On the Isoperimetric Problem for the Laplacian with Robin and Wentzell Boundary Conditions

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Abstract

We consider the problem of minimising the eigenvalues of the Laplacian with Robin boundary conditions $\frac{\partial u}{\partial \nu} + \alpha u = 0$ and generalised Wentzell boundary conditions $\Delta u + \beta \frac{\partial u}{\partial \nu} + \gamma u = 0$ with respect to the domain $\Omega \subset \mathbb{R}^N$ on which the problem is defined. For the Robin problem, when $\alpha > 0$ we extend the Faber-Krahn inequality of Daners [Math. Ann. 335 (2006), 767–785], which states that the ball minimises the first eigenvalue, to prove that the minimiser is unique amongst domains of class C^2. The method of proof uses a functional of the level sets to estimate the first eigenvalue from below, together with a rearrangement of the ball's eigenfunction onto the domain Ω and the usual isoperimetric inequality.

We then prove that the second eigenvalue attains its minimum only on the disjoint union of two equal balls, and set the proof up so it works for the Robin p-Laplacian. For the higher eigenvalues, we show that it is in general impossible for a minimiser to exist independently of $\alpha > 0$. When $\alpha < 0$, we prove that every eigenvalue behaves like $-\alpha^2$ as $\alpha \to -\infty$, provided only that Ω is bounded with C^1 boundary. This generalises a result of Lou and Zhu [Pacific J. Math. 214 (2004), 323–334] for the first eigenvalue.

For the Wentzell problem, we (re-)prove general operator properties, including for the less-studied case $\beta < 0$, where the problem is ill-posed in some sense. In particular, we give a new proof of the compactness of the resolvent and the structure of the spectrum, at least if $\partial\Omega$ is smooth. We prove Faber-Krahn-type inequalities in the general case $\beta, \gamma \neq 0$, based on the Robin counterpart, and for the "best" case $\beta, \gamma > 0$ establish a type of equivalence property between the Wentzell and Robin minimisers for all eigenvalues. This yields a minimiser of the second Wentzell eigenvalue. We also prove a Cheeger-type inequality for the first eigenvalue in this case.

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Statement of originality

This thesis contains no new material which has been accepted for the award of any other degree or diploma. All work in this thesis, except where duly acknowledged otherwise, is believed to be original.

James Kennedy

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Contents

CHAPTER 1

Introduction

No question is so difficult to answer
as that to which the answer is obvious
— George Bernard Shaw

1.1. The isoperimetric problem

The application of isoperimetric inequalities to physical situations is, in a sense, a study of the obvious. That the shape of an object affects its physical properties is so trite as to be hardly worth saying. It is perhaps a more subtle observation that a physical optimum should be attained by an object satisfying some appropriate geometric optimum.

The most basic isoperimetric result states that the ball has the least surface area of all objects of given volume. This result is even said to have been known in some form to the ancient Greeks. Physical intuition thus suggests that the object that minimises heat loss should be perfectly spherical, all other things being equal. Similarly, the fundamental frequency of a vibrating membrane should be lowest when the membrane is circular. This is the famous conjecture of Lord Rayleigh in the 19th Century [**97**].

But the word "conjecture" is telling. It is easy to make such a claim, and even to formulate it mathematically. However experience shows that such "obvious" conjectures are often extremely difficult to prove in all branches of mathematics — although possibly this is not quite what Shaw had in mind. The first full proofs of Rayleigh's conjecture appeared almost 50 years after it was made, with the simultaneous but independent work of Faber [**50**] and Krahn [**78**] in the 1920s. Even then some residual questions remained about the validity of the method they used, and the issue was only completely resolved as late as 1951 with the seminal work of Pólya and Szegö [**96**]. Such problems rank amongst the most interesting, and arguably most challenging, in mathematics, involving a fascinating interplay of analysis and geometry.

Mathematically, both Rayleigh's conjecture and the problem of minimising heat loss reduce to studying the smallest $\mu > 0$ for which the Helmholtz equation

$$\begin{aligned} -\Delta u &= \mu u \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1.1.1}$$

has a solution. Here u can be interpreted as measuring the displacement from rest of the membrane, or the amount of heat present in the object. $\Delta u := \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian of u and $\Omega \subset \mathbb{R}^N$ is some region or domain in N-dimensional space, $N \geq 2$, corresponding to the membrane at rest in Rayleigh's problem or the object suffering heat loss. Imposing the Dirichlet boundary condition $u = 0$ is physically interpreted as having a fixed membrane or a fixed ambient temperature at the surface, or boundary, of the object, and generally represents a convenient mathematical approximation to physical reality. As is well known, the values μ for which (1.1.1) has a solution, called eigenvalues of the Dirichlet Laplacian, form a countable sequence $0 < \mu_1 \leq \mu_2 \leq \ldots \to \infty$. We refer to the corresponding functions solving (1.1.1) as eigenfunctions.

The fundamental frequency of a fixed membrane Ω corresponds exactly to μ_1, while the higher eigenvalues correspond to higher frequencies (also known variously as normal modes, overtones or harmonics). Similarly, the long-term decay of heat in an object Ω behaves like $e^{-\mu_1 t}$ (where $t > 0$ is time). Thus finding the object which optimises either of these physical properties can be mathematically reformulated as finding the domain Ω which minimises μ_1 amongst all domains in $\mathbb{R}^N$ of fixed volume. Thus interpreted, Rayleigh's conjecture, since its proof called the Faber-Krahn inequality or sometimes the theorem of Rayleigh-Faber-Krahn, asserts that $\mu_1(\Omega) \geq \mu_1(B)$, where B is a ball with the same volume of Ω. In fact this inequality is strict if Ω is not "essentially" a ball. (We will give a precise statement of this theorem in Chapter 2.)

Of course there are many mathematically and physically interesting variants of this problem; the 40-year-old survey paper of Payne [**93**] still provides a good introduction. One could study the other eigenvalues of (1.1.1) in the same way, or combinations of eigenvalues, or – of course – other problems. Another starting point in the field is the conjecture (long since proved) of Saint-Venant, which asserts that the torsional rigidity of a beam is greatest when the beam's cross-section is circular (see [**93, 100**]).

In this thesis we will be interested principally in minimisation problems for eigenvalues of the Laplacian, as in (1.1.1), but equipped with two different boundary conditions. The first is the Robin, or third, boundary condition. This is also referred to as the elastically supported membrane case, since for the vibrating membrane model it describes a situation where the displacement on the edge of the membrane is negatively proportional to the rate of change of displacement leaving the membrane. This is as one would expect if the membrane is "elastically supported", that is, not firmly clamped but not perfectly free. The equation is

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1.2}$$

where $\frac{\partial u}{\partial \nu}$ is the outer normal derivate to u on $\partial\Omega$ and $\alpha > 0$ is an arbitrary constant, which we will refer to as the *boundary parameter* of the Robin problem. As with the Dirichlet problem, the solutions (eigenvalues of the Robin Laplacian) $\lambda = \lambda(\Omega, \alpha)$, on a fixed domain Ω, for each given value of α, form a countable set $0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$. The dependence on α (which may also be a function on the boundary or a real number) makes the situation far more interesting. The case $\alpha = 0$ corresponds to Neumann boundary conditions, and, at least formally, $\alpha = \infty$ gives Dirichlet boundary conditions.

The other type of boundary condition is the generalised Wentzell boundary condition, sometimes called the general Wentzell condition or the Wentzell-Robin condition. This is a relatively new boundary condition that has only been intensively studied in the last decade or so; see Chapter 5 for a full description. This is

$$\begin{aligned} -\Delta u &= \Lambda u && \text{in } \Omega, \\ \Delta u + \beta \frac{\partial u}{\partial \nu} + \gamma u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1.1.3}$$

The eigenvalues $\Lambda = \Lambda(\Omega, \beta, \gamma)$ of the Wentzell Laplacian now depend on two parameters (or suitable functions) β and γ as well as Ω. We will study their structure, which is more complicated than for the other boundary conditions, in Chapter 6. One interpretation of this boundary condition, in the model of the vibrating membrane, is that the boundary $\partial\Omega$ is itself affected by vibrations in the membrane Ω and thus contributes to the total kinetic energy of the system. In the model of the heat equation, it models a situation where there is a heat source (if $\beta > 0$) or sink (if $\beta < 0$) on the boundary. (For a full derivation of these, and another interpretation of the Robin boundary condition, see [**63**].)

In this thesis we will prove and study Faber-Krahn-type inequalities for the first eigenvalue of these Robin and Wentzell problems, as well as more general problems of the form

$$\min\{\lambda_k(\Omega, \alpha) : \Omega \subset \mathbb{R}^N \text{ bounded, Lipschitz; } |\Omega|, \alpha \text{ fixed}\}, \tag{1.1.4}$$

for some given $k \geq 1$, or the corresponding minimisation problem for (1.1.3), with $\Lambda_k(\Omega, \beta, \gamma)$ in place of $\lambda_k(\Omega, \alpha)$. Such problems are sometimes called "shape optimisation problems", as in [**28**], but we prefer the term "isoperimetric problems"; hence the title of this thesis and of this section. Such extremal problems in partial differential equations are often given this name, and we feel it better evokes the link with the type of geometric problem to which (1.1.4) is related.

This thesis is essentially divided into two halves, the first devoted to the Robin problem and the second to the Wentzell problem. In the remainder of this chapter we present the standard theory of the Robin Laplacian – that is, existence and regularity of solutions, spectral properties and so forth. This serves both as a mathematical introduction to our problem and a repository of important results that we will need later.

Chapter 2 is devoted to the Faber-Krahn inequality for the first eigenvalue of the Robin problem (1.1.2). The actual inequality had been proven in [**35**]; our contribution is to strengthen the method to prove sharpness of the inequality, that is, that the ball is the *unique* minimiser of the first eigenvalue, at least amongst bounded, C^2-domains (see Theorem 2.1.2). However, for completeness' sake (and to an extent out of mathematical necessity) we sketch the full proof. We also set it up to give an alternative proof in the Dirichlet case, albeit under relatively restricted assumptions on the domain Ω. Instead of the usual symmetrisation arguments, the proof uses a functional of the level sets of the domain Ω, together with a rearrangement argument from the ball onto Ω.

In Chapter 3 we touch on a number of other problems, both solved and unsolved, which are either related to the Robin problem, or else for which insight can be gained from the method used in Chapter 2. So in Section 3.1, we mention the Robin p-Laplacian. Subsequent to the publication of our results in Chapter 2, the Faber-Krahn inequality was generalised to this operator with an adapted proof, so we mention it for the purpose of comparison. Conversely, the open problems presented in Section 3.4, namely the case where $\alpha < 0$ (where now the *maximiser* should be a ball, but this is unproven) and a conjecture due to Pólya on polygons, illustrate the apparent limitations of the method. Section 3.2 compares the results for the functional method with a property of the Robin Laplacian involving supersolutions; the exact nature of the connection probably warrants further exploration. Section 3.3 looks at a different type of inequality for the first Robin eigenvalue, depending only on the geometry of Ω. This is usually called a Cheeger-type inequality, after a corresponding one for the Dirichlet problem. This is a consequence of the functional method of Chapter 2. We remark that it had already been noted by previous authors working in this area [**20, 35**], although we generalise the result slightly.

In Chapter 4 we consider the higher eigenvalues of the Robin problem. We start by proving an inequality for the second eigenvalue, namely that the (unique) minimiser is the domain consisting of the disjoint union of two equal balls, as in the Dirichlet case (see Theorem 4.1.1). In fact we set this up so it works for the p-Laplacian. For the higher eigenvalues, we show that it is in general impossible to find a minimising domain independent of the parameter $\alpha > 0$ in the boundary condition, or equivalently, independent of the volume of the domain for fixed $\alpha > 0$ (see Theorem 4.3.1). When $\alpha < 0$ we prove that every eigenvalue behaves like $-\alpha^2$ as $\alpha \to -\infty$ if the underlying domain is bounded and C^1, independent of the domain's geometry and volume (see Theorem 4.4.1). This generalises a result of [**86**] for the first eigenvalue.

In Chapters 5 and 6 we start on the Wentzell problem (1.1.3). After a brief introduction in Section 5.1, we study basic properties of the associated operator in the remainder of Chapter 5. The operator's behaviour depends greatly on the sign of the parameter β. The case $\beta > 0$ is the better-behaved and far more heavily studied one. In this case, which we deal with in Section 5.2, the appropriately realised operator generates a C_0-semigroup with essentially all the properties of the Dirichlet or Robin problems (it is convenient here to phrase our results in terms of generation properties). Following [**7**], we use form methods, although we generalise the approach to allow $\beta \not\equiv 1$ and $\gamma < 0$. When $\beta < 0$, the operator no longer generates a C_0-semigroup, although it still has compact resolvent, at least if Ω is sufficiently smooth. We now obtain two sequences of eigenvalues, one tending to ∞ and the other to $-\infty$. We study this case in Section 5.3, adapting an operator matrix approach from [**44**] originally designed for the case $\beta > 0$.

Chapter 6 then concentrates specifically on the properties of the eigenvalues of the Wentzell problem. The key observation – although an elementary one – is that every Wentzell eigenvalue and function solves a suitably chosen Robin problem. We exploit this in various guises to prove a number of specific properties of the Wentzell eigenvalues, including a fairly precise description of their structure, of the regularity of the eigenfunctions, and also a number of variational results, continuous dependence on perturbations in the parameters β and γ, and so on, which are similar to the standard results for the Robin case that we present in Section 1.3.

On the content of these two chapters, we remark that some of the results, especially in Section 5.2, are probably not new, or are at best only a marginal improvement on what was previously known. This latter comment applies also to Section 5.3, although our approach there is new. We are unaware of a similar analysis to that in Chapter 6 for the Wentzell problem elsewhere, but the content is quite elementary. These chapters are present primarily to facilitate our study in Chapter 7 of various isoperimetric properties of the Wentzell problem, although it is hoped they may be of some independent interest. In Section 7.1 we obtain Faber-Krahn-type inequalities for the principal eigenvalues as a direct consequence of the Robin Faber-Krahn inequality. In some cases (depending on the sign of β and γ) these rely on the conjectured Faber-Krahn inequality for the Robin problem when $\alpha < 0$ presented in Section 3.4. In Section 7.2, we extend this to prove that in the main case when $\beta, \gamma > 0$, the isoperimetric problem for the kth Wentzell eigenvalue is essentially the same as for the kth Robin eigenvalue. Section 7.3 contains a Cheeger inequality for the first Wentzell eigenvalue, again only in the main case $\beta, \gamma > 0$. This is again a consequence of the corresponding Robin result from Section 3.3, but the form of this inequality is somewhat more interesting.

In Appendix A, we collect a number of important background results that we will use throughout, such as density, trace and compactness theorems for Sobolev spaces, results on operator semigroups and the like. Appendix B contains the proofs of some results from [**35**] that will be needed in Chapter 2. Appendix C is devoted to auxiliary results needed in Chapter 5. While we doubt these are new, it is difficult to find precise references in the literature (perhaps owing to the slight difference in emphasis between harmonic analysis and partial differential equations), so we have included proofs.

Much of the notation used throughout is also described in the appendices. We have tried to use standard modern notation as far as possible throughout; however, here we explicitly mention two slightly non-standard conventions we will be using. First, we will always use λ, λ_n for the eigenvalues of the Robin problem (1.1.2), μ, μ_n for the Dirichlet problem (1.1.1) and Λ, Λ_n for the Wentzell problem (1.1.3).

Second, and perhaps more importantly, by a domain $\Omega \subset \mathbb{R}^N$ we understand an open set, usually bounded, but *not necessarily connected*. The reason for this is that it is often necessary to deal with disconnected domains when considering isoperimetric problems: the domain solving a problem of the form (1.1.4) need not be connected. This assumption does not substantially change the nature of the analysis, but it introduces a few minor annoying technicalities. See also Remark 1.3.2. The new material in this thesis is in the papers and preprints [**38, 39, 75, 76, 77**].

1.2. The Laplacian with Robin boundary conditions

We start out by looking at the Robin problem (1.1.2). In this section we prove basic properties such as existence and uniqueness of solutions. Although all these results are standard and for the most part very well established, for completeness' sake we have included proofs, albeit often brief ones, and/or references to the literature. Throughout this section we will be working with an arbitrary, fixed, bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, and the parameter α in the problem (1.1.2) will be taken as an arbitrary fixed nonzero constant, although assuming $\alpha \in L^\infty(\partial\Omega)$ would require only trivial changes to the results in this section. (Recall that $\alpha = 0$ corresponds to Neumann boundary conditions; see Remark 1.3.9.)

We start with a summary of the theory of weak solutions to (1.1.2) and form methods. For the sake of simplicity we restrict ourselves to the real case and hence, for example, talk about bi- rather than sesquilinear forms. In what follows we will consider the general problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_0,\\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \Gamma_1, \end{aligned} \tag{1.2.1}$$

where $\Omega \subset \mathbb{R}^N$ is a fixed bounded, Lipschitz domain, $f \in L^2(\Omega)$ and Γ_0 and Γ_1 are disjoint subsets of $\partial\Omega$, open and closed in $\partial\Omega$, such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$. Here we allow $\Gamma_0 = \emptyset$ or $\Gamma_1 = \emptyset$, corresponding to pure Robin and pure Dirichlet boundary conditions, respectively; thus (1.2.1) is more general than (1.1.1) or (1.1.2). As we will see, the correct solution space for (1.2.1) is the following Sobolev space.

Definition 1.2.1. For $1 < p < \infty$, let $W_0^{1,p}(\Omega;\Gamma_0)$ be the closure in the $W^{1,p}$-norm of $C_c^\infty(\Omega \cup \Gamma_1)$, the set of all $C^\infty(\overline{\Omega})$ functions whose support is compactly contained away from Γ_0. If $p = 2$, then in preference we will write $H_0^1(\Omega;\Gamma_0)$.

In this chapter we will work only with the case $p = 2$ although we will use the more general space in Chapter 4. The notation is ours, but it is based on the notation $H_0^1(\Omega)$; indeed in this notation $H_0^1(\Omega;\partial\Omega) \equiv H_0^1(\Omega)$, while if $\Gamma_0 = \emptyset$ then $H_0^1(\Omega;\emptyset) \equiv H^1(\Omega)$. Note that $H_0^1(\Omega,\Gamma_0)$ can also be characterised as the space of all $H^1(\Omega)$ functions having zero trace on Γ_0. Note also that $H_0^1(\Omega;\Gamma_0)$ is a Hilbert space with respect to the usual H^1-norm, and more generally $W_0^{1,p}(\Omega;\Gamma_0)$ is a Banach space with respect to the $W^{1,p}$-norm.

Define a bilinear form $Q_\alpha : H_0^1(\Omega;\Gamma_0) \times H_0^1(\Omega;\Gamma_0) \to \mathbb{R}$ by

$$Q_\alpha(u,v) := \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_1} \alpha uv \, d\sigma \tag{1.2.2}$$

for all $u, v \in H^1(\Omega)$, where since there is no danger of confusion we have written u in place of $\operatorname{tr} u = u|_{\Gamma_1}$ in the second integral, and similarly for v (recalling $\operatorname{tr} u = \operatorname{tr} v = 0$ on Γ_0).

A straightforward calculation shows that if $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a classical solution of (1.2.1), then

$$Q_\alpha(u,v) = \int_\Omega fv \, dx = \langle f, v\rangle \tag{1.2.3}$$

for all $v \in H_0^1(\Omega,\Gamma_0)$, where $\langle f, v\rangle$ is the inner product on $L^2(\Omega)$. Conversely, if $u \in H_0^1(\Omega,\Gamma_0)$ satisfies (1.2.3) and $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ then u is a classical solution of (1.2.1). (Both calculations require the divergence theorem, which is valid for Lipschitz domains. See Theorem A4.5.)

If a function $u \in H_0^1(\Omega,\Gamma_0)$ satisfies (1.2.3) we call it a weak solution of the problem (1.2.1). We are only interested in the cases when either $\alpha > 0$ or else $\alpha \le 0$ and $\Gamma_0 = \emptyset$. In either case, the following result is immediate, and we omit the easy proof.

Lemma 1.2.2. *The function* $Q_\alpha : H_0^1(\Omega;\Gamma_0) \times H_0^1(\Omega;\Gamma_0) \to \mathbb{R}$ *is a bounded, symmetric and bilinear form on* $H_0^1(\Omega;\Gamma_0)$*.*

It is also easy, although perhaps less trivial, to show that the form Q_α is bounded from below in the following sense. Such a form is often called an elliptic form.

Lemma 1.2.3. *There exists* $\omega_0 \geq 0$ *such that for every* $\omega \geq \omega_0$*, there exists* $C > 0$ *for which*

$$Q_\alpha(u,u) + \omega\|u\|^2_{L^2(\Omega)} \geq C\|u\|^2_{H^1(\Omega)}$$

for all $u \in H_0^1(\Omega;\Gamma_0)$*. If* $\alpha > 0$ *or* $\Gamma_1 = \emptyset$ *then we may choose* $\omega_0 = 0$*.*

Proof. Since $\partial\Omega$ is Lipschitz it follows from Maz'ja's inequality (see [**88**, Section 4.11]) that there exists $c = c(N, |\Omega|)$ such that

$$\|u\|^2_{L^2(\Omega)} \leq c(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\partial\Omega)})$$

for all $u \in H^1(\Omega)$. Since $H_0^1(\Omega;\Gamma_0)$ is a closed subspace of $H^1(\Omega)$ equipped with the same norm, the conclusion of the lemma now follows easily. Note that if $\Gamma_1 = \emptyset$, $Q_\alpha(u,u) = \int_\Omega |\nabla u|^2\,dx$ is an equivalent norm to $\|u\|^2_{H^1(\Omega)}$ on $H_0^1(\Omega)$. (This follows from Friedrichs' inequality, or alternatively, the Poincaré inequality, which in this case may be viewed as a special case of Maz'ja's inequality.) $\square$

Let $\omega \geq \omega_0$ be as in Lemma 1.2.3, fix $v \in H_0^1(\Omega;\Gamma_0)$ and define an element f_v in the dual space $(H_0^1(\Omega;\Gamma_0))'$ by $f_v(u) = \langle f_v, u\rangle := Q_\alpha(u,v) + \omega\langle u,v\rangle_{L^2(\Omega)}$. We now define a linear operator

$$T_\alpha^\omega : H_0^1(\Omega;\Gamma_0) \to (H_0^1(\Omega;\Gamma_0))' \tag{1.2.4}$$

by $T_\alpha^\omega v := f_v$, that is, $\langle T_\alpha^\omega v, u\rangle = Q_\alpha(u,v) + \omega\langle u,v\rangle_{L^2(\Omega)}$.

Combining Lemmata 1.2.2 and 1.2.3 we see that T_α^ω is bounded and coercive, and so by the Lax-Milgram lemma (Theorem A5.2) is invertible. By definition of T_α^ω, this proves the existence of weak solutions to our problem. Since obviously $(T_\alpha^\omega)^{-1}$ is unique, it also proves that weak solutions are unique. More precisely, we have proved the following theorem.

Theorem 1.2.4. *Let* $\omega \geq 0$ *satisfy the conclusions of Lemma 1.2.3. Then for every* $f \in (H_0^1(\Omega;\Gamma_0))'$ *there exists a unique weak solution (cf.* (1.2.3)*)* $u \in H_0^1(\Omega;\Gamma_0)$ *to* *the problem*

$$\begin{aligned} -\Delta u + \omega u &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \Gamma_1. \end{aligned}$$

We now prove existence of the eigenvalues and eigenfunctions of the problem (1.2.1) (and so of the problem (1.1.2)). Denote by ϕ the dense compact embedding $H_0^1(\Omega;\Gamma_0) \hookrightarrow L^2(\Omega)$ and by ϕ' the induced dense embedding $(L^2(\Omega))' \hookrightarrow (H_0^1(\Omega;\Gamma_0))'$. Making the identification $L^2(\Omega) = (L^2(\Omega))'$, let $R_\alpha^\omega : L^2(\Omega) \to L^2(\Omega)$ be given by $R_\alpha^\omega = \phi \circ (T_\alpha^\omega)^{-1} \circ \phi'$.

Let $S_\alpha : \mathcal{D}(S_\alpha) \to L^2(\Omega)$ be the operator associated with (1.2.1), where $\mathcal{D}(S_\alpha) = \{u \in H_0^1(\Omega;\Gamma_0) : T_\alpha^0 u \in L^2(\Omega)\}$ is the natural domain of S_α. Then it is not hard to prove that S_α is closed. Moreover, R_α^ω is of course the resolvent operator $R(\omega, S_\alpha)$. In particular, we easily get the following properties of the resolvent.

Lemma 1.2.5. *Let $\omega \geq 0$ satisfy the conclusions of Lemma 1.2.3. The operator $R_\alpha^\omega : L^2(\Omega) \to L^2(\Omega)$ is compact and self-adjoint.*

Proof. Compactness follows from compactness of the embedding $\phi : H_0^1(\Omega;\Gamma_0) \to L^2(\Omega)$ while self-adjointness follows from the property $\langle T_\alpha^\omega u, v\rangle = \langle u, T_\alpha^\omega v\rangle = Q_\alpha(u,v) + \omega\langle u, v\rangle_{L^2(\Omega)}$. □

Using the spectral theory for operators with compact resolvent, we obtain the following important result about the eigenvalues of (1.2.1).

Theorem 1.2.6 (Structure of the eigenvalues)**.** *The eigenvalues of* (1.2.1) *are denumerable, and form a sequence $\lambda_1 < \lambda_2 < \lambda_3 < \ldots \to \infty$ with $\lambda_1 > 0$ if $\alpha > 0$. Moreover, the algebraic and geometric multiplicity of each eigenvalue is finite.*

Proof. Since for each α the resolvent $R_\alpha^\omega = R(\omega, S_\alpha)$ is compact for some ω, by [**72**, Theorem III.6.29] the spectrum of the closed operator S_α consists entirely of isolated eigenvalues with finite multiplicities. For the form of the eigenvalues, by [**72**, Theorem III.6.26] the only possible point of accumulation of the eigenvalues of R_α^ω is zero.

If $\alpha > 0$ then we know all eigenvalues must be positive (just use Lemma 1.2.3 and the fact that any eigenvalue λ satisfies $\lambda = Q_\alpha(\psi,\psi)/\|\psi\|_{L^2(\Omega)}$ where ψ is a corresponding eigenfunction); moreover, they are of the form $1/\kappa$, where κ is an eigenvalue of R_α^0 (again, see [**72**, Section III.7]). In particular the eigenvalues of S_α and hence (1.2.1) must have the form claimed in the theorem.

If $\alpha \leq 0$, we look at the operator given by $S_\alpha + \omega I$ for some $\omega > 0$. Then this operator has eigenvalues of the same form as when $\alpha > 0$, and since $\omega I : L^2(\Omega) \to L^2(\Omega)$ is bounded, it will merely shift the spectrum of $S_\alpha + \omega I$ without changing its structure. Hence the eigenvalues of S_α again have the claimed form. $\square$

Remark 1.2.7. (i) Each eigenvalue λ_k, $k \geq 1$ is really a function of α and the domain Ω on which the problem is defined: $\lambda_k = \lambda_k(\Omega, \alpha)$. At times for the sake of brevity we may drop one or both these arguments from our notation if there is no danger of confusion. However, at other times the arguments are important; we will study some of the properties of these functions in the next section.

(ii) The weak theory outlined above has in fact been extended to *arbitrary* open sets $\Omega \subset \mathbb{R}^N$ when $\alpha > 0$. There are (at least) two somewhat different approaches. One way is to use Maz'ja's inequality and Hausdorff measure on the boundary as in [**33**]; the other way is to use the idea of capacities and work with more general measures as in [**11**]. Here we will make no use of these "weak" theories, although the issue will briefly resurface when we look at Wentzell boundary conditions (see Remark 5.2.4(i)).

(iii) It is an immediate and useful consequence of the above theory that each eigenvalue λ_k with eigenfunction $\psi = \psi(\lambda_k)$ can be characterised as $Q_\alpha(\psi, \varphi) = \lambda_k \langle \psi, \varphi \rangle_{L^2(\Omega)}$, that is,

$$\int_\Omega \nabla\psi \cdot \nabla\varphi \, dx + \int_{\Gamma_1} \alpha\psi\varphi \, d\sigma = \lambda_k \int_\Omega \psi\varphi \, dx \tag{1.2.5}$$

for all $\varphi \in H_0^1(\Omega; \Gamma_0)$.

Theorem 1.2.8 (Regularity of the eigenfunctions)**.** *Suppose Ω is Lipschitz. Every eigenfunction ψ of the problem* (1.2.1) *satisfies $\psi \in H^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega)$. If Ω is of class C^2 and $\alpha > 0$, then in addition every $\psi \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ for all $1 < p < \infty$.*

Proof. Interior regularity follows from an easy and standard bootstrapping argument using for example [**48**, Theorem 6.3.2] (which actually works for $\Omega \subset \mathbb{R}^N$ open arbitrary).

For boundary regularity, note that we are assuming Γ_0 and Γ_1 are separated, that is, Γ_0 and Γ_1 are open and closed subsets of $\partial\Omega$. Since boundary regularity is a local property, we may choose neighbourhoods covering $\partial\Omega$ such that in each neighbourhood, either $\Gamma_0 = \emptyset$ or $\Gamma_1 = \emptyset$. Thus without loss of generality it suffices to consider the cases $\partial\Omega = \Gamma_0$ and $\partial\Omega = \Gamma_1$ to prove the remaining assertions. If $\partial\Omega = \Gamma_0$, then it is known $\psi \in C(\overline{\Omega})$ if Ω is only Wiener regular (a much weaker condition than Lipschitz); see for example [**10**]. Assume now for the meantime that Ω is C^2. By [**59**, Theorem 8.34] we have in fact $\psi \in C^{1,\eta}(\overline{\Omega})$ for every $\eta \in (0,1)$ since Ω is $C^{1,\eta}$. That $\psi \in W^{2,p}(\Omega)$ follows from the seminal work of Agmon-Douglis-Nirenberg [**3**]; see [**59**, Theorem 9.19] (with $k = 0$) for a precise statement of this result in current notation.

Now suppose $\Gamma_0 = \emptyset$ and we have a pure Robin problem. Then continuity up to the boundary comes from combining [**33**, Corollary 5.5] with [**108**, Corollary 2.9]) when $\alpha > 0$ and [**36**, Corollary 4.2] when $\alpha \leq 0$. If Ω is C^2 and $\alpha > 0$, then [**4**, Theorem 4.2] implies $\psi \in W^{2,p}(\Omega)$ for all $1 < p < \infty$. We now use a bootstrapping argument via results from [**81**]. By [**81**, Theorem 3.12.1], $\psi \in C^{2,\eta}(\Omega)$ for some $\eta \in (0,1)$. By [**81**, Theorem 10.2.1], $\psi \in C^1(\overline{\Omega})$. □

1.3. Eigenvalue properties of the Robin Laplacian

Here we collect various properties of (1.1.2) that will be important to us in what follows. As with Section 1.2, these are generally well-established, although more specialised. We make the same assumptions on Ω and α as in Section 1.2.

We will be paying special attention to the first eigenvalue λ_1. The results we need are collected in the following theorem. Here, since Ω is fixed, we will be treating the first eigenvalue as a function of $\alpha \in \mathbb{R}$ only, $\lambda_1 = \lambda_1(\alpha)$.

Theorem 1.3.1 (Properties of the first eigenvalue). *Suppose $\Omega \subset \mathbb{R}^N$ is bounded, Lipschitz and connected. Let $\lambda_1 = \lambda_1(\alpha)$ be the first eigenvalue of* (1.1.2) *on Ω, where $\alpha \in \mathbb{R}$. Then*

(i) *$\lambda_1(\alpha)$ is given by the variational characterisation*

$$\begin{aligned}\lambda_1(\alpha) &= \inf_{u \in H^1(\Omega)} \frac{Q_\alpha(u)}{\|u\|^2_{L^2(\Omega)}} \\ &= \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2\,dx + \int_{\partial\Omega} \alpha u^2\,d\sigma}{\int_\Omega u^2\,dx}\end{aligned} \tag{1.3.1}$$

where we have written $Q_\alpha(u)$ for $Q_\alpha(u,u)$. The infimum is attained by any eigenfunction ψ corresponding to $\lambda_1(\alpha)$;

(ii) *the eigenspace of $\lambda_1(\alpha)$ is one-dimensional;*

(iii) *any eigenfunction ψ of $\lambda_1(\alpha)$ does not change sign in Ω;*

(iv) *if in addition Ω is of class C^2 and $\Gamma_0 = \emptyset$, then ψ can be chosen strictly positive in $\overline{\Omega}$;*

(v) *for $\alpha \in \mathbb{R}$ fixed, $\lambda_1(\alpha)$ is the only eigenvalue of* (1.1.2) *having a positive eigenfunction;*

(vi) *as a function of α, $\lambda_1(\alpha)$ is analytic, strictly monotonically increasing and strictly concave (that is, ${\lambda_1}'(\alpha)$ is strictly decreasing) for all $\alpha \in \mathbb{R}$;*

(vii) *as $\alpha \to \infty$, $\lambda_1(\alpha) \to \mu_1$ from below, and ${\lambda_1}'(\alpha) \to 0$;*

(viii) $\lambda_1(0) = 0$ *and*

$$ {\lambda_1}'(0) = \frac{\sigma(\partial\Omega)}{|\Omega|}; \tag{1.3.2} $$

(ix) *finally,*

$$ \frac{\lambda_1(\alpha)}{-\alpha^2} \geq 1 \tag{1.3.3} $$

for all $\alpha < 0$, independent of Ω. There is equality in the limit as $\alpha \to -\infty$ if Ω is C^1.

The ratio in (1.3.1) is usually called the Rayleigh quotient.

Remark 1.3.2. The theorem assumes that Ω is connected. However, we will in general allow our domains to be disconnected. This does not substantially change the problem, however: any disconnected bounded Lipschitz domain Ω will consist of finitely many separated connected components (c.c.s for short), each having Lipschitz boundary. In such a case the eigenvalues of Ω (for any operator or boundary condition) can be found by collecting and reordering the eigenvalues of the c.c.s. For such domains U, V, in a slight abuse of notation we will say $U = V$ if and only if their c.c.s are in bijective correspondence and for each pair $\widetilde{U}, \widetilde{V}$ of c.c.s, there exists a rigid transformation τ such that $\tau(\widetilde{U}) = \widetilde{V}$. In particular, their spectra will coincide. Actually, in general we can allow countably infinitely many separated Lipschitz c.c.s, each one of which is bounded, and this remark is still valid. In such a case, disconnectedness will not affect the regularity of the eigenfunctions (Theorem 1.2.8). We now have

$$\lambda_1(\alpha) = \min\{\lambda_1(\widetilde{\Omega}, \alpha) : \ \widetilde{\Omega} \text{ is a c.c. of } \Omega\}.$$

In particular, (i), (vii) and (ix) hold if Ω is disconnected, and $\lambda_1(\alpha)$ is still continuous, strictly monotonically increasing and strictly concave, with $\lambda_1(0) = 0$. However, it is now possible that λ_1 will be a repeated eigenvalue: $0 < \lambda_1 = \lambda_2$. For each c.c., there will be exactly one eigenvalue $\lambda_p(\Omega)$ with a positive eigenfunction (the first eigenvalue on that c.c.), but we could arrange so that p is arbitrarily large. Moreover, λ_1 is in general no longer analytic as a function of α, and in particular (1.3.2) will fail in general. To see this, suppose Ω is the disjoint union of two c.c.s Ω_1 and Ω_2, where $\sigma(\partial\Omega_1)/|\Omega_1| < \sigma(\partial\Omega_2)/|\Omega_2|$. Then

$$\lim_{\alpha\to 0^-} \frac{\lambda_1(\alpha)}{\alpha} = \frac{\sigma(\partial\Omega_2)}{|\Omega_2|} > \frac{\sigma(\partial\Omega_1)}{|\Omega_1|} = \lim_{\alpha\to 0^+} \frac{\lambda_1(\alpha)}{\alpha}.$$

It is quite possible there could be similar "kinks" elsewhere. Another way to view this is via the theory on analytic perturbations in [**72**, Chapter VII], which we will use below to prove analyticity of λ_1 when Ω is connected. The same theory implies that in this case $\lambda_1(\Omega, \alpha)$ is analytic except at isolated "splitting" points, such as above where $\lambda_1(\Omega_1, \alpha)$ and $\lambda_1(\Omega_2, \alpha)$ cross at $\alpha = 0$.

Proof of Theorem 1.3.1. (i) The characterisation of the first eigenvalue as the infimum of the Rayleigh quotient is standard; see for example [**30**, Chapter VI] or [**62**, Section 1] (note that in the latter the sign of α is switched). For (ii) and (iii), if $\alpha > 0$ then the simplicity and positivity of the first eigenfunction ψ is also standard and can be deduced directly from properties of Q_α (use ψ^+, ψ^- as test functions for positivity); alternatively, see [**33**, Section 5]. If $\alpha < 0$, then we may use the argument in [**36**] to rewrite the problem (1.1.2) as an equivalent Robin problem with positive boundary parameter, albeit with a different (but still uniformly elliptic) form. Our claim then follows from the form properties in this case. Alternatively, see [**60, 62**], or see [**80**] if Ω is piecewise-C^1.

(iv) If Ω is C^2, then positivity up to the boundary follows from a simple argument involving the Hopf maximum principle (see, e.g., [**85**, Section 2] or [**107**, Section 2]), since then the boundary condition holds pointwise everywhere on $\partial\Omega$.

(v) Since our associated operator is a self-adjoint operator on the Hilbert space $L^2(\Omega)$, its eigenfunctions can be chosen to form an orthonormal basis for $L^2(\Omega)$, possibly after a rescaling and after an orthogonalisation of eigenfunctions associated with an eigenvalue of multiplicity larger than 1. Since the first eigenfunction is positive, it follows that every other eigenfunction must change sign. If $\alpha < 0$, we consider the rescaled operator $S_\alpha + \omega I$, $\omega >> 0$, as in the proof of Theorem 1.2.6, which merely shifts the eigenvalues by ω without affecting the eigenfunctions.

(vi) We first prove analyticity. Observe that for any given $u \in H^1(\Omega)$ the form $Q_\alpha(u)$ is analytic in α. This means that the associated family of self-adjoint operators T^0_α is holomorphic of type (B) in the sense of [**72**] (see Section VII.4.2 there. In fact we can actually show that the operators are of type (A) since their domain will be independent of α, but we do not need this). It follows from the theory in [**72**, Sections VII.3 and VII.4] (see Section VII.3.1 in particular) that any finite system of eigenvalues of T^0_α depends locally holomorphically (i.e. analytically) on α; the only possibility we have to rule out is the "splitting of eigenvalues". But this is impossible since it would require an eigenspace of dimension two at the point of splitting. Hence λ_1 depends analytically on α.

Concavity of $\lambda_1(\alpha)$ follows from the characterisation of λ_1 as the infimum of a family of functions $Q_\alpha(u)/\|u\|^2_{L^2(\Omega)}$ which are affine with respect to α. To prove strict concavity, note that if λ_1 is only weakly concave on an interval $(\alpha_1, \alpha_2) \subset \mathbb{R}$ then it must be linear on this interval. Since λ_1 is analytic, unique continuation implies λ_1 must be linear on $\mathbb{R}$. This is impossible since $0 \leq \lambda_1(\alpha) < \mu_1$ for all $\alpha > 0$ by (vii) and (viii). (These parts of (vii) and (viii) follow purely from the variational characterisation and do not use (vi), so there is no circularity.) Thus λ_1 is strictly concave everywhere.

We finally prove monotonicity. Since $Q_{\alpha_1}(u) < Q_{\alpha_2}(u)$ if $\alpha_1 < \alpha_2 \in \mathbb{R}$ for any $u \in H^1(\Omega)$, it is immediate that $\lambda_1(\alpha)$ is (weakly) monotonically increasing. But strict concavity now implies strict monotonicity.

(vii) Clearly $\lambda_1(\alpha) < \mu_1$, as can be seen by comparing variational characterisations. That we actually have $\lambda_1(\alpha) \to \mu_1$ as $\alpha \to \infty$ is noted in [**62**, Section 1]. That ${\lambda_1}'(\alpha) \to 0$ as $\alpha \to \infty$ follows immediately from combining the observations that $\lambda_1(\alpha)$ is monotonically increasing and $\lambda_1(\alpha) \leq \mu_1$ for all $\alpha > 0$.

(viii) It is immediate that $\lambda_1(0) = 0$, since then we have a Neumann problem (see also Remark 1.3.9). Since ${\lambda_1}'(\alpha)$ exists, (1.3.2) follows directly from [**62**, Equation 5], which states that $\lim_{\alpha \to 0} \lambda_1(\alpha)/\alpha = \sigma(\partial\Omega)/|\Omega|$, combined with the observation that $\lambda_1(0) = 0$. See also [**80**, Section 2.1] if $\partial\Omega$ is smooth.

(ix) Finally, (1.3.3) is established using a test function argument in the variational characterisation (1.3.1), adapted from [**60**, Theorem 2.3]. Our claim here is a direct consequence of Lemma 4.4.3, which is in slightly more general form that what we need here; we will omit the proof here to avoid repetition. For the limiting behaviour, proved using different techniques, see [**80**, Theorem 2.2] and [**86**, Theorem 1.1]. □

Remark 1.3.3. It is easy to modify the above results and proofs so that everything remains valid if we consider the problem (1.2.1) with $\alpha > 0$ instead of (1.1.2), albeit with appropriate modifications: for example now the infimum in (1.3.1) is over $H_0^1(\Omega; \Gamma_0)$ rather than $H^1(\Omega)$. Since it is quite standard, we will not go into the proof of this. We note in particular that we still have a simple first eigenvalue with eigenfunction ψ strictly positive in Ω, and that eigenfunction ψ will be strictly positive on Γ_1.

It is also important to know under what circumstances we can "approximate" a given Lipschitz domain Ω with a sequence Ω_n such that $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$. The following basic result will be very useful.

Theorem 1.3.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. There exist sequences of C^∞ domains $V_n \subset \Omega \subset U_n$ such that $\lambda_1(U_n)$, $\lambda_1(V_n) \to \lambda_1(\Omega)$ as $n \to \infty$.*

Proof. Using [**43**, Theorem 5.1] it is possible to approximate Ω from the outside by a sequence of C^∞ domains V_n in such a way that $|V_n| \to |\Omega|$ and [**32**, Theorems 4.4 and 6.2] may be applied to give $\lambda_1(V_n) \to \lambda_1(\Omega)$ as $n \to \infty$. This is also explained in [**35**, Section 4]. For the U_n, we can use exactly the same approximation argument, even though our sequence is now interior rather than exterior to Ω. In particular, [**43**, Theorem 5.1] remains valid, by considering the domain $B \setminus \Omega$, where B is a large ball containing Ω, and Theorems 4.4 and 6.2 of [**32**] still apply in this case, as noted in Remark 5.10 there. □

We next have a couple of analogous results for the higher eigenvalues. Just as many of the properties of the first eigenvalue can be deduced from its variational characterisation (1.3.1), here the so-called minimax formula for the kth eigenvalue will play an important role. This is given by

$$\lambda_k(\alpha) = \max_M \Big(\inf_{0 \neq u \in M} \frac{\int_\Omega |\nabla u|^2 \, dx + \alpha \int_{\partial\Omega} u^2 \, d\sigma}{\int_\Omega u^2 \, dx} \Big), \tag{1.3.4}$$

$\alpha \in \mathbb{R}$, where the maximum is taken over all subspaces M of $H^1(\Omega)$ of codimension $k-1$ (see [**30**, Chapter VI]). The infimum is attained by any eigenfunction associated with λ_k, and the maximal subspace can be obtained by removing the L^2 span of the eigenfunctions associated with the previous $k-1$ eigenvalues. That is, all functions in the maximal subspace M will be orthogonal in L^2 to the first $k-1$ eigenfunctions. (Here we stress it is important that eigenvalues are repeated according to their multiplicities!)

Theorem 1.3.5. *Suppose $\Omega \subset \mathbb{R}^N$ is a fixed bounded, Lipschitz domain. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ and $k \geq 1$. If $\alpha_1 \leq \alpha_2$, then for any $k \geq 1$, $\lambda_k(\alpha_1) \leq \lambda_k(\alpha_2)$. Moreover, for all $\alpha \in \mathbb{R}$, $\lambda_k(\alpha)$ is continuous, and $\lambda_k(\alpha) \leq \mu_k$.*

Proof. Consider the minimax formula for the kth eigenvalue (1.3.4). For fixed M, the infimum appearing in (1.3.4) is a (certainly weakly) monotonically increasing function of α. Hence the same must be true of the maximum over all such M. (This result is explicitly contained in [**30**, Theorem VI.2.6]. Although they only treat the case $N = 2$ it is clear that the dimension of the space will not affect any of the arguments.)

Continuity also follows directly from the minimax formula. That is, since for every $u \in H^1(\Omega)$ the Rayleigh quotient $Q_\alpha(u)$ is continuous with respect to α, for a given subspace M of $H^1(\Omega)$ the same must be true of its infimum; hence the same is also true of of the maximum over all such subspaces.

That $\lambda_k(\alpha) \leq \mu_k$ follows immediately from (1.3.4), since we recover μ_k if we replace $H^1(\Omega)$ by the smaller subspace $H^1_0(\Omega)$. □

In fact the same arguments used in the proof of Theorem 1.3.1 to show analyticity of $\lambda_1(\alpha)$ imply that $\lambda_k(\alpha)$ is now a piecewise-analytic function of α. Even if Ω is connected, there may be splitting (or bifurcation) points. However, it is not clear if λ_k is concave with respect to α, since given $\alpha_1 < \alpha_2$ and $\alpha \in (\alpha_1, \alpha_2)$, there is no reason to expect an eigenfunction ψ associated with $\lambda_k(\alpha)$ will lie in the maximal subspace M for α_i.

We also have the following remarks on the topic of domain monotonicity. We start with the idea of homothety. For any $\Omega \subset \mathbb{R}^N$ and $t > 0$, we can define the domain

$$t\Omega := \{tx \in \mathbb{R}^N : x \in \Omega\}; \tag{1.3.5}$$

then $\partial(t\Omega) = t(\partial\Omega)$ and $|t\Omega| = t^N|\Omega|$. We can think of this as "blowing up" Ω by a factor of t.

Remark 1.3.6. (i) The Dirichlet problem scales well. Making a change of variables $x \mapsto tx$ in the Rayleigh quotient, it is easy to show that $\mu_1(t\Omega) = t^{-2}\mu_1(\Omega)$. In the special case when $N = 2$ this means the product $|t\Omega|\mu_1(t\Omega)$ is invariant with respect to t. Let us now consider the Robin problem. Because of the boundary term in the Rayleigh quotient, the eigenvalues will not scale cleanly with respect to t for fixed α. However, by making the homothety substitution $x \mapsto \alpha x$, we see the problem (1.1.2) is equivalent to the problem

$$\begin{aligned} -\Delta u &= \frac{\lambda}{\alpha^2} u \qquad && \text{in } \alpha\Omega, \\ \frac{\partial u}{\partial \nu} + u &= 0 && \text{on } \partial(\alpha\Omega). \end{aligned} \tag{1.3.6}$$

In particular, instead of considering a fixed domain Ω and varying α, we could assume $\alpha \equiv 1$ (or any other pre-specified constant) and vary $|\Omega|$ (that is, consider the family $t\Omega$). Of course, this only works if we do not change the sign of α or t.

(ii) Another property of the Dirichlet problem is a *domain monotonicity* property: if $U \subset V \subset \mathbb{R}^N$, then $\mu_k(U) \geq \mu_k(V)$ for any $k \geq 1$. This follows from the variational characterisation of μ_k (cf. (1.3.4) but with H_0^1 in place of H^1: then $H_0^1(U)$ may be regarded as a subset of $H_0^1(V)$ by extending functions in $H_0^1(U)$ by zero in $V \setminus U$). However, it is well known that this monotonicity property fails for the Robin problem, even if $k = 1$; see [**62, 94**] for counterexamples. This will be an important point in Section 4.3. We will also prove there that even if $\lambda_k(U, \alpha) \leq \lambda_k(V, \alpha)$ for some $\alpha \in (0, \infty)$, this may not be true for all $\alpha \in (0, \infty)$ (this result is probably not new, but we know of no reference).

Lemma 1.3.7. *Let $B(x, r)$ denote the ball of radius r centred at x. For $\alpha > 0$ fixed the first eigenvalue $\lambda_1(B(x, r))$ is a strictly decreasing, continuous function of $r > 0$.*

Proof. See for example [**22**, Lemma 4.1]. □

Remark 1.3.8. Note that Lemma 1.3.7 remains true, with the same proof, for the p-Laplacian, $1 < p < \infty$ (see Section 3.1).

We conclude with a couple of comments about the Neumann problem.

Remark 1.3.9. We will denote the nth eigenvalue of the Neumann problem

$$\begin{aligned} -\Delta u = \lambda u \qquad & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \qquad & \text{on } \partial\Omega, \end{aligned}$$

which (as has been mentioned) occurs when we set $\alpha = 0$ in (1.1.2), by $\lambda_k(0) = \lambda_k(\Omega, 0)$, since from both a theoretical and practical perspective this seems to be most convenient. Of course, all the results we have listed in this section for the case $\alpha \leq 0$ and $\Gamma_0 = \emptyset$ in (1.2.1) apply here.

Lemma 1.3.10. *If $\Omega \subset \mathbb{R}^N$ is connected, bounded Lipschitz, then $\lambda_2(\Omega, 0) > 0$.*

Proof. This certainly follows from the fact that if Ω is connected then the first Neumann eigenvalue (i.e. 0, in our notation given by $\lambda_1(\Omega, 0)$) is simple and the only eigenfunctions are constant. (To see this, for example, use that any such eigenfunction u satisfies $\Delta u = 0$ in Ω, $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$, that is, u is harmonic and locally constant on $\partial\Omega$; cf. [**59**, Problem 2.2].) $\square$

Chapter 2

An Inequality for the First Eigenvalue of the Robin Laplacian

This chapter is devoted entirely to the study of a Faber-Krahn type inequality for the first eigenvalue of the Robin problem (1.1.2).

2.1. The Faber-Krahn inequality

We start by stating the main theorems we will prove in this chapter. The notation is as in Chapter 1; however, since we will fix $\alpha > 0$ throughout this chapter, we will denote by $\lambda_1(\Omega) > 0$ the first eigenvalue of (1.1.2) on Ω.

Theorem 2.1.1 (Faber-Krahn for Robin problems). *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let B denote a ball having the same volume as Ω. Then $\lambda_1(\Omega) \geq \lambda_1(B)$.*

Theorem 2.1.2 (Sharpness of the inequality). *If in Theorem 2.1.1 the domain Ω is of class C^2, then $\lambda_1(\Omega) = \lambda_1(B)$ if and only if $\Omega = B$ after a translation.*

The first complete proof of Theorem 2.1.1 appeared recently in [**35**]; our contribution here is a proof of Theorem 2.1.2 for the first time.

The original and famous theorem of Faber [**50**] and Krahn [**78**], proved in the 1920s, which resolved the conjecture of Lord Rayleigh [**97**], is as follows.

Theorem 2.1.3 (Faber-Krahn). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and let B denote a ball having the same volume as Ω. Then $\mu_1(\Omega) \geq \mu_1(B)$, with equality if and only if Ω is a ball up to a set of capacity zero.*

See also [**96**]. For a definition and properties of capacity, see [**65**]. This theorem is sometimes referred to as the theorem of Rayleigh-Faber-Krahn. The usual proof uses a technique called Schwarz symmetrisation; see for example [**13, 74, 96**]. Roughly speaking, given a function $u \in W^{1,p}(\Omega) \cap C(\Omega)$ we construct a new function $u^* \in W^{1,p}(B) \cap C(B)$, such that the (upper) level sets $\{x \in B : u^*(x) > t\}$ are concentric balls, and $|\{x \in B : u^*(x) > t\}| = |\{y \in \Omega : u(y) > t\}|$. It can be shown, although this is not trivial, that Schwarz symmetrisation preserves the L^p-norm, but decreases the integral $\int_\Omega |\nabla u|^p \, dx$. In particular, when $p = 2$ we symmetrise the eigenfunction $u = \varphi \in H_0^1(\Omega)$ associated with $\mu_1(\Omega)$. The new function $\varphi^* \in H_0^1(B)$ has smaller Dirichlet integral $\int_\Omega |\nabla \varphi^*|^2 \, dx$. By comparing Rayleigh quotients (cf. (1.3.1) with $H_0^1(\Omega)$ in place of $H^1(\Omega)$), the result follows.

The proof that the inequality is sharp, that is, that the ball is the only minimiser, is harder: one has to show that if Ω is not a ball, then the symmetrisation process strictly decreases the Dirichlet integral. Despite this (or possibly because of it?), the issue of uniqueness of the minimiser is omitted from many of the standard works on the subject, including the seminal book of Pólya and Szegö [**96**]; see the discussion in [**74**, Section II.8].

Given that the Robin problem (1.1.2) is in many ways so similar to the Dirichlet problem (1.1.1), it is natural to ask if the Faber-Krahn inequality holds in this case, and indeed it is claimed Krahn himself asked the question. (This seems to be folklore; see for example the review on MathSciNet of [**20**]. But we cannot find the source.) But the symmetrisation techniques which work in the Dirichlet case do not seem to work here, since in general it is not clear if symmetrisation will decrease the expression $Q_\alpha(u)$ appearing in the Rayleigh quotient (1.3.1).

In fact the first result in the direction of Theorem 2.1.1 was in 1957 with a paper of Payne and Weinberger [**94**], where they proved that $\lambda_1(\Omega) \geq \lambda_1(\tilde{B})$ if $\tilde{B}$ is a ball containing Ω. Unlike in the Dirichlet case this is not an obvious result since the domain monotonicity property fails (see Remark 1.3.6(ii)). Other techniques for estimating $\lambda_1(\Omega)$ from below were described in a paper of Hersch [**67**]. The next major development was the introduction in the 1980s of a new method in [**19, 20**], the ideas for which came from [**67**] and a conformal invariant called extremal length due to Ahlfors and Beurling (see [**5**, Chapter 4]). This involves the use of a functional of the level sets of the first eigenvalue ψ on Ω, which we describe in Section 2.2. This was used to sketch the proof of Theorem 2.1.1 in dimension $N = 2$, albeit with a number of significant details omitted. These were provided just a few years ago in [**35**], which also generalised the result to N dimensions, thus completing the proof of Theorem 2.1.1. As mentioned earlier, our contribution here is to strengthen substantially the method and estimates in

[**20, 35**]. This enables us to give a proof of Theorem 2.1.2 for the first time, thus providing a complete answer to the problem.

We will include the background results from [**35**] leading up to and including the proof of Theorem 2.1.1, both for completeness' sake, and because the exposition would be rather difficult to follow otherwise. We include some of the more important or illustrative proofs from there in this chapter; the others, especially the more technical proofs, have been reproduced in Appendix B. Finally, as in [**35**], we will set the proof up so it works for Dirichlet boundary conditions as well, thus giving a new proof of the sharpness of the inequality in Theorem 2.1.3. Note however that our method only works for Wiener (or Dirichlet) regular bounded domains; see Remark 2.2.2. The new material in this chapter has been published in [**39**]; since publication the results have been generalised and developed in other directions in [**22, 23, 31**].

2.2. A functional H_Ω of the level sets

In order to cover the Robin and Dirichlet cases simultaneously, in this section we will consider the eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} + \alpha u &= 0 && \text{on } \Gamma_1, \end{aligned} \tag{2.2.1}$$

on a given bounded domain $\Omega \subset \mathbb{R}^N$, where Γ_0, Γ_1 are disjoint open and closed subsets of $\partial\Omega$ with $\Gamma_0 \cup \Gamma_1 = \partial\Omega$ (see also (1.2.1)). Here we will assume either Ω is of class C^2, or else Ω is Wiener regular (see Remark 2.2.2) and $\Gamma_1 = \emptyset$. If $\Gamma_1 = \emptyset$, then we have a pure Dirichlet problem, while if $\Gamma_0 = \emptyset$, then we have a pure Robin problem. We will also assume that $\alpha > 0$ is a constant, although for this section and the next we could assume without loss of generality that (for example) $\alpha \in C^1(\Gamma_1)$. Finally, we may assume without loss of generality that Ω is connected. Indeed, for disconnected Ω, given the Faber-Krahn inequality for connected domains it is immediate that $\lambda_1(\Omega) > \lambda_1(B)$. This follows from applying the Faber-Krahn inequality to each connected component of Ω (see Remark 1.3.2) and then using strict monotonicity of $\lambda_1(B)$ with respect to the volume of B (see Lemma 1.3.7).

We introduce the following notation. For open sets $U \subset \Omega$ we denote the interior and exterior boundaries by $\partial_i U := \partial U \cap \Omega$ and $\partial_e U := \partial U \cap \partial\Omega$, respectively (see also (A1.1)). We will be interested in the case where the subsets U are the level sets of the first eigenfunction ψ of (2.2.1). Recall that $\lambda_1(\Omega)$ is simple, and that ψ can be chosen to be strictly positive in Ω (see Remark 1.3.3). We will normalise $\psi \geq 0$ so that $\|\psi\|_\infty = 1$, and denote the level sets of ψ by

$$U_t := \{x \in \Omega : \psi(x) > t\} \tag{2.2.2}$$

and the level surfaces by

$$S_t := \{x \in \Omega : \psi(x) = t\}, \tag{2.2.3}$$

where $t \in [0, 1]$. Note that since $\psi \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ by Theorem 1.2.8, Sard's lemma [**68**, Theorem 3.1.3] implies S_t is, locally, a C^∞ $(N-1)$-manifold inside Ω for almost every $t \in (0, 1)$, although *a priori* the intersection with $\partial\Omega$ could be nasty. Moreover the level sets U_t are open. In particular S_t must coincide with the interior boundary $\partial_i U_t$ of U_t for almost all $t \in (0, 1)$ with respect to Lebesgue measure on $(0, 1)$ (cf. also the comments around (A4.6)).

The principal reason for assuming that Ω is of class C^2 is so that we have the extra regularity of the eigenfunction from Theorem 1.2.8, namely that

$$\psi \in C^\infty(\Omega) \cap C^1(\overline{\Omega}) \cap W^{2,p}(\Omega)$$

for all $p \in (1, \infty)$. If $\Gamma_1 = \emptyset$, then U_t is compactly contained in Ω since the sets $\{x \in \overline{\Omega} : \psi(x) = 0\} \supseteq \partial\Omega$ and $\overline{U}_t = \{x \in \overline{\Omega} : \psi(x) \geq t\}$ are compact and disjoint. In this case, by Sard's lemma, $S_t = \partial U_t$ is a C^∞ manifold for almost every $t \in (0, 1)$. We also set

$$m := \min_{x \in \overline{\Omega}} \psi(x) \geq 0.$$

By Theorem 1.3.1(iv) and Remark 1.3.3, $\psi(x) > 0$ for all $x \in \Gamma_1$, and ψ attains its minimum m on $\partial\Omega$; if $\Gamma_0 = \emptyset$ then $m > 0$, while clearly otherwise $m = 0$. Finally, we observe that $S_t = \emptyset$ if $t \notin (m, 1]$, and $\overline{U}_t \cap \Gamma_0 = \emptyset$ for all $t \in (m, 1)$.

We next recall the following rather technical result from [**35**] concerning the behaviour of the level surfaces S_t, which will needed in the sequel.

Lemma 2.2.1 ([**35**], Lemma 2.3)**.** *The following are true.*

(i) *The function $t \mapsto \sigma(S_t)$ is in $L^1((0, \infty))$.*

(ii) *The* S_t *are of class* C^∞ *and the* U_t *are Lipschitz for almost all* $t \in (m, 1)$.

(iii) *If* $\Gamma_1 \neq \emptyset$, *then there exist* $c > 0$ *and* $t_1 \in (m, 1)$ *such that* $\sigma(S_t) \leq c\sigma(\partial\Omega)$ *for all* $t \in (m, t_1]$.

The proof can be found in Appendix B. We observe that (i) uses the coarea formula, (ii) uses Sard's lemma, and (ii) and (iii) both require Ω to be "reasonably smooth" if $\Gamma_1 \neq \emptyset$. The situation is different if $\Gamma_1 = \emptyset$.

Remark 2.2.2. (i) If $\Gamma_1 = \emptyset$, the only regularity assumption on $\partial\Omega$ we need is that $\psi \in C(\overline{\Omega})$ in order to ensure all level sets U_t are compactly contained in Ω. This assumption on ψ is equivalent to the assumption that Ω is Wiener (or Dirichlet) regular (see, e.g., [**10**]). We remark that this is a far weaker condition than Lipschitz regularity.

(ii) Note also that the actual inequality in Theorem 2.1.3 can be obtained for *all* domains Ω of finite volume, not necessarily bounded, using a standard perturbation argument as in [**35**, Section 4]. The sharpness of the inequality can be obtained for all bounded domains Ω, but it requires a different method from the one above. A proof using symmetrisation is included in [**39**, Section 4]. For arbitrary bounded domains the sharpness only holds up to sets of capacity zero, since removing a set of capacity zero from a domain Ω will not increase its eigenvalues (this is well known; see for example [**24**, Section 2] and the references therein).

With this background material in mind, we are ready to look at the method behind the proof of Theorems 2.1.1 and 2.1.2. The key is the following functional. If $U \subset \Omega$ is open with $\overline{U} \cap \Gamma_0 = \emptyset$, and $\varphi \in C(\Omega)$ is non-negative, then as in [**20, 35**], we let

$$H_\Omega(U, \varphi) := \frac{1}{|U|}\Big(\int_{\partial_i U} \varphi \, d\sigma + \int_{\partial_e U} \alpha \, d\sigma - \int_U |\varphi|^2 \, dx\Big). \tag{2.2.4}$$

Since φ is continuous, each of these integrals makes sense; moreover, we will be working with restricted choices of U and φ for which the last integral is finite. Thus H_Ω will always be well-defined, although $H_\Omega(U, \varphi) = \infty$ is possible. More precisely, the subsets U will be the level sets U_t of ψ; in particular, for almost all t the $\partial_i U$ will be the level surfaces S_t. If $\Gamma_0 = \emptyset$, we also restrict our choice of test functions φ; see Section 2.3.

The main reason why the functional H_Ω is useful is that it can be used to obtain an estimate of the first eigenvalue $\lambda_1(\Omega)$ of (2.2.1) (see Section 2.3). For this, the function $|\nabla\psi|/\psi = |\nabla(\ln\psi)|$ will play an important role. The following proposition motivates the definition of H_Ω. Although this was already proved in [**35**, Proposition 2.1], we include the straightforward but informative proof.

Proposition 2.2.3. *Let ψ be a positive first eigenfunction of the problem* (2.2.1). *Then*

$$H_\Omega(U_t, \frac{|\nabla\psi|}{\psi}) = \lambda_1(\Omega) \tag{2.2.5}$$

for almost all $t \in (m, 1)$.

Proof. Fix $t \in (m, 1)$ such that Lemma 2.2.1(ii) holds for this t. Since $-\Delta\psi = \lambda_1(\Omega)\psi$ in U_t, we have

$$\begin{aligned}\lambda_1(\Omega) &= \frac{1}{|U_t|}\int_{U_t} \frac{-\Delta\psi}{\psi}\,dx \\ &= \frac{1}{|U_t|}\int_{U_t} -\operatorname{div}\Big(\frac{\nabla\psi}{\psi}\Big) - \frac{|\nabla\psi|^2}{\psi^2}\,dx.\end{aligned}$$

Now since $\psi \in W^{2,p}(\Omega)$ for all $p \in (1,\infty)$ and $\psi \geq t > 0$ on $\overline{U}_t$, we have $|\nabla\psi|/\psi \in W^{1,p}(U_t)$ for all $p \in (1,\infty)$. Since U_t is Lipschitz, we may apply the divergence theorem, Theorem A4.5(i), to obtain

$$\int_{U_t} -\operatorname{div}\Big(\frac{\nabla\psi}{\psi}\Big)\,dx = \int_{S_t} -\frac{1}{\psi}\frac{\partial\psi}{\partial\nu_{U_t}}\,d\sigma + \int_{\partial_e U_t} -\frac{1}{\psi}\frac{\partial\psi}{\partial\nu_\Omega}\,d\sigma.$$

But observe that S_t is a smooth level surface of ψ. Hence $|\nabla\psi| = -\frac{\partial\psi}{\partial\nu_{U_t}}$ on S_t. Moreover, by the boundary condition

$$\alpha = -\frac{1}{\psi}\frac{\partial\psi}{\partial\nu_\Omega}$$

on $\partial_e U_t \subset \Gamma_1$. Hence

$$\int_{U_t} -\operatorname{div}\Big(\frac{\nabla\psi}{\psi}\Big)\,dx = \int_{S_t} \frac{|\nabla\psi|}{\psi}\,d\sigma + \int_{\partial_e U_t} \alpha\,d\sigma.$$

Putting this all together yields

$$\lambda_1(\Omega) = \frac{1}{|U_t|}\Big(\int_{S_t}\frac{|\nabla\psi|}{\psi}\,d\sigma + \int_{\partial_e U_t}\alpha\,d\sigma - \int_{U_t}\frac{|\nabla\psi|^2}{\psi^2}\,dx\Big),$$

as required. ☐

Remark 2.2.4. Another way of looking at the above argument is via the weak formulation. That is, we use $1/\psi$ as a test function in the weak form $Q_\alpha(\psi, v) = \lambda_1 \int_\Omega \psi v\, dx$ for all $v \in H^1(\Omega)$. Note that since $\psi \geq m > 0$ on $\overline{\Omega}$, $1/\psi \in H^1(\Omega)$. This yields

$$\int_\Omega \nabla\psi \cdot \nabla\Big(\frac{1}{\psi}\Big) + \int_{\partial\Omega} \alpha\, d\sigma = \lambda_1(\Omega) \int_\Omega dx,$$

or, after rearranging,

$$\lambda_1(\Omega) = \frac{1}{|\Omega|}\Big(\int_{\partial\Omega} \alpha\, d\sigma - \int_\Omega \frac{|\nabla\psi|^2}{\psi^2}\, dx\Big).$$

More generally, to obtain H_Ω for a general level set U_t, we do the same thing for the problem

$$\begin{aligned} -\Delta\psi &= \lambda_1(\Omega)\psi && \text{in } U_t \\ \frac{\partial\psi}{\partial\nu_\Omega} + \alpha\psi &= 0 && \text{on } \partial_e U_t \\ \frac{\partial\psi}{\partial\nu_{U_t}} + \Big(\frac{|\nabla\psi|}{\psi}\Big)\psi &= 0 && \text{on } \partial_i U_t, \end{aligned}$$

since inserting $1/\psi$ into the weak equation for this problem will clearly yield (2.2.5) whenever U_t is Lipschitz (i.e. whenever everything in the above expression makes sense). This does not by itself weaken the smoothness requirement on Ω; however this has subsequently been achieved in [**22**] by combining this idea with a cut-off argument.

We will use Proposition 2.2.3 to obtain a characterisation of $\lambda_1(\Omega)$ in terms of H_Ω for arbitrary $\varphi \in C(\Omega)$ non-negative. For this we need another technical result from [**35**]; a proof is in Appendix B. Given $\varphi \in C(\Omega)$ non-negative we set

$$w := \varphi - \frac{|\nabla\psi|}{\psi}. \tag{2.2.6}$$

We also define

$$F(t) := \int_t^1 \frac{1}{\tau} \int_{S_\tau} w\, d\sigma\, d\tau \tag{2.2.7}$$

for all $t \in (m, 1)$.

Lemma 2.2.5 ([**35**], Lemma 3.3)**.** *Suppose that $\varphi \in C(\Omega)$ is non-negative such that $\varphi \in L^1(U)$ for every open set $U \subset \Omega$ with $\overline{U} \subset \Omega \cup \Gamma_1$. Then F is absolutely* *continuous on* $[\varepsilon, 1)$ *for every* $\varepsilon \in (0,1)$ *and*

$$\frac{d}{dt}F(t) = -\frac{1}{t}\int_{S_t} w\, d\sigma$$

for almost all $t \in (0,1)$.

The proof makes use of the coarea formula (see Theorem A4.6). Note that absolutely continuous functions are differentiable almost everywhere (for a definition and discussion of them, see [**99**, Chapter 7]). We can now give the important characterisation of $\lambda_1(\Omega)$ mentioned earlier.

Theorem 2.2.6. *Let* $\varphi \in C(\Omega)$ *be non-negative. Then*

$$H_\Omega(U_t, \varphi) = \lambda_1(\Omega) - \frac{1}{|U_t|}\Big(\frac{1}{t}\frac{d}{dt}\big(t^2 F(t)\big) + \int_{U_t} |w|^2\, dx\Big) \tag{2.2.8}$$

for almost all $t \in (m, 1)$.

Proof. We use the definition of w to obtain an expression for the difference between $H_\Omega(U_t, \varphi)$ and $H_\Omega(U_t, |\nabla\psi|/\psi)$. First,

$$|\varphi|^2 = \Big(w + \frac{|\nabla\psi|}{\psi}\Big)^2 = |w|^2 + 2w\frac{|\nabla\psi|}{\psi} + \frac{|\nabla\psi|^2}{\psi^2}.$$

Now fix $t \in (m,1)$ such that the results of Section 2.2 hold. We apply the coarea formula (Theorem A4.6), valid for any non-negative measurable, not necessarily integrable, function to obtain

$$\int_{U_t} w\frac{|\nabla\psi|}{\psi} = \int_t^1 \int_{S_\tau} \frac{w}{\psi}\, d\sigma\, d\tau = \int_t^1 \frac{1}{\tau}\int_{S_\tau} w\, d\sigma\, d\tau,$$

where we have also used $\psi = \tau$ on S_τ. Now if we use the definition of H_Ω and the characterisation (2.2.5) of $\lambda_1(\Omega)$, we see that

$$\begin{aligned} H_\Omega(U_t, \varphi) &= \frac{1}{|U_t|}\Big(\int_{S_t} \frac{|\nabla\psi|}{\psi} + w\, d\sigma + \int_{\partial_e U_t} \alpha\, d\sigma - \int_{U_t}\big(w + \frac{|\nabla\psi|}{\psi}\big)^2\, dx\Big) \\ &= \lambda_1(\Omega) + \frac{1}{|U_t|}\Big(\int_{S_t} w\, d\sigma - 2\int_t^1 \frac{1}{\tau}\int_{S_\tau} w\, d\sigma\, d\tau - \int_{U_t} |w|^2\, dx\Big). \end{aligned}$$

Finally, using Lemma 2.2.5, note that

$$\begin{aligned} \frac{1}{t}\frac{d}{dt}\big(t^2F(t)\big) = \frac{1}{t}\big(2tF(t) + t^2F'(t)\big) &= 2F(t) + tF'(t) \\ &= 2\int_t^1 \frac{1}{\tau}\int_{S_t} w\, d\sigma\, d\tau - \int_{S_t} w\, d\sigma. \end{aligned}$$

Since this holds for almost every $t \in (m,1)$, this gives us the desired result. $\square$

2.3. An estimate of the first eigenvalue

In this section we will show how the functional H_Ω can be used to give a lower bound for $\lambda_1(\Omega)$. The estimates we present strengthen the corresponding bounds obtained in [**20, 35**]. The extra information we obtain will allow us to prove the uniqueness of the minimiser, which we will do in the next section.

We start with the case of pure Robin boundary conditions $\Gamma_0 = \emptyset$ in (2.2.1). In this case we restrict our choice of test functions $0 \le \varphi \in C(\Omega)$ to a subset $M_\alpha := M_\alpha(\Omega)$ of $\{u \in C(\Omega) : u \ge 0 \text{ in } \Omega\}$ given by

$$M_\alpha := \{u \in C(\Omega) : u \ge 0 \text{ in } \Omega \text{ and } \limsup_{x \to z} u(x) \le \alpha(z) \text{ for all } z \in \partial\Omega\}.$$

Such functions are called *admissible repartitions* in [**20**]. If $\Gamma_0 \neq \emptyset$ we will continue to use the unrestricted class $\{u \in C(\Omega) : u \ge 0 \text{ in } \Omega\}$. Observe that the restriction in the Robin case is a natural analogue of the unrestricted Dirichlet case, since in the latter case where $\alpha = \infty$, we have "$M_\infty = \{u \in C(\Omega) : u \ge 0 \text{ in } \Omega\}$" in some sense.

Lemma 2.3.1. *If $\alpha \in C^1(\partial\Omega)$, then $M_\alpha \subset L^\infty(\Omega)$.*

Proof. Suppose $u \in M_\alpha$. By definition, given $z \in \partial\Omega$ there exists $r_z > 0$ such that $u(x) \le \alpha(z) + 1$ for all $x \in B(z, r_z) \cap \Omega$. The sets $\{B(z, r_z) : z \in \partial\Omega\}$ form an open cover of $\partial\Omega$. Extract a finite subcover of balls $B_i = B(z_i, r_{z_i})$ centred at z_i, $i = 1, \dots, n$, and let $U = \bigcup_{i=1}^n B_i$. Then $\sup_{x \in U \cap \Omega} u(x) \le \max_i \alpha(z_i) + 1 < \infty$, so $u \in L^\infty(U \cap \Omega)$. Now let $V \subset\subset \Omega$ be such that $\overline{\Omega} \subset U \cup V$. Then $u \in C(\overline{V}) \subset L^\infty(V)$ and so we conclude $u \in L^\infty(\Omega)$. $\square$

It is now clear that $H_\Omega(U_t, \varphi)$ will always be well-defined for $\varphi \in M_\alpha$, since the volume integral in (2.2.4) is always finite. It is of course still possible that $H_\Omega(U_t, \varphi) = \infty$ for some U_t and φ. If $\Gamma_0 \neq \emptyset$, then the situation becomes more interesting if $\overline{U}_t \cap \Gamma_1 \neq \emptyset$ for some $t \in (0, 1)$. In this case we assume $0 \le \varphi \in C(\Omega) \cap L^2(U_t)$ for all $t \in (0, 1)$.

Remark 2.3.2. In the case $\Gamma_0 = \emptyset$, in general $|\nabla\psi|/\psi \in M_\alpha$ if and only if ψ is locally constant on $\partial\Omega$. To see this, suppose $|\nabla\psi|/\psi \in M_\alpha$. Using the boundary condition in (2.2.1),

$$\frac{|\nabla\psi|}{\psi} \le \alpha = -\frac{1}{\psi}\frac{\partial\psi}{\partial\nu_\Omega} \le \frac{|\nabla\psi|}{\psi}$$

at every point on the boundary. The only possibility is that $\frac{\partial\psi}{\partial\nu_\Omega} = -|\nabla\psi|$, which can only be the case if ψ is locally constant on $\partial\Omega$. Conversely, if ψ is locally constant on $\partial\Omega$, then each component of $\partial\Omega$ is a level surface S_t for some t. This implies $\frac{\partial\psi}{\partial\nu_\Omega} = -|\nabla\psi|$ and from the boundary condition $|\nabla\psi|/\psi = \alpha$ on $\partial\Omega$.

The above remark also implies $|\nabla\psi|/\psi \geq \alpha$ on $\partial\Omega$. This leads to the following estimate of $w = \varphi - |\nabla\psi|/\psi$ for $\varphi \in M_\alpha$. It follows from an elementary argument using the compactness of $\partial\Omega$. A proof can be found in Appendix B.

Lemma 2.3.3 ([**35**], Lemma 3.4)**.** *Let $\varphi \in M_\alpha$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $w(x) \leq \varepsilon$ for all $x \in \Omega$ with* $\operatorname{dist}(x, \partial\Omega) < \delta$*.*

In the next theorem we will use the characterisation (2.2.8) of H_Ω to estimate the first eigenvalue from above. That estimate is the key to proving Theorems 2.1.1 and 2.1.2. The theorem we present significantly strengthens [**35**, Theorem 3.1] (which in turn strengthened [**20**, Theorem 1], which first gave the estimate), giving a strict inequality for a larger set of $t \in (m, 1)$.

Theorem 2.3.4. *Suppose $\Gamma_0 = \emptyset$ and let $\varphi \in M_\alpha$. If $\varphi \neq |\nabla\psi|/\psi$, then there exists a set $S \subset (m, 1)$ of positive measure such that*

$$H_\Omega(U_t, \varphi) < \lambda_1(\Omega)$$

for all $t \in S$.

Proof. Assume that $\varphi \neq |\nabla\psi|/\psi$, and suppose for a contradiction that $H_\Omega(U_t, \varphi) \geq \lambda_1(\Omega)$ for almost all $t \in (m, 1)$. By Theorem 2.2.6, we see that

$$\frac{d}{dt}\big(t^2 F(t)\big) \leq -t \int_{U_t} |w|^2 \, dx \leq 0 \tag{2.3.1}$$

for almost all $t \in (m, 1)$. We can also write this as

$$2F(t) + \int_{U_t} |w|^2 \, dx \leq \int_{S_t} w \, d\sigma \tag{2.3.2}$$

for almost all $t \in (m, 1)$. Using the fundamental theorem of calculus for absolutely continuous functions [**99**, Theorem 7.18], (2.3.1) shows that $t^2 F(t)$ is non-increasing on $(m, 1)$. Now $\varphi \neq |\nabla\psi|/\psi$ implies $w \neq 0$, and since $w \in C(\Omega)$,

$$\int_{U_m} |w|^2 \, dx = \int_\Omega |w|^2 \, dx > 0.$$

Moreover, since the U_t are level sets, if $s \geq t$, then

$$\int_{U_s} |w|^2 \, dx \leq \int_{U_t} |w|^2 \, dx,$$

while obviously

$$\int_{U_1} |w|^2 \, dx = 0$$

since $U_1 = \emptyset$ by definition. Hence there exists $t^* \in (m, 1]$ satisfying

$$t^* = \sup\{t \in (m, 1) : \int_{U_t} |w|^2 \, dx > 0\}.$$

But then

$$\frac{d}{dt}\big(t^2 F(t)\big) \leq -t \int_{U_t} |w|^2 \, dx < 0$$

for almost all $t \in (m, t^*)$, so $t^2 F(t)$ is strictly decreasing on (m, t^*).

We showed earlier that $t^2 F(t)$ is non-increasing on $[t^*, 1)$, and that F is continuous with $F(1) = 0$. It follows that there exist $\xi > 0$ and $t_0 \in (m, t^*)$ such that $F(t) > \xi$ for all $t \in (m, t_0]$. By Lemma 2.2.1(iii), there exist $t_1 \in (m, t_0]$ and $c > 0$ such that $\sigma(S_t) < c\sigma(\partial\Omega)$ for all $t \in (m, t_1]$. If we set

$$\varepsilon := \frac{\xi}{c\sigma(\partial\Omega)},$$

then by Lemma 2.3.3, there exists $\delta > 0$ such that $w(x) \leq \varepsilon$ for all $x \in \Omega$ with $\operatorname{dist}(x, \partial\Omega) < \delta$. Since ψ attains a strict minimum on $\partial\Omega$, there exists $t \in (m, t_1]$ such that $\operatorname{dist}(x, \partial\Omega) < \delta$ for all $x \in S_t$. For such t, using (2.3.2), and by our choice of ε, ξ,

$$0 < 2\xi < 2F(t) \leq \int_{S_t} w \, d\sigma \leq \varepsilon\sigma(S_t) \leq \xi,$$

a contradiction. Hence we have shown that if $\varphi \neq |\nabla\psi|/\psi$, then there exists $t \in (m, 1)$ such that $H_\Omega(U_t, \varphi) < \lambda_1(\Omega)$, as claimed. □

We now prove an analogous theorem for the general case $\Gamma_0 \neq \emptyset$ in (2.2.1), which requires a different method of proof. This theorem strengthens [**35**, Theorem 3.5] in the same way that Theorem 2.3.4 strengthens [**35**, Theorem 3.1].

Theorem 2.3.5. *Suppose $\Gamma_0 \neq \emptyset$, and suppose $\varphi \in C(\Omega)$ is non-negative, with $\varphi \in L^2(U_t)$ for every $t \in (0, 1)$. If $\varphi \neq |\nabla\psi|/\psi$, then there exists a set $S \subset (0, 1)$ of positive measure such that*

$$H_\Omega(U_t, \varphi) < \lambda_1(\Omega)$$

for all $t \in S$.

Proof. As with Theorem 2.3.4, assume $\varphi \neq |\nabla\psi|/\psi$, and suppose for a contradiction that $H_\Omega(U_t, \varphi) \geq \lambda_1(\Omega)$ for almost all $t \in (0,1)$. As before, there exists $t^* \in (0,1]$ satisfying

$$t^* = \sup\{t \in (0,1) : \int_{U_t} |w|^2 \, dx > 0\},$$

so that $G(t) := t^2 F(t)$ and $F(t)$ are positive and strictly decreasing on $(0, t^*)$, with $G(t^*) \geq G(1) = 0$. Hence

$$g(t) := \frac{1}{G(t)}$$

is well-defined and strictly increasing on $(0, t^*)$. Now since $F(t) \geq 0$, and

$$G'(t) = \frac{d}{dt}\big(t^2 F(t)\big) \leq -t \int_{U_t} |w|^2 \, dx \leq 0$$

as with (2.3.2), it follows that

$$\int_{S_t} w \, d\sigma \geq 2F(t) + \int_{U_t} |w|^2 \, dx \geq 0$$

for almost all $t \in (0,1)$. Hence, by the coarea formula (Theorem A4.6) and the Cauchy-Schwarz inequality,

$$\begin{aligned} G(t) = t^2 F(t) &= t \int_t^1 \frac{t}{\tau} \int_{S_\tau} w \, d\sigma \, d\tau \\ &\leq t \int_t^1 \int_{S_\tau} w \, d\sigma \, d\tau = t \int_{U_t} w |\nabla\psi| \, dx \\ &\leq t \Big(\int_{U_t} |w|^2 \, dx\Big)^{\frac{1}{2}} \Big(\int_{U_t} |\nabla\psi|^2 \, dx\Big)^{\frac{1}{2}} \end{aligned}$$

for all $t \in (0,1)$. By choice of t^*,

$$G'(t) = \frac{d}{dt}\big(t^2 F(t)\big) < -t \int_{U_t} |w|^2 \, dx$$

for almost all $t \in (0, t^*)$. Combining these inequalities, we get

$$t g'(t) = -\frac{t G'(t)}{(G(t))^2} > \Big(\int_{U_t} |\nabla\psi|^2 \, dx\Big)^{-1}$$

for almost all $t \in (0, t^*)$. Fix $t_1 \in (0, t^*)$. Since the last integral is a decreasing function of t, if we set

$$c := \|\nabla\psi\|_2^{-1},$$

then we have

$$g'(t) > \frac{c}{t}$$

for almost all $t \in (0, t_1]$.

Since G is absolutely continuous and positive on $[\varepsilon, t^*)$ for all $\varepsilon \in (0, t^*)$, so is g. By the fundamental theorem of calculus for such functions [**99**, Theorem 7.18],

$$g(t_1) \geq g(t_1) - g(\varepsilon) = \int_\varepsilon^{t_1} g'(\tau)\, d\tau > c \int_\varepsilon^{t_1} \frac{1}{\tau}\, d\tau = c(\log t_1 - \log \varepsilon)$$

for all $\varepsilon \in (0, t_1]$. This implies that $-\log \varepsilon$ is bounded from above as $\varepsilon \to 0$, which is a contradiction. This completes the proof. $\square$

2.4. Proof of the Faber-Krahn inequality

In this section we will prove Theorems 2.1.1 and 2.1.2 as well as sketch the proof of Theorem 2.1.3. We will keep the latter fairly brief, since the method is essentially the same as for the Robin case, and also since the result is well-known with a different proof.

We will start by looking at the Robin case $\Gamma_0 = \emptyset$. We first need to consider some properties of (2.2.1) when Ω is a ball B of radius R, without loss of generality centred at the origin. Denote the eigenvalue by $\lambda_1(B)$ and the corresponding eigenfunction by ψ^*, as before chosen positive and normlised so that $\|\psi^*\|_\infty = 1$. Clearly ψ^* is radially symmetric; that is, there exists a function $v \in C^1([0, R])$ such that $\psi^*(x) = v(|x|)$. In fact we can describe v explicitly as a positive solution to the ordinary differential equation

$$v''(r) + \frac{N-1}{r} v'(r) + \lambda_1(B) v(r) = 0$$

for $r \in (0, R]$. The boundary condition is then $\frac{v'(r)}{r} + \alpha v(r) = 0$ at $r = R$; by normalisation and symmetry $v(0) = 1$ and $v'(0) = 0$. This may transformed via the substitution $w(r) := r^{\frac{N}{2}-1} v(r)$ into Bessel's equation of order $\frac{N}{2} - 1$ (see [**79**, Section 4.5]). The solution is given by

$$v(r) = c r^{1-\frac{N}{2}} J_{\frac{N}{2}-1}(\sqrt{\lambda_1(B)}\, r), \tag{2.4.1}$$

where J_μ is the Bessel function of the first type of order μ and c is a normalising constant.

Now we set $\varphi^* : \overline{B} \to \mathbb{R}$, $\varphi^* := |\nabla \psi^*| / \psi^*$. Since it is clear that $\nabla \psi^*$ is radially symmetric, the same must be true of φ^*. In particular, by Remark 2.3.2,

$\varphi^* \in M_\alpha(B)$ is constant and identically equal to α on $\partial\Omega$. Moreover, we may write $\varphi^*(x) = g(|x|)$ for some $g \in C([0, R])$. The next lemma uses properties of these Bessel functions. Its proof is in Appendix B.

Lemma 2.4.1 ([**35**], Lemma 4.1)**.** *The function $g : (0, R) \to (0, \infty)$ is strictly increasing.*

As in [**20, 35**], we define a function $\varphi \in M_\alpha(\Omega)$ by constructing a rearrangement of φ^*. We will denote by B_r the ball of radius r centred at the origin, and let $r(t)$ be the radius of the ball with the same volume as $U_t \subset \Omega$, so that $|B_{r(t)}| = |U_t|$. Since Ω and B have the same volume and $U_m = \Omega$ we have $r(m) = R$. Given any $t \in (m, 1]$ we define

$$\varphi(x) := g(r(t))$$

for all $x \in S_t$. Since Ω is a disjoint union of the S_t, $t \in (m, 1]$, the function $\varphi : \Omega \to (0, \infty)$ is well-defined. The following comparison of φ and φ^* uses the isoperimetric inequality; hence we give the proof here.

Lemma 2.4.2 ([**35**], Lemma 4.2)**.** *The function φ constructed above lies in $M_\alpha(\Omega)$ and*

$$\lambda_1(B) = H_B(B_{r(t)}, \varphi^*) \leq H_\Omega(U_t, \varphi) \tag{2.4.2}$$

for all $t \in (m, 1)$.

Proof. Since g is increasing, $\{x \in \Omega : \varphi > t\} = \Omega \setminus \overline{U}_t$ and $\{x \in \Omega : \varphi < t\} = U_t$ are open in Ω for every t, that is, φ is continuous on Ω. Moreover, by construction and since $g > 0$ is increasing we have $0 < \varphi \leq \alpha$, so that $\varphi \in M_\alpha(\Omega)$.

Now we prove (2.4.2). The first equality follows immediately from Proposition 2.2.3. For the inequality, since by construction the level sets of φ^* and φ have the same volume,

$$\int_{U_t} |\varphi|^2 \, dx = \int_{B_{r(t)}} |\varphi^*|^2 \, dx$$

for all $t \in (m, 1]$ by Cavalieri's principle (see [**88**, Section 1.2.3]). Since $|U_t| = |B_{r(t)}|$, the isoperimetric inequality (see for example [**13**]) states that $\sigma(\partial U_t) \geq$

$\sigma(\partial B_{r(t)})$, for every $t \in (m,1]$. Since $\varphi(x) = g(r(t)) \leq \alpha$ for $x \in S_t$,

$$\begin{aligned}\int_{\partial B_{r(t)}} \varphi^* \, d\sigma = g(r(t))\sigma(\partial B_{r(t)}) &\leq g(r(t))\sigma(\partial U_t) \\ &= \int_{S_t} g(r(t))\, d\sigma + \int_{\partial_e U_t} g(r(t))\, d\sigma \\ &\leq \int_{S_t} \varphi \, d\sigma + \int_{\partial_e U_t} \alpha \, d\sigma.\end{aligned}$$

That is, we preserve the volume integral and decrease the surface integral. It follows that

$$\begin{aligned}H_B(B_{r(t)}, \varphi^*) &= \int_{\partial B_{r(t)}} \varphi^* \, d\sigma - \int_{B_{r(t)}} |\varphi^*|^2 \, dx \\ &\leq \int_{S_t} \varphi \, d\sigma + \int_{\partial_e U_t} \alpha \, d\sigma - \int_{U_t} |\varphi|^2 \, dx = H_\Omega(U_t, \varphi),\end{aligned}$$

completing the proof of (2.4.2). □

We also have the following lemmata, which will be used in the proof of Theorem 2.1.2. They will allow us to conclude that almost all the level sets U_t of Ω are concentric balls if $\lambda_1(\Omega) = \lambda_1(B)$. These use the extra information from Theorem 2.3.4.

Lemma 2.4.3. *Suppose that $\lambda_1(\Omega) = \lambda_1(B)$. Then $\varphi = |\nabla \psi|/\psi$ and $H_\Omega(U_t, \varphi) = \lambda_1(B)$ for almost all $t \in (m, 1)$.*

Proof. If $\lambda_1(\Omega) = \lambda_1(B)$, then by Lemma 2.4.2, $\lambda_1(\Omega) = \lambda_1(B) \leq H_\Omega(U_t, \varphi)$ for all $t \in (m, 1)$. Hence by Theorem 2.3.4, $\varphi = |\nabla\psi|/\psi$, and in particular, by Proposition 2.2.3,

$$H_\Omega(U_t, \varphi) = \lambda_1(\Omega) = \lambda_1(B) \tag{2.4.3}$$

for almost all $t \in (m, 1)$. □

Lemma 2.4.4. *Suppose U_t is Lipschitz for some $t \in (m, 1)$. Then $H_\Omega(U_t, \varphi) = \lambda_1(B)$ if and only if U_t is a ball and $\sigma(\partial_e U_t) = 0$.*

Proof. We know that $\lambda_1(B) = H_B(B_{r(t)}, \varphi^*)$ for all $t \in (m, 1)$. Also, by construction, the level sets of φ^* and φ have the same volume. So as in the proof of Lemma 2.4.2 we have

$$\int_{U_t} |\varphi|^2 \, dx = \int_{B_{r(t)}} |\varphi^*|^2 \, dx$$

for all $t \in (m, 1]$. Hence, using the definition of H_Ω,

$$\begin{aligned} H_\Omega(U_t, \varphi) &= \frac{1}{|U_t|}\Big(\int_{S_t} g(r(t))\,d\sigma + \int_{\partial_e U_t} \alpha\,d\sigma - \int_{U_t} |\varphi|^2\,dx\Big) \\ &= \frac{1}{|B_{r(t)}|}\Big(g(r(t))\sigma(S_t) + \alpha\sigma(\partial_e U_t) - \int_{B_{r(t)}} |\varphi^*|^2\,dx\Big). \end{aligned}$$

Now if U_t is a ball with $\sigma(\partial_e U_t) = 0$, then $\sigma(S_t) = \sigma(\partial B_{r(t)})$, and

$$\begin{aligned} H_\Omega(U_t, \varphi) &= \frac{1}{|B_{r(t)}|}\Big(g(r(t))\sigma(\partial B_{r(t)}) - \int_{B_{r(t)}} |\varphi^*|^2\,dx\Big) \\ &= H_B(B_{r(t)}, \varphi^*) = \lambda_1(B), \end{aligned}$$

proving one implication. Conversely, if $H_\Omega(U_t, \varphi) = \lambda_1(B)$, then for this t we have

$$g(r(t))\sigma(S_t) + \alpha\sigma(\partial_e U_t) = g(r(t))\sigma(\partial B_{r(t)}).$$

Since $t \in (m, 1)$, $0 < g(r(t)) < \alpha$. This is only possible if

$$\int_{\partial_e U_t} \alpha\,d\sigma = 0$$

and $\sigma(S_t) = \sigma(\partial B_{r(t)})$. Since $\alpha > 0$ on $\partial\Omega$ by assumption, this means $\sigma(\partial_e U_t) = 0$. Now since ∂U_t is the disjoint union of S_t and $\partial_e U_t$, if $\sigma(\partial_e U_t) = 0$, then we get $\sigma(\partial U_t) = \sigma(\partial B_{r(t)})$. But we know that the ball is the unique minimiser of the isoperimetric inequality, at least among Lipschitz domains [**25**, Theorem 10.2.1]. Hence $U_t = B_{r(t)}$ up to translation. □

Lemma 2.4.5. *Assume $-\Delta u = \lambda u$ in Ω for some $\lambda > 0$. Suppose that for some $t \geq 0$, the level set $\{x \in \Omega : u(x) > t\} = B(x_0, r)$ for some $x_0 \in \Omega$ and $r > 0$. If $u \in C(\overline{B(x_0, r)})$ and $\sigma(\partial_e B(x_0, r)) = 0$, then u is radially symmetric with respect to x_0 in $B(x_0, r)$.*

Proof. Set $v(x) := u(x) - t$. Then

$$-\Delta v = -\Delta u = \lambda u = \lambda(u - t) + \lambda t = \lambda v + \lambda t$$

in $B(x_0, r)$. Since $u \in C(\overline{B(x_0, r)})$ and $\sigma(\partial_e B(x_0, r)) = 0$, we get $u(x) = t$ for all $x \in \partial B(x_0, r)$. So $v = 0$ on $\partial B(x_0, r)$. That is, if we let $f(v) := \lambda v + \lambda t$, then v solves a Dirichlet problem

$$\begin{aligned} -\Delta v &= f(v) && \text{in } B(x_0, r) \\ v &= 0 && \text{on } \partial B(x_0, r) \end{aligned}$$

Now $f(v) = \lambda v + \lambda t \geq 0$ if $v \geq 0$, and by assumption, in $B(x_0, r)$, $u > t$, so $v = u - t > 0$. Hence by [**58**, Corollary 3.5], v is radially symmetric on $B(x_0, r)$ with respect to x_0. □

Remark 2.4.6. (i) By unique continuation of solutions to elliptic equations, if the eigenfunction ψ on Ω satisfies the assumption of Lemma 2.4.5, then ψ on Ω equals the solution given by the Bessel function extending the solution on the ball $B(x_0, r)$. It is possible to use this observation to give a slightly different proof of the sharpness result. Namely, if we know one level set of ψ is a ball, then this is sufficient to conclude that Ω is a ball. We do not go into details here, but the full argument can be found in [**39**].

(ii) We could state Lemma 2.4.5 in greater generality: if the level set has an axis of symmetry, then the solution u will be symmetric with respect to that axis. The proof is the same, except we refer to [**58**, Corollary 3.4].

We can now prove Theorems 2.1.1 and 2.1.2. The proof of the former is divided into two parts: (i) proof that $\lambda_1(B)$ is minimal amongst C^2 domains using the method detailed above; and (ii) proof that $\lambda_1(B)$ is minimal amongst Lipschitz domains using an approximation argument.

Proof of Theorem 2.1.1. (i) Let $\varphi \in M_\alpha(\Omega)$ be the function constructed above. By Theorem 2.3.4 either there exists $t \in (m, 1)$ such that

$$\lambda_1(B) \leq H_B(B_{r(t)}, \varphi^*) \leq H_\Omega(U_t, \varphi) < \lambda_1(\Omega)$$

(where we have also used Lemma 2.4.2) or else $\varphi = |\nabla\psi|/\psi$, in which case we have

$$\lambda_1(B) = H_B(B_{r(t)}, \varphi^*) \leq H_\Omega(U_t, \varphi) = \lambda_1(\Omega)$$

for almost all $t \in (m, 1)$ (again using Lemma 2.4.2, and also Proposition 2.2.3).

(ii) Suppose Ω is a bounded Lipschitz domain and let Ω_n be such that $|\Omega_n| \to |\Omega|$ and $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ as in Theorem 1.3.4. If we let B_n denote the ball centred at the origin such that $|\Omega_n| = |B_n|$, then by part (i) $\lambda_1(\Omega_n) \geq \lambda(B_n)$ for all n. Since $|\Omega_n| \to |\Omega|$, also $|B_n| \to |B|$ as $n \to \infty$ and so $\lambda_1(B_n) \to \lambda_1(B)$. That is,

$$\lambda_1(\Omega) \leftarrow \lambda_1(\Omega_n) \geq \lambda_1(B_n) \to \lambda_1(B)$$

as $n \to \infty$, completing the proof of (ii). □

Proof of Theorem 2.1.2. Again assume Ω is a bounded C^2 domain, and suppose that $\lambda_1(\Omega) = \lambda_1(B)$. By Lemma 2.4.3, $H_\Omega(U_t, \varphi) = \lambda_1(B)$ for almost all $t \in (m, 1)$. Since U_t is Lipschitz for almost every t, by Lemma 2.4.4, U_t is a ball for almost every $t \in (m, 1)$. For every such t, Lemma 2.4.4 also tells us that $\sigma(\partial_e U_t) = 0$, and so by Lemma 2.4.5 the eigenfunction ψ corresponding to $\lambda_1(\Omega)$ is radially symmetric inside U_t, and all interior level sets U_τ, $\tau \in (t, 1]$ are concentric balls. It follows that for all $t \in (m, 1]$, the level sets U_t are concentric balls, and so

$$\Omega = \bigcup_{t \in (m,1]} U_t$$

is a ball. $\square$

We will now deal with the pure Dirichlet boundary condition $\Gamma_1 = \emptyset$. Since we no longer have a Robin or mixed boundary condition, we will denote the first eigenvalue of Ω by $\mu_1(\Omega)$ in order to maintain consistency with (1.1.1) and later sections. We will assume Ω is Wiener regular. If we consider the problem (2.2.1) on the ball B of centre 0 and radius R, then as in the case of Robin boundary conditions, the function $\varphi^* = |\nabla \psi^*|/\psi^* : \overline{B} \to \mathbb{R}$ is radially symmetric and continuous, so that $\varphi^*(x) = g(|x|)$, $g \in C([0, R))$. Moreover, $g : [0, R) \to [0, \infty)$ is strictly increasing with $g(0) = 0$ and $g(r) \to \infty$ as $r \to R$ (we omit the proof). We then define a rearrangement of φ^*, call it $\varphi \in C(\Omega)$, exactly as in the Robin case. Then Lemma 2.4.2 carries over to this case exactly, the only difference to the proof being that $U_t \subset\subset \Omega$, so that $\partial_e U_t = \emptyset$ for all $t \in (0, 1)$. Lemma 2.4.4 also holds with the proof the same except for this detail. In this case Lemma 2.4.3 also carries over without change, and finally Lemma 2.4.5 is directly valid in the Dirichlet case also, since it made no assumptions as to the boundary condition u satisfies.

Proof of Theorem 2.1.3. Let $\varphi \in C(\Omega)$ be the rearrangement of φ^* as above. By Theorem 2.3.5, either $H_\Omega(U_t, \varphi) < \mu_1(\Omega)$ on a set of t-positive measure, or else $\varphi = |\nabla \psi|/\psi$. In the former case, Lemma 2.4.2 implies

$$\mu_1(B) = H_B(B_{r(t)}, \varphi^*) \leq H_\Omega(U_t, \varphi) < \mu_1(\Omega)$$

for all such t. In the latter case, we have

$$\mu_1(B) = H_B(B_{r(t)}, \varphi^*) \leq H_\Omega(U_t, \varphi) = \mu_1(\Omega) \tag{2.4.4}$$

for almost every $t \in (0,1)$, using Proposition 2.2.3. Now if $\mu_1(\Omega) = \mu_1(B)$, the only possibility is that (2.4.4) holds with equality for almost every $t \in (0,1)$. By Lemma 2.4.4, U_t is a ball for almost every $t \in (0,1)$. Using Lemma 2.4.5, every level set is a concentric ball and, as in the proof of Theorem 2.1.2, Ω is a ball. $\square$

CHAPTER 3

Variations and Applications

In this chapter we consider a number of problems related either to the problem or to the method considered in Chapter 2. We will start in Section 3.1 with a short look at the p-Laplacian. Since the publication of the theory in Chapter 2, the method has been developed and extended elsewhere so as to apply to this quasi-linear operator. We will briefly discuss the operator's properties, as well as the generalisation of earlier results, including ours, from the case $p = 2$. We will use some of these in later chapters to present some of our own results in greater generality.

In Section 3.2 we will give an application of the method in Chapter 2 to proving non-existence of certain types of supersolutions to the problem (1.2.1). While our result, Theorem 3.2.1, is not new, the method of proof certainly is and it would be interesting to know how it is connected with the other method of proof involving the Serrin sweeping principle; see [**107**] and [**85**].

In Section 3.3 we look at a different type of isoperimetric inequality for the first eigenvalue of the Robin problem, analogous to Cheeger's inequality for the corresponding Dirichlet problem (cf. [**29**]). This is based on a constant, usually called Cheeger's constant, which in some sense describes the geometry of a given domain.

Finally, in Section 3.4 we consider two open problems. The first is the Robin problem (1.1.2) with $\alpha < 0$, which as we saw in Chapter 1 has many of the same properties as when $\alpha > 0$. The method involving H_Ω used when $\alpha > 0$ seems to break down here and so we present the likely result as a conjecture. The other is a famous conjecture of Pólya on polygonal domains in $\mathbb{R}^2$ with Dirichlet boundary conditions. We consider these two problems mostly to illustrate the limitations of the method in Chapter 2.

3.1. The p-Laplacian with Robin boundary conditions

The eigenvalue problem in this case is

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= \lambda |u|^{p-2}u && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \alpha |u|^{p-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{3.1.1}$$

where $\Omega \subset \mathbb{R}^N$ is bounded, Lipschitz and $\operatorname{div}(|\nabla u|^{p-2}\nabla u) =: \Delta_p u$ is the p-Laplacian of u, $p \in (1, \infty)$. Note that here we assume $\alpha > 0$. When $p = 2$ this reduces to the problem (1.1.2). An eigenvalue λ with eigenfunction ψ is a solution to (3.1.1) (in the weak sense) if it satisfies

$$\int_\Omega |\nabla \psi|^{p-2}\nabla\psi \cdot \nabla\varphi \, dx + \int_{\partial\Omega} \alpha |\psi|^{p-2}\psi\varphi \, d\sigma = \lambda \int_\Omega |\psi|^{p-2}\psi\varphi \, dx \tag{3.1.2}$$

for all $\varphi \in W^{1,p}(\Omega)$. The structure of the set of eigenvalues is not as yet completely known; nor can we expect the eigenfunctions to be quite as regular as in the case $p = 2$. We will summarise below much of what is currently known. Note that everything will remain true with obvious modifications if we replace the boundary condition in (3.1.1) with the Dirichlet condition $u = 0$ on $\partial\Omega$. This is mostly taken from [**82**], and the reader is referred there for the general existence theory for the p-Laplacian.

Proposition 3.1.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded, connected Lipschitz domain and $\alpha > 0$. Then*

(i) *there exists a sequence of eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ of (3.1.1) of the form $0 < \lambda_1 < \lambda_2 \leq \ldots$;*

(ii) *the first eigenvalue $\lambda_1 = \lambda_1(\Omega, \alpha, p) > 0$ satisfies*

$$\lambda_1 = \inf_{\varphi \in W^{1,p}(\Omega)} \frac{\int_\Omega |\nabla\varphi|^p \, dx + \int_{\partial\Omega} \alpha |\varphi|^p \, d\sigma}{\int_\Omega |\varphi|^p \, dx}; \tag{3.1.3}$$

(iii) *the second eigenvalue satisfies*

$$\lambda_2 = \inf\{\lambda > \lambda_1 : \lambda \textit{ is an eigenvalue of } (3.1.1)\};$$

(iv) *the first eigenvalue $\lambda_1 > 0$ is simple and every eigenfunction ψ associated with λ_1 satisfies $\psi > 0$ or $\psi < 0$ in Ω;*

(v) *only eigenfunctions associated with λ_1 do not change sign in Ω;*

(vi) *every eigenfunction ψ of (3.1.1) lies in $W^{1,p}(\Omega) \cap C^{1,\eta}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \eta < 1$.*

The usual method for obtaining the sequence in (i) is a technique known as the Ljusternik-Schnirelman (L-S) principle, and it is an open problem as to whether every eigenvalue is obtainable in this way; in particular (iii) is a non-trivial result.

Proof. Parts (i)-(v) are essentially contained in [**82**]. Although C^1 regularity of Ω is assumed there in order to derive (i) and $C^{1,\theta}$, $0 < \theta < 1$, is assumed for (ii)-(iv), a careful analysis of the proofs shows that only Lipschitz continuity of $\partial\Omega$ is needed, since all background results, including those in the appendices, are valid for Lipschitz domains. (The extra regularity of $\partial\Omega$ is needed only to prove extra boundary regularity of the eigenfunctions.) For (v), first note that by [**37**, Theorem 2.7], every eigenfunction $\psi \in L^\infty(\Omega)$ (see also Section 4 there). But now, as noted in [**22**, Section 2], the arguments in [**81**, pp. 466-7] imply that ψ is Hölder continuous on $\overline{\Omega}$. Also, by [**103**], $\nabla\psi$ is Hölder continuous inside Ω. □

The Faber-Krahn inequality, in the greatest generality known at the time of writing, is as follows.

Theorem 3.1.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain and B a ball with $|B| = |\Omega|$. If $\alpha > 0$, the first eigenvalue $\lambda_1(\Omega)$ of* (3.1.1) *satisfies $\lambda_1(\Omega) \geq \lambda_1(B)$, with equality if and only if $\Omega = B$ after a translation.*

Note that this actually improves our sharpness result in Chapter 2, for all $1 < p < \infty$; hence we will use this version of the theorem in subsequent chapters. This theorem was first proved for smooth domains in [**31**] and then for Lipschitz domains in [**22**]; both were based on [**35, 39**]. In this case the functional becomes

$$H_\Omega(U,\varphi) = \frac{1}{|U|}\Big(\int_{\partial_i U} \varphi\, d\sigma + \int_{\partial_e U} \alpha\, d\sigma - (p-1)\int_U |\varphi|^{\frac{p}{p-1}}\, dx\Big)$$

and the function $|\nabla\psi|^p/|\psi|^p$ satisfies $H_\Omega(U_t, |\nabla\psi|^p/|\psi|^p) = \lambda_1(\Omega)$ for almost all t. The main difficulties to overcome are the lack of regularity of the eigenfunction (note that the assumption that $\psi \in C^2(\Omega)$ (or better) is used extensively in Chapter 2, and when $p \neq 2$ it is possible to find smooth domains for which $\psi \in C^{1,\eta}(\Omega)$ only), the lack of Bessel functions (as in Lemma 2.4.1) and, for the sharpness of the inequality, the lack of the symmetry result on which Lemma 2.4.5 relies. Both [**31**] and [**22**] use the same ideas presented in Chapter 2, but use various methods to overcome these problems. For example, [**22**] uses cut-offs and test function arguments to greatly weaken all the regularity requirements; in particular this allows sharpness to be proved for a broader class of domains.

3.2. An application to supersolutions

There is an interesting minor application of the contradiction argument we used in Chapter 2 to supersolutions of the problem (2.2.1). Roughly speaking, no positive smooth function u can satisfy $-\Delta u > \lambda_1 u$ (plus the boundary condition), where λ_1 is the first eigenvalue of (2.2.1). More precisely, we have the following result. Here we allow $\Gamma_0 = \emptyset$ or $\Gamma_1 = \emptyset$ in (2.2.1).

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded C^2 domain, let λ_1 be the first eigenvalue of the mixed problem* (2.2.1) *and let $\lambda_1 \le c \in C(\Omega)$, with $c(x) \neq \lambda_1$ for some $x \in \Omega$. If $\Gamma_0 = \emptyset$, then assume in addition that $\inf_{x\in\Omega} c(x) > \lambda_1$. Then there is no function $u \in C^2(\overline{\Omega})$ with $u > 0$ on $\Omega \cup \Gamma_1$ such that*

$$\begin{aligned} -\Delta u(x) &= cu(x) && \text{in } \Omega, \\ u &\ge 0 && \text{on } \Gamma_0, \\ \frac{\partial u}{\partial \nu} + \alpha u &\ge 0 && \text{on } \Gamma_1. \end{aligned} \tag{3.2.1}$$

Proof. Suppose first that $\Gamma_0 \neq \emptyset$, and suppose for a contradiction that there exists such a function u. Since $u(x) > 0$ in Ω, we may write $c = -\Delta u(x)/u(x)$ for all $x \in \Omega$. Moreover, using the notation of Chapter 2, we have $\Delta u/u \in C(\overline{U_t})$ for every $t > m = \min_{x\in\overline{\Omega}} \psi(x)$, since $\operatorname{dist}(\overline{U_t}, \Gamma_0) > 0$ and so $u > 0$ on the compact set $\overline{U_t}$. Hence

$$\begin{aligned} \frac{1}{|U_t|} \int c\,dx &= \frac{1}{|U_t|} \int_{U_t} \frac{-\Delta u}{u}\,dx \\ &= \frac{1}{|U_t|} \Big(\int_{U_t} \operatorname{div}\big(-\frac{\nabla u}{u}\big)\,dx - \int_{U_t} \frac{|\nabla u|^2}{u^2}\,dx \Big) \end{aligned}$$

for all $t \in (0,1)$. By Lemma 2.2.1(ii), U_t is Lipschitz for almost every $t \in (0,1)$. Arguing as in the proof of Proposition 2.2.3,

$$\begin{aligned} \int_{U_t} \operatorname{div}\big(-\frac{\nabla u}{u}\big) &= \int_{\partial U_t} -\frac{\nabla u}{u} \cdot \nu_{U_t}\,d\sigma \\ &= \int_{\partial_e U_t} -\frac{\nabla u}{u} \cdot \nu_\Omega\,d\sigma + \int_{\partial_i U_t} -\frac{\nabla u}{u} \cdot \nu_{U_t}\,d\sigma, \end{aligned}$$

where we recall $\partial_e U_t \subset \Gamma_1$ is possibly empty. If $\Gamma_1 \neq \emptyset$, then by assumption

$$-\frac{\nabla u}{u} \cdot \nu_\Omega = -\frac{\partial u}{\partial \nu_\Omega} \cdot \frac{1}{u} \leq \alpha$$

so

$$\int_{\partial_e U_t} -\frac{\nabla u}{u} \cdot \nu_\Omega \, d\sigma \leq \int_{\partial_e U_t} \alpha \, d\sigma.$$

Hence

$$\int_{\partial U_t} -\frac{\nabla u}{u} \cdot \nu_{U_t} \, d\sigma \leq \int_{\partial_e U_t} \alpha \, d\sigma + \int_{\partial_i U_t} \frac{|\nabla u|}{u} \, d\sigma$$

since obviously $-\nabla u \cdot \nu_{U_t} \leq |\nabla u|$. Moreover, since $\lambda_1 \leq c \in C(\Omega)$ and $c \not\equiv \lambda_1$, there exists $t_0 \in (0,1)$ such that

$$\lambda_1 \leq \frac{1}{|U_t|} \int_{U_t} c \, dx$$

for almost all $t \in (0,1)$, with strict inequality for almost all $t \in (0, t_0)$. Putting this all together we have

$$\begin{aligned} \lambda_1 &\leq \frac{1}{|U_t|} \Big(\int_{\partial_e U_t} \alpha \, d\sigma + \int_{\partial_i U_t} \frac{|\nabla u|}{u} \, d\sigma - \int_{U_t} \frac{|\nabla u|^2}{u^2} \, dx \Big) \\ &= H_\Omega(U_t, \frac{|\nabla u|}{u}) \end{aligned}$$

for almost all $t \in (0,1)$, with strict inequality for almost all $t \in (0, t_0)$. Since $0 < u \in C^1(\Omega)$, certainly $|\nabla u|/u \in C(\Omega)$ is non-negative, and lies $L^2(U_t)$ for each t since $u > 0$ on $\overline{U}_t$. In particular, this contradicts Theorem 2.3.5. Hence no such function u can exist.

Now suppose $\Gamma_0 = \emptyset$. Let B_n, $n \in \mathbb{N}$ be a nested sequence of balls compactly contained in Ω whose radii shrink to zero and set $\Omega_n := \Omega \setminus B_n$. Impose Dirichlet boundary conditions on $\partial B_n \subset \partial \Omega_n$ and denote by $\lambda_1(\Omega_n)$ the first eigenvalue of (2.2.1) on Ω_n, with $\Gamma_0 = \partial B_n$ and $\Gamma_1 = \partial \Omega$. Since $\Omega_n \subset \Omega_{n+1}$ for all n and $\Omega = \cup_{n \in \mathbb{N}} \Omega_n$, $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$ as $n \to \infty$ (this result is standard; it follows, for example, from [**34**, Proposition 7.1]). Suppose there exists a function u as described in the theorem. Then since $c > \lambda_1(\Omega)$, for n large enough $-\Delta u \geq \lambda_1(\Omega_n) u$ in Ω_n; moreover, $u > 0$ on ∂B_n. The existence of such a function u on Ω_n contradicts what we showed earlier. □

Remark 3.2.2. (i) If instead in the above argument we assume directly that $-\Delta u \geq \lambda_1 u$ and $\Gamma_0 \neq \emptyset$, then we obtain $\lambda_1 \leq H_\Omega(U_t, |\nabla u|/u)$ for almost all $t \in (0,1)$ and hence $u \equiv \psi$, the first eigenfunction, everywhere in Ω; thus the only supersolution is the first eigenfunction itself. The reason we presented the theorem as above is that this does not work if $\Gamma_0 = \emptyset$. To apply Theorem 2.3.4 instead of Theorem 2.3.5 we would need $u \in M_\alpha(\Omega)$, which is not in general even true of the eigenfunction ψ (see Remark 2.3.2). We would have to replace the condition $\frac{\partial u}{\partial \nu} + \alpha u \geq 0$ on $\Gamma_1 = \partial\Omega$ with $-|\nabla u| + \alpha u \geq 0$ on $\partial\Omega$, which is very restrictive.

(ii) This is not a new result; indeed, it is slightly weaker than some existing results on supersolutions and maximum principles. In particular, if $\Gamma_1 = \emptyset$, then it follows easily from [**107**, Theorem 2] (see also Theorem 3 there), since given some $u > 0$ satisfying the assumptions of Theorem 3.2.1, for any $t \in \mathbb{R}$ the function $v := u - t\psi$ satisfies $\Delta v + \lambda_1 v \leq 0$. By [**107**, Theorem 2], either $u - t\psi = m\psi$ for some $m \in \mathbb{R}$, $u - t\psi \equiv 0$, or $u - t\psi > 0$; the latter being impossible for all $t \in \mathbb{R}$. Hence u is a scalar multiple of ψ. If $\Gamma_1 \neq \emptyset$ we use [**85**, Theorem 1.1] instead (see also the comments on p. 1027 where the relationship between supersolutions and the first eigenvalue is explicitly discussed).

(iii) Related to (ii), there are still two reasons why the theorem is of interest. First, it uses a different method of proof: it would be interesting to see how the two are related and whether any new insights could be obtained from comparing the different approaches. Second, there are various possibilities for weakening the requirements on u and/or Ω, especially when combined with the improved results in [**22**]. As this is off our main topic of interest we do not go into detail here.

3.3. Cheeger's inequality for the Robin Laplacian

Besides the Faber-Krahn inequality, there is another type of inequality for $\lambda_1(\Omega)$ we will consider. Instead of comparing $\lambda_1(\Omega)$ with an eigenvalue of another domain, we will give an *a priori* bound for $\lambda_1(\Omega)$ depending only on the geometry of the domain Ω and the constant α appearing in the boundary condition in (1.1.2). To describe the geometry of the domain we define a constant, often called the *Cheeger constant* of Ω, by

$$h = h(\Omega) := \inf \frac{\sigma(\partial U)}{|U|} > 0, \tag{3.3.1}$$

where the infimum is taken over all sets $U \subset\subset \Omega$, not necessarily connected, smooth or open. This infimum is attained, although any minimiser will have to touch the boundary $\partial\Omega$; see [**73**, Theorem 8]. If no confusion seems likely we will write h in preference to $h(\Omega)$.

A nice overview of the Cheeger constant can be found in [**73**]. It was first introduced in a seminal paper of Cheeger [**29**], where it was proved that the first eigenvalue $\mu_1(M)$ of the Laplace operator on a compact Riemannian manifold M, either without boundary or with Dirichlet boundary conditions, satisfies $\mu_1(M) \geq \frac{1}{4}h(M)^2$. (For manifolds without boundary a slightly different definition of h is used.) Many similar or related results have been established for such manifolds. For example, a partial converse is established in [**27**]: if the Ricci curvature of M is bounded below by $-(N-1)\delta^2$ ($\delta \geq 0$), then $\mu_1(B) \leq 2\delta(N-1)h + 10h^2$. That is, for some (although definitely not all) M, $\mu_1(M)$ is in some sense equivalent to $h(M)^2$. For other results see also [**26**].

If instead we impose Robin boundary conditions, then we can establish an inequality that is similar to, albeit slightly more complicated than, Cheeger's original inequality. For smooth domains in $\mathbb{R}^2$, this was already stated in [**20**]. For C^2 domains in $\mathbb{R}^N$, the result follows from [**35**, Theorem 3.1] as noted in the introduction there. We will generalise the result to general bounded Lipschitz domains. The result could also be obtained from [**22**].

Theorem 3.3.1. *The first eigenvalue $\lambda_1 = \lambda_1(\alpha)$ of* (1.1.2) *on a fixed bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ satisfies*

$$\lambda_1(\alpha) \geq \begin{cases} h\alpha - \alpha^2 & \textit{always;} \\ \dfrac{1}{4}h^2 & \textit{in addition if } \dfrac{1}{2}h \leq \alpha. \end{cases} \tag{3.3.2}$$

Proof. First suppose that Ω is a bounded domain of class C^2. If we take the constant $\alpha \in M_\alpha(\Omega)$ as a test function, then by Theorem 2.3.4, there exists $t \in (m,1)$ such that

$$\begin{aligned}\lambda_1(\Omega) \geq H_\Omega(U_t,\alpha) &= \frac{1}{|U_t|}\Big(\int_{S_t} \alpha\, d\sigma + \int_{\partial_e U_t} \alpha\, d\sigma - \int_{U_t} \alpha^2\, dx\Big) \\ &= \frac{\sigma(U_t)}{|U_t|}\alpha - \alpha^2.\end{aligned}$$

Since $h \leq \sigma(U_t)/|U_t|$, it follows that $\lambda_1(\Omega) \geq h\alpha - \alpha^2$. Now suppose that $\frac{1}{2}h \leq \alpha$. Then $\frac{1}{2}h \in M_\alpha(\Omega)$ and a similar calculation to the one above gives

$$\lambda_1(\Omega) \geq H_\Omega(U_t, \frac{1}{2}h) \geq \frac{\sigma(U_t)}{|U_t|}\,\frac{1}{2}h - \frac{1}{4}h^2 = \frac{1}{4}h^2$$

for some $t \in (m,1)$.

Finally, suppose that Ω is a bounded Lipschitz domain. By Theorem 1.3.4, we can approximate Ω with a sequence of C^∞ domains $\Omega_n \subset \Omega$ such that $\lambda_1(\Omega_n) \to \lambda_1(\Omega)$. Now h is monotonic with respect to the domain, that is, $h(\Omega_n) \geq h(\Omega)$ for all $n \in \mathbb{N}$. Hence for any α we have

$$\lambda_1(\Omega) = \lim_{n\to\infty} \lambda_1(\Omega_n) \geq \limsup_{n\to\infty} h(\Omega_n)\alpha - \alpha^2 \geq h(\Omega)\alpha - \alpha^2$$

since (3.3.2) holds for every Ω_n. If $\frac{1}{2}h(\Omega) \leq \alpha$, then for any $n \in \mathbb{N}$, $\frac{1}{2}h(\Omega) \in M_\alpha(\Omega_n)$ and we may apply Theorem 2.3.4 on Ω_n in the same fashion as earlier to obtain

$$\lambda_1(\Omega_n) \geq h(\Omega_n)\frac{1}{2}h(\Omega) - \frac{1}{4}h(\Omega)^2.$$

Now we use $h(\Omega_n) \geq h(\Omega)$ and pass to the limit. □

Remark 3.3.2. Using the same idea we can recover Cheeger's inequality for the Dirichlet Laplacian, that is, if $\Omega \subset \mathbb{R}^N$ is any Wiener regular bounded domain, then $\mu_1(\Omega) \geq \frac{1}{4}h(\Omega)^2$. The only change to the proof is that we use Theorem 2.3.5 in place of Theorem 2.3.4, and in this case the function $\frac{1}{2}h(\Omega)$ is always in the class of suitable test functions $\{u \in C(\Omega) : u \geq 0\}$. This observation is made in [**20**] for the case $N = 2$ (and for smooth domains, though this is not made explicit there). We leave as an open question what results could be obtained for mixed Dirichlet-Robin problems.

3.4. Two open problems

Here we consider some new problems which might be susceptible to the methods presented in Chapter 2. Perhaps one of the most obvious is the Robin problem (1.1.2) when $\alpha < 0$. We saw in Sections 1.2 and 1.3 that in this case there is still a first eigenvalue $\lambda_1 = \lambda_1(\Omega, \alpha) < 0$, with a well-behaved eigenfunction ψ which can be chosen strictly positive in Ω (see Theorems 1.2.8 and 1.3.1). In this case we expect the ball to have the *largest* first eigenvalue.

Conjecture 3.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let B denote a ball having the same volume as Ω. If $\alpha < 0$, then $\lambda_1(\Omega, \alpha) \leq \lambda_1(B, \alpha)$, with equality if and only if $\Omega = B$ after a translation.*

This conjecture has also appeared in [**1**]. It is supported by the behaviour of $\lambda_1(\alpha)$ at $\alpha = 0$, since given any domain Ω, (1.3.2) together with the isoperimetric inequality implies $\lambda_1(B, \alpha) > \lambda_1(\Omega, \alpha)$ for α sufficiently close to 0 (but importantly how close possibly depending on Ω). This means that the only candidate domain to maximise λ_1 independently of α is B, but it is also possible that the maximiser could change depending on α. (Certainly for $\alpha > 0$ the minimiser of the higher eigenvalues must depend on α; see Section 4.3.) Of course it is also possible that a maximiser does not exist.

Remark 3.4.2. We note in passing why we seek a maximum rather than a minimum for λ_1 in this case. In fact it is immediate from the Rayleigh quotient (1.3.1) (see also [**80**, Equation 2.1]) that we cannot minimise $\lambda_1(\Omega)$ with respect to Ω. Indeed, let Ω_n be a sequence of smooth domains of fixed volume such that the surface measure $\sigma(\partial\Omega_n) \to \infty$. Using any constant as a test function in the Rayleigh quotient for $\lambda_1(\Omega_n)$, we see $\lambda_1(\Omega_n) \le \alpha\sigma(\partial\Omega_n)/|\Omega_n| \to -\infty$ as $n \to \infty$. This argument also shows that we cannot seek to minimise $\lambda_k(\Omega, \alpha)$ with respect to Ω for any given $k \ge 1$ when $\alpha < 0$, since if we let $\widetilde{\Omega}_n$ be the disjoint union of k copies of Ω_n, then $\lambda_k(\widetilde{\Omega}_n) = \lambda_1(\Omega_n) \to -\infty$ as $n \to \infty$.

Unfortunately, the method used when $\alpha > 0$ does not seem to carry across, and we cannot prove the conjecture as yet; there seems to be an asymmetry between the two cases. We do not go into extensive detail, but we illustrate where the method breaks down. The key problem is that the contradiction argument in Theorem 2.3.4 no longer seems to work.

By the maximum principle, the eigenfunction $\psi \in H^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega)$ (chosen positive and normalised so $\|\psi\|_\infty = 1$) now has its minimum in Ω and increases up to its maximum on $\partial\Omega$. That is, $\{x \in \overline{\Omega} : \psi(x) = 1\} \subset \partial\Omega$, while $\min \psi(x) =: m \in \Omega$. Instead of using the level sets $U_t = \{x \in \Omega : \psi(x) > t\}$ it seems more natural to use the interior sub-level sets $V_t := \{x \in \Omega : \psi(x) < t\}$; then $V_m = \emptyset$, $V_1 = \Omega$ and $V_t \subset\subset \Omega$ for $t > m$ sufficiently small. (We could alternatively choose $\psi < 0$ and then have $m \in \partial\Omega$; we could also use the usual level sets U_t, but of course these change nothing of substance.) The functional becomes

$$H_\Omega(V_t, \varphi) = \frac{1}{|V_t|}\Big(\int_{\partial_i V_t} \varphi \, d\sigma + \int_{\partial e V_t} \alpha \, d\sigma - \int_{V_t} |\varphi|^2 \, dx\Big),$$

where we now choose $0 \ge \varphi \in C(\Omega)$. Note that all three integrals are negative. By an argument analogous to the one in Proposition 2.2.3,

$$H_\Omega(V_t, -\frac{|\nabla\psi|}{\psi}) = \lambda_1(\Omega)$$

for a.e. $t \in (m, 1)$. (If we use the U_t, then we will use functions $0 \le \varphi$ and obtain Proposition 2.2.3 verbatim.) For $0 \ge \varphi \in C(\Omega)$ we then set $w := \varphi + |\nabla\psi|/\psi = \varphi - (-|\nabla\psi|/\psi)$ and

$$F(t) := \int_m^t \frac{1}{\tau}\int_{S_\tau} w\,d\sigma\,d\tau; \qquad \frac{d}{dt}F(t) = \frac{1}{t}\int_{S_t} w\,d\sigma.$$

Again, analogous to the case $\alpha > 0$ (see Theorem 2.2.6), for $\varphi \in C(\Omega)$ non-positive

$$H_\Omega(V_t, \varphi) = \lambda_1(\Omega) + \frac{1}{|V_t|}\Big(\frac{1}{t}\frac{d}{dt}\big(t^2F(t)\big) - \int_{V_t} w^2\,dx\Big) \tag{3.4.1}$$

for almost all $t \in (m, 1)$. Now the problem becomes evident: ideally, we would like to show that for some $t \ge 0$ and rearrangement φ of $-|\nabla\psi^*|/\psi^*$ we have

$$\lambda_1(B) \equiv H_B(B_{r(t)}, -\frac{|\nabla\psi^*|}{\psi^*}) \ge H_\Omega(V_t, \varphi) \ge \lambda_1(\Omega)$$

(where all the terms are interpreted in the obvious way), but the latter inequality does not seem to come out of (3.4.1). Indeed, the only obvious contradiction assumption is in fact $H_\Omega(V_t, \varphi) > \lambda_1(\Omega)$, since then

$$\frac{1}{t}\frac{d}{dt}\big(t^2F(t)\big) > \int_{V_t} w^2 \ge 0, \tag{3.4.2}$$

but not even this leads to a contradiction. For, given that $F(m) = 0$, this gives us $F(t) > \xi > 0$ in a neighbourhood of $t = 1$. If we take M_α to be those non-positive $C(\Omega)$ functions φ such that $\limsup_{x\to z}\varphi(x) \ge \alpha = \alpha(z)$ (the obvious analogue of the positive case), then "$w \ge -\varepsilon$" (interpreted appropriately as in Lemma 2.3.3) and this is clearly not going to contradict the definition of F. If instead we take M_α so that $\varphi \le \alpha$ on $\partial\Omega$ and "$w \le \varepsilon$", then expanding out (3.4.2) gives

$$0 < 2\xi < 2F(t) \ge \int_{V_t} w^2\,dx - t^2F'(t)$$

for t close to 1, which still gives no obvious contradiction. As mentioned earlier, we could consider $\varphi \ge 0$ and/or U_t instead, but this does not seem to help.

We next look at a geometrically elegant minimisation problem that is unfortunately rather difficult from an analytic perspective. Consider the Dirichlet problem (1.1.1) on a polygonal domain $\Omega \subset \mathbb{R}^2$. It is a famous conjecture attributed to Pólya that amongst all polygons of a given number of sides n, the regular polygon has the smallest first eigenvalue μ_1 (and is the unique polygon with this property). This is supported by the isoperimetric inequality for such domains: amongst all

n-sided polygons (or n-gons for short) of given area, the regular n-gon has the (strictly) shortest perimeter. The conjecture has been proven for the triangle and the square, but remains open for $n \geq 5$, since the symmetrisation argument used when $n = 3$ or 4 breaks down. (See [**66**, Section 3.2].)

As regards the method from Chapter 2, clearly Theorem 2.3.5 is valid here (unlike the $\alpha < 0$ problem). However, now the rearrangement argument appears to break down. It works on the ball because $|\nabla\psi^*|/\psi^*$ is constant on the level surfaces $B_{r(t)}$ of ψ^*, that is, $|\nabla\psi^*|/\psi^*$ and ψ^* have the *same* level surfaces, namely concentric spheres. Since this is no longer true of the regular n-gons (or n-gons in general), there is no obvious way to construct the rearrangement φ from the regular n-gon's eigenfunction. Of course, it may be possible to construct an alternative rearrangement. Fix n, let P denote the regular n-gon, let ψ^* denote its eigenfunction and let Ω denote any other n-gon. To prove the conjecture, we wish to find a non-negative function $\varphi \in C(\Omega)$ such that

$$H_P(P_t, \frac{|\nabla\psi^*|}{\psi^*}) \leq H_\Omega(U_t, \varphi)$$

for almost all $t \in (0, 1)$, where P_t is the level set of ψ^* satisfying $|P_t| = |U_t|$. To this end it would be important to know if $\sigma(\partial P_t) \leq \sigma(\partial U_t)$ for all, or almost all, $t \in (0, 1)$, but we do not know if this is true.

Remark 3.4.3. It seems likely that in the Robin case (with $\alpha > 0$) the same result is true; that is, that the regular n-gon minimises $\lambda_1(\Omega, \alpha)$ for a given value of $|\Omega|$ and α. This appears to be a new conjecture, but it is an extremely obvious one to make. There is the following evidence in favour of it. First, since $\lambda_1(\Omega, \alpha) \to \mu_1(\Omega)$ as $\alpha \to \infty$, when $n = 3$ or 4 using the Dirichlet result we must have $\lambda_1(P, \alpha) \leq \lambda_1(\Omega, \alpha)$ for α large enough, although how large potentially depending on Ω. Moreover, as with $\alpha < 0$, for $\alpha > 0$ sufficiently small, again possibly depending on Ω, (1.3.2) implies $\lambda_1(P, \alpha) < \lambda_1(\Omega, \alpha)$.

Chapter 4

On the Higher Eigenvalues of the Robin Laplacian

In this chapter we will study the minimisation problem (1.1.4) for the higher eigenvalues of the Robin problem (1.1.2), $\lambda_k = \lambda_k(\Omega, \alpha)$, $k \geq 2$ and $\alpha > 0$. When $k = 2$, we will prove in Sections 4.1 and 4.2 that the unique solution to (1.1.4) for the second eigenvalue $\lambda_2(\Omega)$ (for all $\alpha > 0$) is the disjoint union of two balls of equal volume, the domain which also minimises the second Dirichlet eigenvalue $\mu_2(\Omega)$. We shall call this domain D_2. In this case we will actually prove the result for the p-Laplacian with Robin boundary conditions (see Section 3.1) since the proof is easily adapted to the more general situation.

When $k \geq 3$ (and $p = 2$) we show in Section 4.3 that for many values of k and dimension N, there cannot be a solution of (1.1.4) independent of the boundary parameter $\alpha > 0$, or alternatively, of the volume of the domains $M := |\Omega| > 0$. In order to do so we prove that the domain D_k consisting of the disjoint union of k equal balls $B_1, \dots, B_k$ (see Figure 4.1) is in some sense the only candidate to minimise λ_k for α very small (this is made precise in Theorem 4.3.1).

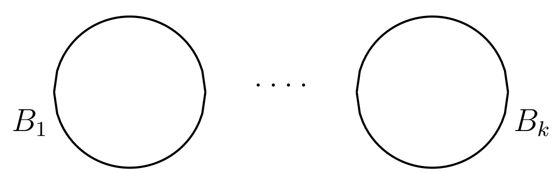

Figure 4.1. The domain D_k

Note that, as with $k = 1$, one could also study the corresponding maximisation problem when $\alpha < 0$ (cf. Remark 3.4.2). As when $k = 1$, we can say very little about the isoperimetric properties of this case, but unlike when $k = 1$, the natural conjectures do not seem to be the mirror images of those for $\alpha > 0$; see Remark 4.1.3(iii). However, we will study the asymptotic behaviour of λ_k as $\alpha \to -\infty$ (cf. Theorem 1.3.1(ix) for when $k = 1$). It turns out that if Ω is C^1, then $\lambda_k(\alpha)/(-\alpha^2) \to 1$ as $\alpha \to -\infty$ for *every* $k \geq 1$. This result, which is quite interesting in its own right, will also be of use to us when we study the Wentzell eigenvalues in Chapter 6.

The material in Sections 4.1, 4.2 and 4.3 is in a paper of the author [**75**]. An earlier and less general formulation of the material in Sections 4.1 and 4.2 was published in [**77**]. Section 4.4 is in [**38**].

4.1. The second eigenvalue

We first consider the second Robin eigenvalue. Our main theorem is as follows. Here we stress that our domains Ω need not be connected. Fix the volume $M = |\Omega|$ and $\alpha > 0$ arbitrary, and let $1 < p < \infty$ in the problem (3.1.1). We also stress that when $p = 2$, (3.1.1) reduces to (1.1.2).

Theorem 4.1.1. *Let $D_2 \subset \mathbb{R}^N$ be given by the disjoint union of two balls of volume $M/2$. For any bounded, Lipschitz domain $\Omega \subset \mathbb{R}^N$, the second eigenvalue $\lambda_2(\Omega) = \lambda_2(\Omega, \alpha, p)$ satisfies $\lambda_2(\Omega) \geq \lambda_2(D_2)$, with equality if and only if $\Omega = D_2$ in the sense of Remark 1.3.2.*

Note that Proposition 3.1.1 guarantees that such a result is actually meaningful for the p-Laplacian. In order to prove this theorem, we will first discuss the corresponding Dirichlet problem, since the proof of Theorem 4.1.1 follows the same lines (but is more complicated). We will defer the proof of the latter to Section 4.2. So we start with the following classical result on the second Dirichlet eigenvalue $\mu_2(\Omega)$. We only consider $p = 2$ here since this is the "classical" case and is included for illustrative purposes, but it makes absolutely no difference to the proof.

Theorem 4.1.2. *Let $\Omega \subset \mathbb{R}^N$ be bounded, open. The second eigenvalue $\mu_2(\Omega)$ of (1.1.1) satisfies $\mu_2(\Omega) \geq \mu_2(D_2)$, with equality if and only if $\Omega = D_2$ up to rigid transformations and sets of capacity zero.*

As with the first eigenvalue μ_1, this result has a somewhat venerable history. It is sometimes attributed to Szëgo, as Pólya does (see [**95**, Remark 5(c)]), although it is also claimed that it goes all the way back to Krahn (see [**66**, Section 1]). The proof is elegant and non-technical. We have based this proof on the sketch in [**66**, Section 4].

Proof of Theorem 4.1.2. Fix $\Omega \subset \mathbb{R}^N$. Suppose that Ω is connected; the proof when Ω is not connected is given in Remark 4.1.3(i). Denote the eigenfunction associated with $\mu_2(\Omega)$ by $\varphi = \varphi(\Omega, \mu_2)$. Then φ must change sign in Ω, since it is orthogonal in $L^2(\Omega)$ to the eigenfunction associated with μ_1, which does not change sign since Ω is connected. Hence the nodal domains Ω^+, Ω^- (see (A4.6)) of $\varphi \in H^1(\Omega) \cap C^\infty(\Omega) \cap C(\overline{\Omega})$ (see Theorem 1.2.8) are both non-empty subsets of Ω. Moreover, $\partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega = \{x \in \Omega : \varphi(x) = 0\}$ (see Appendix A4 and in particular the remarks after (A4.6)). Hence φ satisfies

$$\begin{aligned} -\Delta\varphi &= \mu_2\varphi && \text{in } \Omega^+ \\ \varphi &= 0 && \text{on } \partial\Omega^+. \end{aligned}$$

Since φ does not change sign in Ω^+, $\mu_2(\Omega)$ must be the first Dirichlet eigenvalue of Ω^+, that is, $\mu_2(\Omega) = \mu_1(\Omega^+)$. Similarly, $\mu_2(\Omega) = \mu_1(\Omega^-)$.

Now let B^+ and B^- be balls having the same volume as Ω^+ and Ω^-, respectively. By the usual Faber-Krahn inequality, Theorem 2.1.3, $\mu_1(\Omega^+) \geq \mu_1(B^+)$ and $\mu_1(\Omega^-) \geq \mu_1(B^-)$.

Putting this together, we have $\mu_2(\Omega) \geq \max\{\mu_1(B^+), \mu_1(B^-)\}$. This maximum is minimised over all possible choices of B^+ and B^- when $B^+ = B^-$ have half the volume of Ω. In this case, we have $D_2 = B^+ \cup B^-$, where equality is in the sense of Remark 1.3.2. Since $\mu_2(D_2) = \mu_1(B_1) = \mu_1(B_2)$, we have shown that $\mu_2(\Omega) \geq \mu_2(D_2)$, as claimed.

The uniqueness of the minimiser now follows easily. If Ω is not already the disjoint union of two balls up to a set of capacity zero, then one of the two nodal domains, say Ω^+, will not be a ball. Strictness in the Faber-Krahn inequality, Theorem 2.1.3, applied to this nodal domain gives $\mu_2(\Omega) = \mu_1(\Omega^+) > \mu_1(B^+)$. If $\mu_1(B^+) \geq \mu_2(D_2)$, then we are done. If not, then it must be the case that $|B^+| > \frac{1}{2}|D_2|$. In this case, since $\mu_1(B)$ is strictly monotonically decreasing with respect to $|B|$, $\mu_1(B^-) > \mu_2(D_2)$, so that $\mu_2(\Omega) \geq \mu_1(B^-) > \mu_2(D_2)$. Finally, if Ω is the disjoint union of two balls which are not of equal volume, then it follows immediately from the strict monotonicity of μ_1 that $\mu_2(\Omega) > \mu_2(D_2)$. $\square$

Based on the above proof, the proof of Theorem 4.1.1 should go roughly as follows. If we let ψ denote an eigenfunction of $\lambda_2(\Omega)$, and Ω^+, Ω^- denote the nodal domains of ψ, we would wish to describe $\lambda_2(\Omega)$ as the first eigenvalue of a suitable problem on Ω^+ (and Ω^-) with mixed Robin-Dirichlet boundary conditions $|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu} + \alpha|\psi|^{p-2}\psi = 0$ on $\partial\Omega^+ \cap \partial\Omega$ and $\psi = 0$ on $\partial\Omega^+ \cap \Omega$ (cf. the mixed problem (2.2.1)).

If we then replace the Dirichlet boundary condition on $\partial\Omega^+ \cap \Omega$ with the usual Robin boundary condition, this should shrink the first eigenvalue, and we may then apply the Robin Faber-Krahn inequality, Theorem 2.1.1 or Theorem 3.1.2, to conclude that this is no smaller than $\lambda_1(B^+)$, and do the same with Ω^-. Once we have this inequality, we may proceed in exactly the same fashion as in the Dirichlet case.

This is the rough idea behind the proof of Theorem 4.1.1, but there is a significant complication in the above line of reasoning that renders the proof more difficult. The Robin Faber-Krahn inequality is only known for Lipschitz (or slightly weaker than Lipschitz) domains, and there seem to be significant obstacles to generalising it (see [**22**, Section 6]). This causes a problem here because there is no reason to suppose that the nodal domains of a general Lipschitz domain $\Omega \subset \mathbb{R}^N$ will have Lipschitz boundary. There are two reasons why Ω^+ and Ω^- may not have smooth boundary.

First, we have no control over the behaviour of $\partial\Omega^+$, $\partial\Omega^-$ near where the nodal surface $\{x \in \Omega : \psi(x) = 0\}$ meets $\partial\Omega$. At such points x, this would give us $\frac{\partial \psi}{\partial \nu} = 0$ (however we would wish to interpret this), which implies either $\nabla\psi$ is perpendicular to the outer unit normal, or that $\nabla\psi = 0$ at the point of intersection. The first possibility is fine, since it implies that $\partial\Omega^+$ is Lipschitz at that point, but the second is clearly not. Moreover, there does not seem to be a way to rule out the latter possibility in general. (Contrast this with the behaviour of the first eigenfunction as in Lemma 2.2.1(ii).)

Second, even though ψ will be C^∞ in Ω, we do not know that the nodal surface is a smooth manifold in the interior. Note that Sard's lemma (see for example [**68**, Theorem 3.1.3]) does not suffice, since zero may be in the null set of level surfaces of ψ which are not smooth.

These problems can be overcome by constructing carefully chosen approximating domains (see Section 4.2). To the best of our knowledge, the type of approximation we use is new and might even have other applications. Before we give the proof we make a few observations and remarks.

Remark 4.1.3. (i) We do not require the domains to be connected. This was assumed in the proof of Theorem 4.1.2 and will again be assumed in the proof of Theorem 4.1.1. However, there is an easy (and presumably standard) way to remove this assumption. Suppose Theorem 4.1.1 holds for all connected domains and that Ω is not connected. Then either $\lambda_2(\Omega) = \lambda_2(\widetilde{\Omega})$ for some connected component $\widetilde{\Omega}$ of Ω or there exist disjoint connected components Ω', Ω'' of Ω such that $\lambda_1(\Omega) = \lambda_1(\Omega')$, $\lambda_2(\Omega) = \lambda_1(\Omega'')$. In the former case, if we let $\widetilde{D}_2$ denote a scaled down version of D_2 with $|\widetilde{D}_2| = |\widetilde{\Omega}|$, then Theorem 4.1.1 implies $\lambda_2(\widetilde{\Omega}) > \lambda_2(\widetilde{D}_2)$. Since $|\widetilde{D}_2| < |D_2|$, $\lambda_2(\widetilde{D}_2) > \lambda_2(D_2)$ by Lemma 1.3.7. In the latter case, let B', B'' be balls having the same volume as Ω', Ω'', respectively. Then the Faber-Krahn inequality implies $\lambda_2(\Omega) \geq \max\{\lambda_1(\Omega'), \lambda_1(\Omega'')\} \geq \max\{\lambda_1(B'), \lambda_1(B'')\}$, and the latter is no less than $\lambda_2(D_2)$ via the same argument as in the proof of Theorem 4.1.2 involving B^+, B^-. Finally, if $\lambda_2(\Omega) = \lambda_2(D_2)$ then equality everywhere in the above argument implies via sharpness of the Faber-Krahn inequality that $\Omega' = B'$ and $\Omega'' = B''$, and using strict monotonicity in Lemma 1.3.7 that $|B'| = |B''| = M/2$. Hence $\Omega = D_2$. The same argument works for Dirichlet boundary conditions with trivial modifications.

(ii) It is a physically interesting question to ask if one can find a minimiser amongst all *connected* domains (as pointed out by Henrot [**66**], it's rather hard to play a disconnected drum with one hand). In the Dirichlet case, there is no minimiser: one can find a sequence of connected domains Ω_n with $\mu_2(\Omega_n) \to \mu_2(D_2)$ (for example take "dumbbells" with a handle of shrinking radius). We construct such a sequence of Ω_n below, such that $\lambda_2(\Omega_n) \to \lambda_2(D_2)$ when $p = 2$ (see Example 4.1.4). Hence there is no Robin minimiser amongst connected domains.

(iii) We note as an open problem the case when $\alpha < 0$. As we saw earlier (see Remark 3.4.2), in general we seek a maximiser for $\lambda_k(\Omega, \alpha)$, and when $k = 1$ we conjecture this is the ball (Conjecture 3.4.1). For λ_2, D_2 cannot be the maximiser for all values of $\alpha < 0$ since $\lambda_2(D_2, \alpha) < 0$ for all α, but there exist Ω and α for which $\lambda_2(\Omega, \alpha) > 0$. For example, if $\Omega = B_1$, the ball of radius 1, then by [**80**, Section 2.4] B_1 has $1 + \lfloor |\alpha| \rfloor$ negative eigenvalues. So $\lambda_2(B_1, \alpha) > 0$ if $\alpha > -1$. In fact it seems that B is the best candidate for the maximiser, since $\lambda_2(B, \alpha)$ in the Neumann case $\alpha = 0$ and λ_2 should depend continuously on $\alpha \in \mathbb{R}$ for fixed Ω. In particular for $\alpha < 0$ close to zero we would get $\lambda_1(\Omega, \alpha) < \lambda_1(B, \alpha)$ (how close possibly depending on Ω). Compare with Theorem 4.3.1(i).

(iv) There is another argument for using an approximation method rather than an approach via theory for arbitrary domains to overcome the problems discussed above, albeit very much a matter of taste. The "natural" setting for Robin problems is on Lipschitz domains (cf. Appendix A4 and Remark 1.2.7(ii)), since this is the broadest class of domains where the problem may be understood classically; otherwise the trace inequality fails, ν is not well-defined, etc. Moreover there are several valid alternative theories on non-Lipschitz domains (for example compare [**11**] with [**33**] and also [**23**]). By using the approximation argument we may obtain the result on Lipschitz domains in a more self-contained and elementary manner, although admittedly technical in places.

Example 4.1.4. We construct a sequence of connected domains Ω_n of fixed volume such that $\lambda_2(\Omega_n) \to \lambda_2(D_2)$ when $p = 2$ (see Figure 4.2). Our domains are almost

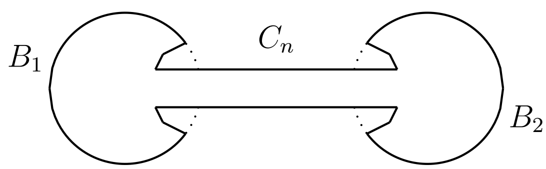

Figure 4.2. The domain Ω_n

identical to the "dumbbells" used in [**66**]. Start with $D_2 = B_1 \cup B_2$ and join B_1 to B_2 with a cylinder C_n of total volume $\frac{1}{n}$. To keep the volume of Ω_n constant, remove part of B_1 and B_2 in a small neighbourhood U_n near where they meet C_n (as in Figure 4.2) in such a way that the resulting boundary is still smooth. It now follows from [**32**, Corollary 3.7] that $\lambda_2(\Omega_n) \to \lambda_2(D_2)$, since the U_n can be chosen in such a way that Assumption 3.2 of [**32**] is satisfied. This construction should work when $p \neq 2$, but we do not know of domain approximation results akin to those in [**32**] for general $1 < p < \infty$.

4.2. Proof of Theorem 4.1.1

Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain. As noted in Remark 4.1.3(i), we may assume without loss of generality that Ω is connected. Its second eigenvalue $\lambda_2(\Omega)$ has an eigenfunction $\psi \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \cap C^{1,\eta}(\Omega)$, $\eta \in (0,1)$ (see Theorem 1.2.8 if $p = 2$ or Proposition 3.1.1 otherwise). Since Ω is connected, ψ changes sign in Ω, so that the open subsets Ω^+ and Ω^- (defined in (A4.6)) are both nonempty. Moreover, $\psi^+, \psi^- \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ (see Appendix A4 and in particular (A4.4) and (A4.5)). Let B^+, B^- be balls having the same volume as

Ω^+, Ω^- respectively. We will show that $\lambda_2(\Omega) > \max\{\lambda_1(B^+), \lambda_1(B^-)\}$, which certainly suffices to prove Theorem 4.1.1 in light of our earlier remarks.

In fact we will first prove $\lambda_2(\Omega) \geq \max\{\lambda_1(B^+), \lambda_1(B^-)\}$ (which proves the inequality in Theorem 4.1.1) and then refine the argument to prove sharpness. We will do it this way because the latter argument is rather technical and this better illustrates the method we use.

Without loss of generality we only consider Ω^+. The key idea is to attach a thin strip near $\partial\Omega$ to Ω^+ to avoid any problems when $\{x \in \Omega : \psi(x) = 0\}$ meets $\partial\Omega$. For $n \geq 1$ let $S_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \delta\}$, where $\delta = \delta(n) > 0$ is chosen such that $|S_n| < 1/(2n)$.

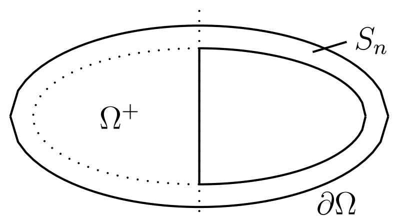

FIGURE 4.3. The sets Ω^+, S_n

Since $\Omega^+ \cup S_n$ may not be smooth, we now approximate from the outside by a smooth domain U_n as follows. By [**49**, Theorem V.20] we can find a domain U_n such that $\Omega^+ \cup S_n \subset U_n \subset \Omega$, $\Gamma_n := \partial U_n \cap \Omega$ is C^∞, and $|U_n \setminus (\Omega^+ \cup S_n)| < 1/(2n)$. (This is equivalent to approximating the non-smooth domain $(\mathbb{R}^N \setminus \Omega) \cup (\Omega^+ \cup S_n)$ from the outside by a sequence of smooth open sets V_m.) Then U_n is Lipschitz, $|U_n \setminus \Omega^+| < 1/n$, and $\partial U_n = \partial\Omega \cup \Gamma_n$ where $\operatorname{dist}(\partial\Omega, \Gamma_n) > \delta(n) > 0$.

We will denote by $W_0^{1,p}(U_n; \Gamma_n)$ the closure in $W^{1,p}(U_n)$ of $C_c^\infty(\overline{U}_n)$, the space of all $C^\infty(\overline{U}_n)$ functions with support compactly contained in $U_n \cup \partial\Omega$. (See also Definition 1.2.1. When $p = 2$ we may think of $H_0^1(U_n)$ as the space of weak solutions to (1.2.1) on U_n, with $\Gamma_n =: \Gamma_0$ and $\partial\Omega =: \Gamma_1$.)

Lemma 4.2.1. *For any $n \geq 1$, $\psi^+ \in W_0^{1,p}(U_n; \Gamma_n)$.*

Proof. In a slight abuse of notation we will not distinguish between ψ^+ on Ω and $\psi^+|_{U_n}$ or $\psi^+|_{\Omega^+}$. Thus $\psi^+ \in W^{1,p}(U_n) \cap C(\overline{U}_n)$ with $\psi^+ = 0$ on Γ_n. Since $\partial\Omega$ and Γ_n are separated our claim follows from a trivial modification of the proof of [**21**, Théorème IX.17] (see also Remarque 20 there; or alternatively cf. [**59**, Section 7.5]). $\square$

We now set

$$\kappa(U_n) := \inf_{\varphi \in W_0^{1,p}(U_n;\Gamma_n)} \frac{\int_{U_n} |\nabla\varphi|^p\,dx + \int_{\partial\Omega} \alpha|\varphi|^P\,d\sigma}{\int_{U_n} |\varphi|^p\,dx}. \tag{4.2.1}$$

The key estimate for $\lambda_2(\Omega)$ is as follows.

Lemma 4.2.2. *For any $n \geq 1$, $\lambda_2(\Omega) \geq \kappa(U_n)$.*

Proof. $\lambda_2(\Omega)$ satisfies

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\varphi\,dx + \int_{\partial\Omega} \alpha|\psi|^{p-2}\psi\varphi\,d\sigma = \lambda_2(\Omega)\int_\Omega |\psi|^{p-2}\psi\varphi\,dx$$

for all $\varphi \in W^{1,p}(\Omega)$. Choosing $\psi^+ \in W^{1,p}(\Omega)\cap C(\overline{\Omega})$ as a test function, we have $\nabla\psi\cdot\nabla\psi^+ = |\nabla\psi^+|^2$ and $\psi\psi^+ = |\psi^+|^2$ pointwise in $\overline{\Omega}$. Since $\|\psi^+\|_p \neq 0$,

$$\lambda_2(\Omega) = \frac{\int_\Omega |\nabla\psi^+|^p\,dx + \int_{\partial\Omega}\alpha|\psi^+|^p\,d\sigma}{\int_\Omega |\psi^+|^p\,dx}. \tag{4.2.2}$$

But the integrands in the volume integrals are nonzero only on $\Omega^+ \subset U_n$. Hence

$$\lambda_2(\Omega) = \frac{\int_{U_n} |\nabla\psi^+|^p\,dx + \int_{\partial\Omega}\alpha|\psi^+|^p\,d\sigma}{\int_{U_n} |\nabla\psi^+|^p\,dx}. \tag{4.2.3}$$

Since $\psi^+ \in W_0^{1,p}(U_n;\Gamma_n)$ by Lemma 4.2.1, comparing (4.2.1) and (4.2.3) yields $\lambda_2(\Omega) \geq \kappa(U_n)$. $\square$

Now impose Robin boundary conditions on Γ_n and consider (1.1.2) on U_n. Since $W_0^{1,p}(U_n;\Gamma_n) \subset W^{1,p}(U_n)$ in the obvious way, by a direct comparison of variational formulae (cf. (4.2.1) and (3.1.3)), $\kappa(U_n) \geq \lambda_1(U_n)$. Now for each n, let B_n be a ball with $|B_n| = |U_n|$. Since ∂U_n is Lipschitz, we may apply the Faber-Krahn inequality (Theorem 3.1.2, which is [**22**, Theorem 1.1]) to obtain $\lambda_1(U_n) \geq \lambda_1(B_n)$. That is,

$$\lambda_2(\Omega) \geq \kappa(U_n) \geq \lambda_1(U_n) \geq \lambda_1(B_n).$$

As $n \to \infty$, $|U_n| \to |\Omega^+|$ and so $|B_n| \to |B^+|$. By Lemma 1.3.7, $\lambda_1(B_n) \to \lambda_1(B^+)$ and thus $\lambda_2(\Omega) \geq \lambda_1(B^+)$, proving the desired inequality.

We now prove sharpness. That is, we will prove $\lambda_2(\Omega) > \lambda_1(B^+)$. Let $\partial_e\Omega^+ := \partial\Omega^+ \cap \partial\Omega$ and $\partial_i\Omega^+ := \partial\Omega^+ \cap \Omega = \partial\Omega^+ \setminus \partial_e\Omega^+$ denote the exterior and interior boundaries of Ω^+, respectively, as in Section 2.2. Note however that here $\partial_i\Omega^+$ will not in general be closed, as it will meet $\partial_e\Omega^+$ somewhere (the above approximation argument being used largely to avoid such potentially bad points). The idea of the sharpness proof is to show that a piece Γ of $\partial_i\Omega^+$ must be smooth, and then impose Robin boundary conditions on Γ, strictly lowering the first eigenvalue of a suitable mixed problem. We then choose the U_n so that $\Gamma_n \supset \Gamma$.

Remark 4.2.3. There are two separate reasons why we should have sharpness of the inequality for connected domains. First, imposing Robin boundary conditions on $\partial_i\Omega^+$ should strictly lower the first eigenvalue. Second, applying the Faber-Krahn inequality to Ω^+ should lower the eigenvalue further since if both Ω^+ and Ω^- are balls of the same volume, then Ω consists of two balls just touching (which is not Lipschitz). Since we are using a domain approximation argument, we cannot exploit the latter principle. Hence we make use of the former observation, although the details are somewhat technical. Consequently we do not actually use sharpness of the Faber-Krahn inequality to obtain sharpness for connected domains.

Lemma 4.2.4. *There exist $x_0 \in \Omega$ and $r > 0$ such that $\psi(x_0) = 0$, $B(x_0, r) \subset\subset \Omega$, $\nabla\psi(x) \neq 0$ for all $x \in B(x_0, r)$, $\psi \in C^\infty(B(x_0, r))$ and $\{x \in B(x_0, r) : \psi(x) = 0\}$ is a surface of class C^∞.*

Proof. We first show we can find $x_0 \in \partial_i\Omega^+$ with $\nabla\psi(x) \neq 0$ in a neighbourhood of x_0. Choose any $x \in \Omega^+$ close to $\partial_i\Omega^+$ and let $\delta_0 := \inf\{\delta > 0 : \partial B(x, \delta) \cap \partial_i\Omega^+ \neq \emptyset\}$. Then $B(x, \delta_0) \subset \Omega^+$ but there exists $x_0 \in \partial B(x, \delta_0) \cap \partial_i\Omega^+$.

We now apply a version of Hopf's lemma for the p-Laplacian due to Vázquez. Since $\psi(x_0) = 0$, $\psi(x) > 0$ in $B(x, \delta_0)$ and $\psi \in C^1(\overline{B(x, \delta_0)})$, by [**104**, Theorem 5] we have $\frac{\partial\psi}{\partial\nu_B}(x_0) < 0$, where ν_B is the outer unit normal to $B(x, \delta_0)$. Hence $\nabla\psi(x_0) \neq 0$, and so by continuity of $\nabla\psi$ there exists a neighbourhood V_0 of x_0 and $m > 0$ such that $|\nabla\psi(x)| \geq m$ for all $x \in V_0$. Inside V_0 set $a(x) := |\nabla\psi(x)|^{p-2}$. Then $0 < m^{p-2} \leq a(x) \in L^\infty(V_0)$. It follows that $\psi \in C^1(\overline{V_0})$ is an eigenfunction of the uniformly elliptic operator $-\operatorname{div}(a(x)\nabla u)$. A standard bootstrapping argument using elliptic regularity theory yields $\psi \in C^\infty(V_0)$. By the implicit function theorem, it follows that the level surface $\{\psi = 0\}$ is, locally, the graph of a C^∞ function inside V_0. □

Fix x_0 and r as in the lemma and set $\Gamma := \partial_i\Omega^+ \cap B(x_0, r/2)$. Then Γ is C^∞ and its surface measure $\sigma(\Gamma) > 0$. Instead of considering $W_0^{1,p}(\Omega^+; \partial_i\Omega^+ \setminus \Gamma)$ it will be easier to work directly with

$$V_0 := \{\varphi \in W^{1,p}(\Omega^+) \cap C(\overline{\Omega^+}) : \varphi = 0 \text{ on } \partial_i\Omega^+ \setminus \Gamma\}.$$

For $\varphi \in V_0$ set

$$Q_p(\varphi) := \frac{\int_{\Omega^+} |\nabla\varphi|^p\,dx + \int_{\partial_e\Omega^+\cup\Gamma} \alpha|\varphi|^p\,d\sigma}{\int_{\Omega^+} |\varphi|^p\,dx} \tag{4.2.4}$$

to be a Rayleigh quotient-type expression, and let

$$\kappa(\Omega^+) := \inf_{\varphi\in V_0} Q_p(\varphi). \tag{4.2.5}$$

Lemma 4.2.5. *We have $\psi^+ \in V_0$ and $\lambda_2(\Omega) = Q_p(\psi^+) = Q_p(\psi)$.*

Proof. Since $\psi^+ = 0$ on all $\partial_i\Omega^+$, it is immediate that $\psi^+ \in V_0$. Since $\psi \equiv \psi^+$ on $\overline{\Omega^+}$, it is also obvious that $Q_p(\psi) = Q_p(\psi^+)$. By (4.2.2),

$$\begin{aligned}\lambda_2(\Omega) &= \frac{\int_\Omega |\nabla\psi^+|^p\,dx + \int_{\partial\Omega} \alpha|\psi^+|^p\,d\sigma}{\int_\Omega |\psi^+|^p\,dx}\\ &= \frac{\int_{\Omega^+} |\nabla\psi^+|^p\,dx + \int_{\partial_e\Omega^+\cup\Gamma} \alpha|\psi^+|^p\,d\sigma}{\int_{\Omega^+} |\psi^+|^p\,dx} = Q_p(\psi^+),\end{aligned}$$

where the second line follows since the volume integrands are nonzero only on Ω^+, and the surface integrand $\alpha|\psi^+|^p \in C(\partial\Omega^+)$ is nonzero only on $\partial_e\Omega^+$. $\square$

Lemma 4.2.6. $\lambda_2(\Omega) > \kappa(\Omega^+)$.

Proof. It is immediate from Lemma 4.2.5 and (4.2.5) that $\lambda_2(\Omega) \geq \kappa(\Omega^+)$. Suppose for a contradiction that there is equality. Then $\lambda_2(\Omega)$ and ψ must satisfy the minimising condition

$$\frac{d}{dt}\Big(\frac{\int_{\Omega^+} |\nabla(\psi - t\varphi)|^p\,dx + \int_{\partial_e\Omega^+\cup\Gamma} \alpha|\psi - t\varphi|^p\,d\sigma}{\int_{\Omega^+} |\psi - t\varphi|^p\,dx}\Big)\Big|_{t=0} = 0$$

for $\varphi \in V_0$ and $t \in \mathbb{R}$. We will sketch the straightforward but tedious evaluation of this derivative. First note that

$$\begin{aligned}\frac{d}{dt}|\nabla(\psi + t\varphi)|^p &= p|\nabla(\psi + t\varphi)|^{p-2}\nabla(\psi + t\varphi)\cdot\nabla\varphi,\\ \frac{d}{dt}|\psi + t\varphi|^p &= p|\psi + t\varphi|^{p-2}(\psi + t\varphi)\varphi.\end{aligned}$$

We now note that we can interchange the order of differentiation and integration for the integrals of $\alpha|\psi + t\varphi|^p, |\psi + t\varphi|^p \in C(\overline{\Omega^+})$. For the gradient, we note the bound

$$\begin{aligned}\big|p|\nabla(\psi + t\varphi)|^{p-2}\nabla(\psi + t\varphi)\cdot\nabla\varphi\big| &\leq p|\nabla(\psi + t\varphi)|^{p-1}|\nabla\varphi|\\ &\leq p\big||\nabla\psi| + |\nabla\varphi|\big|^p\end{aligned}$$

if $|t| \leq 1$. The latter is integrable since $\nabla\psi, \nabla\varphi \in L^p(\Omega^+)$. Thus $\frac{d}{dt}|\nabla(\psi + t\varphi)|^p$ is bounded uniformly with respect to t by an integrable function. This gives us the derivative of each of the integrals. Using the quotient and chain rules as necessary, setting $t = 0$ and solving yields

$$\begin{aligned}\int_{\Omega^+} |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla\varphi\, dx + &\int_{\partial_e\Omega^+\cup\Gamma} \alpha|\psi|^{p-2}\psi\varphi\, d\sigma \\ &= \lambda_2(\Omega)\int_{\Omega^+} |\psi|^{p-2}\psi\varphi\, dx.\end{aligned} \tag{4.2.6}$$

(That is, $\lambda_2(\Omega)$ and ψ are weak solutions of an appropriate eigenvalue problem on Ω^+.) Now since $\partial_i\Omega^+$ is smooth in an open neighbourhood $B(x_0, r)$ of $\overline{\Gamma} \subset \overline{B(x_0, r/2)}$, we can find an open set $U \subset \Omega^+$ Lipschitz with $U \subset\subset B(x_0, r)$ and $\Gamma \subset \partial U$. Let $\varphi \in C_c^\infty(U \cup \Gamma)$. Since every such φ may be regarded as an element of V_0 by extension by 0 outside U, we obtain from (4.2.6) that

$$\int_U |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla\varphi\, dx + \int_\Gamma \alpha|\psi|^{p-2}\psi\varphi\, d\sigma = \lambda_2(\Omega)\int_U |\psi|^{p-2}\psi\varphi\, dx$$

for all $\varphi \in C_c^\infty(U \cup \Gamma)$. We know that $\psi \in C^\infty(\overline{U})$ by Lemma 4.2.4. Hence $-\Delta_p\psi = \lambda_2(\Omega)|\psi|^{p-2}\psi$ pointwise in U. If we multiply by $\varphi \in C_c^\infty(U \cup \Gamma)$, then a simple calculation similar to the one used to obtain (1.2.3) gives

$$\int_U |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla\varphi - \operatorname{div}(|\nabla\psi|^{p-2}\varphi\nabla\psi)\, dx = \lambda_2(\Omega)\int_U |\psi|^{p-2}\psi\varphi\, dx.$$

Applying the divergence theorem on the Lipschitz domain U (see Theorem A4.5),

$$\begin{aligned}\int_U \operatorname{div}(|\nabla\psi|^{p-2}\varphi\nabla\psi)\, dx &= \int_{\partial U} |\nabla\psi|^{p-2}\varphi\nabla\psi \cdot \nu\, d\sigma \\ &= \int_\Gamma |\nabla\psi|^{p-2}\varphi\frac{\partial\psi}{\partial\nu}\, d\sigma\end{aligned}$$

using the compact support of φ on $\partial U \setminus \Gamma$, where ν is the outer unit normal to U on ∂U (coinciding with the normal to Ω^+ on Γ). Plugging this into the above representation and putting everything together it follows that

$$\int_\Gamma \alpha|\psi|^{p-2}\psi\varphi\, d\sigma = -\int_\Gamma |\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu}\varphi\, d\sigma$$

for all $\varphi \in C_c^\infty(U \cup \Gamma)$. Since $C_c^\infty(U \cup \Gamma)$ is dense in $L^q(\Gamma)$ for all $1 < q < \infty$,

$$\alpha|\psi|^{p-2}\psi = -|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu}$$

pointwise σ-almost everywhere in Γ. Hence $\psi \in C^\infty(\overline{U})$ satisfies the boundary condition

$$|\nabla\psi|^{p-2}\frac{\partial\psi}{\partial\nu} + \alpha|\psi|^{p-2}\psi = 0$$

pointwise in Γ. But we know that $\psi = 0$ on Γ, and by Hopf's Lemma [**104**, Theorem 5] applied to the function ψ on the domain U, we have $\frac{\partial\psi}{\partial\nu} > 0$ (and $|\nabla\psi| > 0$) on Γ, which is a contradiction. $\square$

We will now modify the U_n as follows. Choose them as before, but such that $\Gamma \subset \Gamma_n$ for each $n \geq 1$, which we can do since $\partial_i\Omega^+$ is C^∞ in an open neighbourhood of $\overline{\Gamma}$. Then for any $n \geq 1$, U_n is still Lipschitz, $|U_n \backslash \Omega^+| < 1/n$, and since $B(x_0, r) \subset \subset \Omega$, without loss of generality $\operatorname{dist}(\overline{U}_n \setminus \Omega^+, \Gamma) > 0$ as well (see Figure 4.4).

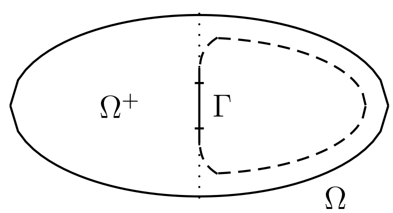

FIGURE 4.4. Ω^+ and U_n. The dotted line represents $\partial_i\Omega^+$ and the dashed line $\Gamma_n = \partial U_n \cap \Omega$.

We need the following result to be able to use the domains U_n. This is in some sense a modification of Lemma 4.2.1.

Lemma 4.2.7. *Let $\varphi \in V_0$ and fix $n \geq 1$. The function $\tilde{\varphi}_n : U_n \to \mathbb{R}$ given by $\tilde{\varphi}_n = \varphi$ in Ω^+, $\tilde{\varphi}_n = 0$ in $U_n \setminus \Omega^+$ lies in $W^{1,p}(U_n)$.*

Proof. Let $\varphi \in V_0$ and $\tilde{\varphi}_n$ be as in the statement of the lemma. Using the lattice properties of V_0 and $W^{1,p}(U_n)$ (cf. [**59**, Lemma 7.6]) we may assume that $\varphi \geq 0$ in Ω^+. For $\xi > 0$ let $\varphi_\xi := (\varphi - \xi)^+ \in V_0$. Then by continuity of φ, there exists an open neighbourhood $U = U(\varphi, \xi)$ of $\partial_i\Omega^+ \setminus \Gamma$ such that $\varphi_\xi \equiv 0$ on $U \cap \overline{\Omega^+}$. Since the intersection of $U_n \setminus \Omega^+$ with $\overline{\Omega^+}$ is contained in $\partial_i\Omega^+ \setminus \Gamma$, we may certainly extend φ_ξ by 0 in $U_n \setminus \Omega^+$ to obtain a function $\tilde{\varphi}_\xi \in W^{1,p}(U_n)$. Since $\tilde{\varphi}_\xi \nearrow \tilde{\varphi}_n$ and

$$\nabla\tilde{\varphi}_\xi(x) \nearrow g(x) := \begin{cases} \nabla\varphi(x) & \text{if } x \in \Omega^+ \\ 0 & \text{if } x \in U_n \setminus \Omega^+ \end{cases}$$

pointwise monotonically in U_n as $\xi \to 0$, it follows easily that $g = \nabla\tilde{\varphi}_n$ and $\tilde{\varphi}_n \in W^{1,p}(U_n)$. $\square$

Now for any $n \geq 1$ and $\varphi \in V_0$, using the variational characterisation of $\lambda_1(U_n)$ given by (3.1.3),

$$\lambda_1(U_n) \leq \frac{\int_{U_n} |\nabla \tilde{\varphi}_n|^p \, dx + \int_{\partial U_n} \alpha |\tilde{\varphi}_n|^p \, d\sigma}{\int_{U_n} |\tilde{\varphi}_n|^p \, dx} = Q_p(\varphi),$$

where $\tilde{\varphi}_n \in W^{1,p}(U_n)$ is the extension of $\varphi \in V_0$ as in Lemma 4.2.7. Hence $\kappa(\Omega^+) \geq \lambda_1(U_n)$ by definition of κ. Let B_n a ball with the same volume as U_n, as before. Then using the Faber-Krahn inequality, $\lambda_1(U_n) \geq \lambda_1(B_n) \to \lambda_1(B^+)$ as $n \to \infty$. Hence

$$\lambda_2(\Omega) > \kappa(\Omega^+) \geq \limsup_{n\to\infty} \lambda_1(U_n) \geq \lambda_1(B^+),$$

which in light of our earlier comments completes the proof.

4.3. Some remarks on the higher eigenvalues

We now return to the case $p = 2$ and the problem (1.1.2), still with $\alpha > 0$. Here we will be considering the problem (1.1.4) for $k \geq 3$. Clearly it would be an impossible task to attempt to identify a minimiser for every $k \geq 3$. Even proving in general the existence of a minimiser is extremely ambitious. To illustrate the difficulties, we will consider briefly what is known in the Dirichlet case. Actually, even in this easier case the general existence of a minimiser is still an open problem. It has been proven in [**28**] that for every $k \geq 3$ one can find a minimiser to the weaker problem of minimising $\mu_k(\Omega)$, $|\Omega| = M$, where Ω is constrained to lie in some fixed "design region" $D \subset \mathbb{R}^N$. Using this result, it has been proven subsequently in [**24**] that a minimiser must exist when $k = 3$. Moreover, once we know a minimiser exists *and is bounded* for $k = 1, \ldots, n$, then results in [**24**] will inductively give us the existence of a (not necessarily bounded) minimiser when $k = n + 1$. When $k = 3$ the minimising domain must be connected in dimension $N = 2$ or 3, but to identify it is still an open problem (although there is a conjecture; see Remark 4.3.4). Indeed, the minimiser is not actually known for *any* $k \geq 3$. Even good candidates have only been identified up to about $k = 5$ or 6 in dimension $N = 2$ or 3 (see [**66**]). Moreover, even in the Dirichlet case it appears that the minimiser may depend on the dimension N for some (most?) k.

In the Robin case, our main result is that a solution to (1.1.4) cannot in general be independent of α, or alternatively, of the volume M. (In addition it seems likely that the dimension N will also affect any solution.) Roughly speaking, we achieve this by what might be thought of as an operator perturbation. More accurately, we vary the value of α in the problem (1.1.2). In particular, as $\alpha \to \infty$, our problem becomes "close" in some sense to the Dirichlet problem, meaning that for α sufficiently large the Robin and Dirichlet minimisers should be the same (if they exist, of course). When α is very close to 0, i.e. the Neumann problem, the domain which we shall denote by D_k, given by the disjoint union of k balls of equal volume, is approximately minimal in a sense to be quantified below. In what follows we will denote by B_m a ball of volume m, so that D_k is the disjoint union of k copies of $B_{M/k}$, and $\lambda_k(D_k, \alpha) = \lambda_1(D_k, \alpha) = \lambda_1(B_{M/k}, \alpha)$. We will set the proof up so it works for a slightly broader class of domains than bounded, Lipschitz.

Theorem 4.3.1. (i) *Let $D_k \subset \mathbb{R}^N$ be the disjoint union of k equal balls. Suppose $\Omega \subset \mathbb{R}^N$ is the disjoint union of countably many connected components Ω_i, $i \in \mathbb{N}$, each bounded and Lipschitz and such that* $\operatorname{dist}(\Omega_i, \Omega_j) \geq \delta$, *say, for some fixed $\delta > 0$ independent of $i \neq j$. If $\Omega \neq D_k$ in the sense of Remark 1.3.2, then there exists $\alpha_\Omega > 0$ possibly depending on Ω such that $\lambda_k(\Omega, \alpha) > \lambda_k(D_k, \alpha)$ for all $\alpha \in (0, \alpha_\Omega)$.*

(ii) *There exist $N \geq 2$ and $k \geq 3$ for which, given $M > 0$, there is no solution to* (1.1.4) *independent of α; equivalently, there is no domain D satisfying $\lambda_k(\Omega, \alpha) \geq \lambda_k(D, \alpha)$ for all $\alpha \in (0, \infty)$ and all Ω.*

(iii) *There exist $N \geq 2$ and $k \geq 3$ for which, given $\alpha > 0$, there is no solution to* (1.1.4) *independent of $|\Omega| = M > 0$.*

Remark 4.3.2. (i) The conclusion of Theorem 4.3.1(ii) and (iii) holds whenever D_k does not minimise the kth Dirichlet eigenvalue μ_k. In particular when $N = 2$ this is true for all $k \geq 3$ (we prove this below) and when $N = 3$ at least for $k = 3$ (for the latter see [**24**, Section 3]).

(ii) An examination of the proof of Theorem 4.3.1(i) shows that the conclusion remains valid whenever the Faber-Krahn inequality holds and is sharp, since the arguments involved are of a very generic nature. The only other property we need is the principle of re-ordering the eigenvalues of connected components (cf. Remark 1.3.2).

(iii) For the same reason, we expect Theorem 4.3.1 to remain valid for the p-Laplacian, without significant modifications to the proof. We do not actually state it for this case because we do not know if the eigenvalues of the p-Laplacian form a discrete set to which we can apply the principles listed in (ii) (see Section 3.1).

(iv) It is easy to see using homothety arguments (see Lemma 1.3.7 and the discussion around it) that no domain with more than k connected components (c.c.s for short) can possibly be a minimiser for λ_k (or μ_k), since there must be some "wastage" with at least one c.c. not contributing to any of the first k eigenvalues. However, this observation does not lead to a simplified proof of case (ii) or (iii) in the proof of Theorem 4.3.1(i). In particular for any number $1 \leq m \leq \infty$ we can find an Ω with m c.c.s such that $\alpha_\Omega < \infty$. To see this, let $k = 3$, $N = 2$, and look at B and D_3. First reduce B slightly in volume to create a smaller ball $\widetilde{B}$ such that still $\lambda_3(\widetilde{B}, \alpha) < \lambda_3(D_3, \alpha)$ for α sufficiently large. Now we may add as many small disjoint balls to $\widetilde{B}$ as we like to obtain Ω. (Such an example could easily be adapted to many other values of k and/or N.) Note also that the proof of Theorem 4.3.1 is made more complicated by the lack of useful properties that the Robin problem satisfies (again, see Remark 1.3.2).

Proof of Theorem 4.3.1(i). There are two cases to consider, depending on how many connected components (c.c.s) Ω has.

(a) Suppose first that Ω has at most $k-1$ c.c.s. If we set $\varepsilon := \min\{\lambda_2(\widetilde{\Omega}, 0) : \widetilde{\Omega}$ is a c.c. of $\Omega\}$, then $\varepsilon > 0$ by Lemma 1.3.10. It follows from continuity in Theorem 1.3.1(vi) that there exists $\tilde{\alpha}_\Omega > 0$ such that

$$\max\{\lambda_1(\widetilde{\Omega}, \alpha) : \widetilde{\Omega} \text{ is a c.c. of } \Omega\} < \varepsilon$$

for all $\alpha \in (0, \tilde{\alpha}_\Omega)$. For all such α, by the pigeonhole principle at least one element of the set $\{\lambda_m(\widetilde{\Omega}, \alpha) : m \geq 2, \widetilde{\Omega}$ is a c.c. of $\Omega\}$ must be one of the first k eigenvalues of Ω, although precisely which m and c.c. may depend on α. In particular, using continuity in Theorem 1.3.1(vi),

$$\begin{aligned} \lambda_k(\Omega, \alpha) &\geq \inf\{\lambda_m(\widetilde{\Omega}, \alpha) : m \geq 2,\, \widetilde{\Omega} \text{ is a c.c. of } \Omega\} \\ &\geq \inf\{\lambda_2(\widetilde{\Omega}, 0) : \widetilde{\Omega} \text{ is a c.c. of } \Omega\} \geq \varepsilon \end{aligned}$$

for all $\alpha \in (0, \tilde{\alpha}_\Omega)$. Since $\lambda_k(D_k, \alpha) = \lambda_1(D_k, \alpha) \to 0$ as $\alpha \to 0$, there exists $\alpha_\Omega \leq \tilde{\alpha}_\Omega$ such that $\lambda_k(D_k, \alpha) < \varepsilon \leq \lambda_k(\Omega, \alpha)$ for all $\alpha \in (0, \alpha_\Omega)$.

(b) Now suppose Ω has at least k c.c.s. We may write Ω as the disjoint union of Ω' and Ω'', where Ω' has $j < \infty$ c.c.s and $|\Omega''| < M/k$ (if $\Omega'' = \emptyset$, then we declare $\lambda_1(\Omega'', \alpha) = \infty$ for all $\alpha > 0$). Consider all possible subsets Ω_i of Ω', where

Ω_i consists of $l_i \le k-1$ c.c.s of Ω' (thus there are fewer than 2^j possible choices of Ω_i). For each i, let $D_{k,i}$ denote a scaled down version of D_k such that $|D_{k,i}| = |\Omega_i|$. Then by case (i) and Lemma 1.3.7, there exists $\alpha_i := \alpha_{\Omega_i}$ such that

$$\lambda_k(\Omega_i, \alpha) > \lambda_k(D_{k,i}, \alpha) \ge \lambda_k(D_k, \alpha) \tag{4.3.1}$$

for all $\alpha \in (0, \alpha_i)$.

Set $\alpha_\Omega := \min_i \alpha_i > 0$, and fix $\alpha \in (0, \alpha_\Omega)$. We will show $\lambda_k(\Omega, \alpha) \ge \lambda_k(D_k, \alpha)$, with equality only if $\Omega = D_k$ in the sense of Remark 1.3.2.

First suppose $\lambda_1(\Omega'', \alpha) \le \lambda_k(\Omega, \alpha)$. Then by the Faber-Krahn inequality, Theorem 2.1.1, and Lemma 1.3.7

$$\lambda_k(\Omega, \alpha) \ge \lambda_1(\Omega'', \alpha) \ge \lambda_1(B_{M/k}) = \lambda_k(D_k, \alpha). \tag{4.3.2}$$

Since $|\Omega''| < M/k$, Lemma 1.3.7 implies that the second inequality in (4.3.2) must be strict.

So assume now that $\lambda_1(\Omega'', \alpha) > \lambda_k(\Omega, \alpha)$. There are two subcases to consider. First, if there are only $l < k$ c.c.s $\Omega_1, \dots, \Omega_l$ of Ω' whose first eigenvalue is smaller than $\lambda_k(\Omega, \alpha)$, setting $\widehat{\Omega}$ to be the disjoint union of $\Omega_1, \dots, \Omega_l$, by (4.3.1) we have

$$\lambda_k(\Omega, \alpha) = \lambda_k(\widehat{\Omega}, \alpha) > \lambda_k(D_k, \alpha)$$

by choice of α_Ω and $\alpha < \alpha_\Omega$. So we are left to consider the case where we can choose k c.c.s Ω_i of Ω' such that $\lambda_1(\Omega_i, \alpha) \le \lambda_k(\Omega, \alpha)$ for all $1 \le i \le k$. Then $\lambda_k(\Omega, \alpha) = \max_{1 \le i \le k} \lambda_1(\Omega_i, \alpha)$. For each i let B_i be a ball with $|B_i| = |\Omega_i|$. By the Faber-Krahn inequality $\lambda_1(\Omega_i, \alpha) \ge \lambda_1(B_i)$ for all i and thus

$$\lambda_k(\Omega, \alpha) \ge \max_i \lambda_1(B_i, \alpha) \ge \lambda_1(B_{M/k}, \alpha) = \lambda_k(D_k, \alpha), \tag{4.3.3}$$

where the second inequality in (4.3.3) follows easily from Lemma 1.3.7 using $\sum_i |B_i| \le |\Omega|$. Finally, if there is equality in (4.3.3), then for every $1 \le i \le k$, $\lambda_1(\Omega_i, \alpha) = \lambda_1(B_i, \alpha) = \lambda_1(B_{M/k}, \alpha)$ and so $\Omega_i = B_i = B_{M/k}$ using sharpness of the Faber-Krahn inequality (see Theorem 3.1.2 and Lemma 1.3.7, respectively). In this case $|\Omega_i| = M/k$, and the only possibility is that Ω consists of k copies of $\Omega_i = B_{M/k}$, so $\Omega = D_k$. $\square$

In order to complete the proof of the theorem and our claim in Remark 4.3.2(i), we will use the following lemma. Recall $\mu_k(\Omega)$ denotes the kth eigenvalue of the Dirichlet Laplacian on Ω.

Lemma 4.3.3. *Let $N = 2$ and fix $k \geq 3$. The domain D_k does not minimise $\mu_k(\Omega)$ amongst all bounded Lipschitz domains in $\mathbb{R}^2$ of given volume.*

Proof. The proof is by an easy induction argument, using results from [**110**]. First note that D_k does not even minimise μ_k amongst all disjoint unions of balls if $3 \leq k \leq 17$ (see [**110**, Section 8]). Now fix $k \geq 4$. We will show that if D_{k+1} minimises μ_{k+1}, then D_j must minimise μ_j for some $3 \leq j \leq k$. For, arguing as in [**110**, Theorem 8.1], D_{k+1} may be written as the disjoint union of open sets U and V, say, where U minimises μ_j and V minimises μ_{k-j+1} (both appropriately scaled) for some integer j between 1 and $k/2$. Now U and V must both be disjoint unions of equal balls, and since the minimiser of μ_j can have at most j c.c.s the only possibility is that $U = D_j$ (rescaled) and $V = D_{k-j+1}$ (also rescaled). Since $k \geq 4$, at least one of j, $k - j + 1$ must be at least 3. By Remark 1.3.6(i) the Dirichlet minimiser is independent of the volume of the domain, so our claim follows. $\square$

Proof of Theorem 4.3.1(ii), (iii) and Remark 4.3.2(i). For (ii), given $k \geq 3$ and $N \geq 2$, suppose that D_k is not the minimiser of μ_k so that there exists a Lipschitz domain V such that $\mu_k(V) < \mu_k(D_k)$. By Theorem 1.3.5 and Theorem 1.3.1(vii) respectively, we have $\lambda_k(V, \alpha) < \mu_k(V)$ and $\lambda_k(D_k, \alpha) = \lambda_1(D_k, \alpha) \to \mu_1(D_k) = \mu_k(D_k)$ as $\alpha \to \infty$. Using continuity, it follows that for α sufficiently large, $\lambda_k(V, \alpha) < \mu_k(D_k, \alpha)$. Hence D_k does not minimise λ_k for all $\alpha \in (0, \infty)$. However, if $U \neq D_k$ is any (Lipschitz) domain which minimises λ_k for some $\tilde{\alpha} \in (0, \infty)$, then by part (i) $\lambda_k(U, \alpha) > \lambda_k(D_k, \alpha)$ for $\alpha < \tilde{\alpha}$ sufficiently small.

For (iii), we note that Theorem 4.3.1(ii) and (iii) are equivalent assertions. Indeed, by Remark 1.3.6, if we allow M to vary between 0 and ∞, by making the homothety substitution $x \mapsto \alpha x$ we may assume $\alpha = 1$. But if we could find a minimiser to the problem

$$\begin{aligned} -\Delta u &= \frac{\lambda_k}{\alpha^2} u \qquad && \text{in } \alpha\Omega, \\ \frac{\partial u}{\partial \nu} + u &= 0 && \text{on } \partial(\alpha\Omega) \end{aligned}$$

(cf. (1.3.6)) independent of $M = |\alpha\Omega| > 0$, then by rescaling back this would give us for some fixed $M > 0$ a minimising solution independent of $\alpha > 0$. (Conversely, having a minimiser for all $\alpha > 0$ would thus give us one for all $M > 0$.) Thus we see (ii) and (iii) are equivalent. Finally, for the remark we note that when $N = 3$,

D_3 does not minimise μ_3 (see [**24**, Section 3]) and when $N = 2$, by Lemma 4.3.3 D_k does not minimise μ_k for any $k \geq 3$, completing the proof. $\square$

Remark 4.3.4. When $N \geq 4$ it is no longer true that $\mu_3(D_3) > \mu_3(B)$. This, along with some numerical evidence, led the authors in [**24**] to conjecture that μ_3 is minimised on B when $N = 2$ or 3 and on D_3 when $N \geq 4$. It therefore seems plausible that when $N \geq 4$, D_3 minimises λ_3 for all $\alpha > 0$.

4.4. The asymptotic behaviour as $\alpha \to -\infty$

Here we will study what happens to $\lambda_k(\alpha)$ in the limit as the boundary parameter $\alpha \to -\infty$ in (1.1.2). Our main result is expressed in the following theorem, which is noteworthy in the sense that the limiting behaviour is the same not only for every domain Ω (the only assumption being C^1 regularity, with no restriction on the volume), but for all the eigenvalues of every such domain.

Theorem 4.4.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded domain of class C^1. Then for every $k \geq 1$ we have*

$$\lim_{\alpha \to -\infty} \frac{\lambda_k(\alpha)}{-\alpha^2} = 1. \tag{4.4.1}$$

(Compare with Theorem 1.3.1(ix).) It was shown in [**80**] that for Ω piecewise-C^1 the first eigenvalue λ_1 has the asymptotic behaviour $\liminf_{\alpha \to -\infty} \lambda_1(\alpha)/(-\alpha^2) \geq 1$, with equality if $\partial\Omega$ is differentially equivalent to a sphere. Moreover, if Ω is a ball, then there are $\lfloor |\alpha| \rfloor + 1$ negative eigenvalues, and they satisfy $\sqrt{-\lambda_k(\alpha)} \sim -\alpha + O(1)$ as $\alpha \to -\infty$. It was subsequently shown in [**86**] that in fact

$$\lim_{\alpha \to -\infty} \frac{\lambda_1(\alpha)}{-\alpha^2} = 1 \tag{4.4.2}$$

if Ω is of class C^1. Related results have been obtained in [**60, 61**]. In fact since $\lambda_k \geq \lambda_1$ this immediately implies that

$$\limsup_{\alpha \to -\infty} \frac{\lambda_k(\alpha)}{-\alpha^2} \leq 1.$$

We will prove $\lambda_k(\alpha_n)/(-{\alpha_n}^2) \to 1$ for an arbitrary sequence $\alpha_n \to -\infty$.

The C^1 assumption in (4.4.2) is optimal: the authors in [**80**] constructed examples of domains with “corners” for which the limit in (4.4.2) is a constant larger than one. Such results were generalised and further studied in [**83**].

We will also prove the following related result on the behaviour of the eigenfunctions of (1.1.2) as $\alpha \to -\infty$. We will use the result of Theorem 4.4.1 to obtain this; however, an analysis of the proof shows that we could replace the C^1 assumption with Lipschitz, provided we know that $\lambda_k(\alpha) \to -\infty$ as $\alpha \to -\infty$. In this section we will always take $\langle\, .\, ,\, .\, \rangle$ to mean the usual inner product on $L^2(\Omega)$ and ψ_k to denote an eigenfunction associated with the kth eigenvalue $\lambda_k(\alpha) = \lambda_k(\Omega, \alpha)$ of (1.1.2), where as usual the λ_k are ordered by increasing size and repeated according to their multiplicities.

Proposition 4.4.2. *Suppose $\Omega \subset \mathbb{R}^N$ is bounded and C^1. Fix $2 \leq p < \infty$ and normalise the eigenfunctions so that $\|\psi_k\|_{L^p(\Omega)} = 1$. Then for each $k \geq 1$ we have*

(i) *$\psi_k \to 0$ in $L^p_{loc}(\Omega)$ as $\alpha \to -\infty$;*
(ii) *$\|\psi_k\|_{L^q(\Omega)} \to 0$ as $\alpha \to -\infty$ for $1 \leq q < p$;*
(iii) *$\|\psi_k\|_{L^r(\Omega)} \to \infty$ as $\alpha \to -\infty$ for $r > p$.*

We defer the proof of Proposition 4.4.2 until later and first discuss the theory related to (1.1.2) that will be needed to prove Theorem 4.4.1. As usual we understand eigenvalues λ and associated eigenfunctions ψ of (1.1.2) in the weak sense, as satisfying $Q_\alpha(\psi, v) = \lambda\langle\psi, v\rangle$ for all $v \in H^1(\Omega)$, where Q_α is the form associated with (1.1.2) given by (1.2.2) (see (1.2.5)). We use the characterisation of the kth eigenvalue as

$$\lambda_k(\alpha) = \inf_{0 \neq v \in M_k} \frac{Q_\alpha(v, v)}{\|v\|^2_{L^2(\Omega)}}, \tag{4.4.3}$$

where M_k is the subspace of $H^1(\Omega)$ obtained by removing the L^2-span of the first $k-1$ eigenfunctions $\psi_1, \ldots, \psi_{k-1}$. (This is the minimax formula (1.3.4) from Chapter 1, but here we will work directly with the maximal subspace which we will call M_k.)

If the eigenfunctions are scaled so that $\|\psi_k\|_{L^2(\Omega)} = 1$ for every $k \geq 1$ (which we will always do for the proof of Theorem 4.4.1), then by standard theory we can choose them so they form a complete orthonormal basis for $L^2(\Omega)$. However for our proof we only really need the orthogonality in $L^2(\Omega)$ of the eigenfunctions ψ_i, which follows immediately from the identity $Q_\alpha(\psi_i, \psi_j) = \lambda_i\langle\psi_i, \psi_j\rangle = \lambda_j\langle\psi_i, \psi_j\rangle$ if $\lambda_i \neq \lambda_j$, and from Gram-Schmidt orthogonalisation applied to a basis of the eigenfunctions of λ_i if it is a repeated eigenvalue. If we set $v_k := v - \sum_{i=1}^{k-1}\langle v, \psi_i\rangle\psi_i$, then $v_k \in M_k$ and so provided $v_k \neq 0$ (that is, provided v is not in the L^2-span of $\psi_1, \ldots, \psi_{k-1}$), we may use v_k as a test function in (4.4.3) to estimate λ_k from above.

We will use the representation (4.4.3), together with an appropriate choice of v and an induction argument, to prove Theorem 4.4.1. Our choice of test function, along with the result of the next lemma, is due to an argument in [**60**] (see Theorem 2.3 there). We will assume throughout that $\Omega \subset \mathbb{R}^N$ is bounded and C^1, although some of the results are valid (with the same proof) for Lipschitz domains. In particular, the next lemma, which proves (1.3.3) from Theorem 1.3.1(ix), is valid for bounded, Lipschitz domains.

Lemma 4.4.3. *Let $d \in \mathbb{R}^N$, $\|d\| = 1$ be any unit vector. Set $u_d(x) := ce^{-\alpha x \cdot d} \in C^\infty(\mathbb{R}^N) \cap H^1(\Omega)$, where $c = c(d, \alpha)$ is a constant chosen so that $\|u_d\|_{L^2(\Omega)} = 1$. Then $Q_\alpha(u_d, u_d) \leq -\alpha^2$ for all $\alpha < 0$.*

Proof. For $x \in \mathbb{R}^N$ writing $x = (x_1, \ldots, x_N)$, we may without loss of generality rotate our coordinate system if necessary so that $d = (0, \ldots, 0, 1)$. In this case $u_d = ce^{-\alpha x_N}$ and $\nabla u_d = (0, \ldots, 0, -c\alpha e^{-\alpha x_N})$. Hence

$$\begin{aligned} Q_\alpha(u_d, u_d) &= \int_\Omega |\nabla u_d|^2 \, dx + \int_{\partial\Omega} \alpha u_d^2 \, d\sigma \\ &= c^2\alpha^2 \int_\Omega e^{-2\alpha x_N} \, dx + c^2\alpha \int_{\partial\Omega} e^{-2\alpha x_N} \, d\sigma. \end{aligned}$$

We will now use the divergence theorem on the vector field $V := (0, \ldots, 0, e^{-2\alpha x_N}) \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and the C^1 (or Lipschitz) domain Ω (see Theorem A4.5). That is, denoting the outer unit normal to Ω by $\nu_\Omega(x) = (\nu_1(x), \ldots, \nu_N(x))$, $x \in \partial\Omega$,

$$\begin{aligned} \int_{\partial\Omega} e^{-2\alpha x_N} \, d\sigma &\geq \int_{\partial\Omega} e^{-2\alpha x_N} \nu_N \, d\sigma = \int_{\partial\Omega} V \cdot \nu_\Omega \, d\sigma \\ &= \int_\Omega \operatorname{div} V \, dx = -2\alpha \int_\Omega e^{-2\alpha x_N} \, dx. \end{aligned}$$

Multiplying through by $\alpha < 0$ and plugging into the expression for $Q_\alpha(u_d, u_d)$,

$$Q_\alpha(u_d, u_d) \leq -\alpha^2 c^2 \int_\Omega e^{-2\alpha x_N} \, dx = -\alpha^2,$$

where the last equality follows since $c = (\int_\Omega e^{-2\alpha x_N} \, dx)^{-\frac{1}{2}}$. $\square$

Remark 4.4.4. An easy calculation shows that the function $u(x) := e^{-\alpha x_N}$ is a positive eigenfunction, with eigenvalue $-\alpha^2$, of the Robin problem (1.1.2) on the half-space $T = \{x \in \mathbb{R}^N : x_N < 0\}$. So in a sense Lemma 4.4.3 is comparing the bounded domain case with the half-space case.

For $d \in \mathbb{R}^N$ a fixed unit vector, set $u_{k+1} := u_d - \sum_{i=1}^k \langle u_d, \psi_i \rangle \psi_i \in M_{k+1}$. We will use u_{k+1} as a trial function in the variational characterisation in order to establish (4.4.1). To that end, we characterise λ_{k+1} inductively in terms of the previous n eigenvalues and functions.

Lemma 4.4.5. *Suppose $u_d \notin span\{\psi_1, \ldots, \psi_k\}$. Then*

$$\lambda_{k+1}(\alpha) \leq \frac{-\alpha^2 - \sum_{i=1}^k \lambda_i(\alpha) \langle u_d, \psi_i \rangle^2}{1 - \sum_{i=1}^k \langle u_d, \psi_i \rangle^2}. \tag{4.4.4}$$

Proof. Since u_d is not a linear combination of the first k eigenfunctions, we can use $u_{k+1} = u_d - \sum_{i=1}^k \langle u_d, \psi_i \rangle \psi_i \not\equiv 0$ as a test function in (4.4.3). A simple calculation using the orthogonality of the eigenfunctions together with the scaling $\|\psi_i\|^2_{L^2(\Omega)} = 1$ shows that

$$0 < \langle u_{k+1}, u_{k+1} \rangle = 1 - \sum_{i=1}^k \langle u_d, \psi_i \rangle^2.$$

We now estimate $Q_\alpha(u_{k+1}, u_{k+1})$. Using the definition of u_{k+1} and the bilinearity of the form Q_α, we see that $Q_\alpha(u_{k+1}, u_{k+1})$ is given by

$$Q_\alpha(u_d, u_d) - 2 \sum_{i=1}^k \langle u_d, \psi_i \rangle \, Q_\alpha(u_d, \psi_i) + \sum_{i=1}^k \sum_{j=1}^k \langle u_d, \psi_i \rangle^2 Q_\alpha(\psi_i, \psi_j).$$

Since $Q_\alpha(u_d, \psi_i) = \lambda_i \langle u_d, \psi_i \rangle$, and since $Q_\alpha(\psi_i, \psi_j) = \lambda_i$ if $i = j$ and 0 otherwise, we obtain

$$Q_\alpha(u_{k+1}, u_{k+1}) = Q_\alpha(u_d, u_d) - \sum_{i=1}^k \lambda_i \langle u_d, \psi_i \rangle^2.$$

Using the estimate of $Q_\alpha(u_d, u_d)$ from Lemma 4.4.3 and combining everything,

$$\lambda_{k+1}(\alpha) \leq \frac{Q_\alpha(u_{k+1}, u_{k+1})}{\|u_{k+1}\|^2_{L^2(\Omega)}} \leq \frac{-\alpha^2 - \sum_{i=1}^k \lambda_i(\alpha) \langle u_d, \psi_i \rangle^2}{1 - \sum_{i=1}^k \langle u_d, \psi_i \rangle^2},$$

establishing (4.4.4). $\square$

Roughly speaking, to prove Theorem 4.4.1 using the characterisation of λ_{k+1} in Lemma 4.4.5 we have to prove that we can find d such $\langle u_d, \psi_i \rangle$ stays small as $\alpha \to -\infty$ for all $1 \leq i \leq k$. To that end we will study the functions u_d more carefully. We start by observing that, for any given $\alpha < 0$, the level sets of u_d are restrictions to Ω of half-planes of the form $\{x \in \mathbb{R}^N : x \cdot d > \kappa\}$, where $\kappa \in \mathbb{R}$. The main place where we will use the assumption that Ω has C^1 boundary is in parts (iii) and (iv) of the next lemma.

Lemma 4.4.6. *Let $d \in \mathbb{R}^N$, $\|d\| = 1$. For $\kappa \in \mathbb{R}$ set*

$$
\begin{aligned}
U_d(\kappa) &:= \{x \in \Omega : x \cdot d > \kappa\}, \\
\kappa_d &:= \sup\{\kappa \in \mathbb{R} : U_d(\kappa) \neq \emptyset\}, \\
K_d &:= \{x \in \overline{\Omega} : x \cdot d = \kappa_d\}.
\end{aligned}
\tag{4.4.5}
$$

Then

(i) *the $U_d(\kappa)$ are open, nested (i.e. $U_d(\kappa) \subset U_d(\kappa')$ if $\kappa > \kappa'$), nonempty if and only if $\kappa < \kappa_d$, and $0 \neq |U_d(\kappa)| \to 0$ as $\kappa \to \kappa_d$ from below;*

(ii) *$\emptyset \neq K_d \subset \partial\Omega$;*

(iii) *if $z \in K_d$, then $d = \nu_\Omega(z)$, the outer unit normal to Ω at z;*

(iv) *if $d \neq e \in \mathbb{R}^N$, $\|e\| = 1$ is another unit vector with $U_e(\kappa)$ and κ_e defined as in (4.4.5), then there exists $\varepsilon > 0$ such that $U_d(\kappa) \cap U_e(\tilde{\kappa}) = \emptyset$ for all $\kappa \in (\kappa_d - \varepsilon, \kappa_d)$ and all $\tilde{\kappa} \in (\kappa_e - \varepsilon, \kappa_e)$.*

Proof. (i) is obvious. For (ii), to show $K_d \neq \emptyset$ we note that $K_d = \cap_{\kappa < \kappa_d} \overline{U_d(\kappa)}$, that is, K_d is the intersection of nested, compact and nonempty sets. That $K_d \subset \partial\Omega$ is immediate from the definitions and the fact that the U_d are open. For (iii), we assume as in the proof of Lemma 4.4.3 that $d = (0, \ldots, 0, 1)$, so that $U_d(\kappa) = \{x \in \Omega : x_N > \kappa\}$ (where we have written $(x_1, \ldots, x_N)$ for $x \in \mathbb{R}^N$). Then $z = (z_1, \ldots, z_N) \in K_d$ means $z_N = \kappa_d$, that is, $z_N = \max\{x_N : x \in \overline{\Omega}\}$. Since Ω is C^1, this means the tangent plane to Ω at any point $z \in K_d$ must be horizontal; thus $\nu_\Omega(z)$ points in the direction x_N, that is, $\nu_\Omega(z) = (0, \ldots, 0, 1)$. For (iv), suppose for a contradiction that there exist $\kappa_j \nearrow \kappa_d$ and $\tilde{\kappa}_j \nearrow \kappa_e$ such that, for each $j \geq 1$, there exists $x_j \in U_d(\kappa_j) \cap U_e(\tilde{\kappa}_j)$. Since $\overline{\Omega}$ is compact, a subsequence of the x_j converges to some $z \in \overline{\Omega}$. Since $x_j \in U_d(\kappa_j)$ and $\cap_{j \geq 1} \overline{U_d(\kappa_j)} = K_d$, we see $z \in K_d$. By a similar argument, $z \in K_e$. This contradicts (iii) since $d \neq e$. $\square$

We now show that for d fixed, all the mass of u_d becomes concentrated in an arbitrarily small region of Ω as $\alpha \to -\infty$.

Lemma 4.4.7. *Let $d \in \mathbb{R}^N$ and $u_d(x) = ce^{-\alpha x \cdot d}$ be as in Lemma 4.4.3 and let $U_d(\kappa)$ and κ_d be as in Lemma 4.4.6. For every $\varepsilon > 0$ and $\kappa' < \kappa_d$ there exists*

$\alpha_\varepsilon := \alpha(\varepsilon, \kappa') < 0$ such that

$$\|u_d\|^2_{L^2(\Omega \setminus U_d(\kappa'))} < \varepsilon \tag{4.4.6}$$

for all $\alpha < \alpha_\varepsilon$.

Proof. Since $u_d(x) \leq ce^{-\alpha\kappa'}$ for all $x \in \Omega \setminus U_d(\kappa')$, we have

$$\|u_d\|^2_{L^2(\Omega \setminus U_d(\kappa'))} \leq ce^{-2\alpha\kappa'}|\Omega|.$$

Choose $\kappa'' \in (\kappa', \kappa_d)$. Then $U_d(\kappa'') \subset U_d(\kappa')$ with $|U_d(\kappa'')| \neq 0$ and

$$1 = \|u_d\|^2_{L^2(\Omega)} \geq \|u_d\|^2_{L^2(U_d(\kappa''))} \geq ce^{-2\alpha\kappa''}|U_d(\kappa'')|.$$

For $\varepsilon > 0$ fixed, choose $\alpha_\varepsilon < 0$ such that

$$e^{-2\alpha_\varepsilon\kappa'}|\Omega| < \varepsilon e^{-2\alpha_\varepsilon\kappa''}|U_d(\kappa'')|, \tag{4.4.7}$$

which we can do since $\kappa' < \kappa''$. Then (4.4.7) will hold uniformly in $\alpha < \alpha_\varepsilon$ and so

$$\|u_d\|^2_{L^2(\Omega \setminus U_d(\kappa'))} < ce^{-2\alpha\kappa'}|\Omega| < \varepsilon ce^{-2\alpha\kappa''}|U_d(\kappa'')| < \varepsilon$$

for all $\alpha < \alpha_\varepsilon$. □

Lemma 4.4.7 implies that for fixed d, $u_d \rightharpoonup 0$ weakly in $L^2(\Omega)$ as $\alpha \to -\infty$. In fact the ψ_i also tend to 0 weakly as $\alpha \to -\infty$ (see Proposition 4.4.2). But this is not enough to show directly that $\langle u_d, \psi_i \rangle$ is uniformly small, since both u_d and ψ_i vary with α. Instead, we will use the following rather technical result concerning the u_d. Since this does not actually use any specific properties of the ψ_i, we set it up so it works for arbitrary L^2-functions φ_i (which we will of course in practice take to be the eigenfunctions ψ_i).

Lemma 4.4.8. *Fix $k \geq 1$ and $\delta > 0$. Suppose we have a sequence $\alpha_n \to -\infty$ and for each $n \geq 1$ a family of k functions $\varphi_i(n) \in L^2(\Omega)$, $1 \leq i \leq k$, such that $\|\varphi_i(n)\|_{L^2(\Omega)} = 1$ for all $1 \leq i \leq k$ and $n \geq 1$. Then there exists a unit vector $d \in \mathbb{R}^N$ and a subsequence $\alpha_{n_l} \to -\infty$ of the α_n such that*

$$\sum_{i=1}^{k} \langle u_d(n_l), \varphi_i(n_l) \rangle^2 \leq \delta \tag{4.4.8}$$

for all $l \in \mathbb{N}$, where $u_d(n_l) = u_d(x, \alpha_{n_l})$ is as in Lemma 4.4.3.

Proof. Fix $k \geq 1$, $\delta > 0$ and a sequence $\alpha_n \to -\infty$. Choose $m \geq 1$ and $\varepsilon > 0$, to be specified precisely later on. Now choose any m distinct $d_j \in \mathbb{R}^N$, $1 \leq j \leq m$, and for each j let $u_j = u_j(x, \alpha_n)$ be as in the statement of the lemma. Now for each j choose a nonempty open set $U_j := U_{d_j}(\kappa_j)$ as in Lemma 4.4.6. By making an appropriate choice of κ_j we may assume the U_j are pairwise disjoint. Using Lemma 4.4.7, we find an $\alpha_\varepsilon < 0$ such that

$$\|u_j\|^2_{L^2(\Omega\setminus U_j)} < \varepsilon$$

for all $\alpha < \alpha_\varepsilon$ and all $1 \leq j \leq m$. By discarding at most finitely many n, we may assume that $\alpha_n < \alpha_\varepsilon$ for all $n \geq 1$. Now for each $n \geq 1$, we have

$$\int_\Omega \sum_{i=1}^k |\varphi_i(n)|^2 \, dx = \sum_{i=1}^k \|\varphi_i(n)\|^2_{L^2(\Omega)} = k.$$

Since the U_j are pairwise disjoint, it follows that for each $n \geq 1$, there exists at least one $j = j_n$ such that

$$\int_{U_{j_n}} \sum_{i=1}^k |\varphi_i(n)|^2 \, dx \leq \frac{k}{m}.$$

For this j_n, using Hölder's inequality, for each $1 \leq i \leq k$ we have

$$\begin{aligned}
|\langle u_{j_n}, \varphi_i(n)\rangle| &\leq \int_{U_{j_n}} |u_j \varphi_i| \, dx + \int_{\Omega\setminus U_{j_n}} |u_j \varphi_i| \, dx \\
&\leq \|u_j\|_{L^2(\Omega)} \Big(\frac{k}{m}\Big)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \|u_j\|_{L^2(\Omega)} \|\varphi_i\|_{L^2(\Omega)} \\
&= \Big(\frac{k}{m}\Big)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}},
\end{aligned}$$

where we have used the bound $\int_{U_{j_k}} |\varphi_i|^2 \, dx \leq k/m$, together with the normalisation $\|u_j\|_{L^2(\Omega)} = \|\varphi_i\|_{L^2(\Omega)} = 1$. We now specify $m \geq 1$ and $\varepsilon > 0$ to be such that

$$k\Big(\Big(\frac{k}{m}\Big)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}}\Big)^2 \leq \delta,$$

noting that this depends only on k and δ.

Squaring the above estimate for $|\langle u_{j_n}, \varphi_i(n)\rangle|$ and summing over i, this implies that for all but finitely many $n \geq 1$, (4.4.8) holds for at least one of the m fixed u_j. By a simple counting argument, there must exist at least one j^* between 1 and m such that (4.4.8) holds for this fixed j^* and infinitely many α_n. This gives us our u_d and (α_{n_l}). □

Proof of Theorem 4.4.1. The proof is by induction on k. The step when $k = 1$ is given by [**86**, Theorem 1.1]. Now fix $k \geq 1$ and suppose we know that for all $1 \leq i \leq k$, $-\lambda_i(\alpha_n)/\alpha_n^2 \to 1$ as $n \to \infty$ for every sequence $\alpha_n \to -\infty$. By an argument from elementary analysis, it suffices to prove that for every such sequence $\alpha_n \to -\infty$, there exists a subsequence $\alpha_{n_l} \to -\infty$ such that $-\lambda_{k+1}(\alpha_{n_l})/\alpha_{n_l}^2 \to 1$ as $l \to \infty$.

So fix a particular sequence $\alpha_n \to -\infty$ and also fix $0 < \delta < 1$. Let u_d satisfy the conclusion of Lemma 4.4.8 for a subsequence which we will still denote by α_n, this $\delta > 0$ and the family of k functions $\psi_i(\alpha_n) =: \varphi_i(n)$, $1 \leq i \leq k$. Then by Lemma 4.4.8 we know that

$$\sum_{i=1}^{k} \langle u_d(\alpha_n), \psi_i(\alpha_n)\rangle^2 \leq \delta \tag{4.4.9}$$

for all $n \geq 1$ and the fixed direction d. In particular, (4.4.9) implies $u_d \notin \operatorname{span}\{\psi_1(\alpha_n), \dots, \psi_k(\alpha_n)\}$ for any $n \geq 1$, since $\delta < 1$. Applying Lemma 4.4.5 to u_d for each $n \geq 1$, we obtain

$$\lambda_{k+1}(\alpha_n) \leq \frac{-\alpha_n^2 - \sum_{i=1}^{k} \lambda_i(\alpha_n)\langle u_d, \psi_i\rangle^2}{1 - \sum_{i=1}^{k}\langle u_d, \psi_i\rangle^2}$$

for every $n \geq 1$. Rearranging gives

$$\frac{\lambda_1(\alpha_n)}{-\alpha_n^2} \geq \frac{\lambda_{k+1}(\alpha_n)}{-\alpha_n^2} \geq \frac{1 - \sum_{i=1}^{k} \frac{\lambda_i(\alpha_n)}{-\alpha_n^2}\langle u_d, \psi_i\rangle^2}{1 - \sum_{i=1}^{k}\langle u_d, \psi_i\rangle^2}. \tag{4.4.10}$$

Using the bound (4.4.9), which holds independently of $n \geq 1$, together with the induction assumption $-\lambda_i(\alpha_n^2)/\alpha_n^2 \to 1$ as $n \to \infty$ for all $i \leq k$ it follows that the last term in (4.4.10) converges to 1 as $n \to \infty$. This establishes the desired limit for $-\lambda_{k+1}(\alpha_n)/\alpha_n^2$, which completes the proof of Theorem 4.4.1. $\square$

We will now give the proof of Proposition 4.4.2. So fix $k \geq 1$ and $p \geq 2$. We first obtain the following interior estimate for ψ_k, from which the proof of the proposition will follow easily.

Lemma 4.4.9. *Under the assumptions of Proposition 4.4.2, if $\varphi \in C_c^\infty(\Omega)$, then*

$$\lambda_k \geq -(p-1)^{-1}\frac{\int_\Omega |\psi_k|^p\, |\nabla\varphi|^2\, dx}{\int_\Omega |\psi_k|^p\, \varphi^2\, dx}$$

for all $\alpha < 0$ and all $k \geq 1$.

Proof. Given $\varphi \in C_c^\infty(\Omega)$, we will use $\phi := \varphi^2|\psi_k|^{p-2}\psi_k$ as a test function in the weak form (1.2.5), that is, $Q_\alpha(\psi_k, v) = \lambda\langle\psi_k, v\rangle$ for all $v \in H^1(\Omega)$. We first note that if $p \geq 2$, then since $\psi_k \in C(\overline{\Omega})$ (see Theorem 1.2.8) we have $\phi \in H^1(\Omega)$ with $\nabla\phi = 2\varphi|\psi_k|^{p-2}\psi_k\nabla\varphi + (p-1)\varphi^2|\psi_k|^{p-2}\nabla\psi_k$. Moreover $\langle\phi, \psi_k\rangle = \int_\Omega \varphi^2|\psi_k|^p\,dx \neq 0$, since ψ_k cannot vanish identically on an open set. For, if it did, by unique continuation of solutions to elliptic equations, it would be identically zero on Ω; see [**12**]. Hence, by completing the square,

$$\begin{aligned}\int_\Omega \nabla\psi_k \cdot \nabla\phi\,dx &= \int_\Omega 2\varphi|\psi_k|^{p-2}\psi_k\nabla\varphi\cdot\nabla\psi_k + (p-1)\varphi^2|\psi_k|^{p-2}|\nabla\psi_k|^2\,dx\\ &= \int_\Omega \left|(p-1)^{\frac{1}{2}}|\psi_k|^{\frac{p}{2}-1}\varphi\nabla\psi_k + (p-1)^{-\frac{1}{2}}|\psi_k|^{\frac{p}{2}-1}\psi_k\nabla\varphi\right|^2 dx\\ &\qquad - \int_\Omega (p-1)^{-1}|\psi_k|^p|\nabla\varphi|^2\,dx.\end{aligned}$$

Substituting this into the weak form for λ_k, and using that $\varphi = \phi \equiv 0$ on $\partial\Omega$,

$$\lambda_k \int_\Omega \varphi^2|\psi_k|^p\,dx = \int_\Omega \nabla\psi_k\cdot\nabla\phi\,dx + \alpha\int_{\partial\Omega}\psi_k\phi\,dx \geq -\int_\Omega (p-1)^{-1}|\psi_n|^p|\nabla\varphi|^2\,dx.$$

Rearranging gives the conclusion of the lemma. □

To prove the proposition, part (i) uses the result of Theorem 4.4.1, that $\lambda_k \to -\infty$ as $\alpha \to -\infty$; parts (ii) and (iii) will follow directly from (i).

Proof of Proposition 4.4.2. (i) Fix $p \geq 2$, $k \geq 1$ and $\Omega_0 \subset\subset \Omega$ and assume $\|\psi_k\|_{L^p(\Omega)} = 1$. Let $\varphi \in C_c^\infty(\Omega)$ be such that $0 \leq \varphi \leq 1$ in Ω and $\varphi \equiv 1$ in Ω_0. Setting $K := (p-1)^{-1}\|\nabla\varphi\|^2_{L^\infty(\Omega)} > 0$, which depends only on p and Ω_0, by Lemma 4.4.9, we have

$$\lambda_k \geq \frac{-K}{\int_{\Omega_0}|\psi_k|^p\,dx}$$

for all $\alpha < 0$. Since $\lambda_k \to -\infty$ as $\alpha \to -\infty$ by Theorem 4.4.1, this forces $\int_{\Omega_0}|\psi_k|^p\,dx \to 0$ as $\alpha \to -\infty$.

(ii) Fix $1 \leq q < p$ and $\varepsilon > 0$. Choose $\Omega_\varepsilon \subset\subset \Omega$ such that $|\Omega \setminus \Omega_\varepsilon|^{\frac{p-q}{p}} < \varepsilon/2$, which we may do since $p > q$. Also choose $\alpha_\varepsilon < 0$ such that

$$\|\psi_k\|^q_{L^p(\Omega_\varepsilon)} < \frac{\varepsilon}{2}|\Omega_\varepsilon|^{\frac{q-p}{p}}$$

for all $\alpha < \alpha_\varepsilon$, which we may do by (i). Noting that p/q and $p/(p-q)$ are dual exponents, Hölder's inequality implies

$$\begin{aligned}
\|\psi_k\|^q_{L^q(\Omega)} &= \int_{\Omega_\varepsilon} |\psi_k|^q\,dx + \int_{\Omega\setminus\Omega_\varepsilon} |\psi_k|^q\,dx \\
&\le \Big(\int_{\Omega_\varepsilon} |\psi_k|^p\,dx\Big)^{\frac{q}{p}} |\Omega_\varepsilon|^{\frac{p-q}{p}} + \Big(\int_{\Omega\setminus\Omega_\varepsilon} |\psi_k|^p\,dx\Big)^{\frac{q}{p}} |\Omega\setminus\Omega_\varepsilon|^{\frac{p-q}{p}} \\
&= \|\psi_k\|^q_{L^p(\Omega_\varepsilon)} |\Omega_\varepsilon|^{\frac{p-q}{p}} + \|\psi_k\|^q_{L^p(\Omega\setminus\Omega_\varepsilon)} |\Omega\setminus\Omega_\varepsilon|^{\frac{p-q}{p}} < \varepsilon
\end{aligned}$$

for all $\alpha < \alpha_\varepsilon$, by choice of Ω_ε and α_ε, and since $\|\psi_k\|^q_{L^p(\Omega\setminus\Omega_\varepsilon)} \le 1$.

(iii) Fix $r > p$. If we normalise ψ_k so that $\|\psi_k\|_{L^r(\Omega)} = 1$, then (ii) implies $\|\psi_k\|_{L^p(\Omega)} \to 0$, so that

$$\frac{\|\psi_k\|_{L^r(\Omega)}}{\|\psi_k\|_{L^p(\Omega)}} \longrightarrow \infty \tag{4.4.11}$$

as $\alpha \to -\infty$. Now re-normalise so that $\|\psi_k\|_{L^p(\Omega)} = 1$. Since this does not affect (4.4.11), in this case $\|\psi_k\|_{L^r(\Omega)} \to \infty$. $\square$

Chapter 5

The Laplacian with Generalised Wentzell Boundary Conditions

For the remainder of this thesis we will consider the Laplacian subject to generalised Wentzell boundary conditions

$$\Delta u + \beta \frac{\partial u}{\partial \nu} + \gamma u = 0 \qquad \text{on } \partial\Omega$$

as in (1.1.3). In this chapter we will prove some basic properties of the generalised Wentzell boundary value problem (or just Wentzell problem for short) similar to those in Section 1.2, including how the boundary condition is actually realised. While these are for the most part not new, we will extend existing results in a couple of directions, principally to the case of negative coefficients, or weights, in the boundary condition. In particular we distinguish between the "good" case $\beta > 0$ in Section 5.2, where the associated operator generates a positive, compact, irreducible analytic semigroup of angle $\pi/2$ on an appropriate L^p-space, and the "bad" case $\beta < 0$ in Section 5.3, where the operator still has compact resolvent, but there are now two sequences of eigenvalues heading to $\pm\infty$. We have placed a number of technical results needed for Section 5.3 in Appendix C. We cannot find proofs of these in the literature so we have included them.

In Chapter 6 we consider the structure and properties of the eigenvalues, analogous to what we did for the Robin problem in Section 1.3. Particular emphasis will be given to how they depend on the boundary coefficients β and γ. This is accomplished via the elementary but useful observation that every Wentzell eigenvalue is that of an appropriate Robin problem. Section 6.2 is devoted to the principal eigenvalues (a generalisation of the "first" eigenvalue), while Section 6.3 looks at the other eigenvalues, and Section 6.4 establishes various properties similar to those established for the Robin problem in Section 1.3. In Chapter 7 we prove isoperimetric inequalities for these eigenvalues, ostensibly our main goal here.

Some of the material in the next three chapters has been published in [**76**]; roughly speaking, Sections 5.2, 5.3, 6.2, 7.1 and 7.3, although some material has been altered (especially in Chapter 5: cf. [**76**, Section 2]). Parts of Sections 6.3 and 7.2 are included in [**75**].

5.1. Some background to the Wentzell problem

The Wentzell boundary value problem, in its modern incarnation, has only been studied substantially in the last few years. A good discussion of the history and interpretation of the Wentzell boundary condition can be found in [**63**]. This boundary condition is often equipped to the heat equation, which becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u && \text{in } \Omega \\ \frac{\partial u}{\partial t}(x,t) &= \beta \frac{\partial u}{\partial \nu}(x,t) + \gamma u(x,t) && \text{on } \partial\Omega, \end{aligned} \tag{5.1.1}$$

$t \in [0,T)$, $0 < T \leq \infty$ (where Δu is the Laplacian of u with respect to the x (space) variables). The main attraction of (5.1.1) is that the operator term on the boundary introduces a dynamic element to the boundary condition.

With this in mind it should not be surprising that most of the study of the operator in its modern incarnation is in terms the semigroup it generates. Indeed, as a result we will phrase some of our results in terms of generation properties (in contrast to the Robin problem in Section 1.2). The generation problem has only been intensively studied since about the turn of this century starting with [**53**]. A nice summary of the mathematical work done since then can be found in [**90**]. Classical existence and uniqueness results for the elliptic problem go back further; in particular to Luo and Trudinger in the late 1980s (see especially [**87**]).

We remark that the actual name for the boundary condition in (5.1.1) is not entirely settled; it is often called Wentzell-Robin rather than generalised Wentzell ("non-generalised" Wentzell in this case referring to $\Delta u = 0$ on $\partial\Omega$). Even the name Wentzell is sometimes spelt Venttsel or Ventcel', or some variant. The boundary condition was originally introduced in a paper of Wentzell (the English version is [**106**]), who under certain conditions wanted to find the most general boundary conditions for which the associated operator generates a Markovian semigroup. The form of the boundary condition usually considered these days – this includes the current context – is somewhat less general than in [**87, 106**].

There are various ways to define an operator associated with the Wentzell problem (1.1.3). Perhaps the most obvious question is how to give meaning to the Laplacian appearing on the boundary. While at first glance it might seem natural to use the Laplace-Beltrami operator, this does not seem to be a common approach. Instead, we will consider several different operators realising (1.1.3). These respectively act on the function space $H^1(\Omega)$, on $C(\overline{\Omega})$, and on a suitable L^p product space. We will be working mostly on H^1 or L^2. On $H^1(\Omega)$ the operator is given by

$$\begin{aligned}\mathcal{D}(\Delta_{H^1}^W) = \{u \in H^1(\Omega) : \Delta u \in H^1(\Omega),\ \frac{\partial u}{\partial \nu} \text{ exists in } L^2(\partial\Omega),\\ (\Delta u)|_{\partial\Omega} + \beta\frac{\partial u}{\partial \nu} + \gamma u|_{\partial\Omega} = 0\}\\ \Delta_{H^1}^W u = \Delta u.\end{aligned} \tag{5.1.2}$$

(For $\frac{\partial u}{\partial \nu}$, see (A4.3).) On $C(\overline{\Omega})$ we can realise the operator as

$$\begin{aligned}\mathcal{D}(\Delta_C^W) = \{u \in C(\overline{\Omega}) \cap H^1(\Omega) : \Delta u \in C(\overline{\Omega}),\ \frac{\partial u}{\partial \nu} \text{ exists in}\\ C(\partial\Omega),\ (\Delta u)|_{\partial\Omega} + \beta\frac{\partial u}{\partial \nu} + \gamma u|_{\partial\Omega} = 0\}\\ \Delta_C^W u = \Delta u\end{aligned} \tag{5.1.3}$$

(see [**7**] or [**108**], or for a different approach [**44**] or [**54**]). In [**54**], the authors work in an L^p space, more precisely in $L^p(\overline{\Omega}, d\mu)$, $d\mu = dx|_\Omega \oplus \frac{d\sigma|_{\partial\Omega}}{\beta}$ (for $\beta > 0$). A slightly different approach, along the lines of [**7**], which we will make some use of, is to define

$$\begin{aligned}\mathcal{D}(\Delta_p^W) = \{(u, u|_{\partial\Omega}) : u \in W^{1,p}(\Omega),\ \Delta u \in L^p(\Omega),\\ \frac{\partial u}{\partial \nu} \text{ exists in } L^p(\partial\Omega, \frac{d\sigma}{\beta})\}\\ \Delta_p^W(u, u|_{\partial\Omega}) = (\Delta u, -\beta\frac{\partial u}{\partial \nu} - \gamma u|_{\partial\Omega}),\end{aligned} \tag{5.1.4}$$

on the same measure space $L^p(\overline{\Omega}, d\mu)$. See also [**90**].

We emphasise that in all the above-mentioned papers, it is assumed $\beta > 0$ (either as a constant or a function), and most assume in addition that $\gamma > 0$. We will be dealing with the cases $\beta < 0$ and/or $\gamma < 0$ as well. The case $\beta < 0$ in particular completely changes the behaviour of the operator(s); for example, it completely destroys the generation properties (see in particular Remark 5.3.8).

Indeed, such boundary conditions are also called reactive type (as in [**105**], which also assumes $\gamma \equiv 0$); see also [**63**, Section 3]. We will explore this in a little more detail in Section 5.3.

Remark 5.1.1. When $p = 2$, the operator given by (5.1.2) is, up to topological isomorphism, the restriction of (5.1.4) to $H^1(\Omega)$. To see this, we set

$$V := \{(u, u|_{\partial\Omega}) : u \in H^1(\Omega)\} \hookrightarrow L^2(\overline{\Omega}, d\mu). \tag{5.1.5}$$

Then V is topologically isomorphic to $H^1(\Omega)$ in the obvious way. Moreover, the restriction of the operator given by (5.1.4) to V is clearly

$$\mathcal{D}(\Delta_2^W|_V) = \{(u, u|_{\partial\Omega}) \in V : \Delta u \in V, \frac{\partial u}{\partial \nu} \text{ exists in } L^2(\partial\Omega, \frac{d\sigma}{\beta})\}$$

$$\Delta_2^W|_V(u, u|_{\partial\Omega}) = (\Delta u, -\beta\frac{\partial u}{\partial \nu} - \gamma u|_{\partial\Omega}),$$

which under the identification $V \cong H^1(\Omega)$ gives $(\Delta u)|_{\partial\Omega} + \beta\frac{\partial u}{\partial \nu} + \gamma u|_{\partial\Omega} = 0$, so that $\Delta_2^W|_V$ is similar to $\Delta_{H^1}^W$ under this identification. Note that for $\beta < 0$, this argument still works, although we cannot consider the natural measure space $L^p(\overline{\Omega}, d\mu)$ and instead work with the measure $dx|_\Omega \oplus d\sigma|_{\partial\Omega}$, since obviously if $\beta < 0$ then $dx|_\Omega \oplus \frac{d\sigma|_{\partial\Omega}}{\beta}$ is not a positive measure. (It may be possible to use indefinite measures, but we do not explore that idea here.)

If we disregard the one-dimensional case $\Omega = (0, 1) \subset \mathbb{R}$ (see for example [**45, 51, 52, 53**]), which we are not interested in here, then the realisation (5.1.2) was probably first studied in [**55**]. There, under the assumption that $\Omega \subset \mathbb{R}^N$ is bounded and C^1, and β, $\gamma \in C(\partial\Omega)$ are nonnegative with β strictly positive, it was shown $\Delta_{H^1}^W$ generates an analytic semigroup on $H^1(\Omega)$. In [**7**], $\Delta_{H^1}^W$ was shown via form methods to generate a compact analytic semigroup on $H^1(\Omega)$ if Ω is Lipschitz, $\beta \equiv 1$ and $0 \leq \gamma \in C(\overline{\Omega})$.

As we have already noted, the principal works on $C(\overline{\Omega})$ are [**44**] (and the related paper [**46**]), [**7, 54, 108**]. Without going into details, if $\partial\Omega$ is sufficiently smooth, then Δ_C^W, or rather a slightly different realisation thereof, has been shown to generate a compact and positive analytic semigroup of angle $\frac{\pi}{2}$ [**44**]. If, however, Ω is only Lipschitz, then analyticity of the generated semigroup currently seems to be open problem. We will use some of the ideas in [**44**] to deal with the case $\beta < 0$, although we will always work with the operators given by (5.1.4) (with $p = 2$) and (5.1.2).

In many of the aforementioned works the Laplacian is replaced by a uniformly elliptic second order differential operator in divergence form. Since we are mostly interested in isoperimetric-type questions – and to keep the exposition simple – we will limit our attention to the Laplacian, and content ourselves with the remark that most of the results in this chapter should be routine to generalise.

5.2. Generation results when $\beta > 0$

In the usual case where $\beta > 0$ the Wentzell Laplacian has some very nice properties, which we summarise in the following theorem. The proof is using form methods as in [**7, 90**], although we allow $\beta \not\equiv 1$ and place fewer restrictions on γ (and, as we noted above, more on the operator – the Laplacian only – though this could be easily weakened). This theorem is probably not new, although this is possibly the first time all the results have been collected in one place and at this level of generality. Moreover this appears to be the first time the form method has been used when $\beta \not\equiv 1$. Recall that $L^2(\overline{\Omega}, d\mu) := L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{d\sigma}{\beta})$.

Theorem 5.2.1. *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, $0 < \beta_0 \leq \beta \in L^\infty(\partial\Omega)$ with β_0 is constant and $\gamma \in L^\infty(\partial\Omega)$. The operator Δ_2^W given by* (5.1.4) *(with $p = 2$) is self-adjoint and generates a positive, compact analytic semigroup of angle $\frac{\pi}{2}$ on $L^2(\overline{\Omega}, d\mu)$. If Ω is connected then in addition this semigroup is irreducible.*

Corollary 5.2.2. *Under the conditions of Theorem 5.2.1, $\Delta_{H^1}^W$ generates a compact analytic semigroup of angle $\frac{\pi}{2}$ on $H^1(\Omega)$. Moreover, $\sigma(\Delta_{H^1}^W) = \sigma(\Delta_2^W)$.*

Remark 5.2.3. The operator Δ_2^W is self-adjoint only on the weighted space $L^2(\overline{\Omega}, d\mu)$, not the unweighted space $L^2(\Omega) \times L^2(\partial\Omega)$ (which we will use in Section 5.3). Similarly, $\Delta_{H^1}^W$ as realised by (5.1.2) is clearly not self-adjoint unless $\beta \equiv 1$.

Proof of Theorem 5.2.1. Fix $\beta > 0$ and set $V := \{(u, u|_{\partial\Omega}) : u \in H^1(\Omega)\}$, which we identify with $H^1(\Omega)$ in the obvious way, and is a Hilbert space equipped with the norm $\|(u, u|_{\partial\Omega})\|_V = \|u\|_{H^1(\Omega)}$ (or indeed any equivalent norm on $H^1(\Omega)$ such as the one induced from Maz'ja's inequality). Also set $H := L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{d\sigma}{\beta})$, a Hilbert space endowed with the product norm. Note that the measure we are imposing on $\partial\Omega$ gives rise to a norm which is equivalent to the usual one on $L^2(\partial\Omega, d\sigma)$. Then $V \hookrightarrow H$ with V dense in H (see [**7**, Proof of Theorem 2.3]).

Define a form on V by $Q_\gamma : H \times H \to \mathbb{R}$,

$$Q_\gamma((u, u|_{\partial\Omega}), (v, v|_{\partial\Omega})) = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \gamma u|_{\partial\Omega} v|_{\partial\Omega} \frac{d\sigma}{\beta}$$

(cf. Section 1.2). Then it is elementary to prove Q_γ is bilinear, bounded (using the Cauchy-Schwarz inequality, the trace inequality and that $\beta \geq \beta_0 > 0$), symmetric and non-negative.

Set $Q_\gamma(u, u|_{\partial\Omega}) := Q_\gamma((u, u|_{\partial\Omega}), (u, u|_{\partial\Omega}))$. Then for $\omega \in \mathbb{R}$,

$$\begin{aligned} Q_\gamma(u, u|_{\partial\Omega}) + \omega \|(u, u|_{\partial\Omega})\|_H^2 &= \int_\Omega |\nabla u|^2 \, dx \\ &+ \int_{\partial\Omega} \gamma |u|_{\partial\Omega}|^2 \frac{d\sigma}{\beta} + \omega \int_\Omega |u|^2 \, dx + \omega \int_{\partial\Omega} |u|_{\partial\Omega}|^2 \frac{d\sigma}{\beta}, \end{aligned}$$

and in particular,

$$Q_\gamma(u, u|_{\partial\Omega}) + \omega \|(u, u|_{\partial\Omega})\|_H^2 \geq \min\{\omega, 1\} \|(u, u|_{\partial\Omega})\|_V^2$$

if we choose ω such that $\gamma + \omega$, $\omega > 0$. Hence the form Q_γ is said to be (H-)elliptic (cf. Lemma 1.2.3). We next wish to show that $-\Delta_2^W$ is the operator associated with Q_γ. The argument is essentially the same as in [**7**, Section 2], although here we do not have $\beta \equiv 1$. Let $-A_\gamma$ be the operator associated with Q_γ. For $(u, u|_{\partial\Omega}) \in \mathcal{D}(A_\gamma)$, let $-A_\gamma(u, u|_{\partial\Omega}) = (f, b) \in H$. Then by definition

$$\begin{aligned} Q_\gamma((u, u|_{\partial\Omega}), (v, v|_{\partial\Omega})) &= \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \gamma u|_{\partial\Omega} v|_{\partial\Omega} \frac{d\sigma}{\beta} \\ &= \int_\Omega f v \, dx + \int_{\partial\Omega} b v|_{\partial\Omega} \frac{d\sigma}{\beta} = \langle (f, b), (v, v|_{\partial\Omega}) \rangle_H \end{aligned}$$

for all $(v, v|_{\partial\Omega})) \in V$, that is, all $v \in H^1(\Omega)$. If we let $v \in H_0^1(\Omega)$, then

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx.$$

Since this is for all $v \in H_0^1(\Omega)$, we have $-\Delta u = f \in L^2(\Omega)$ (see (A4.2)). Hence

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega v \Delta u \, dx = \int_{\partial\Omega} (b - \gamma u|_{\partial\Omega}) v|_{\partial\Omega} \frac{d\sigma}{\beta}$$

for all $v \in H^1(\Omega)$. By definition, this means $\frac{\partial u}{\partial \nu}$ exists in $L^2(\partial\Omega)$ and equals $(b - \gamma u|_{\partial\Omega})/\beta$ (see (A4.3)). Rearranging, this gives $b = \beta \frac{\partial u}{\partial \nu} + \gamma u$, that is, $-A_\gamma(u, u|_{\partial\Omega}) = (-\Delta u, \beta \frac{\partial u}{\partial \nu} + \gamma u)$. Thus $u \in \mathcal{D}(\Delta_2^W)$ and $A_\gamma u = \Delta_2^W u$. Conversely, for $u \in \mathcal{D}(\Delta_2^W)$

we have $u \in V$ and by a similar calculation to the one above $Q_\gamma(u,v) = \langle \Delta_2^W u, v\rangle_H$ for all $v \in V$.

Hence $A_\gamma = \Delta_2^W$. Since A_γ is associated with an elliptic symmetric form, it is self-adjoint and generates an analytic semigroup T_γ of angle $\frac{\pi}{2}$ on H (see, e.g., [**9**, Section 3.7, 3.14 or 7.1] or [**41**, Section XVII.6] and again cf. Section 1.2).

Next we show the semigroup T_γ is positive, which follows fairly easily from the properties of the form domain V. By the first Beurling-Deny criterion it suffices to show that if $(u, u|_{\partial\Omega}) \in V = \mathcal{D}(Q_\gamma)$ then $(u, u|_{\partial\Omega})^+ \in V$ and

$$Q_\gamma((u, u|_{\partial\Omega})^+, (u, u|_{\partial\Omega})^-) \leq 0$$

(see, e.g., [**7**, Section 1]). In fact this is obvious from the lattice properties of $V \cong H^1(\Omega)$. More precisely, if $(u, u|_{\partial\Omega}) \in V$, then $u \in H^1(\Omega)$, so that $u^+ \in H^1(\Omega)$ and so $(u, u|_{\partial\Omega})^+ \in V$. Moreover, for $u \in H^1(\Omega)$ it is standard that $\nabla(u^+)\cdot\nabla(u^-) = 0$ (see [**59**, Lemma 7.6] or Appendix A4) and so $(u|_{\partial\Omega})^+(u|_{\partial\Omega})^- = 0$ also. In particular $Q_\gamma((u, u|_{\partial\Omega})^+, (u, u|_{\partial\Omega})^-) = 0$.

For irreducibility, suppose Ω is connected. We identify H with the measure space $L^2(\Omega\cup\partial\Omega, d\mu)$, $\mu(A) := |A\cap\Omega| + \frac{1}{\beta}\sigma(A\cap\partial\Omega)$. To prove irreducibility, by [**92**, Theorem 2.9] it suffices to show that for every measurable set $A \subset \Omega\cup\partial\Omega$, $1_A V \subset V$ implies $\mu(A) = 0$ or $\mu(\Omega \setminus A) = 0$. (Here by $1_A V \subset V$ we mean if $u \in V$, then the product $1_A(x)u(x)$ is also in V. Note that if Ω has two connected components, say Ω_1 and Ω_2, then $1_{\overline{\Omega}_i} V \subset V$ for $i = 1, 2$ so irreducibility is impossible.)

Set $A_1 := A\cap\Omega$, $A_2 := A\cap\partial\Omega$. Then $1_A V \subset V$ means that for any $u \in H^1(\Omega)$, there exists $v \in H^1(\Omega)$ such that $1_{A_1}u = v$ and $1_{A_2}u|_{\partial\Omega} = v|_{\partial\Omega}$. It is easy to see that in this case $|A_1| = 0$ or $|\Omega \setminus A_1| = 0$. For, since Ω is bounded and Lipschitz, $1_\Omega \in H^1(\Omega)$; hence choosing $u = 1_\Omega$ we obtain $v = 1_{A_1} \in H^1(\Omega)$. The only possibilities are that $1_{A_1} = 1_\Omega$ or $1_{A_1} = 0$ almost everywhere.

Now let $u = 1_\Omega \in H^1(\Omega) \cong V$. If $|A_1| = 0$, then $1_A u = (0, 1_{A_2}) \in V$. Identifying V with $H^1(\Omega)$ this means $1_{A_2} = 0|_{\partial\Omega} = 0$ σ-a.e., that is, $\sigma(A_2) = 0$. If $|\Omega \setminus A_1| = 0$, then $1_A u = (1_\Omega, 1_{A_2}) \in V$. Hence $1_\Omega|_{\partial\Omega} = 1_{A_2}$, so $\sigma(\partial\Omega \setminus A_2) = 0$, and we have proved irreducibility.

To prove compactness, we prove directly that A_γ has compact resolvent. Let $\omega \in \mathbb{R}$ be such that $\tilde{\gamma} := \gamma + \omega > 0$. Then, analogous to Section 1.2 (see in particular Lemma 1.2.5 and Theorem 1.2.6), the form domain $V \cong H^1(\Omega) \hookrightarrow H^1(\Omega) \oplus H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow H$ is compactly embedded into H since $H^1(\Omega) \hookrightarrow L^2(\Omega)$ and

$H^{\frac{1}{2}}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ are both compact by Rellich's theorem. This implies $A_{\tilde{\gamma}}$ has compact resolvent when $\tilde{\gamma} > 0$.

Now define a bounded operator C_ω on H by $C_\omega(u, f) := (0, \omega f)$. Since $A_{\tilde{\gamma}}$ has compact resolvent, $A_\gamma = A_{\tilde{\gamma}} - C_\omega$ also has compact resolvent by a standard perturbation argument (e.g. [**72**, Theorem IV.3.17] will certainly do). Hence we have established all the properties of $A_\gamma = \Delta_2^W$ listed in Theorem 5.2.1. $\square$

Proof of Corollary 5.2.2. As shown in Remark 5.1.1, $\Delta_{H^1}^W$ is, up to topological isomorphism, the part of Δ_2^W in its form domain $V \cong H^1(\Omega)$. Analyticity is now a standard result (see, e.g., [**9**, Sections 3.3, 3.5 and 7.1] and also cf. [**7**, Remark 2.9]). For compactness, we show that the domain $\mathcal{D}(\Delta_2^W)$ is compactly embedded in $V \cong H^1(\Omega)$. We first observe that $\mathcal{D}(\Delta_2^W) \subset H^{\frac{3}{2}}(\Omega)$. To see this, note that every $u \in \mathcal{D}(\Delta_2^W)$ is a solution of the variational problem $\Delta u = f \in L^2(\Omega)$ and $\frac{\partial u}{\partial \nu} = g \in L^2(\partial\Omega)$ for some f and g (namely Δu and $\frac{\partial u}{\partial \nu}$, respectively). By [**70**], $u \in H^{\frac{3}{2}}(\Omega)$.

Now as a Banach space endowed with the graph norm, $\mathcal{D}(\Delta_2^W) \hookrightarrow H$; similarly, $H^{\frac{3}{2}}(\Omega) \hookrightarrow H$ in the obvious way. By the closed graph theorem, $\mathcal{D}(\Delta_2^W) \hookrightarrow H^{\frac{3}{2}}(\Omega)$ (see [**9**, Lemma 3.10.1]). Thus, up to topological isomorphism, we have the inclusions

$$\mathcal{D}(\Delta_2^W) \hookrightarrow H^{\frac{3}{2}}(\Omega) \hookrightarrow H^1(\Omega) \hookrightarrow H, \tag{5.2.1}$$

where the second and third injections are compact. Since for any $\lambda \in \rho(\Delta_2^W)$ we have $R(\lambda, \Delta_2^W)H^1(\Omega) \subset H^{\frac{3}{2}}(\Omega) \subset H^1(\Omega)$, by [**9**, Proposition 3.10.3] we have $\sigma(\Delta_2^W|_{H^1(\Omega)}) = \sigma(\Delta_2^W)$ and $R(\lambda, \Delta_2^W|_{H^1(\Omega)}) = R(\lambda, \Delta_2^W)|_{H^1(\Omega)}$. Moreover for any $\lambda \in \rho(\Delta_2^W)$, $R(\lambda, \Delta_2^W)|_{H^1(\Omega)}$ is compact as a map on $H^1(\Omega)$ as it certainly maps $H^1(\Omega)$ into $\mathcal{D}(\Delta_2^W)$. Since $\Delta_{H^1}^W$ is the restriction of Δ_2^W to $H^1(\Omega)$ (up to topological isomorphism), this completes the proof. $\square$

Note in the above proof that there are several ways we could prove compactness of the resolvent of $\Delta_{H^1}^W$, which is arguably the property which will be of most interest to us in the sequel. For example, since $-\Delta_2^W$ is self-adjoint it is known that there is a unique square root operator, call it $A_\gamma^{\frac{1}{2}}$, such that $(A_\gamma^{\frac{1}{2}})^2 = \Delta_2^W$, and we can characterise the form domain $V = \mathcal{D}(A_\gamma^{\frac{1}{2}})$. Then using properties of Sobolev towers (see [**6**, Theorem V.1.3.8]), we know that the injection of $\mathcal{D}(\Delta_2^W) \hookleftarrow \mathcal{D}(\Delta_{H^1}^W)$ into $\mathcal{D}(A_\gamma^{\frac{1}{2}})$ is compact and dense. Alternatively, we could look at the semigroup induced on the dual space V' via the embedding $H \hookrightarrow V'$ and then exploit the reflexivity of $V = V'$.

Remark 5.2.4. (i) It is possible to define a realisation of the operator Δ_p^W on *arbitrary* domains $\Omega \subset \mathbb{R}^N$, using the form method of Theorem 5.2.1, at least provided $\sigma(K) < \infty$ for every $K \subset \partial\Omega$ compact. Roughly speaking, the idea is to replace the space $V \cong H^1(\Omega)$ with a smaller space on which there are well-defined traces. Precisely, we look at the norm

$$\|u\|_V := \left(\|\nabla u\|^2_{L^2(\Omega,dx)} + \|u|_{\partial\Omega}\|^2_{L^2(\partial\Omega,d\sigma/\beta)}\right)^{\frac{1}{2}}.$$

We denote by $\tilde{V}$ the completion of $V_0 := H^1(\Omega) \cap C^\infty(\Omega) \cap C(\overline{\Omega})$ with respect to this norm. Maz'ja's inequality asserts that for all $u \in V_0$ and hence all $u \in \tilde{V}$, there exists a constant $c = c(N, |\Omega|) > 0$ such that

$$\|u\|_{L^{\frac{2N}{N-1}}(\Omega)} \le c\|u\|_V.$$

Note however that this identification is not necessarily unique: see [**11**]. In this case we can replace $\tilde{V}$ by a closed subspace as in [**33**, Remark 3.2(d)]. Thus we obtain $\tilde{V} \hookrightarrow H$ and we can proceed. This has been done very recently in [**8**, Section 4.5] where the authors amongst other things obtain an analytic semigroup on $\tilde{V}$, or more accurately the closed subspace they call $H^1_{b,\sigma}(\Omega)$. (My thanks to Dr. ter Elst for drawing this to my attention.)

(ii) A seemingly open problem is to construct a theory of the p-Laplacian with Wentzell boundary conditions, which should be

$$\begin{aligned} -\Delta_p u &= f && \text{in } \Omega, \\ \Delta_p u + \beta|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \gamma|u|^{p-2}u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{5.2.2}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian of u (see Section 3.1 and cf. Section 4.1). No work appears to have been done on this problem to date, but we do not attempt a further exploration here.

5.3. The resolvent and spectrum when $\beta < 0$

We will now consider the case when $\beta < 0$. As we noted in Section 5.1, here we do not expect the Wentzell Laplacian to have any nice generation properties. In fact, it has been proved recently in [**105**], essentially using form methods, that in the case $\gamma \equiv 0$ the problem (1.1.3) has discrete spectrum, with its eigenvalues of the form Λ_k, where $\Lambda_0 = 0$, $\Lambda_{2k-1} \to \infty$ and $\Lambda_{2k} \to -\infty$ as $k \to \infty$ (see Theorem 3 there). The proof uses a different form to the one used in Section 5.2, given by $Q(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx$, on a carefully chosen subspace of $H^1(\Omega)$ on which Q is an inner product. A similar result is obtained in [**14**] for the eigenvalues of the problem $-\Delta u + Vu = \lambda u$ in Ω, $\frac{\partial u}{\partial \nu} - \beta^{-1}\lambda u = 0$ on $\partial\Omega$ (given in our notation), where $V(x)$ is a suitable potential and $\beta < 0$. This is based on the study of a related, ill-posed, heat equation and uses another form method.

We will be aiming to achieve a similar result in the more general case when $\gamma \in L^\infty(\partial\Omega)$, but we will use perturbation methods and results based on [**44, 46**] which, at least at an heuristic level, allow us to see what gives rise to the two sequences of eigenvalues. We start by considering the operator given by (5.1.4), where we will always take $p = 2$. We will also consider Δ_2^W as an operator on the product space $L^2(\Omega) \times L^2(\partial\Omega) \cong L^2(\Omega) \oplus L^2(\partial\Omega)$. Note that the measure on this space is equivalent to the one on the space $L^2(\overline{\Omega}, d\mu)$ used above. In particular all the results we are interested in, such as compactness, analyticity (or lack thereof) etc. are unaffected. Throughout this section we will always assume that our domains Ω are bounded, even if this is not explicitly stated. Moreover, since they will always be at least Lipschitz, they can be assumed to be connected without loss of generality (see Remark 1.3.2). In addition our main results require Ω to be fairly smooth (see Remark 5.3.3).

Theorem 5.3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,1}$. The operator Δ_2^W given by* (5.1.4) *with $p = 2$ has compact resolvent for all $\beta < 0$ and $\gamma \in L^\infty(\partial\Omega)$.*

Corollary 5.3.2. *Under the assumptions of Theorem 5.3.1, $\Delta_{H^1}^W$ given by* (5.1.2) *has compact resolvent and $\sigma(\Delta_{H^1}^W) = \sigma(\Delta_2^W)$.*

Proof of Corollary 5.3.2. The argument is the same as in the proof of Corollary 5.2.2. Namely, noting that $\mathcal{D}(\Delta_2^W)$ is unchanged from when $\beta > 0$, we still have $\mathcal{D}(\Delta_2^W) \subset H^{\frac{3}{2}}(\Omega)$ and hence obtain (5.2.1) with H replaced by $L^2(\Omega) \times L^2(\partial\Omega)$. Using [**9**, Proposition 3.10.3] in the same way as earlier, and noting that Remark 5.1.1 still applies, the result follows easily. $\square$

Remark 5.3.3. (i) The regularity assumption in Theorem 5.3.1 and Corollary 5.3.2 is quite strong. Unfortunately, this assumption is central to the method we use. The key place where it is used is to show that the Dirichlet Laplacian on $L^2(\Omega)$,

Δ^D, has domain $\mathcal{D}(\Delta^D) \subset H^2(\Omega)$ (see Lemma C1.1 for the details). Actually, all we need is the (slightly) weaker requirement that $\mathcal{D}(\Delta^D) \subset H^s(\Omega)$ for some $s > 3/2$. Unfortunately for general Lipschitz and C^1 domains, the best estimate we can expect is $H^{\frac{3}{2}}(\Omega)$; see [**69**]. However, there are some other types of domains for which this result is valid. Our proofs can be easily modified to handle convex C^1 domains and polygonal domains in $\mathbb{R}^2$, for example (see [**64**, Theorem 3.2.1.2] and [**64**, Theorem 4.3.1.4], respectively, for the key H^2-estimate, although we omit the details). Polygonal domains are of particular interest for isoperimetric problems; cf. Section 3.4.

(ii) Although we assume here that $\beta < 0$ is constant, it is possible that our method could be generalised to allow $\beta \in C(\partial\Omega)$ strictly negative, say, using multiplicative perturbation arguments for generators of analytic semigroups; cf. [**44**, Remark 1.2].

For the proof of Theorem 5.3.1 we cannot simply conclude using an argument similar to that in Corollary 5.3.2 that Δ_2^W has compact resolvent. Although $\mathcal{D}(\Delta_2^W)$ injects compactly into $L^2(\Omega) \times L^2(\partial\Omega)$, we also need to know that $(\lambda I - \Delta_2^W)$ is invertible for some $\lambda \in \mathbb{C}$. To do this we would have to show that the operator Δ_2^W is a closed operator from $L^2(\Omega) \times L^2(\partial\Omega)$ to itself. In light of our results, this is clearly true, but there does not seem to be an easy way to prove this directly. Instead, we will roughly speaking, use the method of [**44**], although with some variations and a moderate addition. In particular [**44**] only deals with the case $\beta > 0$ and uses a realisation of Δ^W on $C(\overline{\Omega})$. We will have to build up some rather heavy machinery to do this, so a direct proof that Δ_2^W is closed would lead to a much simpler, more elementary proof of Theorem 5.3.1. However, the method we use below allows us to gain some insight into the behaviour of the operator and the structure of its spectrum (see Remark 5.3.8). Many of the background results we will need for this are collected in Appendix C. We first represent the operator Δ_2^W in the following way. Define an operator as follows (cf. [**44**, eq. (1.4)]).

$$\begin{aligned} \mathcal{D}(\mathcal{A}) = \{(u,f) \in L^2(\Omega) \times L^2(\partial\Omega) : u \in \mathcal{D}(\Delta_{\max}), \\ f \in \mathcal{D}(N),\ u - Pf \in \mathcal{D}(\Delta^D)\} \\ \mathcal{A}\begin{pmatrix} u \\ f \end{pmatrix} = \begin{pmatrix} \Delta^D & 0 \\ B & -\beta N + C \end{pmatrix} \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ f \end{pmatrix}, \end{aligned} \tag{5.3.1}$$

where $\Delta_{\max} : L^2(\Omega) \to L^2(\Omega)$ is the maximal Laplacian on $L^2(\Omega)$

$$\mathcal{D}(\Delta_{\max}) = \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega)\}$$
$$\Delta_{\max} u = \Delta u,$$

$\Delta^D : L^2(\Omega) \to L^2(\Omega)$ is the Dirichlet Laplacian on $L^2(\Omega)$ given by

$$\mathcal{D}(\Delta^D) = H_0^1(\Omega) \cap \mathcal{D}(\Delta_{\max})$$
$$\Delta^D u = \Delta u$$

(associated with (1.2.1) when $\Gamma_1 = \emptyset$; see [**9**, Example 7.2.1]), P is the operator taking a function $f \in L^2(\partial\Omega)$ to the harmonic function $u \in L^2(\Omega)$ satisfying $u|_{\partial\Omega} = f$ in an appropriate sense (see Theorem C2.1), $N : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is the Dirichlet-to-Neumann operator given by

$$\begin{aligned} \mathcal{D}(N) &= \{f \in L^2(\partial\Omega) : \frac{\partial}{\partial\nu}(Pf) \in L^2(\partial\Omega)\} \\ Nf &= \frac{\partial}{\partial\nu}(Pf), \end{aligned} \tag{5.3.2}$$

(see also Theorem C3.1 as well as (A4.3)), $B : L^2(\Omega) \to L^2(\partial\Omega)$ is the operator

$$\begin{aligned} \mathcal{D}(B) &= \{u \in H^1(\Omega) : \frac{\partial u}{\partial\nu} \in L^2(\partial\Omega)\} \\ Bu &= -\beta\,\frac{\partial u}{\partial\nu} \end{aligned} \tag{5.3.3}$$

and finally $C : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is the bounded operator $Cf = -\gamma f$. Also, by slight abuse of notation, I is the identity on either $L^2(\Omega)$ or $L^2(\partial\Omega)$ as appropriate.

Lemma 5.3.4. $\Delta_2^W = \mathcal{A}$ *in the sense of operators.*

Proof. First suppose $u \in \mathcal{D}(\Delta_2^W)$. Then $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$, so $u \in \mathcal{D}(\Delta_{\max})$. Set $f := \operatorname{tr} u$, *a priori* in $H^{\frac{1}{2}}(\partial\Omega)$. But since $\frac{\partial u}{\partial\nu} \in L^2(\partial\Omega)$, in fact $f \in \mathcal{D}(N)$. To see this, let $v \in \mathcal{D}(\Delta^D)$ be the solution of $\Delta v = \Delta u$ in Ω, $v = 0$ on $\partial\Omega$. By Lemma C1.1, $\frac{\partial v}{\partial\nu} \in L^2(\partial\Omega)$. Then $w := u - v$ solves $\Delta w = 0$ in Ω, $\operatorname{tr} w = \operatorname{tr} u = f$ on $\partial\Omega$ (i.e. $w = Pf$) and $\frac{\partial w}{\partial\nu} = \frac{\partial u}{\partial\nu} - \frac{\partial v}{\partial\nu} \in L^2(\partial\Omega)$.

Thus we have $u \in \mathcal{D}(\Delta_{\max})$, $f \in \mathcal{D}(N)$, and $\operatorname{tr}(u - Pf) = 0$, i.e. $u - Pf \in \mathcal{D}(\Delta^D)$, implying $(u, f) \in \mathcal{D}(\mathcal{A})$. Hence $\mathcal{D}(\Delta_2^W) \subset \mathcal{D}(\mathcal{A})$.

For the other containment, suppose $(u, f) \in \mathcal{D}(\mathcal{A})$. Then $u \in \mathcal{D}(\Delta_{\max})$, so that $u \in H^1(\Omega)$ and $\Delta u \in L^2(\partial\Omega)$. Since $u - Pf \in \mathcal{D}(\Delta^D)$, $\operatorname{tr}(u - Pf) = 0$ giving $\operatorname{tr} u = f$. Moreover, $u - Pf \in D(\Delta^D)$ implies $\frac{\partial}{\partial\nu}(u - Pf) \in L^2(\partial\Omega)$ (see

Lemma C1.1). Since $f \in \mathcal{D}(N)$, we also have $\frac{\partial}{\partial\nu}(Pf) \in L^2(\partial\Omega)$ by definition and so $\frac{\partial u}{\partial\nu}$ exists and is in $L^2(\partial\Omega)$. Hence $u \in \mathcal{D}(\Delta_2^W)$, and so $\mathcal{D}(\mathcal{A}) \subset \mathcal{D}(\Delta_2^W)$.

Moreover, for a corresponding pair $(u, f) \in \mathcal{D}(\mathcal{A})$, $u \in \mathcal{D}(\Delta_2^W)$,

$$\mathcal{A}\begin{pmatrix} u \\ f \end{pmatrix} = \begin{pmatrix} \Delta^D & 0 \\ B & -\beta N + C \end{pmatrix}\begin{pmatrix} I & -P \\ 0 & I \end{pmatrix}\begin{pmatrix} u \\ f \end{pmatrix}$$

$$= \begin{pmatrix} \Delta(u - Pf) \\ Bu - BPf - \beta N + Cf \end{pmatrix} = \begin{pmatrix} \Delta u \\ Bu + C \operatorname{tr} u \end{pmatrix} = \Delta_2^W u,$$

noting that $\Delta(Pf) = 0$, $BP = -\beta N$ and $f = \operatorname{tr} u$. $\square$

This motivates defining an operator on $L^2(\Omega) \times L^2(\partial\Omega)$ by

$$T := \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix}.$$

Lemma 5.3.5. *The operator T is a bounded invertible linear operator on $L^2(\Omega) \times L^2(\partial\Omega)$. Its inverse is given by*

$$T^{-1} = \begin{pmatrix} I & P \\ 0 & I \end{pmatrix},$$

and this is also bounded and linear on $L^2(\Omega) \times L^2(\partial\Omega)$.

Proof. For concreteness' sake, we will equip $L^2(\Omega) \times L^2(\partial\Omega)$ with the sum norm, although any equivalent norm would work. That T and T^{-1} are well-defined, bounded and linear follows immediately from Theorem C2.1. Moreover, it is trivial to check that $TT^{-1} = T^{-1}T$ is the identity on $L^2(\Omega) \times L^2(\partial\Omega)$. $\square$

Set

$$W := \begin{pmatrix} \Delta^D & 0 \\ B & -\beta N + C \end{pmatrix},$$

so that $\mathcal{A} = WT$. Then it is immediate that $\mathcal{A}$ is similar to the operator $TWTT^{-1} = TW$ given by

$$TW = \begin{pmatrix} I & -P \\ 0 & I \end{pmatrix}\begin{pmatrix} \Delta^D & 0 \\ B & -\beta N + C \end{pmatrix}$$

(cf. [**44**, pp. 551-2]). Note that $\mathcal{D}(TW) = \mathcal{D}(\Delta^D) \times \mathcal{D}(N)$, as is clear from the characterisation of $\mathcal{D}(\mathcal{A})$.

We also introduce a perturbation operator

$$S_\varepsilon := \begin{pmatrix} \varepsilon I & 0 \\ 0 & I \end{pmatrix}$$

on $L^2(\Omega) \times L^2(\partial\Omega)$, where $\varepsilon > 0$. Clearly S_ε is bounded and invertible with bounded inverse ${S_\varepsilon}^{-1} = S_{\varepsilon^{-1}}$.

So we will finally consider the operator on $L^2(\Omega) \times L^2(\partial\Omega)$ given by

$$\begin{aligned} \mathcal{D}(S_\varepsilon T W {S_\varepsilon}^{-1}) &= \mathcal{D}(\Delta^D) \times \mathcal{D}(N) \\ S_\varepsilon T W {S_\varepsilon}^{-1} &= \begin{pmatrix} \Delta^D - PB & \varepsilon P(\beta N - C) \\ \varepsilon^{-1} B & -\beta N + C \end{pmatrix} \\ &= \begin{pmatrix} \Delta^D & 0 \\ 0 & -\beta N \end{pmatrix} + \begin{pmatrix} -PB & \varepsilon P(\beta N - C) \\ \varepsilon^{-1} B & C \end{pmatrix} \\ &=: \mathcal{A}_0 + \mathcal{B}_\varepsilon \end{aligned}$$

(cf. [**44**, p. 552]), which is similar to $\mathcal{A}$ and hence Δ_2^W for every $\varepsilon > 0$. The attraction of this representation is that the operator $\mathcal{A}_0$ is very well behaved (in the words of Engel [**44**] it represents a "decoupling" of the interior and the boundary dynamics) and the perturbing operator $\mathcal{B}_\varepsilon$, for sufficiently small ε, is sufficiently well bounded with respect to $\mathcal{A}_0$ that its properties are preserved under the perturbation.

More precisely, we know both $-N$ and Δ^D generate compact analytic semigroups of angle $\frac{\pi}{2}$ on $L^2(\partial\Omega)$ and $L^2(\Omega)$, respectively. (For N, use Theorem C3.1. For Δ^D, this is a standard result – for example, use [**9**, Proposition 7.1.1] with $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ and scalar product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$.) In particular, both operators are *sectorial* in the following sense (cf. [**9**, p. 166], [**47**, Theorem 2.10] and [**44**, Definition A.2]).

Definition 5.3.6. Let A be a closed linear operator on a Banach space X with dense domain $\mathcal{D}(A)$. Then A is *sectorial* (of angle θ) if there exist $\theta \in (0, \frac{\pi}{2}]$ and $r \geq 0$ such that the modified sector

$$\Sigma_{\theta, r} := \{z \in \mathbb{C} : |\arg z| < \frac{\pi}{2} + \theta\} \cap \{z \in \mathbb{C} : |z| > r\}$$

is contained in $\rho(A)$ and for every $\varepsilon \in (0, \theta)$ there exists $M_\varepsilon > 0$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|}$$

for all $\lambda \in \Sigma_{\theta,r} \setminus \{0\}$.

Then the operator A generates a (bounded) analytic semigroup of angle θ if and only if it is sectorial of angle θ (see, e.g., [**47**, Theorem 4.6]; see also Appendix A5). In particular, the following lemma is immediate.

Lemma 5.3.7. *Let Ω be of class $C^{1,1}$. If $\beta > 0$, the operator $\mathcal{A}_0$ generates a compact analytic semigroup of angle $\frac{\pi}{2}$ on $L^2(\Omega) \times L^2(\partial\Omega)$. If $\beta < 0$, then $\mathcal{A}_0$ has compact resolvent.*

We can now our prove main theorem. Note that when $\beta > 0$ we are in the same situation as in [**44**, Lemma A.4]; this would give us another way to obtain (most of) Theorem 5.2.1. Of course, here we are only interested in $\beta < 0$. For convenience we will be taking the sum norm on $L^2(\Omega) \times L^2(\partial\Omega)$.

Proof of Theorem 5.3.1. To prove that Δ_2^W has compact resolvent, by similarity it suffices to prove that $S_\varepsilon TWS_\varepsilon^{-1}$ has compact resolvent for some $\varepsilon > 0$. In order to do this, we first show that $\mathcal{B}_\varepsilon$ is $\mathcal{A}_0$-bounded with bound $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. To do this, first observe that since P and C are bounded, B is relatively Δ^D-bounded with bound 0 and $P(\beta N - C)$ is obviously relatively bounded with respect to $-\beta N$, there exist constants a, b, $C_\varepsilon > 0$ such that

$$\begin{aligned}
\|PBu\|_{L^2(\Omega)} &\leq \varepsilon\|\Delta^D u\|_{L^2(\Omega)} + C_\varepsilon\|u\|_{L^2(\Omega)},\\
\|\varepsilon^{-1}Bu\|_{L^2(\partial\Omega)} &\leq \varepsilon\|\Delta^D u\|_{L^2(\Omega)} + C_\varepsilon\|u\|_{L^2(\Omega)},\\
\|P(\beta N - C)f\|_{L^2(\Omega)} &\leq a\|\beta N f\|_{L^2(\partial\Omega)} + b\|f\|_{L^2(\partial\Omega)},\\
\|Cf\|_{L^2(\partial\Omega)} &\leq \varepsilon\|\beta N f\|_{L^2(\partial\Omega)} + C_\varepsilon\|f\|_{L^2(\partial\Omega)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left\|\mathcal{B}_\varepsilon \begin{pmatrix} u \\ f \end{pmatrix}\right\| &= \| - PBu + \varepsilon P(\beta N - C)f\|_{L^2(\Omega)} + \|\varepsilon^{-1}Bu + Cf\|_{L^2(\partial\Omega)}\\
&\leq \|PBu\|_{L^2(\Omega)} + \varepsilon\|P(\beta N - C)f\|_{L^2(\Omega)}\\
&\qquad\qquad + \varepsilon^{-1}\|Bu\|_{L^2(\partial\Omega)} + \|Cf\|_{L^2(\partial\Omega)}\\
&\leq 2\varepsilon\|\Delta^D u\|_{L^2(\Omega)} + (1+a)\varepsilon\|\beta N f\|_{L^2(\partial\Omega)}\\
&\qquad\qquad + 2C_\varepsilon\|u\|_{L^2(\Omega)} + (\varepsilon b + C_\varepsilon)\|f\|_{L^2(\partial\Omega)}\\
&\leq \delta_\varepsilon\left\|\mathcal{A}_0 \begin{pmatrix} u \\ f \end{pmatrix}\right\| + K_\varepsilon\left\|\begin{pmatrix} u \\ f \end{pmatrix}\right\|,
\end{aligned}$$

where $\delta_\varepsilon := \max\{2\varepsilon, (1+a)\varepsilon\}$, $K_\varepsilon := \max\{2C_\varepsilon, \varepsilon b + C_\varepsilon\}$ (cf. [**44**, Lemma A.4]. Note that if $\beta > 0$, then following [**44**] we easily get that $\mathcal{A}_0 + \mathcal{B}_\varepsilon$ generates a compact analytic semigroup of angle $\frac{\pi}{2}$ for $\varepsilon > 0$ small enough).

If $\beta < 0$, to show that $\mathcal{A}_0 + \mathcal{B}_\varepsilon$ has compact resolvent for some $\varepsilon > 0$, by [**72**, Theorem IV.3.17] it suffices to show there exists $\lambda \in \rho(\mathcal{A}_0)$ such that

$$\delta_\varepsilon \|\mathcal{A}_0 R(\lambda, \mathcal{A}_0)\| + K_\varepsilon \|R(\lambda, \mathcal{A}_0)\| < 1. \tag{5.3.4}$$

Now since Δ^D and $-|\beta|N$ are sectorial of angle $\frac{\pi}{2}$, there exist $C > 0$ and $r > 0$ such that

$$\|R(\lambda, \Delta^D)\| + \|R(\lambda, -|\beta|N)\| \leq \frac{C}{|\lambda|} \tag{5.3.5}$$

for all $\lambda \in \{z \in \mathbb{C} : |\arg z| \leq \frac{3\pi}{4}\} \cap \{z \in \mathbb{C} : |z| > r\}$, say. In particular (5.3.5) holds for Δ^D on $|\arg z| = \frac{3\pi}{4}$ and $-|\beta|N$ on $|\arg z| = \frac{\pi}{4}$, and so for $-\beta N = +|\beta|N$ on $|\arg z| = \frac{3\pi}{4}$. Hence, still using the sum norm,

$$\begin{aligned}
\|\mathcal{A}_0 R(\lambda, \mathcal{A}_0)\| &= \|\Delta^D R(\lambda, \Delta^D)\| + \|(-\beta N) R(\lambda, -\beta N)\| \\
&\leq 2\|I\| + |\lambda| (\|R(\lambda, \Delta^D)\| + \|R(\lambda, -\beta N)\|) \\
&\leq 2 + 2C
\end{aligned}$$

for $\lambda \in \mathbb{C}$ sufficiently large with $|\arg \lambda| = \frac{3\pi}{4}$. Since also

$$\|R(\lambda, \mathcal{A}_0)\| = \|R(\lambda, \Delta^D)\| + \|R(\lambda, -\beta N)\| \leq \frac{2C}{|\lambda|},$$

for such λ we have

$$\delta_\varepsilon \|\mathcal{A}_0 R(\lambda, \mathcal{A}_0)\| + K_\varepsilon \|R(\lambda, \mathcal{A}_0)\| \leq \delta_\varepsilon (2 + 2C) + K_\varepsilon \frac{2C}{|\lambda|}.$$

If we choose $\varepsilon > 0$ such that $\delta_\varepsilon < \frac{1}{2}(\frac{1}{2+2C})$, then choosing $|\lambda| > \max\{4K_\varepsilon C, r\}$ we see (5.3.5) is satisfied. Hence for this ε, $\mathcal{A}_0 + \mathcal{B}_\varepsilon$ has compact resolvent, and thus so too does the similar operator Δ_2^W. $\square$

Remark 5.3.8. Observe that the eigenvalues of $\mathcal{A}_0$, $\sigma(\mathcal{A}_0)$, may be written as the disjoint union of $\sigma(\Delta^D)$, which gives us a sequence heading to $-\infty$, and $\sigma(\beta N)$, which gives us a sequence *heading in the opposite direction*, to $+\infty$, and this only arises if we flip the sign of β. (That is, if $\beta > 0$, then both sequences head in the same direction and $\sigma(\mathcal{A}_0)$ is essentially contained in a half-line, as in [**44**].) This "double sequence" structure is the same as that found in [**105**], and provides what might be considered an heuristic explanation for the phenomenon.

We would of course like to make this heuristic explanation rigorous. More precisely, we would like to show that $\sigma(\Delta_2^W)$ (which we now know to be denumerable) has the same form as $\sigma(\mathcal{A}_0)$. Equivalently, it suffices to do this for the similar operator $\mathcal{A}_0 + \mathcal{B}_\varepsilon$ (for any $\varepsilon > 0$). This becomes a question about the stability of the spectrum of $\mathcal{A}_0$ under the relatively bounded perturbation $\mathcal{B}_\varepsilon$, which is unfortunately rather difficult to answer by these abstract means. As a first step we have the following observation, which seeks to gain the most possible out of the method of proof of Theorem 5.3.1 (in particular, we may use basically any ray through the origin in place of $\arg z = 3\pi/4$, which was chosen only to illustrate the principle).

Proposition 5.3.9. *Let $\mathscr{R}$ be any ray through the origin of the complex plane except for $\mathbb{R}$. There exists $\varepsilon_0 = \varepsilon_0(\mathscr{R}) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, there exists $K = K(\mathscr{R}, \varepsilon)$ such that $\lambda \in \rho(\mathcal{A}_0 + \mathcal{B}_\varepsilon)$ if $\lambda \in \mathscr{R}$ and $|\lambda| \geq K$.*

Proof. This follows from an easy modification of the proof of Theorem 5.3.1. Given $\mathscr{R}$, let $C = C(\mathscr{R}) > 0$ satisfy the analyticity estimate (5.3.5) with the argument of $\mathscr{R}$ in place of $\frac{3\pi}{4}$. Now let $\varepsilon_0 > 0$ be such that $\delta_{\varepsilon_0} < \frac{1}{2}(\frac{1}{2+2C})$ for this C. Then obviously this same estimate holds for all $\varepsilon \in (0, \varepsilon_0)$. For any such ε, if we choose as before $\lambda \in \mathscr{R}$ satisfying $|\lambda| > \max\{4K_\varepsilon C, r\}$ (for our new values of K_ε and C), then for this ε and λ, $\lambda \in \rho(\mathcal{A}_0)$ and the estimate (5.3.4) is satisfied, implying $\lambda \in \rho(\mathcal{A}_0 + \mathcal{B}_\varepsilon)$. $\square$

(Clearly, we have not sought the optimal value for ε_0 or K in the above proof.) However, this tells us nothing about where the spectrum actually is. Ideally, we would like to use [**72**, Theorem IV.3.18], which says that if we can draw a rectifiable, simple closed curve Γ in the complex plane around a given large eigenvalue of $\mathcal{A}_0$, such that the important estimate (5.3.5) holds for all $\lambda \in \Gamma$, then $\mathcal{A}_0 + \mathcal{B}_\varepsilon$ must have an eigenvalue enclosed by Γ. The difficulty is, somewhat ironically, that because such a curve has to cross the real line (which is the area of interest) the resolvent estimate (5.3.5) breaks down. It may be possible to overcome this by exploiting the spectral gap between $0 \in \rho(-\beta N)$ and $\mu_1 \in \mathcal{D}(\Delta^D)$ where a resolvent estimate should hold. However, we will obtain the result using a very different approach, namely the fixed point method of identifying eigenvalues we will introduce in Chapter 6, combined with the properties of the Robin eigenvalues we proved in Section 4.4 (see Remark 6.2.6 and Section 6.3).

For the sake of comparison, we next briefly consider what happens when we apply the method directly to the operator on $H^1(\Omega)$ rather than considering it as the restriction of Δ_2^W. In this case we use more closely the method of [**46**] (an abstraction of [**44**]) for when $\beta > 0$, and combine this with the perturbation method we used in the proof of Theorem 5.3.1 to deal with the case $\beta < 0$. The price, however, is that we now need to assume $\partial\Omega$ is very smooth in order to use the method. We make this all precise with the following result, although we do not go into extensive detail.

Proposition 5.3.10. *Let Ω be of class $C^{2,1}$. The operator $\Delta_{H^1}^W$ given by* (5.1.2) *has compact resolvent for all $\beta < 0$ and $\gamma \in L^\infty(\partial\Omega)$.*

Proof. We first wish to show that if $\beta > 0$ then [**46**, Theorem 3.1] is valid. In this case $X = H^1(\Omega)$, $\partial X = H^{\frac{1}{2}}(\partial\Omega)$ by the trace theorem [**64**, Theorem 1.5.1.3] and $L : \partial X \to X$ is the trace operator, $A_m = \Delta_{\max}$, and B and C are as defined above. It is easy to see (cf. (5.1.2) and [**46**, Eq. (2.1)]) that in this case $A = \Delta_{H^1}^W$. Moreover, $L_0 = (L|_{\ker A_M})^{-1} = P : H^{\frac{1}{2}}(\partial\Omega) \to \ker \Delta_{\max} \subset H^1(\Omega)$.

Then condition (i) of [**46**, Theorem 3.1] is satisfied by Proposition C2.5. For condition (ii), we require that the restriction of $\Delta_{\max}$ to $\ker \mathrm{tr} = H_0^1(\Omega)$, that is, that $\Delta_{\max}|_{H_0^1(\Omega)}$ has compact resolvent and is sectorial of angle $\frac{\pi}{2}$, but this is well-known (for example, use [**9**, Example 7.2.1] to obtain this for Δ^D on $L^2(\Omega)$ and then use that $\Delta_{\max}|_{H_0^1(\Omega)}$ is the part of Δ^D in the form domain $H_0^1(\Omega)$). For (iii), we have to prove that the operator B defined by (5.3.3) is relatively $\Delta_{\max}|_{H_0^1(\Omega)}$-bounded with bound zero. Since Ω is of class $C^{2,1}$, we have $\mathcal{D}(\Delta^D|_{H_0^1(\Omega)}) \subset H^3(\Omega)$ using [**59**, Theorem 9.19], since in this case $\Delta u \in H^1(\Omega)$ for $u \in \mathcal{D}(\Delta^D|_{H_0^1(\Omega)})$. In particular we obtain the estimate $\|u\|_{H^3(\Omega)} \leq K(\|\Delta u\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)})$ where $K > 0$ is independent of $u \in H^3(\Omega)$. Also, B is now a bounded operator from $H^2(\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega) = \partial X$, so that $\mathcal{D}(\Delta_{\max}|_{H_0^1(\Omega)}) = \mathcal{D}(\Delta^D|_{H_0^1(\Omega)}) \subset \mathcal{D}(B)$. Since in fact $H^3(\Omega)$ embeds compactly in $H^2(\Omega)$ by Rellich's theorem it follows from Ehrling's Lemma (see [**98**, Theorem 6.99]) that for all $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq \varepsilon \|\Delta u\|_{H^1(\Omega)} + C\|u\|_{H^1(\Omega)}$$

for all $u \in \mathcal{D}(\Delta_{\max}|_{H_0^1(\Omega)}) \subset H^3(\Omega)$, giving the desired result. (Compare this with Lemma C1.1.)

Finally, in this case, the negative of the Dirichlet-to-Neumann map N generates a compact analytic semigroup of angle $\frac{\pi}{2}$ by Theorem C3.2. Hence (iv) is satisfied by $-\beta N$ if $\beta > 0$. In particular, by [**46**, Theorem 3.1] if $\beta > 0$ then the operator $\Delta^W_{H^1}$ generates a compact analytic semigroup of angle $\frac{\pi}{2}$ on $H^1(\Omega)$. If however $\beta < 0$, then we may replace [**46**, Lemma A.4] with the same argument as in the proof of Theorem 5.3.1 to conclude that $\Delta^W_{H^1}$ still has compact resolvent. $\square$

Remark 5.3.11. In [**76**, Proposition 2.6], we used the same method as in the proof of Proposition 5.3.10, and essentially for the same purpose, but following [**44**] we were working in the space $C(\overline{\Omega})$ rather than $H^1(\Omega)$. In fact the result from [**76**] tells us that for Ω of class $C^{2,\eta}$, a realisation of Δ^W on $C(\overline{\Omega})$ (different from the one in (5.1.3); instead using the setup of [**44**]) has compact resolvent for all $\beta < 0$ and $\gamma \in C(\partial\Omega)$.

CHAPTER 6

The Eigenvalues of the Wentzell Laplacian

Here we wish to study the eigenvalues of the Wentzell problem (1.1.3) in more detail. After looking at some general spectral properties, we will consider the form and structure of the principal eigenvalues and the other eigenvalues separately. We will conclude with a few elementary variational and monotonicity properties akin to those considered in Section 1.3.

6.1. General remarks

We start by showing that the spectrum of the Wentzell problem is essentially independent of the many realisations of the operator we considered in Chapter 5. More precisely, the next lemma, when combined with the results of Chapter 5, shows that, regardless of the sign of β, $\gamma \neq 0$, if Ω is a bounded, Lipschitz domain, then $\sigma(\Delta_2^W) = \sigma_p(\Delta_2^W) = \sigma_p(\Delta_{H^1}^W)$ (where $\sigma_p(A)$ is the point spectrum of A). Moreover, if Ω is sufficiently smooth (Lipschitz if $\beta > 0$ or $C^{1,1}$ will certainly do if $\beta < 0$), then in addition $\sigma_p(\Delta_{H^1}^W) = \sigma(\Delta_{H^1}^W)$. Note when considering Δ_2^W given by (5.1.4) that it does not matter whether we use the measure $dx \oplus \frac{d\sigma}{\beta}$ (which we can do if $\beta > 0$) or $dx \oplus d\sigma$ (which we must do if $\beta < 0$) on $L^2(\Omega) \oplus L^2(\partial\Omega)$, since this will not affect the spectrum of Δ_2^W. From now on we will restrict our attention to the case where $\beta, \gamma \neq 0$ are constants rather than possibly L^∞-functions, although this will not always be necessary.

Lemma 6.1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain and assume $\beta, \gamma \neq 0$ are constant. Then $\sigma_p(\Delta_2^W) = \sigma_p(\Delta_{H^1}^W)$.*

Proof. The inclusion $\sigma_p(\Delta_{H^1}^W) \subset \sigma_p(\Delta_2^W)$, taking into account the isomorphism described in Remark 5.1.1, is obvious. For the other inclusion, suppose $\Lambda \in \sigma_p(\Delta_2^W)$, and let u be an associated eigenfunction. This means that $-\Delta_2^W(u, u|_{\partial\Omega}) = \Lambda(u, u|_{\partial\Omega})$, that is, $-\Delta u = \Lambda u \in H^1(\Omega)$ and $\beta\frac{\partial u}{\partial\nu} + \gamma u = \Lambda u$ on $\partial\Omega$, that is, $(\Delta u)|_{\partial\Omega} + \beta\frac{\partial u}{\partial\nu} + u|_{\partial\Omega} = 0$. Hence $\Lambda \in \sigma_p(\Delta_{H^1}^W)$. □

So from now on, we will speak only of the eigenvalues "of the Wentzell Laplacian", i.e. the problem (1.1.3), since it does not matter which operator we use. Indeed, the regularity theory we consider below implies that we also have $\sigma_p(\Delta^W_{H^1}) = \sigma_p(\Delta^W_C)$, where Δ^W_C is given by (5.1.3).

We will largely be interested in eigenvalues of the following type.

Definition 6.1.2 (Principal eigenvalue)**.** For any of the operators under consideration, we shall call an eigenvalue Λ *principal* if its eigenspace is one-dimensional and its eigenfunction can be chosen strictly positive in Ω; in particular, a principal eigenvalue will be isolated.

Note that for a a self-adjoint operator which is bounded from below, such as the Dirichlet or Robin Laplacian, the first eigenvalue, which is equal to the spectral bound, is the unique principal eigenvalue (at least if Ω is connected. See Appendix A5). For the Wentzell Laplacian, we will see that there may be anywhere from zero to two principal eigenvalues, depending on β and γ. Thus the notion of a principal eigenvalue generalises the idea of a first eigenvalue. On this point, we note that our definition is stronger than what is often used, due to the requirement on the eigenspace. However, Lemma 6.1.3 shows that this is not actually a restriction at all, at least in our case, since if Ω is connected then *any* positive eigenfunction of (1.1.3) lies in a one-dimensional eigenspace. Note that if $\beta > 0$, then the Wentzell Laplacian is self-adjoint (at least on the right space; see Remark 5.2.3) and hence, as in the Dirichlet and Robin cases, we automatically get exactly one such principal eigenvalue (see also Remark 6.2.2). We will use a different approach to obtain and study the principal (and other) eigenvalues. The attraction, both of our approach and of the definition, is that they work equally well in the non-self-adjoint case $\beta < 0$. Our method is based on the following elementary identification of every Wentzell eigenvalue (resp. function) as the eigenvalue (resp. function) of an appropriately chosen Robin problem.

Lemma 6.1.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain. Suppose u is an eigenfunction of* (1.1.3)*, with eigenvalue Λ_u. Then u is an eigenfunction, with eigenvalue Λ_u, of the Robin problem* (1.1.2) *with boundary parameter $\alpha := \frac{\gamma-\Lambda_u}{\beta} \in \mathbb{R}$. If in addition Ω is connected and u is positive in Ω, then Λ_u is the first eigenvalue of* (1.1.2) *for this value of α, and the eigenspace of Λ_u is one-dimensional.*

Proof. First suppose u is an eigenfunction of (1.1.3), with eigenvalue Λ_u. Since $-\Delta u = \Lambda_u u$ (weakly) in Ω and $\Delta u \in H^1(\Omega)$, by taking traces we see u satisfies the boundary condition

$$\frac{\partial u}{\partial \nu} + \frac{\gamma - \Lambda_u}{\beta} u = 0 \qquad \text{on } \partial\Omega. \tag{6.1.1}$$

Setting $\alpha := \frac{\gamma - \Lambda_u}{\beta} \in \mathbb{R}$, it follows immediately that u is an eigenfunction with eigenvalue Λ_u of (1.1.2) for this α.

Now suppose u is positive. By Theorem 1.3.1, since Ω is connected there is exactly one positive eigenfunction v (unique up to scalar multiples) of (1.1.2), and $\lambda_1 = \lambda_1(\Omega, \alpha)$ is its associated eigenvalue. The only possibility is that $u = kv$ for some constant $k > 0$ and $\Lambda_u = \lambda_1(\Omega, \alpha)$. Similarly, if u_1 and u_2 are two positive eigenfunctions of (1.1.3) for Λ_u, then by considering Λ_u as the first eigenvalue of (1.1.2), we must have $u_1 = m u_2$ for some $m > 0$. $\square$

Corollary 6.1.4 (Regularity of eigenfunctions)**.** *Every eigenfunction u of* (1.1.3) *lies in $H^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega)$. If Ω is of class C^2 and $\beta, \gamma > 0$, then in addition $u \in W^{2,p}(\Omega) \cap C^1(\overline{\Omega})$ for every $1 < p < \infty$.*

Proof. This follows immediately from Lemma 6.1.3 and Theorem 1.2.8. $\square$

Remark 6.1.5. (i) We can go further than Lemma 6.1.3 in making explicit the link between the eigenvalues of the Robin problem (1.1.2) and those of the Wentzell problem (1.1.3). More precisely, we may identify every Wentzell eigenvalue via a type of fixed point argument involving the Robin problem. If, given Ω, β, γ and $k \geq 1$, we can find $\alpha \in \mathbb{R}$ such that $\alpha = \beta^{-1}(\gamma - \lambda_k(\alpha))$, then it is clear from (6.1.1) that for this value of α, $\lambda_k = \lambda_k(\Omega, \alpha)$ will be an eigenvalue $\Lambda(\Omega, \beta, \gamma)$ of (1.1.3); moreover, every eigenvalue of (1.1.3) can be written in this way. This means that the Wentzell eigenvalues are in exact correspondence with the points of intersection of the family of curves $g_k(\alpha) := \beta^{-1}(\gamma - \lambda_k(\alpha))$, $k \geq 1$, with the fixed point line $f(\alpha) = \alpha$. In particular, by Lemma 6.1.3, all principal eigenvalues $\Lambda(\beta, \gamma)$ will be given by $\lambda_1(\alpha)$, where $g_1(\alpha) = \beta^{-1}(\gamma - \lambda_1(\alpha)) = \alpha$. The next two sections are dedicated to exploring this in more detail.

(ii) It appears this fixed point idea has not really been used in the study of the Wentzell problem before. However, something similar was used to study an analogous problem in ordinary differential equations in a series of papers by Binding, Browne and Watson (see for example [**16, 17, 18**]). The authors call their problems Sturm-Liouville problems with eigenparameter-dependent boundary conditions, and use various techniques to transform the boundary conditions into simpler forms. (My thanks to Prof. Goldstein for drawing this to my attention.)

6.2. The principal eigenvalues

This section is devoted to studying the principal eigenvalues of (1.1.3). In particular, we wish to classify the number and sign of them that a given domain Ω possesses, depending on the sign of the parameters β and γ. In this section, since we are dealing with principal eigenvalues we will assume for simplicity and without particular loss of generality that Ω is connected (see Remark 1.3.2). Our main result for the section is the following theorem. Recall that, by definition, every principal eigenvalue for us is isolated (that is, is the only eigenvalue in an open $\mathbb{C}$-neighbourhood and has one-dimensional eigenspace).

Theorem 6.2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected domain satisfying the minimal regularity assumptions of Section 5.2 or 5.3 as appropriate.*

(i) *If β, $\gamma > 0$ there exists a unique principal eigenvalue $\Lambda_1 = \Lambda_1(\Omega, \beta, \gamma)$ of the problem* (1.1.3)*, which satisfies $0 < \Lambda_1 < \Lambda$ for every other eigenvalue Λ of* (1.1.3)*.*

(ii) *If $\beta > 0$, $\gamma < 0$, there exists a unique principal eigenvalue Λ_1 of* (1.1.3) *which now satisfies $\Lambda_1 < 0$ and $\Lambda_1 < \Lambda$ for every other eigenvalue Λ of* (1.1.3)*.*

(iii) *If β, $\gamma < 0$, there exist two principal eigenvalues Λ_1^- and Λ_1^+ satisfying $\Lambda_1^- < 0 < \Lambda_1^+$. Moreover, there is no other eigenvalue $\Lambda \in [\Lambda_1^-, \Lambda_1^+]$.*

(iv) *If $\beta < 0$, $\gamma > 0$, then*

  (a) *for every $\beta \in (-\frac{\sigma(\partial\Omega)}{|\Omega|}, 0)$, there exists a unique $\gamma^* = \gamma^*(\Omega, \beta) > 0$ such that there are two positive principal eigenvalues $0 < \Lambda_1^- < \Lambda_1^+$ for $0 < \gamma < \gamma^*$, one ($0 < \Lambda_1^- = \Lambda_1^+$) for $\gamma = \gamma^*$, and no principal eigenvalues for $\gamma > \gamma^*$. As $\beta \to -\frac{\sigma(\partial\Omega)}{|\Omega|}$, $\gamma^* \to 0$;*

  (b) *if $\beta = -\frac{\sigma(\partial\Omega)}{|\Omega|}$, there is no principal eigenvalue for any $\gamma > 0$;*

  (c) *for every $\beta \in (-\infty, -\frac{\sigma(\partial\Omega)}{|\Omega|})$, there exists a unique $\gamma^{**} = \gamma^{**}(\Omega, \beta) > 0$ such that there are two negative principal eigenvalues $\Lambda_1^- < \Lambda_1^+ < 0$ for $0 < \gamma < \gamma^{**}$, one for $\gamma = \gamma^{**}$ and no principal eigenvalues for $\gamma > \gamma^{**}$. As $\beta \to -\frac{\sigma(\partial\Omega)}{|\Omega|}$, $\gamma^{**} \to 0$.*

If for a given pair β, γ there are two principal eigenvalues $\Lambda_1^- < \Lambda_1^+$, then there does not exist another eigenvalue $\Lambda \in [\Lambda_1^-, \Lambda_1^+]$.

To find the principal eigenvalue(s) of (1.1.3) we will use the fixed point argument described in Remark 6.1.5(i), involving the function $g_1(\alpha) = \beta^{-1}(\gamma - \lambda_1(\alpha))$ This will make heavy use of the properties of $\lambda_1(\alpha)$ listed in Theorem 1.3.1. We consider four different cases corresponding to those in Theorem 6.2.1. Case (iv) will be considered separately.

Remark 6.2.2. For (i) and (ii), there is a generic method available to show the existence of the principal eigenvalue (see Remark 6.2.6(i)), which would also allow the same conclusion to be obtained under far more general assumptions on β and γ; say, $\beta_0 \leq \beta \in L^\infty(\partial\Omega)$, where β_0 is a positive constant, and $\gamma \in L^\infty(\partial\Omega)$. Our fixed point method, apart from working in all cases, also allows us to relate the Wentzell eigenvalues to the Robin ones in a way that will be useful for the isoperimetric inequalities in Chapter 7.

Proposition 6.2.3. *The following statements are true.*

(i) *If β, $\gamma > 0$, then the function g_1 defined above has exactly one fixed point α. In this case $\alpha > 0$.*
(ii) *If $\beta > 0$, $\gamma < 0$, then g_1 has exactly one fixed point α. In this case $\alpha < 0$.*
(iii) *If β, $\gamma < 0$, then g_1 has exactly one positive fixed point and exactly one negative fixed point.*

Proof. (i) The function $g_1(\alpha) = \beta^{-1}(\gamma - \lambda_1(\alpha))$ is on $[0, \infty)$, with $g_1(0) = \beta^{-1}\gamma > 0$. Since $\beta > 0$ and $\lambda_1'(\alpha) > 0$ everywhere (Theorem 1.3.1(vi)), it is strictly monotonically decreasing, so $g_1(\beta^{-1}\gamma) < \beta^{-1}\gamma$. By the intermediate value theorem, $g_1(\alpha) = \alpha$ for some $\alpha \in (0, \beta^{-1}\gamma)$. (See Figure 6.1.) The fixed point α is unique because $g_1(\alpha)$ is decreasing everywhere.

(ii) The argument is essentially the same as in case (i). Since $g_1(\alpha)$ is continuous and monotonically decreasing on $(-\infty, 0]$, with $g_1(0) = \beta^{-1}\gamma < 0$, we have $g_1(\beta^{-1}\gamma) > \beta^{-1}\gamma$. By the intermediate value theorem there is a fixed point in $(\beta^{-1}\gamma, 0)$. As in case (i), uniqueness follows immediately since $g_1(\alpha)$ is decreasing.

(iii) We start with the positive fixed point. Since $g_1(0) = \beta^{-1}\gamma > 0$ and $\lim_{\alpha\to\infty} g_1(\alpha) = \beta^{-1}(\gamma - \mu_1(\Omega)) \in (\beta^{-1}\gamma, \infty)$, by the intermediate value theorem $g_1(\alpha) = \alpha$ for some $\alpha \in (0, \infty)$. Since $\beta < 0$, g_1 is monotonically increasing, as

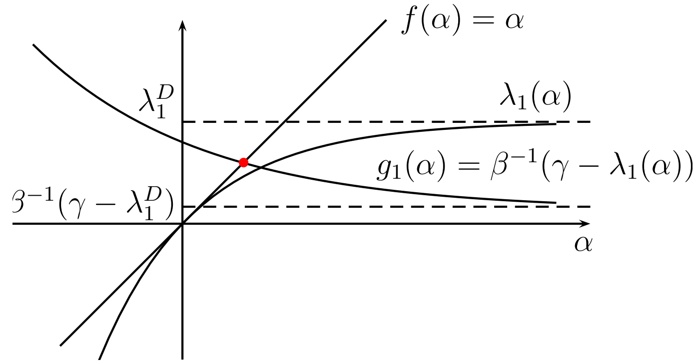

FIGURE 6.1. The fixed point when $\beta > 0$, $\gamma > 0$. Heuristically, the case $\beta > 0$, $\gamma < 0$ is similar.

well as strictly concave (that is, ${g_1}'(\alpha)$ is strictly decreasing everywhere). So there can only be one fixed point in $[0, \infty)$. (See Figure 6.2.)

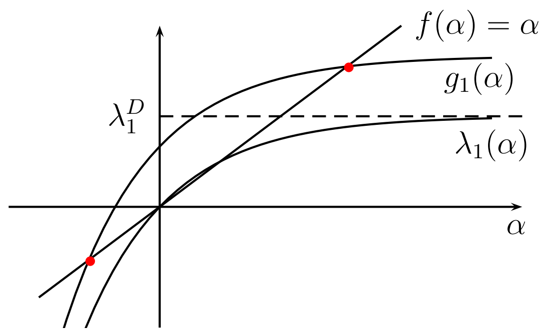

FIGURE 6.2. There are two fixed points when $\beta < 0$, $\gamma < 0$. The case $\beta < 0$, $\gamma > 0$ is similar, although more complicated.

For the negative fixed point, the same concavity argument tells us there can be at most one in $(-\infty, 0]$. We have $g_1(0) > 0$. By (1.3.3), $g_1(\alpha) \leq \beta^{-1}(\gamma + \alpha^2)$ for $\alpha < 0$, so certainly $g_1(\alpha) < \alpha$ for α large enough. $\square$

Remark 6.2.4. We note that the smoothness assumptions we have been making in cases (iii) and (iv) are necessary only to ensure that at least one realisation of the Wentzell Laplacian has compact resolvent. The formal calculations always work, even if Ω is only Lipschitz. For the rest of this section we will not explicitly make further reference to the smoothness assumptions we are making.

Before we consider case (iv), we make the following observation about the nature of the principal eigenvalues we are finding. Note that part (ii) below applies whenever $\beta < 0$, that is, to cases (iii) and (iv) of Theorem 6.2.1.

Proposition 6.2.5. (i) *In cases (i) and (ii) of Proposition 6.2.3, the unique principal eigenvalue is the first, or smallest, eigenvalue of* (1.1.3)*.*

(ii) *Suppose $\beta < 0$ and there are two fixed points $\alpha^- < \alpha^+$. Denote the two associated principal eigenvalues of* (1.1.3) *by $\Lambda_1^- := \lambda_1(\alpha^-) < \Lambda_1^+ := \lambda_1(\alpha^+)$. Then there does not exist an eigenvalue of* (1.1.3) *in $(\Lambda_1^-, \Lambda_1^+)$.*

Proof. (i) In both these cases we have $\beta > 0$. Denote the unique principal eigenvalue by $\lambda_1(\alpha^*)$, where $\alpha^* = \beta^{-1}(\gamma - \lambda_1(\alpha^*))$ is the unique fixed point. Suppose for a contradiction that there exists an eigenvalue $\Lambda \in (-\infty, \lambda_1(\alpha^*))$. Then Λ would be an eigenvalue, say the nth, of a Robin problem (1.1.2) with boundary parameter $\tilde{\alpha} := \beta^{-1}(\gamma - \Lambda) \in \mathbb{R}$. That is, we can write $\Lambda = \lambda_n(\tilde{\alpha})$. Then we have $\alpha^* = \beta^{-1}(\gamma - \lambda_1(\alpha^*)) < \beta^{-1}(\gamma - \lambda_n(\tilde{\alpha})) = \tilde{\alpha}$ using the contradiction assumption. Now since $\beta > 0$ we know by Theorem 1.3.1 that $\beta^{-1}(\gamma - \lambda_1(\alpha))$ is decreasing everywhere and so $\beta^{-1}(\gamma - \lambda_1(\alpha)) < \alpha$ for all $\alpha > \alpha^*$. In particular, $\beta^{-1}(\gamma - \lambda_1(\tilde{\alpha})) < \tilde{\alpha}$. But since $\beta > 0$, $\beta^{-1}(\gamma - \lambda_n(\alpha)) \leq \beta^{-1}(\gamma - \lambda_1(\alpha))$ for all $n \geq 1$ and all $\alpha \in \mathbb{R}$. In particular, $\tilde{\alpha} = \beta^{-1}(\gamma - \lambda_n(\tilde{\alpha})) \leq \beta^{-1}(\gamma - \lambda_1(\tilde{\alpha})) < \tilde{\alpha}$, a contradiction.

(ii) Similarly, suppose there exists an eigenvalue $\Lambda \in (\Lambda_1^-, \Lambda_1^+)$, and write Λ as the nth eigenvalue of (1.1.2) so that $\Lambda = \lambda_n(\tilde{\alpha})$. Now $g_1(\alpha) = \beta^{-1}(\gamma - \lambda_1(\alpha)) > \alpha$ for all $\alpha \in (\alpha^-, \alpha^+)$, since $g_1(\alpha)$ is strictly concave with $g_1(\alpha) = \alpha$ at $\alpha = \alpha^-, \alpha^+$. By definition of the fixed points, we have $\tilde{\alpha} \in (\alpha^-, \alpha^+)$, so that $\beta^{-1}(\gamma - \lambda_1(\tilde{\alpha})) > \tilde{\alpha}$. But for all $\alpha \in \mathbb{R}$, $\beta^{-1}(\gamma - \lambda_n(\alpha)) \geq \beta^{-1}(\gamma - \lambda_1(\alpha))$, since $\beta < 0$. In particular, $\tilde{\alpha} = \beta^{-1}(\gamma - \lambda_n(\tilde{\alpha})) > \tilde{\alpha}$, a contradiction. □

Remark 6.2.6. (i) There is a generic argument available to show that when $\beta > 0$ there is unique principal eigenvalue which is also the first (cf. Remark 6.2.2). By a well known theorem of Courant and Hilbert [**30**, Section VI.6] the first eigenvalue of any self-adjoint second order differential equation with arbitrary homogeneous boundary conditions is principal. Since our operator is self-adjoint exactly when $\beta > 0$ (since it is associated with an elliptic form – combine the proof of Theorem 5.2.1 with, for example, the theory in [**9**, Section 7.1]), when we combine this with the uniqueness of the principal eigenvalue, the result follows. Equivalently, if an operator generates a compact, positive and irreducible semigroup, then by the Krein-Rutman theorem, that is, infinite-dimensional Perron-Frobenius theory, its spectral bound is a real number and this is the unique eigenvalue which has a strictly positive eigenfunction which is unique up to scalar multiples. (See [**42**, Chapter 7] or Appendix A5. Cf. also Section 6.4, where there is some variational theory.)

(ii) When there are two principal eigenvalues, we will show in Section 6.3 that, just as the larger is at the base of a sequence tending to ∞, the smaller one is always at the head of a sequence of negative eigenvalues tending to $-\infty$, at least if Ω has sufficiently smooth boundary (cf. Remark 5.3.8 and the discussion following it). This seems to be a rather difficult result to obtain, as noted in [**105**]. Our fixed point method gives an independent proof to the ones used in [**14, 105**] (which studied slightly different or less general problems). However, since we will use the asymptotic estimates obtained in Section 4.4 our method of proof cannot really be considered simpler than the ones used in [**14, 105**].

We now consider case (iv). In this case, since $\beta < 0$, the function $g(\alpha)$ has the same slope as in case (iii), only shifted down (cf. Figures 6.2 and 6.3). Heuristically, this gives rise to two essentially different types of behaviour. If $|\beta|$ is large, so that g_1 is flat near $\alpha = 0$, then we can expect negative fixed points (as in Figure 6.3). If $|\beta|$ is small, then g_1 is steep near $\alpha = 0$ and we can expect positive fixed points. Note that we expect no more than two fixed points, and these should have the same sign. For some values of $\beta < 0$, we expect fewer than two.

We wish to formalise these ideas. We start with the following simple but useful observation, which was implicitly used in the proof of Proposition 6.2.5(ii).

Lemma 6.2.7. *Let $\beta < 0$ and $\gamma \neq 0$. If $g_1(\alpha)$ has two fixed points $\alpha^- < \alpha^+$, then $\{\alpha \in \mathbb{R} : g_1(\alpha) > \alpha\} = (\alpha^-, \alpha^+)$. Moreover, if there exists $\tilde{\alpha} \in \mathbb{R}$ for which $g_1(\tilde{\alpha}) > \tilde{\alpha}$, then there exist exactly two fixed points $\alpha^- < \tilde{\alpha} < \alpha^+$.*

Proof. By Theorem 1.3.1, $g_1(\alpha)$ is strictly increasing and strictly concave everywhere, so the first statement follows immediately. The second statement follows from exactly the same argument as in the proof of Proposition 6.2.3(iii), only with $\tilde{\alpha}$ in place of $\alpha = 0$. $\square$

We now wish to establish the existence of values of β, γ for which there is exactly one fixed point. These can be considered "borderline cases". They occur when the curve $g_1(\alpha)$ is tangent to the fixed point line at the point of intersection (see Figure 6.3).

Lemma 6.2.8. *For all $\beta \in (-\frac{\sigma(\partial\Omega)}{|\Omega|}, 0)$ there exists $\gamma^* = \gamma^*(\Omega, \beta) > 0$ such that there exists exactly one positive solution α^* to the fixed point problem $\alpha = \beta^{-1}(\gamma^* - \lambda_1(\alpha))$. Moreover, for all $\beta \in (-\infty, -\frac{\sigma(\partial\Omega)}{|\Omega|})$, there exists $\gamma^{**}(\Omega, \beta) > 0$ for which there exists exactly one negative fixed point α^{**}.*

Proof. Given β, we wish to find γ such that there exists a point of intersection at which the curve $g_1(\alpha)$ is tangent to the fixed point line $f(\alpha) = \alpha$. Since g_1 is strictly concave, this point of intersection, if it exists, must be unique. Thus we are looking for γ and α for which, given β, $\alpha = \beta^{-1}(\gamma - \lambda_1(\alpha))$ and ${g_1}'(\alpha) = 1$. The latter is equivalent to ${\lambda_1}'(\alpha) = -\beta$.

For the positive fixed point, by Theorem 1.3.1, ${\lambda_1}'(\alpha)$ is a strictly monotonically decreasing surjection from $(0, \infty)$ to $(0, \frac{\sigma(\partial\Omega)}{|\Omega|})$. Hence given any $\beta \in (-\frac{\sigma(\partial\Omega)}{|\Omega|}, 0)$, there indeed exists $\alpha^* \in (0, \infty)$ for which ${\lambda_1}'(\alpha^*) = -\beta$.

Choosing $\gamma^* := \beta\alpha^* + \lambda_1(\alpha^*)$ gives us the desired value of γ. If $\beta > -\frac{\sigma(\partial\Omega)}{|\Omega|}$, then by (1.3.2), ${g_1}'(0) > 1$ and g_1 must lie below the fixed point line. In particular, $g_1(0) = \beta^{-1}\gamma^* < 0$, implying $\gamma^* > 0$.

For the negative fixed point, using the estimate (1.3.3), ${\lambda_1}'(\alpha)$ maps $(-\infty, 0)$ onto $(\frac{\sigma(\partial\Omega)}{|\Omega|}, \infty)$, and so given $\beta \in (-\infty, \frac{\sigma(\partial\Omega)}{|\Omega|})$, there again exist $\alpha^{**} \in (-\infty, 0)$ and $\gamma^{**} = \beta\alpha^{**} + \lambda_1(\alpha^{**}) > 0$ satisfying the requirements of the lemma. $\square$

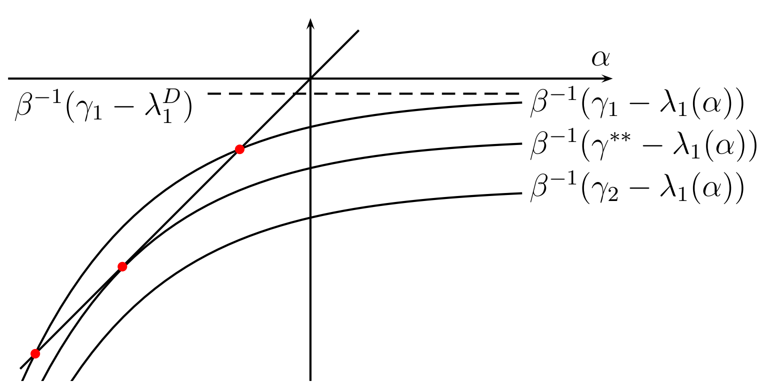

FIGURE 6.3. The family of curves obtained by fixing β and varying γ. In this case, since $|\beta|$ is small, we are searching for γ^{**} not γ^*.

In fact, since ${\lambda_1}'(\alpha)$ is monotonic and hence a bijection, α^* and α^{**}, and hence γ^* and γ^{**}, must be unique. This is formalised in the following classification, which also establishes the existence of two principal eigenvalues for some β and γ.

Lemma 6.2.9. *Fix $\beta < 0$ and suppose γ^* is any value of $\gamma > 0$ for which there is exactly one fixed point. Then there are two positive fixed points for the pair β, γ if $\gamma \in (0, \gamma^*)$ and no fixed points if $\gamma \in (\gamma^*, \infty)$. Similarly, if there is exactly one negative fixed point for some $\gamma^{**} > 0$, then there are two negative fixed points if $\gamma \in (0, \gamma^{**})$ and no fixed points if $\gamma \in (\gamma^{**}, \infty)$.*

Proof. We prove the statement for γ^*; the proof for γ^{**} is essentially the same, and is omitted. First suppose $\gamma \in (0, \gamma^*)$. Then $\beta^{-1}(\gamma - \lambda_1(\alpha)) > \beta^{-1}(\gamma^* - \lambda_1(\alpha))$ for all $\alpha > 0$ and in particular $\beta^{-1}(\gamma - \lambda_1(\alpha^*)) > \alpha^*$, where α^* is the fixed point from Lemma 6.2.8. By Lemma 6.2.7 there must therefore be exactly two fixed points $\alpha^- < \alpha^+$ for the pair β, γ. Clearly $\alpha^+ > \alpha^* > 0$, while since $\beta^{-1}\gamma = g_1(0) < 0$ and $g_1(\alpha^*) > \alpha^*$ we must have $\alpha^- \in (0, \alpha^*)$. Hence both fixed points are positive.

Suppose now that $\gamma > \gamma^*$. Since γ^* satisfies $\beta^{-1}(\gamma^* - \lambda_1(\alpha)) \leq \alpha$ for all $\alpha \in \mathbb{R}$, γ satisfies $\beta^{-1}(\gamma - \lambda_1(\alpha)) < \alpha$ for all $\alpha \in \mathbb{R}$, so there cannot be any fixed point. □

We also consider the special case when $\beta = -\frac{\sigma(\partial\Omega)}{|\Omega|}$. The case $\beta = -\frac{\sigma(\partial\Omega)}{|\Omega|}$ and $\gamma = 0$ is called the "critical case" in [**105**], where the analysis is much harder. This also corresponds to the "resonance case" in [**14**].

Lemma 6.2.10. *When $\beta = -\frac{\sigma(\partial\Omega)}{|\Omega|}$, there are no fixed points for any $\gamma < 0$.*

Proof. For this value of β, by Theorem 1.3.1, ${g_1}'(0) = 1$, while $g_1(0) = \beta^{-1}\gamma < 0$. Since ${g_1}'(\alpha)$ is strictly decreasing on $\mathbb{R}$, it follows that $g_1(\alpha) \leq g_1(0) + \alpha < \alpha$ for all $\alpha \in \mathbb{R}$, and hence there are no fixed points. □

Remark 6.2.11. (i) As $\beta \searrow -\frac{\sigma(\partial\Omega)}{|\Omega|}$, ${\lambda_1}'(\alpha^*) = -\beta \nearrow \frac{\sigma(\partial\Omega)}{|\Omega|}$, implying $\alpha^* \to 0$ and so $\gamma^* = \lambda_1(\alpha^*) + \beta\alpha^* \to 0$. Similarly, as $\beta \nearrow -\frac{\sigma(\partial\Omega)}{|\Omega|}$, ${\lambda_1}'(\alpha^{**}) \searrow \frac{\sigma(\partial\Omega)}{|\Omega|}$, implying $\alpha^{**} \to 0$ and $\gamma^{**} \to 0$. (See also Figure 7.1.)

(ii) We could of course fix γ and search for $\beta^*(\Omega, \gamma)$. However, it seems more natural to do it our way since β is in a sense more important for determining the structure of the eigenvalues: changing γ merely shifts the spectrum but changing β might change its structure (cf. Remark 6.2.6(ii) or Lemma 6.2.10).

If we combine Propositions 6.2.3 and 6.2.5, Lemmata 6.2.8, 6.2.9 and 6.2.10 and Remark 6.2.11(i) with our earlier remarks, we see we have proved Theorem 6.2.1.

Remark 6.2.12. We finish by explicitly mentioning what happens if Ω is not connected, since this will be important in the sequel. In this case Ω will consist of finitely many connected components, each bounded and with the appropriate degree of regularity (Lipschitz or $C^{1,1}$). As usual, it is clear that the eigenvalues (and functions) of Ω can be found by collecting and rearranging the eigenvalues (functions) of all the individual connected components. Moreover, since the fixed point argument is unaffected (Remark 6.1.5(i) does not assume that Ω is connected), in cases (i)-(iii), the only difference will be the number of principal eigenvalues (as with all other boundary conditions), but case (iv) is more complicated. We defer considering this in more detail until the next section.

6.3. The other eigenvalues

Here we wish to use the ideas introduced at the start of the chapter to describe and classify the other eigenvalues of the Wentzell problem (1.1.3) depending on the different signs of β and γ. Naturally, we will use Theorem 1.3.5 in place of Theorem 1.3.1. Since the former contains less information than the latter, we can say correspondingly less about Λ_k relative to Λ_1.

We first wish to establish a complement to the statement on principal eigenvalues in Lemma 6.1.3, that is, that the kth eigenvalue of the Wentzell Laplacian is just the kth eigenvalue (for the *same* k) of the Robin problem (1.1.2) for suitable α. Unlike in the case of a principal eigenvalue, depending on the sign of β it is not exactly clear what we mean by "the kth eigenvalue of (1.1.3)" and so we need a more involved classification. If $\beta < 0$ we obtain two sequences of eigenvalues, one heading in each direction away from the origin. We keep the case numbers from Theorem 6.2.1. For this we do not assume that Ω is connected.

Theorem 6.3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain, and fix $\beta, \gamma \in \mathbb{R} \setminus \{0\}$.*

(i), (ii) *If $\beta > 0$, then there exists a sequence of eigenvalues $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots$, where $\Lambda_k = \Lambda_k(\Omega, \beta, \gamma)$. For every $k \geq 1$ there exists $\alpha_k = \alpha(\Omega, \beta, \gamma, k) \in \mathbb{R}$ such that $\Lambda_k = \lambda_k(\alpha_k)$. Moreover $\alpha_k \to -\infty$ and $\Lambda_k \to \infty$ as $k \to \infty$.*

(iii) *Suppose $\beta < 0$ and $\gamma < 0$. Then there exist sequences $0 < \Lambda_1^+ \leq \Lambda_2^+ \leq \Lambda_3^+ \leq \ldots \to \infty$, and $0 > \Lambda_1^- \geq \Lambda_2^- \geq \ldots \to -\infty$, where $\Lambda_k^\pm = \Lambda_k^\pm(\Omega, \beta, \gamma)$ satisfies $\Lambda_k^\pm = \lambda_j(\alpha_k)$ for some $j \leq k$ and some $\alpha_k \in \mathbb{R}$.*

(iv) *Suppose* $\beta < 0$ *and* $\gamma > 0$. *Then there exists a denumerable set of eigenvalues* $\Lambda_m = \Lambda_m(\Omega, \beta, \gamma)$, *where* $m \in \mathbb{Z}$, *and such that* $\Lambda_m \to \pm\infty$ *as* $m \to \pm\infty$.

Remark 6.3.2. (i) In cases (iii) and (iv), unlike in (i) and (ii) we cannot in general say that $\Lambda_k^{\pm} = \lambda_k(\alpha_k)$ for some $\alpha_k \in \mathbb{R}$ because we do not know that λ_k is a concave function of α (see Theorem 1.3.5 and the comments following it). This means that we cannot rule out multiple solutions (i.e. more than two) to the fixed point equation $\beta^{-1}(\gamma - \lambda_k(\alpha)) = \alpha$, leading to multiple Wentzell eigenvalues associated with λ_k.

(ii) Case (iv) becomes even more complicated if Ω is not connected; in particular $g_k(\alpha)$ will tend not to be differentiable at $\alpha = 0$ (see Remark 1.3.2). As an example, suppose Ω is the disjoint union of Ω_1 and Ω_2, with $-\sigma(\partial\Omega_1)/|\Omega_1| < \beta < -\sigma(\partial\Omega_2)/|\Omega_2|$. Then there will be no fixed points associated with g_1. It could then be arranged so that there are four eigenvalues associated with g_2, say, of which two are principal (and of different signs) and two are not.

(iii) We remark however that if Ω is connected it is easy to see from the proof of Theorem 6.3.1 that, whenever we have principal eigenvalues $\Lambda_1^- \leq \Lambda_1^+$, we can find sequences of eigenvalues $\Lambda_1^+ < \Lambda_2^+ \leq \ldots \to \infty$ and $\Lambda_1^- > \Lambda_2^- \geq \ldots \to -\infty$.

(iv) We observe again (cf. Remark 6.1.5(ii)) that the ideas here are of a similar flavour to the ordinary differential equations studied in [**16, 17**] and related papers.

Proof. The proof is routine, and uses an easy induction argument. First note that, as in the case of the principal eigenvalues, by Remark 6.1.5(i), for any $\beta, \gamma \in \mathbb{R}\backslash\{0\}$ the set of fixed points $\{\alpha \in \mathbb{R} : \beta^{-1}(\gamma - \lambda_k(\alpha)) = \alpha$ for some $k \geq 1\}$ is in exact correspondence with the set of Wentzell eigenvalues for this β, γ. Let $g_k : \mathbb{R} \to \mathbb{R}$, $g_k(\alpha) = \beta^{-1}(\gamma - \lambda_k(\alpha))$.

We also note that if for some value of $\beta, \gamma \in \mathbb{R} \setminus \{0\}$ we have $\Lambda_1(\Omega) = \Lambda_2(\Omega)$, then it follows there exists $\alpha \in \mathbb{R}$ such that $\Lambda_1(\Omega) = \lambda_1(\Omega, \alpha) = \lambda_2(\Omega, \alpha) = \Lambda_2(\Omega)$. In case (iii) we replace $\Lambda_{\{1,2\}}$ with $\Lambda_{\{1,2\}}^{\pm}$; in case (iv) we are assuming Ω is connected so this cannot happen to the principal eigenvalues.

(i), (ii) For every $k \geq 1$ the curve $g_k(\alpha)$ is continuous and monotonically decreasing (see Theorem 1.3.5). Hence for all $k \geq 1$ there exists a unique $\alpha_k \in \mathbb{R}$ such that $g_k(\alpha_k) = \alpha_k$; this of course will be a Wentzell eigenvalue.

Moreover, since $\lambda_k(\alpha) \leq \lambda_{k+1}(\alpha)$ for all $\alpha \in \mathbb{R}$ and $\beta > 0$, $g_{k+1}(\alpha) \geq g_k(\alpha)$ for all k and α and so we must have $\alpha_k \geq \alpha_{k+1}$ for all k. Since $\beta^{-1}(\gamma - \lambda_k(\alpha_k)) = \alpha_k$ by construction, and $\beta > 0$, this implies $\lambda_k(\alpha_k) \leq \lambda_{k+1}(\alpha_{k+1})$ for all $k \geq 1$. Hence by induction Λ_k exists and equals $\lambda_k(\alpha_k)$.

Finally, since our operator is self-adjoint we know from abstract operator theory that $\Lambda_k \to \infty$ as $k \to \infty$. However we will prove this directly. Note that $\alpha_k \to -\infty$ implies $\Lambda_k \to \infty$ since $\Lambda_k = \lambda_k(\alpha_k) = \gamma - \beta\alpha_k$. Suppose that $\alpha_k \nrightarrow -\infty$. By monotonicity there exists $\tilde{\alpha} \in \mathbb{R}$ such that $\alpha_n \searrow \tilde{\alpha}$. Since $\beta^{-1}(\gamma - \lambda_k(\alpha_k)) = \alpha_k \geq \tilde{\alpha}$, by monotonicity $\beta^{-1}(\gamma - \lambda_k(\tilde{\alpha})) \geq \tilde{\alpha}$. In particular $\lambda_k(\tilde{\alpha}) \leq \gamma - \beta\tilde{\alpha}$ for all $k \geq 1$, contradicting $\lambda_k(\tilde{\alpha}) \to \infty$ as $k \to \infty$ by Theorem 1.2.6.

(iii) The argument is similar. We know there exists $\alpha_1^+ > 0$ such that $\Lambda_1^+ = \lambda_1(\alpha_1^+)$. Moreover for all $k \geq 1$, $g_k(0) = \beta^{-1}(\gamma - \lambda_k(0)) > 0$, while g_k is continuous and monotonically increasing with $\lim_{\alpha \to \infty} g_k(\alpha) \leq \beta^{-1}(\gamma - \mu_k)$ (see Theorem 1.3.5). In particular for every $k \geq 1$ there is at least one fixed point $\alpha_k^+ \in (0, \infty)$ corresponding to a Wentzell eigenvalue, which means $\lambda_k(\alpha_k^+)$ corresponds to $\Lambda_j = \Lambda_j(\alpha_k)$ for some $j \geq k$. To show that $\Lambda_k^+ \to \infty$, for each k let α_k^+ be the *smallest* fixed point of $g_k(\alpha)$. Then $g_{k+1}(\alpha) \geq g_k(\alpha)$ for all k and α implies $\alpha_{k+1}^+ \geq \alpha_k^+$. Since $\alpha_k^+ \to \infty$ implies the Wentzell eigenvalue $\Lambda_j(\alpha_k^+)$ (not necessarily the kth) tends to ∞, assume for a contradiction that $\alpha_k^+ \to \tilde{\alpha}$. Then once again this implies the sequence $\lambda_k(\tilde{\alpha}) \leq \gamma - \beta\tilde{\alpha}$ is bounded from above as $k \to \infty$, a contradiction.

Now we consider the negative eigenvalues. Again $g_k(\alpha)$ is continuous and monotonically increasing, with $g_k(0) > 0$. By Theorem 4.4.1, there exists $\tilde{\alpha}_k < 0$ such that $g_k(\tilde{\alpha}_k) < \tilde{\alpha}_k$. Thus we get at least one Wentzell eigenvalue for each curve g_k. Every other property follows in a manner entirely analogous to that of the positive case.

(iv) We note that there exists $k_0 \geq 1$ sufficiently large such that $g_{k_0}(0) = \beta^{-1}(\gamma - \lambda_{k_0}(0)) > 0$. Hence as in (iii) we will obtain at least two fixed points for g_{k_0}. By an argument essentially the same as in (iii), we will also obtain at least two for each g_k with $k \geq k_0$. The proof that $\Lambda_k \to \pm\infty$ is essentially the same as in (iii) as well. $\square$

Finally, for completeness' sake, in case (iv) we note that if $\lambda_k(\alpha)$ is not concave, then it is not really possible to define analogues of γ^* and γ^{**} for $k > 1$ (see

Lemmata 6.2.8 and 6.2.9 and Remark 6.2.11). If in fact it turns out $\lambda_k(\alpha)$ is concave and C^1 (which except in special cases requires Ω to be connected), then γ_k^* and γ_k^{**} should be defined as follows. For $\beta \in (-\lambda_k{}'(0), 0)$, define

$$\gamma_k^* := \sup\{\gamma > 0 : \beta^{-1}(\gamma - \lambda_k(\alpha)) = \alpha \text{ for some } \alpha \in \mathbb{R}\}. \tag{6.3.1}$$

Similarly, for $\beta \in (-\infty, -\lambda_n{}'(0))$, define

$$\gamma_k^{**} := \sup\{\gamma > 0 : \beta^{-1}(\gamma - \lambda_k(\alpha)) = \alpha \text{ for some } \alpha \in \mathbb{R}\}. \tag{6.3.2}$$

(Recall that $\lambda_1{}'(0) = \sigma(\partial\Omega)/|\Omega|$.) Under these assumptions g_k has at least one fixed point, and hence there is at least one Wentzell eigenvalue corresponding to λ_k, if $\beta \in (-\lambda_k{}'(0), 0)$ and $\gamma \in (0, \gamma_k^*]$, or if $\beta \in (-\infty, -\lambda_k{}'(0))$ and $\gamma \in (0, \gamma_k^{**}]$; while there are no fixed points if $\beta \in (-\lambda_k{}'(0), 0)$ and $\gamma \in (\gamma_k^*, \infty)$, $\beta = -\lambda_k{}'(0)$ or if $\beta \in (-\infty, -\lambda_k{}'(0))$ and $\gamma \in (\gamma_k^{**}, \infty)$.

6.4. Variational and monotonicity properties

Here we look at further results concerning the Wentzell eigenvalues (mostly the principal ones), with particular reference to variational, continuity and monotonicity properties. Let $\Omega \subset \mathbb{R}^N$ be a fixed bounded, Lipschitz domain throughout, not necessarily connected; when dealing with $\beta < 0$ assume in addition that Ω is of class $C^{1,1}$ if compactness of the resolvent is desired.

We start with some elementary variational theory. We fix $\beta, \gamma \in \mathbb{R} \setminus \{0\}$. The first result is essentially a weak version of the fixed point argument.

Lemma 6.4.1. *Suppose* Λ *is an eigenvalue of* (1.1.3) *with corresponding eigenfunction* ψ. *Then* $\Lambda \neq 0$ *if and only if*

$$\Lambda = \frac{\int_\Omega |\nabla\psi|^2 + \int_{\partial\Omega} \gamma\psi^2 \,\frac{d\sigma}{\beta}}{\int_\Omega \psi^2\,dx + \int_{\partial\Omega} \psi^2 \,\frac{d\sigma}{\beta}}, \tag{6.4.1}$$

while $\Lambda = 0$ *if and only if* γ/β *is an eigenvalue* $\xi < 0$ *of the Steklov problem*

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \xi u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{6.4.2}$$

The variational theory for the Steklov problem is similar to – and to an extent can be deduced from – that of the Robin problem in Section 1.3, with $\lambda = 0$ and $\xi = \alpha < 0$.

Proof. We have $\Lambda = \Lambda(\Omega, \beta, \gamma) = \lambda_k(\Omega, \frac{\gamma - \Lambda}{\beta})$ for some $k \geq 1$, with ψ being an eigenfunction of λ_k. This means that

$$\Lambda = \lambda_k = \frac{\int_\Omega |\nabla \psi|^2 \, dx + \int_{\partial\Omega} \frac{\gamma - \Lambda}{\beta} \psi^2 \, d\sigma}{\int_\Omega \psi^2 \, dx}. \tag{6.4.3}$$

It is easy to see by rearranging that this is equivalent to (6.4.1) if and only if $\Lambda \neq 0$. If $\Lambda = 0$, then from (6.4.3) we see that γ/β and ψ are variational solutions to the problem (6.4.2). (Equivalently, we could work in the classical framework and substitute $\Lambda = 0$ into (1.1.3) interpreted classically.) Similarly, if γ/β solves (6.4.2) with corresponding eigenfunction ψ, then we see immediately that it solves the corresponding Wentzell problem with eigenvalue given by $\Delta u = 0u = 0$ (in either the classical or the variational sense). $\square$

If $\beta > 0$ then we can still obtain a corresponding variational characterisation for Λ_1 as follows.

Proposition 6.4.2. *Given Ω, $\beta > 0$ and $\gamma \neq 0$, suppose Λ_1 is the first eigenvalue of* (1.1.3). *Then*

$$\Lambda_1 = \inf_{u \in H^1(\Omega)} Q_W(u) := \inf_{u \in H^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx + \int_{\partial\Omega} \gamma u^2 \, \frac{d\sigma}{\beta}}{\int_\Omega u^2 \, dx + \int_{\partial\Omega} u^2 \, \frac{d\sigma}{\beta}}, \tag{6.4.4}$$

with the infimum being attained by any eigenfunction.

Proof. It is immediate from Lemma 6.4.1 that Λ_1 is no smaller than this infimum. To prove there is equality, suppose for a contradiction that there exists $u \in H^1(\Omega)$ with $\Lambda_1 > Q_W(u) =: \lambda \in \mathbb{R}$. Rearranging the expression $\lambda = Q_W(u)$ and using the assumption that $\lambda < \Lambda_1$ gives

$$\begin{aligned} \lambda = \frac{\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \frac{\gamma - \lambda}{\beta} u^2 \, d\sigma}{\int_\Omega u^2 \, dx} &\geq \frac{\int_\Omega |\nabla u|^2 + \int_{\partial\Omega} \frac{\gamma - \Lambda_1}{\beta} u^2 \, d\sigma}{\int_\Omega u^2 \, dx} \\ &\geq \lambda_1\big(\Omega, \frac{\gamma - \Lambda_1}{\beta}\big) = \Lambda_1, \end{aligned}$$

a contradiction. It is clear from Lemma 6.4.1 that the infimum is attained by any eigenfunction. $\square$

Remark 6.4.3. (i) Observe that the Rayleigh-type quotient in (6.4.4) is of the form

$$\frac{Q_\gamma(u, u|_{\partial\Omega})}{\|(u, u|_{\partial\Omega})\|^2_{L^2(\overline{\Omega}, d\mu)}},$$

where Q_γ is the form from Theorem 5.2.1, and $L^2(\overline{\Omega}, d\mu) \ni (u, u|_{\partial\Omega})$ is the weighted space $L^2(\Omega, dx) \oplus L^2(\partial\Omega, \frac{d\sigma}{\beta})$ from Section 5.2.

(ii) We cannot give a similar characterisation of the principal eigenvalue(s) when $\beta < 0$, since then the form is not semi-bounded. See also [**14**], which discusses a problem of a very similar flavour.

As in the Robin case, the eigenvalues are in general "smooth" functions of the parameters β and γ. This is fairly easy to see by elementary means, but as with the Robin problem we will use the general and powerful theory in [**72**, Chapter VII]. Actually, this would allow us to cover without too much trouble the degenerate cases $\beta = 0$ or $\gamma = 0$ as well.

Lemma 6.4.4. *Denote by $\Delta_2^W(\beta,\gamma)$ the operator given by* (5.1.4) *with $p = 2$ for given $\beta, \gamma \neq 0$ and fix $\beta_0, \gamma_0 \neq 0$. Then*

(i) *the family of operators $\Delta_2^W(\beta_0,\gamma)$ is holomorphic of type (A) in the sense of* [**72**] *with respect to the parameter γ, for all $\gamma \in \mathbb{R}$;*

(ii) *$\Delta_2^W(\beta,\gamma_0)$ is locally holomorphic of type (A) with respect to $\beta \in \mathbb{R}$.*

Recall that when considering (5.1.4) we use the measure $dx \oplus d\sigma$ if $\beta < 0$ (see Remark 5.1.1), but this makes no difference for our purposes here.

Proof. We will use [**72**, Theorem VII.2.6]. First note that $\mathcal{D} := \mathcal{D}(\Delta_2^W(\beta,\gamma))$ is independent of β, γ. Given $\beta_0, \gamma_0 \in \mathbb{R} \setminus \{0\}$, we write

$$Tu := \Delta_2^W(\beta_0,\gamma_0)u = (\Delta u, -\beta_0 \frac{\partial u}{\partial \nu} - \gamma_0 u)$$

for $u \in \mathcal{D} = \mathcal{D}(T) = \mathcal{D}(\Delta_2^W(\beta,\gamma))$. Then T is closed by Theorem 5.2.1 if $\beta > 0$ or by Theorem 5.3.1 if $\beta < 0$.

(i) We define a new operator by $T^{(1)}(u, f) = (0, -f)$, $\mathcal{D}(T^{(1)}) = L^2(\Omega) \oplus L^2(\partial\Omega) \supset \mathcal{D}$. (Here we are following the notation of [**72**].) Since $T^{(1)}$ is bounded, it is certainly T-bounded with bound 0. By [**72**, Theorem VII.2.6] (see also Remark VII.2.7 there),

$$T(\gamma) := T + \gamma T^{(1)}$$

defined as a family of operators on $\mathcal{D}$, already closed by Theorems 5.2.1 and 5.3.1, is holomorphic of type (A) for all $\gamma \in \mathbb{R}$.

(ii) Similarly, define $T^{(2)}u = (0, -\frac{\partial u}{\partial \nu})$ for $u \in \mathcal{D}$. Clearly $T^{(2)}$ is T-bounded, with T-bound no larger than β_0. Hence for $|\beta| < \beta_0^{-1}$, the already closed family of operators

$$T(\beta) = T + \beta T^{(2)}$$

on $\mathcal{D}$ is holomorphic of type (A) for $\beta \in B(\beta_0, \beta_0^{-1})$. This gives us local holomorphicity for $\beta \in \mathbb{R} \setminus \{0\}$. $\square$

Proposition 6.4.5. *The following statements are true.*

(i) *Given $\beta > 0$, every Wentzell eigenvalue $\Lambda = \Lambda(\gamma)$ of* (1.1.3) *depends continuously on $\gamma \in \mathbb{R}$.*

(ii) *Given $\gamma \neq 0$, every eigenvalue $\Lambda = \Lambda(\beta)$ of* (1.1.3) *depends continuously on $\beta > 0$.*

(iii) *If Ω is connected and Λ is a principal eigenvalue, the functions in (i) and (ii) are analytic.*

Proof. As we have already seen, considering the eigenvalues of (1.1.3) is equivalent to considering the eigenvalues of Δ_2^W (or $\Delta_{H^1}^W, \dots$). With this in mind parts (i) and (ii) follow from [**72**, Theorem VII.1.8]. For, since $\beta > 0$, Δ_2^W is a self-adjoint operator on a Hilbert space (see Remark 6.2.6(i)); hence by [**72**, Section VII.3.1] every eigenvalue Λ is an analytic function of γ (resp. β), with the possible exception of "splitting points", which will not affect continuity.

For part (iii), using the same theory as in (i) and (ii) we only have to rule out the "splitting" behaviour. Since Ω is connected, we know that every principal eigenvalue has one-dimensional eigenspace for all $\beta, \gamma \neq 0$ (this actually follows from our definition; see also Lemma 6.1.3); hence splitting is impossible (cf. Theorem 1.3.1(vi)). $\square$

Remark 6.4.6. It is a property of holomorphic families of type (A) that *all* members of the family have compact resolvent, or else none do [**72**, Theorem VII.2.4]. This opens up the following potential method for obtaining compactness of the resolvent for Δ_2^W when $\beta < 0$ and Ω is only Lipschitz, not $C^{1,1}$. The idea is, first, to obtain compactness of the resolvent of $\Delta_2^W(\beta_0, \gamma)$ for $\beta_0 < 0$ very small. This would use the method of Section 5.3, but only requires that $\frac{\partial u}{\partial \nu}$ is relatively bounded with respect to the Dirichlet Laplacian (not necessarily with bound zero), which can be shown for Lipschitz domains; β_0 then replaces the perturbation parameter ε. Using the holomorphicity argument of Lemma 6.4.4, we then obtain that the *closure* of Δ_2^W has compact resolvent for all β in a neighbourhood of β_0. We can then extend this "local" property to cover all $\beta < 0$ as we did in Lemma 6.4.4. So we conclude that the closure of $\Delta_2^W(\beta,\gamma)$ has compact resolvent for all $\beta < 0$ if Ω is Lipschitz. Of course, the problem in the first place was that we do not know *a priori* Δ_2^W is closed in general, even though it is extremely likely (see Section 5.3).

Recall that $t\Omega = \{tx \in \mathbb{R}^N : x \in \Omega\}$ (see (1.3.5)). It is easy to use the variational characterisation (6.4.4) to show that $\Lambda_1(t\Omega,\beta,\gamma)$ decreases (strictly) monotonically as t increases for fixed $\beta,\gamma > 0$ (compare with Lemma 1.3.7 and [**22**, Lemma 4.1]).

Proposition 6.4.7. *Suppose $\Omega \subset \mathbb{R}^N$ is bounded, Lipschitz and $\beta > 0$ and $\gamma > 0$ are fixed. Then the function $t \mapsto \Lambda_1(t\Omega,\beta,\gamma)$ is strictly decreasing on $(0,\infty)$.*

Proof. By rescaling if necessary it suffices to prove that $\Lambda_1(\Omega) := \Lambda_1(\Omega,\beta,\gamma) < \Lambda_1(t\Omega,\beta,\gamma) =: \Lambda_1(t\Omega)$ if $t \in (0,1)$. So fix $t \in (0,1)$ and let ψ_t be an eigenfunction associated with $\Lambda_1(t\Omega)$. For $x \in \Omega$, set $\varphi(x) := \psi_t(tx)$. Then $\varphi \in H^1(\Omega)$ with $\nabla\varphi(x) = t\nabla\psi_t(tx)$, and

$$\begin{aligned}
\Lambda_1(\Omega) \le Q_W(\varphi) &= \frac{\int_\Omega |\nabla\varphi|^2\,dx + \int_{\partial\Omega}\gamma\varphi^2\,\frac{d\sigma}{\beta}}{\int_\Omega \varphi^2\,dx1 + \int_{\partial\Omega}\varphi^2\,\frac{d\sigma}{\beta}}\\
&= \frac{t^2\int_{t\Omega}|\nabla\psi_t|^2\,dx + t\int_{\partial(t\Omega)}\gamma\psi_t^2\,\frac{d\sigma}{\beta}}{\int_{t\Omega}\psi_t^2\,dx + t\int_{\partial(t\Omega)}\psi_t^2\,\frac{d\sigma}{\beta}}\\
&\le \frac{t\Big(t\int_{t\Omega}|\nabla\psi_t|^2\,dx + \int_{\partial(t\Omega)}\gamma\psi_t^2\,\frac{d\sigma}{\beta}\Big)}{t\Big(\int_{t\Omega}\psi_t^2\,dx + \int_{\partial(t\Omega)}\psi_t^2\,\frac{d\sigma}{\beta}\Big)} < \Lambda_1(t\Omega)
\end{aligned}$$

since $t < 1$ and ψ_t is not constant in Ω. $\square$

CHAPTER 7

Inequalities for the Wentzell Laplacian

Having established the spectral theory for the Wentzell Laplacian in Chapters 5 and 6, which for the most part turns out to be either very similar to, or dependent on, that of the Robin problem, we now wish to establish similar isoperimetric estimates for its eigenvalues. This is generally easy to do using the same type of elementary techniques used in Sections 6.2 and 6.3.

7.1. On the principal eigenvalues

We first seek to establish the main Faber-Krahn-type inequality for the principal eigenvalues of (1.1.3). The optimising domain will again be the ball. What makes this somewhat more interesting is the multitude of principal eigenvalues to consider, depending on the sign of β and γ in (1.1.3). Unfortunately, part of what we prove depends on Conjecture 3.4.1. Our main theorem is the following. The cases and notation are the same as in Section 6.2. Note that we allow Ω to be disconnected, except in case (iv) (see Remark 6.3.2). For (iii) and (iv), we can replace the $C^{1,1}$ assumption with the alternate assumptions listed in Remark 5.3.3.

Theorem 7.1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, Lipschitz domain and B be a ball having the same measure as Ω. Then*

(i) *if β, $\gamma > 0$, then $\Lambda_1(B) \leq \Lambda_1(\Omega)$.*

If in addition Ω is of class $C^{1,1}$, then

(iii) *if β, $\gamma < 0$, then $\Lambda_1^+(B) \leq \Lambda_1^+(\Omega)$;*

(iv) *if $\beta < 0$, $\gamma > 0$, Ω is connected and Ω and B have two positive principal eigenvalues, then $\Lambda_1^-(\Omega) \leq \Lambda_1^-(B) \leq \Lambda_1^+(B) \leq \Lambda_1^+(\Omega)$.*

In all the above inequalities, B is the unique domain (up to translation) for which there is equality.

Remark 7.1.2. Suppose in addition to the assumptions of Theorem 7.1.1 that Conjecture 3.4.1 holds. Using the case numbers from Theorem 6.2.1, we have

(ii) if $\beta > 0$, $\gamma < 0$, then $\Lambda_1(\Omega) \le \Lambda_1(B)$;

(iii) if β, $\gamma < 0$, then $\Lambda_1^-(B) \le \Lambda_1^-(\Omega)$;

(iv) if $\beta < 0$, $\gamma > 0$, Ω is connected and Ω and B have two negative eigenvalues, then these satisfy

$$\Lambda_1^-(B) \le \Lambda_1^-(\Omega) \le \Lambda_1^+(\Omega) \le \Lambda_1^+(B).$$

In all the above inequalities B is, up to translation, the unique domain for which there is equality.

Implicit in the above statements about case (iv) is the following statement about the borderline cases where there is only one principal eigenvalue. In the notation of Section 6.2, we wish to know how γ^* and γ^{**} depend on Ω. We assume that Ω is connected here, since γ^* and γ^{**} are in general only well-defined in this case. Before we state the result, we observe that by the isoperimetric inequality (see [**13, 25**]), $-\frac{\sigma(\partial\Omega)}{|\Omega|} \le -\frac{\sigma(\partial B)}{|B|} < 0$, with equality if and only if Ω is a ball.

Proposition 7.1.3. *Under the assumptions of Theorem 7.1.1, if in addition Ω is of class $C^{1,1}$ and connected, then for any given $\beta \in (-\frac{\sigma(\partial B)}{|B|}, 0)$, we have $\gamma^*(B,\beta) \le \gamma^*(\Omega,\beta)$, with equality if and only if Ω is a ball.*

Remark 7.1.4. In Proposition 7.1.3, if in addition Conjecture 3.4.1 holds, if $\beta \in (-\infty, -\frac{\sigma(\partial\Omega)}{|\Omega|})$, then $\gamma^{**}(\Omega,\beta) \le \gamma^{**}(B,\beta)$, again with equality if and only if Ω is a ball.

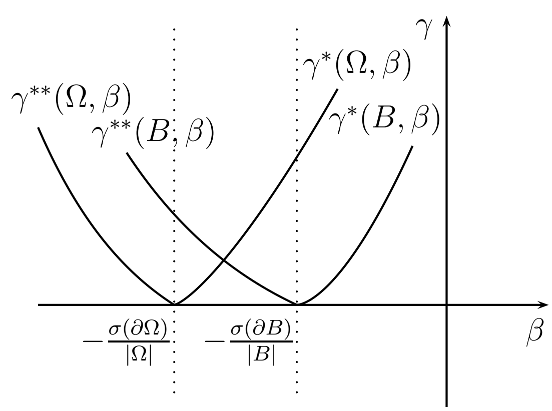

FIGURE 7.1. The dependence of γ^*, γ^{**} on β and the domain.

Remark 7.1.5. (i) In other words, Proposition 7.1.3 and Remark 7.1.4 say that B is the essentially unique minimiser (with respect to set inclusion) of the set of values of β, γ for which (1.1.3) has two positive principal eigenvalues, and maximiser of the set for which (1.1.3) has two negative principal eigenvalues.

(ii) Our earlier comment (Remark 6.2.4) that the assumption on the smoothness of Ω is only to guarantee the compactness of the associated operator still applies in this section (see also Remark 5.3.3).

Before we give the proof of these results, we fix our notation. We will return to writing $\lambda_1(\Omega, \alpha)$ rather than just $\lambda_1(\alpha)$ for the first Robin eigenvalue of Ω with boundary parameter α. If there is just one principal Wentzell eigenvalue, we will denote it by $\Lambda_1(\Omega)$, and we will let $\alpha_\Omega := \beta^{-1}(\gamma - \Lambda_1(\Omega))$, so that $\Lambda_1(\Omega) \equiv \lambda_1(\Omega, \alpha_\Omega)$. Importantly, we can make this identification even if Ω is not connected; see Lemma 6.1.3 and Remark 6.2.12. We will thus use the two interchangeably without further comment. Similarly, if there are two principal eigenvalues, we will call them $\Lambda_1^-(\Omega) < \Lambda_1^+(\Omega)$. We will set $\alpha_\Omega^- := \beta^{-1}(\gamma - \Lambda_1^-(\Omega))$ so that $\Lambda_1^-(\Omega) \equiv \lambda_1(\Omega, \alpha_\Omega^-)$, and do the same for α_Ω^+, and the same again for B.

In this notation, the Faber-Krahn inequality for the Robin Laplacian, Theorem 2.1.1, tells us that

$$\lambda_1(B, \alpha) \leq \lambda_1(\Omega, \alpha) \tag{7.1.1}$$

for all $\alpha > 0$, with equality in (7.1.1) only if $\Omega = B$ in the sense of Remark 1.3.2 by [**22**, Theorem 1.1] (see Theorem 3.1.2). Similarly, for the negative eigenvalues, Conjecture 3.4.1 tells us that $\lambda_1(\Omega, \alpha) \leq \lambda_1(B, \alpha)$ for all $\alpha < 0$, with equality if and only if $\Omega = B$.

Finally, bearing in mind Remark 7.1.5(ii), we will not make any explicit statements about the regularity of Ω we are assuming in the proofs, having already stated these above.

Proof of Theorem 7.1.1 and Remark 7.1.2. (i) Using the Robin Faber-Krahn inequality in the form (7.1.1) when $\alpha = \alpha_\Omega$, we get $\lambda_1(\Omega, \alpha_\Omega) \geq \lambda_1(B, \alpha_\Omega)$ (where we recall $\lambda_1(B, \alpha_\Omega)$ is the first Robin eigenvalue of B with boundary parameter α_Ω). Suppose for a contradiction that $\Lambda_1(\Omega) < \Lambda_1(B)$. Since $\beta > 0$,

$$\alpha_\Omega = \beta^{-1}(\gamma - \Lambda_1(\Omega)) > \beta^{-1}(\gamma - \Lambda_1(B)) = \alpha_B.$$

By monotonicity of $\lambda_1(B, \alpha)$ with respect to α, $\lambda_1(B, \alpha_\Omega) > \lambda_1(B, \alpha_B)$, and so

$$\Lambda_1(B) < \lambda_1(B, \alpha_\Omega) \leq \lambda_1(\Omega, \alpha_\Omega) = \Lambda_1(\Omega) < \Lambda_1(B),$$

a contradiction.

(ii) The proof is similar to (i), only now we use Conjecture 3.4.1 to obtain

$$\lambda_1(\Omega, \alpha_\Omega) \leq \lambda_1(B, \alpha_\Omega) < 0.$$

Supposing that $\lambda_1(B, \alpha_B) < \lambda_1(\Omega, \alpha_\Omega)$, it follows since $\beta > 0$ that $\alpha_\Omega < \alpha_B$, and so $\lambda_1(B, \alpha_\Omega) \leq \lambda_1(B, \alpha_B)$, again giving us a contradiction.

(iii) In this case there are two inequalities, one for each principal eigenvalue $\lambda_1(\Omega, \alpha_\Omega^-) < 0 < \lambda_1(\Omega, \alpha_\Omega^+)$. For the positive eigenvalue, using (7.1.1) when $\alpha = \alpha_B^+$, and since $\beta > 0$,

$$\beta^{-1}(\gamma - \lambda_1(\Omega, \alpha_B^+)) \geq \beta^{-1}(\gamma - \lambda_1(B, \alpha_B^+)) = \alpha_B^+.$$

By Lemma 6.2.7 applied to Ω, $\alpha_B^+ \in [\alpha_\Omega^-, \alpha_\Omega^+]$ and in particular $\alpha_B^+ \leq \alpha_\Omega^+$. (Note that Lemma 6.2.7 remains valid with the same proof if Ω is not connected, since g_1 is still strictly increasing and strictly concave; see Remark 1.3.2.) By definition, this means

$$\beta^{-1}(\gamma - \lambda_1(B, \alpha_B^+)) \leq \beta^{-1}(\gamma - \lambda_1(\Omega, \alpha_\Omega^+)).$$

Hence $\Lambda_1^+(B) \leq \Lambda_1^+(\Omega)$. For the negative eigenvalue, using Conjecture 3.4.1,

$$\alpha_\Omega^- = \beta^{-1}(\gamma - \lambda_1(\Omega, \alpha_\Omega^-)) \leq \beta^{-1}(\gamma - \lambda_1(B, \alpha_\Omega^-)).$$

Applying Lemma 6.2.7 as above we see that $\alpha_\Omega^- \in [\alpha_B^-, \alpha_B^+]$ and in particular $\alpha_B^- \leq \alpha_\Omega^-$. Hence as above $\Lambda_1^-(B) \leq \Lambda_1^-(\Omega)$.

(iv) Given $\beta < 0$, $\gamma > 0$, assume that both Ω and B actually have two eigenvalues. Noting that the sign of γ played no role in (iii) except to determine the sign of the appropriate α's and Λ's, we may repeat almost verbatim the argument in (iii) to reach the conclusion of (iv).

Finally, we prove that the ball is the unique minimiser, or maximiser as appropriate, of all the inequalities listed in Theorem 7.1.1 and Remark 7.1.2. The proof is the same in all cases, and an immediate consequence of the sharpness of the Robin Faber-Krahn inequality.

Suppose, in any of the cases, that $\Lambda_1(\Omega) = \Lambda_1(B)$. Set

$$\alpha := \beta^{-1}(\gamma - \Lambda_1(\Omega)) = \beta^{-1}(\gamma - \Lambda_1(B)).$$

Then both problems, i.e. those on Ω and B, have the same Robin boundary parameter α. Hence we may directly apply [**22**, Theorem 1.1] if $\Lambda_1(\Omega) > 0$ and hence $\alpha > 0$, or Conjecture 3.4.1 otherwise, to conclude $\Omega = B$ after a translation. $\square$

Proof of Proposition 7.1.3 and Remark 7.1.4. First, fix $\beta \in (-\frac{\sigma(\partial B)}{|B|}, 0)$. We wish to prove that $\gamma^*(\Omega, \beta) \geq \gamma^*(B, \beta)$. Let α_B^* be the unique fixed point associated with $\gamma^*(B, \beta)$ on B. Then by (7.1.1) applied when $\alpha = \alpha_B^* > 0$, $0 < \lambda_1(B, \alpha_B^*) \leq \lambda_1(\Omega, \alpha_B^*)$, that is,

$$0 < \alpha_B^* = \beta^{-1}(\gamma^*(B, \beta) - \lambda_1(B, \alpha_B^*)) \leq \beta^{-1}(\gamma^*(B, \beta) - \lambda_1(\Omega, \alpha_B^*)).$$

Hence by Lemma 6.2.7, there must be at least one fixed point for $\gamma^*(B, \beta)$ on Ω, and so $\gamma^*(B, \beta) \leq \gamma^*(\Omega, \beta)$ by Lemma 6.2.9. For $\beta \in (-\infty, -\frac{\sigma(\partial\Omega)}{|\Omega|})$ the argument that $\gamma^{**}(B, \beta) \leq \gamma^{**}(\Omega, \beta)$ is essentially the same, only using Conjecture 3.4.1 in place of (7.1.1). Finally, if Ω is not a ball, then the inequality (7.1.1) is sharp for all $\alpha > 0$ and we obtain the strict inequality $\beta^{-1}(\gamma^*(B, \beta) - \lambda_1(\Omega, \alpha^*(B)) > \alpha^*(B)$ and so Lemmata 6.2.7 and 6.2.9 yield $\gamma^*(\Omega, \beta) < \gamma^*(B, \beta)$ (and similarly for γ^{**}). $\square$

7.2. On the other eigenvalues

Let us start with the observation that the arguments used in Section 7.1 to prove Theorem 7.1.1 are in a sense very generic: they rely only on the abstract properties of the Robin eigenvalue $\lambda_1(\Omega, \alpha)$ as a function of α, together with the Robin Faber-Krahn inequality. As such, we can easily generalise such arguments to the higher Wentzell eigenvalues $\Lambda_k(\Omega)$. In what follows we will only consider the case $\beta, \gamma > 0$, since this is the most important case and since considerably less can be said about the others as they all rely to a varying degree on the Robin eigenvalues $\lambda_k(\Omega, \alpha)$ when $\alpha < 0$.

So fix $\beta, \gamma > 0$ arbitrary. Our primary goal is to prove the following theorem, which basically says that the minimisation problems for the Robin Laplacian when $\alpha > 0$ and Wentzell Laplacian when $\beta, \gamma > 0$ are essentially the same. Recall that D_k is the disjoint union of k equal balls, $k \geq 2$.

Theorem 7.2.1. *Let $\beta, \gamma > 0$ and $k \geq 2$ be fixed, let $D \subset \mathbb{R}^N$ be a bounded Lipschitz domain, and let $D_k \subset \mathbb{R}^N$ be as in Section 4.3.*

(i) *Suppose that for every bounded Lipschitz $\Omega \subset \mathbb{R}^N$ we have*

$$\lambda_k(D, \alpha) \leq \lambda_k(\Omega, \alpha) \tag{7.2.1}$$

for all $\alpha \in (0, \gamma/\beta)$. Then

$$\Lambda_k(D, \beta, \gamma) \le \Lambda_k(\Omega, \beta, \gamma) \tag{7.2.2}$$

for all such Ω. Conversely, if (7.2.2) holds, then (7.2.1) holds for some $\alpha \in (0, \gamma/\beta)$.

(ii) *If (7.2.1) is sharp for all $\alpha \in (0, \gamma/\beta)$, then so is (7.2.2) for this β, γ. If (7.2.2) is sharp, then (7.2.1) holds and is sharp for some $\alpha \in (0, \gamma/\beta)$.*

(iii) *Suppose $\Omega \subset \mathbb{R}^N$ is bounded, Lipschitz. There exists $\alpha_\Omega > 0$ possibly depending Ω such that $\Lambda_k(\Omega, \beta, \gamma) > \Lambda_k(D_k, \beta, \gamma)$ for all β, γ with $\gamma/\beta < \alpha_\Omega$.*

(iv) *If for some k and N the conclusion of Theorem 4.3.1(ii) holds, then there does not exist a bounded, Lipschitz domain $D \subset \mathbb{R}^N$ such that $\Lambda_k(\Omega, \beta, \gamma) \ge \Lambda_k(D, \beta, \gamma)$ for all such Ω and all $\beta, \gamma > 0$.*

(v) *For any bounded, Lipschitz $\Omega \subset \mathbb{R}^N$ and any $\beta, \gamma > 0$, we have $\Lambda_2(\Omega, \beta, \gamma) \ge \Lambda_2(D_2, \beta, \gamma)$, with equality if and only if $\Omega = D_2$.*

In order to prove this theorem, we start with the observation that the fixed point α satisfying $0 < \Lambda_1(\Omega, \beta, \gamma) = \gamma - \alpha\beta$ is positive: $\alpha > 0$. This follows from Proposition 6.2.3(i) combined with Proposition 6.2.5(i). In particular, this gives the bound $\Lambda_1(\Omega, \beta, \gamma) < \gamma$ always, *independent of the volume of* Ω. This yields the following result, which is true in particular for D_k.

Lemma 7.2.2. *Suppose Ω has at least k connected components (c.c.s). Then $\Lambda_k(\Omega, \beta, \gamma) < \gamma$.*

Proof. We have $\Lambda_k(\Omega) \le \min\{\lambda_1(\widetilde{\Omega})\} < \gamma$, where the minimum is taken over all c.c.s $\widetilde{\Omega}$ of Ω. $\square$

Since β, γ are fixed we will now write $\Lambda_k(\Omega, \beta, \gamma) = \Lambda_k(\Omega)$ if there is no danger of confusion. The following lemma contains the core of the argument in the proof of Theorem 7.2.1.

Lemma 7.2.3. *Let $\beta, \gamma > 0$ be given and $U, V \subset \mathbb{R}^N$ bounded, Lipschitz.*

(i) *If $\Lambda_k(U) < \gamma$, then for $\alpha := (\gamma - \Lambda_k(U))/\beta$,*

$$\lambda_k(U, \alpha) \ge \lambda_k(V, \alpha) \tag{7.2.3}$$

implies

$$\Lambda_k(U) \geq \Lambda_k(V). \tag{7.2.4}$$

If the equality in (7.2.3) *is strict, then it is also strict in* (7.2.4).

(ii) *Suppose* $\Lambda_k(V) < \gamma$ *and let* $\alpha := (\gamma - \Lambda_k(V))/\beta$. *If* (7.2.4) *holds (resp. is strict), then* (7.2.3) *holds (resp. is strict) for this* α.

Proof. (i) Suppose that (7.2.3) holds but (7.2.4) fails. Using Theorem 6.3.1(i) and (7.2.3) respectively,

$$\begin{aligned}\Lambda_k(U) &= \lambda_k(U, \frac{\gamma - \Lambda_k(U)}{\beta}) \\ &\geq \lambda_k(V, \frac{\gamma - \Lambda_k(U)}{\beta}) \geq \lambda_k(V, \frac{\gamma - \Lambda_k(V)}{\beta}) = \Lambda_k(V),\end{aligned}$$

where the second inequality follows from Theorem 1.3.5 since $\gamma - \Lambda_k(U) \geq \gamma - \Lambda_k(V)$ by the contradiction assumption. Hence $\Lambda_k(U) \geq \Lambda_k(V)$, contradicting the assumption that (7.2.4) fails. Now suppose (7.2.3) is strict and the contradiction assumption becomes $\Lambda_k(U) \leq \Lambda_k(V)$. Since the first inequality in the above line of reasoning is now strict, we still obtain a contradiction as nothing else changes. Hence we cannot have equality in (7.2.4).

(ii) Now suppose that (7.2.4) holds and that (7.2.3) fails. Interchanging the roles of U and V, we may argue essentially exactly as in (i) to obtain the desired conclusion (and do similarly for strictness). □

Proof of Theorem 7.2.1. (i) Suppose D satisfies (7.2.1). Choose a minimising sequence $(\Omega_m)_{m\in\mathbb{N}}$ for Λ_k. By Lemma 7.2.2, we may assume $\Lambda_k(\Omega_m) < \gamma$ for all m, so that $(\gamma - \Lambda_k(\Omega_m))/\beta \in (0, \gamma/\beta)$ and thus (7.2.1) holds for these values of α. Fixing $m \in \mathbb{N}$, we may apply Lemma 7.2.3(i) with Ω_m in place of U and D in place of V to conclude $\Lambda_k(\Omega_m) \geq \Lambda_k(D)$. Since $(\Omega_m)_{m\in\mathbb{N}}$ was a minimising sequence, D must minimise $\Lambda_k(\Omega)$. For the converse, suppose D satisfies (7.2.2). Since $\Lambda_k(D) < \gamma$ by Lemma 7.2.2, it follows directly from Lemma 7.2.3(ii) that D satisfies (7.2.1) for $\alpha = (\gamma - \Lambda_k(D))/\beta$.

(ii) Sharpness in both directions now follows immediately from strictness of the inequalities in Lemma 7.2.3.

(iii) Fix $\Omega \neq D_k$. By Theorem 4.3.1(i), there exists $\alpha_\Omega > 0$ such that $\lambda_k(\Omega, \alpha) > \lambda_k(D_k, \alpha)$ for all $\alpha \in (0, \alpha_\Omega)$. If β, γ are fixed with $\gamma/\beta < \alpha_\Omega$, then we have

$\lambda_k(\Omega, \alpha) > \lambda_k(D_k, \alpha)$ for $\alpha = (\gamma - \Lambda_k(\Omega))/\beta$ in particular. Since also $\Lambda_k(D_k) < \gamma$ by Lemma 7.2.2, without loss of generality we may assume $\Lambda_k(\Omega) < \gamma$ (otherwise $\Lambda_k(\Omega) \geq \gamma > \Lambda_k(D)$ and we are done). But in this case it follows from Lemma 7.2.3(i) (with $\Omega = U$) that $\Lambda_k(\Omega) > \Lambda_k(D_k)$ anyway.

(iv) Let k and N be such that the conclusion of Theorem 4.3.1(ii) holds. By (iii) it suffices to show there exist $\beta, \gamma > 0$ and a domain Ω with $\Lambda_k(\Omega, \beta, \gamma) < \Lambda_k(D_k, \beta, \gamma)$. Choose Ω and $\alpha^* > 0$ such that $\lambda_k(\Omega, \alpha^*) < \lambda_k(D_k, \alpha^*)$.

Now we may write $\Lambda_k(D_k, \beta, \gamma) = \Lambda_1(D_k, \beta, \gamma) = \gamma - \alpha\beta$, where α satisfies $(\gamma - \lambda_1(D_k, \alpha))\beta = \alpha$. Since $\lambda_1(D_k, \alpha)$ is continuous and monotonic with respect to α, an elementary argument shows that by fixing β and varying γ, we may obtain every $\alpha > 0$ as a solution to $(\gamma - \lambda_1(D_k, \alpha))\beta = \alpha$ for some $\beta, \gamma > 0$. Now choose β, γ such that $\Lambda_k(D_k, \beta, \gamma) = \gamma - \alpha^*\beta$. For this β, γ, we may apply Lemma 7.2.3(i) with $U = D_k$ and $V = \Omega$ to conclude $\Lambda_k(D_k, \beta, \gamma) > \Lambda_k(\Omega, \beta, \gamma)$.

(v) This follows immediately from (i) and (ii) combined with Theorem 4.1.1. $\square$

7.3. A variant of Cheeger's inequality

Recall from Section 3.3 the Cheeger-type inequality Theorem 3.3.1, which in essence said that the first Robin eigenvalue $\lambda_1(\Omega, \alpha)$ could be bounded from below by a suitable combination of α and the Cheeger constant of Ω (see (3.3.1)).

We wish to establish an analogous inequality in the case of Wentzell boundary conditions. For this we will restrict our attention to the case β, $\gamma > 0$ in the boundary condition in (1.1.3). In what follows, as in Section 3.3 we will abbreviate $h(\Omega)$ as h if there is no danger of confusion.

Theorem 7.3.1. *The first eigenvalue $\Lambda_1(\Omega)$ of* (1.1.3) *on a fixed bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ satisfies*

$$\Lambda_1(\Omega) \geq \begin{cases} \frac{1}{4}h^2 & \text{if } h \leq \sqrt{\beta^2 + 4\gamma} - \beta; \\ p(h) & \text{if } h \geq \sqrt{\beta^2 + 4\gamma} - \beta, \end{cases} \tag{7.3.1}$$

where $p(h) := \gamma + \frac{\beta}{2}(\sqrt{(\beta + h)^2 - 4\gamma} - (\beta + h))$.

Remark 7.3.2. (i) The bound described in (7.3.1) is not as messy as it may appear. For $h \leq \sqrt{\beta^2 + 4\gamma} - \beta$, as long as $p(h)$ is well-defined (that is, $h \geq 2\sqrt{\gamma} - \beta$), we have $\frac{1}{4}h^2 \geq p(h)$, with equality only at $h = \sqrt{\beta^2 + 4\gamma} - \beta$. In fact the bound in (7.3.1) is a C^1 (but not C^2) monotonically increasing function of h (see Lemma 7.3.10 below). Moreover, as $h \to \infty$, $p(h) \to \gamma$ from below (Lemma 7.3.9), which is the best we can expect since we always have $\Lambda_1(\Omega) < \gamma$ (see Lemma 7.2.2).

(ii) One potential use for Theorem 7.3.1 is the following. Fixing β, γ, if we have a sequence of Lipschitz domains Ω_n of fixed volume, but with $h(\Omega_n) \to \infty$, by Theorem 7.3.1 we have $\Lambda_1(\Omega_n) \to \gamma$. For such a sequence the bound (7.3.1) gives a better estimate of $\Lambda_1(\Omega_n)$ than the Faber-Krahn inequality, Theorem 7.1.1, since if $|B| = |\Omega_n|$, then $\Lambda_1(B) < \gamma$ is fixed, and so eventually $\Lambda_1(\Omega_n) \geq p(h(\Omega_n)) > \Lambda_1(B)$. For completeness' sake, we will construct such a sequence in Example 7.3.3.

Example 7.3.3. We construct a sequence Ω_n for which $|\Omega_n| = 1$ for all n, but $h(\Omega_n) \to \infty$. Let Ω_n be a sequence of rectangles in $\mathbb{R}^2$ of length n and width $\frac{1}{n}$. Then by [**73**, Remark 13] the minimising domain for $h(\Omega_n)$, call it D_n, certainly contains the union of all largest balls (those of radius $\frac{1}{2n}$) in Ω_n. Thus ∂D_n and $\partial\Omega_n$ must coincide on the longer sides of the rectangle for all but a piece of length $\frac{1}{2n}$ on each end, so that $\sigma(D_n) \geq 2(n - 2\frac{1}{2n})$. Since $|D_n| \leq 1$, we get $h(\Omega_n) \geq 2(n - \frac{1}{n})$. In particular $h(\Omega_n) \to \infty$ as $n \to \infty$.

The proof of Theorem 7.3.1 is based on Theorem 3.3.1 together with the usual identification of the first Wentzell eigenvalue as the first eigenvalue of a suitable Robin problem. We divide the proof into a number of steps. Since from now on we will only be considering a fixed domain Ω, we shall abbreviate $\Lambda_1(\Omega)$, $\lambda_1(\Omega, \alpha)$ and $h(\Omega)$ as Λ_1, $\lambda_1(\alpha)$ and h, respectively.

Lemma 7.3.4. *Suppose* $h \geq \max\{0, 2\sqrt{\gamma} - \beta\}$. *Then* $\Lambda_1 \in (0, q(h)] \cup [p(h), \infty)$, *where* $q(h) := \gamma + \frac{\beta}{2}(-\sqrt{(\beta+h)^2 - 4\gamma} - (\beta + h))$.

Proof. Set $\alpha = (\gamma - \Lambda_1)/\beta$ and make the usual identification $\Lambda_1 = \lambda_1(\alpha)$. By Theorem 3.3.1, $\lambda_1(\alpha)$ satisfies (3.3.2). So we may substitute this value of α into the inequality $\lambda_1(\alpha) \geq h\alpha - \alpha^2$ and replace λ_1 by Λ_1 to obtain the quadratic relation

$$\Lambda_1^2 + (\beta h - 2\gamma + \beta^2)\Lambda_1 + \gamma^2 - \beta\gamma h \geq 0, \tag{7.3.2}$$

which is valid for all $h > 0$. The associated equation has real roots if $h \geq 2\sqrt{\gamma} - \beta$, and these are given by $q(h)$, $p(h)$. Bearing in mind $\Lambda_1 > 0$, the solution to the inequality (7.3.2) is thus $\Lambda_1 \in (0, q(h)] \cup [p(h), \infty)$. $\square$

Lemma 7.3.5. *If* $\Lambda_1 \le \gamma - \frac{1}{2}\beta h$*, then* $h \le \sqrt{\beta^2 + 4\gamma} - \beta$*.*

Proof. If $\alpha = (\gamma - \Lambda_1)/\beta$, then the condition $\frac{1}{2}h \le \alpha$ in (3.3.2) may be rewritten as $\Lambda_1 \le \gamma - \frac{1}{2}\beta h$.

In particular, combining this with Theorem 3.3.1 we get $\frac{1}{4}h^2 \le \gamma - \frac{1}{2}\beta h$. This is equivalent to the quadratic inequality $h^2 + 2\beta h - 4\gamma \le 0$, which has solution $h \in [-\sqrt{\beta^2 + 4\gamma} - \beta, \sqrt{\beta^2 + 4\gamma} - \beta]$. Since h must be positive, this shows that if $\lambda_1 \le \gamma - \frac{1}{2}\beta h$, then $h \in (0, \sqrt{\beta^2 + 4\gamma} - \beta]$. □

Lemma 7.3.6. *We have*

$$\Lambda_1 \ge \min\{\gamma - \frac{1}{2}\beta h, \frac{1}{4}h^2\}, \tag{7.3.3}$$

valid for any $h > 0$*.*

Proof. If $\Lambda_1 \le \gamma - \frac{1}{2}\beta h$, then by the proof of Lemma 7.3.5, $\frac{1}{4}h^2 \le \alpha$ and so by Theorem 3.3.1 $\Lambda_1 \equiv \lambda_1(\alpha) \ge \frac{1}{4}h^2$. Equation (7.3.3) now follows. □

Lemma 7.3.7. *If* $h \le \sqrt{\beta^2 + 4\gamma} - \beta$*, then* $\Lambda_1 \ge \frac{1}{4}h^2$*.*

Proof. If $h \in (0, \sqrt{\beta^2 + 4\gamma} - \beta]$, then working backwards through the proof of Lemma 7.3.5, h satisfies the inequality $h^2 + 2\beta h - 4\gamma \le 0$, or, rearranged, $\frac{1}{4}h^2 \le \gamma - \frac{1}{2}\beta h$. The assertion of the Lemma now follows from (7.3.3). □

Lemma 7.3.8. *If* $h > 0$ *also satisfies* $h \ge 2\sqrt{\gamma} - \beta$*, then* $\Lambda_1 \ge p(h)$*.*

Proof. We use (7.3.3) to show that the case $\Lambda_1 \le q(h)$ in Lemma 7.3.4 is impossible. So suppose in addition to the assumptions of the Lemma that $\Lambda_1 \le q(h)$.

If $\Lambda_1 \ge \gamma - \frac{1}{2}\beta h$, then

$$\gamma - \frac{1}{2}\beta h \le q(h) = \gamma - \frac{1}{2}\beta h - \frac{1}{2}\beta\,(\beta + \sqrt{(\beta + h)^2 - 4\gamma}),$$

an immediate contradiction. Similarly, if $\Lambda_1 \ge \frac{1}{4}h^2$, then $\frac{1}{4}h^2 \le q(h)$. Using the definition of q, and rearranging, we get

$$h^2 + 2\beta h + 2\beta^2 - 4\gamma \le -2\beta\sqrt{(\beta + h)^2 - 4\gamma}\ (\le 0),$$

but $h^2 + 2\beta h + 2\beta^2 - 4\gamma = (\beta + h)^2 - 4\gamma + \beta^2 > 0$ since we are assuming $\beta > 0$ and $h \ge 2\sqrt{\gamma} - \beta$. Hence we see the assumption $\Lambda_1 \le q(h)$ contradicts Lemma 7.3.6. □

Since $2\sqrt{\gamma} - \beta < \sqrt{\beta^2 + 4\gamma} - \beta$, Lemma 7.3.7 and Lemma 7.3.8 prove Theorem 7.3.1. It remains to prove the claims made in Remark 7.3.2(i).

Lemma 7.3.9. *The function $p : [2\sqrt{\gamma} - \beta, \infty) \to \mathbb{R}$ is a smooth, monotonically increasing function of h with $p(h) < \gamma$ and $p(h) \to \infty$ as $h \to \infty$.*

Proof. Observe that $p(h)$ has the form $p(h) = \gamma - \frac{\beta}{2} f(\beta + h)$, where $f(x) := x - \sqrt{x^2 - k}$ (k constant) is monotonically decreasing. Hence smoothness and monotonicity of $p(h)$ are immediate. The other claims follow since $0 < f(x) \to 0$ as $x \to \infty$. $\square$

Lemma 7.3.10. *We have $\frac{1}{4}h^2 \geq p(h)$, with equality only at $h = \sqrt{\beta^2 + 4\gamma} - \beta$, and at that point $p'(h) = \frac{1}{2}h$ and $p''(h) < \frac{1}{2}$.*

Proof. When $h = \sqrt{\beta^2 + 4\gamma} - \beta$, we have $p(h) = \frac{1}{4}h^2 = \gamma - \frac{1}{2}\beta h = \gamma + \frac{1}{2}\beta^2 - \frac{1}{2}\beta\sqrt{\beta^2 + 4\gamma}$.

Now set $F : [2\sqrt{\gamma} - \beta, \infty) \to \mathbb{R}$, $F(h) := \frac{1}{4}h^2 - p(h)$. Then F is C^∞ on $(2\sqrt{\gamma} - \beta, \infty)$, and a somewhat tedious calculation shows that

$$F'(h) = \frac{1}{2}h + \frac{1}{2}\beta\Big(1 - \frac{\beta + h}{\sqrt{(\beta + h)^2 - 4\gamma}}\Big)$$

is zero if and only if $h = \sqrt{\beta^2 + 4\gamma} - \beta$. Moreover, it can be shown (by an even more tedious calculation, which we will not do here) that

$$F''(\sqrt{\beta^2 + 4\gamma} - \beta) = \frac{1}{2} + \frac{2\gamma}{\beta^2},$$

which is positive for all β, $\gamma > 0$. $\square$

Appendix A

Notation and Background Results

Here we collect some important properties and results such as on operator theory or vector calculus of particular importance to us. This is of course not intended to be anything like a comprehensive treatment, and while we have tried to find precise references for the results we cite we will not be including any proofs.

A1. General remarks on notation

As far as possible, we have tried to keep our notation standard. Here we list a few slightly non-standard conventions we have used.

We use the notation $U \subset\subset V$, particularly when U and V are open, to indicate that U is compactly contained in $V \subset \mathbb{R}^N$; that is, $\overline{U}$ is compact and contained in $\operatorname{int} V$. If we write $U \subset V$, there may be equality; we may write $U \subseteq V$ if we wish to emphasise this possibility. Given open sets $U, V \subset \mathbb{R}^N$ with $U \subset V$, we define the interior and exterior boudaries of U relative to V by

$$\partial_i U := \partial U \cap V \qquad \text{and} \qquad \partial_e U := \partial U \cap \partial V, \tag{A1.1}$$

respectively, so that $\partial U = \partial_i U \cup \partial_e U$, and $\partial_i U \cap \partial_e U = \emptyset$.

We only work with two types of measure: N-dimensional volume (or Lebesgue measure) and $(N-1)$-dimensional surface (or Hausdorff) measure. If $U \subset \mathbb{R}^N$ is a set, then $|U|$ denotes its N-dimensional Lebesgue measure and $\sigma(U)$ its $(N-1)$-dimensional Hausdorff measure, scaled so that it coincides with $(N-1)$-dimensional Lebesgue measure on all sets where both are well-defined. (For a definition of Hausdorff measure, see [**56**, Section 2.10].)

By $L^p(X, d\mu)$ we understand the usual L^p-spaces, $1 \leq p \leq \infty$. In practice we usually have $X \subseteq \mathbb{R}^N$ and μ either Lebesgue or Hausdorff measure. We use $L^p_{\text{loc}}(X)$ to denote the set of all $f : X \to \mathbb{R}$ such that for all $x \in X$ there exists an open neighbourhood U_x of x such that $f \in L^p(U_x)$. Similarly, for $U \subseteq \mathbb{R}^N$ we understand $C(U)$ and $C^k(U)$ as, respectively, the space of continuous and k-times continuously differentiable functions $f : U \to \mathbb{R}$ (or $\mathbb{R}^N$). For $f \in C(U)$, we define the support of f, $\operatorname{supp} f$, to be the closure in U of the set $\{x \in U : f(x) \neq 0\}$. We also say $f \in C_c^k(U)$, $k \in \mathbb{N} \cup \{\infty\}$ if $f \in C^k(U)$ and $\operatorname{supp} f \subset\subset U$. The Hölder space $C^{0,\eta}(U)$, $\eta \in (0,1]$, is the space of all $f \in C(U)$ such that

$$\sup_{x,y \in U} \frac{|f(x) - f(y)|}{\|x - y\|^\eta} < \infty.$$

In particular, if $\eta = 1$, then $f \in C^{0,1}(U)$ is said to be *Lipschitz continuous*. The *Lipschitz constant* of f is the infimum of all constants K such that

$$|f(x) - f(y)| \leq K\|x - y\| \tag{A1.2}$$

for all x, $y \in U$. Also $C^{k,\eta}(U)$, $k \in \mathbb{N}$, is the space of all $C^k(U)$ functions f such that all kth order derivatives of f lie in $C^{0,\eta}(U)$. We note here the following very famous property of Lipschitz functions.

Theorem A1.1 (Rademacher's Theorem)**.** *Let $f : U \to \mathbb{R}$ be Lipschitz, where $U \subset \mathbb{R}^N$. Then f is (Fréchet) differentiable almost everywhere with respect to N-dimensional Lebsegue measure and $\|\nabla f\|_{L^\infty(U)} \leq K$, where K is any constant satisfying* (A1.2)*.*

Proof. Noting that every Lipschitz continuous function is absolutely continuous (see [**99**, Chapter 7]) on every line segment in U, the claim follows from [**99**, Theorem 7.18]. Alternatively, see [**89**, Chapter 3]. □

Following [**30**] (see p. 48 there), if X is a subset of $\mathbb{R}^N$ or a manifold, then we say a function $f : X \to \mathbb{R}$ is *piecewise continuous* if X may be subdivided into finitely many subdomains X_i such that $f|_{X_i} : X_i \to \mathbb{R}$ is continuous and for all $z \in \partial X_i$ and $X_i \ni x_n \to z$, $f(x_n) \to f(z)$ as $n \to \infty$. We will take piecewise-C^k to mean *continuous* functions whose derivatives of order up to $k - 1$ exist and are piecewise continuous.

Finally, given a function $f : X \to Y$ we use the notation $\operatorname{graph} f$ to stand for the set of all points $(x, f(x)) \in X \times Y$, that is, the graph of f.

A2. Classes of domains

Let $\Omega \subset \mathbb{R}^N$ be an open set, not necessarily bounded. If Ω is the set on which our problem is defined, then we will tend to call it a domain. We generally assume Ω is bounded (see Definition A2.5), although *not necessarily connected*. Our main definition of the regularity class of a domain follows Nečas [**91**] and Adams [**2**].

Definition A2.1. We will say a domain $\Omega \subset \mathbb{R}^N$ is of class C^k, resp. $C^{k,\eta}$, where $0 \leq k \leq \infty$ and $0 < \eta \leq 1$, if for every point $z \in \partial\Omega$ there exists an open neighbourhood $U_z \subset \mathbb{R}^N$ containing z and a local coordinate system $(x_1, \dots, x_N)$ such that $z = 0$ in this coordinate system and inside U_z, $\partial\Omega$ is the graph of some C^k, resp. $C^{k,\eta}$, function f of $N-1$ variables $x_1, \dots x_{N-1}$. We also require $\Omega \cap U_z = \{x = (x_1, \dots, x_N) \in U_z : x_N < f(x_1, \dots, x_{N-1})\}$ (that is, Ω is locally "on one side" of $\partial\Omega$). If Ω is of class $C^{0,1}$, we will say Ω is *Lipschitz*.

Remark A2.2. (i) We write C^k in preference to $C^{k,0}$.

(ii) There is a common alternative definition of C^k, $C^{k,\eta}$. A domain can be called of class C^k (or $C^{k,\eta}$) if for every $z \in \partial\Omega$ there exist an open neighbourhood U_z of z and C^k (or $C^{k,\eta}$) transformations $\Phi : U_z \to \mathbb{R}^N$, $\Phi^{-1} : \Phi(U_z) \to U_z$ such that Φ maps $\partial\Omega$ onto the plane $x_N = 0$ in $\Phi(U_z) \subset \mathbb{R}^N$ (where we write $x = (x_1, \dots, x_N) \in \mathbb{R}^N$). This definition is sometimes thought of as the "manifold" approach. For $C^{k,\eta}$ domains, $k \geq 1$, $\eta \in [0, 1]$ it can be shown using the implicit function theorem that the two definitions are equivalent (see [**64**, Section 1.2]). However, for $C^{0,\eta}$ domains they are not (see [**57**, Appendix II] or again [**64**, Section 1.2]). In particular, this definition of Lipschitz is weaker than the one we will use, given in Definition A2.1. For a detailed (and rather entertaining) discussion of various different definitions of classes of domains and domain properties see [**57**].

Definition A2.3. We will say a domain $\Omega \subset \mathbb{R}^N$ is *piecewise-C^k* (for $1 \leq k \leq \infty$) if Ω is Lipschitz and $\partial\Omega$ is locally the graph (in the sense of Definition A2.1) of a piecewise-C^k function.

Example A2.4. All polyhedral domains in $\mathbb{R}^N$ are piecewise-C^∞.

Definition A2.5. We say a domain $\Omega \subset \mathbb{R}^N$ is *bounded* if there exists $r > 0$ such that $\Omega \subset B(0, r)$.

A3. Weak derivatives and Sobolev spaces.

Here we introduce our notation concerning weak derivatives and our characterisations of various Sobolev spaces. For a proper treatment of this subject, we refer to [**2, 88**]. Alternatively, almost every standard reference on partial differential equations has a chapter on Sobolev spaces; see for example [**48**, Chapter 5], [**59**, Chapter 7], [**64**, Chapter 1], [**84**, Chapter 1] or [**98**, Section 6.4].

Let $\Omega \subseteq \mathbb{R}^N$ be open, not necessarily bounded. We say a locally integrable function $u : \Omega \to \mathbb{R}$ has a weak (or generalised) αth partial derivative g of order $|\alpha| := \sum_i \alpha_i$, where $\alpha = (\alpha_1, \ldots \alpha_k) \in \mathbb{N}^N$, and write $g = \frac{\partial^{|\alpha|} u}{\partial x^\alpha} := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}$, if

$$\int_\Omega u \, \frac{\partial^{|\alpha|} \varphi}{\partial x^\alpha} \, dx = (-1)^{|\alpha|} \int_\Omega \varphi g \, dx \tag{A3.1}$$

for all test functions $\varphi \in C_c^\infty(\Omega)$. If such a function u has all first order weak partial derivatives, we denote its gradient vector by $\nabla u := (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N})$. We sometimes use $D^\alpha u$ to denote the αth weak partial derivative of u. Weak partial derivatives are unique up to sets of zero N-Lebesgue measure.

Fix $k \in \mathbb{N}$ and $1 \le p < \infty$. For $\Omega \subseteq \mathbb{R}^N$ we set $W^{1,1}_{\mathrm{loc}}(\Omega)$ to be

$$\{u \in L^1_{\mathrm{loc}}(\Omega) : \frac{\partial u}{\partial x_i} \text{ exists and is in } L^1_{\mathrm{loc}}(\Omega),\ i = 1, \ldots, N\}. \tag{A3.2}$$

Note that $C^1(\Omega) \subset W^{1,1}_{\mathrm{loc}}(\Omega)$. We also set $W^{k,p}(\Omega)$ to be the Sobolev space

$$\begin{aligned} \{u \in L^p(\Omega) : &\text{ all weak partial derivatives of } u \\ &\qquad \text{up to order } k \text{ exist and are in } L^p(\Omega)\ \}. \end{aligned} \tag{A3.3}$$

This is a Banach space with respect to the norm

$$\|u\|_{W^{k,p}(\Omega)} := \Big(\sum_{0 \le |\alpha| \le k} \int_\Omega \Big| \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}} \Big|^p \, dx \Big)^{\frac{1}{p}}, \tag{A3.4}$$

where the sum is over all multi-indices α with $|\alpha| = \sum \alpha_i \le k$. By notational convention the 0th order derivative is u itself. Hence the first summand in (A3.4) is $\int_\Omega |u|^p \, dx$, and $W^{0,p}(\Omega) \equiv L^p(\Omega)$. We could alternatively take one of the several equivalent norms, for example $\sum_{0 \le |\alpha| \le k} \| \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}} \|_p$. This makes no difference for our purposes. When $p = 2$, $H^k(\Omega) := W^{k,2}(\Omega)$ is a Hilbert space with inner product given by

$$\langle u, v \rangle = \int_\Omega uv \, dx + \sum_{|\alpha| \le k} \int_\Omega \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}} \, \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}} \, dx. \tag{A3.5}$$

As a special and important case, when $k = 1$ the space $H^1(\Omega)$ has inner product given by

$$\langle u, v \rangle = \int_\Omega uv \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx.$$

If instead of considering real-valued functions $u : \Omega \to \mathbb{R}$ we look at vector-valued $u : \Omega \to \mathbb{R}^M$, we still denote the corresponding Sobolev space by $W^{k,p}(\Omega)$ as shorthand for $W^{k,p}(\Omega, \mathbb{R}^M)$. (For example, "$\nabla u \in H^1(\Omega)$" means every component $\frac{\partial u}{\partial x_i} \in H^1(\Omega)$, $1 \leq i \leq N$.)

Remark A3.1. (i) The space $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is always dense in $W^{k,p}(\Omega)$ (with respect to the norm (A3.4); see [**98**, Section 6.4.3]). If Ω is Lipschitz, then in addition $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$. However, for Ω arbitrary this latter statement is not true in general. (See, e.g., [**59**, Section 7.6].) Note that $C^k(\overline{\Omega}) \subset W^{k,p}(\Omega)$ for all k and p, that is, classically differentiable functions are (locally) weakly differentiable and their classical and weak derivatives coincide.

(ii) In light of (i) we could also characterise $W^{k,p}(\Omega)$ as the closure with respect to the norm given by (A3.4) of the space $C^\infty(\Omega)$ (and/or $C^\infty(\overline{\Omega})$ if Ω is Lipschitz). Some authors take this as their definition and it is a useful characterisation for us.

(iii) There are other characterisations of $W^{k,p}(\Omega)$ which are equivalent to ours for Lipschitz domains. We will introduce another definition of $W^{k,p}(\Omega)$ below that allows us to define this space for general $k \in \mathbb{R}^+$.

We denote by $W_0^{k,p}(\Omega)$ (or $H_0^k(\Omega)$ if $p = 2$) the closure of the space of test functions of compact support $C_c^\infty(\Omega)$ with respect to the $W^{k,p}$-norm on Ω. Then $W_0^{k,p}(\Omega)$ is a closed subspace of $W^{k,p}(\Omega)$. If $\Omega = \mathbb{R}^N$ then $W_0^{k,p}(\mathbb{R}^N) = W^{k,p}(\mathbb{R}^N)$ but otherwise we have strict inclusion. Here we are only interested in the case $k = 1$. Then for a broad class of domains including those with Lipschitz boundary, $W_0^{1,p}(\Omega)$ is the space of $W^{1,p}(\Omega)$ functions which have zero trace on $\partial\Omega$ (see Theorem A4.1). Alternatively, $\varphi \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ implies $\varphi = 0$ on $\partial\Omega$; see [**15**].

We also consider Sobolev spaces $W^{s,p}(\Omega)$ for some $s \notin \mathbb{N}$. Here we are mostly interested in $p = 2$. So for negative integers, we define the space $H^{-k}(\Omega)$, $k \in \mathbb{N}$, as being the set of all linear functionals on $H_0^k(\Omega)$. (See [**98**, Section 6.4.9].) For $s \in \mathbb{R}^+$ not an integer, $1 < p < \infty$ and $\Omega \subseteq \mathbb{R}^N$, following Grisvard [**64**, Section 1.3], writing $s = k + \varepsilon$ where $k \in \mathbb{N}$ and $\varepsilon \in (0, 1)$ we denote by $W^{s,p}(\Omega)$ the space of all functions $u \in W^{k,p}(\Omega)$ such that

$$\int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+\varepsilon p}}\, dx\, dy < \infty, \tag{A3.6}$$

for every multi-index α with $|\alpha| = k$. This is a Banach space with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \Big(\|u\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \int_{\Omega\times\Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{N+\varepsilon p}}\, dx\, dy \Big)^{\frac{1}{p}}.$$

We can also denote by $W_0^{s,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to the $W^{s,p}$-norm.

When $p = 2$ there is another common definition of $H^s(\Omega)$ via the rate of decay of a function's Fourier transform. Precisely, letting $\widehat{u}$ denote the Fourier transform of u we define $H^s(\mathbb{R}^N)$ as the space of all distributions u for which

$$\|u\|^2_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} (1 + |x|^2)^s |\widehat{u}(x)|^2 \, dx < \infty \tag{A3.7}$$

We then let $H^s(\Omega)$ be the restriction of $H^s(\mathbb{R}^N)$ to Ω. This approach is used in [**69**], and there is a reasonably detailed discussion of this and related spaces in Sections 2 and 3 there. A standard reference here is [**84**, Chapter 1]. Note that $W^{s,2}(\mathbb{R}^N)$ as defined by (A3.6) and $H^s(\mathbb{R}^N)$ as defined by (A3.7) are equal spaces with equivalent norms, and the same is true for $W^{s,2}(\Omega)$ and $H^s(\Omega)$ when Ω is Lipschitz (see [**64**, Section 1.3] or [**89**, Chapter 3]). Hence when $p = 2$ it does not matter which of the two definitions we take.

We also consider the boundary spaces $W^{s,p}(\partial\Omega)$. For Ω bounded, Lipschitz and $1 < p < \infty$, we can define the space $L^p(\partial\Omega)$ ($\equiv W^{0,p}(\partial\Omega)$) in the usual way as the set of all p-integrable functions $f : \partial\Omega \to \mathbb{R}$ equipped with the obvious norm. For bounded domains of class $C^{k,1}$ we can define the spaces $W^{s,p}(\partial\Omega)$ for any $|s| \le k + 1$. Let U_z be any neighbourhood and f any function satisfying the requirements of Definition A2.1. Let $F(x) = (x_1, \ldots, x_{N-1}, f(x_1, \ldots, x_{N-1}))$. A distribution u on $\partial\Omega$ is in $W^{s,p}(\partial\Omega)$ if $u \circ F \in W^{s,p}(U_z \cap F^{-1}(\partial\Omega \cap U_z))$ for every such U_z and f. If $s \in (0, 1)$, one possible norm is

$$\|u\|_{W^{s,p}(\partial\Omega)} := \Big(\int_{\partial\Omega} |u|^p \, dx + \int_{\partial\Omega\times\partial\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N-1+sp}} \, d\sigma(x) \, d\sigma(y) \Big)^{\frac{1}{p}}.$$

(See [**64**, Section 1.3.3].)

A4. Some properties of domains and Sobolev spaces.

Here we collect many useful properties of Lipschitz domains and Sobolev spaces. Results we state here we generally use without proof throughout the body of the thesis. Suppose $\Omega \subset \mathbb{R}^N$ is a bounded, Lipschitz domain. It follows from Rademacher's theorem that Ω has a well-defined outward-pointing unit normal vector $\nu = \nu_\Omega(z)$ at σ-almost every $z \in \partial\Omega$. This normal is compatible with all the usual properties such as the divergence (Gauss-Green) theorem, the definition of boundary spaces and outward normal derivatives (as in the Robin boundary condition), which we will list in this subsection. In this sense Lipschitz domains are the "natural" domains on which to consider problems such as (1.1.2) and (1.1.3) since too many properties fail if we look for a substantially weaker class of domains.

We will start with some useful and important general properties of Sobolev spaces on Lipschitz domains. The first is the idea of traces, which is necessary for the version of the divergence theorem we will use. For any $u \in W^{1,p}(\Omega)$, we can define the trace of u, which we will denote by either $u|_{\partial\Omega}$ or else $\operatorname{tr} u$ (the latter if we want to emphasise it is a trace) as the unique function in $L^p(\partial\Omega)$ satisfying the following theorem (see, e.g., [**48**, pp. 257-61] or [**64**, Chapter 1]). We do not use γu for the trace of u to avoid potential confusion with the parameter appearing in the Wentzell boundary condition.

Theorem A4.1 (Trace theorem)**.** *Suppose $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$ is bounded, Lipschitz. There is a unique bounded linear operator $\operatorname{tr} : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ such that $\operatorname{tr} u = u|_{\partial\Omega}$ if $u \in W^{1,p} \cap C(\overline{\Omega})$. Moreover, tr is compact as a map from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$.*

In fact we can say a little more: tr is a bounded surjection with right inverse (that is, $\operatorname{tr} \circ (\operatorname{tr})^{-1}$ is the identity) from $W^{1,p}(\Omega)$ to $W^{1-\frac{1}{p},p}(\partial\Omega)$ with kernel exactly $W_0^{1,p}(\Omega)$ (that is, $W_0^{1,p}(\Omega)$ consists exactly of those $W^{1,p}(\Omega)$ functions with zero trace). In particular we may write

$$W^{1-\frac{1}{p},p}(\partial\Omega) \simeq W^{1,p}(\Omega)/W_0^{1,p}(\Omega).$$

Actually, the trace theorem is valid for a greater range of Sobolev spaces, even for Lipschitz domains.

Theorem A4.2 (Trace theorem II)**.** *Suppose $\Omega \subset \mathbb{R}^N$ is bounded, Lipschitz. The trace operator in Theorem A4.1 extends to a bounded linear operator from $H^s(\Omega)$ to $H^{s-\frac{1}{2}}(\partial\Omega)$ for every $\frac{1}{2} < s < \frac{3}{2}$.*

Proof. This is well known; see for example [**89**, Theorem 3.38]. □

It is clear from the definition that $W^{k,p}(\Omega)$ embeds continuously in $W^{j,p}(\Omega)$ if $k \geq j \geq 0$. It is an important result that these embeddings are compact. The same is true of the boundary spaces, although we will only prove this under more limited conditions.

Theorem A4.3 (Rellich's theorem). *Suppose $\Omega \subset \mathbb{R}^N$ is a bounded, Lipschitz domain.*

(i) *If $s \geq 0$ and $1 < p < \infty$, then the embedding $W^{s+\varepsilon,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ is compact for all $\varepsilon > 0$.*

(ii) *If $0 \leq s < s+\varepsilon \leq 1$, then $H^{s+\varepsilon}(\partial\Omega) \hookrightarrow H^s(\partial\Omega)$ is compact.*

Proof. For (i), we refer to [**64**, Theorem 1.4.3.2]. For (ii), we note that s and $s+\varepsilon$ are in the correct range so that the boundary spaces are well-defined. We use the trace theorem to obtain a continuous mapping of $H^{s+\varepsilon}(\partial\Omega)$ into $H^{s+\frac{1}{2}+\varepsilon}(\Omega)$. By (i) and the trace theorem the map $H^{s+\frac{1}{2}+\varepsilon}(\Omega) \hookrightarrow H^{s+\frac{1}{2}}(\Omega) \to H^s(\partial\Omega)$ is compact (since the composition of a compact and a continuous map is compact). Putting the three maps together, we obtain a compact mapping from $H^{s+\varepsilon}(\partial\Omega)$ to $H^s(\partial\Omega)$. □

Another important domain property is the ability to extend functions in $W^{k,p}(\Omega)$ to functions in $W^{k,p}(\mathbb{R}^N)$. In the other direction, the restrictions of $W^{k,p}(\mathbb{R}^N)$ functions to Ω trivially lie in $W^{k,p}(\Omega)$.

Theorem A4.4 (Extension theorem). *Take $\Omega \subset \mathbb{R}^N$ Lipschitz and $1 < p < \infty$.*

(i) *If $\partial\Omega$ is bounded, then for any $k \in \mathbb{N}$ there exists a bounded operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)$ such that $Eu|_\Omega = u$ for all $u \in W^{k,p}(\Omega)$.*

(ii) *The same conclusion holds for any $s > 0$ in place of k if Ω is bounded.*

In both cases E can be chosen independently of k.

See for example [**98**, Theorem 6.88] and [**69**, Theorem 2.3] for (i), and [**64**, Theorem 1.4.3.1] for (ii). In the special case where our functions have zero trace, we can continuously embed $W_0^{k,p}(\Omega) \hookrightarrow W^{k,p}(\mathbb{R}^N)$ by setting $\varphi \in W^{k,p}(\Omega)$ to be zero outside Ω. This is for general $\Omega \subset \mathbb{R}^N$.

If $f = (f_1, \ldots, f_N) : \Omega \to \mathbb{R}^N$ is in $W^{1,1}_{\mathrm{loc}}(\Omega) \equiv W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, $\Omega \subseteq \mathbb{R}^N$ arbitrary, then we can define $\operatorname{div} f := \sum_{i=1}^N \frac{\partial f_i}{\partial x_i}$. Note in part (i) of the following theorem that f on $\partial\Omega$ is understood in the sense of traces.

Theorem A4.5 (Divergence theorem for Lipschitz domains). (i) *Let $\Omega \subset \mathbb{R}^N$ be bounded and Lipschitz and suppose $f : \Omega \to \mathbb{R}^N$ is in $W^{1,1}(\Omega)$. Then*

$$\int_\Omega \operatorname{div} f \, dx = \int_{\partial\Omega} f \cdot \nu \, d\sigma, \tag{A4.1}$$

where $\nu = \nu_\Omega$ is the outer unit normal to Ω.

(ii) *The conclusion of (i) remains true if* Ω *is Lipschitz but not necessarily bounded, and* $f \in C_c^1(\mathbb{R}^N, \mathbb{R}^N)$.

Proof. For (i), take $v \equiv 1$ in [**91**, Théorème 3.1.1]. Since $v = 1 \in W^{1,q}(\Omega)$ for all $1 \leq q \leq \infty$, we may take $u = f_i \in W^{1,1}(\Omega)$. Now sum over $i = 1, \dots, N$. For (ii), see [**49**, Section 5.8]. □

We now extend the definition of some classical concepts to the case of weakly differentiable functions. If $u \in W^{2,p}(\Omega)$ for some $1 \leq p < \infty$ (or if $u \in C^2(\Omega)$) then we write $\Delta u := \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$. In particular, $\Delta u = \operatorname{div}(\nabla u)$ if it exists. If $u \in H^1(\Omega)$ then we may define $\Delta u \in L^2(\Omega)$ in the sense of distributions as the function $f \in L^2(\Omega)$, if it exists, satisfying

$$-\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \tag{A4.2}$$

for all $v \in H_0^1(\Omega)$. Not every $u \in H^1(\Omega)$ will have Δu existing in this sense, but if u is sufficiently smooth then a simple calculation shows that this Δu coincides with the classical definition.

If Ω is Lipschitz with outer unit normal given by $\nu = \nu(x)$, then motivated by Green's first identity (see [**59**, Section 2.4]) we may also define the outer normal derivative $\frac{\partial u}{\partial \nu}$ of $u \in \{w \in H^1(\Omega) : \Delta w \in L^2(\Omega)\}$ in the weak sense as the function $b \in L^2(\partial\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega v \Delta u \, dx = \int_{\partial\Omega} bv \, d\sigma \tag{A4.3}$$

for all $v \in H^1(\Omega)$, if such a function exists. Again, if everything is sufficiently smooth then this coincides with the usual definition of $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$.

We next have a useful property of $W^{1,p}$ functions that can be regarded as a type of lattice property. For $u \in W^{1,p}(\Omega)$, $1 < p < \infty$, set

$$u^+(x) := \max\{u(x), 0\}, \qquad u^-(x) := \max\{-u(x), 0\} \tag{A4.4}$$

Then we have $u = u^+ - u^-$ and $|u| = u^+ + u^-$. It is an important and useful fact that in this case $u^+, u^- \in W^{1,p}(\Omega)$, with

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u > 0 \\ 0 & \text{if } u \leq 0, \end{cases} \tag{A4.5}$$

with an analogous formula for ∇u^-; see [**59**, Lemma 7.6]. In fact this is true for any locally integrable continuous function with locally integrable weak derivatives. Moreover if $u \in W^{1,p}(\Omega) \cap C(\Omega)$ then $u^+, u^- \in W^{1,p}(\Omega) \cap C(\Omega)$ also.

If $u \in W^{1,p}(\Omega) \cap C(\Omega)$ (for us usually an eigenfunction on Ω associated with some operator), then its nodal domains are

$$\begin{aligned} \Omega^+ &:= \{x \in \Omega : u(x) > 0\} \\ \Omega^- &:= \{x \in \Omega : u(x) < 0\} \end{aligned} \tag{A4.6}$$

Note that since u is continuous, these are both open subsets of Ω. The nodal surface of u is the set $\{x \in \Omega : u(x) = 0\}$. Under some conditions on u, for example if u is (sub-, super-) harmonic, then the nodal surface will coincide with the sets $\partial\Omega^+ \cap \Omega$ and $\partial\Omega^- \cap \Omega$, and the nodal surface is a genuine surface, since $u(x) = 0$ on a set of positive measure then implies $u \equiv 0$ in Ω. However, for an arbitrary $W^{1,p}(\Omega) \cap C(\Omega)$ function (even a $C^\infty(\overline{\Omega})$ function) it is possible that its nodal "surface" could be much larger and have nonzero N-dimensional measure. For everything above we could replace 0 with an arbitrary "level" $c \in \mathbb{R}$ and all the results would continue to hold. In this case $\{x \in \Omega : u(x) > t\}$ is usually called the (upper) level set of u (of level t) and $\{x \in \Omega : u(x) = t\}$ the (interior) level surface.

Our last theorem in this section is an important result linking the level surfaces of a function with an appropriate volume integral. More broadly, this can be viewed as a key theorem linking geometry and analysis.

Theorem A4.6 (Coarea formula)**.** *Let $\Omega \subset \mathbb{R}^N$. Suppose $u : \Omega \to \mathbb{R}$ is Lipschitz continuous and $\phi \in L^1(\mathbb{R}^N)$. Then*

$$\int_\Omega \phi\, |\nabla u|\, dx = \int_0^\infty \int_{\{|u(x)|=t\}} \phi\, d\sigma\, dt.$$

In particular, $\sigma(\{u(x) = t\}) < \infty$ for almost all $t \in (0, \infty)$.

Proof. See [**88**, Section 1.2.4] or [**49**, Section 3.4.2]. □

Remark A4.7. In the above theorem we could alternatively assume that ϕ is measurable and non-negative.

A5. Operator theory and semigroups.

Here we collect some basic results from operator theory and functional analysis, along with the notation we will use. Many books cover this material; see for example [**9**, Appendix B] or [**98**, Chapter 7], or the dedicated and seminal book [**72**]. We refer to them for the proofs of the statements we make below.

For X a Banach space over $\mathbb{R}$ or $\mathbb{C}$ and A an operator (linear mapping) from X to itself, we denote by $\mathcal{D}(A) = \{x \in X : Ax \text{ exists in } X\}$ the domain of A. If A is closed, that is, $\{(x, Ax) : x \in \mathcal{D}(A)\} \subset X \times X$ is closed with respect to the norm induced on $X \times X$, then $\mathcal{D}(A)$ is a Banach space with respect to the graph norm $\|x\|_A := \|x\|_X + \|Ax\|_X$. Moreover A is bounded, that is, $\|A\| := \sup\{\|Ax\|_X : x \in \mathcal{D}(A) \text{ and } \|x\|_X \leq 1\} < \infty$, if and only if A is continuous.

For the following spectral properties, we assume X is a Banach space over $\mathbb{C}$; if X is over $\mathbb{R}$ then we consider the complexification of X and A. We denote by $\rho(A)$ the resolvent set of A given by

$$\{\lambda \in \mathbb{C} : (\lambda I - A) : \mathcal{D}(A) \to X \text{ is bijective and bounded}\}$$

If $\rho(A) \neq \emptyset$, then A is closed. For $\lambda \in \rho(A)$, the resolvent operator $R(\lambda, A) := (\lambda I - A)^{-1} : X \to \mathcal{D}(A)$ is well-defined, linear and bounded. If for some $\lambda \in \rho(A)$ the map $R(\lambda, A)$ is compact as a map from X to itself, then this is true for all $\lambda \in \rho(A)$ and we say A has compact resolvent. We denote by $\sigma(A) = \mathbb{C} \setminus \rho(A)$ the spectrum of A and by $\sigma_p(A) \subseteq \sigma(A)$ the point spectrum, that is, the set of eigenvalues of A. If A has compact resolvent, then $\sigma(A) = \sigma_p(A)$ is a denumerable set. In this case we say A has discrete spectrum.

We next state versions of the well-known Riesz representation theorem and Lax-Milgram lemma. Proofs can be found in [**40**].

Theorem A5.1 (Riesz representation theorem)**.** *Let H be a Hilbert space and H' its dual space. There exists an isometric isomorphism from H to H', such that for every $u \in H$ there exists a unique $f_u \in H'$ with $\langle u, v\rangle = (f_u, v)$ for all $v \in H'$, and $\|u\|_H = \|f_u\|_{H'}$. Here $\langle\, .\, ,\, .\,\rangle$ denotes the duality pairing between H and H', and $(\, .\, ,\, .\,)$ the inner product on H.*

Theorem A5.2 (Lax-Milgram lemma)**.** *Let H be a Hilbert space and H' its dual space. Suppose $A : H \to H'$ is a bounded linear operator, and there exists $b > 0$ such that $\langle Au, u\rangle \geq b\|u\|_H^2$ for all $u \in H$. Then A is invertible.*

We now have some basic results on (one-parameter, linear) semigroups. We refer to [**9, 42, 47**] (see also [**72**, Chapter 9]) for full explanations and proofs.

Definition A5.3. Given a Banach space X a (one-parameter) C_0*-semigroup* T on X is a family of operators $T(t)$, $t \in \mathbb{R}^+ = [0, \infty)$, where for each t, $T(t) : X \to X$ satisfies

(i) $T(.)x : \mathbb{R}^+ \to X$ is continuous for each $x \in X$;
(ii) $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$;
(iii) $T(0) = I$, the identity on X.

If $A : \mathcal{D}(A) \to X$ is a closed linear operator, $\mathcal{D}(A) \subset X$, we say A generates the semigroup T if

$$\mathcal{D}(A) = \{x \in X : \lim_{t \to 0} \frac{1}{t}(T(t)x - x) \text{ exists in } X\},$$
$$Ax = \lim_{t \to 0} \frac{1}{t}(T(t)x - x).$$

Whenever we say semigroup, we always mean C_0-semigroup even if this is not explicitly stated. If A is the generator of T, then $T(t)x \in \mathcal{D}(A)$ and $AT(t)x = T(t)Ax$ for all $t \geq 0$. Moreover for every $x \in X$, $u(t) = T(t)x \in C^1(\mathbb{R}^+, X)$ is the unique solution to the abstract Cauchy problem

$$\frac{\partial u}{\partial t} = Au(t) \qquad \text{if } t \geq 0,$$
$$u(0) = x.$$

In particular, if A is a second-order elliptic operator defined on some domain $\Omega \subset \mathbb{R}^N$, then T, if it exists, is the solution to the corresponding heat equation.

Definition A5.4. Let $T : \mathbb{R}^+ \times X \to X$ be a (C_0-)semigroup and A its generator. We say T is:

(i) *analytic* (of angle $\theta \in (0, \pi/2]$) if it has an analytic extension to the complex sector $\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta + \pi/2\}$ which is bounded on $\Sigma_{\theta'} \cap \{z \in \mathbb{C} : |z| \leq 1\}$ for all $\theta' \in (0, \theta)$ (cf. Definition 5.3.6). Note that many authors use the term "holomorphic" rather than "analytic";
(ii) *positive* if $X = L^p(\Omega, d\mu)$, where (Ω, Σ, μ) is a finite measure space, and for each $f \in X$ with $f \geq 0$ σ-a.e., we have $T(t)f \geq 0$ σ-a.e., for some (equivalently, all) $t \in \mathbb{R}^+$;

(iii) *irreducible* if $X = L^p(\Omega, d\mu)$ and for each $\omega \in \Sigma$ and $t \in \mathbb{R}^+$, $T(t)L^p(\omega) \subset L^p(\omega)$. Here $L^p(\omega) := \{f \in L^p(\Omega, d\mu) : f = 0 \ \mu\text{-a.e. on } \Omega \setminus \omega\}$.

Also, $T(t)$ is compact as an operator from X to X for some $t \in (0, \infty)$ if and only if it is compact for all $t \in (0, \infty)$. Moreover, T is compact if and only if its generator A has compact resolvent and T is immediately norm continuous, that is, continuous as an operator from $(0, \infty)$ into the space of bounded linear operators on X. If T is analytic, then it is automatically immediately norm continuous.

There are many equivalent characterisations as well as necessary and sufficient conditions for T to have all or any of these properties; the books [**9**] and [**47**] are in no small part devoted to studying these. The following result is of particular interest to us. If A generates a positive, irreducible semigroup on some L^p space and has compact resolvent, then its spectral bound $s(A) := \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$ is finite (by de Pagter's theorem). Moreover, $s(A) \in \sigma(A)$ and there exists $0 < u \in \mathcal{D}(A)$ such that $Au = s(A)u$ (Krein-Rutman). Finally, the eigenspace of u is one-dimensional and in fact u can be chosen strictly positive almost everywhere in Ω. In words, the generator of such a semigroup has a principal eigenvalue at the edge of its spectrum, and the corresponding eigenfunction is unique up to scalar multiples.

Appendix B

Background Results on the Functional H_Ω

Here for the convenience of the reader we reproduce the proofs of some of the lemmata from [**35**] that were used in Section 2.2. The notation we use throughout is the same as in that section.

Proof of Lemma 2.2.1. (i) The coarea formula (see Theorem A4.6) states that

$$\|\nabla\psi\|_{L^1(\Omega)} = \int_0^\infty \sigma(S_t)\,dt,$$

and in particular the function $\sigma(S_t)$ is t-integrable. (Note that $\sigma(S_t) = 0$ if $t \notin (m, 1)$.)

(ii) As was already noted in Section 2.2, the S_t are C^∞ for almost all t by Sard's lemma [**68**, Theorem 3.1.3] since $\psi \in C^\infty(\Omega)$. If $\Gamma_1 = \emptyset$, then since $U_t \subset\subset \Omega$ there is nothing left to prove. So suppose $\Gamma_1 \neq \emptyset$. Since Ω is C^2 we need to show that ∂U_t is the graph of a Lipschitz function where S_t and $\partial\Omega$ meet, that is, near $\overline{S}_t \cap \Gamma_1$ (recalling that the level sets are compactly contained away from Γ_0).

So fix $x_0 \in \overline{S}_t \cap \Gamma_1$ and $t \in (m, 1)$. Without loss of generality assume $x_0 = 0$ and choose a coordinate system $(x_1, \dots, x_N) =: (x', x_N)$ such that $\nu_\Omega(x_0)$ points in the direction x_N. For $\delta > 0$ set

$$Q_\delta := \{x \in \mathbb{R}^N : |x_i| < \delta \text{ for } 1 \leq i \leq N\} \tag{B.1}$$

to be the cube of radius 2δ centred at $x_0 = 0$ and

$$R_\delta := \{x \in Q_\delta : x_N = 0\} \tag{B.2}$$

to be its intersection with the hyperplane $\{x_N = 0\}$. Since Ω is C^2, by definition there exists $\delta > 0$ and $\varphi : R_\delta \to \mathbb{R}$ of class C^2 such that

$$\Omega \cap Q_\delta = \{(x', x_N) \in R_\delta \times (-\delta, \delta) : x_N < u(x')\}.$$

Now by Theorem A4.4 ψ can be extended to a function $\tilde{\psi} \in W^{2,p}(\mathbb{R}^N)$ having compact support, for any $p \geq 1$ (see also [**59**, Theorem 7.25]). By standard embedding theorems if $p > N$ then $\tilde{\psi} \in C^1(\mathbb{R}^N)$. Since $\psi > 0$ and $\alpha > 0$ on

Γ_1, by the boundary condition $\nabla\tilde{\psi}(x_0)\cdot\nu_\Omega(x_0) = -\alpha\psi(x_0) < 0$ and in particular $\nabla\tilde{\psi}(x_0) \neq 0$. Hence by the implicit function theorem, near x_0 $\tilde{S}_t := \{x \in \mathbb{R}^N : \tilde{\psi}(x) = t\}$ is a C^1 surface, and, setting $\tilde{U}_t := \{x \in \mathbb{R}^N : \tilde{\psi}(x) > t\}$, we have $\tilde{S}_t = \partial\tilde{U}_t$.

Note that the outer unit normal to $\tilde{U}_t$ at x_0 points in the direction of $-\nabla\psi(x_0)$; since $\nabla\tilde{\psi}(x_0)\cdot\nu_\Omega(x_0) < 0$, $\nabla\tilde{\psi}(x_0)$ has a nonzero component in the direction of $\nu(x_0)$. Hence by the implicit function theorem we may also represent $\partial\tilde{U}$ as the graph of a function over the same coordinate axes. That is, there exists $\zeta > 0$ and $\phi \in C^1(R_\zeta)$ such that

$$\tilde{U}_t \cap Q_\zeta = \{(x', x_N) \in R_\zeta \times (-\zeta,\zeta) : x_N < \phi(x')\}.$$

Now set $\varepsilon := \min\{\delta,\zeta\}$ and $g := \min\{\varphi,\phi\}$. Then

$$\tilde{U}_t \cap \Omega \cap Q_\varepsilon = \{(x', x_N) \in R_\varepsilon \times (-\varepsilon,\varepsilon) : x_N < g(x')\}.$$

Since g is Lipschitz continuous as the minimum of two C^1 functions, $U_t = \tilde{U}_t \cap \Omega$ is Lipschitz near x_0.

(iii) Suppose $\Gamma_1 \neq \emptyset$. Denote by $W \subset \overline{\Omega}$ the set of points on which ψ attains its minimum, that is, $W = \{x \in \overline{\Omega} : \psi(x) = m\}$. As noted in Section 2.2, the maximum principle implies $W \subset \partial\Omega$; moreover by Hopf's lemma $\frac{\partial u}{\partial\nu} < 0$ on W. Since Ω has sufficiently smooth boundary, $\nabla\psi$ is continuous up to the boundary and ν is smooth as well. Hence there exists $\xi > 0$ such that

$$-\frac{\partial\psi}{\partial\nu}(z) = -\nu(z)\cdot\nabla\psi(z) \geq \xi > 0$$

for all $z \in W$. Using uniform continuity on compact sets, there exists $\delta_0 > 0$ such that

$$-\nu(z)\cdot\nabla\tilde{\psi}(x) \geq \frac{\xi}{2} > 0 \qquad \text{(B.3)}$$

for all $z \in W$ and $x \in B(z,\delta_0)$, where $\tilde{\psi}$ is the extension of ψ used in (ii). By a similar argument there exists $\delta_1 > 0$ such that $\nu(x)\cdot\nu(z) \geq \frac{1}{2}$ for all $z \in W$ and $x \in B(z,\delta_1) \cap \partial\Omega$.

Choose $0 < \delta \leq \min\{\delta_0,\delta_1\}$, so that for each $z \in W$ there is an open cube $Q_{z,\delta}$ centred at z, such that $Q_{z,\delta}$ has the form (B.1) if our coordinate system is chosen so that $\nu(z)$ points in the direction x_N. Then $R_{z,\delta}$ is as in (ii), and there exists $u \in C^2(R_{z,\delta})$ such that (B.2) holds. Since $\{Q_{z,\delta} : z \in W\}$ is an open cover of the compact set W, we may extract a finite subcover $\{Q_{z_i,\delta}\}$, where $z_i \in W$, $1 \le i \le n$, say.

Now since ψ attains a strict minimum on W, by continuity of ψ there exists $t_1 \in (m, 1)$ such that

$$S_t \subset V := \cup_{i=1}^n Q_{z_i,\delta}$$

for all $t \in (m, t_1)$. Fix $t \in (m, t_1)$ and for the meantime also fix a particular cube $Q_\delta := Q_{z_j,\delta}$. Choose our coordinate system so that $\nu(z)$ points in the x_N direction as described above. Then (B.3) says that

$$-\frac{\partial\tilde{\psi}}{\partial x_N}(x) \ge \frac{\xi}{2} > 0$$

for all $x \in Q_\delta$. If $S_t \cap Q_\delta \ne \emptyset$, then by the implicit function theorem this implies there exists $D \subset R_\delta = R_{z_j,\delta}$ and a C^1 function $v : D \to \mathbb{R}$ such that $S_t \cap Q_\delta$ is the graph of v. Then

$$\sigma(S_t \cap Q_\delta) = \int_D \sqrt{1 + |\nabla v(x')|^2}\, dx'.$$

Since S_t is a level surface of $\tilde{\psi}$, its normal is given by

$$(-\nabla v(x'), 1) = \Big(\frac{\partial\tilde{\psi}}{\partial x_N}(x', v(x'))\Big)^{-1} \nabla\tilde{\psi}(x', v(x')).$$

Using (B.3),

$$\sigma(S_t \cap Q_\delta) = \int_{R_\delta} \Big|\frac{\partial\tilde{\psi}}{\partial x_N}(x', v(x'))\Big|^{-1} |\nabla\tilde{\psi}(x', v(x'))\, dx' \le \frac{2}{\xi}\|\nabla\tilde{\psi}\|_{L^\infty(\mathbb{R}^N)}\sigma(R_\delta).$$

Since $\delta > 0$ was chosen so that $\partial\Omega \cap Q_\delta$ is the graph of a C^2 function u, we certainly have $\sigma(R_\delta) \le \sigma(\partial\Omega \cap Q_\delta) \le \sigma(\partial\Omega)$. In particular this means that

$$\sigma(S_t \cap Q_\delta) \le \frac{2}{\xi}\|\nabla\tilde{\psi}\|_{L^\infty(\mathbb{R}^N)}\sigma(\partial\Omega).$$

Summing over all n cubes and using that $\sigma(S_t) \le \sum_{i=1}^n \sigma(S_t \cap Q_{z_i,\delta})$ yields the result. □

Proof of Lemma 2.2.5. Fix $\varepsilon \in (0, 1)$ and $\varphi \in C(\Omega) \cap L^1(U_\varepsilon)$. By the coarea formula (Theorem A4.6),

$$\int_\varepsilon^1 \frac{1}{\tau} \int_{S_\tau} \varphi \, d\sigma d\tau = \int_{U_\varepsilon} \varphi \frac{|\nabla\psi|}{\psi}\, dx < \infty,$$

with an analogous statement being true if we replace φ with $|\nabla\psi|/\psi$. Since both are non-negative, we can define $f(\tau) \in L^1((0,1))$ by

$$f(\tau) := \frac{1}{\tau}\int_{S_\tau} w\, d\sigma = \frac{1}{\tau}\int_{S_\tau} \varphi\, d\sigma - \frac{1}{\tau}\int_{S_\tau} \frac{|\nabla\psi|}{\psi}\, d\sigma.$$

It follows that

$$F(t) = \int_t^1 f(\tau)\, d\tau$$

is absolutely continuous and hence differentiable almost everywhere on $[\varepsilon, 1)$, with

$$\frac{d}{dt}F(t) = -f(t) = -\frac{1}{t}\int_{S_t} w\, d\sigma$$

for almost every $t \in (\varepsilon, 1)$ (see [**99**, Theorem 7.18]). Since $\varepsilon \in (0,1)$ was arbitrary this completes the proof. □

Proof of Lemma 2.3.3. This uses a simple compactness argument. Given $\varphi \in M_\alpha$, fix $\varepsilon > 0$. Using that $|\nabla\varphi|/\varphi \in C(\overline{\Omega})$, where $\overline{\Omega}$ is compact, there exists δ_0 such that

$$\Big|\frac{|\nabla\psi(x)|}{\psi(x)} - \frac{|\nabla\psi(y)|}{\psi(y)}\Big| < \frac{\varepsilon}{2} \tag{B.4}$$

for all $x, y \in \overline{\Omega}$ such that $|x - y| < \delta_0$. Now, since $\limsup_{x\to z}\varphi(x) \le \alpha(z)$ for $x \in \Omega$ and $z \in \partial\Omega$ by assumption, there exists $r_z = r(z) > 0$ such that $\sup_{x\in B(z,r_z)\cap\Omega}\varphi(x) < \alpha(z) + \varepsilon/2$. That is,

$$\varphi(x) - \alpha(z) < \frac{\varepsilon}{2} \tag{B.5}$$

for all $x \in B(z, r_z)\cap\Omega$. Since the $B(z, r_z)$ cover the compact set $\partial\Omega$ we may extract a finite subcover $B(z_i, r_{z_i})$, $1 \le i \le n$, say. We then choose $\delta \le \min\{r_{z_1}, \ldots, r_{z_n}, \delta_0\}$ to be such that $x \in \cup_{i=1}^n B(z_i, r_{z_i})$ if $\operatorname{dist}(x, \partial\Omega) < \delta$.

Now fix $x \in \Omega$ such that $\operatorname{dist}(x, \partial\Omega) < \delta$. We may suppose that $x \in B(z_j, r_{z_j})$. Since $\alpha(z_j) \le |\nabla\psi(z_j)|/\psi(z_j)$ (see Remark 2.3.2), (B.4) and (B.5) imply

$$\begin{aligned} w(x) &= \varphi(x) - \frac{|\nabla\psi(x)|}{\psi(x)} \\ &= \varphi(x) - \alpha(z_j) + \alpha(z_j) - \frac{|\nabla\psi(x)|}{\psi(x)} \\ &< \frac{\varepsilon}{2} + \frac{|\nabla\psi(z_j)|}{\psi(z_j)} - \frac{|\nabla\psi(x)|}{\psi(x)} < \varepsilon. \end{aligned}$$

Since x was arbitrary this completes the proof. □

Proof of Lemma 2.4.1. Since ψ^* is decreasing in the radial direction and by elementary properties of Bessel functions (see [**109**, p. 45]) we have

$$|\nabla\psi^*| = -v'(r) = c\sqrt{\lambda_1(B)}\, r^{1-\frac{N}{2}} J_{\frac{N}{2}}(\sqrt{\lambda_1(B)}\, r)$$

for all $r \in [0, R)$. Combining this with (2.4.1),

$$g(r) = \sqrt{\lambda_1(B)}\, \frac{J_{\frac{N}{2}}(\sqrt{\lambda_1(B)}\, r)}{J_{\frac{N}{2}-1}(\sqrt{\lambda_1(B)}\, r)}$$

for all $r \in (0, R)$. Let j_n, $n \in \mathbb{N}$, be the positive zeros of $J_{\frac{N}{2}-1}$. Then by [**109**, p. 498], if $r \neq j_n$ for any $n \in \mathbb{N}$, then

$$\frac{J_{\frac{N}{2}}(r)}{J_{\frac{N}{2}-1}(r)} = -\sum_{n=0}^{\infty}\Big(\frac{1}{r-j_n} + \frac{1}{r+j_n}\Big).$$

In particular, since each term is a decreasing function of r on the interval (j_n, j_{n+1}) (for any $n \in \mathbb{N}$), the whole series must be decreasing and hence g is strictly increasing. $\square$

Appendix C

On Harmonic Functions and Associated Operators

This appendix is devoted wholly to proving a number of technical results concerning harmonic functions and the like that were used in Section 5.3. While we do not think these results are new, we cannot find a precise reference to them so we include proofs. Wherever possible, we set up our results so they are valid for general C^1 or Lipschitz domains, so that our arguments in Section 5.3 will be valid for the broadest possible class of domains (see Remark 5.3.3). We will always assume, without further comment, that Ω is bounded.

C1. The normal derivative

Here we consider the function $B : L^2(\Omega) \to L^2(\partial\Omega)$ given by (5.3.3), that is,

$$\mathcal{D}(B) = \{u \in H^1(\Omega) : \frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)\}$$
$$Bu = -\beta \frac{\partial u}{\partial \nu}$$

where $\frac{\partial u}{\partial \nu}$ is defined as in (A4.3). We wish to prove that B is relatively Δ^D-bounded with bound 0. This is the key place where the assumption that Ω is of class $C^{1,1}$ will be needed, since without it, the perturbation idea that was essential to the proof of Theorem 5.3.1 will no longer work. This argument still works if Ω is C^1 and convex or polygonal in $\mathbb{R}^2$; see Remark 5.3.3.

Lemma C1.1. *For Ω of class $C^{1,1}$, the operator B is relatively Δ^D-bounded with bound* 0.

Remark C1.2. Lemma C1.1 implies in particular that $\mathcal{D}(\Delta^D) \subset \mathcal{D}(B)$, that is, every $u \in \mathcal{D}(\Delta^D)$ has a weak outer normal derivative $\frac{\partial u}{\partial \nu} \in L^2(\partial\Omega)$.

Proof. Firstly, we note that if Ω is of class $C^{1,1}$, then by [**64**, Theorem 2.2.2.3 and Corollary 2.2.2.4], $\mathcal{D}(\Delta^D) \subset H^2(\Omega)$ and moreover the embedding $\mathcal{D}(\Delta^D) \hookrightarrow H^2(\Omega)$ is continuous, that is, $\|u\|_{H^2(\Omega)} \leq K(\|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$ for some $K > 0$ independent of $u \in \mathcal{D}(\Delta^D)$. By Rellich's theorem, $H^2(\Omega)$ embeds compactly in

$H^k(\Omega)$ for $k \in (\frac{3}{2}, 2)$. It follows that for every $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that

$$\|u\|_{H^k(\Omega)} \leq \varepsilon \|\Delta u\|_{L^2(\Omega)} + C\|u\|_{L^2(\Omega)} \tag{C1.1}$$

for all $u \in H^k(\Omega)$. (This result is sometimes called Ehrling's Lemma; see [**98**, Theorem 6.99].)

Now we note that B is bounded as an operator from $H^k(\Omega)$ to $L^2(\partial\Omega)$ for all $k \in (\frac{3}{2}, 2)$. For, if $u \in H^k(\Omega)$, then by [**64**, Theorem 1.5.1.2] (a higher order version of the trace theorem) $\frac{\partial u}{\partial \nu}$ exists in $H^{k-\frac{3}{2}}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$ with norm dominated by $\|u\|_{H^k(\Omega)}$.

That is, there exists $\tilde{K} > 0$ independent of u such that $\|Bu\|_{L^2(\partial\Omega)} \leq \tilde{K}\|u\|_{H^k(\Omega)}$. Since in particular $\mathcal{D}(\Delta^D) \subset \mathcal{D}(B)$, the latter containing $H^k(\Omega)$ for every $k > \frac{3}{2}$, the conclusion of the lemma follows immediately from combining this with (C1.1). □

C2. On the homogeneous Dirichlet problem

Here we consider the map $P : L^2(\partial\Omega) \to L^2(\Omega)$ used in Section 5.3, which is related to the homogeneous Dirichlet problem

$$\begin{aligned} -\Delta u &= 0 \qquad \text{in } \Omega, \\ u &= f \qquad \text{on } \partial\Omega, \end{aligned}$$

for $f \in L^2(\Omega)$. We will assume throughout that Ω is of class C^1 only. We will also assume for the meantime that $2 \leq p < \infty$, as this makes no difference from the case $p = 2$. We start by defining the non-tangential maximal function N_α of $u \in L^p(\Omega) \cap C(\Omega)$ by

$$\begin{aligned} \Gamma_\alpha(z) &:= \{x \in \Omega : \operatorname{dist}(x, z) \leq (1+\alpha)\operatorname{dist}(x, \partial\Omega)\} \\ N_\alpha(u)(z) &:= \sup\{|u(x)| : x \in \Gamma_\alpha(z)\}, \end{aligned} \tag{C2.1}$$

where $\alpha > 0$ is fixed, arbitrary.

Let $f \in L^p(\partial\Omega)$ for some $2 \leq p < \infty$. Then by [**71**, Corollary 3.2] there is a unique harmonic function $u \in L^p(\Omega) \cap C^2(\Omega)$ such that u converges non-tangentially σ-a.e. to f and for any $\alpha > 0$, there exists a constant $C = C(p, \alpha)$ such that

$$\|N_\alpha(u)\|_{L^p(\partial\Omega)} \leq C(p, \alpha)\|f\|_{L^p(\Omega)}. \tag{C2.2}$$

(In fact this result holds for Lipschitz Ω. By standard results we also have $u \in C^\infty(\Omega)$ even in this case, but we will not need this.) The map $P : L^p(\partial\Omega) \to L^p(\Omega)$ is then defined by $Pf = u$.

Theorem C2.1. *For $p = 2$ and Ω of class C^1 the map $P : L^2(\partial\Omega) \to L^2(\Omega)$ is well-defined, linear and bounded.*

That P is well-defined and linear follows immediately from [**71**]. As for boundedness, it is well-known that if $f \in L^2(\partial\Omega)$, then $u \in H^{\frac{1}{2}}(\Omega)$ (see, e.g., [**71**, Section 6.3]); however, this does not immediately imply P is bounded, since it is not clear if P is closed as an operator from $L^2(\partial\Omega)$ to $H^{\frac{1}{2}}(\Omega)$. Moreover, while we very strongly expect this is true we cannot find any reference for this, at least for our class of domains (though see [**101**, p. 236] and the references therein). To complete the proof of Theorem C2.1, we will prove here that $P : L^2(\partial\Omega) \to L^2(\Omega)$ is bounded.

To do so we will first establish an estimate near the boundary using (C2.2), and then an interior estimate. Cover $\partial\Omega$ by a collection of k local coordinate systems: choose $z_i \in \partial\Omega$ and cubes $Q_i = (-b_i, b_i)^N$ containing z_i, $i = 1, \ldots, k$, such that for each i, in the local coordinate system $x_1, \ldots, x_N$, inside Q_i $\partial\Omega$ is the graph of a C^1 function $\varphi_i = \varphi_i(x_1, \ldots, x_{N-1})$, and such that $\partial\Omega \subset \bigcup_{i=1}^k Q_i$. Then for each i, $\|\nabla\varphi_i\|_{L^\infty(Q_i)}$ is bounded.

For $z \in \partial\Omega \cap Q_i$, let

$$V_z := \{x = (x_1, \ldots, x_N) \in \Omega : z = (x_1, \ldots, x_{N-1}, \varphi_i(x_1, \ldots x_{N-1}))\}$$

be the vertical line passing through x contained in $\Omega \cap Q_i$.

Lemma C2.2. *In the framework described above, there exists $\alpha = \alpha(i)$ such that for all $z \in \operatorname{graph}\varphi_i$, the set $V_z \subset \Gamma_\alpha(z)$.*

Proof. Suppose that Ω is of class C^1, fix i and suppose for a contradiction that for all $\alpha > 0$ there exists $z(\alpha) \in \operatorname{graph}\varphi_i$ such that $V_z \not\subset \Gamma_\alpha(z)$.

Choose $\alpha > \|\nabla\varphi_i\|_{L^\infty(Q_i)}$. Then there exists $z \in \operatorname{graph}\varphi_i$ and $x \in V_z$ with $(1+\alpha)\operatorname{dist}(x, \partial\Omega) < \operatorname{dist}(x, z)$.

Let $w \in \partial\Omega$ be such that $\operatorname{dist}(x, w) < \frac{1}{1+\alpha}\operatorname{dist}(x, z)$. We will write $w' = (w_1, \ldots, w_{N-1}, 0)$ and $z' = (z_1, \ldots, z_{N-1}, 0)$ (so that $z = (z', \varphi_i(z'))$ and similarly

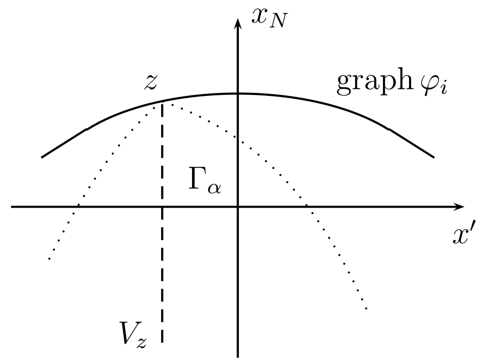

Figure C.1. V_z and Γ_α

for w). Let m be the absolute value of the gradient of the line joining w and z. Then $m \leq \|\nabla\varphi_i\|_{L^\infty(Q_i)}$.

We will show that we must have $m > \alpha$, which contradicts $m \leq \|\nabla\varphi_i\|_{L^\infty(Q_i)} < \alpha$. To this end, suppose without loss of generality that $\operatorname{dist}(x, z) = 1$. Then $\operatorname{dist}(x, w) \leq \frac{1}{1+\alpha}$. As in Figure C2, let $y = (z_1, \ldots, z_{N-1}, \varphi_i(w_N))$. If we let $b = \operatorname{dist}(w, y)$ and $a = \operatorname{dist}(y, z)$, then we have $m = a/b$.

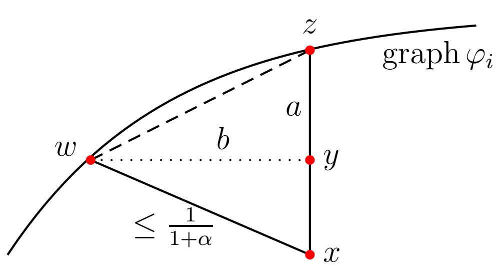

Figure C.2.

We will minimise the ratio $m = a/b$. First note that for m to be minimal, we must have $w_N > x_N$ (so that the angle wxz is acute). Next note that $b^2 \leq \frac{1}{(1+\alpha)^2} - (1-a)^2$. Then

$$m = \frac{a}{b} \geq \frac{a}{\sqrt{\frac{1}{(1+\alpha)^2} - (1-a)^2}}, \tag{C2.3}$$

where $0 < a \leq 1$ and $1 - a < \frac{1}{1+\alpha}$, that is, $a > 1 - \frac{1}{1+\alpha} > 0$. We claim that the right hand side of (C2.3) is minimised over the appropriate range of a, α when

$$m = \sqrt{(1+\alpha)^2 - 1} = \sqrt{\alpha^2 + 2\alpha} > \alpha.$$

Indeed, an elementary calculation shows that

$$\frac{a}{\sqrt{\frac{1}{(1+\alpha)^2} - (1-a)^2}} \geq \sqrt{(1+\alpha)^2 - 1}$$

if and only if

$$\frac{(\alpha+1)^2}{(\alpha+1)^2 - 1} a^2 - 2a + \frac{(\alpha+1)^2 - 1}{(\alpha+1)^2} \geq 0. \tag{C2.4}$$

Now another elementary calculation shows this quadratic expression has a repeated root at $a = 1 - \frac{1}{(\alpha+1)^2}$, showing that the expression is positive semi-definite and so (C2.4) indeed holds for our range of a, α, proving our claim.

In particular, we conclude that

$$\alpha \leq m \leq \|\nabla\varphi_i\|_{L^\infty(Q_i)} < \alpha,$$

a contradiction. □

The next result can be stated for $2 \leq p < \infty$.

Lemma C2.3. *There exist $\Omega_0 \subset\subset \Omega$, $\alpha > 0$ and $K_0 = K_0(p, \alpha) > 0$ such that, on $\Omega \setminus \overline{\Omega}_0$,*

$$\|u\|_{L^p(\Omega\setminus\overline{\Omega}_0)} \leq K_0 \|N_\alpha(u)\|_{L^p(\partial\Omega)} \tag{C2.5}$$

for all $u \in L^p(\Omega \setminus \overline{\Omega}_0)$.

Note that for some $u \in L^p(\Omega \setminus \overline{\Omega}_0)$, the right hand side of (C2.5) may be ∞.

Proof. Choose a local covering of $\partial\Omega$ with cubes $Q_i = (-b_i, b_i)^N$ as in Lemma C2.2, set

$$\Omega_0 := \Omega \setminus \Big(\bigcup_{i=1}^{k} \overline{Q}_i\Big)$$

and choose $\alpha > 0$ satisfying the conclusion of Lemma C2.2 for every i. Now for $u \in L^p(\Omega \setminus \overline{\Omega}_0)$, write

$$\begin{aligned}\int_{\Omega\cap Q_i} |u|^p\,dx &= \int_{\mathbb{R}^{N-1}\cap Q_i} \int_{-b_i}^{\varphi_i(x')} |u|^p\,dx_N\,dx' \\ &\leq \int_{\mathbb{R}^{N-1}\cap Q_i} |N_\alpha(u)(x', \varphi_i(x'))|^p 2b_i\,dx',\end{aligned}$$

where we have replaced u on V_z by the constant function $|N_\alpha(u)(z)| \geq |u(x)|$ for $x \in V_z$, and also $\varphi_i(x')$ by b_i. Hence

$$\begin{aligned}\int_{\Omega\cap Q_i} |u|^p\,dx &\leq \int_{\mathbb{R}^{N-1}\cap Q_i} 2b_i|N_\alpha(u)(x',\varphi_i(x'))|^p\sqrt{1+|\nabla\varphi_i|^2(x')}\,dx' \\ &= 2b_i\|N_\alpha(u)\|^p_{L^p(\partial\Omega\cap Q_i)} \leq 2b_i\|N_\alpha(u)\|^p_{L^p(\partial\Omega)}.\end{aligned}$$

Summing over all k cubes,

$$\begin{aligned}\|u\|^p_{L^p(\Omega\setminus\overline{\Omega}_0)} = \int_{\Omega\setminus\overline{\Omega}_0} |u|^p\,dx &\leq \sum_{i=1}^k \int_{\Omega\cap Q_i} |u|^p\,dx \\ &\leq k\max_i 2b_i\|N_\alpha(u)\|^p_{L^p(\partial\Omega)}.\end{aligned}$$

□

Combining Lemma C2.3 with (C2.2), we see that there exist $\Omega_0 \subset\subset \Omega$, $\alpha > 0$ and $C_0 > 0$ such that

$$\|Pf\|_{L^p(\Omega\setminus\overline{\Omega}_0)} \leq C_0\|f\|_{L^p(\partial\Omega)} \tag{C2.6}$$

for all $f \in L^p(\partial\Omega)$. We will now prove an interior estimate directly using the Poisson integral formula. While this should be valid for general $2 \leq p < \infty$, our proof only works in the case $p = 2$. It is immediate that Theorem C2.1 follows from (C2.6) and Lemma C2.4.

Lemma C2.4. *Under the assumptions of Theorem C2.1, suppose that $\Omega_0 \subset\subset \Omega$. Then there exists a constant $C > 0$ depending only on Ω and Ω_0 such that*

$$\|Pf\|_{L^2(\Omega_0)} \leq C\|f\|_{L^2(\partial\Omega)} \tag{C2.7}$$

for all $f \in L^2(\partial\Omega)$.

Proof. Without loss of generality, we may assume that $0 \leq f \not\equiv 0$ (that is, f is non-negative and not 0 σ-almost everywhere). If $f = 0$, then $Pf = 0$ and there is nothing to prove. If $f \in L^2(\partial\Omega)$ is not non-negative everywhere, then write $f = f^+ - f^-$, where $f^+, f^- \geq 0$. Since P is linear, writing $u = Pf = Pf^+ - Pf^-$ and, assuming (C2.7) holds for non-negative functions, we have

$$\begin{aligned}\|u\|_{L^2(\Omega_0)} &\leq \|Pf^+\|_{L^2(\Omega_0)} + \|Pf^-\|_{L^2(\Omega_0)} \\ &\leq C\|f^+\|_{L^2(\partial\Omega)} + C\|f^-\|_{L^2(\partial\Omega)} \leq 2C\|f\|_{L^2(\partial\Omega)}.\end{aligned}$$

So now suppose $0 \leq f \in L^1(\partial\Omega) \supset L^2(\partial\Omega)$ and $f > 0$ on a set of σ-positive measure and write $u = Pf \in C^\infty(\Omega)$ (for the fact that we can do this for $f \in L^1(\partial\Omega)$ we refer to [**71**]). By the maximum principle, we must have $u(x) > 0$ for all $x \in \Omega$, since otherwise the harmonic function u would attain an interior minimum. Given $\Omega_0 \subset\subset \Omega$, by the Harnack inequality there exists a constant $k > 0$ depending only on Ω and Ω_0 such that

$$v(x) \leq kv(y)$$

for every positive harmonic function v on Ω and all $x, y \in \overline{\Omega_0}$ (see [**59**, Theorem 2.5] or [**71**, p. 13]). For our harmonic function $u = Pf$, for each $x \in \Omega$, we may write

$$u(x) := \int_{\partial\Omega} f \, d\omega^x,$$

where ω^x is the harmonic measure on $\partial\Omega$ at x (see [**71**]). Note that $f \in L^1(\partial\Omega, d\omega^x)$ for all $x \in \Omega$ (this is immediate from the definition). This means that if we fix $x_0 \in \Omega_0$, then we have

$$\frac{1}{k}\int_{\partial\Omega} f \, d\omega^x \leq \int_{\partial\Omega} f \, d\omega^{x_0} \leq k \int_{\partial\Omega} f \, d\omega^x$$

for all $f \in L^1(\partial\Omega)$ and all $x \in \overline{\Omega_0}$. Now since ω^x and ω^{x_0} are mutually absolutely continuous (see [**71**, Theorem 3.1]), by the Radon-Nikodym theorem there exists $h \in L^1(\partial\Omega, d\omega^{x_0})$ depending on $x \in \Omega$ such that $d\omega^x = hd\omega^{x_0}$. Hence

$$\int_{\partial\Omega} f\Big(\frac{h}{k}\Big) \, d\omega^{x_0} \leq \int_{\partial\Omega} f \, d\omega^{x_0}$$

for all $0 \leq f \in L^1(\partial\Omega, d\omega^{x_0})$. This means the function h may be identified with a linear functional on $L^1(\partial\Omega, d\omega^{x_0})$ with norm no more than 1. Identifying the dual of $L^1(\partial\Omega, d\omega^{x_0})$ with $L^\infty(\partial\Omega, d\omega^{x_0})$, this implies

$$\|h\|_{L^\infty(\partial\Omega, d\omega^{x_0})} \leq k.$$

Since ω^{x_0} and σ are mutually absolutely continuous (again, see [**71**, Theorem 3.1]), they have the same sets of measure 0 and hence give rise to the same L^∞-norms. That is,

$$\|h\|_{L^\infty(\partial\Omega, d\sigma)} = \|h\|_{L^\infty(\partial\Omega, d\omega^{x_0})} \leq k,$$

where the same $k > 0$ works for all $x \in \overline{\Omega_0}$. Writing $d\omega^{x_0} = gd\sigma$, we actually have $g \in L^2(\partial\Omega, d\sigma)$ by [**71**, Theorem 3.1(b)]. Now fix $f \in L^2(\partial\Omega) = L^2(\partial\Omega, d\sigma)$. Then for all $x \in \overline{\Omega_0}$, by the Cauchy-Schwarz inequality we have

$$\begin{aligned} u(x) = \int_{\partial\Omega} f\, d\omega^x &= \int_{\partial\Omega} f h(x) g\, d\sigma \\ &\leq \|h\|_{L^\infty(\partial\Omega, d\sigma)} \|f\|_{L^2(\partial\Omega, d\sigma)} \|g\|_{L^2(\partial\Omega, d\sigma)} \\ &\leq k \|g\|_{L^2(\partial\Omega, d\sigma)} \|f\|_{L^2(\partial\Omega, d\sigma)}, \end{aligned}$$

where k is the constant from the Harnack inequality depending only on Ω and Ω_0. Note also that g depends only on x_0 and hence on Ω_0. Squaring and integrating over Ω_0,

$$\|u\|^2_{L^2(\Omega_0)} \leq k^2 |\Omega_0| \|g\|_{L^2(\partial\Omega, d\sigma)} \|f\|_{L^2(\partial\Omega, d\sigma)}.$$

Setting $C := k|\Omega_0|^{\frac{1}{2}} \|g\|_{L^2(\partial\Omega, d\sigma)}$, which depends only on Ω and Ω_0, gives the desired estimate. $\square$

We also wish to consider the restriction of P to $H^{\frac{1}{2}}(\partial\Omega)$. It is known that for $s \in [0,1]$, P maps $H^s(\partial\Omega)$ into $H^{s+\frac{1}{2}}(\Omega)$ if Ω is Lipschitz (see [**69**, p. 165]). In particular, for Lipschitz Ω, the restriction of P to $H^{\frac{1}{2}}(\partial\Omega)$, which for simplicity we will still denote by P, is a linear operator $P : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$. Note that we now have that $\operatorname{tr}(Pf) = f$ for $f \in H^{\frac{1}{2}}(\partial\Omega)$ (see, e.g., [**69**, Theorem 5.1] if $N \geq 3$, or else use that the trace operator has a right continuous inverse [**64**, Theorem 1.5.1.3]). We will now give a proof that P is in fact bounded from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^1(\Omega)$. All Proposition C2.5 really says is that the set of harmonic functions in $H^1(\Omega)$ is closed in the H^1-norm, which is well known.

Proposition C2.5. *The operator $P : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is bounded. Here Ω is Lipschitz.*

Proof. By the closed graph theorem, it suffices to prove P is closed. So suppose $f_n \to f$ in $H^{\frac{1}{2}}(\partial\Omega)$ and $Pf_n \to u$ in $H^1(\Omega)$. Since $\mathcal{D}(P) = H^{\frac{1}{2}}(\partial\Omega)$ we only need to show $Pf = u$.

Now since $\Delta P f_n = 0$ for all n, for any $v \in C_c^\infty(\Omega)$ we have

$$0 = \int_\Omega v \Delta P f_n\, dx = -\int_\Omega \nabla P f_n \cdot \nabla v\, dx \longrightarrow -\int_\Omega \nabla u \cdot \nabla v\, dx$$

as $n \to \infty$ since $Pf_n \to u$ in $H^1(\Omega)$. Hence $\int_\Omega \nabla u \cdot \nabla v\, dx = 0$ for all $v \in C_c^\infty(\Omega)$, so that $\Delta u = 0$ in the sense of distributions.

Also, by the trace theorem (Theorem A4.2), $\operatorname{tr}(Pf_n) \to \operatorname{tr} u$ in $H^{\frac{1}{2}}(\partial\Omega)$, that is, $f = \operatorname{tr} u$. Since u solves $\Delta u = 0$ in Ω and $\operatorname{tr} u = f$, we conclude $u = Pf$. $\square$

C3. The Dirichlet-to-Neumann operator

Here we prove some properties of the Dirichlet-to-Neumann operator $N : L^2(\partial\Omega) \to L^2(\partial\Omega)$ given by (5.3.2), that is,

$$\mathcal{D}(N) = \{f \in L^2(\partial\Omega) : \frac{\partial}{\partial\nu}(Pf) \in L^2(\partial\Omega)\}$$
$$Nf = \frac{\partial}{\partial\nu}(Pf),$$

where for the rest of the section we will assume Ω is Lipschitz. We will use form methods to study the properties of N. For another approach using boundary layer techniques see Sections 7.11 and 12.c of [**102**].

Theorem C3.1. *The operator $-N$ generates a compact analytic semigroup of angle $\frac{\pi}{2}$ on $L^2(\partial\Omega)$ for Ω Lipschitz.*

Proof. Define a form on $H^{\frac{1}{2}}(\partial\Omega)$ by

$$Q(f, g) := \int_\Omega \nabla(Pf) \cdot \nabla(Pg)\, dx.$$

Then Q is bi- (sesqui-) linear, symmetric, non-negative and by Proposition C2.5 is bounded.

Moreover, Q is $L^2(\partial\Omega)$-elliptic in the sense that for all $\lambda > 0$ there exists $C = C(\lambda)$ such that

$$Q(f, f) + \lambda\|f\|^2_{L^2(\partial\Omega)} \geq C\|f\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \tag{C3.1}$$

for all $f \in H^{\frac{1}{2}}(\partial\Omega)$. To see this, fix $\lambda > 0$. Then by Maz'ja's inequality [**88**, Section 4.11], and using that $\operatorname{tr}(Pf) = f$, we have

$$\begin{aligned} Q(f, f) + \lambda\|f\|^2_{L^2(\partial\Omega)} &= \|\nabla(Pf)\|^2_{L^2(\Omega)} + \lambda\|\operatorname{tr}(Pf)\|^2_{L^2(\partial\Omega)} \\ &\geq \min\{\lambda, 1\}C_1(\Omega)\|Pf\|^2_{L^{\frac{2N}{N-2}}(\Omega)} \\ &\geq \min\{\lambda, 1\}C_2(\Omega)\|Pf\|^2_{L^2(\Omega)}. \end{aligned}$$

In particular,

$$\begin{aligned} Q(f, f) + \lambda\|f\|_{L^2(\partial\Omega)} &\geq C_3(\lambda, \Omega)(\|\nabla(Pf)\|^2_{L^2(\Omega)} + \|Pf\|^2_{L^2(\Omega)}) \\ &\geq C_4(\lambda, \Omega)\|f\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \end{aligned}$$

by the trace inequality. This establishes (C3.1).

Next we establish that N is the operator associated with Q. That operator, call it $\tilde{N}$, is given by

$$
\begin{aligned}
&\mathcal{D}(\tilde{N}) = \{f \in \mathcal{D}(Q) = H^{\frac{1}{2}}(\partial\Omega) :\ \text{there exists } h \in L^2(\partial\Omega) \\
&\qquad\qquad \text{such that } Q(f,g) = \langle h, g\rangle_{L^2(\partial\Omega)} \text{ for all } g \in H^{\frac{1}{2}}(\partial\Omega)\} \\
&\tilde{N}f = g.
\end{aligned}
\tag{C3.2}
$$

So suppose that $f \in \mathcal{D}(N)$. We certainly then have $f \in H^{\frac{1}{2}}(\partial\Omega) = \mathcal{D}(Q)$ (for example $\frac{\partial}{\partial\nu}(Pf) \in L^2(\partial\Omega)$ implies $Pf \in H^{\frac{3}{2}}(\Omega)$ by [**70**]; see also [**69**, Theorem 5.15]). Then for any $g \in H^{\frac{1}{2}}(\partial\Omega)$,

$$
\begin{aligned}
\int_\Omega \nabla(Pf)\cdot\nabla(Pg) &= \int_\Omega (Pg)\Delta(Pf)\,dx + \int_\Omega \nabla(Pf)\cdot\nabla(Pg)\,dx \\
&= \int_{\partial\Omega} g\frac{\partial}{\partial\nu}(Pf)\,dx,
\end{aligned}
$$

that is, $Q(f,g) = \langle \frac{\partial}{\partial\nu}(Pf), g\rangle_{L^2(\partial\Omega)}$.

This shows that if $f \in \mathcal{D}(N)$, then f satisfies the domain condition (C3.2) and $Q(f,g) = \langle Nf, g\rangle_{L^2(\partial\Omega)}$ for all $g \in H^{\frac{1}{2}}(\partial\Omega)$; thus $N \subset \tilde{N}$ in the sense of operators. For the converse, let $f \in \mathcal{D}(\tilde{N})$, write $\tilde{N}f = h$ and for $v \in H^1(\Omega)$ arbitrary write $v = u + Pg$, where $u \in H^1_0(\Omega)$ and $g \in H^{\frac{1}{2}}(\Omega)$. Then $Pf \in H^1(\Omega)$ by Proposition C2.5, $\Delta(Pf) = 0$ in the sense of distributions, and $\int_\Omega \nabla(Pf)\cdot\nabla u\,dx = 0$ since $\operatorname{tr} u = 0$, and so

$$
\begin{aligned}
\int_\Omega v\Delta(Pf) + \int_\Omega \nabla(Pf)\cdot\nabla v\,dx = 0 &+ \int_\Omega \nabla(Pf)\cdot\nabla u\,dx \\
&+ \int_\Omega \nabla(Pf)\cdot\nabla(Pg)\,dx = \int_{\partial\Omega} h\operatorname{tr} v\,dx
\end{aligned}
$$

for all $v \in H^1(\Omega)$, where for the last step we used the definition of $\tilde{N}$. By definition $h = \frac{\partial}{\partial\nu}(Pf)$ (see (A4.3)), that is, $f \in \mathcal{D}(N)$ and $\tilde{N}f = h = Nf$. Hence $N = \tilde{N}$ is associated with Q.

Now it follows that N has compact resolvent since the form domain $H^{\frac{1}{2}}(\partial\Omega)$ embeds compactly into $L^2(\partial\Omega)$ by Rellich's theorem. It also follows by standard theory (see for example Section 7.1 or Section 3.14 of [**9**]) that $-N$ generates an analytic semigroup of angle $\frac{\pi}{2}$. $\square$

We can also consider the restriction of N to the form domain $H^{\frac{1}{2}}(\partial\Omega)$. In this case we immediately know that $-N$ still generates an analytic semigroup of angle

$\frac{\pi}{2}$. If Ω is sufficiently smooth we have $N : H^{k+1}(\partial\Omega) \to H^k(\partial\Omega)$ for any $k \in \mathbb{R}^+$ (see [**102**, Section 7.11]).

Theorem C3.2. *Let Ω be Lipschitz. The part of $-N$ in $H^{\frac{1}{2}}(\partial\Omega)$, given by the negative of*

$$\begin{aligned} \mathcal{D}(N) &= \{f \in H^{\frac{1}{2}}(\partial\Omega) : \frac{\partial}{\partial\nu}(Pf) \in H^{\frac{1}{2}}(\partial\Omega)\} \\ Nf &= \frac{\partial}{\partial\nu}(Pf), \end{aligned} \tag{C3.3}$$

generates a compact analytic semigroup of angle $\frac{\pi}{2}$ on $H^{\frac{1}{2}}(\partial\Omega)$.

Proof. That $-N$ generates an analytic semigroup of angle $\frac{\pi}{2}$ on its form domain $H^{\frac{1}{2}}(\partial\Omega)$ is a standard result (cf. the proof of Corollary 5.2.2). For compactness, since $-N$ is associated with a symmetric form it is self-adjoint and hence there exists a square root operator $A = (-N)^{\frac{1}{2}}$ (that is, $A^2 = -N$) such that $\mathcal{D}(A)$ is the form domain $H^{\frac{1}{2}}(\partial\Omega)$ (see [**72**, Theorems V.3.35 and VI.2.23]). By properties of Sobolev towers, $\mathcal{D}(A^2) = \mathcal{D}(N)$ embeds compactly (and densely) in $\mathcal{D}(A) = H^{\frac{1}{2}}(\partial\Omega)$ (see [**6**, Theorem V.1.3.8]). In particular the domain of the part of $-N$ in $H^{\frac{1}{2}}(\partial\Omega)$ certainly embeds compactly in $H^{\frac{1}{2}}(\partial\Omega)$. Since in addition $R(\lambda, -N|_{H^{\frac{1}{2}}(\partial\Omega)})$ is well-defined and closed for $\lambda \in \rho(-N)$ (see [**9**, Proposition 3.10.3]), it must be compact. $\square$

References

[1] Mini-Workshop: Shape Analysis for Eigenvalues. *Oberwolfach Rep.*, 4(2):995–1025, 2007. Abstracts from the mini-workshop held April 8–14, 2007, Organized by Dorin Bucur, Giuseppe Buttazzo and Antoine Henrot, Oberwolfach Reports. Vol. 4, no. 2.

[2] Robert A. Adams. *Sobolev spaces.* Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[3] S. Agmon, A. Douglis, and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.

[4] Shmuel Agmon. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.*, 15:119–147, 1962.

[5] Lars V. Ahlfors. *Conformal invariants: topics in geometric function theory.* McGraw-Hill, New York, 1973.

[6] Herbert Amann. *Linear and quasilinear parabolic problems. Vol. I.* Abstract linear theory, volume 89 of *Monographs in Mathematics.* Birkhäuser Boston Inc., Boston, MA, 1995.

[7] W. Arendt, G. Metafune, D. Pallara, and S. Romanelli. The Laplacian with Wentzell-Robin boundary conditions on spaces of continuous functions. *Semigroup Forum*, 67(2):247–261, 2003.

[8] W. Arendt and A. F. M. ter Elst. Sectorial forms and degenerate differential operators. *arXiv*, 2008. arXiv:0812.3944v1.

[9] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 2001.

[10] Wolfgang Arendt and Philippe Bénilan. Wiener regularity and heat semigroups on spaces of continuous functions. In *Topics in nonlinear analysis*,

volume 35 of *Progr. Nonlinear Differential Equations Appl.*, pages 29–49. Birkhäuser, Basel, 1999.

[11] Wolfgang Arendt and Mahamadi Warma. The Laplacian with Robin boundary conditions on arbitrary domains. *Potential Anal.*, 19(4):341–363, 2003.

[12] N. Aronszajn. A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order. *J. Math. Pures Appl. (9)*, 36:235–249, 1957.

[13] Catherine Bandle. *Isoperimetric inequalities and applications*, volume 7 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass., 1980.

[14] Catherine Bandle, Joachim von Below, and Wolfgang Reichel. Parabolic problems with dynamical boundary conditions: eigenvalue expansions and blow up. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 17(1):35–67, 2006.

[15] Markus Biegert and Mahamadi Warma. Regularity in capacity and the Dirichlet Laplacian. *Potential Anal.*, 25(3):289–305, 2006.

[16] Paul A. Binding, Patrick J. Browne, and Bruce A. Watson. Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. II. *J. Comput. Appl. Math.*, 148(1):147–168, 2002.

[17] Paul A. Binding, Patrick J. Browne, and Bruce A. Watson. Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter. I. *Proc. Edinb. Math. Soc. (2)*, 45(3):631–645, 2002.

[18] Paul A. Binding, Patrick J. Browne, and Bruce A. Watson. Transformations between Sturm-Liouville problems with eigenvalue dependent and independent boundary conditions. *Bull. London Math. Soc.*, 33(6):749–757, 2001.

[19] Marie-Hélène Bossel. Longueurs extrémales et fonctionnelles de domaine. *Complex Variables Theory Appl.*, 6(2-4):203–234, 1986.

[20] Marie-Hélène Bossel. Membranes élastiquement liées: extension du théorème de Rayleigh-Faber-Krahn et de l'inégalité de Cheeger. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(1):47–50, 1986.

[21] Haïm Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. Théorie et applications. Masson, Paris, 1983.

[22] Dorin Bucur and Daniel Daners. An alternative approach to the Faber-Krahn inequality for Robin problems. *Calc. Var. Partial Differential Equations*,

37:75–86, 2010.

[23] Dorin Bucur and Alessandro Giacomini. A variational approach to the isoperimetric inequality for the Robin eigenvalue problem. Preprint.

[24] Dorin Bucur and Antoine Henrot. Minimization of the third eigenvalue of the Dirichlet Laplacian. *R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci.*, 456(1996):985–996, 2000.

[25] Yu. D. Burago and V. A. Zalgaller. *Geometric inequalities*, volume 285 of *Grundlehren der Mathematischen Wissenschaften*. Translated from the Russian by A. B. Sosinskiĭ, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1988.

[26] Peter Buser. On Cheeger's inequality $\lambda_1 \geq h^2/4$. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 29–77. Amer. Math. Soc., Providence, R.I., 1980.

[27] Peter Buser. A note on the isoperimetric constant. *Ann. Sci. École Norm. Sup. (4)*, 15(2):213–230, 1982.

[28] Giuseppe Buttazzo and Gianni Dal Maso. An existence result for a class of shape optimization problems. *Arch. Rational Mech. Anal.*, 122(2):183–195, 1993.

[29] Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton Univ. Press, Princeton, N. J., 1970.

[30] R. Courant and D. Hilbert. *Methods of mathematical physics. Vol. I.* Interscience Publishers, New York, N.Y., 1953.

[31] Qiuyi Dai and Yuxia Fu. Faber-Krahn inequality for Robin problem involving p-Laplacian. *arXiv*, 2009. arXiv:0912.0393v1

[32] E. N. Dancer and D. Daners. Domain perturbation for elliptic equations subject to Robin boundary conditions. *J. Differential Equations*, 138(1):86–132, 1997.

[33] Daniel Daners. Robin boundary value problems on arbitrary domains. *Trans. Amer. Math. Soc.*, 352(9):4207–4236, 2000.

[34] Daniel Daners. Dirichlet problems on varying domains. *J. Differential Equations*, 188(2):591–624, 2003.

[35] Daniel Daners. A Faber-Krahn inequality for Robin problems in any space dimension. *Math. Ann.*, 335(4):767–785, 2006.

[36] Daniel Daners. Inverse positivity for general Robin problems on Lipschitz domains. *Arch. Math.*, 92:57–69, 2009.

[37] Daniel Daners and Pavel Drábek. A priori estimates for a class of quasi-linear elliptic equations. *Trans. Amer. Math. Soc.*, 361(12):6475–6500, 2009.

[38] Daniel Daners and James B. Kennedy. On the asymptotic behaviour of the eigenvalues of a Robin problem with a large parameter. *Differential Integral Equations*, 23(7-8):659–669, 2010.

[39] Daniel Daners and James Kennedy. Uniqueness in the Faber-Krahn inequality for Robin problems. *SIAM J. Math. Anal.*, 39(4):1191–1207, 2007/08.

[40] Robert Dautray and Jacques-Louis Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 2.* Functional and variational methods. Translated from the French by Ian N. Sneddon. Springer-Verlag, Berlin, 1988.

[41] Robert Dautray and Jacques-Louis Lions. *Mathematical analysis and numerical methods for science and technology. Vol. 5.* Evolution problems. I. Translated from the French by Alan Craig. Springer-Verlag, Berlin, 1992.

[42] Edward Brian Davies. *One-parameter semigroups*, volume 15 of *London Mathematical Society Monographs*. Academic Press, London, 1980.

[43] Pavel Doktor. Approximation of domains with Lipschitzian boundary. *Časopis Pěst. Mat.*, 101(3):237–255, 1976.

[44] Klaus-Jochen Engel. The Laplacian on $C(\overline{\Omega})$ with generalized Wentzell boundary conditions. *Arch. Math. (Basel)*, 81(5):548–558, 2003.

[45] Klaus-Jochen Engel. Second order differential operators on $C[0,1]$ with Wentzell-Robin boundary conditions. In *Evolution equations*, volume 234 of *Lecture Notes in Pure and Appl. Math.*, pages 159–165. Dekker, New York, 2003.

[46] Klaus-Jochen Engel and Genni Fragnelli. Analyticity of semigroups generated by operators with generalized Wentzell boundary conditions. *Adv. Differential Equations*, 10(11):1301–1320, 2005.

[47] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[48] Lawrence C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[49] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[50] G. Faber. Beweiss dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt. *Sitzungsbericht der bayrischen Akademie der Wissenschaften zu München*, pages 169–172, 1923.

[51] Angelo Favini, Gisèle R. Goldstein, Jerome A. Goldstein, and Silvia Romanelli. General Wentzell boundary conditions, differential operators and analytic semigroups in $C[0,1]$. *Bol. Soc. Parana. Mat. (3)*, 20(1-2):93–103 (2003), 2002.

[52] Angelo Favini, Gisèle Ruiz Goldstein, Jerome A. Goldstein, Enrico Obrecht, and Silvia Romanelli. General Wentzell boundary conditions and analytic semigroups on $W^{1,p}(0,1)$. *Appl. Anal.*, 82(9):927–935, 2003.

[53] Angelo Favini, Gisèle Ruiz Goldstein, Jerome A. Goldstein, and Silvia Romanelli. C_0-semigroups generated by second order differential operators with general Wentzell boundary conditions. *Proc. Amer. Math. Soc.*, 128(7):1981–1989, 2000.

[54] Angelo Favini, Gisèle Ruiz Goldstein, Jerome A. Goldstein, and Silvia Romanelli. The heat equation with generalized Wentzell boundary condition. *J. Evol. Equ.*, 2(1):1–19, 2002.

[55] Angelo Favini, Gisèle Ruiz Goldstein, Jerome A. Goldstein, Enrico Obrecht, and Silvia Romanelli. The Laplacian with generalized Wentzell boundary conditions. In *Evolution equations: applications to physics, industry, life sciences and economics (Levico Terme, 2000)*, volume 55 of *Progr. Nonlinear Differential Equations Appl.*, pages 169–180. Birkhäuser, Basel, 2003.

[56] Herbert Federer. *Geometric measure theory*. volume 153 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York, 1969.

[57] L. E. Fraenkel. On regularity of the boundary in the theory of Sobolev spaces. *Proc. London Math. Soc. (3)*, 39(3):385–427, 1979.

[58] L. E. Fraenkel. *An introduction to maximum principles and symmetry in elliptic problems*, volume 128 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.

[59] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften*. Second edition, Springer-Verlag, Berlin, 1983.

[60] Tiziana Giorgi and Robert Smits. Eigenvalue estimates and critical temperature in zero fields for enhanced surface superconductivity. *Z. Angew. Math. Phys.*, 58(2):224–245, 2007.

[61] Tiziana Giorgi and Robert Smits. Bounds and monotonicity for the generalized Robin problem. *Z. Angew. Math. Phys.*, 59(4):600–618, 2008.

[62] Tiziana Giorgi and Robert G. Smits. Monotonicity results for the principal eigenvalue of the generalized Robin problem. *Illinois J. Math.*, 49(4):1133–1143 (electronic), 2005.

[63] Gisèle Ruiz Goldstein. Derivation and physical interpretation of general boundary conditions. *Adv. Differential Equations*, 11(4):457–480, 2006.

[64] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[65] Juha Heinonen, Tero Kilpeläinen, and Olli Martio. *Nonlinear potential theory of degenerate elliptic equations*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1993.

[66] Antoine Henrot. Minimization problems for eigenvalues of the Laplacian. *J. Evol. Equ.*, 3(3):443–461, 2003. Dedicated to Philippe Bénilan.

[67] Joseph Hersch. Sur la fréquence fondamentale d'une membrane vibrante: évaluations par défaut et principe de maximum. *Z. Angew. Math. Phys.*, 11:387–413, 1960.

[68] Morris W. Hirsch. *Differential Topology*. Graduate Texts in Mathematics, No. 33. Springer-Verlag, New York, 1976.

[69] David Jerison and Carlos E. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.

[70] David S. Jerison and Carlos E. Kenig. The Neumann problem on Lipschitz domains. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):203–207, 1981.

[71] David S. Jerison and Carlos E. Kenig. Boundary value problems on Lipschitz domains. In *Studies in partial differential equations*, volume 23 of *MAA Stud. Math.*, pages 1–68. Math. Assoc. America, Washington, DC, 1982.

[72] Tosio Kato. *Perturbation theory for linear operators.* volume 132 of *Grundlehren der Mathematischen Wissenschaften.* Second edition, Springer-Verlag, Berlin, 1976.

[73] B. Kawohl and V. Fridman. Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant. *Comment. Math. Univ. Carolin.*, 44(4):659–667, 2003.

[74] Bernhard Kawohl. *Rearrangements and convexity of level sets in PDE*, volume 1150 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1985.

[75] J. B. Kennedy. On the isoperimetric problem for the higher eigenvalues of the Robin and Wentzell Laplacians. *Z. Agnew. Math. Phys.*, to appear. Preprint arXiv:0910.3966v1.

[76] James Kennedy. A Faber-Krahn inequality for the Laplacian with generalised Wentzell boundary conditions. *J. Evol. Equ.*, 8(3):557–582, 2008.

[77] James Kennedy. An isoperimetric inequality for the second eigenvalue of the Laplacian with Robin boundary conditions. *Proc. Amer. Math. Soc.*, 137(2):627–633, 2009.

[78] E. Krahn. Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises. *Math. Ann.*, 94(1):97–100, 1925.

[79] Erwin Kreyszig. *Advanced engineering mathematics.* Eighth edition, John Wiley & Sons Inc., New York, 1999.

[80] A. A. Lacey, J. R. Ockendon, and J. Sabina. Multidimensional reaction diffusion equations with nonlinear boundary conditions. *SIAM J. Appl. Math.*, 58(5):1622–1647 (electronic), 1998.

[81] Olga A. Ladyzhenskaya and Nina N. Ural′tseva. *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York, 1968.

[82] An Lê. Eigenvalue problems for the p-Laplacian. *Nonlinear Anal.*, 64(5):1057–1099, 2006.

[83] Michael Levitin and Leonid Parnovski. On the principal eigenvalue of a Robin problem with a large parameter. *Math. Nachr.*, 281(2):272–281, 2008.

[84] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I.* Translated from the French by P. Kenneth. Volume 181 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, New York, 1972.

[85] Julián López-Gómez. Classifying smooth supersolutions for a general class of elliptic boundary value problems. *Adv. Differential Equations*, 8(9):1025–1042, 2003.

[86] Yuan Lou and Meijun Zhu. A singularly perturbed linear eigenvalue problem in C^1 domains. *Pacific J. Math.*, 214(2):323–334, 2004.

[87] Yousong Luo and Neil S. Trudinger. Linear second order elliptic equations with Venttsel′ boundary conditions. *Proc. Roy. Soc. Edinburgh Sect. A*, 118(3-4):193–207, 1991.

[88] Vladimir G. Maz'ja. *Sobolev spaces.* Translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985.

[89] William McLean. *Strongly elliptic systems and boundary integral equations.* Cambridge University Press, Cambridge, 2000.

[90] Delio Mugnolo and Silvia Romanelli. Dirichlet forms for general Wentzell boundary conditions, analytic semigroups, and cosine operator functions. *Electron. J. Differential Equations*, pages No. 118, 20 pp. (electronic), 2006.

[91] Jindřich Nečas. *Les méthodes directes en théorie des équations elliptiques.* Masson et Cie, Paris, 1967.

[92] El Maati Ouhabaz. *Analysis of heat equations on domains*, volume 31 of *London Mathematical Society Monographs Series.* Princeton University Press, Princeton, NJ, 2005.

[93] L. E. Payne. Isoperimetric inequalities and their applications. *SIAM Rev.*, 9:453–488, 1967.

[94] L. E. Payne and H. F. Weinberger. Lower bounds for vibration frequencies of elastically supported membranes and plates. *J. Soc. Indust. Appl. Math.*, 5:171–182, 1957.

[95] G. Pólya. On the characteristic frequencies of a symmetric membrane. *Math. Z.*, 63:331–337, 1955.

[96] G. Pólya and G. Szegö. *Isoperimetric Inequalities in Mathematical Physics.* Annals of Mathematics Studies, no. 27. Princeton University Press, Princeton, NJ, 1951.

[97] Lord Rayleigh. *The Theory of Sound.* Macmillan, London, 1877.

[98] Michael Renardy and Robert C. Rogers. *An introduction to partial differential equations*, volume 13 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1993.

[99] Walter Rudin. *Real and complex analysis.* Third edition, McGraw-Hill Book Co., New York, 1987.

[100] B. de Saint-Venant. Mémoire sur la torsion des prismes. *Mémoires présentés par divers savants à l'Académie des Sciences*, 14:233–560, 1856.

[101] Elias M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, NJ, 1970.

[102] Michael E. Taylor. *Partial differential equations. II.* Qualitative studies of linear equations. Volume 116 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1996.

[103] Peter Tolksdorf. Regularity for a more general class of quasilinear elliptic equations. *J. Differential Equations*, 51(1):126–150, 1984.

[104] J. L. Vázquez. A strong maximum principle for some quasilinear elliptic equations. *Appl. Math. Optim.*, 12(3):191–202, 1984.

[105] Juan Luis Vázquez and Enzo Vitillaro. Heat equation with dynamical boundary conditions of reactive type. *Comm. Partial Differential Equations*, 33(4-6):561–612, 2008.

[106] A. D. Ventcel′. On boundary conditions for multi-dimensional diffusion processes. *Theor. Probability Appl.*, 4:164–177, 1959.

[107] Wolfgang Walter. A theorem on elliptic differential inequalities with an application to gradient bounds. *Math. Z.*, 200(2):293–299, 1989.

[108] Mahamadi Warma. The Robin and Wentzell-Robin Laplacians on Lipschitz domains. *Semigroup Forum*, 73(1):10–30, 2006.

[109] G. N. Watson. *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, Cambridge, 1944.

[110] Sven Andreas Wolf and Joseph B. Keller. Range of the first two eigenvalues of the Laplacian. *Proc. Roy. Soc. London Ser. A*, 447(1930):397–412, 1994.