View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Sydney eScholarship

Faculty of Economics and Business
The University of Sydney

OME WORKING PAPER SERIES

Bayesian time-varying quantile forecasting for Value-at-Risk in financial markets

Richard Gerlach
Faculty of Economics and Business
The University of Sydney

Cathy W.S. Chen
Graduate Institute of Statistics and Actuarial Science
Feng Chia University

Nancy Y. C. Chan
Graduate Institute of Statistics and Actuarial Science
Feng Chia University

Abstract

Recently, Bayesian solutions to the quantile regression problem, via the likeli-hood of a Skewed-Laplace distribution, have been proposed. These approaches are extended and applied to a family of dynamic conditional autoregressive quantile models. Popular Value at Risk models, used for risk management in finance, are extended to this fully nonlinear family. An adaptive Markov chain Monte Carlo sampling scheme is adapted for estimation and inference. Simulation studies illustrate favourable performance, compared to the standard numerical optimization of the usual nonparametric quantile criterion function, in finite samples. An empirical study generating Value at Risk forecasts for ten major financial stock indices finds significant nonlinearity in dynamic quantiles and evidence favoring the proposed model family, for lower level quantiles, compared to a range of standard parametric volatility models, a semi-parametric smoothly mixing regression and some nonparametric risk measures, in the literature.

August 2009

OME Working Paper No: 01/2009
http://www.econ.usyd.edu.au/ome/research/working_papers

Bayesian time-varying quantile forecasting for Value-at-Risk in financial markets

Richard H. Gerlach[a], Cathy W. S. Chen[b] [*], and Nancy Y. C. Chan[b]

[a]Econometrics and Business Statistics, University of Sydney, Australia.

Tel: 612 9351 3944. Fax: 612 9351 6409.

Email: R.Gerlach@econ.usyd.edu.au.

[b]Graduate Institute of Statistics and Actuarial Science, Feng Chia University, Taiwan.

Tel: 886 4 2451 7250 ext. 4412. Fax: 886 4 2451 7092.

Email: chenws@fcu.edu.tw.

[*]Corresponding author is Cathy W. S. Chen. Email: chenws@fcu.edu.tw.

Abstract

Recently, Bayesian solutions to the quantile regression problem, via the likelihood of a Skewed-Laplace distribution, have been proposed. These approaches are extended and applied to a family of dynamic conditional autoregressive quantile models. Popular Value at Risk models, used for risk management in finance, are extended to this fully nonlinear family. An adaptive Markov chain Monte Carlo sampling scheme is adapted for estimation and inference. Simulation studies illustrate favorable performance, compared to the standard numerical optimization of the usual non-parametric quantile criterion function, in finite samples. An empirical study generating Value at Risk forecasts for ten major financial stock indices finds significant nonlinearity in dynamic quantiles and evidence favoring the proposed model family, for lower level quantiles, compared to a range of standard parametric volatility models, a semi-parametric smoothly mixing regression and some non-parametric risk measures, in the literature.

Keywords: CAViaR model; Asymmetric; Skew-Laplace distribution; Value-at-Risk; GARCH; Regression quantile.

1 Introduction

The quantile regression problem was first proposed and solved by Koenker and Bassett (1978). The problem involved estimating the parameters of a standard linear regression model, whose aim was to specify the regression line for the response variable measured at a specific quantile of the distribution, rather than at the mean of the response. Applications of this method have been wide ranging and include: labour and wage economics (e.g. Machado and Mata, 2005); medicine (e.g. Wei *et al.* 2006); ecology (e.g. Cade and Noon, 1996), among many others. The quantile regression solution involved the criterion function that, when minimised, returned the 'optimal' quantile regression estimator. This criterion required no distributional assumption, as such quantile regression is often regarded as a non-parametric method, despite mostly parametric forms being assumed for the regression relationship between the response and the regressors; most examples of its use thus make it semi-parametric in nature.

Recently, various authors have noted that the quantile regression criterion function is related to the likelihood for the Skewed-Laplace (SL) distribution; see e.g. Yu and Moyeed (2001) and Tsionas (2003). This discovery allows likelihood estimation, which has motivated Bayesian solutions to this problem. These proposals, see also Yu and Zhang (2005) and Geraci and Bottai (2007), all involve Markov chain Monte Carlo (MCMC) computational methods due to the non-standard form of the SL likelihood. We extend these MCMC methods to be adaptive, slightly extending the method in Chen and So (2006), to cover a family of nonlinear dynamic autoregressive semi-parametric conditional quantile models.

Value-at-Risk (VaR) forecasting is required by financial institutions worldwide (see Basel II: http://www.bis.org/publ/bcbsca.htm) for capital allocation and risk management. Thus, in recent years, volatility forecasting and measures of market risk have been very important for financial institutions. VaR forecasts the minimum loss over a given time interval, under normal market conditions, at a given confidence level, for an investment portfolio (Jorion, 1996); VaR is thus a function of the quantiles of an asset return distribution. Many competing econometric and time series models/methods have been used in the literature to forecast quantiles, and hence VaR (see Kuester et al, 2006 for a review). Engle and Manganelli (2004) proposed to use dynamic quantile regression to model VaR directly, introducing some conditional autoregressive VaR (CAViaR) dynamic quantile models. However, Manganelli and Engle (2004) noted that the classical numerical optimisation procedure used for estimation in this model family could be problematic or inefficient at very low quantiles, and presumably at "low" sample sizes as well, where "low" also depends on the relevant quantile level. This paper develops a Bayesian MCMC estimator for CAViaR models, taking advantage of the likely gains in efficiency that numerical integration, used to estimate the posterior mean, can display over the standard numerical differentiation based approaches in Engle and Manganelli(2004).

GARCH-type models, see Engle (1982) and Bollerslev (1986), with parametric specified error distributions, are also popular in VaR forecasting, see e.g. Chen and So (2006). Manganelli and Engle (2004) found that the basic CAViaR model could out-perform standard GARCH and historical simulation approaches for forecasting VaR from data simulated from a range of dynamic models, especially for fat-tailed error processes. We show that this result also holds for some real market return data, at the 1% quantile level. Giacomini and Komunjer (2005) developed an encompassing test for forecast models and find that a CAViaR model is most useful at the 1% quantile level, but that a simple GARCH model with Gaussian errors outperforms the CAViaR model at the 5% level, for the US S&P500 return data from 1985-2001. We extend these results in this paper across ten international market indices, including the US S&P500, finding that this conclusion generalises across the markets considered, in a more up-to-date time period. Further, Geweke and Keane (2007) propose a smoothly mixing regression approach to semi-parametrically and flexibly estimate predictive densities for observations, that can capture some types of heteroscedasticity. This approach outperformed GARCH models for the US S&P 500 index from 1995-1999. Finally, many financial institutions simply employ sample return percentiles, called historical simulation, to estimate VaR. We consider all the above-mentioned VaR forecasting approaches in this paper.

Black (1976) first discovered the asymmetric volatility phenomenon in financial markets. Many nonlinear models have been proposed to capture this trait, including some simple nonlinear CAViaR specifications. In this paper we extend the existing CAViaR model forms into a fully nonlinear family of dynamic models, in the spirit of threshold GARCH modelling (see Zakoian, 1994 and Brooks, 2001). We employ an adaptive MCMC sampling scheme, extending work by Yu and Moyeed (2001), Tsionas (2003), Geraci and Bottai (2007) and Chen and So (2006), to facilitate efficient Bayesian estimation, inference and forecasting for the proposed dynamic quantile model family. The proposed model and MCMC methods are examined firstly through a simulation study, and secondly through application to various financial market stock indices in a study of VaR forecasting. The simulation study illustrates favourable estimation performance, in terms of efficiency, compared with numerically optimising the quantile criterion function. The empirical study puts forward evidence of strong nonlinearity in dynamic quantiles in financial markets, supporting the extended fully nonlinear CAViaR model. Secondly, the VaR forecasting study illustrates that CAViaR models performed favourably compared to RiskMetrics and general GARCH estimators at VaR forecasting during our forecast period of 2005 to 2007, especially at the 1% quantile while simpler models, including GARCH and RiskMetrics did well at the 5% quantile level.

Section 2 discusses dynamic quantiles and the link with the SL distribution. Section 3 proposes the extended family of dynamic autoregressive quantile models. Section 4 presents the MCMC methods employed here. Sections 5 and 6 respectively present the simulation and empirical studies. Section 7 concludes.

2 Dynamic quantiles and Value at Risk

This section discusses the general dynamic quantile problem and the relation to the SL distribution.

2.1 General problem

The general dynamic quantile regression problem may be written:

$$y_t = f_t(\boldsymbol{\beta}, \boldsymbol{x}_{t-1}) + u_t, \tag{1}$$

where y_t is a dynamic observation at time t; $\boldsymbol{x}_{t-1}$ is a set of explanatory variables, which could include lagged values of the response y_{t-k}; $k > 0$; $\boldsymbol{\beta}$ are the unknown parameters and u_t is an unknown error term, with a generally unspecified distribution. The exogenous variables are lagged so as to accommodate forecasting, which is highly relevant in the financial examples we consider. The function $f_t(.)$ defines the dynamic link between the response y_t and the explanatory information $\boldsymbol{x}_{t-1}$. It is usually linear in the parameters and $\boldsymbol{x}$, an aspect that will be extended here. The conditional quantile at level α is then:

$$q_\alpha(y_t|\boldsymbol{\beta}, \boldsymbol{x}_{t-1}) = f_t(\boldsymbol{\beta}_\alpha, \boldsymbol{x}_{t-1})$$

where $\boldsymbol{\beta}_\alpha$ is the solution to:

$$\min_{\boldsymbol{\beta}} \sum_t \rho_\alpha \left(y_t - f_t(\boldsymbol{\beta}, \boldsymbol{x}_{t-1})\right). \tag{2}$$

The function $\rho(.)$ is a loss function, specified as $\rho_\alpha(u) = u(\alpha - I(u < 0))$.

2.2 The Skewed-Laplace connection

Yu and Moyeed (2001) and Tsionas (2003) illustrated the link between the solution to the quantile estimation problem and the likelihood for the SL distribution, as follows. The SL location-scale family ($SL(\mu, \tau, \alpha)$) of distributions has density function:

$$p_\alpha(u) = \frac{\alpha(1-\alpha)}{\tau} \exp\left[-\rho_\alpha\left(\frac{u-\mu}{\tau}\right)\right], \tag{3}$$

where μ is the mode and $\tau > 0$ is a scale parameter. If it is assumed, in model (1), that $u_t \sim SL(0, \tau, \alpha)$ and is i.i.d., then the likelihood function becomes:

$$L_\alpha(\boldsymbol{\beta}, \tau; \boldsymbol{y}, \boldsymbol{X}) \propto \tau^{-n} \exp\left\{-\tau^{-1} \sum_{t=1}^{n} (y_t - f_t(\boldsymbol{\beta}))\left[\alpha - I_{(-\infty,0)}(y_t - f_t(\boldsymbol{\beta}))\right]\right\}. \tag{4}$$

Since (2) is contained in the exponent of the likelihood, the maximum likelihood estimate for $\boldsymbol{\beta}$ is equivalent to the quantile estimator in (2).

It is important to emphasize that, though we treat (4) exactly as a likelihood function, it is not actually assumed that $\boldsymbol{y}$ follows a Skewed-Laplace distribution. The likelihood is only employed as it leads to an estimator that is mathematically equivalent to (2). This allows us to consider powerful computational methods, such as adaptive MCMC algorithms, that employ numerical integration (which can be made arbitrarily accurate), instead of numerical optimisation. Authors such as Yu and Moyeed (2001) and others, have illustrated that accurate estimation and inference was achieved under this approach. We further investigate and add to these findings for dynamic nonlinear quantile models.

2.3 Value at Risk

The Basel Capital Accord, originally signed by the Group of Ten countries in 1988, requires Authorized Deposit-taking Institutions (ADIs) to hold sufficient capital to provide a cushion against unexpected losses. Value-at-Risk (VaR) is a procedure designed to forecast the worst expected loss over a given time interval under normal market conditions, at a given confidence level α (Jorion, 1996). That is,

$$\alpha = Pr(y_t < -VaR_t | \boldsymbol{y}^{1,t-1})$$

The VaR is thus proportional to a quantile in the conditional one-step-ahead forecast distribution for the observations. Manganelli and Engle (2004) categorised VaR methods as:

1. **Parametric**: makes a fully parametric distributional and model form assumption. e.g. GARCH with Gaussian errors.

2. **Non-parametric**: minimal distributional and dynamic assumptions; e.g. historical simulation (i.e. using past sample quantiles).

3. **Semi-parametric**: some assumptions are made, either about the error distribution, or the model dynamics, but not both; e.g. quantile regression (CAViaR).

Monte Carlo (MC) simulation methods are commonly used in all three classes above. For instance, multi-step ahead VaR estimates are not available in closed form for many models when estimated from daily returns (e.g. GARCH models). MC methods are used in these cases, or similar bootstrap-type methods could be used in a truly non-parametric situation. MCMC methods, as used in this paper, fit into this class also.

A popular method for VaR estimation is RiskMetrics (RM), proposed by J.P. Morgan in 1996, where an IGARCH(1,1) process, with no mean equation, is employed. For expositional purposes, the standard GARCH(1,1) model is:

$$\begin{aligned} y_t &= \mu + a_t \ ; \ a_t = \sigma_t \epsilon_t, \quad \epsilon_t \overset{\text{i.i.d.}}{\sim} D(0,1) \\ \sigma_t^2 &= \alpha_1 + \alpha_2 \sigma_{t-1}^2 + \alpha_3 a_{t-1}^2, \end{aligned}$$

with the IGARCH(1,1) RM model a special case that sets: $D \equiv N(0,1)$, $\mu = \alpha_0 = 0$, $\alpha_1 = 1 - \beta_1$ and $\beta_1 = 0.94$ (for daily data); and is thus non-stationary in volatility. Standard GARCH theory (e.g. see Tsay, 2005) allows closed form solutions to the one-step-ahead quantiles of $y_t|\boldsymbol{y}^{1,t-1}$, given the parameter values and based on parametric errors. The standard GARCH(1,1) with Gaussian errors is labeled GARCH-n, while the GARCH(1,1) with standardised Student-t errors is denoted GARCH-t.

2.4 CAViaR models for VaR

Engle and Manganelli (2004) proposed various dynamic quantile functions $f(.)$, called conditional autoregressive Value at Risk (CAViaR) models. We initially discuss three of their specifications:

Indirect GARCH(1,1)(IG):

$$f_t(\boldsymbol{\beta}) = [\beta_1 + \beta_2 f_{t-1}^2(\boldsymbol{\beta}) + \beta_3 y_{t-1}^2]^{1/2}. \tag{5}$$

This equation is exactly equivalent to the dynamic quantile function for a GARCH(1,1) model with an i.i.d. symmetric error distribution and mean $\mu = 0$. The model thus allows efficient estimation for GARCH(1,1) quantiles with unspecified error distribution. This is an advantage: GARCH models are estimated under a parametric error distribution. However, it is well known that GARCH models tend to over-estimate volatility in general (e.g. see Chen, Gerlach and Lin, 2008) and over-react to large return shocks (since they are squared). As such we prefer the absolute value model types below:

Symmetric Absolute Value (SAV):

$$f_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 f_{t-1}(\boldsymbol{\beta}) + \beta_3 |y_{t-1}| \,. \tag{6}$$

The quantile again responds symmetrically to the lagged return y_{t-1}. This equation is equivalent to the quantile function for a standard deviation GARCH model (e.g. see Zakoian, 1994), where:

$$\sigma_t = \gamma_1 + \gamma_2 \sigma_{t-1} + \gamma_2 |a_{t-1}|.$$

The two CAViaR models SAV and IG have symmetric responses to positive and negative observations. To account for financial market asymmetry, via the leverage effect (Black, 1976), the SAV model was extended in Engle and Manganelli (2004) to:

Asymmetric Slope (AS):

$$f_t(\boldsymbol{\beta}) = \beta_1 + \beta_2 f_{t-1}(\boldsymbol{\beta}) + \left(\beta_3 I_{(y_{t-1}>0)} + \beta_4 I_{(y_{t-1}<0)}\right) |y_{t-1}|\,, \tag{7}$$

where the dynamic quantile function can respond differently to positive and negative responses. Such a threshold nonlinear model is similar in spirit to the GJR-GARCH model (Glosten, Jaganathan and Runkle, 1993) or EGARCH (Nelson, 1991), where asymmetry is modelled by adding one parameter only, and hence the types of asymmetry captured are limited. Again the AS model corresponds to a standard deviation GJR-GARCH model with mean $\mu = 0$ where:

$$\sigma_t = \delta_1 + \delta_2 \sigma_{t-1} + \delta_3 I_{(a_{t-1}>0)} |a_{t-1}| + \delta_4 I_{(a_{t-1}<0)} |a_{t-1}|.$$

We extend these CAViaR models in Section 3 to capture more flexible and complete asymmetric responses, via more general threshold nonlinear forms.

3 Proposed nonlinear dynamic quantile family

Li and Li (1996) and Brooks (2001) extend the simple 1st generation ARCH and GARCH models to be fully threshold nonlinear: i.e. all parameters in the volatility (and mean) equations were allowed to change between regimes, based on an observed threshold variable. These models were natural extensions of the original threshold autoregressive (TAR) model of Tong (1978), so as to allow fully threshold nonlinear dynamic volatility (and mean). In this same spirit, it is natural to extend the SAV and AS (CAViaR) models to the:

Threshold CAViaR (T-CAViaR):

$$f_t(\boldsymbol{\beta}) = \begin{cases} \beta_1 + \beta_2 f_{t-1}(\boldsymbol{\beta}) + \beta_3 |y_{t-1}|, & z_{t-1} \leq r \\ \beta_4 + \beta_5 f_{t-1}(\boldsymbol{\beta}) + \beta_6 |y_{t-1}|, & z_{t-1} > r \end{cases} \quad (8)$$

Here $\boldsymbol{z}$ is an observed threshold variable which could be exogenous, or self-exciting i.e. $z_t = y_t$ and r is the threshold value, typically set as $r = 0$, or estimated, though empirically many estimates in the literature are not significant from zero; as such we fix $r = 0$, which also makes the T-CAViaR a direct extension of the AS-CAViaR model in 7. We consider both exogenous and self-exciting thresholds in this paper. The exogenous threshold is the return on the US S&P500 index and the model using that threshold is denoted as T-CAViaR-x. The self-exciting model is denoted T-CAViaR.

Here each parameter in the dynamic quantile function can respond differently to positive and negative responses. We call this the T-CAViaR family since it includes the SAV ($r = \infty$) and AS ($r = 0$, β_4=β_1 and β_5=β_2) CAViaR models as special cases. Once again, this model is the semi-parametric equivalent of the standard deviation T-GARCH model with mean $\mu = 0$:

$$\sigma_t = \begin{cases} \delta_1 + \delta_2 \sigma_{t-1} + \delta_3 |a_{t-1}|, & z_{t-1} \leq 0 \\ \delta_4 + \delta_5 \sigma_{t-1} + \delta_6 |a_{t-1}|, & z_{t-1} > 0 \end{cases} \quad (9)$$

We choose to focus on this absolute value CAViaR model-type. However, a corresponding T-CAViaR-IG model could be specified as:

Threshold Indirect-GARCH (T-CAViaR-IG):

$$f_t(\boldsymbol{\beta}) = \begin{cases} \left[\beta_1 + \beta_2 f_{t-1}^2(\boldsymbol{\beta}) + \beta_3 y_{t-1}^2\right]^{1/2}, & z_{t-1} \leq r \\ \left[\beta_4 + \beta_5 f_{t-1}^2(\boldsymbol{\beta}) + \beta_6 y_{t-1}^2\right]^{1/2}, & z_{t-1} > r \end{cases} \quad (10)$$

We limit focus in this paper to the TCAV model family (8). These models are non-parametric in their error specifications, but simply extend the existing forms for the dynamic function $f(.)$ to be fully threshold nonlinear: i.e. they are semi-parametric.

Dynamic models typically have constraints or restrictions on the parameters for stationarity (or positivity of dynamic variances). However, such restrictions are difficult to locate or derive for CAViaR models, and we choose not to set any in this paper, as in Engle and Manganelli (2004).

4 Bayesian methods

Bayesian methods generally require the specification of a likelihood function and a prior distribution. The likelihood function for the T-CAViaR is completely specified by (4) and (8). We now specify the prior distribution.

4.1 Prior and Posterior densities

We choose the prior to be uninformative over the possible region for the regression-type parameters $\boldsymbol{\beta}$, The joint prior is thus:

$$\pi(\boldsymbol{\beta}) \propto 1.$$

which is equivalent to a flat prior on $\boldsymbol{\beta}$ over the real line, in six dimensions. This prior distribution can have no effect on parameter estimation, in this metric for $\boldsymbol{\beta}$, since it is constant. All estimation will thus be based on the likelihood function alone. Yu and Moheed (2001) showed that the posterior in $\boldsymbol{\beta}$ was proper, under this improper prior, for general quantile regression models.

Using (4), (8), plus the standard Jeffreys prior $\pi(\tau) \propto \tau^{-1}$, the joint posterior density for $\boldsymbol{\beta}, \tau|\boldsymbol{y}$ is:

$$\begin{aligned} p(\boldsymbol{\beta}, \tau|\boldsymbol{y}) &\propto p(\boldsymbol{\beta}, \tau) L_\alpha(\boldsymbol{\beta}, \tau; \boldsymbol{y}) \\ &\propto \tau^{-(n+1)} \exp\{-\tau^{-1} \sum_{t=2}^{n} \rho_\alpha \left(y_t - f_t(\boldsymbol{\beta})\right)\}, \end{aligned} \tag{11}$$

which is in the form of an inverse gamma density in τ. Since estimation of τ is not relevant in VaR forecasting, this parameter was integrated out of $p(\boldsymbol{\beta}, \tau|\boldsymbol{y})$, to obtain the marginal posterior for $\boldsymbol{\beta}|\boldsymbol{y}$, via direct integration:

$$\begin{aligned} p(\boldsymbol{\beta}|\boldsymbol{y}) &= \int p(\boldsymbol{\beta}, \tau|\boldsymbol{y}) d\tau \\ &\propto \left[\sum_{t=2}^{n} \rho_\alpha \left(y_t - f_t(\boldsymbol{\beta})\right)\right]^{-n}, \end{aligned} \tag{12}$$

using the fact that the inverse gamma density integrates to 1. This posterior is not in a form permitting direct inference on $\boldsymbol{\beta}$. We thus turn to computational methods for estimation and inference.

4.2 Adaptive MCMC sampling using Metropolis methods

If a sample could be directly simulated from $p(\boldsymbol{\beta}|\boldsymbol{y})$ then it could be chosen to be an independent Monte Carlo sample. However, this is not possible given the non-standard form, so instead a dependent (Markov chain) Monte Carlo sample is obtained from (12) via the Metropolis and Metropolis-Hastings (MH) (Metropolis *et al.*, 1953; Hastings, 1970) algorithms. To speed convergence and to allow optimal mixing properties in the chain, an adaptive MCMC algorithm for $\boldsymbol{\beta}|\boldsymbol{y}$ is employed that combines a random walk Metropolis (RW-M) and an independent kernel (IK-) MH algorithm.

We extend the sampling scheme from Chen and So (2006), who used Gaussian proposal densities. Such a proposal can get 'stuck' in local modes and take a large number of iterates to move out of that area of the posterior. To improve on this aspect, for the burn-in period iterations, a Student-t proposal distribution, with low degrees of freedom (e.g. df $= 5$), is employed in a RW-M algorithm. The scale matrix, which might initially be chosen as e.g. diagonal with positive values, is subsequently tuned to achieve optimal acceptance rates, between 15% and 50%, as recommended in Gelman *et al.* (1996). After the burn-in period, the sample mean vector and sample var-cov matrix are formed using these M iterates of $\boldsymbol{\beta}$. These are subsequently employed in the sampling period (iterations $M + 1$ to N) as the mean and scale matrix for another Student-t proposal distribution (again with low degrees of freedom) in an IK-MH algorithm.

This adaptive proposal updating procedure will speed mixing in the posterior distribution, over that for the simple RW-M method, as long as the burn-in period has 'covered' the posterior distribution. See Chen and So (2006) for more details of the Gaussian proposal case, but the Student-t proposals employed here will further assist in achieving coverage and mixing, over the Gaussian, for both the burn-in and sampling periods, by lowering the probability of getting stuck in local modes for long periods and allowing for occasional large jumps around the posterior space. We extensively examined trace plots and autocorrelation function (ACF) plots, from multiple runs of the MCMC sampler, from differing starting points, for each element of $\boldsymbol{\beta}$, so as to confirm convergence and to infer adequate coverage. We also employed Gelman's 'R' statistic, which is typically very close to 1 for our MCMC iterates, implying fast mixing and good convergence properties, for all parameters in this sampling scheme.

5 Simulation Study

The results from a simulation study, focusing on the full T-CAV model only, are now discussed. Results from an extended study employing the AS and SAV models are available from the authors on request: the results are very similar to those presented here. The simulation study serves to demonstrate an example of the comparative efficiency of the proposed MCMC method and the usual numerically optimised quantile method (2), both in parameter estimation and in quantile estimation/forecasting. While accurate results can be expected under both methods, a comparison is relevant for a number of reasons. Non-informative (flat) Bayesian priors are employed, so that as sample size increases Bayesian posterior mean and the standard maximum likelihood (ml) estimator will converge. However, low quantile levels ($\alpha = 0.05, 0.01$) are examined here and it is well known that standard errors for quantile regression estimates increase, for fixed sample size n, as quantile levels depart from $\alpha = 0.5$. There are two possible sources for differences between Bayesian and ml estimators for a CAViaR model: (i) the likelihood ($\equiv$ posterior here) in $\boldsymbol{\beta}$ is skewed; i.e. the central limit theorem may not be a good approximation, for the chosen n, and the posterior may not be symmetric. In this case the Bayesian posterior mean will be different to the posterior mode ($\equiv$ mle and classical estimator here); and (ii) both estimators are numerical in nature. The posterior mean uses numerical integration, which can be made arbitrarily accurate, this accuracy controlled by the number of MC iterates employed. However the mle employs a numeric search algorithm, effectively employing numerical differentiation. This procedure will become less accurate as the dimension of the model grows, especially for "smaller" sample sizes, and may give rise to a less efficient estimator due to this numerical inaccuracy. The level of numerical accuracy of the mle if difficult to control, except by increasing n, which is fixed here.

A specific choice of parametric error distribution is needed in order to simulate data for quantile models. We focus on the GARCH family for this purpose, choosing the equivalent, to the full T-CAViar model, standard deviation T-GARCH model in 9.

5.1 Simulation Set-up and Results

Samples of size $n = 2000$ are simulated from the T-GARCH model with Student-t errors. The full T-CAV model is then fit to this dataset, once using the MCMC method in Section 4 and secondly using the standard optimisation routine, employing the 'fminsearch' routine in Matlab software, to numerically minimise (2). The Matlab code employed was adapted, and extended for the full T-CAV model, from freely available code written by Simone Manganelli; downloadable from
http://www.simonemanganelli.org/Simone/Research.html.
VaR estimates for the in-sample data points, as well as a 1-step-ahead forecast for VaR at time $n + 1 = 2001$ were also calculated. The T-GARCH-t model is specified as:

$$\begin{aligned}
y_t &= a_t, \\
a_t &= \sigma_t \varepsilon_t, \quad \varepsilon_t \overset{\text{i.i.d.}}{\sim} t_6^*, \\
\sigma_t &= \begin{cases} 0.2 + 0.03|a_{t-1}| + 0.95\sigma_{t-1}, & \text{if} \quad a_{t-1} \leq 0, \\ 0.05 + 0.15|a_{t-1}| + 0.75\sigma_{t-1}, & \text{if} \quad a_{t-1} > 0. \end{cases}
\end{aligned}$$

Here ν is the degrees of freedom, set equal to 6, and t^* represents the Student-t distribution, standardised to have unit variance. The true 1-step-ahead α level quantile is then $q_\alpha(y_{t+1}|\boldsymbol{\beta}) = \sigma_{t+1} T_6^{-1}(\alpha)\sqrt{\frac{4}{6}}$, where T_6^{-1} is the inverse cdf for the Student-t distribution.

Following Basel II risk management guidelines, quantile levels of $\alpha = 0.01, 0.05$ were considered. 400 datasets were simulated from this model, with results summarised in Table 1. It is straightforward to show that the corresponding parameters of the T-CAViaR model are: $\beta_1(\alpha) = 0.2T_6^{-1}(\alpha)\sqrt{\frac{4}{6}}$; $\beta_2 = 0.95$; $\beta_3(\alpha) = 0.03T_6^{-1}(\alpha)\sqrt{\frac{4}{6}}$; while $\beta_4(\alpha) = 0.05T_6^{-1}(\alpha)\sqrt{\frac{4}{6}}$; $\beta_5 = 0.75$ and $\beta_3(\alpha) = 0.15T_6^{-1}(\alpha)\sqrt{\frac{4}{6}}$; giving the true parameter values in Table 1.

The total MCMC sample size was $N = 40000$, with a burn-in of $M = 15000$ iterations. Initial MCMC iterates were randomly set in the interval $(0, 1)$ for each parameter. For the standard numerical estimator, a grid of starting values were chosen for each parameter (from $[-1, 1]$) and then the simplex method (default in Matlab's 'fminsearch' function) was used to find parameter estimates, starting from each combination of grid values, that minimised (2); the final estimates were those that gave the global minimum value of (2)

Table 1: Summary statistics for the T-CAV model for simulated data from a T-GARCH-t model.

MCMC		α=1%				α=5%		
par.	True	Mean	Std.	95% C.I.	True	Mean	Std.	95% C.I.
β_1	-0.513	-0.532	**0.524**	(-1.62, 0.59)	-0.317	-0.333	0.180	(-0.72, 0.03)
β_2	0.95	**0.946**	**0.147**	(0.64, 1.25)	0.95	0.941	0.084	(0.77, 1.11)
β_3	-0.077	-0.079	**0.224**	(-0.58, 0.27)	-0.048	**-0.047**	**0.073**	(-0.22, 0.10)
β_4	-0.128	**-0.231**	**0.460**	(-1.29, 0.33)	-0.079	-0.106	**0.148**	(-0.51, 0.12)
β_5	0.75	**0.719**	**0.145**	(0.40, 0.95)	0.75	-0.734	**0.080**	(0.54, 0.86)
β_6	-0.385	-0.407	**0.190**	(-0.83,-0.10)	-0.238	-0.255	**0.079**	(-0.43,-0.13)
$\beta_1 - \beta_4$	-0.385	**-0.301**	**0.853**	(-1.60, 1.99)	-0.238	**-0.227**	**0.290**	(-0.80, 0.45)
$\beta_2 - \beta_5$	0.20	**0.227**	**0.233**	(-0.19, 0.77)	0.20	**0.207**	**0.133**	(-0.04, 0.50)
$\beta_3 - \beta_6$	0.308	0.329	**0.279**	(-0.24, 0.85)	0.190	0.208	**0.106**	(0.00, 0.41)
$\widehat{VaR}_{n+1}$	-3.909	**-3.922**	1.449	(-7.76,-1.76)	-2.417	**-2.415**	**0.845**	(-4.61,-1.05)
VaR_{n+1}		-3.909	1.349	(-7.28,-1.71)		-2.417	0.834	(-4.50,-1.06)
$\widehat{VaR}_{n+1} - VaR_{n+1}$	0	**-0.013**	0.631	(-1.36, 1.10)	0	**0.002**	**0.223**	(-0.41, 0.45)
MAD		**0.432**	**0.154**	(0.18, 0.79)		**0.162**	**0.054**	(0.08, 0.29)
MedAD		**0.327**	**0.111**	(0.14, 0.58)		**0.124**	**0.045**	(0.06, 0.23)
RMSE		**0.572**	**0.285**	(0.20, 1.24)		**0.207**	**0.072**	(0.09, 0.37)
Quantile		α=1%				α=5%		
par.	True	Mean	Std.	95% C.I.	True	Mean	Std.	95% C.I.
β_1	-0.513	**-0.496**	0.541	(-1.55, 0.77)	-0.317	**-0.320**	**0.179**	(-0.68, 0.07)
β_2	0.95	0.955	0.153	(0.65, 1.29)	0.95	**0.948**	**0.083**	(0.77, 1.12)
β_3	-0.077	**-0.076**	0.233	(-0.62, 0.28)	-0.048	-0.041	0.074	(-0.21, 0.10)
β_4	-0.128	-0.246	0.529	(-1.74, 0.32)	-0.079	**-0.096**	0.152	(-0.50, 0.13)
β_5	0.75	0.718	0.158	(0.33, 0.94)	0.75	**0.743**	0.081	(0.54, 0.87)
β_6	-0.385	**-0.402**	0.195	(-0.85,-0.09)	-0.238	**-0.248**	**0.079**	(-0.42,-0.11)
$\beta_1 - \beta_4$	-0.385	-0.250	0.938	(-1.63, 2.21)	-0.238	-0.225	0.303	(-0.79, 0.54)
$\beta_2 - \beta_5$	0.20	0.237	0.252	(-0.16, 0.80)	0.20	**0.205**	0.139	(-0.04, 0.52)
$\beta_3 - \beta_6$	0.308	**0.327**	0.288	(-0.23, 0.84)	0.190	0.207	0.108	(-0.00, 0.42)
$\widehat{VaR}_{t+1}$	-3.909	-3.924	**1.443**	(-7.74,-1.76)	-2.417	-2.401	0.854	(-4.60,-1.06)
VaR_{t+1}		-3.909	1.349	(-7.28,-1.71)		-2.417	0.834	(-4.50,-1.06)
$\widehat{VaR}_{n+1} - VaR_{n+1}$	0	-0.015	**0.629**	(-1.35, 1.08)	0	0.017	0.233	(-0.44, 0.48)
MAD		0.445	0.162	(0.19, 0.79)		0.166	0.056	(0.08, 0.30)
MedAD		0.336	0.117	(0.14, 0.60)		0.127	0.046	(0.06, 0.23)
RMSE		0.590	0.293	(0.22, 1.35)		0.213	0.076	(0.10, 0.40)

across all starting positions. This method was far slower than the MCMC, for the full TCAViaR model. The MCMC method typically took just over 1 minute (i.e. $65 - 70$ seconds) at $n = 2000$, while the standard optimisation method took around 5-6 minutes, for each dataset. However, for the AS and SAV models, the MCMC method took 60-65 seconds (AS) and 45-50 seconds (SAV), while the standard estimator took only 25-30 seconds (AS) and 5-10 seconds (SAV), in Matlab on a standard desktop PC. Adding extra parameters to the model makes a significantly greater impact to running time for the standard quantile estimator, since it involves a grid search.

For the MCMC method, Table 1 reports the average of the 400 posterior means for each parameter, their standard deviation (Std.) from each true value, as well as the 95% interval for the estimates. The same summaries are shown for the true (simulated) VaR at $t = n + 1 = 2001$, the MCMC 1 step-ahead forecast VaR for $t = 2001$, and the difference between forecast and true VaR by MCMC. This same information is also shown for the traditional quantile parameter estimates and forecasts of VaR_{n+1}. Finally, the measures mean absolute deviation (MAD), median absolute deviation (MedAD) and root mean square error (RMSE) were used to assess the competing MCMC and traditional quantile in-sample estimates of the VaR, for all 2000 observations. For the traditional quantile estimator, the average estimate (mle) is shown, as are the standard deviations of the mles from the true value and a 95% interval over the 400 parameter estimates.

We consider estimation bias and precision (efficiency). Regarding bias, both methods parameter estimates average close to the true values across both quantile levels. Values highlighted in bold are closer to the true value: i.e. optimal among the two methods. Clearly, the methods are comparable regarding estimation bias. However, regarding precision, the MCMC estimates are almost all closer in squared error terms to the true value than the standard quantile estimators. While the differences are mostly marginal, the agreement across parameters in this aspect is striking. Clearly the MCMC estimator is more precise than the standard estimator for this model, over the 400 datasets generated, at $n = 2000$. We have found this result also holds for different parameter value sets and across the AS and SAV models also. Further, this result also carries over to VaR estimation in sample, with all accuracy measures favouring the MCMC estimates. However, 1 step-ahead forecast VaR results are mixed and comparable between the two methods. In short, the MCMC method should be the preferred estimator, at least for the full T-CAViaR model, as it gives comparable levels of bias and marginally but clearly better precision, more efficiency, in parameter and in-sample VaR estimation at this sample size.

This is an interesting result given that asymptotically, Bayesian estimation under diffuse priors and standard estimates should yield the same results. An examination of MCMC posterior distributions for each parameter $\boldsymbol{\beta}$ usually revealed significant negative skewness, but not significant excess kurtosis, allowing a Jarque-Bera test to reject independent normality, suggesting that indeed the posterior mean could be different to the mle at $n = 2000$. However, since the MCMC and standard parameter estimates are highly similar in terms of bias, it is more likely that it is the inherent difference in numerical estimation paradigms, i.e. numerical integration vs numerical optimisation, that is driving the observed differences in efficiency of estimation. This is likely a sample size issue, since the function (2) weights each observation error $y_t - f_t(\boldsymbol{\beta})$ by α for positive errors and by $1 - \alpha$ for negative errors. For $\alpha = 0.01$ and $n = 2000$ then, ≈ 1980 errors or sample observations would receive the very small weight of $\alpha = 0.01$, while only ≈ 20 of the errors or sample points would receive the much higher weight of $1 - \alpha = 0.99$. As expected, for $\alpha = 0.01$, the MCMC and ml estimators do converge by about $n = 10000$ for this model, while for $\alpha = 0.05$ convergence is clear by around $n = 5000$, as found by further simulations not reported here.

6 Testing VaR models

A common non-test criterion to compare VaR models is the rate of violation, defined as the proportion of observations for which the actual return is more extreme than the forecasted VaR level, over the forecast period. The violation rate is:

$$\text{VRate} = \frac{\sum_{t=n+1}^{n+m} I(y_t < -\text{VaR}_t)}{m},$$

where n is the learning sample size and m is the forecast or test sample size. A forecast model's VRate should be close to the nominal level α. We employed the ratio VRate/α, to help compare the competing models, where models with VRate/$\alpha \approx 1$ are most desirable. However, as in Wong and So (2003), when VRate $< \alpha$, risk and loss estimates are conservative (higher than actual), while alternatively, when VRate $> \alpha$, risk estimates are lower than actual and financial institutions may not allocate sufficient capital to cover likely future losses. Here solvency outweighs profitability and for models where VRate/α

are equidistant from 1, lower rates are preferred; e.g. VRate/$\alpha = 0.9$ is preferred to VRate/$\alpha = 1.1$.

We further consider three standard hypothesis-testing methods for evaluating and testing the accuracy of VaR models: the unconditional coverage (UC) test of Kupiec (1995): a likelihood ratio test that the true violation rate equals α; the conditional coverage (CC) test of Christoffersen (1998): a joint test, combining a likelihood ratio test for independence of violations and the unconditional coverage test; and the Dynamic Quantile (DQ) test of Engle and Manganelli (2004). Both the CC and DQ are joint tests where the null hypothesis consists of: independence of a model's violations, equivalently correct conditional violation rate for a given model, combined with a correct unconditional violation rate. The DQ test is well-known to be more powerful than the CC test, see Berkowitz, Christoffersen and Pelletier (2007). These tests are now standard; we refer readers to the original papers for details.

Table 2: Summary statistics: stock index returns for ten stock markets.

	Japan	France	Germany	Italy	U.K.
Mean	0.016	-0.0033	0.0027	0.0034	0.0014
Variance	1.955	2.05	2.692	1.226	1.2903
Skewness	-0.080	-0.0411	-0.0993	-0.384	-0.223
	(0.21)	(0.51)	(0.11)	(<0.001)	(0.0003)
Excess kurtosis	1.518	3.504	3.103	5.069	3.946
	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)
	Canada	U.S.	Taiwan	Hong Kong	Australia
Mean	0.025	0.0062	0.0309	0.0201	0.037
Variance	0.760	1.156	2.109	1.3540	0.443
Skewness	-0.465	0.162	-0.018	-0.313	-0.566
	(<0.001)	(0.010)	(0.78)	(<0.001)	(<0.001)
Excess kurtosis	3.842	2.864	1.867	3.772	3.796
	(<0.001)	(<0.001)	(<0.001)	(<0.001)	(<0.001)

Note: P-values based on asymptotic normality are listed, under a null hypothesis of 0.

7 Empirical Results

We considered ten daily international stock market indices: the S&P 500 (US); FTSE 100 (UK); CAC 40 (France); Dax 30 (Germany); Milan MIBTel Index (Italy); Toronto SE 300 (Canada); AORD All ordinaries index (Australia); Nikkei 225 Index (Japan); TSEC weighted index (Taiwan) and the HANG SENG Index (Hong Kong). Data were obtained from Datastream International and covered the period from January 1, 2001 to January 19, 2007. The log return series were generated by taking logarithmic differences of the daily price index, $y_t = (ln(P_t) - ln(P_{t-1})) \times 100$, where P_t is the closing price index on day t.

The full sample was divided into a learning sample: January 1, 2001 to January 10, 2005; and a forecast sample: the 500 trading days from January 11, 2005 to early to mid-January, 2007. Small differences in end-dates across markets occurred due to different market-specific non-trading days. Table 2 shows summary statistics from the full sample of these market indices including sample mean, variance, skewness and kurtosis. All ten return series display the standard properties of daily asset return data: they are heavy-tailed and mostly negatively skewed.

7.1 TCAViaR model estimation results

Tables 3-4 show parameter estimates for the full T-CAViaR model applied to the entire series for each market. The MCMC burn-in sample size was again 15000 iterations, followed by a sampling period of 25000 iterations. To assess mixing and convergence, the MCMC method was run from five different, randomly generated, starting positions, for each market, at $\alpha = 0.01, 0.05$. Each of the six parameters $(\beta_1, \ldots, \beta_6)$ had at least one starting value on either side of its reported posterior mean in Table 4. Convergence to the same posterior distribution was clear in all five runs for each parameter, in each case well before the end of the burn-in sample. A typical example of the inefficiency factors 'R' (see Gelman et al, 2005, pg 296) for each parameter over these five runs were 1.029, 1.034, 1.019, 1.041, 1.032 and 1.006; all highlighting fast mixing and clear and efficient convergence for the proposed sampling scheme.

Table 3: MCMC parameter estimates from T-CAV model fit to returns from major market indices.

		α=1%			α=5%		
	parameter	Mean	std.	95% C.I.	Mean	std.	95% C.I.
Japan	β_1	-0.374	0.054	(-0.470,-0.272)	-0.045	0.031	(-0.108, 0.011)
	β_2	0.771	0.019	(0.739, 0.805)	0.905	0.017	(0.871, 0.937)
	β_3	-0.378	0.018	(-0.412,-0.344)	-0.206	0.015	(-0.233,-0.177)
	β_4	0.012	0.036	(-0.056, 0.077)	-0.031	0.027	(-0.086, 0.023)
	β_5	0.927	0.014	(0.902, 0.954)	0.920	0.015	(0.890, 0.948)
	β_6	-0.203	0.015	(-0.235,-0.175)	-0.099	0.018	(-0.134,-0.063)
	β_1-β_4	-0.386	0.084	(-0.531,-0.232)	-0.015	0.053	(-0.124, 0.089)
	β_2-β_5	-0.157	0.029	(-0.207,-0.102)	-0.015	0.028	(-0.072, 0.039)
	β_3-β_6	-0.175	0.020	(-0.212,-0.137)	-0.106	0.021	(-0.149,-0.067)
France	β_1	-0.200	0.020	(-0.235,-0.160)	0.054	0.015	(0.024, 0.083)
	β_2	0.897	0.010	(0.878, 0.918)	0.972	0.007	(0.958, 0.986)
	β_3	-0.253	0.016	(-0.287,-0.220)	-0.232	0.013	(-0.259,-0.205)
	β_4	0.0024	0.022	(-0.042, 0.046)	-0.100	0.015	(-0.129,-0.071)
	β_5	0.904	0.008	(0.891, 0.921)	0.910	0.009	(0.892, 0.926)
	β_6	-0.115	0.021	(-0.157,-0.077)	0.032	0.014	(0.006, 0.059)
	β_1-β_4	-0.202	0.039	(-0.277,-0.129)	0.155	0.029	(0.096, 0.212)
	β_2-β_5	-0.007	0.013	(-0.029, 0.020)	0.062	0.013	(0.036, 0.088)
	β_3-β_6	-0.138	0.016	(-0.172,-0.107)	-0.264	0.020	(-0.307,-0.224)
Germany	β_1	-0.182	0.043	(-0.266,-0.109)	0.010	0.028	(-0.048, 0.054)
	β_2	0.882	0.016	(0.851, 0.908)	0.979	0.020	(0.935, 1.011)
	β_3	-0.303	0.022	(-0.353,-0.265)	-0.171	0.018	(-0.211,-0.141)
	β_4	-0.048	0.031	(-0.102, 0.014)	-0.075	0.019	(-0.107,-0.038)
	β_5	0.922	0.010	(0.903, 0.942)	0.928	0.010	(0.912, 0.950)
	β_6	-0.051	0.014	(-0.076,-0.024)	0.031	0.012	(0.007, 0.053)
	β_1-β_4	-0.134	0.073	(-0.274,-0.008)	0.086	0.046	(-0.010, 0.157)
	β_2-β_5	-0.041	0.024	(-0.086, 0.002)	0.051	0.027	(-0.011, 0.097)
	β_3-β_6	-0.253	0.028	(-0.315,-0.204)	-0.202	0.019	(-0.247,-0.171)
Italy	β_1	0.286	0.038	(0.208, 0.354)	-0.002	0.014	(-0.027, 0.027)
	β_2	1.070	0.011	(1.045, 1.091)	0.996	0.008	(0.980, 1.010)
	β_3	-0.469	0.014	(-0.495,-0.444)	-0.155	0.013	(-0.179,-0.131)
	β_4	-0.519	0.033	(-0.589,-0.451)	-0.055	0.011	(-0.078,-0.036)
	β_5	0.713	0.012	(0.689, 0.743)	0.911	0.009	(0.894, 0.926)
	β_6	0.036	0.012	(0.011, 0.057)	0.026	0.015	(-0.006, 0.052)
	β_1-β_4	0.805	0.070	(0.665, 0.937)	0.053	0.025	(0.009, 0.101)
	β_2-β_5	0.357	0.022	(0.304, 0.394)	0.085	0.013	(0.060, 0.113)
	β_3-β_6	-0.504	0.015	(-0.530,-0.476)	-0.181	0.021	(-0.222,-0.143)
UK	β_1	0.103	0.020	(0.061, 0.137)	0.029	0.019	(-0.008, 0.065)
	β_2	1.019	0.014	(0.994, 1.040)	0.966	0.001	(0.947, 0.985)
	β_3	-0.260	0.032	(-0.318,-0.213)	-0.279	0.015	(-0.308,-0.249)
	β_4	-0.169	0.017	(-0.200,-0.127)	-0.081	0.016	(-0.112,-0.048)
	β_5	0.853	0.014	(0.827, 0.883)	0.906	0.011	(0.883, 0.928)
	β_6	-0.045	0.025	(-0.098, 0.0001)	0.063	0.014	(0.036, 0.093)
	β_1-β_4	0.272	0.035	(0.192, 0.334)	0.110	0.034	(0.041, 0.175)
	β_2-β_5	0.167	0.019	(0.126, 0.208)	0.060	0.019	(0.021, 0.098)
	β_3-β_6	-0.215	0.025	(-0.268,-0.167)	-0.342	0.021	(-0.384,-0.301)

Table 4: MCMC parameter estimates from T-CAV model fit to returns from major market indices.

		α=1%			α=5%		
	Parameter	Mean	std.	95% C.I.	Mean	std.	95% C.I.
Canada	β_1	-0.143	0.025	(-0.186,-0.095)	-0.027	0.020	(-0.064,0.013)
	β_2	0.937	0.014	(0.915,0.960)	0.956	0.014	(0.926,0.985)
	β_3	-0.128	0.011	(-0.151,-0.105)	-0.149	0.014	(-0.177,-0.120)
	β_4	0.012	0.018	(-0.022,0.044)	-0.034	0.016	(-0.067,-0.004)
	β_5	0.965	0.014	(0.943,0.989)	0.922	0.012	(0.898,0.947)
	β_6	0.014	0.020	(-0.022,0.054)	-0.024	0.015	(-0.054,0.005)
	β_1-β_4	-0.154	0.042	(-0.225,-0.075)	0.008	0.034	(-0.061,0.078)
	β_2-β_5	-0.028	0.027	(-0.072,0.017)	0.034	0.023	(-0.014,0.078)
	β_3-β_6	-0.142	0.020	(-0.182,-0.111)	-0.125	0.020	(-0.164,-0.086)
US	β_1	-0.121	0.027	(-0.180,-0.067)	0.033	0.022	(-0.011,0.072)
	β_2	0.921	0.016	(0.887,0.954)	1.024	0.014	(0.996,1.052)
	β_3	-0.140	0.016	(-0.174,-0.112)	-0.076	0.010	(-0.095,-0.056)
	β_4	0.085	0.024	(0.036,0.135)	-0.079	0.019	(-0.113,-0.043)
	β_5	1.007	0.010	(0.983,1.030)	0.929	0.013	(0.905,0.954)
	β_6	-0.029	0.021	(-0.074,0.009)	0.035	0.009	(0.019,0.053)
	β_1-β_4	-0.206	0.051	(-0.317,-0.103)	0.113	0.039	(0.033,0.182)
	β_2-β_5	-0.086	0.023	(-0.130,-0.032)	0.095	0.025	(0.045,0.142)
	β_3-β_6	-0.112	0.015	(-0.151,-0.081)	-0.112	0.014	(-0.141,-0.087)
Taiwan	β_1	-0.076	0.126	(-0.290,0.249)	-0.100	0.031	(-0.161,-0.044)
	β_2	0.884	0.036	(0.825,0.980)	0.904	0.015	(0.873,0.934)
	β_3	-0.439	0.030	(-0.501,-0.380)	-0.240	0.020	(-0.277,-0.200)
	β_4	-0.758	0.149	(-0.964,-0.295)	-0.033	0.026	(-0.084,0.018)
	β_5	0.775	0.037	(0.721,0.889)	0.901	0.017	(0.867,0.934)
	β_6	0.070	0.019	(0.030,0.098)	-0.058	0.013	(-0.084,-0.032)
	β_1-β_4	0.682	0.237	(0.148,1.136)	-0.067	0.051	(-0.170,0.032)
	β_2-β_5	0.109	0.062	(-0.032,0.223)	0.004	0.025	(-0.047,0.053)
	β_3-β_6	-0.509	0.037	(-0.583,-0.437)	-0.183	0.022	(-0.223,-0.138)
HK	β_1	-0.003	0.102	(-0.188,0.214)	-0.103	0.033	(-0.169,-0.040)
	β_2	0.896	0.035	(0.836,0.969)	0.903	0.017	(0.872,0.938)
	β_3	-0.468	0.045	(-0.559,-0.367)	-0.240	0.020	(-0.277,-0.198)
	β_4	-0.969	0.093	(-1.144,-0.718)	-0.029	0.030	(-0.089,0.029)
	β_5	0.723	0.027	(0.672,0.792)	0.902	0.019	(0.865,0.938)
	β_6	0.089	0.014	(0.063,0.113)	-0.059	0.014	(-0.087,-0.033)
	β_1-β_4	0.966	0.130	(0.703,1.252)	-0.074	0.058	(-0.189,0.035)
	β_2-β_5	0.173	0.041	(0.090,0.266)	0.001	0.030	(-0.056,0.061)
	β_3-β_6	-0.557	0.050	(-0.653,-0.433)	-0.181	0.022	(-0.225,-0.139)
Australia	β_1	-0.242	0.016	(-0.275,-0.211)	-0.196	0.036	(-0.267,-0.126)
	β_2	0.771	0.010	(0.751,0.793)	0.798	0.038	(0.723,0.870)
	β_3	-0.496	0.021	(-0.538,-0.460)	-0.201	0.024	(-0.252,-0.157)
	β_4	0.001	0.015	(-0.030,0.030)	0.005	0.019	(-0.033,0.042)
	β_5	0.983	0.019	(0.947,1.023)	0.904	0.020	(0.865,0.943)
	β_6	0.116	0.018	(0.079,0.148)	-0.047	0.019	(-0.084,-0.011)
	β_1-β_4	-0.243	0.028	(-0.297,-0.188)	-0.200	0.052	(-0.305,-0.096)
	β_2-β_5	-0.200	0.020	(-0.238,-0.160)	-0.106	0.051	(-0.209,-0.005)
	β_3-β_6	-0.612	0.023	(-0.659,-0.565)	-0.154	0.029	(-0.211,-0.102)

Tables 3-4 indicate that the 10 markets roughly fit into three categories in terms of the parameter estimates obtained. Consider first $\alpha = 0.01$. Japan, France, Germany, Canada, US and Australia display similar estimates across parameters: a significantly negative intercept ($\beta_1 < 0$), medium positive persistence (β_2), significantly less than 1, and a large negative effect of the lagged return ($\beta_3 < 0$), all in the negative regime; while exhibiting small insignificant intercepts ($\beta_4 \approx 0$), higher positive persistence ($\beta_5 > \beta_2$) and smaller, mostly still negative and significant, lagged return effect ($0 > \beta_6 > \beta_3$) in the positive regime. For these markets, quantiles become more extreme (negative) following negative returns and are positively but not too strongly persistent (except the US), more so following positive returns. In contrast, the markets in Italy and the UK have significantly positive intercepts ($\beta_1 > 0$) and very strong positive persistence ($\beta_2 > 1$), with estimates above 1, significantly so for Italy, in the negative regime. The effects ($\beta_3 < \beta_6 < 0$) are similar to the first market grouping. In the positive regime, intercepts are now significantly negative ($\beta_4 < 0$), with lower positive persistence ($\beta_5 < \beta_2$). For these two markets, dynamic quantiles still become more extreme (negative) following negative returns but now are strongly positively persistent, more so following negative returns. Finally, Taiwan and Hong Kong have insignificant intercepts in the negative regime, as well as very strong lagged return effects ($\beta_3 << 0$). In the positive regime, intercepts are very high and negative ($\beta_4 << 0$) but lagged return effects are smaller and significantly positive ($|\beta_6| < |\beta_3|; \beta_6 > 0$). Quantile persistence in these markets is comparatively medium.

Tables 3-4 also show estimates and inferences for differences between corresponding parameters in the positive and negative regimes. These estimates indicate the level and direction of asymmetries between the two regimes. All markets show clear evidence of significant asymmetry in response to negative and positive returns, in their dynamic quantiles. The markets fell into the same three classes in their nonlinear behaviour. Firstly, the lagged return effect is consistently and significantly asymmetric across all ten markets. Dynamic quantiles become more extreme (negative) following negative returns than positive returns. For the first group above, each parameter difference is negative, but usually significantly so only for the intercept and the lagged return effect. For the second and third groups, intercepts and persistence are significantly higher in the negative regime.

7.2 Forecasting VaR study

VaR is forecasted 1 day ahead for each day in the forecast sample of 500 returns, using a range of competing models from all three quantile estimation classes: non-parametric, semi-parametric and fully parametric. Many financial institutions use sample return percentiles (called historical simulation) to forecast VaR; we follow their approach and employ a short-term (ST, last 25 days) and a long-term (LT, last 100 days) percentile: these are non-parametric estimates. We compare two full T-CAV models, one with local return threshold (TCAViaR, labelled TCAV), the other with US market return threshold (T-CAViaR, lebelled TCAVx) with its two nested versions: AS and SAV, as well as a range of popular GARCH specifications, including GARCH-n (G-n), GARCH-t (G-t); GJR-GARCH (GJR) and IGARCH (IG). Except for the RiskMetrics (RM) model, where the parameter was set to 0.94, all models were re-estimated in order to forecast each day's quantile, using MCMC methods. Details for the MCMC algorithms for the models just listed can be found in Chen, Gerlach and So (2006). We also considered the MCMC estimated method of Geweke and Keane (2007) (GK), that employs a smooth mixture of Gaussian regression models. This method required a long pre-sample period of returns: we employed an extra 1000 daily returns, pre-2001, to employ this method and used all the same settings listed in Geweke and Keane (2007). For each day y_{n+t} in the forecast sample then, each semi and parametric model used the entire previous data set $(y_1, \ldots, y_{n+t-1})$ as observations in an MCMC sampling scheme to estimate parameters, which were then used to generate a forecast of the next day's α-level quantile; thus for each day 25000 (post burn-in) VaR forecasts were generated, one for each MCMC iterate, for each model. The Geweke and Keane (GK) method used an additional 1000 daily returns from pre-2001 in this process. These forecasts were simply averaged to give the posterior mean quantile forecast for each day. Naturally this was quite time consuming: each MCMC run took just under 1 minute, so two years of forecasts (500 days) took approximately 8 hours to produce, for each model, on a standard modern desktop PC.

Table 5 shows the ratios of the observed VRate to the true nominal level, for $\alpha = 0.01, 0.05$, across all twelve models/methods and ten markets. The results are quite different from $\alpha = 0.01$ to 0.05. First, at $\alpha = 0.01$ it is immediately apparent that in all markets, except France, one of the CAViaR models ranked (at least equal) first, with VRate/α closest to 1. Further, the long and short-term historical simulation methods, together with the full set of GARCH models, all had ratios mostly above 1 across the ten markets, at $\alpha = 0.01$. This indicates that risk is being consistently under-estimated by this group of models. The GK method seems highly variable in its performance across the markets, at $\alpha = 0.01$.

A different story is apparent at $\alpha = 0.05$. Here the models are much closer in performance, with first place rankings spread across models over the ten markets, and ratios closer to 1 (in location and spread) across all models, except the GK method and the short-term percentile method (ST) that both consistently under-estimated risk once again. We examine the models forecast performance in more detail below.

Figures 2 and **??** illustrate some of the models/methods for the Italian MIBTel index returns in the forecast period at $\alpha = 0.01, 0.05$ respectively. For this market at $\alpha = 0.01$ the four CAViaR models ranked first-fourth in VRate/0.01 with 1 (SAV) and 0.8 (AS, TCAV, TCAVx), followed by the GARCH and GK models with ratios, all significantly different to 1, of 3 (GJR and GK) or 3.2, while the ST method had a ratio of 3.8. Figure 2 highlights that the TCAViaR model was clearly more extreme (and accurate) in its quantile forecasts in this market. The GARCH models' forecasts, all similar to the GJR shown, were consistently too close to the return series for a 1% quantile level. The GK method, while demonstrating that indeed it can capture heteroscedasticity, seemed too smooth in its dynamic quantiles for this market, and did not react to high volatility periods as much as the other models/methods did, its' quantile forecasts being less extreme than all other methods in highly volatile return periods and being generally much smoother at all times in this sample. The ST method is clearly not optimal for this market either, partly since 25 days is not sufficient to estimate a 1% sample quantile. ST's forecasts are far less smooth than all other models in this case.

Figure **??** shows the VaR forecasts for $\alpha = 0.05$, again for Italy, and for the same four methods. Now we see that the ST method, at least in low-medium volatility periods, is approximating the GARCH (all similar to the GJR shown) models' forecasts, which are all quite similar to the TCAViAR model. However, the ST method "recovers" slowest after extreme returns or high volatility periods, since any extreme return will affect the sample percentile for exactly 25 days. The GK method again seems quite different to the other methods, it is smoother and less reactive to extreme returns: though it can clearly capture some types of heteroscedasticity, it has struggled with the specific type represented by this data set, at both $\alpha = 0.05, 0.01$.

Table 6 displays summary statistics for the observed ratios VRate/α for each model across the ten markets, using the numbers in Table 5. The results confirm the discussion and illustrations above. At $\alpha = 0.01$ the SAV, AS and TCAV (self-exciting) models are most favoured across all criteria. In particular, their typical ratios VRate/α are very close to 1, both in location and spread, and they usually finish in the top three models for each market on this criteria. The GARCH models considered typically under-estimate risk, with ratios above 1, across markets, as do the GK and historical simulation methods. The TCAV model has the lowest standard deviation, average ratio second closest to 1, and was ranked in the top 3 models for all ten markets. The SAV ranked first in six markets, average ratio was closest to 1, had second lowest deviation in ratios and finished in the top 3 models in seven markets.

At $\alpha = 0.05$, all the models except the short-term 25 day percentile (ST) have ratios with location close to 1 across markets, with small standard deviation (except the GK method). The GARCH and CAViaR models seem quite comparable, and are hard to distinguish between, at this quantile level, the the TCAV model had average ratio closest to 1, followed by GJR and AS. However, once again the SAV finished first in top rankings with four, and equal first with RM in finishing in the top 3 models in five markets. The fully threshold TCAV models both had slightly larger standard deviation in ratios compared to the GARCH, ST, LT, AS and SAV models.

Table 5: Ratio of VRate/α at $\alpha = 0.01$, 0.05 for each model across the ten markets.

model	Japan	France	Germany	Italy	U.K.	Canada	US	Taiwan	HK	Australia
$\alpha = 0.01$										
ST	**4.6**	**4.2**	**4.8**	**3.8**	**4.6**	**5**	**4**	**4.4**	**3.4**	**4.6**
LT	1.2	1.6	1.8	**2.4**	1.6	1.8	1.6	1.4	1.4	**2**
G-n	1.4	1.2	1.6	**3.0**	1.2	**2.4**	1.4	1.6	1.6	**2.8**
G-t	1.4	1	1.6	**3.0**	1.2	**2**	1	1.2	1.4	1.8
GJR	1.2	1	1.8	**3.2**	1	1.6	1	1.4	1.6	**2**
IG	1.6	1.6	**2**	**3.2**	**2.2**	**2.6**	1.8	1.8	1.6	**3**
RM	1.6	1.8	1.6	**3.0**	**2.2**	**2.6**	1.6	**2**	1.8	**2.6**
GK	**0.2**	**0.2**	**0.2**	**3.0**	**2.0**	**5.0**	0.8	1	0.8	**9.2**
SAV	0.8	0.4	0.6	1	1	1.8	1.2	1	1	1.4
AS	0.8	0.6	0.8	0.8	1	1.4	1.6	0.6	1.2	1.8
T-CAV	0.8	0.8	1.2	0.8	1	1.6	1	1	0.8	1.4
T-CAVx	1.4	1.4	1.6	0.8	1.6	**2.2**	–	1	0.4	**3.8**
$\alpha = 0.05$										
ST	1.32	1.4	**1.48**	1.4	**1.56**	**1.52**	1.2	1.44	1.36	**1.52**
LT	1.04	1.08	1.16	1.24	1.08	1.4	1.04	1.2	1.08	1.28
G-n	0.96	1.04	1.2	1.08	1.08	1.32	0.84	0.84	1.16	1.32
G-t	1	1.16	1.24	1.2	1.08	1.32	0.88	0.88	1.2	1.32
GJR	0.96	1.16	1.16	1	1	1.32	0.84	0.72	1	1.24
IG	1	1.16	1.12	1.16	1.12	1.24	0.92	1	1.2	1.2
RM	0.92	1.12	1.12	1.16	1.04	1.24	0.92	1	1.24	1.16
GK	**0.36**	**0.4**	**0.24**	1.36	1.12	**2.2**	**0.6**	**0.48**	**0.6**	**3**
SAV	0.76	1.04	1	1	0.88	1.12	0.72	0.68	0.84	1.08
AS	0.8	1.32	1.12	1.12	1.16	1.04	**0.64**	**0.6**	**0.6**	1.24
TCAV	0.84	1.32	0.96	1.2	1.12	1.12	**0.6**	**0.56**	1.08	1.36
TCAV-x	0.92	1.2	0.96	1.2	1.04	1.44	–	0.76	0.92	**1.76**

Note: Boxes indicate closest to 1 in that market), bold indicates the model is rejected by the unconditional coverage test (at a 5% level), for each market.

To help distinguish between the better models at each quantile level, Table 7 shows summary statistics for each model's rank in terms of how close its' VRate/α ratio is to 1 in each market. Naturally we prefer models with ratios close to 1, while again for ratios that are equidistant from 1, less than 1 is preferred to above 1. The average, median, standard deviation and range of the forecast ranks for each model over the ten markets are displayed in Table 7. At $\alpha = 1\%$, the self-exciting TCAViaR model has by far the lowest mean rank, the 2nd lowest median rank, as well as the 2nd smallest standard deviation in rank of VRate/α across the models (ST only ranks 11 or 12 explaining why it has the smallest Std.). In fact, the TCAV model ranked in the top 3 in every market, and thus has the smallest rang in ranks. The SAV and then AS CAViaR models were next best on these measures, with SAV having the best median rank of 1. The GARCH-n, GARCH-t, and the GJR-GARCH and TCAVx models, typically ranked just behind the TCAV, SAV and AS CAViaR models in 4th-7th placing, followed by the long-term percentile, Geweke and Keane methods; while RM, IG and short-term percentile methods typically ranked lowest in each market. Clearly the TCAV model has forecasted dynamic quantiles comparatively well over this forecast sample and for the models considered, at $\alpha = 0.01$. Further, the CAViaR models did best as a group, followed by the stationary GARCH specifications, the GARCH-t's fat-tails outperforming the GARCH-n as expected, then the GK and remaining methods. RiskMetrics and IG did not do well at this quantile level at all.

Table 6: Summary statistics for VRate/α at $\alpha = 0.01, 0.05$ for each model across the ten markets.

	α=1%					α=5%				
	Mean	Median	Std.	1st	Top 3	Mean	Median	Std.	1st	Top 3
ST	**4.34**	**4.5**	0.49	0	0	**1.42**	**1.4**	0.11	0	0
LT	1.72	1.7	0.33	0	0	1.16	1.1	0.12	0	2
G-n	1.82	1.6	0.66	0	0	1.08	1.1	0.17	1	3
G-t	1.56	1.4	0.60	2	3	1.13	1.2	0.16	0	2
GJR	1.58	1.5	0.67	3	4	1.04	1.0	0.18	3	4
IG	2.14	1.9	0.60	0	0	1.11	1.1	0.10	2	4
RM	2.08	1.9	0.50	0	0	1.09	1.1	0.12	1	5
GK	2.24	0.9	**2.88**	1	2	1.04	0.6	**0.91**	0	0
SAV	1.02	1.0	0.39	6	7	0.91	0.9	0.16	4	5
AS	1.04	0.9	0.40	4	6	0.96	1.1	0.28	1	1
TCAV	1.04	1.0	0.28	5	10	1.02	1.1	0.28	0	3
TCAV-x	1.58	1.4	0.98	1	1	1.13	1.0	0.31	0	3

Note: Boxes indicate the favoured model, bold indicates the least favoured model, in each column.

For $\alpha = 0.05$ there is a different story. The RiskMetrics method now has the lowest average and median rank across markets, closely followed by the IGARCH and GJR-GARCH models. However, the SAV model finished with the equal best median rank, but highest standard deviation in rank, while GARCH-n had the best standard deviation in ranks across markets. These four models are still hard to separate at this quantile level, but perhaps the GJR-GARCH would be marginally favoured considering both Tables 6

Table 7: Summary statistics for model ranks, in terms of VRate/α, at $\alpha = 0.01, 0.05$ across the ten markets.

	α=1%				α=5%			
	Mean	Median	Std.	Range	Mean	Median	Std.	Range
ST	**11.75**	**12.0**	0.48	1	10.60	11.0	1.43	5
LT	6.65	7.0	2.14	6	6.20	5.5	3.36	11
G-n	6.20	6.0	1.69	5	4.80	4.5	[2.30]	8
G-t	4.30	5.0	2.00	5	5.30	5.0	2.58	9
GJR	4.95	4.5	3.70	9	3.90	4.0	2.38	[6]
IG	9.20	9.5	1.40	5	3.90	4.0	2.33	7
RM	8.40	9.0	2.32	7	[3.80]	[3.5]	2.39	8
GK	7.00	9.0	**4.09**	**11**	**11.10**	**11.5**	1.29	4
SAV	2.50	[1.0]	2.17	6	4.40	[3.5]	**3.60**	9
AS	3.10	2.5	2.28	6	7.00	8.5	3.46	10
TCAV	[1.60]	1.5	[0.70]	[2]	6.70	8.0	3.16	8
TCAV-x	5.33	6.0	2.74	9	5.89	6.0	3.37	9

Note: Boxes indicate the favoured model, bold indicates the least favoured model, in each column.

and 7. The TCAViaR models now typically rank in the middle (5th, 6th or 7th) of the models.

To summarise, the fully nonlinear self-exciting TCAViaR model seems to be favoured at dynamic quantile forecasting when $\alpha = 0.01$, while four models are hard to separate at $alpha = 0.05$. We now consider formally testing these models as a final point of comparison.

Table 8 counts the number of rejections for each model, over the 10 markets, for each of the three tests considered: unconditional coverage (UC), conditional coverage (CC) and the dynamic quantile (DQ) test, where four lags were used as in Engle and Manganelli (2004). At $\alpha = 0.01$ most of the models are rejected in most of the markets, by the DQ test. The CAViaR models fare best as a group, with the self-exciting TCAViaR model rejected in the least number of markets for all three tests, and for all tests combined (four rejections). The GK method is only rejected in six markets, while all the GARCH models are rejected in at least eight markets; ST, RM and IG are rejected in all ten markets. At $\alpha = 0.05$ the historical simulation (ST, LT) and GK methods are rejected in almost all markets, while the CAViaR models are rejected in two or three markets each and the GARCH and RM models fare best with only 1 or 2 rejections across markets.

Table 8: Counts of model rejections for three standard quantile coverage tests, across the ten markets.

	α=1%				α=5%			
Model	UC p	CC p	DQ_4 p	Total	UC p	CC p	DQ_4 p	Total
ST	**10**	**10**	**10**	**10**	5	5	**9**	**9**
LT	2	2	9	9	0	2	7	7
G-n	3	3	9	9	0	0	1	1
G-t	2	1	9	9	0	0	2	2
GJR	2	1	8	8	0	0	1	1
IG	5	4	**10**	**10**	0	0	1	1
RM	5	4	**10**	**10**	0	0	1	1
GK	7	4	6	9	**8**	**7**	8	**9**
SAV	0	0	7	7	0	0	2	2
AS	0	0	5	5	3	0	1	3
TCAV	0	0	4	4	2	1	1	3
TCAVx	2	0	6	6	0	0	2	2

Note: Boxes indicate the favoured model, bold indicates the least favoured model, in each column.

In summary, the CAViaR family of models were highly competitive at worst, and far more accurate at best, at dynamic quantile VaR forecasting, compared to a range of popular and well-known VaR methods. At $\alpha = 0.05$ most of the models performed roughly similarly with no clear standout, though RiskMetrics and the CAViaR SAV model finished best in mean and median ranking respectively, in terms of violation rates, across markets; and the RM, GJR and G-n models were rejected the least across all VaR forecast methods. The full TCAViaR model, though still performing well in overall violation rate across markets, perhaps is an unnecessarily complex model at quantile level 0.05. However, the self-exciting TCAViaR model performed in a comparably superior fashion to the other models, followed by the remaining CAViaR specifications, for $\alpha = 0.01$. It had by far the lowest mean rank, in terms of violation rates, and was rejected the least number of times across markets. The RiskMetrics, IGARCH, stationary GARCH, GK and historical simulation methods were simply not competitive with the CAViaR models at this more extreme quantile.

8 Conclusion

The CAViaR model for dynamic quantile estimation is extended to a fully nonlinear family. Bayesian MCMC methods were adapted to this family, employing the well-known link between the quantile estimation criterion function and the Skewed-Laplace density. MCMC estimation proved favourably accurate and efficient, in simulations from a GARCH model, compared to the standard numerically optimised quantile criterion function at the quantile levels 0.05 and 0.01. Empirical results for ten major market index returns reveal significant nonlinearity and asymmetry in response to negative and positive lagged shocks, as expected. Further, a VaR forecasting study revealed that the fully nonlinear self-exciting TCAViaR model compared most favourably in terms of violation rates and independence of violations, to more parsimonious CAViaR models. A range of well-known GARCH models, including RiskMetrics, historical simulation and a semiparametric smoothly mixing regression approach were not competitive at the quantile level 0.01, across the ten markets. However, at level 0.05, most models forecast reasonably similarly, results marginally favouring RiskMetrics, GJR-GARCH and the simple CAViaR SAV specification.

References

Black, F. (1976), "Studies in stock price volatility changes." *Proceedings of the American Statistical Association.* Business and Economic Statistics Section, 177-181.

Bollerslev, T. (1986), "Generalized autoregressive conditional heteroskedasticity." *Journal of Econometrics*, **31**, 307-327.

Brooks, C. (2001), "A double-threshold GARCH model for the French Franc/Deutschmark exchange rate." *Journal of Forecasting*, **20**, 135-143.

Cade, B and Noon, B (1996) A gentle introduction to quantile regression for ecologists, *Frontiers in Ecology and the Environment*, 412-420.

Chen C.W.S, Gerlach, R. and Lin, E.M.H. (2008) "Volatility Forecasting Using Threshold Heteroskedastic Models of the Intra-day Range" *Computational Statistics and Data Analysis*, Special Issue on Statistical and Computational Methods in Finance, **52**, 2990-3010

Chen, C. W. S., Gerlach, R. and So, M.K.P. (2006) "Comparison of non-nested asymmetric heteroskedastic models," *Computational Statistics and Data Analysis*, Special Issue on Nonlinear Modelling and Financial Econometrics, **51**, 4, 2164-2178.

Chen, C. W. S., and So, M. K. P. (2006), "On a threshold heteroscedastic model." *International Journal of Forecasting*, **22**, 73-89.

Christoffersen. P. (1998), "Evaluating interval forecasts." *International Economic Review*, **39**, 841-862.

Engle, R. F. (1982), "Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflations." *Econometrica*, **50**, 987-1007.

Engle, R. F. and Manganelli, S. (2004), "CAViaR: Conditional autoregressive value at risk by regression quantiles." *Journal of Business and Economic Statistics*, **22**, 367-381.

Gelman, A., Carlin, J.B., Stern, H.S. and Rubin, D. B. (2005), Bayesian Data Analysis, 2nd Edition, Chapman & Hall: Boca Raton, FL.

Gelman, A., Roberts, G.O., and GILKS, W.R. (1996), "Efficient Metropolis jumping rules." *Bayesian Statistics*, **5** 599-608. Clarendon, Oxford.

Geraci, M. and Bottai, M. (2007), "Quantile regression for longitudinal data using the asymmetric Laplace distribution." *Biostatistics*, **8**, 140-154.

Geweke, J. and Keane, M. (2007), "Smoothly mixing regressions." *Journal of Econometrics*, **138**, 252-290.

Giacomini, R. and Komunjer, I. (2005), "Evaluation and Combination of Conditional Quantile Forecasts." *Journal of Business & Economic Statistics*, **23**, 416-431.

Glosten, L. R., Jagannathan, R. and Runkle, D. E. (1993), "On the relation between the expected value and the volatility of the nominal excess return on stock." *Journal of Finance*, **48**, 1779-1801.

Hastings,W.K. (1970), "Monte-Carlo sampling methods using Markov chains and their applications" *Biometrika*, **57**, 97V109.

Jorion, P. (1996), "Risk: measuring the risk in Value at Risk." *Financial Analysis Journal*, **52**, 47-56.

Koenker, R. and Bassett, G.(1978), "Regression Quantiles." *Econometrica*, **1**, 33-50.

Kuester, K., Mittnik, S., and Paolella, M.S. (2006), "Value-at-risk prediction: A comparison of alternative strategies." *Journal of Financial Econometrics*, **4**, 53-89.

Kupiec, P. H. (1995), "Techniques for verifying the accuracy of risk measurement models." *The Journal of Derivatives*, **3**, 73-84.

Li, C.W. and Li,W.K. (1996), "On a double-threshold autoregressive heteroscedastic time series model." *Journal of Applied Econometrics*, **11**, 253V274.

Machado, J.A.F. and J. Mata (2005), "Counterfactual decomposition of changes in wage distributions using quantile regression." *Journal of Applied Econometrics*,**20**, 445-65.

Manganelli, S. and Engle, R. F. (2004), "A comparison of value-at-risk models in finance." Szegö G., ed., *Risk measures for the 21st century*, Wiley: Chichester.

Metropolis, N., Rosenbluth, A. W., Rosenbluth, M. N., Teller, A. H., and Teller, E. (1953), "Equation of state calculations by fast computing machines." *J. Chem. Phys*, **21**, 1087-1092.

Morgan, J. P. (1996), RiskMetrics, *Technical Document*, *Fourth Edition*, *New York*.

Nelson, Daniel B. (1991), "Conditional Heteroskedasticity in Asset Returns: A New Approach Econometrica." *Econometrica*, **59**, 347-370.

Tong, H. (1978), "On a threshold model." In *Pattern Recognition and Signal Processing*, ed. Chen, C.H., Sijho and Noordho, Amsterdam.

Tsay, R.S. (2005), Analysis of Financial Time Series. second ed.Wiley, Hoboken.

Tsionas, E. G. (2003), "Bayesian quantile inference." *Journal of Statistical Computation and Simulation*, **9**, 659-674.

Wei, Y., Pere, A., Koenker, R., and He, X. (2006), "Quantile regression methods for reference growth charts." *Statistics in Medicine*, **25**, 1369-1382.

Wong, C.M. and So, K.P. (2003), "On conditional moments of GARCH models, with applications to multiple period value at risk estimation." *Statistica Sinica*, **13**, 1015-1044.

Yu, K. and Moyeed, R.A. (2001), "Bayesian quantile regression." *Statistics and Probability Letters*, **54**, 437-447.

Yu, K. and Zhang, J. (2005), "Distributions and Applications - A Three-parameter Asymmetric Laplace Distribution and Its Extension." *Communications in Statistics-Theory and Methods*, **34**, 1867-1879.

Zakoian, J. (1994), "Threshold Heteroskedastic Model." *Journal of Economic Dynamics and Control*, **18**, 931-955.

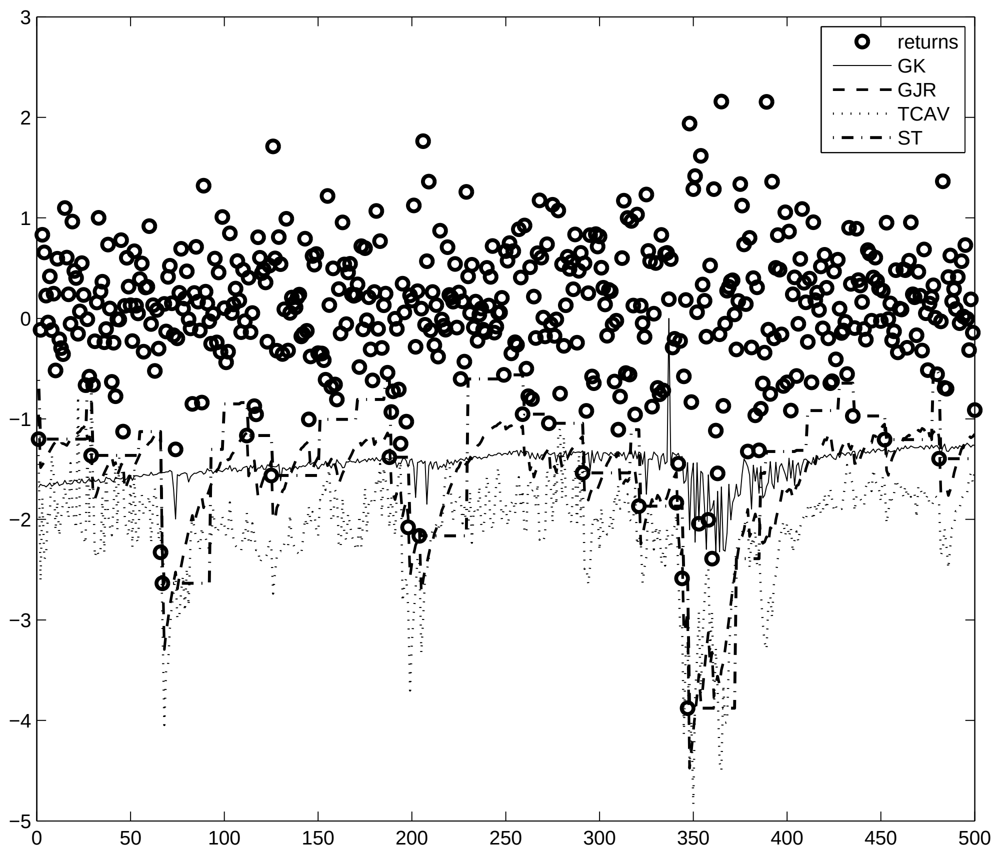

Figure 1: Italian MIBTel index returns Jan, 2005 to Jan, 2007 (circles), together with four sets of forecasted VaR series at $\alpha = 0.01$. Series are GK, GJR, TCAV and ST

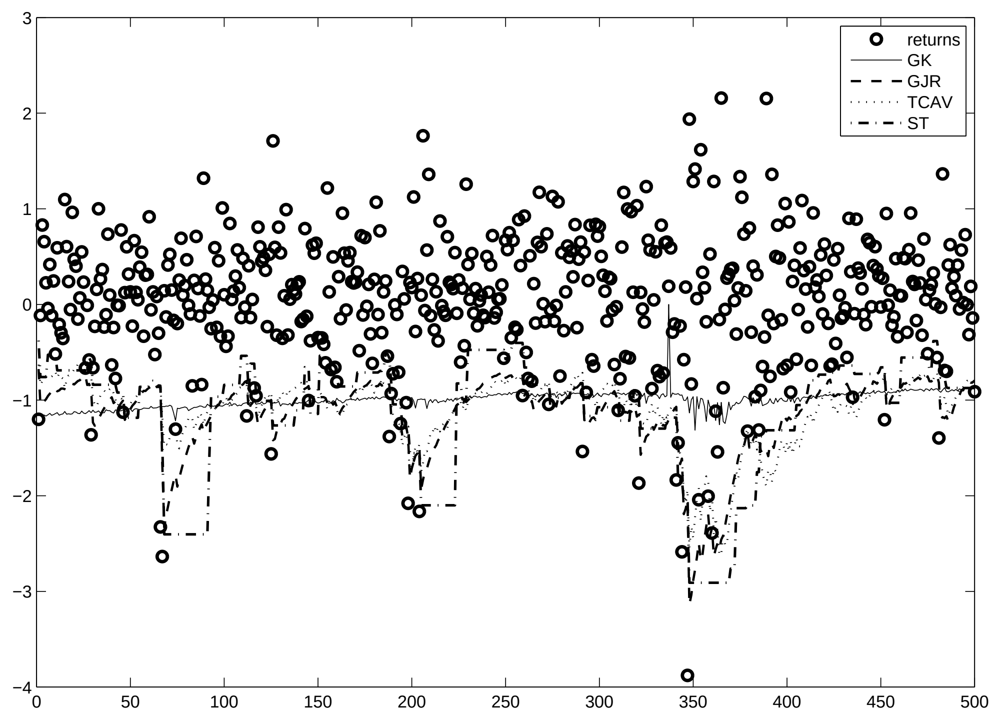

Figure 2: Italian MIBTel index returns Jan, 2005 to Jan, 2007 (circles), together with four sets of forecasted VaR series at $\alpha = 0.05$. Series are GK, GJR, TCAV and ST