s_i(p_0))/2.$$

Thus for ε in $\widetilde{E}_0$, there is an increase in dispatch if $\widetilde{s}$ is used of at least $\Delta(\delta_1, \delta_2)$ which we define as

$$\min(s_j(p_0+\delta_1)-s_j(p_0+\delta_2), s_j(p_0-\delta_2)-s_j(p_0+\delta_2)+(s_i(p_0+\delta_2)-s_i(p_0))/2).$$

From part (b) and (8) we have, for $p \in (p_0 - \delta_2, p_0 + \delta_1)$,

$$p - C_j'(s_j(p)) \geq \frac{s_j(p_0 - \delta_2)}{U - D'(p_{\max})}.$$

Notice that as δ_2 approaches zero the value of the right hand side increases rather then decreasing. For small enough δ_2 we have

$$p - C_j'(s_j(p)) \geq \delta_0 = \frac{s_j(p_0)}{2(U - D'(p_{\max}))}.$$

Hence $C_j'(\widetilde{q}_j) < p_0 + \delta_1 - \delta_0$ and so $C_j'(q) < p_0 + \delta_1 - \delta_0$ for $q \in (q_j, \widetilde{q}_j)$.

So

$$\pi_j(\widetilde{s}_j, s_i, \varepsilon) > \widetilde{q}_j\widetilde{p} - (p_0 + \delta_1 - \delta_0)(\widetilde{q}_j - q_j) - C_j(q_j)$$
$$> p_0 q_j - C_j(q_j) + \widetilde{q}_j(\widetilde{p} - p_0) + (\delta_0 - \delta_1)\Delta(\delta_1, \delta_2).$$

Now $\widetilde{p}$ approaches p_0 as $\delta_2 \to 0$ and we see that $\pi_j(\widetilde{s}_j, s_i, \varepsilon) > \pi_j(s_j, s_i, \varepsilon) + \delta_0\Delta(\delta_1, \delta_2)/2$ for δ_1 and δ_2 small enough. Because of the jump in s_i we know that $(s_i(p_0 + \delta_2) - s_i(p_0))/2$ is bounded below as δ_2 goes to zero. So $\Delta(\delta_1, \delta_2)$ will be $s_j(p_0+\delta_1) - s_j(p_0+\delta_2) > 0$ for δ_1 and δ_2 small enough. If ε is chosen so that $p(s_i, s_j, \varepsilon)$ is in $(p_0+\delta_2, p_0+\delta_1]$ then prices may drop by as much as $\delta_1+\delta_2$. However the probability of this happening is given by the probability that ε falls in an interval of length $s_j(p_0+\delta_1) + s_i(p_0+\delta_1) - s_j(p_0+\delta_2) - s_i(p_0+\delta_2)$. Overall if we take $\delta_2 = \delta_1^2$ and allow both to approach zero then the maximum decrease in expected profit from this component is of order δ_1^2, whereas the probability of ε in $\widetilde{E}_0$ is bounded below by a constant and so the increase in expected profit through this is of order δ_1. Hence we have an improvement in moving to $\widetilde{s}_j$ for δ_1 small enough, contradicting the optimality of s_j. ■

3 An example with no strong SFE

We consider a particular case of the duopoly problem when firm i has fixed marginal costs $C_i'(x) = c_i$, $i = 1, 2$ and we label the firms so that $c_1 < c_2$. We also suppose that there is a linear demand function $D(p) = A - bp$ so that total demand is given by $D = A - bp + \varepsilon$. Finally we will take the demand shock ε as uniformly distributed on $[0, X]$. As before we write $\bar{q}_i$ for the capacity of firm i. We consider a problem in which $\bar{q}_2$ is smaller than $\bar{q}_1$. In fact we let $\bar{q}_1$ be large enough that this capacity constraint is never reached. We will also assume a large enough value of X, so that the demand shock can be high enough for firm 2 to reach its capacity.

In this case we can develop an analytical solution for the equilibrium. We begin by assuming that there is a strong supply function equilibrium in order to demonstrate that for some parameter choices no strong supply function equilibrium exists. Then we will derive a supply function equilibrium which is not a strong SFE for this example: it has a vertical segment for firm 1 covering the range of prices in which firm 2 reaches its capacity.

Using the results of AH we know that, if there is a strong supply function equilibrium in a duopoly where only firm 2 has a capacity constraint that applies, then this will have three regions:

A: $p_0 < p < c_2$: firm 1 supplying alone;
B: $c_2 < p < p_2$: both firms supplying;
C: $p_2 < p < p_3$: firm 2 at capacity, firm 1 acts as a monopoly supplier.

We begin by characterizing the solution in Region B. In this case the equation (6) for the supply functions becomes

$$s_1(p) = (p - c_1)\left(s_2'(p) + b\right), \tag{9}$$
$$s_2(p) = (p - c_2)\left(s_1'(p) + b\right). \tag{10}$$

Thus

$$s_1'(p) = (p - c_1)s_2''(p) + s_2'(p) + b$$

and

$$s_2(p) = (p - c_2)\left((p - c_1)s_2''(p) + s_2'(p) + 2b\right).$$

We can solve this second order ODE for s_2 and from it derive s_1. After some manipulation we obtain

$$\begin{aligned} s_1(p) &= -(p-c_1)\,b\left(\log(p-c_1)\right) + bc_2 - bc_1 + K_1(p-c_1) \\ &\quad +K_2\left((\log(p-c_1))(p-c_1) - (\log(p-c_2))(p-c_1) + c_1 - c_2\right), \\ s_2(p) &= -(p-c_2)\,b\ln(p-c_1) + K_1(p-c_2) \\ &\quad +K_2\left((\log(p-c_1))(p-c_2) - (\log(p-c_2))(p-c_2) + c_1 - c_2\right). \end{aligned}$$

From Theorem 1 we know that $s_2(c_2) = 0$, and hence $K_2 = 0$. Thus

$$s_1(p) = -(p-c_1)\,b\log(p-c_1) + b(c_2-c_1) + K_1(p-c_1), \tag{11}$$

$$s_2(p) = -(p-c_2)\,b\log(p-c_1) + K_1(p-c_2). \tag{12}$$

Note that with this structure of solution, the differential equation for s_1 shows that the monopoly solution $s_1(p) = b(p-c_1)$ occurs in both region (A) and region (C). As p approaches c_2 from below $s_1(p)$ approaches $(c_2-c_1)b$. To ensure that s_1 is an increasing function (and does not jump down at price c_2) we need

$$(c_2-c_1)\left(K_1 - b\log(c_2-c_1)\right) \geq 0 \tag{13}$$

i.e. $K_1 \geq b\log(c_2-c_1)$.

We also need to ensure monotonicity of s_1 at the price p^* where firm 2 meets its capacity constraint. Considering (9) we can see that to avoid a jump down in s_1 at p^* we must have $s_2'(p) \leq 0$ as p approaches p^* from below. Since s_2 must be increasing this implies that $\lim_{p\to p^*} s_2'(p) = 0$ where the limit is taken from below. This argument is given in more detail in AH. Now

$$s_2'(p) = -b\frac{(p-c_2)}{(p-c_1)} - b\log(p-c_1) + K_1,$$

so the condition $s_2'(p^*) = 0$ implies

$$\left(K_1 - b\ln(p^*-c_1)\right)(p^*-c_1) = b\,(p^*-c_2). \tag{14}$$

Thus from (12)

$$s_2(p^*) = (p^*-c_2)\left(K_1 - b\ln(p^*-c_1)\right) = \frac{b\,(p^*-c_2)^2}{(p^*-c_1)}.$$

Since we know that s_2 is continuous at p^*, we have $s_2(p^*) = \bar{q}_2$. This is a quadratic equation with solution

$$p^* = c_2 + \frac{\bar{q}_2}{2b} + \frac{1}{2b}\sqrt{\bar{q}_2^2 + 4b(c_2-c_1)\bar{q}_2}$$

(the other root is smaller than c_2). Putting this value into (14) gives the value of K_1 and hence the complete solution. Note that (13) will be automatically satisfied.

In general we have

$$s_1'(p) = -b - b\log(p - c_1) + K_1,$$
$$s_2'(p) = -b\frac{(p - c_2)}{(p - c_1)} - b\log(p - c_1) + K_1 > s_1'(p),$$

so that at the point where $s_2'(p) = 0$ we have already reached a part of the s_1 curve that is decreasing. As this is not possible we have ruled out the possibility of a supply function equilibrium in pure strategies if the higher price firm reaches its capacity first and each of the regions (A), (B) and (C) occurs.

At the lowest demand shock, the demand is given by $A - bp$. The condition for region (A) to occur is that with this demand the market clears below price c_2. Since the supply function offer for firm 1 below price c_2 is $s_1(p) = b(p - c_1)$, the market clears at a price of $p = (A + bc_1)/(2b)$ and the condition is

$$A < b(2c_2 - c_1). \tag{15}$$

Now consider the possibility that firm 2 does not reach its capacity (or just reaches its capacity at maximum demand). Thus region (C) does not occur. In this case the highest price must be less than the price, say p^H, at which s_1 starts to decrease. From $s_1'(p^H) = 0$ we have

$$p^H = \exp\left(\frac{K_1}{b} - 1\right) + c_1.$$

At this price the amount supplied by firm 2 is

$$\begin{aligned} s_2(p^H) &= -\left(p^H - c_2\right)(K_1 - b) + K_1\left(p^H - c_2\right) \\ &= b(\exp(\frac{K_1}{b} - 1) + c_1 - c_2). \end{aligned}$$

Thus we need to have

$$\bar{q}_2 \geq b(\exp(\frac{K_1}{b} - 1) + c_1 - c_2).$$

This equation puts a bound on K_1 which in turn limits the total amount that can be supplied at these prices. Note that

$$\begin{aligned} s_1(p^H) &= -\left(p^H - c_1\right) b\log\left(p^H - c_1\right) + b(c_2 - c_1) + K_1\left(p^H - c_1\right) \\ &= b\left(\exp(\frac{K_1}{b} - 1) + c_2 - c_1\right) \\ &\leq \bar{q}_2 + 2b(c_2 - c_1). \end{aligned}$$

At the maximum demand shock, the demand is $A+X-pb \geq A+X-p^H b$. On the other hand the maximum supply if region C does not occur is less than

$$\begin{aligned} s_1(p^H)+s_2(p^H) &= b\left(p^H+c_2-2c_1\right)+b\left(p^H-c_2\right) \\ &\leq 3\bar{q}_2-2bc_1+3bc_2-p^H b. \end{aligned}$$

Hence every solution includes region C if the following condition holds:

$$A+X > 3\bar{q}_2-2bc_1+3bc. \tag{16}$$

Now we consider a specific example by taking $c_1=20$, $c_2=30$, $\bar{q}_1=1500$, $\bar{q}_2=300$, $A=300$, $b=10$, $X=1200$ (the choice of $\bar{q}_1$ is not important provided it is large enough to ensure that firm 1 does not exhaust its capacity). We can confirm that all three regions occur with these parameters (and so there is no strong SFE) by checking (15) and (16). Notice that this scenario is not unrealistic. We can suppose that prices are given in euros per megawatt hour. There is a smaller and more expensive (gas) generator competing with a large amount of coal fired generation. There is sufficient capacity here to ensure that the market can always be supplied. The price elasticity might be derived from a non-strategic fringe generator or from imports.

Now we show how to construct an equilibrium for this example. The equilibrium supply function for firm 1 includes a vertical segment. We let $\sigma_1(p,K_1)$, $\sigma_2(p,K_1)$ be the pair of supply function solutions given by (11) and (12). These must match the equilibrium when both supply functions are increasing. We define the supply functions

$$\begin{aligned} s_1(p) &= \sigma_1(p,K_1) \text{ for } c_2 < p \leq \gamma \\ &= \sigma_1(\gamma,K_1) \text{ for } \gamma < p \leq c_1+(\sigma_1(\gamma,K_1)/b) \\ &= (p-c_1)b \text{ otherwise.} \end{aligned}$$

and

$$\begin{aligned} s_2(p) &= 0 \text{ for } p \leq c_2, \\ &= \sigma_2(p,K_1) \text{ for } c_2 < p \leq \gamma, \\ &= (p-c_2)b \text{ for } \gamma < p \leq c_2+\bar{q}_2/b), \\ &= \bar{q}_2 \text{ for } p > c_2+\bar{q}_2/b. \end{aligned}$$

This solution has two free parameters: K_1 and γ: K_1 determines the length of the horizontal segment at price c_2 and γ gives the point at which the vertical segment starts. We will choose these parameters to satisfy two additional conditions: first that s_2 is continuous at γ and second that the optimality conditions (Lemma 1 (b)) are satisfied over the vertical segment for s_1. The first condition implies

$$\sigma_2(\gamma,K_1)=(\gamma-c_2)b.$$

The vertical section goes from γ to $c_1+s_1(\gamma)/b$ so the second condition can be written

$$\int_\gamma^{c_1+(s_1(\gamma)/b)} Z_1(s_1(\gamma),p)dp = 0 \tag{17}$$

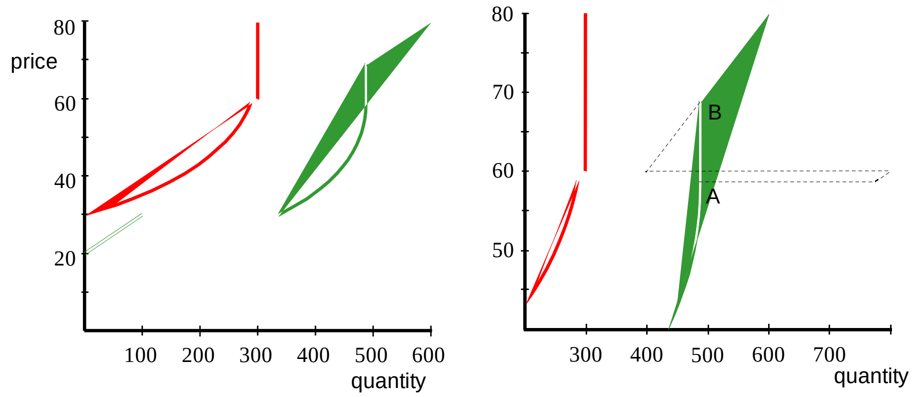

Fig. 1 Equilibrium solution including a vertical segment

where, from (7),

$$Z_1(q,p) = f(q + s_2(p) - A + bp)[(p - c_1)(s_2'(p) + b) - q].$$

Now $f(x) = 1/X$ in the range $(0, X)$, so provided dispatch can occur at point (q, p) we have

$$Z_1(q,p) = \frac{1}{X}(2(p - c_1)b - q) \text{ for } p \in (\gamma,\ c_2 + \bar{q}_2/b),$$
$$Z_1(q,p) = \frac{1}{X}((p - c_1)b - q) \text{ for } p > c_2 + \bar{q}_2/b.$$

Thus the integrand in (17) starts at $(1/X)(2(\gamma - c_1)b - s_1(\gamma))$ (which is positive) and increases to $(1/X)(2(c_2 + \bar{q}_2/b - c_1)b - s_1(\gamma))$ then it jumps down to $(1/X)((c_2 + \bar{q}_2/b - c_1)b - s_1(\gamma))$ (which is negative) and then increases to zero. To achieve the condition (17) we must have

$$\begin{aligned} &(c_2 + \bar{q}_2/b - \gamma)\,(2(\gamma - c_1)b + 2(c_2 + \bar{q}_2/b - c_1)b - 2s_1(\gamma)) \\ &= (c_1 + s_1(\gamma)/b - c_2 - \bar{q}_2/b)\,(s_1(\gamma) - (c_2 + \bar{q}_2/b - c_1)b)\,. \end{aligned}$$

Together with (11) and (12) this gives enough relationships to work out the values of K_1 and γ.

Now we can calculate the equilibrium solution for the example. We obtain $\gamma = 58.730$ (so $s_1(\gamma) = 487.3$) and $K_1 = 46.566$. Figure 1 shows the equilibrium solution with on the right an expanded version where the dashed line shows the boundary between positive and negative values of Z_1. The vertical segment goes from the point marked A to the point marked B and in this section does not follow the best response $Z_1 = 0$ curve.

4 Existence of a Supply Function Equilibrium

Now we return to a consideration of the more general asymmetric duopoly using the assumptions of section 2. We assume that the range of demand shocks is sufficiently wide that at low demand shocks just one firm produces and for high demand shocks at least one of the firms is at its capacity.

From the differential equations we can define the monopoly solution $s_i^*(p)$, which solves the equation

$$s_i^*(p) = -[p - C_i'(s_i^*(p))]D'(p), \; i \neq j. \tag{18}$$

Note that s_i^* takes the value zero at the price $C_i'(0)$. Moreover

$$s_i^{*\prime}(p) = -[p - C_i'(s_i^*(p))]D''(p) - [1 - C_i''(s_i^*(p))s_i^{*\prime}(p)]D''(p) \tag{19}$$

So

$$s_i^{*\prime}(p) = \left(\frac{-D''(p)}{-D'(p)}\right) \frac{s_i^*(p) - D'(p)}{1 - C_i''(s_i^*(p))D''(p)} > 0.$$

From now on, and without loss of generality we suppose that $C_1'(0) < C_2'(0)$. For convenience we write $p_2 = C_2'(0)$. From Theorem 1 (a) we know that $s_2(p) = 0$ for $p \leq p_2$ and hence firm 1 makes an optimal response of $s_1(p) = s_1^*(p)$ in this range. From part (c) there may be a jump in the s_1 value at p_2 but there is no discontinuity in s_2. Thus at $p = p_2$ we expect the solution to start to follow the unique solution to the ODEs with initial conditions $s_1(p_2) = \alpha \geq s_1^*(p_2)$ and $s_2(p_2) = 0$. It is useful to index the solutions to the ODE system by α, the value that s_1 jumps up to at price p_2.

Lemma 2 *The solutions to the ODEs (indexed by α) $s_i^{(\alpha)}(p)$ are continuous increasing functions of α for each p.*

Proof

AH show (Lemma 4) that the solutions to the ODE system are ordered, so if $0 < \alpha_1 < \alpha_2$, then since $s_i^{(\alpha_1)}(C_2'(0)) \leq s_i^{(\alpha_2)}(C_2'(0))$ and this inequality is strict for $i = 1$, then $s_i^{(\alpha_1)}(p) \leq s_i^{(\alpha_2)}(p)$ throughout the price range over which the ODEs hold. Thus we have shown that the solutions can never decrease as functions of α.

AH also show (Lemma 5) that the solutions to the ODEs are unique. This is enough to show continuity as a function of α. For suppose that there is some p_0 and α_0 with $\lim_{\delta \searrow 0} s_i^{(\alpha_0 - \delta)}(p_0) \neq \lim_{\delta \searrow 0} s_i^{(\alpha_0 + \delta)}(p_0)$ for $i = 1$ or $i = 2$. We take $s_i^{(A)}(p) = \lim_{\delta \searrow 0} s_i^{(\alpha_0 - \delta)}(p)$, $i = 1, 2$ and $s_i^{(B)}(p) = \lim_{\delta \searrow 0} s_i^{(\alpha_0 + \delta)}(p)$, $i = 1, 2$. Then note that since all slopes are bounded the supply functions $s_i^{(A)}$ and $s_i^{(B)}$ must also satisfy the ODE system. Moreover they have the same initial conditions but differ at the price p_0, which contradicts uniqueness. Thus no such p_0 and α_0 can be found and continuity is established. ∎

We define $p_i^* = (s_i^*)^{-1}(\bar{q}_i)$ so that p_i^* is the price at which the monopoly solution for firm i hits the capacity bound. We let k be the firm with the smaller value for p_i^* and let h be the other firm. So we have $p_k^* \leq p_h^*$.

We will define $p_h^X(q)$ for $q \in [s_h^*(p_k^*), \bar{q}_h]$ as the price at which a vertical section may start for s_h. We let $p_h^*(\cdot)$ be the inverse of s_h^*, so if there is a vertical section it will end at the price $p_h^*(q)$ where $s_h^*(p_h^*(q)) = q$. With this notation $p_h^* = p_h^*(\bar{q}_h)$. Along the vertical segment at q we have

$$\begin{aligned} Z_h(q,p) &= f(q + s_k^*(p) - D(p))\,[(p - C_h'(q))\,(s_k^{*\prime}(p) - D'(p)) - q] \text{ for } p \le p_k^* \\ &= f(q + \bar{q}_k - D(p))\,[-(p - C_h'(q))D'(p) - q] \text{ for } p > p_k^* \end{aligned}$$

We require the integral $\int_{p_h^X(q)}^{p_h^*(q)} Z_h(q,p)dp = 0$ which can be seen as a definition of $p_h^X(q)$: it is the value of $p_0 < p_k^*$ which makes $\int_{p_0}^{p_h^*(q)} Z_h(q,p)dp = 0$. Note that the integral has two components. The integral between p_0 and p_k^* is decreasing as $q_h^*(p_0)$ increases.

For $p < p_h^*(q)$ we will have $s_h^*(p) < q$. Now

$$s_h^*(p) = -[p - C_h'(s_h^*(p))]D'(p) > -[p - C_h'(q)]D'(p)$$

and so $-(p - C_h'(q))D'(p) - q < 0$, and $Z_h(q,p) < 0$ for $p \in (p_k^*, p_h^*(q))$ (and increases to zero at $p_h^*(q)$).

Let $q_h^Y(p)$ be given by the solution to:

$$q_h^Y(p) = (p - C_h'(q_h^Y(p)))\,(s_k^{*\prime}(p) - D'(p))\,. \tag{20}$$

So we have $Z_h(q_h^Y(p), p) = 0$ and $Z_h > 0$ between the line defined by $q_h^Y(p)$ and p_k^*. Note that $q_h^Y(p)$ is simply the supply function that is an optimal response to $s_k^*(p)$.

Observe that when $q = s_h^*(p_1^*)$, and $p = p_k^*$ then

$$\begin{aligned} (p - C_h'(q))\,(s_k^{*\prime}(p) - D'(p)) - q &= (p_k^* - C_h'(s_h^*(p_k^*)))\,(s_k^{*\prime}(p) - D'(p)) - s_h^*(p_k^*) \\ &= (p_k^* - C_h'(s_h^*(p_k^*)))s_k^{*\prime}(p) > 0, \end{aligned}$$

from which we can deduce that $q_h^Y(p_k^*) > s_h^*(p_k^*)$.

To find $p_h^X(q)$ we extend the integral downwards from p_k^* until either the overall integral is zero or until $q_h^Y(p)$ is reached (when further extension will make the overall integral larger rather than smaller.) We have $p_h^X(s_h^*(p_k^*)) = p_k^*$ and from continuity and the fact that $q_h^Y(p_k^*) > s_h^*(p_k^*)$, there is at least some range of q values until either $q_h^Y(p)$ is reached, or $q = \bar{q}_h$.

We will show that one of three different equilibrium solutions will occur. The different cases are illustrated in Figure 2, which shows a situation where $k = 2$ (i.e. $p_2^* < p_1^*$), but the alternative with $k = 1$ is very similar.

We need to make additional assumptions in order to prove existence.

Assumption 2
(a) $C_i'''(x) \ge 0$ and $D'''(p) \le 0$;
(b) $-2D''(p)\max(\bar{q}_1, \bar{q}_2) < (-D'(p))^2$ for all p.

First we establish a preliminary lemma that determines the direction in which $s_i^{(\alpha)}$ may cross s_i^*.

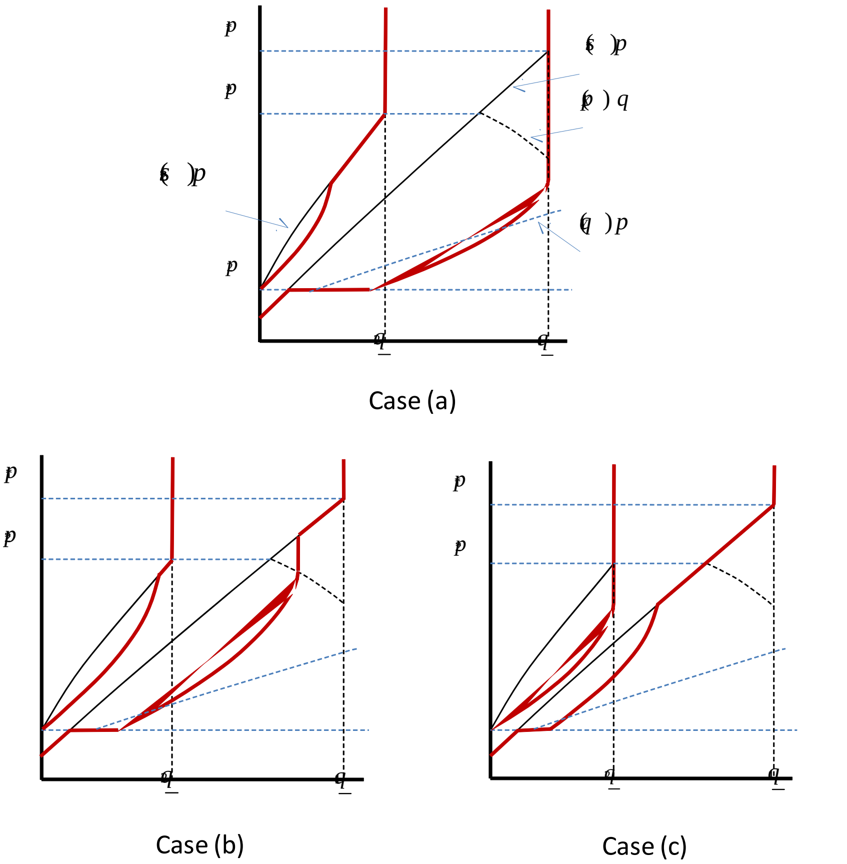

Fig. 2 Three different types of equilibrium

Lemma 3 *Under Assumptions 1 and 2, if* $s_i^{(\alpha)}(p) = s_i^*(p)$ *for some* $p > p_2$ *then* $s_i^{(\alpha)}(p-\delta) > s_i^*(p-\delta)$ *and* $s_i^{(\alpha)}(p_Z+\delta) < s_i^*(p_Z+\delta)$ *for* δ *small enough. Moreover if* $s_j^{(\alpha)\prime}(p) = 0$ *then* $s_j^{(\alpha)}(p) \le q_j^Y(p)$

Proof:

We consider a price p_Q at which $s_i^{(\alpha)}(p_Q) = s_i^*(p_Q)$. Suppose first that the two derivatives match: $s_i^{(\alpha)\prime}(p_Q) = s_i^{*\prime}(p_Q)$. We will show that $s_i^{(\alpha)\prime\prime}(p_Q) < s_i^{*\prime\prime}(p_Q)$, which is enough to rule out certain kinds of tangent behavior. Write $Q = s_i^{(\alpha)}(p_Q)$ and $\beta = s_i^{(\alpha)\prime}(p_Q)$ and let $j \neq i$. Since $s_i^{(\alpha)}(p_Q) = s_i^*(p_Q)$, we have

$$Q = -[p_Q - C_i'(Q)]D'(p_Q) = [p_Q - C_i'(Q](s_j^{(\alpha)\prime}(p_Q) - D'(p_Q)),$$

and hence $s_j^{(\alpha)\prime}(p_Q) = 0$. From (19) we know that

$$\beta = -[p_Q - C_i'(Q)]D''(p_Q) - [1 - C_i''(Q)\beta]D'(p_Q).$$

Similarly, as $\beta = s_i^{(\alpha)\prime}(p_Q)$

$$\begin{aligned}\beta &= [p_Q - C_i'(Q))](s_j^{(\alpha)\prime\prime}(p) - D''(p)) + [1 - C_i''(Q)s_i^{(\alpha)\prime}(p_Q)](s_j^{(\alpha)\prime}(p_Q) - D'(p_Q)) \\ &= [p_Q - C_i'(Q))](s_j^{(\alpha)\prime\prime}(p) - D''(p)) - [1 - C_i''(Q)\beta]D'(p_Q).\end{aligned}$$

Comparing the two expressions for β shows that $s_j^{(\alpha)\prime\prime}(p_Q) = 0$.

Differentiating (6) we obtain

$$s_j^{(\alpha)\prime\prime}(p) = \frac{[p - C_i'(s_i^{(\alpha)}(p))]s_i^{(\alpha)\prime}(p) - s_i^{(\alpha)}(p)[1 - C_i''(s_i^{(\alpha)}(p))s_i^{(\alpha)\prime}(p)]}{[p - C_i'(s_i^{(\alpha)}(p))]^2} + D''(p). \tag{21}$$

Since this is zero at p_Q we can deduce that

$$[1 - C_i''(Q)\beta] = [p_Q - C_i'(Q)]\frac{\beta + D''(p_Q)[p_Q - C_i'(Q)]}{Q}.$$

But $Q = [p_Q - C_i'(Q)](-D'(p_Q))$ so

$$[1 - C_i''(Q)\beta] = \frac{\beta}{(-D'(p_Q))} + \frac{D''(p_Q)Q}{(-D'(p_Q))^2}. \tag{22}$$

Now we use the equivalent expression to (21) for $s_i^{(\alpha)\prime\prime}(p)$ and the observation that $s_j^{(\alpha)\prime}(p_Q) = 0$ to deduce

$$s_i^{(\alpha)\prime\prime}(p_Q) = -\frac{s_j^{(\alpha)}(p_Q)}{[p_Q - C_j'(Q)]^2} + D''(p_Q) < D''(p_Q).$$

Now

$$\begin{aligned}s_i^{*\prime\prime}(p) = -[p - C_i'(s_i^*(p))]D'''(p) - 2[1 - C_i''(s_i^*(p))s_i^{*\prime}(p)]D''(p) \\ + [C_i''(s_i^*(p))s_i^{*\prime\prime}(p) + C_i'''(s_i^*(p))(s_i^{*\prime}(p))^2]D'(p).\end{aligned}$$

So

$$\begin{aligned}&s_i^{*\prime\prime}(p_Q)(1 - C_i''(Q)D'(p_Q)) \\ &= -[p_Q - C_i'(Q)]D'''(p_Q) - 2[1 - C_i''(Q)\beta]D''(p_Q) - C_i'''(Q)\beta^2 D'(p_Q) \\ &> -2[1 - C_i''(Q)\beta]D''(p_Q)\end{aligned}$$

using Assumption 2 (a). But from (22)

$$[1 - C_i''(Q)\beta] > \frac{D''(p_Q)Q}{(-D'(p_Q))^2}.$$

So

$$-2[1 - C_i''(Q)\beta] < -2\frac{D''(p_Q)Q}{(-D'(p_Q))^2} < 1 < (1 - C_i''(Q)D'(p_Q))$$

from Assumption 2 (b). Hence

$$\begin{aligned} -2[1 - C_i''(Q)\beta]D''(p_Q) &> (1 - C_i''(Q)D'(p_Q))D''(p_Q) \\ &> (1 - C_i''(Q)D'(p_Q))s_i^{(\alpha)\prime\prime}(p_Q) \end{aligned}$$

and we have established that $s_i^{(\alpha)\prime\prime}(p_Q) < s_i^{*\prime\prime}(p_Q)$ as claimed.

We need to treat the cases $i = 1$ and $i = 2$ slightly differently. We show that there will be at least one value for α which achieves a crossing for s_2 for p close to p_2. We do this by finding values of α_1, α_2 such that $s_2^{(\alpha_1)}(p) < s_2^*(p)$ and $s_2^{(\alpha_2)}(p) > s_2^*(p)$ will hold for p close enough to $C_2'(0)$.

Now

$$s_2^{(\alpha)\prime}(p_2) = s_1^{(\alpha)}(p_2)/(p_2 - C_1'(s_1^{(\alpha)}(p_2)) + D'(p_2) = \alpha/(p_2 - C_1'(\alpha)) + D'(p_2)$$

and

$$s_2^{*\prime}(p_2) = -[p_2 - C_2'(0)]D''(p_2) - [1 - C_2''(0)s_2^{*\prime}(p_2)]D'(p_2).$$

The first term is zero, so

$$s_2^{*\prime}(p_2) = \frac{-D'(p_2)}{1 - C_2''(0)D'(p_2)}. \tag{23}$$

Consider the equation

$$\frac{\alpha}{(p_2 - C_1'(\alpha))} = -D'(p_2)\left(1 + \frac{1}{1 - C_2''(0)D'(p_2)}\right) \tag{24}$$

The left hand side is increasing in α and so there is a single solution α_0 which makes $s_2^{(\alpha_0)\prime}(p_2) = s_2^{*\prime}(p_2)$. Moreover for $\alpha > \alpha_0$ we have $s_2^{(\alpha)\prime}(p_2) > s_2^{*\prime}(p_2)$. We can also use the analysis above with $p_Q = p_2$ to show that $s_2^{(\alpha_0)\prime\prime}(p_2) < s_2^{*\prime\prime}(p_2)$. This is enough (by continuity) to show the existence of the values α_1, α_2 that we wanted with $\alpha_1 = \alpha_0$ and α_2 slightly larger. Thus for p close enough to p_2 we will have $s_2^{(\alpha_0)}(p) < s_2^*(p) < s_2^{(\alpha_2)}(p)$ which shows the existence of an $\alpha > \alpha_0$ with $s_2^{(\alpha)}(p) = s_2^*(p)$

The argument for $i = 1$ is simpler since for any $\alpha > s_1^*(p_2)$ the solution starts with $s_1^{(\alpha)}(p) > s_1^*(p)$. It is possible that there is no crossing for any values of $\alpha > s_1^*(p_2)$ but if there is a crossing then the first one is in the direction we wish.

So for both $i = 1$ and $i = 2$ we have crossings that occur in the right direction. and from Lemma 2 above we know that there can be at most one such crossing at each value of p. It remains to ensure that all such crossings (other than at $p = p_2$) take place in the same direction. The alternative we want to rule out is that for higher values of p the functions may cross again. If this were to happen there would (at the boundary between p values where crossing takes place in different directions) be a price p_Q at which $s_i^{(\alpha)}(p_Q) = s_i^*(p_Q)$, $s_i^{(\alpha)\prime}(p_Q) = s_i^{*\prime}(p_Q)$ and $s_i^{(\alpha)}(p) > s_i^*(p)$ for p close but not equal to p_Q. But the result above that $s_i^{(\alpha)\prime\prime}(p_Q) < s_i^{*\prime\prime}(p_Q)$ rules out this possibility.

The second statement in the lemma follows straightforwardly. If $s_j^{(\alpha)\prime}(p) = 0$ then $s_i^{(\alpha)}(p) = s_i^*(p)$. Because of the result on the direction of crossings this implies $s_i^{(\alpha)\prime}(p) \leq s_i^{*\prime}(p)$ and hence from the definition of q_j^Y and the equation (6) for s_j we have established that $s_j^{(\alpha)}(p) \leq q_j^Y(p)$. ■

The idea behind our existence proof is to track what happens as α increases. The lowest value of α is α_0 given by the solution to (24). Comparing (20) and (23) with (24) we can see that $\alpha_0 = q_1^Y(p_2)$. Observe that

$$s_1^*(p_2) = (-D'(p_2))[p_2 - C_1'(s_1^*(p_2))].$$

Since $\alpha_0/(p_2 - C_1'(\alpha_0)) > -D'(p_2)$ and the left hand side is increasing as a function of α, this shows $\alpha_0 > s_1^*(p_2)$ which is a requirement for $s_1^{(\alpha_0)}$ to be non-decreasing at p_2. Moreover (using l'Hopital) we have

$$\begin{aligned}
s_1^{(\alpha_0)\prime}(p_2) &= s_2^{(\alpha_0)}(p_2)/(p_2 - C_2'(s_2^{(\alpha_0)}(p_2))) + D'(p_2) \\
&= s_2^{(\alpha_0)\prime}(p_2)/(1 - C_2''(0)s_2^{(\alpha_0)\prime}(p_2)) + D'(p_2) \\
&= s_2^{*\prime}(p_2)/(1 - C_2''(0)s_2^{*\prime}(p_2)) + D'(p_2) = 0.
\end{aligned}$$

Choosing $\alpha > \alpha_0$ leads to $s_1^{(\alpha)\prime}(p_2) > 0$.

Theorem 2 *Under Assumptions 1 and 2 there will be a pure strategy supply function equilibrium for the duopoly.*

Proof:

We start by dealing with a special case when $\alpha_0 \geq \bar{q}_1$ or equivalently

$$\frac{\bar{q}_1}{(p_2 - C_1'(\bar{q}_1))} \leq -D'(p_2)\left(1 + \frac{1}{1 - C_2''(0)D'(p_2)}\right). \tag{25}$$

In this case there will be an equilibrium with $s_1(p) = \bar{q}_1$ and $s_2(p) = s_2^*(p)$ for $p > p_2$. We can check that this solution is an equilibrium since the condition is equivalent to $q_1^Y(p_2) \geq \bar{q}_1$ and so the optimal response to $s_2^*(p)$ is $s_1(p) = \bar{q}_1$.

The result of Lemmas 2 and 3 is enough with continuity to show that the crossing point increases with α in a continuous way. Now we let $p_W(\alpha) = \min(p_{1W}(\alpha), p_{2W}(\alpha))$ where $p_{iW}(\alpha)$ is the price (greater than p_2) at which the crossing occurs: thus $s_i^{(\alpha)}(p_{iW}(\alpha)) = s_i^*(p_{iW}(\alpha)) > 0$. Our earlier discussion shows that $s_2^{(\alpha_0)}(p)$ remains less than $s_2^*(p)$. This is enough to show that either $p_{2W}(\alpha)$ is defined for all $\alpha > \alpha_0$ or there is a range of α values so that $p_{2W}(\alpha)$ can take all values between p_2 and p_2^*. So even if there is no crossing for $s_1^{(\alpha)}$ and $p_{1W}(\alpha)$ is not defined, we still have $p_W(\alpha)$ well defined.

We will search for an equilibrium solution with one of three situations occurring: (a) $p_W(\alpha) = p_{kW}(\alpha)$, $p_{kW}(\alpha) < p_h^X(\bar{q}_h)$ and $s_h^{(\alpha)}(p_{kW}(\alpha)) = \bar{q}_h$; (b) $p_W(\alpha) = p_{kW}(\alpha)$ and $p_{kW}(\alpha) = p_h^X(s_h^{(\alpha)}(p_{kW}(\alpha)))$; or (c) $p_W(\alpha) = p_{hW}(\alpha)$ and $s_k^{(\alpha)}(p_{hW}(\alpha)) = \bar{q}_k$ (these are the three cases shown in Figure 3)

Case (b) involves a vertical section that has been constructed to satisfy the required first order optimality conditions and this matches the example of the previous section. We can check the sign of the Z function away from the solution and see that this will indeed be a Nash equilibrium with global optimality. Also it is not hard to check that the solution corresponding to case (c) involves each firm making an optimal response to the supply function of the other.

Case (a) is more complex. The vertical section at $\bar{q}_h$ includes part with $p > p_h^*$ where the optimal response to firm k is higher than $\bar{q}_h$ and so having $s_h^{(\alpha)}(p) = \bar{q}_h$ is certainly optimal. $s_h^{(\alpha)}$ also includes a section from p_k^* to p_h^* where firm k is at capacity and the optimal response is a supply function $s_h^*(p)$ which is lower than $\bar{q}_h$. Observe that for $p \in (p_{kW}(\alpha), p_k^*)$ we must have $q_h^Y(p) > \bar{q}_h$. The reason is that we can increase α and track $s_h^{(\alpha)}(p_{kW}(\alpha))$. This increases with α and remains at $\bar{q}_h$ or higher, and hence (using Lemma 3) $q_h^Y(p_{kW}(\alpha)) > s_h^{(\alpha)}(p_{kW}(\alpha)) \geq \bar{q}_h$. Thus throughout this region $Z_h > 0$. Now since $p_{kW}(\alpha) < p_h^X(\bar{q}_h)$ we know from the definition of p_h^X that $\int_{p_{kW}(\alpha)}^{p_h^*} Z_h(q,p)dp > 0$, and so does not satisfy the optimality condition of Lemma 1 (b).The implication of this inequality is that an improvement can be made by shifting the vertical segment to the right (increasing $s_h^{(\alpha)}(p)$) but this is impossible because the supply function is already at its capacity limit. The only feasible perturbations are those involving a shift to the left of the lower part of the vertical segment, but since $\int_{p_{kW}(\alpha)}^{p} Z_h(q,p)dp > \int_{p_{kW}(\alpha)}^{p_h^*} Z_h(q,p)dp > 0$ for all $p > p_k^*$, and moreover $\int_{p_{kW}(\alpha)}^{p} Z_h(q,p)dp > 0$ for $p < p_k^*$, then all such shifts will only make the overall profit smaller. Using this argument we can see that this case will also deliver a Nash equilibrium.

Notice that $p_W(\alpha)$ is a continuous function of α and so $s_i^{(\alpha)}(p_W(\alpha))$ is also continuous as a function of α for $i = 1, 2$. When $\alpha = \alpha_0$ (defined by (24)) then $s_k^{(\alpha)}(p_W(\alpha)) < \bar{q}_k$ and (except in two special cases) $s_h^{(\alpha)}(p_W(\alpha)), p_W(\alpha)$ is in the region Γ defined in the (q,p) plane by

$$\Gamma = \{(q,p) : q \leq \bar{q}_h,\ p \leq p_h^X(q), \text{and } q \leq q_h^Y(p)\},$$

and where the constraint involving $p_h^X(q)$ only applies at q values for which this function is defined. The first special case is when (25) holds and has been discussed already. A second special case occurs when $h = 1$ and $p_1^X(q) = p_2$ for a value of q less than α_0. In this case we have $p_1^X(q_X) = p_2$ for $q_X \leq \alpha_0$ and we construct an equilibrium solution by starting a vertical segment in the supply function s_1 at p_2. We have $s_1(p) = q_X$ and $s_2(p) = \min(s_2^*(p), \bar{q}_2)$ for $p \in (p_2, p_1^*(q_X))$. The optimality check for this equilibrium is similar to that for case (b).

Excluding these special cases we let α^* be the lowest value of α at which either $s_k^{(\alpha)}(p_W(\alpha^*)) = \bar{q}_k$ or the point $(s_h^{(\alpha)}(p_W(\alpha)), p_W(\alpha))$ crosses the boundary of the region Γ. (We use this 'geometrical' definition to avoid having to assume monotonicity in either of the functions p_h^X or q_h^Y). Note that

$s_1^{(\alpha)}(p_W(\alpha)) > \alpha$ and so by taking α large enough we can guarantee that there is a value of α with $s_1^{(\alpha)}(p_W(\alpha)) > \bar{q}_1$ and so α^* is well-defined.

Consider the case that $s_k^{(\alpha)}(p_W(\alpha^*)) = \bar{q}_k$. We look at the two possibilities arising from the definition of p_W. Suppose that $p_{kW}(\alpha^*) < p_{hW}(\alpha^*)$; then $\bar{q}_k = s_k^{(\alpha)}(p_{kW}(\alpha^*)) = s_k^*(p_{kW}(\alpha^*))$ and so $p_{kW}(\alpha^*) = p_k^*$. But since $p_{kW}(\alpha^*) < p_{hW}(\alpha^*)$, we have $s_h^{(\alpha)}(p_{kW}(\alpha^*)) > s_h^*(p_{kW}(\alpha^*)) = s_h^*(p_k^*) = q_h^*(p_k^*)$, which contradicts the definition of α^*. Thus our supposition is wrong and $p_{kW}(\alpha^*) \geq p_{hW}(\alpha^*)$. So $s_k^{(\alpha)}(p_{hW}(\alpha^*)) = \bar{q}_k$ which establishes the condition we require for case (c).

Now consider the case in which α^* is defined by $(s_h^{(\alpha)}(p_W(\alpha)), p_W(\alpha))$ leaving the region Γ. Note that the crossing cannot involve $q_h^Y(p)$ since if $s_h^{(\alpha)}(p_W(\alpha)) = q_h^Y(p_W(\alpha))$ then $s_h^{(\alpha)\prime}(p_W(\alpha)) = 0$ and we get a contradiction from Lemma 3. So we have either $s_h^{(\alpha^*)}(p_W(\alpha^*)) = \bar{q}_h$ or $p_W(\alpha^*) = p_h^X(s_2^{(\alpha^*)}(p_W(\alpha^*)))$. Again there are two possibilities arising from the definition of p_W. Suppose that $p_{kW}(\alpha^*) > p_{hW}(\alpha^*)$ then $s_h^{(\alpha)}(p_{2W}(\alpha^*)) = s_h^*(p_{2W}(\alpha^*))$ and either $s_h^*(p_{hW}(\alpha^*)) = \bar{q}_h$ (which implies $p_{hW}(\alpha^*) = p_h^*$) or $p_W(\alpha^*) = p_h^X(s_h^*(p_{hW}(\alpha^*)))$ (which implies $s_h^*(p_{hW}(\alpha^*)) > s_h^*(p_k^*)$). So in either case $p_{hW}(\alpha^*) > p_k^*$ which is a contradiction and hence we must have $p_{kW}(\alpha^*) \leq p_{hW}(\alpha^*)$ and so either $s_h^{(\alpha)}(p_{kW}(\alpha^*)) = \bar{q}_h$ or $p_{kW}(\alpha^*) = p_h^X(s_h^{(\alpha)}(p_{kW}(\alpha^*)))$ and in either case we have the appropriate condition for an equilibrium. ■

5 Uniqueness of a Supply Function Equilibrium

In this section we ask whether there may be more than one supply function equilibrium. The argument is in two parts: first we show that the only equilibria are of the form we constructed, and then we establish that there can only be one such equilibria. To establish uniqueness we need to make further assumptions. As before we write p_2 for $C_2'(0) > C_1'(0)$.

Assumption 3

(a) The functions $p_i^X(q)$ are monotonic decreasing for the range of q values for which they are defined;

(b) The functions $q_i^Y(p)$ are strictly monotonic increasing in p

(c) $D(p_2) + \underline{\varepsilon} < s_1^*(p_2)$, $D(p_k^*) + \bar{\varepsilon} > \bar{q}_1 + \bar{q}_2$ and $p_k^* < p_h^*$.

Part (c) of this assumption is required in order to ensure that the range of possible demand shocks is wide enough and the condition that p_k^* is strictly less than p_h^* ensures that both firms cannot reach their capacity at the same price in equilibrium.

Theorem 3 *Under Assumptions 1, 2 and 3 there is a unique supply function equilibrium.*

Proof.

We begin by showing that any SFE has an initial monopoly segment, followed (possibly) by a horizontal segment, followed by a segment in which both price and quantity increase. The final segment has one firm at its capacity and the other firm acting as a monopoly. However this may be preceded by a penultimate segment in which one firm's supply function is vertical, while the other firm first acts as a monopoly and then reaches its capacity.

From Theorem 1 we know that there can only be a horizontal segment at p_2 and that at prices below p_2 firm 2 makes zero offer and so firm 1 offers its monopoly supply function $s_1^*(p)$. At prices higher than p_2 there is a segment in which the supply functions satisfy the first order conditions (6) unless perhaps there is immediately a vertical segment.

We want to show there cannot be a vertical segment unless it takes place over a range of prices at which the other firm hits its capacity bound. Suppose otherwise: then we have $s_i(p)$ constant over a range $p \in (p_a, p_b)$ with s_i, s_j satisfying the first order conditions (6) for p just above p_b or just below p_a. Since s_j makes an optimal response to s_i we have $s_j(p) = s_j^*(p)$ for $p \in (p_a, p_b)$ provided $s_j^*(p)$ is less than $\bar{q}_j$ throughout this price range. Since neither firm has a horizontal segment $s_i(p_b^+) = s_i(p_b)$ and $s_j(p_b^+) = s_j^*(p_b)$. This also implies that $s_j'(p_b^+) \geq s_j^{*\prime}(p_b)$ as if s_j becomes lower than s_j^* this leads to a non-monotonic solution for s_i. Hence $s_i(p_b^+) \geq q_i^Y(p_b^+)$. In the same way we can deduce that the limit as p approaches p_a from below of $s_j'(p)$ is no greater than $s_j^{*\prime}(p_a)$. Hence $s_i(p_a) \leq q_i^Y(p_a)$. But as $s_i(p_a) = s_i(p_b^+)$ this contradicts Assumption 3 (b).

This establishes that any SFE has to have the same form as the SFE shown to exist in Theorem 2. Now consider the possibility that there are two such equilibria corresponding to α values α_A and α_B with $\alpha_A < \alpha_B$. Using the AH result on the ordering of SFE's with different starting values we can deduce that $p_{iW}(\alpha_B) > p_{iW}(\alpha_A)$, $i = 1, 2$. Thus the different possibilities for the SFE associated with α_A all produce contradictions. If $s_2^{(\alpha_A)}(p_{1W}(\alpha_A)) = \bar{q}_2$ then $s_2^{(\alpha_B)}(p_{1W}(\alpha_B)) > \bar{q}_2$; if $s_1^{(\alpha_A)}(p_{2W}(\alpha_A)) = \bar{q}_1$ then $s_1^{(\alpha_B)}(p_{2W}(\alpha_B)) > \bar{q}_1$; and if there is a vertical segment with $p_{iW}(\alpha_A) = p_j^X(s_j^{(\alpha_A)}(p_{iW}(\alpha_A)))$ then $s_j^{(\alpha_B)}(p_{iW}(\alpha_B)) > s_j^{(\alpha_A)}(p_{iW}(\alpha_A))$ and so using Assumption 3 (a) $p_{iW}(\alpha_B) > p_j^X(s_j^{(\alpha_B)}(p_{iW}(\alpha_B)))$ and there can be no equivalent vertical segment for the solution corresponding to α_B.

Finally we consider the two special cases discussed in the proof of Theorem 2. Observe that a solution in which $s_1(p_2^+) = \bar{q}_1$ (so s_1 jumps up to its capacity at p_2) can only occur when $\alpha_0 \geq \bar{q}_1$. If not then $q_i^Y(p_2) < \bar{q}_1$ so $Z_1(\bar{q}_1, p_2) < 0$ and we get a contradiction from Lemma 1 (c) (there is an improving perturbation that sets $s_1(p) = \bar{q}_1 - \delta$ for $p \in (p_2, p_2 + \delta)$). In the same way a solution in which there is a vertical section starting at price p_2 (so $p_1^X(q_X) = p_2$) can only occur when $q_X \leq \alpha_0$. If $q_X > \alpha_0$ then $Z_1(q_X, p_2) < 0$ and again we have a contradiction from Lemma 1 (c). ∎

6 Discussion

In this paper we have focussed on pure strategy equilibrium. Recent work by Escobar and Jofre [6] has demonstrated that for this type of problem there will be a mixed strategy equilibrium even if a pure strategy equilibrium does not exist. Our results show that, under very general conditions on the problem parameters, such mixed strategy equilibria are not required in the duopoly case.

In setting up this problem we make some restrictive assumptions on the allowable supply functions S: we assume that supply functions have only a finite number of pieces and that the derivatives are bounded. Both these assumptions will have the effect of allowing a sequence of solutions in S with a limit that is not in S. Thus S is not compact and we may be concerned about existence of optimal solutions or equilibria in this setting. However our results show that under the conditions given we can indeed find a SFE satisfying these conditions. Nevertheless we have not ruled out such pathological solutions and so our proof of uniqueness leaves open the possibility of other SFE occurring with either an infinite number of pieces or unbounded derivatives.

It is natural to ask what happens when there are more than two firms. We conjecture that there will exist a supply function equilibrium under the same conditions that apply for a duopoly. At the start the solution is determined by the length of the horizontal segment that is introduced at the second lowest value of $C_i'(0)$. This solution can be traced across prices at which one of the firms reaches its capacity limit and when just two firms are left at quantities less than their capacities then the analysis goes through in the same way as for the duopoly case. However there are complications which can arise when marginal costs for different firms are very different. For example we might have a group of firms competing in one price range leaving only one monopoly firm at the highest price within this range, and then find another group of firms entering at a higher price range. And this pattern could be repeated many times over. In such cases the sections in which there is just one firm operating between its capacity bounds (and hence offering the monopoly solution) serve to divide up the problem into independent subproblems in each of which a unique SFE solution can be found using exactly the approach of this paper.

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