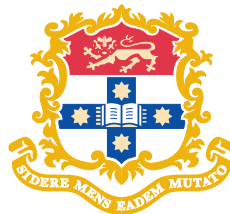


ASYMPTOTICS OF HIGHER-ORDER
PAINLEVÉ EQUATIONS

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ABSTRACT

We undertake an asymptotic study of a second Painlevé hierarchy based on the Jimbo-Miwa Lax pair in the limit as the independent variable approaches infinity. The hierarchy is defined by an infinite sequence of non-linear ordinary differential equations, indexed by order, with the classical second Painlevé equation as the first member. We investigate general and special asymptotic behaviours admitted by each equation in the hierarchy. We show that the general asymptotic behaviour is described by two related hyperelliptic functions, where the genus of the functions increases with each member of the hierarchy, and we prove that there exist special families of solutions which are represented by algebraic formal power series. For specific values of the constants which appear in the higher-order second Painlevé equations, exact solutions are also constructed.

Particular attention is given to the fourth-order analogue of the classical second Painlevé equation. In this case, the general asymptotic behaviour is given to leading-order by two related genus-2 hyperelliptic functions. These functions are characterised by four complex parameters which depend on the independent variable through the perturbation terms of the leading-order equations, and we investigate how these parameters change with respect to this variable. We also show that the fourth-order equation admits two classes of algebraic formal power series and that there exist families of true solutions with these behaviours in specified sectors of the complex plane, as well as unique solutions in extended sectors.

To complement our asymptotic study of higher-order Painlevé equations, we consider a new setting in which classical Painlevé equations arise. We study reaction-diffusion equations with quadratic and cubic source terms, with a spatio-temporal dependence included in those terms, and show that solutions of these equations are given by first and second Painlevé transcendents.

PREFACE

This work was conducted between July 2004 and February 2009 at the University of Sydney under the supervision of Nalini Joshi. Some results presented here have been previously published. Chapters 2 and 8 are based on the following collaborations with Nalini Joshi, where I am the primary contributor:

- Existence and uniqueness of Tronquée solutions of the fourth-order Jimbo-Miwa second Painlevé equation. *Proceedings of the American Mathematical Society*, 137(6): 2005-2014, 2009;
- New exact solutions of spatially and temporally varying reaction-diffusion equations. *Analysis and Applications*, 6(4): 371-381, 2008.

The contents of this thesis is original work of which I am the sole author. The use of existing works is explicitly and duly acknowledged in the text.

Sydney, June 2009

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INTRODUCTION

This thesis investigates general and special asymptotic behaviours admitted by higher-order Painlevé equations. These equations, and associated Painlevé hierarchies, appear in a range of problems in applied mathematics and mathematical physics, as well as in pure mathematical research. We study a second Painlevé hierarchy based on the Jimbo-Miwa Lax pair in the limit as the independent variable approaches infinity. Our work builds upon the foundations of established asymptotic results for the classical second Painlevé equation. Before performing our analysis, we begin with a survey of these classical results and an outline of the precise problem that we will address in this thesis.

1.1 CLASSICAL RESULTS

The asymptotics of the classical second Painlevé equation

$$P_{\text{II}} : \quad y_{xx} = 2y^3 + xy + \alpha, \quad |x| \rightarrow \infty, \quad \alpha = \text{constant},$$

have been widely studied. P_{II} first appeared in the classification work of Painlevé in 1900-02 [54, 55], and in 1913-14, Boutroux [7, 8] reported all possible local asymptotic behaviours of the P_{II} transcendent near infinity. Boutroux gave the transformation $z = \frac{2}{3}x^{3/2}$, $y(x) = \sqrt{x}u(z)$ which maps P_{II} to

$$u_{zz} = 2u^3 + u + \frac{1}{3z}(2\alpha + 3u) + \frac{u}{9z^2}. \quad (1.1)$$

The generic solution of equation (1.1), to leading-order in the limit as $|z| \rightarrow \infty$, is described by a Jacobi elliptic function which satisfies

$$v_z^2 = v^4 + v^2 + E,$$

for constant E . It follows that $v(z)$ is implicitly defined by the integral equation

$$\int_0^{v(z)} \frac{ds}{\sqrt{P(s; E)}} = z - \phi, \quad P(s; E) = s^4 + s^2 + E,$$

where ϕ is the position of a zero of $v(z)$ and the path of integration avoids the singularities of the integrand. The leading-order elliptic function solutions $v(z)$ are characterised by the parameters E and ϕ .

To describe the asymptotic behaviour of a particular solution $u(z)$ of equation (1.1), beyond the local neighbourhood of the initial point z_0 at which it is considered, it is necessary to investigate how the parameters E and ϕ change as z is varied near this point. Boutroux¹ considered the discrete change in E as z is moved from z_0 along a sequence of points $\{z_n\}$, $n \geq 0$, each separated by a period of the leading-order elliptic function ω_n , as defined at that point:

$$\omega_n = z_{n+1} - z_n, \quad \omega_n = \oint_{\rho} \frac{ds}{\sqrt{P(s; E_n)}}, \quad (1.2)$$

where $E_n \equiv E(z_n)$ for $E(z_n) = u_z(z_n) - u(z_n)^4 - u(z_n)^2$, and ρ is any contour enclosing two of the branch points of the integrand. He found

$$E_{n+1} - E_n = -\tilde{\omega}_n z_n^{-1} + \mathcal{O}(z_n^{-2}), \quad \tilde{\omega}_n = \oint_{\rho} \sqrt{P(s; E_n)} ds, \quad (1.3)$$

and argued that at any point z_n , for which the sequence $\{z_n\}$ admits infinite extension, E is determined from the transcendental equation

$$\oint_{\rho} \sqrt{s^4 + s^2 + E} = 0, \quad (1.4)$$

that is, $\tilde{\omega} = 0$, where $\tilde{\omega}$ is equal to $\tilde{\omega}_n$ (1.3) with $E_n = E$. This result expands the leading-order elliptic function estimate $v(z)$ beyond the local neighbourhood of the initial point z_0 to a larger domain of order z . An extended overview of these results is given in [22] and [37].

The complete description of the generic elliptic function asymptotics of P_{II} was obtained by Joshi and Kruskal [31, 32]. They solved the connection problem for P_{II} , relating the asymptotic behaviour of a solution along different paths of approach to infinity. By investigating the parameters E and ϕ through the method of multiple-scales, they showed that the solution $u(z)$ of equation (1.1) satisfies

$$\int_0^{u(z)} \frac{ds}{\sqrt{P(s; E)}} = z + \mu\omega_1 + \nu\omega_2 + t, \quad t = \mathcal{O}(z^{-1}), \quad (1.5)$$

in the interior of each quadrant in the z -plane. In this description, E is determined from (1.4), μ and ν are complex constants, and ω_1 and ω_2

¹ Boutroux performed this calculation for P_{I} , however the approach for P_{II} is identical.

denote the two periods,

$$\omega_i = \oint_{\rho_i} \frac{ds}{\sqrt{P(s; \mathbf{E})}}, \quad i = 1, 2, \quad (1.6)$$

of the elliptic function behaviour.² These asymptotic results are globally valid - the constants μ and ν , and the periods ω_1 and ω_2 , undergo specified jumps across the real and imaginary axes.³

While the generic elliptic function solution is known to possess poles in every neighbourhood of infinity in the complex plane, there are special families of solutions that do not have poles in every subsector of such a neighbourhood. Boutroux proved the existence of two one-parameter families of solutions of P_{II} that he called *tronquée*, and of six unique solutions called *tri-tronquée*. The solutions are asymptotic to

$$y(x) = \left(\frac{-x}{2}\right)^{1/2} \left(1 + \mathcal{O}(x^{-3/2(1-\epsilon)})\right), \quad (1.7)$$

or

$$y(x) = \frac{-\alpha}{x} \left(1 + \mathcal{O}(x^{-3/2(1-\epsilon)})\right), \quad (1.8)$$

for $\epsilon > 0$, in particular annular sectors of the plane as $|x| \rightarrow \infty$. There exists $x_0 \neq 0$, such that the *tronquée* solutions are pole-free in the sectors

$$\Omega_k = \{x \in \mathbb{C} \mid |x| > |x_0|, (k-1)\pi/3 < \arg x < (k+1)\pi/3\},$$

where $k = 0, \dots, 5$. Boutroux identified that each such sector is bisected by a ray given by $\arg x = k\pi/3$, and that no poles are present beyond a circle of some finite radius along such a ray. Similarly, the *tri-tronquée* solutions are pole-free in extended sectors $\Omega_k \cup \Omega_{k+2}$ within which lie three bisectors of the original sectors.

Boutroux also pointed out that the elliptic asymptotics degenerate into trigonometric asymptotics for $\arg x$ in a small neighbourhood of the rays $\arg x = k\pi/3$. Note that the boundaries of the sectors of validity of each description (1.5), given by the real and imaginary axes in the z -plane,

² Note that ω_1 and ω_2 (1.6) denote individual periods at any point z with \mathbf{E} defined by (1.4), whereas ω_n (1.2) denotes either of the elliptic function periods at the point z_n with $\mathbf{E}_n = \mathbf{E}(z_n)$.

³ Although we do not solve a connection problem in this work, we mention Joshi and Kruskal (J&K) because they utilised a *direct method* to derive their results, rather than solving the associated linear problem. We take a similar approach throughout this work, deriving results from the non-linear equations rather than the linear problems. We also note that the work of J&K builds upon that of McCoy, Tracy, and Wu [46, 47], Ablowitz and Segur [1], and Hastings and McLeod [25] who solved the connection problem for a special case of P_{II} , and that in both [1, 25] results from inverse scattering are used.

correspond to these special rays in the x -plane via the Boutroux transformation $z = \frac{2}{3}x^{3/2}$. This shows the connection between the domains of validity of the general and special asymptotic behaviours.

1.2 JIMBO-MIWA SECOND PAINLEVÉ HIERARCHY

P_{II} is one of six classical Painlevé equations ($P_{\text{I}} - P_{\text{VI}}$) which are regarded as completely integrable because they can be solved through an associated system of linear equations. P_{II} has two well-known linear problems, or Lax pairs, given by Flaschka and Newell (FN) in [21], and Jimbo and Miwa (JM) in [28]. The FN and JM linear problems for P_{II} are both given by 2×2 matrix systems of the form

$$\partial_\lambda \Psi = F\Psi, \quad (1.9a)$$

$$\partial_x \Psi = G\Psi, \quad (1.9b)$$

where the coefficient matrices F and G are analytic in x and rational in λ . The compatibility of the linear problem (1.9) is $\partial_{\lambda x} \Psi = \partial_{x\lambda} \Psi$, and this yields the following condition:

$$\partial_x F - \partial_\lambda G + [F, G] = 0, \quad (1.10)$$

where $[,]$ is the usual matrix commutator. In each case (FN or JM) the constraint (1.10) is satisfied by P_{II} or an equivalent system. Importantly, the matrices F in the two problems differ in their dependence on the spectral parameter λ ; the FN equation (1.9a) has an irregular singularity of rank three at infinity and a regular singularity at zero, whereas the JM equation (1.9a) has a single rank three singularity at infinity. Given this singularity structure there does not exist an algebraic gauge transformation between these two systems, however an invertible integral transformation between the FN and JM linear problems is given in [34].

P_{II} is also the first member of two infinite hierarchies of ODEs, which arise as symmetry reductions of two different PDE hierarchies: the modified KdV hierarchy, and a dispersive water wave hierarchy. The reduction of these equations and their associated linear problems are given explicitly in [29] and [23], respectively. For P_{II} , it is important to recognise that the two reductions lead to each of the two different linear problems given by Flaschka and Newell, and Jimbo and Miwa (see [24]). We therefore distinguish the two hierarchies by those names, and denote them FN $P_{\text{II}}^{(n)}$ and JM $P_{\text{II}}^{(n)}$.

There are many recent investigations of higher-order analogues of

Painlevé equations. Cosgrove [15, 16] completed a classification study of a class of fourth- and fifth-order ODEs with the Painlevé property, and reported equations that define new higher-order transcendents, as well as equations that are solved in terms of known special functions. Other studies have focussed on the extension of the numerous properties of the classical Painlevé equations to the equation hierarchies. For the Flaschka-Newell second Painlevé hierarchy, Bäcklund transformations and special integrals were studied in [13], rational solutions and their related special polynomials in [12, 17], and the Hamiltonian structure of the hierarchy in [45].

The asymptotic behaviour of the Jimbo-Miwa second Painlevé hierarchy has been examined by the group of Kawai and Takei, for example in [35, 50]. They study the asymptotic behaviour, for an introduced large parameter, of instanton (0-parameter) solutions of JM $P_{\text{II}}^{(n)}$, as well as $P_1^{(n)}$ and FN $P_{\text{II}}^{(n)}$. Their method of *exact WKB analysis* is based on Borel summation of WKB solutions. In this work, we also undertake an asymptotic study the Jimbo-Miwa hierarchy, but we consider a different asymptotic limit and a different method of analysis.

We concentrate on the Jimbo-Miwa hierarchy in the limit as the independent variable approaches infinity. We initially focus on the second member of the Jimbo-Miwa hierarchy

$$\text{JM } P_{\text{II}}^{(2)} : \quad \begin{aligned} u_{xx} - 3uu_x + u^3 + 6uv + 4g_3x &= 0, \\ v_{xx} + 3uv_x + 3v^2 + 3u^2v &= 4\alpha_2, \end{aligned} \quad (1.11)$$

and then we consider the general class of equations: for each integer $n \geq 1$, JM $P_{\text{II}}^{(n)}$ is defined by two n th-order ODEs

$$\text{JM } P_{\text{II}}^{(n)} : \quad \mathbf{a}_n + 2^n g_{n+1}(x, 0)^T = 2^n (0, \alpha_n)^T, \quad (1.12)$$

where $g_{n+1} \neq 0$ and α_n are constants, \mathbf{a}_n is defined recursively by

$$\mathbf{a}_n = \mathbf{r}\mathbf{a}_{n-1}, \quad \mathbf{a}_0 = (u, v)^T, \quad (1.13)$$

and \mathbf{r} is the matrix operator

$$\mathbf{r} = \begin{pmatrix} u - \partial_x & 2 \\ 2v - \partial_x^{-1}v_x & u + \partial_x - \partial_x^{-1}u_x \end{pmatrix}. \quad (1.14)$$

The first member of the hierarchy

$$\text{JM P}_{\text{II}}^{(1)} : \quad \begin{aligned} -u_x + u^2 + 2v + 2g_2x &= 0, \\ v_x + 2uv &= 2\alpha_1, \end{aligned} \quad (1.15)$$

is equivalent to P_{II} in u , and P_{34} (of [26]) in v . Similarly, the system $\text{JM P}_{\text{II}}^{(2)}$ (1.11) is equivalent to a fourth-order scalar equation in u

$$\begin{aligned} u_{xxxx} &= \frac{2u_x u_{xxx}}{u} + \frac{3u_{xx}^2}{2u} - \frac{2u_x^2}{u_{xx}}u + 5u^2 u_{xx} + \frac{8g_3x u_{xx}}{u} + \frac{5}{2}uu_x^2 \\ &- \frac{8g_3x u_x^2}{u^2} + \frac{8g_3u_x}{u} - \frac{5}{2}u^5 - 8g_3x u^2 - 12u(2\alpha_2 + g_3) + \frac{8g_3^2x^2}{u}. \end{aligned} \quad (1.16)$$

It is interesting to note that this equation also appears in other studies. In particular, it appears in a study of caustic-type limits of PDEs [36], and in [41] in a discussion of higher-order equations which admit hyperelliptic functions as asymptotic behaviours. Equation (1.16) is quite different to the other fourth-order analogue of P_{II} , or the second member of the Flaschka-Newell hierarchy,

$$\text{FN P}_{\text{II}}^{(2)} : \quad y_{xxxx} - 10y^2 y_{xx} - 10y y_x^2 + 6y^5 = xy + \alpha_2. \quad (1.17)$$

Thus, an important open question (to be revisited) is whether these two equations are related through a transformation of variables, or even whether they admit the same types of qualitative asymptotic behaviours in the limit as $|x| \rightarrow \infty$.

1.2.1 Hyperelliptic asymptotics

In addition to the elliptic asymptotics of P_{II} , the first five classical Painlevé equations all have general asymptotic behaviours described by elliptic functions. Furthermore, there are a number of studies which indicate that this asymptotic structure admits a natural extension to higher-order Painlevé equations. Drach [18] demonstrated that the stationary reductions of the KdV hierarchy are solved in terms of hyperelliptic functions, where the genus of the functions increases with each member of the hierarchy. In [29], Joshi reviewed this construction and showed that the large parameter limit of $\text{FN P}_{\text{II}}^{(n)}$ is solved by the stationary solutions of the KdV hierarchy. Littlewood [41] investigated a particular fourth-order Painlevé transcendent through its associated linear problem, and formulated solutions with hyperelliptic (genus-2) asymptotics. Equation (1.16) is explicitly identified in [41] as possessing hyperelliptic

asymptotics, however the form of the hyperelliptic solution is not given.

In this work we study the general asymptotic behaviour of each equation in the Jimbo-Miwa hierarchy, in the large independent variable limit, and show that the leading-order behaviour of JM $P_{\text{II}}^{(n)}$ is described by two related hyperelliptic functions of genus- n . We construct the explicit form of the asymptotics in each case via the method of Drach [18] applied to the compatibility condition of the associated scalar linear problem. The construction of the generic asymptotics of the hierarchy would not be possible without linear theory - the compatibility condition of the linearisation scheme is used critically in the derivation of these results.

Matrix and scalar linear problems for JM $P_{\text{II}}^{(n)}$ are both given in [23], where the process for moving between the two types of problems is also discussed. The scalar linear problem for JM $P_{\text{II}}^{(n)}$ is given by the system

$$\psi_{xx} = f\psi, \quad (1.18a)$$

$$\psi_\lambda = 2h_n\psi_x - (h_n)_x\psi, \quad (1.18b)$$

where the coefficient functions f and h_n are given by

$$f = \left(\lambda - \frac{1}{2}u\right)^2 + \frac{1}{2}u_x - v, \quad (1.19a)$$

$$\tilde{h}_n = \lambda^n + 2^{-n} \sum_{i=0}^{n-1} \mathbf{a}_{i,1} (2\lambda)^{n-1-i}, \quad \tilde{h}_n = g_{n+1}h_n, \quad (1.19b)$$

where $\mathbf{a}_{i,1}$ denotes the first component of \mathbf{a}_i (1.13), and (u, v) solves JM $P_{\text{II}}^{(n)}$ (1.12) with constant g_{n+1} . The compatibility of (1.18) is $\partial_{\lambda xx}\psi = \partial_{\lambda xx}\psi$ and this yields the following condition:

$$(h_n)_{xxx} - 4f(h_n)_x - 2f_x h_n + f_\lambda = 0. \quad (1.20)$$

For $n = 1$ and $n = 2$, (1.19b) implies

$$\tilde{h}_1 = \lambda + \frac{1}{2}u, \quad \tilde{h}_1 = g_2 h_1, \quad (1.21)$$

$$\tilde{h}_2 = \lambda^2 + \frac{1}{2}u\lambda + \frac{1}{4}(-u_x + u^2 + 2v), \quad \tilde{h}_2 = g_3 h_2, \quad (1.22)$$

and the constraint (1.20) is satisfied by the systems (1.15) and (1.11), respectively.

The generic asymptotic results that we derive for JM $P_{\text{II}}^{(n)}$ lead to complex analytic information about the solutions to the equations. This is particularly useful for the Painlevé equations because the solutions are highly transcendental functions whose explicit analytic properties have been difficult to deduce.

1.2.2 Algebraic asymptotics

Further to the study of the generic asymptotics of JM $P_{\text{II}}^{(n)}$ we consider special asymptotic behaviours admitted by the equations in this hierarchy. As mentioned, Painlevé equations and hierarchies appear in many areas of mathematical physics, and are of particular interest in the study of string equations of certain matrix models [48] and of gradient catastrophe in wave equations, including the focusing non-linear Schrödinger equation [20]. Remarkably, the physically important solutions in these settings turn out to be those that are pole-free, or *tronquée*, in certain sectors.

As well as the *tronquée* solutions derived by Boutroux for P_{I} and P_{II} , *tronquée*-type solutions of FN $P_{\text{II}}^{(n)}$ were considered by Joshi and Mazzocco [33]. In this work, we derive all possible algebraic asymptotic behaviours admitted by JM $P_{\text{II}}^{(n)}$. For the fourth-order equation, solutions with these behaviours are shown to extend the notion of Boutroux's *tronquée* and *tri-tronquée* solutions to higher-*tronquée* cases. We construct the solutions by direct analysis of the governing non-linear equations using the method of dominant balances [5]. To address questions of existence and uniqueness we utilise classical results from asymptotic theory.

1.2.3 Special solutions

As a corollary to the asymptotic study of the hierarchy we investigate special solutions of JM $P_{\text{II}}^{(n)}$ which are valid for particular values of the parameter α_n . For arbitrary values of the constants, solutions of the Painlevé equations cannot be written in terms of classical special functions (by definition). However when the parameters take special values, each equation $P_{\text{II}} - P_{\text{VI}}$ admits special solutions which are expressed in terms of such functions. It is well known that the second Painlevé equation admits two classes of special solutions. When α equals an integer, P_{II} can be solved by rational functions, and when α equals a half-integer, P_{II} has a one-parameter family of solutions written in terms of Airy functions. In each class, an infinite sequence of solutions is generated from compositions of Bäcklund transformations applied to two seed, or *immediately accessible*, solutions [51]. We investigate seed solutions of JM $P_{\text{II}}^{(n)}$ which extend these two special solutions through the hierarchy.

1.3 NEW APPLICATION

The widespread appearance of the Painlevé transcendents in applied mathematics has afforded them the status of new non-linear special functions. Since the prominent work of Ablowitz and Segur [1], which demonstrated the connection between Painlevé equations and completely integrable systems, there has been significant research in the area of Painlevé analysis.⁴ In addition to the asymptotic study described above, we explore a new connection between the first and second Painlevé equations and variable coefficient reaction-diffusion equations.

We consider generalised Fisher and Nagumo-type reaction-diffusion equations, which include a spatio-temporal dependence in the coefficient functions of their source terms. These equations are ubiquitous in the study of biological and physical systems, and in this work we show that solutions to such reaction-diffusion equations can be written in terms of the classical first and second Painlevé transcendents. An original feature of our analysis is that the coefficient functions are also solutions of differential equations, including the Painlevé equations. We do not assume any special boundary or initial conditions, so that the solutions we derive can be used to investigate the dynamics of any system of interest in the investigated class.

1.4 SYNOPSIS

This document is divided into four parts. The first two parts are devoted to the asymptotic study of the Jimbo-Miwa hierarchy and Part [iii](#) (Chapter [8](#)) is dedicated to the new application, described in [§1.3](#). An extended introduction to this topic is given in Chapter [8](#), which is largely self-contained. Part [iv](#) (Chapter [9](#)) offers brief concluding remarks.

In Part [i](#) we focus on the fourth-order system $\text{JM } P_{\text{II}}^{(2)}$ ([1.11](#)). We show in Chapter [2](#) that $\text{JM } P_{\text{II}}^{(2)}$ admits special asymptotic behaviours given by algebraic formal power series. The size and orientation of the sectors of validity of these asymptotic behaviours, and the number of free parameters in the asymptotic descriptions, are all given explicitly. In Chapter [3](#) we consider the general dominant asymptotic behaviour of $\text{JM } P_{\text{II}}^{(2)}$ and derive leading-order solutions expressed in terms of genus-2 hyperelliptic functions. We also analyse the behaviour of the parameters in the defining hyperelliptic functions, thus making the first steps toward

⁴ In the introduction to the recent special edition of *J. Phys. A*, celebrating 100 years of P_{VI} [[14](#)], Clarkson et al provide an excellent overview of the current state of the area.

extending the locally valid leading-order asymptotics to a wider domain of validity.

In Part [ii](#) we continue this dual asymptotic analysis of the general and special behaviours for the entire Jimbo-Miwa hierarchy. Further insight into the structure of the hierarchy and associated linear problem is provided in [Chapter 4](#). This underpins the asymptotic analysis of the subsequent chapters; in [Chapter 5](#) we examine special asymptotic behaviours of JM $P_{\text{II}}^{(n)}$ and in [Chapter 6](#) we interrogate the general asymptotic behaviour. Concluding Part [ii](#), in [Chapter 7](#) we present special solutions of the hierarchy which are valid for particular values of the parameter.

Part I

ASYMPTOTIC STUDY OF THE
FOURTH-ORDER JIMBO-MIWA SECOND
PAINLEVÉ EQUATION

2.1 ABSTRACT

In this chapter we begin our asymptotic study of the fourth-order Jimbo-Miwa second Painlevé equation. We prove that there are two families of algebraic formal power series solutions and that there exist true solutions with these behaviours in sectors σ of the complex plane. Given σ we also prove that there exists a wider sector $\Sigma \supset \sigma$ in which there exists a unique solution in each family. These provide the analogue of Boutroux's tri-tronquée solutions for the classical second Painlevé equation. Surprisingly, they also extend beyond the tri-tronquée solutions, in that we find penta-, hepta-, ennea-, and hendeca-tronquée solutions. We proceed by direct analysis of the non-linear equations and provide explicit details on the asymptotic expansion of the solutions, as well as the size and orientation of the sectors of validity of these descriptions.

The plan of the chapter is as follows. We construct formal solutions in §2.2 and prove their existence in §2.3. In §2.4, we prove that a subset of these solutions is unique and suggest a naming convention of these new solutions in §2.5. We conclude with a discussion in §2.6, where we note the differences between the asymptotic behaviours admitted by the Flaschka-Newell and Jimbo-Miwa fourth-order second Painlevé equations.

2.2 FORMAL SOLUTIONS

Proposition 2.1.¹ In the limit as $|x| \rightarrow \infty$, the system of equations JM $P_{\text{II}}^{(2)}$ (1.11) admits two families of formal solutions:

$$u_{A,j,f} = \omega_j (-4g_3x)^{1/3} \sum_{i=0}^{\infty} \frac{a_i}{(x^{1/3})^{4i}}, \quad (2.2a)$$

$$v_{A,j,f} = \frac{4\alpha_2}{3\omega_j^2 (-4g_3x)^{2/3}} \sum_{i=0}^{\infty} \frac{b_i}{(x^{1/3})^{4i}}, \quad (2.2b)$$

¹ Note that we use the same counter for propositions, remarks, theorems, and equations in this document.

and

$$u_{B,j,f} = \omega_j \left(\frac{4g_3x}{5} \right)^{1/3} \sum_{i=0}^{\infty} \frac{d_i}{(x^{1/3})^{4i}}, \quad (2.3a)$$

$$v_{B,j,f} = -\omega_j^2 \left(\frac{4g_3x}{5} \right)^{2/3} \sum_{i=0}^{\infty} \frac{e_i}{(x^{1/3})^{4i}}, \quad (2.3b)$$

where $w_j^3 = 1$, $j = 1, 2, 3$, $a_0 = b_0 = d_0 = e_0 = 1$, and a_i , b_i , d_i , and e_i are constants determined by substitution.

Proof. The solutions in Proposition 2.1 are readily constructed by making use of the Boutroux transformation of variables

$$u = x^{1/3}U, \quad v = x^{2/3}V, \quad z = \frac{3}{4}x^{4/3}, \quad (2.4)$$

in the system (1.11), where U and V are functions of z . This change of variables is chosen such that there is a maximum number of dominant terms in the transformed equations (see [31]).² Written in Boutroux coordinates,

$$\begin{aligned} U_{zz} - 3UU_z + U^3 + 6UV + 4g_3 &= \frac{3}{4z}(U^2 - U_z) + \frac{U}{8z^2}, \\ V_{zz} + 3UV_z + 3V^2 + 3U^2V &= \frac{3\alpha_2}{z} - \frac{1}{4z}(6UV + 5V_z) + \frac{V}{8z^2}, \end{aligned} \quad (2.5)$$

the equations reveal which terms are dominant in the limit as $|z| \rightarrow \infty$. In this case two algebraic leading-order balances are possible:

$$\text{Case A : } (U, V) = (\mathcal{O}(1), \mathcal{O}(z^{-1})),$$

$$\text{Case B : } (U, V) = (\mathcal{O}(1), \mathcal{O}(1)).$$

Formal solutions of (2.5) corresponding to these balances take the form

$$\text{Case A : } U(z) = c_0 \sum_{i=0}^{\infty} u_i z^{-i}, \quad V(z) = \frac{\alpha_2}{c_0^2 z} \sum_{i=0}^{\infty} v_i z^{-i}, \quad (2.6)$$

$$\text{Case B : } U(z) = c_1 \sum_{i=0}^{\infty} u_i z^{-i}, \quad V(z) = -c_1^2 \sum_{i=0}^{\infty} v_i z^{-i}, \quad (2.7)$$

where c_0 and c_1 are given by

$$c_0^3 = -4g_3, \quad c_1^3 = 4g_3/5, \quad (2.8)$$

and u_i, v_i are constants determined by substitution in each case with

² We have already encountered a Boutroux change of variables for P_{II} , and we will often use this kind of transformation in our subsequent work.

$u_0 = v_0 = 1$. Returning to the original coordinates via (2.4), these formal solutions are written as in Proposition 2.1, where the subscripts A and B match the cases defined above. \square

2.3 EXISTENCE OF TRUE SOLUTIONS

Proposition 2.9. Given any $x_0 \in \mathbb{C}$ such that $|x_0| > 1$, define the sectors

$$\sigma_{A,k,\beta} = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{k\pi}{4} < \arg(x - x_0) < \frac{(k + \beta)\pi}{4} \right\},$$

$$\sigma_{B,k} = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{(2k + 1)\pi}{8} < \arg(x - x_0) < \frac{(2k + 7)\pi}{8} \right\},$$

where $\beta = 1$ or 2 , and $k = 0, \dots, 7$. Then the following hold:

1. There exist two-parameter solutions of JM $P_{\text{II}}^{(2)}$ whose asymptotic behaviour as $|x| \rightarrow \infty$ is given by (2.2) or (2.3) in the respective sectors $\sigma_{A,k,\beta}$ and $\sigma_{B,k}$.
2. There exist one-parameter solutions of JM $P_{\text{II}}^{(2)}$ whose asymptotic behaviour as $|x| \rightarrow \infty$ is given by (2.2) in the sectors $\sigma_{A,k,3}$.

Proof. We prove the result in the transformed z coordinates, and then apply (2.4) to obtain the required results in terms of x . To prove the existence of actual solutions with behaviour (2.6) or (2.7), we employ Wasow's main asymptotic existence theorem, given below.

Theorem 2.10. [62] Let S be an open sector of the complex z -plane with vertex at the origin and a positive central angle not exceeding $\pi/(q + 1)$ (q a nonnegative integer). Let $F(z, Y)$ be an n -dimensional vector function of z and the n -dimensional vector Y with the following properties.

- A. $F(z, Y)$ is a polynomial in the components Y_j of Y , $j = 1, \dots, n$, with coefficients that are holomorphic in z in the region $z \in S$ and $0 < z_0 < |z| < \infty$, for z_0 constant.
- B. The coefficients of the polynomial $F(z, Y)$ have asymptotic series in powers of z^{-1} , as $z \rightarrow \infty$, in S .
- C. If $F_j(z, Y)$ denotes the components of $F(z, Y)$ then all eigenvalues λ_j , $j = 1, \dots, n$ of the Jacobian matrix

$$\left\{ \lim_{\substack{x \rightarrow \infty \\ x \in S}} \left(\frac{\partial F_j}{\partial Y_k} \Big|_{Y=0} \right) \right\}, \quad (2.11)$$

are different from zero.

D. The differential equation

$$z^{-q} \frac{dY}{dz} = F(z, Y) \quad (2.12)$$

is formally satisfied by a power series of the form

$$\sum_{r=1}^{\infty} y_r z^{-r}. \quad (2.13)$$

Then there exists, for sufficiently large $z \in S$, a solution of $y = \phi(z)$ of (2.12) such that, in every proper subsector of S ,

$$\phi(z) \sim \sum_{r=1}^{\infty} y_r z^{-r}, \quad z \rightarrow \infty. \quad (2.14)$$

The hypotheses of this theorem require that the system of equations under consideration is formally satisfied by an asymptotic expansion of the form (2.13), where we note that the expansion begins with the index 1 and not 0. Given the expansions (2.6) and (2.7), we therefore make the following change of variables.

Case A: Let (U, V) be a solution of (2.5) which is asymptotic to (2.6), and define $\tilde{U} = U - c_0$ and $\tilde{V} = V$. Then, (\tilde{U}, \tilde{V}) solves the system

$$\begin{aligned} \tilde{U}_{zz} - 3c_0 \tilde{U}_z + 3c_0^2 \tilde{U} + 6c_0 \tilde{V} + \left(3\tilde{U} \tilde{U}_z + 6\tilde{U} \tilde{V} + \tilde{U}^3 \right) &= \mathcal{O}(z^{-1}), \\ \tilde{V}_{zz} + 3c_0 \tilde{V}_z + 3c_0^2 \tilde{V} + \left(3\tilde{U} \tilde{V}_z + 3\tilde{V}^2 + 3\tilde{U}^2 \tilde{V} + 6c_0 \tilde{U} \tilde{V} \right) &= \mathcal{O}(z^{-1}), \end{aligned} \quad (2.15)$$

and has a formal expansion given by

$$\tilde{U}(z) = \sum_{i=1}^{\infty} \tilde{u}_i z^{-i}, \quad \tilde{V}(z) = \sum_{i=1}^{\infty} \tilde{v}_i z^{-i}, \quad (2.16)$$

for constants \tilde{u}_i and \tilde{v}_i .

Case B: Let (U, V) be a solution of (2.5) which is asymptotic to (2.7), and define $\tilde{U} = U - c_1$ and $\tilde{V} = V + c_1^2$. Then, (\tilde{U}, \tilde{V}) solves the system

$$\begin{aligned} \tilde{U}_{zz} - 3c_1 \tilde{U}_z + 6c_1 \tilde{V} - 3c_1^2 \tilde{U} \\ + \left(3\tilde{U} \tilde{U}_z + 6\tilde{U} \tilde{V} + \tilde{U}^3 + 3c_1 \tilde{U}^2 \right) &= \mathcal{O}(z^{-1}), \\ \tilde{V}_{zz} + 3c_1 \tilde{V}_z - 3c_1^2 \tilde{V} - 6c_1^3 \tilde{U} \\ + \left(3\tilde{U} \tilde{V}_z + 3\tilde{V}^2 + 3\tilde{U}^2 \tilde{V} + 6c_1 \tilde{U} \tilde{V} - 3c_1^2 \tilde{U}^2 \right) &= \mathcal{O}(z^{-1}), \end{aligned} \quad (2.17)$$

and has a formal solution given by (2.16).

Following Wasow's theorem, we rewrite each system of two second-order equations (2.15) and (2.17), as a system of four first-order equations

$$\frac{d\mathbf{Y}}{dz} = \mathbf{F}(z, \mathbf{Y}), \quad \mathbf{F}(z, \mathbf{Y}) = \mathbf{F}_0(\mathbf{Y}) + \sum_{i=1}^{\infty} \frac{1}{z^i} \mathbf{F}_i(\mathbf{Y}),$$

where the components of the vector \mathbf{Y} are $Y_1 = \tilde{U}$, $Y_2 = \tilde{V}$, $Y_3 = \tilde{U}_z$, and $Y_4 = \tilde{V}_z$. We then construct the Jacobian of $\mathbf{F}(z, \mathbf{Y})$ evaluated at $\mathbf{Y} = 0$ as $|z| \rightarrow \infty$. For Case A we find

$$\mathbf{J}_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3c_0^2 & -6c_0 & 3c_0 & 0 \\ 0 & -3c_0^2 & 0 & -3c_0 \end{pmatrix}, \quad (2.18)$$

and similarly for Case B,

$$\mathbf{J}_B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3c_1^2 & -6c_1 & 3c_1 & 0 \\ 6c_1^3 & 3c_1^2 & 0 & -3c_1 \end{pmatrix}. \quad (2.19)$$

Let λ_A, λ_B , denote the eigenvalues of the matrices $\mathbf{J}_A, \mathbf{J}_B$. Scaling $\mu = \lambda_A/c_0$ we obtain

$$\text{Case A : } \mu_1 = \sqrt{3} \exp(i\pi/6) = -\overline{\mu_2} = -\mu_3 = \overline{\mu_4}, \quad (2.20)$$

where $i = \sqrt{-1}$ and the overbar denotes complex conjugation, and scaling $\nu = \lambda_B/c_1$ we find

$$\begin{aligned} \text{Case B : } \nu_1 &= \sqrt{3(5 - \sqrt{5})}/2 = -\nu_3, \\ \nu_2 &= \sqrt{3(5 + \sqrt{5})}/2 = -\nu_4. \end{aligned} \quad (2.21)$$

Since each eigenvalue is non-zero, all the hypotheses of Wasow's theorem 2.10 are fulfilled. This proves the existence of true solutions with behaviours that are valid in sectors in the complex z -plane with a central angle less than π . Applying the Boutroux transformation (2.4) gives corresponding sectors of validity in the x -plane. These sectors have an angular opening less than $3\pi/4$, as given by $\sigma_{A,k,\beta}$ and $\sigma_{B,k}$ in Proposition 2.9.

We now turn to the determination of the precise sectors of validity. These sectors are determined by exponential terms which occur beyond all orders in the asymptotic description of the solutions (see [27]). Let (U_0, V_0) be a solution of (2.5) with asymptotic behaviour (2.6) or (2.7) and perturb this solution as follows:

$$U = U_0 + \widehat{U}, \quad V = V_0 + \widehat{V}, \quad (2.22)$$

where $(\widehat{U}, \widehat{V}) \ll 1$. Substitute (2.22) into equations (2.5). Using the asymptotic behaviour of (U_0, V_0) , given by (2.6) or (2.7), we find that the perturbation term must satisfy equations (2.15) and (2.17) respectively, with $(\widetilde{U}, \widetilde{V}) \mapsto (\widehat{U}, \widehat{V})$ in each case. Since $(\widehat{U}, \widehat{V}) \ll 1$, the linear terms in the equations are dominant; and hence to determine the asymptotic behaviour of $(\widehat{U}, \widehat{V})$ it is sufficient to consider only these terms. For Case A these are

$$\begin{aligned} \widehat{U}_{zz} - 3c_0 \widehat{U}_z + 3c_0^2 \widehat{U} + 6c_0 \widehat{V} &= \mathcal{O}(z^{-1}), \\ \widehat{V}_{zz} + 3c_0 \widehat{V}_z + 3c_0^2 \widehat{V} &= \mathcal{O}(z^{-1}), \end{aligned}$$

and for Case B,

$$\begin{aligned} \widehat{U}_{zz} - 3c_1 \widehat{U}_z + 6c_1 \widehat{V} - 3c_1^2 \widehat{U} &= \mathcal{O}(z^{-1}), \\ \widehat{V}_{zz} + 3c_1 \widehat{V}_z - 3c_1^2 \widehat{V} - 6c_1^3 \widehat{U} &= \mathcal{O}(z^{-1}). \end{aligned}$$

Proceeding as above, we write these systems of second-order equations as systems of first-order equations

$$\frac{dY}{dz} = K(z)Y, \quad K(z) = J + \sum_{i=1}^{\infty} \frac{1}{z^i} K_i, \quad (2.23)$$

where Y is a column vector with components $Y_1 = \widehat{U}$, $Y_2 = \widehat{V}$, $Y_3 = \widehat{U}_z$, and $Y_4 = \widehat{V}_z$, and $K(z)$ is a matrix of asymptotically expanded coefficients, where the Jacobian matrix J is defined by J_A (2.18) or J_B (2.19) in each case. The asymptotic behaviour of solutions to (2.23) is found using the following theorem.

Theorem 2.24. [62] Let S be an open sector of the z -plane with vertex at the origin and a positive central angle not exceeding $\pi/(q+1)$. Let $K(z)$ be an n -by- n matrix function holomorphic in S for $z_0 \leq z < \infty$ and

admitting in S a uniformly asymptotic power series

$$K(z) \sim \sum_{r=0}^{\infty} A_r z^{-r}, \quad z \rightarrow \infty, \quad z \in S.$$

Assume that all eigenvalues λ_j , $j = 1, \dots, n$ of A_0 are distinct. Then the differential equation

$$z^{-q} \frac{dY}{dz} = K(z)Y,$$

possesses a fundamental matrix solution of the form

$$Y(z) = H(z)z^D \exp(Q(z)). \quad (2.25)$$

Here $Q(z)$ is a diagonal matrix whose diagonal elements are polynomials of degree $q + 1$. The leading term of $Q(z)$ is

$$\frac{z^{q+1}}{q+1} \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n).$$

D is a constant diagonal matrix, and the matrix $H(z)$ has in S an asymptotic expansion

$$H(z) \sim \sum_{i=0}^{\infty} H_i z^{-i}, \quad |z| \rightarrow \infty, \quad \det H_0 \neq 0.$$

By Theorem 2.24, the asymptotic behaviour of solutions Y to (2.23) is given by (2.25) with $q = 0$ and $n = 4$. Extracting the asymptotic behaviour of $(\widehat{U}, \widehat{V})$ from Y given by (2.25), we obtain

$$(\widehat{U}(z), \widehat{V}(z))^T \sim \sum_{i=1}^4 C_i \exp(\lambda_i z), \quad (2.26)$$

where λ_i are the distinct eigenvalues of J , the Jacobian matrix given in (2.23), and C_i is a vector with two arbitrary constant components.

This resultant asymptotic behaviour for $(\widehat{U}, \widehat{V})$ must be consistent with the original assumption that $(\widehat{U}, \widehat{V}) \ll 1$. This is the case only if $\Re(\lambda_i z) < 0$ or $C_i = 0$, for each i in (2.26). The sectors defined by those z for which $\Re(\lambda_i z) < 0$ are

$$S_{i,l} = \{z \in \mathbb{C} \mid a_i + 2l\pi < \arg(z) < a_i + (2l+1)\pi\}, \quad l \in \mathbb{Z}, \quad (2.27)$$

where $a_i = \pi/2 - \phi_i$ and $\phi_i = \arg \lambda_i$. In each sector $S_{i,l}$ the corresponding constant C_i is free. If two sectors overlap, then in the sector of overlap

two constants are free. The orientation of the sectors is determined by a_i , and hence by the eigenvalues λ_i . Thus, using the specific values of a_i together with the transformation (2.4) we obtain the sectors $\sigma_{A,k,\beta}$ and $\sigma_{B,k}$, and the corresponding number of free parameters of the solutions in these sectors. This concludes the proof of Proposition 2.9. \square

Remark 2.28. Note that the eigenvalues of J denoted λ in (2.26) and thereafter correspond to the eigenvalues of J_A which are λ_A in Case A, and the eigenvalues of J_B which are λ_B in Case B. Also, while there appear to be no visible free parameters in the asymptotic expansions (2.2) and (2.3), the perturbation (2.26) implies that solutions with these behaviours have parameters hidden beyond all orders as coefficients of exponentially small terms.

In order to make clear the statement in Proposition 2.9, let us now construct the sectors of validity of some one- and two- parameter solutions of $JM P_{II}^{(2)}$ whose asymptotic behaviour as $|x| \rightarrow \infty$ is given by (2.2).

Example 2.29. Consider the eigenvalues for Case A. Recall that $\lambda_{A,i} = c_0 \mu_i$ where μ_i are given by (2.20). Since $c_0^3 = -4g_3$ (2.8) we have three sub-cases to explore, corresponding to the roots c_0 . These roots are 1. $\sqrt[3]{4g_3}e^{i\pi}$, 2. $\sqrt[3]{4g_3}e^{-i\pi/3}$, and 3. $\sqrt[3]{4g_3}e^{i\pi/3}$. Choose g_3 to be real and positive, without loss of generality, and let $\kappa = \sqrt{3}\sqrt[3]{4g_3}$. Set $\lambda_A \equiv \lambda$. Then the eigenvalues corresponding to the first two roots c_0 are

$$\text{Sub-case 1 : } \lambda_1 = \kappa \exp(-5i\pi/6) = -\bar{\lambda}_2 = -\lambda_3 = \bar{\lambda}_4,$$

$$\text{Sub-case 2 : } \lambda_1 = \kappa \exp(-i\pi/6) = \lambda_3 \quad \lambda_3 = \kappa \exp(i\pi/2) = -\lambda_4,$$

and we have omitted Sub-case 3 as it is not required for this example. The sectors of validity (2.27) are expressed in terms of $a_i = \pi/2 - \phi_i$, where $\phi_i = \arg \lambda_i$. For the sub-cases above we have

$$\text{Sub-case 1 : } a_1 = 4\pi/3, a_2 = 2\pi/3, a_3 = \pi/3, a_4 = -\pi/3,$$

$$\text{Sub-case 2 : } a_1 = 2\pi/3, a_2 = 0, a_3 = -\pi/3, a_4 = \pi.$$

Consider the sectors given by (2.27) in the x -plane (using (2.4)). For simplicity, take $l = 0$ and include a superscript to denote the sub-case we are considering. For Sub-case 1 we have

$$S_{1,0}^{(1)} = \{x \in \mathbb{C} \mid \pi < \arg(x) < 7\pi/4\},$$

$$S_{2,0}^{(1)} = \{x \in \mathbb{C} \mid \pi/2 < \arg(x) < 5\pi/4\},$$

$$\begin{aligned} S_{3,0}^{(1)} &= \{x \in \mathbb{C} \mid \pi/4 < \arg(x) < \pi\}, \\ S_{4,0}^{(1)} &= \{x \in \mathbb{C} \mid 7\pi/4 < \arg(x) < \pi/2\}. \end{aligned}$$

In the region $A := S_{2,0}^{(1)}$ we have $\Re(\lambda_2 z) < 0$ while $\Re(\lambda_1 z) > 0$, $\Re(\lambda_3 z) > 0$, and $\Re(\lambda_4 z) > 0$. Thus c_2 is free while $c_1 = c_3 = c_4 = 0$. In the region $B := S_{2,0}^{(1)} \cap S_{3,0}^{(1)}$ we have $\Re(\lambda_2 z) < 0$ and $\Re(\lambda_3 z) < 0$ while $\Re(\lambda_1 z) > 0$ and $\Re(\lambda_4 z) > 0$. Thus c_2 and c_3 are free while $c_1 = c_4 = 0$.

For Sub-case 2 we have

$$\begin{aligned} S_{1,0}^{(2)} &= \{x \in \mathbb{C} \mid \pi/2 < \arg(x) < 5\pi/4\}, \\ S_{2,0}^{(2)} &= \{x \in \mathbb{C} \mid 0 < \arg(x) < 3\pi/4\}, \\ S_{3,0}^{(2)} &= \{x \in \mathbb{C} \mid 7\pi/4 < \arg(x) < \pi/2\}, \\ S_{4,0}^{(2)} &= \{x \in \mathbb{C} \mid 3\pi/4 < \arg(x) < 3\pi/2\}. \end{aligned}$$

In the region $C := S_{1,0}^{(2)} \cap S_{2,0}^{(2)}$ we have $\Re(\lambda_1 z) < 0$ and $\Re(\lambda_2 z) < 0$ while $\Re(\lambda_3 z) > 0$ and $\Re(\lambda_4 z) > 0$ so that c_1 and c_2 are free while $c_3 = c_4 = 0$.

Each of the regions we have highlighted: A , B , and C , is bounded on one side by $\arg x = \pi/2$. While region B has twice the angular width of C , each allows a two-parameter solution of the form (2.2). Furthermore, in region A there exists a one-parameter family of solutions asymptotic to (2.2).

Denote the solutions defined in Proposition 2.9 as follows:

$$(u_{A,j}, v_{A,j}) \underset[x \in \sigma_{A,k,3}]{|x| \rightarrow \infty} \sim (u_{A,j,f}, v_{A,j,f}), \quad (u_{B,j}, v_{B,j}) \underset[x \in \sigma_{B,k}]{|x| \rightarrow \infty} \sim (u_{B,j,f}, v_{B,j,f}).$$

2.4 UNIQUE SOLUTIONS

Proposition 2.30. Given $x_0 \neq 0$, the true solutions $(u_{A,j}, v_{A,j})$ and $(u_{B,j}, v_{B,j})$ are unique in the respective sectors

$$\begin{aligned} \Sigma_{A,k,j} &= \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{k\pi}{4} < \arg(x - x_0) < \frac{(k + \beta_{k,j})\pi}{4} \right\}, \\ \Sigma_{B,k} &= \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{(2k+1)\pi}{8} < \arg(x - x_0) < \frac{(2k+13)\pi}{8} \right\}, \end{aligned}$$

where $k = 0, \dots, 7$. For each k , j takes two values $j_1, j_2 \in \{1, 2, 3\}$, $j_1 \neq j_2$, and $\beta_{k,j_1} = 4$ and $\beta_{k,j_2} = 5$. If $j \in \{j_1, j_2\}$ for both k and $k+1$ then $\beta_{k,j} = 4$ and $\beta_{k+1,j} = 5$.

Proof. We prove the result in the transformed z coordinates and then apply (2.4) to obtain the required results in terms of x . For some choice

of i and l , define the following two sectors:

$$\widehat{S}_{i,l,\pm\epsilon} = \{z \in \mathbb{C} \mid a_i \pm \epsilon + 2l\pi < \arg z < a_i \pm \epsilon + (2l+1)\pi\},$$

which are related to $S_{i,l}$ (2.27) by an angular rotation by $\pm\epsilon$, $\epsilon \ll 1$. Let (U_1, V_1) and (U_2, V_2) be two solutions whose asymptotic behaviour is known to be (2.6) or (2.7) in the respective sectors $\widehat{S}_{i,l,+\epsilon}$ and $\widehat{S}_{i+1,l,-\epsilon}$. Define the overlap of these two sectors as

$$\widetilde{S}_{i,l,\epsilon} = \widehat{S}_{i,l,+\epsilon} \cap \widehat{S}_{i+1,l,-\epsilon}.$$

We have two pieces of information we can exploit. Firstly, the two solutions have the same asymptotic behaviour for $z \in \widetilde{S}_{i,l,\epsilon}$. Thus,

$$(W, T) := (U_1 - U_2, V_1 - V_2) \underset{\substack{|z| \rightarrow \infty \\ z \in \widetilde{S}_{i,l,\epsilon}}}{=} o(z^{-m}), \quad (2.31)$$

for all positive integers m . Secondly, since (U_1, V_1) and (U_2, V_2) are both solutions of (2.5), their difference (W, T) must satisfy the system of linear ODEs

$$\begin{aligned} W_{zz} - 3U_1W_z + (-3U_{2,z} + 6V_2 + U_1^2 + U_1U_2 + U_2^2)W + 6U_1T \\ = \frac{3}{4z}((U_1 + U_2)W - W_z) + \frac{1}{8z^2}W, \\ T_{zz} + U_2T_z + (3(U_1 + U_2)V_1 + V_{1,z})W + (3(V_1 + V_2) + 3U_2^2)T \\ = -\frac{1}{4z}(6V_1W + 6U_2T + 5T_z) + \frac{1}{8z^2}T. \end{aligned} \quad (2.32)$$

In this system, the asymptotic expansions of the coefficients of (W, T) , and derivatives of these terms, are known; both (U_1, V_1) and (U_2, V_2) are given by (2.6) or (2.7) for Case A and Case B respectively. Thus (2.32) is a linear system which can be rewritten in the form (2.23) and hence solved by (2.25). We recall that in (2.23), J is the Jacobian matrix defined by J_A (2.18) or J_B (2.19) in each case. Extracting the asymptotic behaviour of (W, T) from Y given by (2.25) we obtain

$$(W, T)^T \sim \sum_{r=1}^4 C_r \exp(\lambda_r z), \quad z \in \widetilde{S}_{i,l,\epsilon}, \quad (2.33)$$

where λ_r are the eigenvalues of J_A or J_B , and C_r are free vector constants. The two expressions for the asymptotic behaviour of (W, T) , (2.31) and (2.33), must be consistent. Since $a_i = \pi/2 - \phi_i$ and $\phi_i = \arg \lambda_i$, it is clear that for each r in (2.33) there exists some z in $\widetilde{S}_{i,l,\epsilon}$ for which $\Re(\lambda_r z) > 0$.

Thus (2.31) and (2.33) are consistent only if each C_r in (2.33) is set to zero. This gives $(U_1, V_1) = (U_2, V_2)$ in $\tilde{S}_{i,l,\epsilon}$. The sector of validity for the asymptotic behaviour can be analytically continued into the extended sector $S_{i,l} \cup S_{i+1,l}$. \square

2.5 TRONQUÉE SOLUTIONS

The families of solutions defined by Proposition 2.9 are the analogue of Boutroux's tronquée solutions of P_{II} , while the 24 solutions defined by Proposition 2.30 (16 asymptotic to (2.2), 8 to (2.3)) are the analogue of the tri-tronquée solutions. The properties of the solutions of JM $P_{II}^{(2)}$ are summarised in Table 1. In particular, the number of special rays internal to the sectors of validity are given with a corresponding name suggested, in the spirit of Boutroux's tronquée and tri-tronquée solutions.

Table 1: Tronquée solutions of JM $P_{II}^{(2)}$

Case A					
Parameters	2	2	1	0	0
Sector Opening	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$
Internal Rays	1	3	5	7	9
Tronquée Prefix	-	tri-	penta-	hepta-	ennea-
Case B					
Parameters	2	0			
Sector Opening	$3\pi/4$	$3\pi/2$			
Internal Rays	5	11			
Tronquée Prefix	penta-	hendeca-			

2.6 DISCUSSION

This chapter has demonstrated the existence of new types of tronquée solutions of JM $P_{II}^{(2)}$ which extend the results obtained by Boutroux for the classical first and second Painlevé equations. Interestingly, we note that while the alternative fourth-order second Painlevé equation FN $P_{II}^{(2)}$ (1.17) does not admit these new types of tronquée solutions, it does admit another variation: bi-tronquée solutions (in the nomenclature of §2.5).

A further qualitative difference between the two fourth-order equations is the form of the asymptotic expansions of the solutions. In contrast to the expansions for JM $P_{II}^{(2)}$ given by (2.2) and (2.3), those for FN $P_{II}^{(2)}$

(1.17) are given in [33] as

$$y(x) = \left(\frac{x}{6}\right)^{1/4} \left(1 + \mathcal{O}(x^{-5/4(1-\epsilon)})\right), \quad (2.34)$$

or

$$y(x) = -\frac{2\alpha}{5x} \left(1 + \mathcal{O}(x^{-5/4(1-\epsilon)})\right), \quad (2.35)$$

for $\epsilon > 0$. In particular, we note that an asymptotic expansion of the form $y(x) \sim x^{-1}$, which is admitted by P_{II} and each FN $P_{\text{II}}^{(n)}$ is not valid for either $u(x)$ or $v(x)$ in JM $P_{\text{II}}^{(2)}$. This expansion does arise in the next member of the Jimbo-Miwa hierarchy, JM $P_{\text{II}}^{(3)}$:

$$\begin{aligned} -u_{3x} + 2v_{xx} + 4uu_{xx} + 3u_x^2 - 6vu_x - 6u^2u_x \\ + u^4 + 12u^2v + 6v^2 + 8xg_4 = 0, \\ v_{3x} + 4uv_{xx} + 2vu_{xx} + 2u_xv_x + 6vv_x + 6u^2v_x \\ + 4u^3v + 12uv^2 = 8\alpha_3. \end{aligned}$$

This system admits three possible algebraic expansions of the form

$$u(x) = u_0 \left(1 + \mathcal{O}(x^{-5/4(1-\epsilon)})\right), \quad v(x) = v_0 \left(1 + \mathcal{O}(x^{-5/4(1-\epsilon)})\right),$$

with leading-order behaviours given by³

$$u_0 = (-8g_4x)^{1/4}, \quad v_0 = -2\alpha_3(-8g_4x)^{-3/4}, \quad (2.36)$$

$$u_0 = (24g_4x/7)^{1/4}, \quad v_0 = -\frac{1}{3}(24g_4x/7)^{1/2}, \quad (2.37)$$

$$u_0 = \frac{-(2\alpha_3 + g_4)}{(4g_4x)}, \quad v_0 = (-4g_4x/3)^{1/2}. \quad (2.38)$$

It is straightforward to show that the asymptotic expansions for $u(x)$ in the system of equations JM $P_{\text{II}}^{(2m-1)}$ are comparable to those of FN $P_{\text{II}}^{(m)}$, given in [33].

Another interesting result follows from the degeneracy of u_0 given in (2.38). If $\alpha_3 = -g_4/2$, then a special solution of the system JM $P_{\text{II}}^{(3)}$ is

$$u(x) = 0, \quad \widehat{v}_{xx} = 6\widehat{v}^2 + 2g_4x, \quad \alpha_3 = -g_4/2,$$

where $\widehat{v} = -v/2$, and the equation for $\widehat{v}(x)$ is equivalent to the classical first Painlevé equation. In Chapter 7 we show that similar results hold between all the odd ($n \geq 3$) members of the Jimbo-Miwa hierarchy and the equations of the first Painlevé hierarchy.

³ These behaviours are derived in Chapter 5 via the method presented in §2.2.

In the next chapter we continue our Boutroux-inspired study of $\text{JM P}_{\text{II}}^{(2)}$. Instead of the special algebraic asymptotics considered thus far, we turn our attention to the general asymptotic behaviour of the transcendent.

3.1 ABSTRACT

In this chapter we study the general asymptotic behaviour of the fourth-order Jimbo-Miwa second Painlevé equation. We show that the general asymptotic behaviour at any point is given to leading-order by two related hyperelliptic functions of genus-2. There are four parameters in our description of the solution: two energy parameters E_1 and E_2 , and two phase parameters. We concentrate on E_1 and E_2 and investigate how they evolve as the independent variable is changed near the point at infinity. E_1 and E_2 are shown to be bounded, and special values of these parameters for which the hyperelliptic behaviour degenerates are identified. We also examine the relationship between E_1 and E_2 and some complete hyperelliptic integrals, and derive a set of PDEs satisfied by these integrals. As a corollary to our asymptotic study in the large independent variable limit, we also report large parameter asymptotics of JM $P_{\text{II}}^{(2)}$.

The plan of the chapter is as follows. We construct leading-order solutions in §3.2 and examine the impact of the correction terms in §3.3. In §3.4 we derive some relations between hyperelliptic integrals and a set of PDEs satisfied by these integrals. Degenerate behaviour is considered in §3.5 and we report the large parameter asymptotics in §3.6. We conclude with a discussion in §3.7.

3.2 LEADING-ORDER ASYMPTOTICS

Proposition 3.1. Let $\gamma_1(z)$ and $\gamma_2(z)$ be defined by the inversion of the hyperelliptic integrals

$$\int_0^{\gamma_1} \frac{ds}{\sqrt{P(s)}} + \int_0^{\gamma_2} \frac{ds}{\sqrt{P(s)}} = c_0, \quad (3.2a)$$

$$\int_0^{\gamma_1} \frac{s ds}{\sqrt{P(s)}} + \int_0^{\gamma_2} \frac{s ds}{\sqrt{P(s)}} = z + c_1, \quad (3.2b)$$

where P is the sixth-degree polynomial

$$P(s) = 4s^6 + 4gs^3 + \frac{1}{2}E_1s + \frac{1}{4}(E_2 + 4g_3^2), \quad (3.3)$$

and c_0, c_1, E_1 and E_2 are constants to leading-order. In the limit as $|x| \rightarrow \infty$ the general asymptotic behaviour of JM $P_{\text{II}}^{(2)}$ (1.11) is given in terms of Boutroux coordinates (2.4) where $U(z)$ and $V(z)$ denote the following functions:

$$U = -2(\gamma_1 + \gamma_2) + \mathcal{O}(z^{-1}), \quad (3.4a)$$

$$V = -(\gamma_1 + \gamma_2)_z - 2(\gamma_1^2 + \gamma_1\gamma_2 + \gamma_2^2) + \mathcal{O}(z^{-1}). \quad (3.4b)$$

3.2.1 Proof of Proposition 3.1 by direct analysis

Proof. To construct the asymptotics of (1.11) for large x we make the Boutroux change of variables (2.4) so that the governing equations are given by (2.5). Let $A_{2,1}$ and $A_{2,2}$ denote the terms

$$A_{2,1} = U_{zz} - 3UU_z + U^3 + 6UV, \quad (3.5a)$$

$$A_{2,2} = V_{zz} + 3UV_z + 3V^2 + 3U^2V, \quad (3.5b)$$

and $K_{2,1}$ and $K_{2,2}$ denote the terms

$$K_{2,1} = \frac{3}{4z}(U^2 - U_z) + \frac{U}{8z^2}, \quad (3.6a)$$

$$K_{2,2} = -\frac{1}{4z}(6UV + 5V_z) + \frac{V}{8z^2}. \quad (3.6b)$$

Then (2.5) becomes

$$A_{2,1} + 4g_3 = K_{2,1}, \quad (3.7a)$$

$$A_{2,2} = 3\alpha_2 z^{-1} + K_{2,2}. \quad (3.7b)$$

Use the following integrating factors:

$$(M_1, N_1) = (V_z, U_z), \quad (3.8a)$$

$$(M_2, N_2) = ((V_z + 2UV)_z, (-U_z + U^2 + 2UV)_z), \quad (3.8b)$$

to define two (leading-order) first integrals

$$E_i = \int M_i (A_{2,1} + 4g_3) + N_i A_{2,2} dz, \quad i = 1, 2. \quad (3.9)$$

The integrating factors (3.8) are chosen so that each expression (3.9) is given locally, that is, without integrals. For example, setting $n = 1$ in (3.9) gives the integrand

$$\begin{aligned} & M_1 (A_{2,1} + 4g_3) + N_1 A_{2,2} \\ &= U_{zz}V_z + U_z V_{zz} + 6UVV_z + 3U_z V^2 + U^3 V_z + 3U^2 U_z V + 4g_3 V_z, \\ &= \partial_z (U_z V_z + 3UV^2 + U^3 V + 4g_3 V), \end{aligned}$$

such that E_1 is given by

$$E_1 = U_z V_z + 3UV^2 + U^3 V + 4g_3 V. \quad (3.10a)$$

Similarly, E_2 is given by

$$\begin{aligned} E_2 &= V_z^2 + V_z (U^3 + 6UV - UU_z) + 2V^3 - 3V^2 (U_z - 3U^2) \\ &\quad + V (2U^4 - 3U^2 U_z + U_z^2) + 4g_3 (V_z + 2UV). \end{aligned} \quad (3.10b)$$

The generic leading-order asymptotic behaviour of (3.7) is determined from the system

$$A_{2,1} + 4g_3 = 0, \quad A_{2,2} = 0, \quad (3.11)$$

since $K_{2,1}$ and $K_{2,2}$ can be shown to be $\mathcal{O}(z^{-1})$.¹ Together with the definition (3.9), equations (3.11) imply that E_1 and E_2 are constant to leading-order.

We require a change of variables which maps the system (3.10) to a canonical set of equations for which we can identify the solution. Suppose $U(z)$ and $V(z)$ are given by

$$U = -2(\Gamma_1 + \Gamma_2), \quad (3.12a)$$

$$V = -\frac{1}{2}(U_z + U^2) + 2\Gamma_1 \Gamma_2, \quad (3.12b)$$

for two functions $\Gamma_1(z)$ and $\Gamma_2(z)$. Then U_z is given by the derivative of (3.12a), and V_z is given by

$$\begin{aligned} V_z &= -\frac{1}{2}(U_{zz} + 2UU_z) + 2(\Gamma_1 \Gamma_2)_z, \\ &= -\frac{1}{2}(UU_z - U^3 - 6UV - 4g_3) + 2(\Gamma_1 \Gamma_2)_z - \frac{1}{2}K_{2,1}, \end{aligned} \quad (3.13)$$

¹ The calculations in this chapter are entirely formal; when estimates of asymptotic size are made we do not provide proofs of our results. Given that we go on to study a hierarchy of equations, the formal method is advantageous as it is easily applicable to all the equations in the hierarchy. Rigorous justification of asymptotic results for the classical second Painleve equation are given in [30], for example.

where we have replaced the second derivative U_{zz} using its definition from (3.7a) with (3.5a), and $\mathcal{K}_{2,1}$ is given by (3.6a). Sequentially replace V_z , V , U_z then U in (3.10) to yield

$$\begin{aligned} E_1 &= 2(\Gamma_1 - \Gamma_2)(\Gamma_1 - \Gamma_2)_z(\Gamma_1 + \Gamma_2)_z \\ &\quad - \frac{8}{(\Gamma_1 - \Gamma_2)}((\Gamma_1^6 - \Gamma_2^6)) + g_3(\Gamma_1^3 - \Gamma_2^3) + \mathcal{T}_1, \end{aligned} \quad (3.14a)$$

$$\begin{aligned} E_2 &= 4(\Gamma_1 - \Gamma_2)\left(\Gamma_1(\Gamma_2)_z^2 - (\Gamma_1)_z^2\Gamma_2\right) \\ &\quad - \frac{16\Gamma_1\Gamma_2}{(\Gamma_1 - \Gamma_2)}((\Gamma_1^5 - \Gamma_2^5)) + g_3(\Gamma_1^2 - \Gamma_2^2) - 4g_3^2 + \mathcal{T}_2, \end{aligned} \quad (3.14b)$$

where $(\Gamma_1)_z$ and $(\Gamma_2)_z$ denote the derivatives of Γ_1 and Γ_2 with respect to z . The terms \mathcal{T}_1 and \mathcal{T}_2 can be shown to be $\mathcal{O}(z^{-1})$, and they take the form

$$\mathcal{T}_1 = -\mathcal{K}_{2,1}(\Gamma_1 + \Gamma_2)_z, \quad \mathcal{T}_2 = \frac{1}{4}\mathcal{K}_{2,1}(\mathcal{K}_{2,1} + 8(\Gamma_1\Gamma_2)_z).$$

To leading-order as $|z| \rightarrow \infty$ the general asymptotic behaviour of the system (3.14) is determined from

$$\begin{aligned} E_1 &= 2(\Gamma_1 - \Gamma_2)(\Gamma_1 - \Gamma_2)_z(\Gamma_1 + \Gamma_2)_z \\ &\quad - \frac{8}{(\Gamma_1 - \Gamma_2)}((\Gamma_1^6 - \Gamma_2^6)) + g_3(\Gamma_1^3 - \Gamma_2^3), \end{aligned} \quad (3.15a)$$

$$\begin{aligned} E_2 &= 4(\Gamma_1 - \Gamma_2)\left(\Gamma_1(\Gamma_2)_z^2 - (\Gamma_1)_z^2\Gamma_2\right) \\ &\quad - \frac{16\Gamma_1\Gamma_2}{(\Gamma_1 - \Gamma_2)}((\Gamma_1^5 - \Gamma_2^5)) + g_3(\Gamma_1^2 - \Gamma_2^2) - 4g_3^2, \end{aligned} \quad (3.15b)$$

since \mathcal{T}_1 and \mathcal{T}_2 are $\mathcal{O}(z^{-1})$. Multiply (3.15a) by $2\Gamma_1$ and add the resulting equation to (3.15b) to eliminate $(\Gamma_2)_z^2$. Solve for $(\Gamma_1)_z^2$ to yield

$$(\Gamma_1)_z^2 = \frac{P(\Gamma_1)}{(\Gamma_1 - \Gamma_2)^2}, \quad (3.16a)$$

where P is the polynomial (3.3). Similarly, multiply (3.15a) by $2\Gamma_2$ and add the resulting equation to (3.15b) to eliminate $(\Gamma_1)_z^2$. Solve for $(\Gamma_2)_z^2$ to yield

$$(\Gamma_2)_z^2 = \frac{P(\Gamma_2)}{(\Gamma_2 - \Gamma_1)^2}. \quad (3.16b)$$

In each equation (3.16), take square roots with the following choice of

signs:

$$(\Gamma_1)_z = \frac{\sqrt{P(\Gamma_1)}}{(\Gamma_1 - \Gamma_2)}, \quad (3.17a)$$

$$(\Gamma_2)_z = \frac{\sqrt{P(\Gamma_2)}}{(\Gamma_2 - \Gamma_1)}. \quad (3.17b)$$

Divide (3.17a) by $\sqrt{P(\Gamma_1)}$ and (3.17b) by $\sqrt{P(\Gamma_2)}$, and add the resulting equations to give

$$\frac{(\Gamma_1)_z}{\sqrt{P(\Gamma_1)}} + \frac{(\Gamma_2)_z}{\sqrt{P(\Gamma_2)}} = 0. \quad (3.18a)$$

Similarly, divide (3.17a) by $\sqrt{P(\Gamma_1)}/\Gamma_1$ and (3.17b) by $\sqrt{P(\Gamma_2)}/\Gamma_2$, and add the resulting equations to give

$$\frac{\Gamma_1 (\Gamma_1)_z}{\sqrt{P(\Gamma_1)}} + \frac{\Gamma_2 (\Gamma_2)_z}{\sqrt{P(\Gamma_2)}} = 1. \quad (3.18b)$$

Integrate (3.18a) and (3.18b) (with respect to z) to obtain

$$\int_0^{\Gamma_1} \frac{ds}{\sqrt{P(s)}} + \int_0^{\Gamma_2} \frac{ds}{\sqrt{P(s)}} = c_0, \quad (3.19a)$$

$$\int_0^{\Gamma_1} \frac{s ds}{\sqrt{P(s)}} + \int_0^{\Gamma_2} \frac{s ds}{\sqrt{P(s)}} = z + c_1, \quad (3.19b)$$

where c_0 and c_1 are arbitrary constants of integration.

On comparison of (3.2) and (3.19) we conclude that

$$\Gamma_1 = \gamma_1 + \mathcal{O}(z^{-1}), \quad \Gamma_2 = \gamma_2 + \mathcal{O}(z^{-1}), \quad (3.20)$$

where the correction terms arise because the system of equations (3.15), from which (3.19) is derived, is correct only to leading-order. Substitute (3.20) into (3.12) to obtain the solution (3.4). \square

We now present an alternative proof of Proposition 3.1 which proceeds via analysis of the compatibility condition (1.20) of the associated linear problem (1.18). This approach follows that of Drach [18] (see also [29]), and has the advantage of explaining a number of steps in our previous proof. For example, the ansatz (3.12) is explicitly identified via this approach. In Chapter 6, Drach's method is used to construct solutions of every equation in the Jimbo-Miwa hierarchy, and so it is prudent to see the method applied to a particular case here.

3.2.2 Proof of Proposition 3.1 by the method of Drach

Proof. Consider the compatibility condition of the scalar linear problem for JM $P_{\text{II}}^{(2)}$ given in Chapter 1 by equation (1.20) with $n = 2$, where the functions f and h_2 are defined by (1.19a) and (1.22), respectively. Recall the Boutroux change of variables (2.4), and in addition let the spectral variable be transformed as follows:

$$\lambda = x^{1/3}\mu. \quad (3.21)$$

After applying the change of variables (2.4) and (3.21), equation (1.20), for $n = 2$, is mapped to

$$(\mathcal{H}_2)_{zzz} - 4\mathcal{F}(\mathcal{H}_2)_z - 2\mathcal{F}_z\mathcal{H}_2 = \mathcal{R}_2, \quad (3.22)$$

where \mathcal{F} and \mathcal{H}_2 are given by

$$\mathcal{F} = \left(\mu - \frac{1}{2}U\right)^2 + \frac{1}{2}U_z - V, \quad (3.23a)$$

$$\tilde{\mathcal{H}}_2 = \mu^2 + \frac{1}{2}U\mu + \frac{1}{4}(-U_z + U^2 + 2V), \quad \tilde{\mathcal{H}}_2 = g_3\mathcal{H}_2. \quad (3.23b)$$

The right-hand side of (3.22) can be shown to be $\mathcal{O}(z^{-1})$ and takes the form

$$\mathcal{R}_2 = \frac{t_1}{z} + \frac{t_2}{z^2} + \frac{t_3}{z^3} + \frac{t_4}{z^4}, \quad (3.24)$$

where $t_i = r_i\mu + s_i$ for $i = 1, 2, 3$, with

$$\begin{aligned} r_1 &= (16g_3)^{-1} (10U_{zzz} - 18V_{zz} - 21UU_{zz} + 6UV_z - 7U_z^2 \\ &\quad + 30VU_z + 6U^2U_z - 24V^2 - 6U^2V + 3U^4 + 12g_3U), \\ s_1 &= (8g_3)^{-1} (-3(2U_{zz} - 5UU_z + 6UV + U^3 + 4g_3)), \\ r_2 &= (64g_3)^{-1} (-5U_{zz} + 4V_z + UU_z + 3U^3), \\ s_2 &= (32g_3)^{-1} (5U_z - 3U^2), \\ r_3 &= (128g_3)^{-1} (15U_z - 8V - U^2), \\ s_3 &= (64g_3)^{-1} (-5U), \end{aligned}$$

and $t_4 = (64g_3)^{-1}(-5U)$.

To leading-order as $|z| \rightarrow \infty$ the general asymptotic behaviour of equation (3.22) is determined from

$$(\mathcal{H}_2)_{zzz} - 4\mathcal{F}(\mathcal{H}_2)_z - 2\mathcal{F}_z\mathcal{H}_2 = 0, \quad (3.25)$$

since \mathcal{R}_2 is $\mathcal{O}(z^{-1})$. Multiply equation (3.25) by $2\mathcal{H}_2$ and integrate with respect to z to yield

$$2\tilde{\mathcal{H}}_2(\tilde{\mathcal{H}}_2)_{zz} - (\tilde{\mathcal{H}}_2)_z^2 - 4\mathcal{F}(\tilde{\mathcal{H}}_2)^2 + \tilde{P}(\mu) = 0, \quad (3.26)$$

where $\tilde{\mathcal{H}}_2$ is defined by (3.23b), and $\tilde{P}(\mu)$ is an arbitrary integration function. Given \mathcal{F} and $\tilde{\mathcal{H}}_2$ are polynomial in μ , equation (3.26) implies that \tilde{P} is polynomial in μ . In addition, on relating the highest degree terms in this equation we find that \tilde{P} is of degree 6 so that

$$\tilde{P}(\mu) = \sum_{i=0}^6 p_i \mu^i, \quad p_5 = 0, \quad (3.27)$$

for constants p_i .

To find the coefficients of the polynomial \tilde{P} , substitute (3.23) and (3.27) into equation (3.26) and balance powers of μ . The highest degree coefficients are given by

$$\mu^6: \quad p_6 = 4, \quad (3.28a)$$

$$\mu^4: \quad p_4 = 0, \quad (3.28b)$$

where the equation at μ^5 is automatically satisfied since we set $p_5 = 0$ in (3.27). At lower powers of μ we find

$$\mu^3: \quad p_3 = -A_{2,1}, \quad (3.29a)$$

$$\mu^2: \quad p_2 = 2^{-1} (\partial_z A_{2,1} - 2A_{2,2}), \quad (3.29b)$$

where $A_{2,1}$ and $A_{2,2}$ are given by (3.5). These terms also satisfy the leading-order equations (3.11), so that the coefficients (3.29) simplify as follows:

$$p_3 = 4g_3, \quad p_2 = 0. \quad (3.30)$$

At the two lowest powers of μ we have

$$\mu: \quad p_1 = 2^{-2} \left\{ 2E_1 + U (\partial_z A_{2,1} - 2A_{2,2}) - 2V (A_{2,1} + 4g_3) \right\}, \quad (3.31a)$$

$$\mu^0: \quad p_0 = 2^{-3} \left\{ 2E_2 + \frac{1}{2} A_{2,1}^2 + (-U_z + U^2 + 2V) (\partial_z A_{2,1} - 2A_{2,2}) - 2(V_z + 2UV) (A_{2,1} + 4g_3) \right\}, \quad (3.31b)$$

where E_1 and E_2 are given by (3.10) and are constant to leading-order.

For these coefficients, equations (3.11) imply

$$p_1 = \frac{1}{2}E_1, \quad (3.32a)$$

$$p_0 = \frac{1}{4}(E_2 + 4g_3^2). \quad (3.32b)$$

Substitute the full set of coefficients (3.28), (3.30) and (3.32), into the generic form of the polynomial (3.27) to find $\tilde{P} = P$ (3.3). Thus equation (3.26) is rewritten as

$$2\tilde{\mathcal{H}}_2(\tilde{\mathcal{H}}_2)_{zz} - (\tilde{\mathcal{H}}_2)_z^2 - 4\mathcal{F}(\tilde{\mathcal{H}}_2)^2 + P(\mu) = 0. \quad (3.33)$$

Notice that at points where $\tilde{\mathcal{H}}_2$ vanishes, equation (3.33) implies $(\tilde{\mathcal{H}}_2)_z^2 = P(\mu)$. Thus to construct hyperelliptic behaviour it is useful to rewrite $\tilde{\mathcal{H}}_2$ in the following factorised form:

$$\tilde{\mathcal{H}}_2 = (\mu - \mathcal{G}_1(z))(\mu - \mathcal{G}_2(z)), \quad (3.34)$$

for two functions $\mathcal{G}_1(z)$ and $\mathcal{G}_2(z)$, and then evaluate equation (3.33) at the zeros of $\tilde{\mathcal{H}}_2$. Set $\mu = \mathcal{G}_1$ and $\mu = \mathcal{G}_2$ in (3.33) to give

$$(\mathcal{G}_1 - \mathcal{G}_2)^2 (\mathcal{G}_1)_z^2 = P(\mathcal{G}_1), \quad (3.35a)$$

$$(\mathcal{G}_2 - \mathcal{G}_1)^2 (\mathcal{G}_2)_z^2 = P(\mathcal{G}_2), \quad (3.35b)$$

respectively. Divide each equation (3.35) by $(\mathcal{G}_1 - \mathcal{G}_2)^2$ to yield the system (3.16) with $\Gamma_1 = \mathcal{G}_1$ and $\Gamma_2 = \mathcal{G}_2$. Given (3.20) it follows that

$$\mathcal{G}_1 = \gamma_1 + \mathcal{O}(z^{-1}), \quad \mathcal{G}_2 = \gamma_2 + \mathcal{O}(z^{-1}). \quad (3.36)$$

Substitute (3.36) into (3.34) to yield

$$\tilde{\mathcal{H}}_2 = (\mu - \gamma_1(z))(\mu - \gamma_2(z)), \quad (3.37)$$

up to $\mathcal{O}(z^{-1})$. Compare the two definitions of $\tilde{\mathcal{H}}_2$ (3.23b) and (3.37), to obtain the leading-order relations

$$-(\gamma_1 + \gamma_2) = \frac{1}{2}U, \quad \gamma_1\gamma_2 = \frac{1}{4}(-U_z + U^2 + 2V),$$

from which (3.4) follows. \square

Remark 3.38. Equations (3.32) are particularly useful as they define the first-integrals E_1 and E_2 (3.10) to leading-order. If we multiply equation (3.32a) by 2 and then differentiate, the integrating factors M_1 and N_1 (3.8a) are equal to the coefficients of U_{zz} and V_{zz} , respectively. Similarly

for (M_2, N_2) .

Proposition 3.1 shows that the general asymptotic behaviour of JM $P_{\text{II}}^{(2)}$ is described by two related hyperelliptic functions of genus-2. While the individual functions $\gamma_1(z)$ and $\gamma_2(z)$ are multivalued, each having moveable quadratic branch points, the symmetric combinations (3.4a) and (3.4b) of these functions and their derivatives are meromorphic [15]. These globally meromorphic functions, constructed from functions given by the inversion of hyperelliptic integrals, *are* hyperelliptic functions.

To avoid the multivalued description of γ_1 and γ_2 these functions are generally considered as functions of two complex variables (see [4, 43], for example). To see this, replace c_0 and $z + c_1$ on the right-hand side of equations (3.2) with ζ_1 and ζ_2 respectively, and use these new variables as arguments for γ_1 and γ_2 ,

$$\int_0^{\gamma_1} \frac{ds}{\sqrt{P(s)}} + \int_0^{\gamma_2} \frac{ds}{\sqrt{P(s)}} = \zeta_1, \quad (3.39a)$$

$$\int_0^{\gamma_1} \frac{s ds}{\sqrt{P(s)}} + \int_0^{\gamma_2} \frac{s ds}{\sqrt{P(s)}} = \zeta_2. \quad (3.39b)$$

In this formulation, γ_1 and γ_2 are single valued functions of ζ_1 and ζ_2 , and possess a system of four independent periods (ω_j, Ω_j) where

$$\omega_j = \oint_{\rho_j} \frac{ds}{\sqrt{P(s)}}, \quad \Omega_j = \oint_{\rho_j} \frac{s ds}{\sqrt{P(s)}}, \quad j = 1, 2, 3, 4, \quad (3.40)$$

and ρ_j is a contour which encloses two of the branch points of $1/\sqrt{P(s)}$. Let $(\omega, \Omega) \equiv (\omega_j, \Omega_j)$ denote any of the four period systems with corresponding $\rho \equiv \rho_j$. Then the functions γ_1 and γ_2 satisfy the following periodic relations:

$$\begin{aligned} \gamma_1(\zeta_1, \zeta_2) &= \gamma_1(\zeta_1 + \omega, \zeta_2 + \Omega), \\ \gamma_2(\zeta_1, \zeta_2) &= \gamma_2(\zeta_1 + \omega, \zeta_2 + \Omega). \end{aligned} \quad (3.41)$$

Indeed, any symmetric combination of these functions also has this property.

Given that we are considering the asymptotic behaviour of an ODE, we will retain the single variable description given in Proposition 3.1. In [41], it is suggested that the second independent variable may manifest through one of the parameters α_2 or g_3 , however we do not pursue this idea here.

3.3 NEXT TO LEADING-ORDER ASYMPTOTICS

Consider the four parameters c_0 , c_1 , E_1 and E_2 which characterise the description of the leading-order hyperelliptic function solutions given in Proposition 3.1. Since the full governing equations are (3.7), on differentiating the expressions for the parameters E_i (3.9) the correction terms give

$$\frac{dE_i}{dz} = M_i K_{2,1} + N_i (-3\alpha_2 z^{-1} + K_{2,2}), \quad i = 1, 2,$$

or, in powers of z^{-1} :

$$\frac{dE_1}{dz} = -\frac{\mathcal{S}_1}{4z} + \frac{\mathcal{S}_2}{8z^2}, \quad \frac{dE_2}{dz} = -\frac{\mathcal{S}_3}{4z} + \frac{\mathcal{S}_4}{8z^2},$$

where \mathcal{S}_1 and \mathcal{S}_2 are given by

$$\begin{aligned} \mathcal{S}_1 &= 8U_z V_z - 3U^2 V_z + 6UVU_z - 12\alpha_2 U_z, \\ \mathcal{S}_2 &= U_z V + UV_z, \end{aligned}$$

and \mathcal{S}_3 and \mathcal{S}_4 are given by

$$\begin{aligned} \mathcal{S}_3 &= 10V_z^2 + 2UV_z (-4U_z + 4U^2 + 21V) \\ &\quad + 3V (2U_z^2 - 3U_z V - 7U^2 U_z + 15U^2 V + 5U^4) \\ &\quad + 4g_3 (5V_z + 6UV) - 12\alpha_2 (2V_z - UU_z + 6UV + U^3 + 4g_3), \\ \mathcal{S}_4 &= 2VV_z - U^2 V_z + UU_z V + 3UV^2 - 2U^3 V + 4g_3 V. \end{aligned}$$

Therefore as z changes the two parameters E_1 and E_2 will change. Since c_0 and c_1 , defined by (3.2), are written in terms of E_1 and E_2 through the polynomial P , these parameters also change with z . Thus each hyperelliptic function behaviour is only locally-valid.

To extend this domain of validity, we need to understand how the parameters in the local description change when the initial point, at which they are considered, is varied. To analyse these changes, we first examine the discrete change in E_1 and E_2 as z is varied near the point at infinity. We perform this calculation in three steps. To begin, we study the change in E_1 and E_2 as z is varied between any two points z_0 and z_1 . We assume no relation between the points, or between the values of the functions that we consider at these points.

Proposition 3.42. Let $z_0 \gg 1$ and $z_1 \gg 1$ denote two points in \mathbb{C} and

set $d_0 = z_1 - z_0$. Then the parameter E_1 satisfies

$$\begin{aligned} \Delta E_1(z_0) &= -\frac{3}{4z_0} \left(H_1(z_1) - H_1(z_0) \right) - \frac{5}{4} \left(\frac{d_0}{z_0} \right) E_1(z_0) \\ &\quad - \frac{6g_3}{z_0} \left\{ \int_{\gamma_1(z_0)}^{\gamma_1(z_1)} \frac{s^3 ds}{\sqrt{P(s)}} + \int_{\gamma_2(z_0)}^{\gamma_2(z_1)} \frac{s^3 ds}{\sqrt{P(s)}} \right\} + \mathcal{O}(z_0^{-2}), \end{aligned} \quad (3.43)$$

where $\Delta E_1(z_0) = E_1(z_1) - E_1(z_0)$ and $H_1 = F_1 - 2g_3U$ where

$$F_1 = -4\alpha_2 U + \frac{1}{3} (UV_z + 2VU_z), \quad (3.44)$$

and the parameter E_2 satisfies

$$\begin{aligned} \Delta E_2(z_0) &= -\frac{3}{4z_0} \left(H_2(z_1) - H_2(z_0) \right) - \frac{3}{2} \left(\frac{d_0}{z_0} \right) \left(E_2(z_0) + 8g_3^2 \right) \\ &\quad - \frac{24g_3}{z_0} \left\{ \int_{\gamma_1(z_0)}^{\gamma_1(z_1)} \frac{s^4 ds}{\sqrt{P(s)}} + \int_{\gamma_2(z_0)}^{\gamma_2(z_1)} \frac{s^4 ds}{\sqrt{P(s)}} \right\} + \mathcal{O}(z_0^{-2}), \end{aligned} \quad (3.45)$$

where $\Delta E_2(z_0) = E_2(z_1) - E_2(z_0)$ and

$$H_2 = F_2 - 2g_3 (2V - 2U_z + U^2), \quad (3.46)$$

$$\begin{aligned} F_2 &= -4\alpha_2 (-U_z + U^2 + 2V) \\ &\quad + \frac{1}{3} (-U^3V + 9UV^2 - U^2V_z + 4VV_z + 8g_3V). \end{aligned} \quad (3.47)$$

Proof. Although the difference equations in the proposition are written in the transformed coordinates, they are more simply derived by returning to the original coordinates. The governing equations are (1.11), or

$$\mathbf{a}_{2,1} + 4g_3x = 0, \quad (3.48a)$$

$$\mathbf{a}_{2,2} = 4\alpha_2, \quad (3.48b)$$

where $\mathbf{a}_{2,1}$ and $\mathbf{a}_{2,2}$ denote the terms

$$\mathbf{a}_{2,1} = u_{xx} - 3uu_x + u^3 + 6uv, \quad (3.49)$$

$$\mathbf{a}_{2,1} = v_{xx} + 3uv_x + 3u^2v + 3v^2. \quad (3.50)$$

Given (3.48) it follows that

$$m_1 (\mathbf{a}_{2,1} + 4g_3x) + n_1 (\mathbf{a}_{2,2} - 4\alpha_2) = 0, \quad (3.51a)$$

$$m_2 (\mathbf{a}_{2,1} + 4g_3x) + n_2 (\mathbf{a}_{2,2} - 4\alpha_2) = 0, \quad (3.51b)$$

for any (m_1, n_1) and (m_2, n_2) . Informed by (3.8) we choose

$$(m_1, n_1) = (v_x, u_x), \quad (3.52a)$$

$$(m_2, n_2) = ((v_x + 2uv)_x, (-u_x + u^2 + 2uv)_x), \quad (3.52b)$$

then integrate (3.51a) to yield

$$\left[u_x v_x + 3v^2 u + u^3 v + 4g_3 x v - 4\alpha_2 u \right]_{x_0}^{x_1} = 4g_3 \int_{x_0}^{x_1} v \, dx, \quad (3.53a)$$

and integrate (3.51b) to give

$$\begin{aligned} & \left[v_x^2 - u u_x v_x + 6u v v_x + u^3 v_x + v u_x^2 - 3v^2 u_x \right. \\ & \quad \left. - 3u^2 v u_x + 2v^3 + 9u^2 v^2 + 2u^4 v + 4g_3 x (v_x + 2uv) \right. \\ & \quad \left. - 4\alpha_2 (-u_x + u^2 + 2v) \right]_{x_0}^{x_1} = 4g_3 \int_{x_0}^{x_1} v_x + 2uv \, dx, \end{aligned} \quad (3.53b)$$

for any choice of x_0 and x_1 . This process is familiar from the first proof of Proposition 3.1 in §3.2.1.

Apply the Boutroux transformation (2.4) to equation (3.53a), but leave the coefficients in terms of x as follows:

$$\left[x^{5/3} E_1(z) + x^{1/3} G_1(z) = \mathcal{O}(x^{-1}) \right]_{x_0}^{x_1}, \quad (3.54a)$$

and similarly for equation (3.53b),

$$\left[x^2 E_2(z) + x^{2/3} G_2(z) = \mathcal{O}(x^{-2/3}) \right]_{x_0}^{x_1}. \quad (3.54b)$$

In (3.54a) and (3.54b), E_1 and E_2 are defined by (3.10) and

$$G_i(z) = F_i(z) - 4g_3 l_i(z), \quad i = 1, 2, \quad (3.55)$$

where F_1 and F_2 are given by (3.44) and (3.47) respectively, and l_1 and l_2 denote the integrals

$$\begin{aligned} l_1(z) &= \int^z V(\zeta) \, d\zeta, \\ l_2(z) &= \int^z V_\zeta(\zeta) + 2U(\zeta)V(\zeta) \, d\zeta. \end{aligned}$$

Evaluate (3.54a) at the two endpoints and divide by $x_1^{5/3}$ to give

$$\begin{aligned} & \left(\mathbf{E}_1(z_1) - \left(\frac{x_0}{x_1} \right)^{5/3} \mathbf{E}_1(z_0) \right) \\ & + \frac{1}{x_1^{4/3}} \left(\mathbf{G}_1(z_1) - \left(\frac{x_0}{x_1} \right)^{1/3} \mathbf{G}_1(z_0) \right) = \mathcal{O} \left(x_1^{-8/3}, x_0^{-1} x_1^{-5/3} \right). \end{aligned} \quad (3.56a)$$

Similarly, evaluate (3.54b) and divide by x_1^2 to give

$$\begin{aligned} & \left(\mathbf{E}_2(z_1) - \left(\frac{x_0}{x_1} \right)^2 \mathbf{E}_2(z_0) \right) \\ & + \frac{1}{x_1^{4/3}} \left(\mathbf{G}_1(z_1) - \left(\frac{x_0}{x_1} \right)^{2/3} \mathbf{G}_1(z_0) \right) = \mathcal{O} \left(x_1^{-8/3}, x_0^{-2/3} x_1^{-2} \right). \end{aligned} \quad (3.56b)$$

Given the relationship between x and z (2.4), and given $d_0 = z_1 - z_0$, we can rewrite the coefficients in equations (3.56a) and (3.56b) in terms of d_0 and z_0 only. Equations (3.56a) and (3.56b) become

$$\Delta \mathbf{E}_1(z_0) = \frac{-3}{4z_0} \left(\mathbf{G}_1(z_1) - \mathbf{G}_1(z_0) \right) - \frac{5}{4} \left(\frac{d_0}{z_0} \right) \mathbf{E}_1(z_0) + \mathcal{O} \left(z_0^{-2} \right), \quad (3.57a)$$

$$\Delta \mathbf{E}_2(z_0) = \frac{-3}{4z_0} \left(\mathbf{G}_2(z_1) - \mathbf{G}_2(z_0) \right) - \frac{3}{2} \left(\frac{d_0}{z_0} \right) \mathbf{E}_2(z_0) + \mathcal{O} \left(z_0^{-2} \right). \quad (3.57b)$$

The left-hand sides of (3.57a) and (3.57b) are precisely the differences we want to investigate. Let us turn our attention to the terms \mathbf{G}_i (3.55). For \mathbf{G}_1 we have

$$\begin{aligned} \mathbf{G}_1(z_1) - \mathbf{G}_1(z_0) &= (\mathbf{F}_1(z_1) - \mathbf{F}_1(z_0)) - 4g_3 (l_1(z_1) - l(z_0)), \\ &= (\mathbf{F}_1(z_1) - \mathbf{F}_1(z_0)) - 4g_3 \int_{z_0}^{z_1} V(\zeta) d\zeta, \end{aligned} \quad (3.58)$$

where \mathbf{F}_1 is given by (3.44) and the integrand V can be recast to leading-order as follows:

$$V(\zeta) = \frac{1}{2} U_\zeta - 2(\gamma_1 - \gamma_2)^{-1} (\gamma_1^3 - \gamma_2^3),$$

by (3.4) and then as

$$V(\zeta) = \frac{1}{2} U_\zeta - 2 \left\{ \frac{\gamma_1^3(\gamma_1)_\zeta}{\sqrt{P(\gamma_1)}} + \frac{\gamma_2^3(\gamma_2)_\zeta}{\sqrt{P(\gamma_2)}} \right\},$$

using (3.17) with (3.20). Integrate to yield

$$\int_{z_0}^{z_1} V(\zeta) d\zeta = \frac{1}{2} [U]_{z_0}^{z_1} - 2 \int_{\gamma_1(z_0)}^{\gamma_1(z_1)} \frac{s^3 ds}{\sqrt{P(s)}} - 2 \int_{\gamma_2(z_0)}^{\gamma_2(z_1)} \frac{s^3 ds}{\sqrt{P(s)}}.$$

Similarly, for G_2 we have

$$\begin{aligned} G_2(z_1) - G_2(z_0) &= (F_2(z_1) - F_2(z_0)) - 4g_3 \int_{z_0}^{z_1} V_\zeta(\zeta) + 2U(\zeta)V(\zeta) d\zeta, \end{aligned} \quad (3.59)$$

where the integral can be rewritten as follows:

$$\begin{aligned} &\int_{z_0}^{z_1} V_\zeta(\zeta) + 2U(\zeta)V(\zeta) d\zeta \\ &= \frac{1}{2} [2V - 2U_z + U^2]_{z_0}^{z_1} - 2g_3 d_0 - 4 \int_{\gamma_1(z_0)}^{\gamma_1(z_1)} \frac{s^4 ds}{\sqrt{P(s)}} - 4 \int_{\gamma_2(z_0)}^{\gamma_2(z_1)} \frac{s^4 ds}{\sqrt{P(s)}}, \end{aligned}$$

to leading-order. Substitute the expanded integrals into (3.58) and (3.59), then substitute the resulting expressions for G_i into equations (3.57a) and (3.57b) to obtain (3.43) and (3.45), respectively. \square

Proposition 3.42 provides a first expression for the discrete change in E_1 and E_2 as z is varied between two arbitrary points z_0 and z_1 . We use this result at points z_0, z_1 where γ_1 and γ_2 take on the same values.

Proposition 3.60. Assume that there exist two points $z_1 \gg 1$ and $z_0 \gg 1$ in \mathbb{C} for which

$$\gamma_1(z_0) = \gamma_1(z_1), \quad \gamma_2(z_0) = \gamma_2(z_1). \quad (3.61)$$

Then the parameters E_1 and E_2 satisfy

$$\Delta E_1 = -\frac{1}{z_0} \left\{ \frac{5}{4} d_0 E_1 + 12g_3 \Psi_3 \right\} + \mathcal{O}(z_0^{-2}), \quad (3.62a)$$

$$\Delta E_2 = -\frac{1}{z_0} \left\{ \frac{3}{2} d_0 (E_2 + 8g_3^2) + 48g_3 \Psi_4 \right\} + \mathcal{O}(z_0^{-2}), \quad (3.62b)$$

where $d_0 = z_1 - z_0$,

$$\Psi_i = \oint_{\rho} \frac{s^i ds}{\sqrt{P(s)}}, \quad i = 3, 4, \quad (3.63)$$

and each term on the right-hand side of (3.62) is evaluated at z_0 .

Proof. Given γ_1 and γ_2 satisfy (3.2), on differentiating we have

$$\frac{(\gamma_1)_z}{\sqrt{P(\gamma_1)}} + \frac{(\gamma_2)_z}{\sqrt{P(\gamma_2)}} = 0, \quad \frac{\gamma_1(\gamma_1)_z}{\sqrt{P(\gamma_1)}} + \frac{\gamma_2(\gamma_2)_z}{\sqrt{P(\gamma_2)}} = 1,$$

and hence

$$(\gamma_1)_z = \frac{\sqrt{P(\gamma_1)}}{(\gamma_1 - \gamma_2)}, \quad (\gamma_2)_z = \frac{\sqrt{P(\gamma_1)}}{(\gamma_2 - \gamma_1)}. \quad (3.64)$$

To leading-order, the right-hand sides of each equation in (3.64) only depend on γ_1 and γ_2 , thus (3.61) implies

$$(\gamma_1)_z \Big|_{z=z_1} = (\gamma_1)_z \Big|_{z=z_0}, \quad (\gamma_2)_z \Big|_{z=z_1} = (\gamma_2)_z \Big|_{z=z_0},$$

up to $\mathcal{O}(z_0^{-1})$. By (3.4), U , V and U_z can all be written exclusively in terms of γ_1 , γ_2 , $(\gamma_1)_z$ and $(\gamma_2)_z$. Given (3.4b) and (3.64), V_z can also be expressed this way. Thus

$$\begin{aligned} U(z_0) &= U(z_1), & U_z(z_0) &= U_z(z_1), \\ V(z_0) &= V(z_1), & V_z(z_0) &= V_z(z_1), \end{aligned} \quad (3.65)$$

up to $\mathcal{O}(z_0^{-1})$.

Consider equations (3.43) and (3.45) subject to (3.65). In these equations, the terms H_1 and H_2 are expressed in terms of U , V , U_z and V_z such that (3.65) implies

$$H_1(z_1) - H_1(z_0) = \mathcal{O}(z_0^{-1}), \quad H_2(z_1) - H_2(z_0) = \mathcal{O}(z_0^{-1}). \quad (3.66)$$

In addition, the integral terms in (3.43) and (3.45) simplify as follows:

$$\int_{\gamma_j(z_0)}^{\gamma_j(z_1)} \frac{s^i ds}{\sqrt{P(s)}} = \Psi_i, \quad j = 1, 2, \quad i = 3, 4, \quad (3.67)$$

where Ψ_3 and Ψ_4 are defined by (3.63). Given (3.66) and (3.67), the difference equations (3.43) and (3.45) are rewritten as (3.62a) and (3.62b), respectively. \square

Proposition 3.60 provides a second expression for the difference in E_1 and E_2 , between two points z_0 and z_1 where the functions γ_1 and γ_2 satisfy (3.61). Return to the defining integrals (3.2). Although the functions $\gamma_1(z)$ and $\gamma_2(z)$ are not periodic in z in general, they do take the same value when we simultaneously add to z and c_0 the corresponding periods Ω_j and ω_j (3.40). Since c_0 does not appear explicitly in the

description of the parameters E_1 and E_2 we set

$$\gamma_1(z_0) = \gamma_1(z_0 + \Omega), \quad \gamma_2(z_0) = \gamma_2(z_0 + \Omega), \quad (3.68)$$

in these expressions, with an implied updating of the parameter c_0 . Thus there does exist a point z_1 such that (3.61) holds, namely $z_1 = z_0 + \Omega$. In (3.68), we use Ω to denote any of the four components Ω_j of the period system (ω_j, Ω_j) (3.40).² We do the same for ω , such that these terms are defined by

$$\omega = \oint_{\rho} \frac{ds}{\sqrt{P(s)}}, \quad \Omega = \oint_{\rho} \frac{s ds}{\sqrt{P(s)}}, \quad (3.69)$$

where $\rho \equiv \rho_j$ denotes any of the four independent contours enclosing two of the branch points of the integrands. We use this notation in the remainder of the chapter.

Given (3.68) we can rewrite (3.62) in terms of complete elliptic integrals using the following definitions. Let $\tilde{\omega}$ and $\tilde{\Omega}$ denote two hyperelliptic integrals of the second kind

$$\tilde{\omega} = \oint_{\rho} \sqrt{P(s)} ds, \quad \tilde{\Omega} = \oint_{\rho} s \sqrt{P(s)} ds. \quad (3.70)$$

These integrals satisfy the differential relations

$$\Omega = 4 \frac{\partial \tilde{\omega}}{\partial E_1}, \quad \omega = 8 \frac{\partial \tilde{\omega}}{\partial E_2}, \quad (3.71a)$$

$$\Psi_2 = 4 \frac{\partial \tilde{\Omega}}{\partial E_1}, \quad \Omega = 8 \frac{\partial \tilde{\Omega}}{\partial E_2}, \quad (3.71b)$$

where Ψ_2 denotes another hyperelliptic integral of the first kind given by (3.63) with $i = 2$.

Proposition 3.72. Let $z_0 \gg 1$ and $z_1 \gg 1$ denote two points in \mathbb{C} where (3.68) holds. Then E_1 and E_2 satisfy the difference equations

$$\Delta E_1(z_0) = -\frac{1}{z_0} \left\{ 8\tilde{\omega} + T_1 \right\} + \mathcal{O}(z_0^{-2}), \quad (3.73a)$$

$$\Delta E_2(z_0) = -\frac{4}{z_0} \left\{ 10\tilde{\Omega} + T_2 \right\} + \mathcal{O}(z_0^{-2}), \quad (3.73b)$$

where T_1 and T_2 are given by

$$T_1 = -5E_1 \frac{\partial \tilde{\omega}}{\partial E_1} - 12(E_2 + 4g_3^2) \frac{\partial \tilde{\omega}}{\partial E_2},$$

² This means there are four points z_1 for which $z_1 = z_0 + \Omega$, however we do not differentiate between these points, or the different periods, in this work.

$$T_2 = -10E_1 \frac{\partial \tilde{\Omega}}{\partial E_1} - 3(3E_2 + 8g_3^2) \frac{\partial \tilde{\Omega}}{\partial E_2},$$

and each term on the right-hand side of (3.73) is evaluated at z_0 .

Proof. Since $z_1 = z_0 + \Omega$ and $d_0 = z_1 - z_0$ we can replace d_0 with Ω in equations (3.62) to give

$$\Delta E_1 = -\frac{1}{z_0} \left\{ \frac{5}{4} \Omega E_1 + 12g_3 \Psi_3 \right\} + \mathcal{O}(z_0^{-2}), \quad (3.74a)$$

$$\Delta E_2 = -\frac{1}{z_0} \left\{ \frac{3}{2} \Omega (E_2 + 8g_3^2) + 48g_3 \Psi_4 \right\} + \mathcal{O}(z_0^{-2}). \quad (3.74b)$$

Let us now construct equations satisfied by Ψ_3 and Ψ_4 using integration by parts applied to $\tilde{\omega}$ and $\tilde{\Omega}$. To do so it is useful to note the following:

$$P'(s) = \frac{1}{s} \left\{ 6P(s) - 12g_3 s^3 - \frac{5}{2} E_1 s - 6 \left(\frac{1}{4} E_2 + g_3^2 \right) \right\},$$

which is a consequence of (3.3). Apply integration by parts to $\tilde{\omega}$,

$$\begin{aligned} \tilde{\omega} &= \left[s \sqrt{P(s)} \right]_\rho - \frac{1}{2} \oint_\rho \frac{s P'(s) ds}{\sqrt{P(s)}}, \\ &= -\frac{1}{2} \left\{ 6\tilde{\omega} - 12g_3 \Psi_3 - \frac{5}{2} E_1 \Omega - 6 \left(\frac{1}{4} E_2 + g_3^2 \right) \omega \right\}, \end{aligned} \quad (3.75)$$

and then to $\tilde{\Omega}$,

$$\begin{aligned} \tilde{\Omega} &= \left[\frac{1}{2} s^2 \sqrt{P(s)} \right]_\rho - \frac{1}{4} \oint_\rho \frac{s^2 P'(s) ds}{\sqrt{P(s)}}, \\ &= -\frac{1}{4} \left\{ 6\tilde{\Omega} - 12g_3 \Psi_4 - \frac{5}{2} E_1 \Psi_2 - 6 \left(\frac{1}{4} E_2 + g_3^2 \right) \Omega \right\}. \end{aligned} \quad (3.76)$$

On the right-hand side of (3.75) replace the terms Ω and ω with derivatives of $\tilde{\omega}$ using (3.71a), and in (3.76) replace the terms Ψ_2 and Ω with derivatives of $\tilde{\Omega}$ using (3.71b). This yields the following expressions:

$$\tilde{\omega} = \frac{1}{4} \left\{ 6g_3 \Psi_3 + 5E_1 \frac{\partial \tilde{\omega}}{\partial E_1} + 6(E_2 + 4g_3^2) \frac{\partial \tilde{\omega}}{\partial E_2} \right\}, \quad (3.77)$$

$$\tilde{\Omega} = \frac{1}{5} \left\{ 6g_3 \Psi_4 + 5E_1 \frac{\partial \tilde{\Omega}}{\partial E_2} + 6(E_2 + 4g_3^2) \frac{\partial \tilde{\Omega}}{\partial E_2} \right\}. \quad (3.78)$$

Solve (3.77) for Ψ_3 and substitute the result into (3.74a). Replace Ω using (3.71a) to give (3.73a). Solve (3.78) for Ψ_4 and substitute the result into (3.74b). Replace Ω using (3.71b) to give (3.73b). \square

Thus we have derived the discrete change in E_1 and E_2 as z is varied from some initial point z_0 through a distance of a period of the leading-

order hyperelliptic function, as defined at that point.

Remark 3.79. Recall the equations and definitions in (1.2), (1.3) and (1.4) for the classical second Painlevé equation. The difference equations (3.73) for E_1 and E_2 are analogous to (1.3) for the single parameter E in the leading-order elliptic function of P_{II} . In the case of P_{II} , Boutroux used (1.3), together with the fact that E is bounded, to derive the condition (1.4) from which E is determined. The condition can be derived in an alternative way as follows.

The difference equation in (1.3) implies that E is slowly-varying up to $\mathcal{O}(z_n^{-2})$, that is, it changes by very small amounts over short distances and only varies significantly over large z . Hence the distance of a period $\omega_n = z_{n+1} - z_n$ is small on the scale of the independent variable. Thus if we divide the difference equation in (1.3) by ω_n then the resulting left-hand side can be considered as an averaged derivative over a large distance, in the sense of Whitham [63] (see also [38, 42]). That is,

$$\frac{dE}{dz} = -\frac{\tilde{\omega}}{\omega z} + \mathcal{O}(z^{-2}), \quad (3.80)$$

where we use the same notation E to represent the averaged quantity, and ω and $\tilde{\omega}$ are equal to ω_n and $\tilde{\omega}_n$, respectively, with $E_n = E$. Set $E_n = E$ in the definitions in (1.2) and (1.3). Then it follows that $d\tilde{\omega}/dE = \omega$. Thus on multiplying equation (3.80) by ω and dividing by $\tilde{\omega}$ we obtain $d(\log \tilde{\omega})/dz \sim -z^{-1}$ from which it follows that $\tilde{\omega} \rightarrow 0$ as $|z| \rightarrow \infty$.

Let us conclude our study of E_1 and E_2 by following an analogous argument based on the averaging method to show that these parameters are bounded.

Proposition 3.81. E_1 and E_2 are bounded.

Proof. It was shown in Proposition 3.72 that E_1 and E_2 satisfy the difference equations (3.74). These equations imply that E_1 and E_2 are slowly varying up to $\mathcal{O}(z_0^{-2})$, such that if we divide these equations by $\Omega = z_1 - z_0$ to give

$$\frac{\Delta E_1}{\Omega} = -\frac{1}{z_0} \left\{ \frac{5}{4} E_1 + 12g_3 \frac{\Psi_3}{\Omega} \right\} + \mathcal{O}(z_0^{-2}), \quad (3.82a)$$

$$\frac{\Delta E_2}{\Omega} = -\frac{1}{z_0} \left\{ \frac{3}{2} \left(E_2 + 8g_3^2 \right) + 48g_3 \frac{\Psi_4}{\Omega} \right\} + \mathcal{O}(z_0^{-2}), \quad (3.82b)$$

then the left-hand side of (3.82a) and (3.82b) can be considered as averaged derivatives for E_1 and E_2 , as discussed above. This leads to the

ODEs

$$\frac{dE_1}{dz} = -\frac{1}{z} \left\{ \frac{5}{4} E_1 + 12g_3 \frac{\Psi_3}{\Omega} \right\} + \mathcal{O}(z^{-2}), \quad (3.83a)$$

$$\frac{dE_2}{dz} = -\frac{1}{z} \left\{ \frac{3}{2} (E_2 + 8g_3^2) + 48g_3 \frac{\Psi_4}{\Omega} \right\} + \mathcal{O}(z^{-2}), \quad (3.83b)$$

where we have used the same notation E_1 and E_2 to represent the averaged quantities.

For some $z \in \mathbb{C}$ assume that $E_1 \gg 1$ and consider (3.83a) subject to this condition. In particular, consider the integrals Ψ_3 and Ω . Let the integral variable in the definition of Ψ_3 (3.63) and Ω (3.69) be scaled as $s = E_1^{1/5} t$. Then the polynomial (3.3) which appears in these definitions is given by

$$P(t) = E_1^{6/5} \left\{ \frac{1}{2} t (8t^5 + 1) + \mathcal{O}(|E_1|^{-3/5}) \right\},$$

and Ψ_3 and Ω take the respective forms

$$\begin{aligned} \Psi_3 &= E_1^{1/5} \oint_{\rho} \frac{t^3 dt}{\sqrt{\frac{1}{2} t (8t^5 + 1)}} + \mathcal{O}(|E_1|^{-2/5}), \\ \Omega &= E_1^{-1/5} \oint_{\rho} \frac{t dt}{\sqrt{\frac{1}{2} t (8t^5 + 1)}} + \mathcal{O}(|E_1|^{-4/5}). \end{aligned}$$

The ratio of these terms is $\Psi_3/\Omega = \mathcal{O}(|E_1|^{2/5})$. Therefore for $E_1 \gg 1$, equation (3.83a) gives

$$\frac{dE_1}{dz} = -\frac{5}{4z} E_1 + \mathcal{O}(z^{-2}, |E_1|^{2/5}).$$

This equation implies $E_1 \sim z^{-5/4}$ from which the first result follows.

Similarly, for some $z \in \mathbb{C}$ and bounded g_3 , assume that $E_2 \gg 1$ and consider (3.83b) subject to this condition. Scale the integration variable in the definition of Ψ_4 (3.63) and Ω (3.69) as $s = E_2^{1/6} t$, so that in this case the polynomial (3.3) is

$$P(t) = E_2 \left\{ \frac{1}{4} (16t^6 + 1) + \mathcal{O}(|E_2|^{-1/2}) \right\},$$

and the integrals Ψ_4 and Ω take the respective forms

$$\Psi_4 = E_2^{1/3} \oint_{\rho} \frac{t^4 dt}{\sqrt{\frac{1}{4} (16t^6 + 1)}} + \mathcal{O}(|E_2|^{-1/6}),$$

$$\Omega = E_2^{-1/6} \oint_{\rho} \frac{t dt}{\sqrt{\frac{1}{4}(16t^6 + 1)}} + \mathcal{O}\left(|E_2|^{-2/3}\right).$$

The ratio of these terms is $\Psi_4/\Omega = \mathcal{O}(|E_2|^{1/2})$. Therefore for $E_2 \gg 1$, equation (3.83b) gives

$$\frac{dE_2}{dz} = -\frac{3}{2z}E_2 + \mathcal{O}\left(z^{-2}, |E_2|^{1/2}\right).$$

This equation implies $E_2 \sim z^{-3/2}$ from which the second result follows. \square

3.4 RELATIONS BETWEEN HYPERELLIPTIC INTEGRALS

In this section we construct PDEs which are satisfied by the complete elliptic integrals given in (3.70), with $\tilde{\omega}(E_1, E_2)$ and $\tilde{\Omega}(E_1, E_2)$ considered as functions of the parameters through the polynomial (3.3). Define the complete hyperelliptic integrals of the third-kind

$$\Phi_i = \oint_{\rho} \frac{s^i ds}{P(s)^{3/2}}, \quad i = 0, \dots, 5. \quad (3.84)$$

Then we have the following relations:

$$\Phi_{i+1} = -4 \frac{\partial \Psi_i}{\partial E_1}, \quad \Phi_i = -8 \frac{\partial \Psi_i}{\partial E_2}, \quad (3.85a)$$

where Ψ_i is given by (3.63) for $i = 0, \dots, 5$, with $\omega \equiv \Psi_0$ and $\Omega \equiv \Psi_1$ by (3.69). Together, the relations (3.71) and (3.85a) imply

$$\Phi_1 = -32 \frac{\partial^2 \tilde{\omega}}{\partial E_1 \partial E_2}, \quad \Phi_0 = -64 \frac{\partial^2 \tilde{\omega}}{\partial E_2^2}, \quad (3.85b)$$

$$\Phi_2 = -32 \frac{\partial^2 \tilde{\Omega}}{\partial E_1 \partial E_2}, \quad \Phi_1 = -64 \frac{\partial^2 \tilde{\Omega}}{\partial E_2^2}. \quad (3.85c)$$

Return to equations (3.77) and (3.78). We want to express Ψ_3 and Ψ_4 in terms of $\tilde{\omega}$ and $\tilde{\Omega}$ using integration by parts. To do so, it is useful to note the following:

$$P'(s) = \frac{1}{s^4} \left\{ 6s^3 P(s) - 3g_3 P(s) - \frac{5}{2} E_1 s^4 - 6 \left(\frac{1}{4} E_2 - g_3^2 \right) s^3 + \frac{3}{2} g_3 E_1 s + 3g_3 \left(\frac{1}{4} E_1 + g_3^2 \right) \right\},$$

which is a consequence of (3.3). Apply integration by parts to Ψ_3 to give

$$\begin{aligned}\Psi_3 &= \left[\frac{s^4}{4\sqrt{P(s)}} \right]_\rho + \frac{1}{8} \oint_\rho \frac{s^4 P'(s) ds}{P(s)^{3/2}}, \\ &= \frac{1}{8} \left\{ 6\Psi_3 - 3g_3\Omega - \frac{5}{2}E_1\Phi_4 - 6\left(\frac{1}{4}E_2 - g_3^2\right)\Phi_3 \right. \\ &\quad \left. + \frac{3}{2}g_3E_1\Phi_1 + 3g_3\left(\frac{1}{4}E_2 + g_3^2\right)\Phi_0 \right\}.\end{aligned}$$

Next, integrate Ψ_4 by parts to give

$$\begin{aligned}\Psi_4 &= \left[\frac{s^5}{5\sqrt{P(s)}} \right]_\rho + \frac{1}{10} \oint_\rho \frac{s^5 P'(s) ds}{P(s)^{3/2}}, \\ &= \frac{1}{10} \left\{ 6\Psi_4 - 3g_3\Psi_2 - \frac{5}{2}E_1\Phi_5 - 6\left(\frac{1}{4}E_2 - g_3^2\right)\Phi_4 \right. \\ &\quad \left. + \frac{3}{2}g_3E_1\Phi_2 + 3g_3\left(\frac{1}{4}E_2 + g_3^2\right)\Phi_1 \right\}.\end{aligned}$$

In these expressions for Ψ_3 and Ψ_4 , replace Ω using (3.71a), Ψ_2 using (3.71b), and each Φ_i using (3.85) to yield

$$\begin{aligned}\Psi_3 &= \frac{1}{2} \left\{ -12g_3 \frac{\partial \tilde{\omega}}{\partial E_1} + 10E_1 \frac{\partial \Psi_3}{\partial E_1} + 12(E_2 - 4g_3^2) \frac{\partial \Psi_3}{\partial E_2} \right. \\ &\quad \left. - 48g_3 \frac{\partial^2 \tilde{\omega}}{\partial E_1 \partial E_2} - 48g_3(E_2 + 4g_3^2) \frac{\partial^2 \tilde{\omega}}{\partial E_2^2} \right\},\end{aligned}\tag{3.86}$$

$$\begin{aligned}\Psi_4 &= \frac{1}{4} \left\{ -12g_3 \frac{\partial \tilde{\Omega}}{\partial E_1} + 10E_1 \frac{\partial \Psi_4}{\partial E_1} + 12(E_2 - 4g_3^2) \frac{\partial \Psi_4}{\partial E_2} \right. \\ &\quad \left. - 48g_3 \frac{\partial^2 \tilde{\Omega}}{\partial E_1 \partial E_2} - 48g_3(E_2 + 4g_3^2) \frac{\partial^2 \tilde{\Omega}}{\partial E_2^2} \right\}.\end{aligned}\tag{3.87}$$

Solve (3.77) for Ψ_3 and substitute the result, and derivatives, into (3.86). Similarly, solve (3.78) for Ψ_4 and substitute the result, and derivatives, into (3.87). The resulting equations are

$$(D_1 + 5D_2 + 4)\tilde{\omega} = 0,\tag{3.88a}$$

$$(D_1 + 7D_2 + 10)\tilde{\Omega} = 0,\tag{3.88b}$$

where D_1 and D_2 denote the partial differential operators

$$\begin{aligned}D_1 &= 25E_1^2 \frac{\partial^2}{\partial E_1^2} + 12E_1(5E_2 + 12g_3^2) \frac{\partial^2}{\partial E_1 E_2} \\ &\quad + 36E_2(E_2 + 4g_3^2) \frac{\partial^2}{\partial E_2^2} + 144g_3^2 \frac{\partial}{\partial E_2},\end{aligned}\tag{3.89}$$

$$D_2 = 5E_1 \frac{\partial}{\partial E_1} - 6E_2 \frac{\partial}{\partial E_2}.\tag{3.90}$$

The PDEs for $\tilde{\omega}$ and $\tilde{\Omega}$ are analogous to the Legendre ODE derived in [31] (Appendix B.2) for the complete elliptic integral $\tilde{\omega}$ related to P_{II} . In that case, the singular points of the ODE correspond to special values of the parameter E for which the generic elliptic behaviour degenerates to trigonometric behaviour. In the PDEs for $\tilde{\omega}$ and $\tilde{\Omega}$ (3.88) this kind of singular behaviour manifests on the characteristic curves. For (3.88a) and (3.88b) the characteristic curves are

$$c_{\pm}E_1 = \left(\pm 4g_3 + \sqrt{5E_2 + 36g_3^2} \right)^{2/3} \left(\mp 6g_3 + \sqrt{5E_2 + 36g_3^2} \right), \quad (3.91)$$

for any constants c_{\pm} .

Observe that, given (3.89) and (3.90), the PDEs (3.88) are almost equidimensional, or of Cauchy-Euler type. Substitute the ansatz

$$\tilde{\omega} = E_1^m E_2^n, \quad (3.92)$$

into equation (3.88a), and divide the resulting equation by $\tilde{\omega}$ given by (3.92). We find

$$(n^2 - 16n + 4) + (m + n) \left\{ \frac{144g_3^2 n}{E_2} + 10(5m + 7n - 10) \right\} = 0. \quad (3.93)$$

Setting $m = -n$ in (3.93) we find that n must satisfy $n^2 - 16n + 4 = 0$, and thus a special class of solutions to (3.88a) is given by

$$\tilde{\omega} = \kappa_1 E_1^{-n_+} E_2^{n_+} + \kappa_2 E_1^{-n_-} E_2^{n_-}, \quad n_{+,-} = 2 \left(4 \pm \sqrt{15} \right), \quad (3.94)$$

for two arbitrary constants κ_1 and κ_2 . Similarly, a special class of solutions to (3.88b) is given by

$$\tilde{\Omega} = \mathcal{K}_1 E_1^{-p_+} E_2^{p_+} + \mathcal{K}_2 E_1^{-p_-} E_2^{p_-}, \quad p_{+,-} = 9 \pm \sqrt{71}, \quad (3.95)$$

where \mathcal{K}_1 and \mathcal{K}_2 are arbitrary constants. It is surprising that E_1 and E_2 are arbitrary in these special solutions. Given that $\tilde{\omega}$ and $\tilde{\Omega}$ are defined by (3.70), we would expect that any exact expression of these terms would correspond to a degeneracy of the integrand for special values of E_1 and E_2 , as discussed below.

3.5 DEGENERATE SOLUTIONS

The hyperelliptic solution of JM $P_{\text{II}}^{(2)}$ admit several cases of degeneracy to simpler functions. For special values of E_1 and E_2 the polynomial P

(3.3) has double or triple roots, and in these cases some of the branch points in the hyperelliptic integrals merge. To find these degeneracies, consider the derivatives of the polynomial (3.3),

$$\begin{aligned} P'(s) &= 12s^2(2s^3 + g_3) + \frac{1}{2}E_1, & P^{(3)}(s) &= 24(20s^3 + g_3), \\ P''(s) &= 24s(5s^3 + g_3), & P^{(4)}(s) &= 1440s^2. \end{aligned}$$

Since $g_3 \neq 0$, it is clear that there are no possible sextic, quintic, or quartic roots. Triple roots occur when the second derivative equals zero, that is, when $s = 0$ or $s^3 = -g_3/5$. Substitute $s = 0$ into the conditions $P'(s) = 0$ and then $P(s) = 0$ to give

$$E_1 = 0, \quad E_2 = -4g_3^2. \quad (3.96a)$$

Substitute $s = (-g_3/5)^{1/3}$ into $P'(s) = 0$ and $P(s) = 0$ to yield

$$E_1 = 2g_3(-36/5)(-g_3/5)^{1/3}, \quad E_2 = -36/5, \quad (3.96b)$$

Double roots occur when $P'(s)$ and $P(s)$ both equal zero. If we let s_2 denote a double root then

$$E_1 = -24s_2^2(2s_2^3 + g_3), \quad E_2 = 16s_2^3(5s_2^3 + 2g_3) - 4g_3^2. \quad (3.96c)$$

Recall the PDEs for $\tilde{\omega}$ and $\tilde{\Omega}$ (3.88). The special values of E_1 and E_2 (3.96a) correspond to a vanishing of all the highest derivative terms in the PDEs for $\tilde{\omega}$ and $\tilde{\Omega}$ (3.88). The values in (3.96b) correspond to the parabolic case, where there is only one family of characteristics for the PDEs instead of the two given by (3.91). The special values (3.96c) correspond to the c_- characteristic with $c_- = (-5/3)(5/2)^{2/3}$.

Subject to the conditions (3.96), the polynomial (3.3) takes the following simplified forms. For $s_3 = 0$ and $s_3 = (-g_3/5)^{1/3}$,

$$P(s) = 4(s - s_3)^3(s^3 + 3s_3s^2 + 6s_3^2s + (10s_3^3 + g_3)), \quad (3.97)$$

and for any double root s_2 ,

$$P(s) = (s - s_2)^2(a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0), \quad (3.98)$$

where the coefficients in the second factor are

$$a_4 = 4, \quad a_3 = 8s_2, \quad a_2 = 12s_2^2, \quad a_1 = 16s_2^3 + 4g_3, \quad a_0 = 20s_2^4 + 8s_2g_3.$$

When P is given by (3.97) or (3.98), the hyperelliptic functions that we

have constructed degenerate to elliptic functions.

3.6 LARGE PARAMETER LIMIT

In addition to the large independent variable asymptotics considered thus far, we can also construct the asymptotics of JM $P_{\text{II}}^{(2)}$ in the limit as $|\alpha_2| \rightarrow \infty$. In this case, the generic solutions of JM $P_{\text{II}}^{(2)}$ take the form

$$u = \alpha_2^{1/4} U(z), \quad v = \alpha_2^{1/2} V(z), \quad z = \alpha_2^{1/4} x,$$

where $U(z)$ and $V(z)$ are given by

$$U = -2(\tilde{\gamma}_1 + \tilde{\gamma}_2) + \mathcal{O}(\alpha_2^{-1}), \quad (3.99a)$$

$$V = -(\tilde{\gamma}_1 + \tilde{\gamma}_2)_z - 2(\tilde{\gamma}_1^2 + \tilde{\gamma}_1 \tilde{\gamma}_2 + \tilde{\gamma}_2^2) + \mathcal{O}(\alpha_2^{-1}), \quad (3.99b)$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are determined from the inversion of the hyperelliptic integrals

$$\begin{aligned} \int_0^{\tilde{\gamma}_1} \frac{ds}{\sqrt{Q(s)}} + \int_0^{\tilde{\gamma}_2} \frac{ds}{\sqrt{Q(s)}} &= c_2, \\ \int_0^{\tilde{\gamma}_1} \frac{s ds}{\sqrt{Q(s)}} + \int_0^{\tilde{\gamma}_2} \frac{s ds}{\sqrt{Q(s)}} &= z + c_3, \end{aligned}$$

where the polynomial Q is given by

$$Q(s) = 4s^6 - 4s^2 + \frac{1}{2}\tilde{\mathbb{E}}_1 s + \frac{1}{4}\tilde{\mathbb{E}}_2, \quad (3.100)$$

and c_2 , c_3 , $\tilde{\mathbb{E}}_1$ and $\tilde{\mathbb{E}}_2$ are constants to leading-order.

These solutions can be derived from equation (3.26) as in §3.2.2 for the large z limit. Although we will not provide full details of the construction of the solutions (3.99), we will show how to derive (3.100). Recall that in equation (3.26) the coefficients of the generic polynomial \tilde{P} (3.27) are given by (3.28), (3.29), and (3.31), and to determine these coefficients precisely we used the governing leading-order equations (3.11) to simplify expressions (3.29) and (3.31). In the large α_2 limit, the leading-order equations are $A_{2,1} = 0$ and $A_{2,2} = 4$ so that in this case, equations (3.28), (3.29), and (3.31) yield

$$p_6 = 4, \quad p_4 = 0, \quad p_3 = 0, \quad p_2 = -4, \quad p_1 = \frac{1}{2}\tilde{\mathbb{E}}_1, \quad p_0 = \frac{1}{4}\tilde{\mathbb{E}}_2,$$

where \tilde{E}_1 and \tilde{E}_2 are related to E_1 and E_2 by

$$\tilde{E}_1 = E_1 - 4g_3V + 4U, \quad (3.101)$$

$$\tilde{E}_2 = E_2 - 4g_3(V_z + 2UV) + 4(-U_z + U^2 + 2V). \quad (3.102)$$

Hence \tilde{P} (3.27) is equal to Q (3.100).

3.7 DISCUSSION

This chapter presented the leading-order general asymptotic behaviour of JM $P_{\text{II}}^{(2)}$ in terms of two related genus-2 hyperelliptic functions. We have shown that the elliptic function behaviour of the classical second Painlevé transcendent admits a natural extension to the higher-order equations in the Jimbo-Miwa hierarchy. However, unlike their elliptic counterparts, the explicit calculation of the hyperelliptic functions via inversion of the defining integrals is difficult to deduce, in particular for computational usage (see [59], for example).

In further analogy with the study of the classical equation, we examined the two parameters E_1 and E_2 which characterise the leading-order hyperelliptic function solutions. Our analysis of the discrete change in E_1 and E_2 , as z is varied across a period of the hyperelliptic functions, provides a starting point for the extension of the derived locally-valid solutions to a wider domain of asymptotic validity. Further investigations of the degeneration of the hyperelliptic behaviour for special values of E_1 and E_2 would also be of interest for future study. In particular, there should be a connection between the special rays identified in the analysis of the tronquée solutions in Chapter 2 and these behaviours.

With regard to the remaining investigations in this thesis, we note that the integrating factors (m_i, n_i) (3.52) can be rewritten in terms of the components of lower-order members of the Jimbo-Miwa hierarchy. The definition (1.13) implies $\mathbf{a}_0 = (\mathbf{a}_{0,1}, \mathbf{a}_{0,2})^T$ and $\mathbf{a}_1 = (\mathbf{a}_{1,1}, \mathbf{a}_{1,2})^T$ where

$$\mathbf{a}_{0,1} = u, \quad \mathbf{a}_{0,2} = v, \quad \mathbf{a}_{1,1} = -u_x + u^2 + 2v, \quad \mathbf{a}_{1,2} = v_x + 2uv,$$

and $\mathbf{a}_1 + 2g_2(1, 0)^T = 2\alpha_1(0, 1)^T$ defines JM $P_{\text{II}}^{(1)}$ (1.15). In this notation, the integrating factors (3.52) are

$$(m_1, n_1) = (\mathbf{a}_{0,2}, \mathbf{a}_{0,1})_x,$$

$$(m_2, n_2) = (\mathbf{a}_{1,2}, \mathbf{a}_{1,1})_x,$$

where the subscript x implies differentiation of both elements. A similar

result holds for the integrating factors (M_i, N_i) , expressed in Boutroux coordinates. The reason for this rewriting is that these expressions can be generalised for the entire hierarchy of equations, to be explored in Chapter 6.

This chapter concludes our investigation of JM $P_{\text{II}}^{(2)}$. Part i provides an extension of the classical results for P_{II} to this fourth-order analogue. We now turn our attention to the asymptotic study of the entire Jimbo-Miwa hierarchy.

Part II

ASYMPTOTIC STUDY OF THE JIMBO-MIWA
SECOND PAINLEVÉ HIERARCHY

STRUCTURE OF THE HIERARCHY AND
ASSOCIATED LINEAR PROBLEM

4.1 ABSTRACT

In this chapter we construct the equations that will be used to determine the asymptotic behaviour of the Jimbo-Miwa hierarchy. For the fourth-order system $\text{JM P}_{\text{II}}^{(2)}$ considered in Part [i](#), the governing non-linear equations and the coefficient terms in the associated linear problem are all given explicitly. The analogous equations and terms for $\text{JM P}_{\text{II}}^{(n)}$ are defined recursively. In [§4.2](#) we derive a more explicit form of the system of governing non-linear equations $\text{JM P}_{\text{II}}^{(n)}$ from the recursive definition given in [Chapter 1](#). We apply a Boutroux transformation of variables to this new system in [§4.3](#) to yield the equations studied in [Chapter 5](#). In [§4.4](#) we apply the same change of variables, with an additional transformation of the spectral parameter, to the compatibility condition of the scalar linear problem for $\text{JM P}_{\text{II}}^{(n)}$. This provides the equation that we study in [Chapter 6](#).

4.2 MORE EXPLICIT FORM

Consider \mathbf{a}_n in the definition of the hierarchy [\(1.12\)](#). This expression satisfies the recurrence relation and initial condition [\(1.13\)](#) from which we can generate \mathbf{a}_1 and \mathbf{a}_2 ,

$$\begin{aligned}\mathbf{a}_1 &= (-u_x + u^2 + 2v, v_x + 2uv)^T, \\ \mathbf{a}_2 &= (u_{xx} - 3uu_x + u^3 + 6uv, v_{xx} + 3uv_x + 3v^2 + 3u^2v)^T,\end{aligned}$$

and an infinite sequence of such expressions, each of which is a collection of non-derivative and derivative terms. Let

$$\mathbf{b}_n = \mathbf{b}_n(u, v), \quad \mathbf{c}_n = \mathbf{c}_n(u, \dots, u_n, v, \dots, v_n), \quad (4.1)$$

denote these terms, where $u_m = \partial_{mx}u$ and $v_m = \partial_{mx}v$, such that \mathbf{a}_n can be decomposed as follows:

$$\mathbf{a}_n = \mathbf{b}_n + \mathbf{c}_n. \quad (4.2)$$

Substitute (4.2) into (1.13), noting (4.1), to give

$$\mathbf{b}_{n+1} = r_1 \mathbf{b}_n, \quad (4.3)$$

$$\mathbf{c}_{n+1} = r_2 \mathbf{b}_n + r \mathbf{c}_n, \quad (4.4)$$

where the initial conditions are $\mathbf{b}_0 = (u, v)^T$ and $\mathbf{c}_0 = 0$, r is given by (1.14), and r_1 and r_2 are the operators

$$r_1 = r - r_2, \quad r_2 = \begin{pmatrix} -\partial_x & 0 \\ 0 & \partial_x \end{pmatrix}. \quad (4.5)$$

Given (4.3) and (4.4) we propose forms for \mathbf{b}_n and \mathbf{c}_n , and test their validity.

Proposition 4.6. Let $\mathbf{b}_n = (\mathbf{b}_{n,1}, \mathbf{b}_{n,2})^T$. Then,

$$\mathbf{b}_{n,1} = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,i} u^{n+1-2i} v^i, \quad (4.7a)$$

$$\mathbf{b}_{n,2} = \sum_{i=1}^{\lfloor n/2 \rfloor + 1} b_{n,i} u^{n+2-2i} v^i, \quad (4.7b)$$

where $a_{n,i}$ and $b_{n,i}$ are defined by

$$a_{n,i} = \frac{(n+1)!}{i!^2(n+1-2i)!}, \quad i = 0, \dots, (n+1)/2, \quad (4.8a)$$

$$b_{n,i} = \frac{(n+1)!}{i!(i-1)!(n+2-2i)!}, \quad i = 1, \dots, (n+2)/2. \quad (4.8b)$$

Proof. The proof is by induction, using (4.3). The calculation is straightforward after noting the relations

$$\begin{aligned} a_{n+1,i} &= a_{n,i} + 2b_{n,i}, & i &= 0, \dots, (n+2)/2, \\ b_{n+1,i+1} &= (2 - (i+1)^{-1}) a_{n,i} + b_{n,i+1}, & i &= 0, \dots, (n+1)/2, \end{aligned}$$

which follow from (4.8) with the end point conditions $b_{n,0} = 0$, $a_{n,j} = 0$, and $b_{n,k} = 0$, for $j = (n+2)/2$ and $k = (n+3)/2$. \square

Proposition 4.9. Let $\mathbf{u}_n = ((-1)^n u_n, v_n)^T$,

$$\mathbf{c}_n = \mathbf{u}_n + \mathbf{d}_n, \quad (4.10)$$

and $\mathbf{d}_n = (\mathbf{d}_{n,1}, \mathbf{d}_{n,2})^T$. Then,

$$\mathbf{d}_{n,1} = \sum_{\mathbf{k}} c_{k_0, \dots, k_{n-1}, l_0, \dots, l_{n-1}} u_0^{k_0} \dots u_{n-1}^{k_{n-1}} v_0^{l_0} \dots v_{n-1}^{l_{n-1}}, \quad (4.11a)$$

$$\mathbf{d}_{n,2} = \sum_{\widehat{\mathbf{k}}} \widehat{c}_{\widehat{k}_0, \dots, \widehat{k}_{n-1}, \widehat{l}_0, \dots, \widehat{l}_{n-1}} u_0^{\widehat{k}_0} \dots u_{n-1}^{\widehat{k}_{n-1}} v_0^{\widehat{l}_0} \dots v_{n-1}^{\widehat{l}_{n-1}}, \quad (4.11b)$$

where \mathbf{k} and $\widehat{\mathbf{k}}$ denote the following conditions:

$$\mathbf{k} : \langle k \rangle + \langle l \rangle = n + 1, \quad k_0 \leq n - 1, \quad l_0 \leq \lceil n/2 \rceil - 1, \quad (4.12)$$

$$\widehat{\mathbf{k}} : \langle \widehat{k} \rangle + \langle \widehat{l} \rangle = n + 2, \quad \widehat{k}_0 \leq n - 1, \quad \widehat{l}_0 \leq \lceil n/2 \rceil - 1, \quad (4.13)$$

c and \widehat{c} are constants, and k, l, \widehat{k} , and \widehat{l} are multi-indices with norms

$$\langle k \rangle = \sum_{i=0}^{n-1} (i+1)k_i, \quad \langle l \rangle = \sum_{i=0}^{n-1} (i+2)l_i, \quad (4.14)$$

$$\langle \widehat{k} \rangle = \sum_{i=0}^{n-1} (i+1)\widehat{k}_i, \quad \langle \widehat{l} \rangle = \sum_{i=0}^{n-1} (i+2)\widehat{l}_i. \quad (4.15)$$

Proof. The proof is by induction, using (4.4). Given the generic form provided for \mathbf{d}_n , the proof is straightforward - all that needs to be checked are the conditions on the norms (4.12) and (4.13). \square

Remark 4.16. While the generic form provided for \mathbf{d}_n (4.11) is sufficient to conduct most of the following calculations, knowing the term highlighted below will assist in the determination of the leading-order asymptotics. Substitute (4.10) into (4.4) to give

$$\begin{aligned} \mathbf{u}_{n+1} &= \mathbf{r}_2 \mathbf{u}_n, \\ \mathbf{d}_{n+1} &= \mathbf{r}_1 (\mathbf{u}_n + \mathbf{d}_n) + \mathbf{r}_2 (\mathbf{b}_n + \mathbf{d}_n). \end{aligned} \quad (4.17)$$

Send n to $(n-1)$ in (4.17) and substitute (4.5). Consider the second component of \mathbf{d}_n ,

$$\mathbf{d}_{n,2} = \partial_x \mathbf{b}_{n-1,2} + t_1, \quad (4.18)$$

where t_1 represents the omitted terms in $\mathbf{d}_{n,2}$ for which the precise form is not required. Let n be odd and set $j = (n+1)/2$. Then, substitute

(4.7b) into equation (4.18) to yield

$$\begin{aligned} \mathbf{d}_{n,2} &= \partial_x b_{n-1,j} v^j + t_2, \\ &= j b_{n-1,j} v^{j-1} v_x + t_2, \\ &= \frac{1}{2} b_{n,j} v^{j-1} v_x + t_2, \end{aligned}$$

where we have used the identity $2j b_{n-1,j} = b_{n,j}$, which follows from (4.8b), to simplify, and $t_2 = t_1 + \partial_x (\mathbf{b}_{n-1,2} - b_{n-1,j} v^j)$.

The above propositions and remark provide a more explicit form of the term \mathbf{a}_n in the governing equations (1.12). However while written in these coordinates it is not clear which terms in the equations balance in the limit as $|x| \rightarrow \infty$. This is rectified by the following change of variables.

4.3 BOUTROUX TRANSFORMATION

Let $U_m = \partial_{mz} U$ and $V_m = \partial_{mz} V$ denote the m th-order derivatives of $U(z)$ and $V(z)$ with respect to z .

Proposition 4.19. For each $n \geq 1$, the Boutroux transformation

$$u = x^{1/(n+1)} U(z), \quad (4.20a)$$

$$v = x^{2/(n+1)} V(z), \quad (4.20b)$$

$$z = \beta x^{(n+2)/(n+1)}, \quad \beta = (n+1)/(n+2), \quad (4.20c)$$

maps JM $\mathbb{P}_{\text{II}}^{(n)}$ (1.12) to

$$\mathbf{A}_n + 2^n g_{n+1}(1, 0)^T = 2^n \beta \alpha_n z^{-1} (0, 1)^T + \mathbf{K}_n, \quad (4.21)$$

where \mathbf{A}_n equals \mathbf{a}_n (1.13) subject to

$$x = z, \quad u_m = U_m, \quad v_m = V_m, \quad (4.22)$$

and \mathbf{K}_n is an $\mathcal{O}(z^{-1})$ polynomial in z^{-1} .

Proof. Substitute (4.2) and (4.10) into (1.12) to give

$$\mathbf{u}_n + \mathbf{d}_n + \mathbf{b}_n + 2^n g_{n+1}(x, 0)^T = 2^n (0, \alpha_n)^T. \quad (4.23)$$

We must consider the transformation of the three terms \mathbf{u}_n , \mathbf{d}_n , and \mathbf{b}_n . Directly substitute u and v given by (4.20) into the components of \mathbf{b}_n

(4.7) to give

$$\mathbf{b}_n = \mathbf{x}_n \mathbf{B}_n, \quad \mathbf{x}_n = \begin{pmatrix} x & 0 \\ 0 & x^{(n+2)/(n+1)} \end{pmatrix}, \quad (4.24)$$

where \mathbf{B}_n equals \mathbf{b}_n subject to (4.22).

Let us now show by induction that

$$u_m = \sum_{i=0}^m f_{m,i} x^{(i+1)/(n+1)-(m-i)} U_i, \quad (4.25)$$

$$v_m = \sum_{i=0}^m g_{m,i} x^{(i+2)/(n+1)-(m-i)} V_i, \quad (4.26)$$

for constants $f_{m,i}$, $g_{m,i}$ where $f_{m,m} = g_{m,m} = 1$. For $m = 1$, set $f_{1,0} = (n+1)^{-1}$ and $g_{1,0} = 2(n+1)^{-1}$ to obtain the result. Let

$$j_{m,i} = \left(\frac{i+1}{n+1} \right) - (m-i),$$

and suppose (4.25) is valid for some $m \geq 1$. For $(m+1)$:

$$\begin{aligned} u_{m+1} &= \sum_{i=0}^m f_{m,i} \partial_x (x^{j_{m,i}} U_i), \\ &= \sum_{i=0}^m f_{m,i} \left(j_{m,i} x^{j_{m,i}-1} U_i + x^{j_{m,i}+1/(n+1)} U_{i+1} \right), \\ &= \sum_{i=0}^m f_{m,i} \left(j_{m,i} x^{j_{m+1,i}} U_i + x^{j_{m+1,i}+1} U_{i+1} \right), \\ &= \sum_{i=0}^{m+1} f_{m+1,i} x^{j_{m+1,i}} U_i, \end{aligned}$$

where we have used $\partial_x = x^{1/(n+1)} \partial_z$, by equation (4.20c), and have set $f_{m+1,i} = f_{m,i} j_{m,i} + f_{m,i-1}$. Similarly for v_m .

For $m = n$, equations (4.25) and (4.26) imply that

$$\mathbf{u}_n = \mathbf{x}_n (\mathbf{U}_n - \mathbf{H}_n), \quad (4.27)$$

where \mathbf{U}_n equals \mathbf{u}_n subject to (4.22), \mathbf{x}_n is given in (4.24), and $\mathbf{H}_n =$

$-((-1)^n \widehat{\mu}_n, \widehat{\nu}_n)^T$ where

$$\widehat{\mu}_n = \sum_{i=0}^{n-1} f_{n,i} \left(x^{(n+2)/(n+1)} \right)^{(i-n)} U_i = \sum_{i=0}^{n-1} \widehat{f}_{n,i} z^{(i-n)} U_i, \quad (4.28a)$$

$$\widehat{\nu}_n = \sum_{i=0}^{n-1} g_{n,i} \left(x^{(n+2)/(n+1)} \right)^{(i-n)} V_i = \sum_{i=0}^{n-1} \widehat{g}_{n,i} z^{(i-n)} V_i. \quad (4.28b)$$

The coefficients $f_{n,i}$, $g_{n,i}$ and $\widehat{f}_{n,i}$, $\widehat{g}_{n,i}$ are related by

$$\beta^{(i-n)} \widehat{f}_{n,i} = f_{n,i}, \quad \beta^{(i-n)} \widehat{g}_{n,i} = g_{n,i},$$

and z and β are defined in (4.20c). It is clear from (4.28) that H_n is of $\mathcal{O}(z^{-1})$ and is polynomial in z^{-1} .

Consider the transformation of each term $u_m^{k_m}$ and $v_m^{l_m}$ in $\mathbf{d}_{n,1}$ where $0 \leq m \leq (n-1)$. For a given m , let i_m denote the index in the sum (4.25), and j_m the index in the sum (4.26) such that $0 \leq i_m \leq m$ and $0 \leq j_m \leq m$. Let $\gamma_{1,m}$ and $\gamma_{2,m}$ be the exponents of x in $u_m^{k_m}$ and $v_m^{l_m}$ respectively, subject to the transformation (4.20). Then (4.25) implies that

$$\begin{aligned} \gamma_{1,m} &= \left(\frac{i_m + 1}{n+1} - (m - i_m) \right) k_m, \\ &= \left(\frac{m+1}{n+1} \right) k_m - \left(\frac{n+2}{n+1} \right) (m - i_m) k_m, \end{aligned} \quad (4.29)$$

and (4.26) implies that

$$\gamma_{2,m} = \left(\frac{m+2}{n+1} \right) l_m - \left(\frac{n+2}{n+1} \right) (m - j_m) l_m. \quad (4.30)$$

Let γ_3 be the exponent of x in a generic term in $\mathbf{d}_{n,1}$. This term is given by the sum of $\gamma_{1,m}$ and $\gamma_{2,m}$ over all possible m ,

$$\gamma_3 = \sum_{m=0}^{n-1} (\gamma_{1,m} + \gamma_{2,m}).$$

Substitute (4.29) and (4.30) into this expression and simplify using (4.12) and (4.14) to give

$$\gamma_3 = 1 - \left(\frac{n+2}{n+1} \right) \kappa_1,$$

where κ_1 denotes the sum

$$\kappa_1 = \sum_{m=1}^{n-1} \left\{ (m - i_m) k_m + (m - j_m) l_m \right\}. \quad (4.31)$$

Similarly, let the exponent of x in a generic term in $\mathbf{d}_{n,2}$ be γ_4 . Then,

$$\gamma_4 = \frac{n+2}{n+1} - \left(\frac{n+2}{n+1} \right) \kappa_2, \quad (4.32)$$

where κ_2 denotes the sum

$$\kappa_2 = \sum_{m=1}^{n-1} \left\{ (m - \hat{i}_m) \hat{k}_m + (m - \hat{j}_m) \hat{l}_m \right\}, \quad (4.33)$$

and $0 \leq \hat{i}_m \leq m$ and $0 \leq \hat{j}_m \leq m$.

Thus, subject to the Boutroux transformation (4.20), every term in \mathbf{d}_n has the prefactor

$$\begin{aligned} (x^{\gamma_3}, x^{\gamma_4})^T &= \mathbf{x}_n \left(x^{\gamma_3-1}, x^{\gamma_4-(n+2)/(n+1)} \right)^T, \\ &= \mathbf{x}_n \left(\beta^{\kappa_1} z^{-\kappa_1}, \beta^{\kappa_2} z^{-\kappa_2} \right)^T, \end{aligned} \quad (4.34)$$

where z and β are defined in (4.20c), and \mathbf{x}_n is given in (4.24). If $i_m, j_m, \hat{i}_m, \hat{j}_m \equiv m$ for all $m = 0, \dots, (n-1)$, then $\kappa_1 = \kappa_2 = 0$. Thus,

$$\mathbf{d}_n = \mathbf{x}_n (\mathbf{D}_n - \mathbf{J}_n), \quad (4.35)$$

where \mathbf{D}_n equals \mathbf{d}_n subject to (4.22), and \mathbf{J}_n denotes all additional terms from the chain rule described above, where κ_1 and κ_2 are non-zero. Thus by (4.31) and (4.33), for those elements in \mathbf{J}_n we have $\kappa_1 \in \mathbb{N}$ and $\kappa_2 \in \mathbb{N}$, and hence \mathbf{J}_n is an $\mathcal{O}(z^{-1})$ polynomial in z^{-1} .

Substitute (4.24), (4.27), and (4.35) into (4.23). Left-multiply the resulting expression by \mathbf{x}_n^{-1} to recover

$$\mathbf{U}_n + \mathbf{D}_n + \mathbf{B}_n + 2^n g_{n+1}(1, 0)^T = 2^n \beta \alpha_n z^{-1} (0, 1)^T + \mathbf{K}_n, \quad (4.36)$$

where $\mathbf{K}_n = \mathbf{H}_n + \mathbf{J}_n$. As in (4.2) and (4.10) set $\mathbf{A}_n = \mathbf{B}_n + \mathbf{C}_n$ and $\mathbf{C}_n = \mathbf{U}_n + \mathbf{D}_n$. The result (4.21) follows directly from (4.36). \square

Remark 4.37. Given the definition of \mathbf{A}_n in terms of \mathbf{a}_n in Proposition 4.19, equation (1.13) implies that \mathbf{A}_n satisfies the recurrence relation and initial condition

$$\mathbf{A}_{n+1} = \mathbf{R}\mathbf{A}_n, \quad \mathbf{A}_0 = (U, V)^T, \quad (4.38)$$

where the operator R equals r (1.14) subject to (4.22),

$$R = \begin{pmatrix} U - \partial_z & 2 \\ 2V - \partial_z^{-1}V_z & U + \partial_z - \partial_z^{-1}U_z \end{pmatrix}. \quad (4.39)$$

The expanded form of the transformed equations (4.36) provides almost enough detail to determine the leading-order algebraic asymptotics in the next chapter. In order to completely determine these behaviours we require further insight into the term $K_{n,2}$ where $K_n = (K_{n,1}, K_{n,2})^T$.

Lemma 4.40. For n even, $K_{n,2}$ does not include a term of the form $\beta_1 V^{\beta_2} z^{-1}$ and for all n , $K_{n,2}$ does not include a term of the form $\beta_3 U^{\beta_4} z^{-1}$ for non-zero constants $\beta_1, \beta_2, \beta_3$ and β_4 .

Proof. Recall that $K_n = H_n + J_n$. Given $H_n = -((-1)^n \hat{\mu}_n, \hat{\nu}_n)^T$ where $\hat{\mu}_n$ and $\hat{\nu}_n$ are given by (4.28), it is clear that $H_{n,2}$ does not contain the terms given in the proposition. Consider $J_{n,2}$. Equation (4.34) implies that the generic exponent of z in $J_{n,2}$ is $-\kappa_2$. If $\kappa_2 = 1$ then equation (4.33) imposes the following condition on $\hat{i}_m, \hat{j}_m, \hat{k}_m$, and \hat{l}_m :

$$\sum_{m=1}^{n-1} \left\{ (m - \hat{i}_m) \hat{k}_m + (m - \hat{j}_m) \hat{l}_m \right\} = 1. \quad (4.41)$$

Assume that there exists a term in $J_{n,2}$ of $\mathcal{O}(z^{-1})$ and containing V only; that is, derivatives of V , U , and derivatives of U do not appear. Then for all m , $\hat{k}_m = 0$ (as all u terms and derivatives must be absent before transformation), and $\hat{j}_m = 0$ (as all V derivatives are absent). Thus by equation (4.41) we have

$$\sum_{m=1}^{n-1} m \hat{l}_m = 1. \quad (4.42)$$

Since $\hat{l}_m \geq 0$, equation (4.42) is satisfied only if $\hat{l}_1 = 1$ and $\hat{l}_m = 0$ for $m = 2, \dots, (n-1)$. Under these conditions equation (4.15) implies

$$\langle \hat{k} \rangle = 0, \quad \langle \hat{l} \rangle = 2\hat{l}_0 + 3.$$

Then, by equation (4.13)

$$\hat{l}_0 = \frac{n-1}{2}, \quad \hat{l}_0 \leq \lceil n/2 \rceil - 1, \quad (4.43)$$

which holds only for n odd. Thus for n even there is no such term in $J_{n,2}$, and hence $K_{n,2}$.

Assume that there exists a term in $J_{n,2}$ of $\mathcal{O}(z^{-1})$ and containing U only. Then for all m , $\widehat{l}_m = 0$ (as all v terms and derivatives must be absent before transformation) and $\widehat{i}_m = 0$ (as all U derivatives are absent). Thus by equation (4.41) we have

$$\sum_{m=1}^{n-1} m\widehat{k}_m = 1. \quad (4.44)$$

Since $\widehat{k}_m \geq 0$, equation (4.44) is satisfied only if $\widehat{k}_1 = 1$ and $\widehat{k}_m = 0$ for $m = 2, \dots, (n-1)$. Under these conditions equation (4.15) implies

$$\langle \widehat{k} \rangle = \widehat{k}_0 + 2, \quad \langle \widehat{l} \rangle = 0.$$

Then, by equation (4.13)

$$\widehat{k}_0 = n, \quad \widehat{k}_0 \leq n-1,$$

which is a contradiction. Thus there is no such term in $J_{n,2}$, and hence $K_{n,2}$. \square

Remark 4.45. Consider the term in $d_{n,2}$ which we highlighted in Remark 4.16. Subject to (4.20) this term transforms as follows:

$$\frac{1}{2}b_{n,j}v^{j-1}v_x = x^{(n+2)/(n+1)}\frac{1}{2}b_{n,j}\left(V^{j-1}V_z + \frac{2}{(n+2)z}V^j\right), \quad (4.46)$$

where we recall that n is odd and $j = (n+1)/2$. This term contributes $\beta_1 V^{\beta_2} z^{-1}$ to $K_{n,2}$ where

$$\beta_1 = -\frac{b_{n,j}}{n+2}, \quad \beta_2 = \frac{(n+1)}{2}. \quad (4.47)$$

Note that the exponent of v in $\frac{1}{2}b_{n,j}v^{j-1}v_x$ is consistent with the necessary condition (4.43) derived in Lemma 4.40.

Together, Proposition 4.19, Lemma 4.40 and Remark 4.45 provide all the necessary details about the transformed system of governing equations that is required in Chapter 5. The equations for analysis in Chapter 6 are obtained below.

4.4 LINEAR PROBLEM

In this section we transform the compatibility condition of the scalar linear problem for JM $P_{\Pi}^{(n)}$ given in Chapter 1 by equation (1.20), where

the functions f and h_n are defined by (1.19). This process is similar to that presented in §3.2.2 for the fourth-order equation.

Proposition 4.48. For each $n \geq 1$, the extended Boutroux transformation given by (4.20) and

$$\mu = x^{-1/(n+1)}\lambda, \quad (4.49)$$

maps equation (1.20) to

$$(\mathcal{H}_n)_{zzz} - 4\mathcal{F}(\mathcal{H}_n)_z - 2\mathcal{F}_z\mathcal{H}_n = \mathcal{R}_n, \quad (4.50)$$

where \mathcal{F} and \mathcal{H}_n equal f and h_n , respectively, subject to (4.22), and $\mathcal{R}_n = \mathcal{O}(z^{-1})$.

Proof. Consider h_n (1.19b) and the components $\mathfrak{a}_{i,1}$ of this term. Proposition 4.19 provides that, subject to the change of variables (4.20), $\mathfrak{a}_{n,1}$ is given by $\mathfrak{a}_{n,1} = x(\mathbf{A}_{n,1} - \mathbf{K}_{n,1})$. Following the argument given in the proof of Proposition 4.19, it is straightforward to show that the same change of variables applied to $\mathfrak{a}_{i,1}$ gives

$$\mathfrak{a}_{i,1} = x^{(i+1)/(n+1)}(\mathbf{A}_{i,1} - \mathbf{K}_{i,1}), \quad i = 0, \dots, n.$$

Hence, subject to (4.20) and (4.49), h_n (1.19b) is mapped to

$$h_n = x^{n/(n+1)}(H_n + t_1), \quad (4.51)$$

where t_1 is given by

$$\tilde{t}_1 = -2^{-n} \sum_{i=0}^{n-1} \mathbf{K}_{i,1} (2\mu)^{n-1-i}, \quad \tilde{t}_1 = g_{n+1}t_1,$$

and is order z^{-1} since each $\mathbf{K}_{i,1} = \mathcal{O}(z^{-1})$. Similarly, subject to (4.20) and (4.49), f (1.19a) is mapped to

$$f = x^{2/(n+1)}(\mathcal{F} + t_2), \quad t_2 = \frac{U}{2(n+2)z}. \quad (4.52)$$

The change of independent variables (4.20c) and (4.49) imply the relations

$$\partial_\lambda = x^{-1/(n+1)}\partial_\mu, \quad \partial_x = x^{1/(n+1)} \left\{ \partial_z - \frac{\mu}{(n+2)z} \partial_\mu \right\},$$

such that, given (4.51) and (4.52), it follows that the derivatives of these

terms are mapped to

$$\begin{aligned} (h_n)_x &= x((\mathcal{H}_n)_z + t_3), & f_x &= x^{3/(n+1)}(\mathcal{F}_z + t_5), \\ (h_n)_{xxx} &= x^{(n+3)/(n+1)}((\mathcal{H}_n)_{zzz} + t_4), & f_\lambda &= x^{1/(n+1)}(\mathcal{F}_\mu + t_6), \end{aligned}$$

where t_3, t_4, t_5, t_6 are all $\mathcal{O}(z^{-1})$. Substitute these transformed expressions into (1.20) to recover (4.50) where

$$\mathcal{R}_n = -t_4 + 4(t_2(\mathcal{H}_n)_z + t_3\mathcal{F}) - 2(t_1\mathcal{F}_z - t_5\mathcal{H}_n) - \beta z^{-1}(\mathcal{F}_\mu + t_6),$$

for β and z given by (4.20c). \square

The dominant terms in equation (4.50) are

$$(\mathcal{H}_n)_{zzz} - 4\mathcal{F}(\mathcal{H}_n)_z - 2\mathcal{F}_z\mathcal{H}_n = 0, \quad (4.53)$$

since $\mathcal{R}_n = \mathcal{O}(z^{-1})$. Multiply (4.53) by $2\mathcal{H}_n$ and integrate to give

$$2\tilde{\mathcal{H}}_n(\tilde{\mathcal{H}}_n)_{zz} - (\tilde{\mathcal{H}}_n)_z^2 - 4\mathcal{F}(\tilde{\mathcal{H}}_n)^2 + \tilde{P}(\mu) = 0, \quad \tilde{\mathcal{H}}_n = g_{n+1}\mathcal{H}_n, \quad (4.54a)$$

where $\tilde{P}(\mu)$ is an arbitrary integration function and

$$\mathcal{F} = \mu^2 + \frac{1}{2}U\mu + \frac{1}{4}(-U_z + U^2 + 2V), \quad (4.54b)$$

$$\tilde{\mathcal{H}}_n = \mu^n + 2^{-n} \sum_{i=0}^{n-1} A_{i,1} (2\mu)^{n-1-i}. \quad (4.54c)$$

The system (4.54) is analysed in Chapter 6.

5.1 ABSTRACT

In Chapter 2 we found new types of tronquée solutions of the fourth-order Jimbo-Miwa second Painlevé equation. These solutions were shown to be asymptotic to algebraic formal power series in particular sectors of the complex plane. In this chapter we derive solutions with algebraic asymptotics for the entire Jimbo-Miwa hierarchy. We find three possible classes of solutions and for each class we prove that true solutions with these behaviours exist in sectors of the complex plane of a specified angular opening. For one case, we obtain explicit results which detail the precise sectors of validity and the number of free parameters in the asymptotic descriptions that are valid in these sectors. At each level n of the hierarchy we find families of m -parameter solutions for $m = 1, \dots, n$, as well as unique solutions.

Many of the elements of analysis will be familiar from Chapter 2. We construct three types of formal solutions in §5.2, and prove existence of true solutions in §5.3. In §5.4 and §5.5 we derive sectors of validity first for families of solutions, and then for unique solutions within these families. We conclude with a discussion in §5.6. Throughout this chapter n is an integer greater than unity.

5.2 FORMAL SOLUTIONS

Theorem 5.1. In the limit as $|x| \rightarrow \infty$, JM $P_{\text{II}}^{(n)}$ admits three possible classes of formal solutions of the form

$$u_f = u_0 \sum_{i=0}^{\infty} u_{n,i} x^{-i(n+2)/(n+1)}, \quad (5.2a)$$

$$v_f = v_0 \sum_{i=0}^{\infty} v_{n,i} x^{-i(n+2)/(n+1)}, \quad (5.2b)$$

for constants $u_{n,i}$, $v_{n,i}$ and $u_{n,0} = v_{n,0} = 1$, where

$$\text{Case A : } (u_0, v_0) = (u_{A,j}, v_{A,j}), \quad (5.3a)$$

$$\text{Case B : } (u_0, v_0) = (u_{B,j}, v_{B,j}), \quad n \geq 2, \quad (5.3b)$$

$$\text{Case C : } (u_0, v_0) = (u_{C,j}, v_{C,j}), \quad n \text{ odd}, \quad (5.3c)$$

for $j = 1, \dots, (n+1)$. The leading-order behaviours are given by

$$u_{A,j} = \omega_j (-2^n g_{n+1} x)^{1/(n+1)}, \quad (5.4a)$$

$$v_{A,j} = \frac{2^n \alpha_n}{(n+1) \omega_j^n} \left(\frac{-1}{2^n g_{n+1} x} \right)^{n/(n+1)}, \quad (5.4b)$$

$$u_{B,j} = \omega_j \left(\frac{-2^n g_{n+1} x}{\mathcal{S}_n(c_3)} \right)^{1/(n+1)}, \quad (5.5a)$$

$$v_{B,j} = c_3 \omega_j^2 \left(\frac{-2^n g_{n+1} x}{\mathcal{S}_n(c_3)} \right)^{2/(n+1)}, \quad (5.5b)$$

$$u_{C,j} = \frac{-(2\alpha_n + g_{n+1})}{g_{n+1}(n+1)x}, \quad (5.6a)$$

$$v_{C,j} = \omega_j^2 \left(\frac{-2^n g_{n+1} x}{a_{n,(n+1)/2}} \right)^{2/(n+1)}, \quad (5.6b)$$

where α_n and g_{n+1} are the arbitrary constants in the hierarchy definition. $\omega_j^{n+1} = 1$ and c_3 denotes any solution of

$$\mathcal{T}_n(c_3) = 0, \quad \mathcal{T}_n(c_3) = \sum_{i=0}^{\lfloor n/2 \rfloor} b_{n,i+1} c_3^i, \quad (5.7)$$

where $\mathcal{S}_n(c_3)$ is the polynomial

$$\mathcal{S}_n(c_3) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_{n,i} c_3^i. \quad (5.8)$$

The constants $a_{n,i}$ and $b_{n,i}$ are defined in (4.8).

Proof. Consider the governing equations in Boutroux coordinates given by (4.21). In the limit as $|z| \rightarrow \infty$ the generic asymptotic balance for this equation is $A_n \sim -2^n(g_{n+1}, 0)^T$. However, if we demand that U and V have algebraic asymptotics in this limit then

$$U \gg U_1 \gg U_2 \gg \dots \gg U_n,$$

$$V \gg V_1 \gg V_2 \gg \dots \gg V_n.$$

That is, the functions U and V are larger than their derivatives. Hence the dominant terms in the governing equations are the non-derivative terms. In the expanded expression of the transformed governing equations given by (4.36) these terms are denoted B_n .

On inspection of (4.36) we deduce three possible algebraic leading-order balances,

$$(U, V) = (\mathcal{O}(1), \mathcal{O}(z^{-1})), \quad (5.9a)$$

$$(U, V) = (\mathcal{O}(1), \mathcal{O}(1)), \quad n \geq 2, \quad (5.9b)$$

$$(U, V) = (\mathcal{O}(z^{-1}), \mathcal{O}(1)), \quad n \text{ odd}, \quad (5.9c)$$

Let $B_n = (B_{n,1}, B_{n,2})^T$. Then, corresponding to the three cases above we obtain the following asymptotic relations. The first equation in the system (4.36) implies

$$a_{n,0}U^{n+1} \sim -2^n g_{n+1}, \quad (5.10a)$$

$$B_{n,1} \sim -2^n g_{n+1}, \quad (5.10b)$$

$$a_{n,j}V^j \sim -2^n g_{n+1}, \quad (5.10c)$$

where $j = (n+1)/2$ and the terms on the left-hand side of (5.10a) and (5.10c) are extracted from $B_{n,1}$ (that is, (4.7a) with (4.22)). For n even, $B_{n,1}$ does not contain a term without U , and hence relation (5.10c) (and balance (5.9c)) is not valid in this case.

The second equation of (4.36) implies the asymptotic relations

$$b_{n,1}U^n V \sim 2^n \beta \alpha_n z^{-1}, \quad (5.11a)$$

$$B_{n,2} \sim 0, \quad (5.11b)$$

$$b_{n,j}UV^j \sim (2^n \beta \alpha_n - \beta_1 V^j) z^{-1}, \quad (5.11c)$$

where $j = (n+1)/2$, β is given in (4.20c), and the terms on the left-hand side of (5.11a) and (5.11c) are extracted from $B_{n,2}$ (that is, (4.7b) with (4.22)). (5.11b) is taken to mean that the multiple terms in $B_{n,2}$ balance with one another. For $n = 1$, $B_{n,2}$ contains only one term, and hence relation (5.11b) (and balance (5.9b)) is not valid in this case. The term $\beta_1 V^j$ in (5.11c) is specified in Remark 4.45. Lemma 4.40 implies that there is no additional term on the right-hand side of (5.11a).

Relations (5.10a) and (5.11a) imply leading-order asymptotics

$$(U_A, V_A) = (c_0, c_1 z^{-1}), \quad (5.12)$$

where c_0 and c_1 denote the following constants:

$$c_0 = (-2^n g_{n+1})^{1/(n+1)}, \quad (5.13a)$$

$$c_1 = \frac{-\alpha_n c_0}{(n+2)g_{n+1}}. \quad (5.13b)$$

Relations (5.10b) and (5.11b) imply leading-order asymptotics

$$(U_B, V_B) = (c_2, c_2^2 c_3), \quad (5.14)$$

where c_2 denotes the following constant:

$$c_2 = \left(\frac{-2^n g_{n+1}}{\mathcal{S}_n(c_3)} \right)^{1/(n+1)}, \quad (5.15)$$

where c_3 and $\mathcal{S}_n(c_3)$ are defined in (5.7) and (5.8). Finally, relations (5.10c) and (5.11c) imply leading-order asymptotics

$$(U_C, V_C) = (c_4 z^{-1}, c_5), \quad (5.16)$$

where c_4 and c_5 denote the following constants:

$$c_4 = \frac{-1}{n+2} \left(1 + \frac{(n+1)\alpha_n}{j!g_{n+1}} \right), \quad (5.17a)$$

$$c_5 = \left(\frac{-2^n (j!)^2 g_{n+1}}{(n+1)!} \right)^{1/j}. \quad (5.17b)$$

where $j = (n+1)/2$.

By Proposition 4.19, all terms in equations (4.36) are polynomial in z^{-1} . Hence formal solutions corresponding to these leading-order behaviours are of the form

$$U = U_\chi \sum_{i=0}^{\infty} U_{n,i} z^{-i}, \quad V = V_\chi \sum_{i=0}^{\infty} V_{n,i} z^{-i}, \quad (5.18)$$

where χ denotes A, B, or C corresponding to (5.12), (5.14), and (5.16) respectively. $U_{n,i}, V_{n,i}$ are constants with $U_{n,0} = V_{n,0} = 1$. Returning to the original coordinates via (4.20) gives the three classes of solution in Theorem 5.1. \square

Remark 5.19. In Theorem 5.1, for cases A and B there are $(n+1)$ leading-order behaviours corresponding to the different roots of $\omega^{n+1} = 1$. For Case C there are $(n+1)/2$, where n is odd. It is necessary to distinguish between these behaviours in discussions of uniqueness. In addition, given c_3 denotes any root of the polynomial $\mathcal{T}_n(c_3)$ (5.7) there are possibly

$\lfloor n/2 \rfloor$ choices of this constant in the description of Case B.

5.3 EXISTENCE OF TRUE SOLUTIONS

Let $\mathbf{U} = (U, V)^T$ and $\mathbf{U}_m = \partial_{mx}\mathbf{U}$ with $\mathbf{U}_0 \equiv \mathbf{U}$, unless otherwise stated.

Theorem 5.20. There exist true solutions of JM $\mathbb{P}_{\text{II}}^{(n)}$ whose asymptotic behaviour as $|x| \rightarrow \infty$ is given by (5.2) with

A. (5.3a) and (5.4)

B. (5.3b) and (5.5)

C. (5.3c) and (5.6)

in sectors of the complex x -plane with an angular opening less than $\theta = (n+1)/(n+2)\pi$.

Proof. The proof follows from the subsequent propositions. Propositions 5.65, 5.69, and 5.75 (corresponding to cases A, B, C), show that the assumptions of Proposition 5.21 are fulfilled. The results of Proposition 5.21, subject to the inversion of the Boutroux transformation (4.20), imply the theorem. \square

We first determine conditions for the existence of true solutions of JM $\mathbb{P}_{\text{II}}^{(n)}$ which are asymptotic to generic algebraic formal power series, rather than one of the specific cases identified in Theorem 5.1. We proceed in the Boutroux coordinate system for which the governing equations are (4.21) and the formal solutions take the form (5.18).

Proposition 5.21. Let \mathbf{U} be a solution of (4.21) which is asymptotic to (5.18) and define

$$\tilde{\mathbf{U}} = \mathbf{U} - \mathring{\mathbf{U}}, \quad (5.22)$$

where $\mathring{\mathbf{U}}$ denotes the $\mathcal{O}(1)$ terms of \mathbf{U} . Let

$$\mathring{\mathbf{A}}_n = \mathbf{A}_n \big|_{\mathbf{U}=\mathring{\mathbf{U}}}, \quad (5.23)$$

and let $\tilde{\mathbf{A}}_n$ denote the Fréchet derivative of $\mathring{\mathbf{A}}_n$ with respect to $\mathring{\mathbf{U}}$, applied to $\tilde{\mathbf{U}}$ as follows:

$$\tilde{\mathbf{A}}_n = \mathring{\mathbf{A}}'_n \tilde{\mathbf{U}}. \quad (5.24)$$

Let $\tilde{\mathbf{A}}_n$ be written as

$$\tilde{\mathbf{A}}_n = \sigma^n \tilde{\mathbf{U}}_n - \sum_{i=1}^n \tilde{\mathbf{J}}_i \tilde{\mathbf{U}}_{i-1}, \quad \sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (5.25)$$

$$\tilde{\mathbf{J}}_i = \begin{pmatrix} (-1)^n \mathbf{J}_{j_n, j_i} & (-1)^n \mathbf{J}_{j_n, k_i} \\ \mathbf{J}_{k_n, j_i} & \mathbf{J}_{k_n, k_i} \end{pmatrix}, \quad i = 1, \dots, n, \quad (5.26)$$

where $j_i = 2i - 1$ and $k_i = 2i$. Finally, let \mathbf{J} denote the $2n \times 2n$ matrix

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \vdots & \vdots & 0 & 1 \\ \mathbf{J}_1 & \dots & & & \mathbf{J}_n & \end{pmatrix}, \quad (5.27)$$

where each \mathbf{J}_i is a 2×2 matrix with components

$$\mathbf{J}_i = \begin{pmatrix} \mathbf{J}_{j_n, j_i} & \mathbf{J}_{j_n, k_i} \\ \mathbf{J}_{k_n, j_i} & \mathbf{J}_{k_n, k_i} \end{pmatrix}, \quad i = 1, \dots, n. \quad (5.28)$$

If all the eigenvalues of \mathbf{J} are non-zero then there exist true solutions of (4.21) which are asymptotic to (5.18) in sectors of the complex z -plane with an angular opening less than π .

Proof. As in the proof of Proposition 2.9, to prove the existence of true solutions we employ Wasow's existence theorem 2.10. Recall that the hypotheses of this theorem require that the system of equations under consideration is formally satisfied by an asymptotic expansion of the form (2.13), where the expansion begins with the index 1 and not 0. Given our expansions are of the form (5.18) we require the change of variables (5.22) in order to use the theorem.

Substitute (5.22) into (4.21) to give

$$\mathring{\mathbf{A}}_n + \tilde{\tilde{\mathbf{A}}}_n + 2^n g_{n+1}(1, 0)^T = \tilde{\mathbf{K}}_n, \quad (5.29)$$

where $\mathring{\mathbf{A}}_n$ is given by (5.23), $\tilde{\tilde{\mathbf{A}}}_n = \tilde{\mathbf{A}}_n + \mathcal{O}(\tilde{\mathbf{U}}^2)$ where $\tilde{\mathbf{A}}_n$ is given by (5.24), and $\tilde{\mathbf{K}}_n = \mathcal{O}(z^{-1})$. Substitute (5.22) into (5.18) to give

$$\tilde{\mathbf{U}}(z) = \sum_{i=1}^{\infty} \tilde{u}_i z^{-i}, \quad \tilde{\mathbf{V}}(z) = \sum_{i=1}^{\infty} \tilde{v}_i z^{-i}. \quad (5.30)$$

By definition, \mathring{A}_n satisfies the system

$$\mathring{A}_n + 2^n g_{n+1}(1, 0)^T = 0. \quad (5.31)$$

Then equations (5.29) and (5.31) imply that \tilde{U} solves the system

$$\tilde{\tilde{A}}_n = \tilde{K}_n, \quad \tilde{K}_n = \mathcal{O}(z^{-1}), \quad \tilde{\tilde{A}}_n = \tilde{A}_n + \mathcal{O}(\tilde{U}^2), \quad (5.32)$$

and has a formal expansion with components given by (5.30). These expansions are of the required form for the theorem.

Rewrite the system of two n th-order equations (5.32) as a system of $2n$ first-order equations $\partial_z Y = F$, where the vector F admits an expansion

$$F(z, Y) = F_0(Y) + \sum_{i=1}^{\infty} \frac{1}{z^i} F_i(Y), \quad (5.33)$$

and the components of Y are

$$y_{2k-1} = \tilde{U}_{k-1}, \quad y_{2k} = \tilde{V}_{k-1}, \quad k = 1, \dots, n.$$

Following Theorem 2.10, construct the Jacobian of F evaluated at $Y = 0$ as $|z| \rightarrow \infty$,

$$\lim_{|z| \rightarrow \infty} \left(\frac{\partial F_j}{\partial y_k} \Big|_{Y=0} \right) = \frac{\partial F_{0,j}}{\partial y_k} \Big|_{Y=0}, \quad (5.34)$$

where each F_j denotes a component of F with $j = 1, \dots, 2n$.

Since the expression in (5.34) is evaluated at $Y = 0$ only those terms in F_0 which are linear in Y will contribute. In view of the above construction, F_0 is derived from the $\mathcal{O}(1)$ terms in equation (5.32), that is, from $\tilde{\tilde{A}}_n$. In addition, the terms which are linear in Y are derived from the terms in (5.32) which are linear in \tilde{U} , that is, \tilde{A}_n . Finally, given equations (5.26) - (5.28) it is clear that

$$\frac{\partial F_{0,j}}{\partial y_k} \Big|_{Y=0} = J.$$

Hence, if each eigenvalue of J (5.27) is non-zero then all the hypotheses of Theorem 2.10 are fulfilled. This proves existence of true solutions with behaviours that are valid in sectors of the complex z -plane with a central angle less than π . \square

The conditions of Proposition 5.21 need to be tested for the three classes of solution given in Theorem 5.1. Before we perform such a test

we need to know the matrix J or equivalently the expression $\tilde{\mathbb{A}}_n$. The next proposition provides two recurrence relations for $\tilde{\mathbb{A}}_n$ which can be used to examine any proposed forms of this term.

Proposition 5.35. Let $\mathring{\mathbb{R}}$ and $\tilde{\mathbb{R}}$ denote the matrix operators

$$\mathring{\mathbb{R}} = \mathbb{R} |_{\mathbb{U}=\mathring{\mathbb{U}}}, \quad (5.36)$$

$$\tilde{\mathbb{R}} = \left(\mathring{\mathbb{R}}\tilde{\mathbb{U}} \right)', \quad (5.37)$$

where \mathbb{R} is given by (4.39) and $'$ denotes the Fréchet derivative with respect to $\mathring{\mathbb{U}}$. Then, $\tilde{\mathbb{A}}_n$ satisfies the recurrence relations

$$\tilde{\mathbb{A}}_{n+1} = \mathring{\mathbb{R}}\tilde{\mathbb{A}}_n + \tilde{\mathbb{R}}\mathring{\mathbb{A}}_n, \quad (5.38)$$

$$\tilde{\mathbb{A}}_{n+2} = \mathring{\mathbb{R}}^2\tilde{\mathbb{A}}_n + \mathring{\mathbb{R}}\tilde{\mathbb{R}}\mathring{\mathbb{A}}_n, \quad (5.39)$$

where $\mathring{\mathbb{A}}_n$ is given by (5.23).

Proof. Replace n with $(n+1)$ in equations (5.23) and (5.24) to give

$$\mathring{\mathbb{A}}_{n+1} = \mathbb{A}_{n+1} |_{\mathbb{U}=\mathring{\mathbb{U}}}, \quad (5.40a)$$

$$\tilde{\mathbb{A}}_{n+1} = \mathring{\mathbb{A}}'_{n+1}\tilde{\mathbb{U}}, \quad (5.40b)$$

and substitute (5.23), (5.36) and (5.40a) into the recurrence relation in (4.38) evaluated at $\mathbb{U} = \mathring{\mathbb{U}}$ to yield

$$\mathring{\mathbb{A}}_{n+1} = \mathring{\mathbb{R}}\mathring{\mathbb{A}}_n. \quad (5.41)$$

Given (5.41), rewrite equation (5.40b) as follows:

$$\begin{aligned} \tilde{\mathbb{A}}_{n+1} &= \left(\mathring{\mathbb{R}}\mathring{\mathbb{A}}_n \right)' \tilde{\mathbb{U}}, \\ &= \mathring{\mathbb{R}}\mathring{\mathbb{A}}'_n \tilde{\mathbb{U}} + \left(\mathring{\mathbb{R}}\tilde{\mathbb{U}} \right)' \mathring{\mathbb{A}}_n, \\ &= \mathring{\mathbb{R}}\tilde{\mathbb{A}}_n + \tilde{\mathbb{R}}\mathring{\mathbb{A}}_n, \end{aligned}$$

where we have used (5.24) and (5.37) to simplify. This equation is precisely (5.38). Repeat the process to yield (5.39). \square

Remark 5.42. Given (4.39), (5.36), and (5.37), $\tilde{\mathbb{R}}$ is given by

$$\tilde{\mathbb{R}} = \begin{pmatrix} \tilde{\mathbb{U}} & 0 \\ 2\tilde{\mathbb{V}} - \partial_z^{-1}\tilde{\mathbb{V}}_z & \tilde{\mathbb{U}} - \partial_z^{-1}\tilde{\mathbb{U}}_z \end{pmatrix}. \quad (5.43)$$

Recall that cases A and B are valid for all $n \geq 2$ (Case A is also valid

for $n = 1$) whereas Case C is valid only for n odd. We therefore use the two-point recurrence relation (5.38) to test the proposed form of $\tilde{\mathbf{A}}_n$ for cases A and B, and the three-point recurrence relation (5.39) for Case C.

Proposition 5.44. Case A: $\tilde{\mathbf{A}}_n$ is given by (5.25) subject to

$$\begin{aligned} J_{j_n, j_i} &= (-1)^i f_{n, i}, & J_{j_n, k_i} &= -2\gamma_{i+1} f_{n, i+1}, \\ J_{k_n, j_i} &= 0, & J_{k_n, k_i} &= -f_{n, i}. \end{aligned} \quad (5.45)$$

for $i = 1, \dots, n$, where $f_{n, i}$ and γ_i are defined as

$$f_{n, i} = \frac{(n+1)!}{i!(n+1-i)!} c_0^{n+1-i}, \quad i = 1, \dots, (n+1), \quad (5.46)$$

$$\gamma_i = \frac{1}{2} (1 + (-1)^i), \quad (5.47)$$

and c_0 satisfies (5.13a).

Proof. The leading-order asymptotics in this case are given by (5.12) so that $\mathring{\mathbf{U}} = (c_0, 0)^T$. Given the definition of $\mathring{\mathbf{A}}_n$ (5.23), and using the asymptotic relations (5.10a) and (5.11a), we find $\mathring{\mathbf{A}}_n = (c_0^{n+1}, 0)^T$. Thus the recurrence relation for $\tilde{\mathbf{A}}_n$ (5.38) is rewritten as

$$\tilde{\mathbf{A}}_{n+1} = \mathring{\mathbf{R}} \tilde{\mathbf{A}}_n + c_0^{n+1} \mathring{\mathbf{U}}, \quad \mathring{\mathbf{R}} = \begin{pmatrix} c_0 - \partial_z & 2 \\ 0 & c_0 + \partial_z \end{pmatrix}, \quad (5.48)$$

where we have used (5.43). Note the equation: $f_{n+1, i+1} = f_{n, i} + c_0 f_{n, i+1}$, which follows from (5.46). Then it is straightforward to show by induction, using (5.48), that $\tilde{\mathbf{A}}_n$ is given by (5.25) subject to (5.45). \square

Proposition 5.49. Case B: $\tilde{\mathbf{A}}_n$ is given by (5.25) subject to

$$\begin{aligned} J_{j_n, j_i} &= (-1)^i \mathcal{Q}_{n, i}, & J_{j_n, k_i} &= -2\gamma_{i+1} \mathcal{Q}_{n, i+1}, \\ J_{k_n, j_i} &= -2c_2^2 c_3 \gamma_{i+1} \mathcal{Q}_{n, i+1}, & J_{k_n, k_i} &= -\mathcal{Q}_{n, i}. \end{aligned} \quad (5.50)$$

for $i = 1, \dots, n$, where γ_i is defined by (5.47), and c_2 and c_3 by (5.15) and (5.7) respectively. For $i = 1, \dots, n$,

$$\mathcal{Q}_{n, i} = \binom{n+1}{i} c_2^{n+1-i} \mathcal{P}_{n, i}, \quad (5.51)$$

$$\mathcal{P}_{n, i} = \sum_{j=0}^{n-i} d_{n, i, j} c_3^j, \quad (5.52)$$

$$d_{n, i, j} = \frac{([i/2])!(n+1-i)!}{(j + [i/2])! j! (n+1-2j-i)!}, \quad (5.53)$$

with $\mathcal{Q}_{n, n+1} = 1$.

Proof. The leading-order asymptotics in this case are given by (5.14) so that $\dot{\mathbf{U}} = (c_2, c_2^2 c_3)^T$. Given the definition of $\dot{\mathbf{A}}_n$ (5.23), and using the asymptotic relations (5.10b) and (5.11b), we find $\dot{\mathbf{A}}_n = (\dot{\mathbf{A}}_{n,1}, \dot{\mathbf{A}}_{n,2})^T$ has components

$$\dot{\mathbf{A}}_{n,1} = \mathcal{Q}_{n,0}, \quad \dot{\mathbf{A}}_{n,2} = \frac{2}{n+2} c_2^2 c_3 \mathcal{Q}_{n+1,2},$$

where $\mathcal{Q}_{n,i}$ is given by (5.51). Thus the recurrence relation for $\tilde{\mathbf{A}}_n$ (5.38) is rewritten as

$$\tilde{\mathbf{A}}_{n+1} = \mathring{\mathbf{R}} \tilde{\mathbf{A}}_n + \mathcal{Q}_{n,0} \tilde{\mathbf{U}}, \quad \mathring{\mathbf{R}} = \begin{pmatrix} c_2 - \partial_z & 2 \\ 2c_2^2 c_3 & c_2 + \partial_z \end{pmatrix}, \quad (5.54)$$

where we have used (5.43). Note the equation:

$$\mathcal{Q}_{n+1,i} = \mathcal{Q}_{n,i-1} + c_2 \mathcal{Q}_{n,i} + 4c_2^2 c_3 \gamma_{i+1} \mathcal{Q}_{n,i+1}, \quad (5.55)$$

which follows from (5.51). Then it is straightforward to show by induction, using (5.54), that $\tilde{\mathbf{A}}_n$ is given by (5.25) subject to (5.50). \square

Proposition 5.56. Case C: $\tilde{\mathbf{A}}_n$ is given by (5.25) subject to

$$\begin{aligned} J_{j_n, j_i} &= \gamma_i e_{n,i}, & J_{j_n, k_i} &= -2\gamma_{i+1} e_{n,i+1}, \\ J_{k_n, j_i} &= -2c_5 \gamma_{i+1} e_{n,i+1}, & J_{k_n, k_i} &= -\gamma_i e_{n,i}. \end{aligned} \quad (5.57)$$

for $i = 1, \dots, n$, where γ_i is defined by (5.47), c_5 by (5.17b), and

$$e_{n,i} = \frac{2^{n+1-i} (n/2)! c_5^{(n+1-i)/2}}{((i-1)/2)! ((n+1-i)/2)!}, \quad i = 1, \dots, (n+1). \quad (5.58)$$

Proof. The leading-order asymptotics in this case are given by (5.16) so that $\dot{\mathbf{U}} = (0, c_5)^T$. Given the definition of $\dot{\mathbf{A}}_n$ (5.23), and using the asymptotic relations (5.10c) and (5.11c), we find $\dot{\mathbf{A}}_n = (e_{n,0}, 0)^T$, where $e_{n,i}$ is defined by (5.58). Thus the recurrence relation for $\tilde{\mathbf{A}}_n$ (5.39) is rewritten as

$$\tilde{\mathbf{A}}_{n+2} = \mathring{\mathbf{R}}^2 \tilde{\mathbf{A}}_n + e_{n,0} (\sigma \tilde{\mathbf{U}}_1 + \tau \tilde{\mathbf{U}}), \quad (5.59)$$

where σ is defined in (5.25), $\mathring{\mathbf{R}}^2$ and τ are given by

$$\mathring{\mathbf{R}}^2 = \begin{pmatrix} \partial_z^2 + 4c_5 & 0 \\ 0 & \partial_z^2 + 4c_5 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 2 \\ 2c_5 & 0 \end{pmatrix},$$

and we have used (5.43) and

$$\mathring{R} = \begin{pmatrix} -\partial_z & 2 \\ 2c_5 & \partial_z \end{pmatrix}.$$

Note the equation: $e_{n+2,i} = e_{n,i-2} + 4c_5e_{n,i}$, which follows from (5.58). Then it is straightforward to show by induction, using (5.59), that \tilde{A}_n is given by (5.25) subject to (5.57). \square

Propositions 5.44, 5.49, and 5.56 provide the form of \tilde{A}_n for the three classes of solution of interest. Given \tilde{A}_n we can use equations (5.26) - (5.28) to construct J. In order to interrogate the assumptions of Proposition 5.21 we turn our attention to the eigenvalues of J.

Proposition 5.60. Let the eigenvalues of J be λ and set

$$\rho_1 = -\lambda^n + \sum_{i=1}^n J_{j_n, j_i} \lambda^{i-1}, \quad (5.61a)$$

$$\rho_2 = -\lambda^n + \sum_{i=1}^n J_{k_n, k_i} \lambda^{i-1}, \quad (5.61b)$$

$$\rho_3 = \sum_{i=1}^n J_{j_n, k_i} \lambda^{i-1}, \quad (5.61c)$$

$$\rho_4 = \sum_{i=1}^n J_{k_n, j_i} \lambda^{i-1}. \quad (5.61d)$$

Then the characteristic equation of J is

$$\rho(\lambda) = 0, \quad \rho = \rho_1\rho_2 - \rho_3\rho_4. \quad (5.62)$$

Proof. Let $\mathcal{M} = J - \lambda\mathcal{I}$ and let $\widehat{\mathcal{M}} = \widehat{J} - \lambda\mathcal{I}$ denote the upper-triangular form of \mathcal{M} , where \widehat{J} has the same structure as J given in (5.27), but with $\widehat{J}_j \equiv 0$ for $j = 1, \dots, (n-1)$, and

$$\widehat{J}_n = \begin{pmatrix} \widehat{J}_{j_n, j_n} & \widehat{J}_{j_n, k_n} \\ 0 & \widehat{J}_{k_n, k_n} \end{pmatrix}.$$

By elementary operations on J, we find that the components of \widehat{J}_n are

$$\widehat{J}_{j_n, j_n} = \lambda^{1-n} (\lambda^n + \rho_1), \quad (5.63a)$$

$$\widehat{J}_{j_n, k_n} = \lambda^{1-n} \rho_3, \quad (5.63b)$$

$$\widehat{J}_{k_n, k_n} = \lambda^{1-n} (\lambda^n + \rho_2 - \rho_3\rho_4\rho_1^{-1}), \quad (5.63c)$$

where ρ_1, ρ_2, ρ_3 , and ρ_4 are defined by (5.61). The characteristic equation for J is simply $\det \mathcal{M} = \det \widehat{\mathcal{M}} = 0$ which is

$$\lambda^{2(n-1)} \left(\widehat{J}_{j_n, j_n} - \lambda \right) \left(\widehat{J}_{k_n, k_n} - \lambda \right) = 0. \quad (5.64)$$

Substitute (5.63) into (5.64) to obtain the result (5.62). \square

Surprisingly, for Case A we can solve the $(2n)$ th-degree characteristic polynomial to obtain an explicit expression for the eigenvalues.

Proposition 5.65. Case A: the eigenvalues of J are

$$\lambda = \pm 2ic_0 \sin \theta_j \exp i\theta_j, \quad \theta_j = \frac{\pi j}{(n+1)}, \quad (5.66)$$

where $i = \sqrt{-1}$, $j = 1, \dots, n$, and c_0 is given in (5.13a).

Proof. Substitute (5.45) into (5.61a) and (5.61b) to give, respectively,

$$\begin{aligned} (-1)^{n-1} \rho_1 &= (-1)^n \lambda^n + \sum_{i=1}^n (-1)^{i-1} f_{n,i} \lambda^{i-1}, \\ -\rho_2 &= \lambda^n + \sum_{i=1}^n f_{n,i} \lambda^{i-1}, \end{aligned}$$

and substitute (5.45) into (5.61d) to give $\rho_4 = 0$. For such ρ_j , (5.62) implies that the characteristic equation for the eigenvalues is

$$\left\{ (-1)^n \lambda^n + \sum_{i=1}^n (-1)^{i-1} f_{n,i} \lambda^{i-1} \right\} \left\{ \lambda^n + \sum_{i=1}^n f_{n,i} \lambda^{i-1} \right\} = 0.$$

Use the definition of $f_{n,i}$ (5.46) to rewrite this equation as

$$\left((1 + \lambda_a)^{n+1} - 1 \right) \left((1 - \lambda_a)^{n+1} - 1 \right) = 0, \quad \lambda_a \neq 0, \quad (5.67)$$

where $\lambda_a = \lambda/c_0$.

Equation (5.67) implies that

$$1 \pm \lambda_a = \exp 2i\theta_j, \quad (5.68)$$

where θ_j is defined in (5.66) and $i = \sqrt{-1}$. Rearrange (5.68) to give

$$\begin{aligned} \pm \lambda_a &= \exp 2i\theta_j - 1, \\ &= \exp i\theta_j (\exp i\theta_j - \exp(-i\theta_j)), \\ &= 2i \sin \theta_j \exp i\theta_j. \end{aligned}$$

From this expression use $\lambda = c_0\lambda_a$ to obtain the result (5.66). \square

Proposition 5.69. Case B: the eigenvalues of J are all non-zero.

Proof. Assume that J has a zero eigenvalue, that is $\rho(0) = 0$. Set $\lambda = 0$ in (5.61) and substitute (5.50) to give

$$\rho_1(0) = (-1)^{n+1}\mathcal{Q}_{n,1}, \quad (5.70a)$$

$$\rho_2(0) = -\mathcal{Q}_{n,1}, \quad (5.70b)$$

$$\rho_3(0) = 2(-1)^{n+1}\mathcal{Q}_{n,2}, \quad (5.70c)$$

$$\rho_4(0) = -2c_2^2c_3\mathcal{Q}_{n,2}, \quad (5.70d)$$

where $\mathcal{Q}_{n,1}$ and $\mathcal{Q}_{n,2}$ are defined by equation (5.51). Substitute (5.70) into equation (5.62) and take the square root to give

$$\mathcal{Q}_{n,1} = \pm 2c_2\sqrt{c_3}\mathcal{Q}_{n,2}. \quad (5.71)$$

Eliminate c_2 from (5.71) using (5.51) to give

$$\mathcal{P}_{n,1} = \pm n\sqrt{c_3}\mathcal{P}_{n,2}, \quad (5.72)$$

where $\mathcal{P}_{n,1}$ and $\mathcal{P}_{n,2}$ are defined by (5.52).

Let us now show that c_3 is real and negative. Recall the definition of c_3 (5.7). Firstly, $c_3 \neq 0$ since $b_{n,1} \neq 0$ by (4.8b). Next, rewrite the polynomial $\mathcal{T}_n(c_3)$ (5.7) using (4.8b) to give

$$\mathcal{T}_n(c_3) = c_3^{n/2} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(n+1)!}{(j+1)!j!(n+2j)!} \left(\frac{1}{\sqrt{c_3}} \right)^{n-2j}. \quad (5.73)$$

Set $\alpha = -(n+1)$ and rewrite (5.73) as follows:

$$\mathcal{T}_n(c_3) = (-1)^n c_3^{n/2} \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{\Gamma(\alpha+n-j)}{j!(n+2j)!} \left(\frac{1}{\sqrt{c_3}} \right)^{n-2j}.$$

Then formula (22.3.4) of [3] implies

$$\mathcal{T}_n(c_3) = (-1)^n c_3^{n/2} C_n^{(\alpha)}(x), \quad x = \frac{1}{\sqrt{c_3}}, \quad \alpha = -(n+1), \quad (5.74)$$

where $C_n^{(\alpha)}(x)$ denotes the n th Gegenbauer polynomial with parameter α . Theorem 3 of [19] (see also [57]) states: for $\alpha < (1-n)$, all roots of $C_n^{(\alpha)}(x)$ lie on the imaginary axis. Given (5.74) this implies that all roots of $\mathcal{T}_n(c_3)$ are real and negative (since $c_3 \neq 0$).

Since c_3 is negative $\sqrt{c_3}$ is purely imaginary. All other terms in (5.72) are real. Thus (5.72) holds only if $\mathcal{P}_{n,1} = \mathcal{P}_{n,2} = 0$. It is straightforward to show that $\gcd(\mathcal{T}_n, \mathcal{P}_{n,1}) = 1$ and hence $\mathcal{P}_{n,1} \neq 0$, since $\mathcal{T}_n(c_3) = 0$. Thus equation (5.72) is not consistent and hence our initial assumption is incorrect. \square

Proposition 5.75. Case C: the eigenvalues of J are all non-zero.

Proof. Substituting (5.57) into (5.61) gives

$$\rho_1 = -\lambda p(\lambda) = \rho_2, \quad \rho_3 = -\lambda p(\lambda), \quad \rho_4 = -2c_5 p(\lambda), \quad (5.76)$$

where $p(\lambda)$ is given by

$$p(\lambda) = \sum_{i=0}^{(n-1)/2} e_{n,2i+2} \lambda^{2i}. \quad (5.77)$$

Given (5.76), equation (5.62) implies that the characteristic polynomial in this case is

$$(\lambda^2 + 4c_5) p(\lambda)^2 = 0. \quad (5.78)$$

By (5.17b) and (5.58), c_5 and $e_{n,2}$ are non-zero and thus $\lambda = 0$ is not a root of equation (5.78). \square

5.4 SECTORS OF VALIDITY

The results of Proposition 5.21 imply the existence of true solutions in sectors of the complex plane of a particular size. We now investigate the orientation of the sectors and the number of free parameters in the asymptotic description of the true solutions in these sectors. For the remainder of the chapter we choose g_{n+1} to be real and positive.

Theorem 5.79. Given any $x_0 \in \mathbb{C}$ such that $|x_0| > 1$, for each $n \geq 1$ and $m = 1, \dots, n$, define the sectors

$$\begin{aligned} \hat{\sigma}_{m,k} &= \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{k\pi}{(n+2)} < \arg(x - x_0) < \frac{(k + \beta_{n,m})\pi}{(n+2)} \right\}, \\ \sigma_{m,k} &= \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{k\pi}{(n+2)} < \arg(x - x_0) < \frac{(k + \beta_{n,m} + 1)\pi}{(n+2)} \right\}, \end{aligned}$$

where $k = 0, \dots, (2n-1)$ and $\beta_{n,m} = (n-m) + 1$. Then the following hold:

1. There exist families of m -parameter solutions of JM $P_{\text{II}}^{(n)}$, for $m = 2, \dots, n$, whose asymptotic behaviour as $|x| \rightarrow \infty$ is given by (5.2) with (5.3a) and (5.4) in the sectors $\hat{\sigma}_{m,k}$ and $\sigma_{m,k}$.
2. There exists a family of one-parameter solutions of JM $P_{\text{II}}^{(n)}$ whose asymptotic behaviour as $|x| \rightarrow \infty$ is given by (5.2) with (5.3a) and (5.4) in the sectors $\sigma_{1,k}$.

Proof. The proof follows from Propositions 5.65 (above) and 5.80 (below). In particular, Proposition 5.65 provides an explicit expression for the eigenvalues of J in this case. From this expression it is clear that the eigenvalues are distinct, which in turn shows that the assumption of Proposition 5.80 is fulfilled. The results of Proposition 5.80 subject to the Boutroux transformation (4.20) imply the theorem. \square

As in the previous section, we present our argument in the transformed Boutroux coordinates.

Proposition 5.80. If J (5.27) is diagonalisable, then solutions of (4.21) which are asymptotic to (5.18) are valid in the following sectors of the complex z -plane:

$$S_{i,l} = \{z \in \mathbb{C} \mid \varphi_i + 2l\pi < \arg(z) < \varphi_i + (2l+1)\pi\}, \quad l \in \mathbb{Z}, \quad (5.81)$$

where $\varphi_i = \pi/2 - \phi_i$, $\phi_i = \arg \lambda_i$, and λ_i are the eigenvalues of J . In each sector there is at least one free parameter hidden beyond all orders in the asymptotic description of the solution.

Proof. Let U_0 be a formal solution of (4.21) with components given by (5.18). Perturb this solution as follows:

$$U = U_0 + \hat{U}, \quad \hat{U} \ll 1, \quad (5.82)$$

and recall that \hat{U} denotes the $\mathcal{O}(1)$ part of U_0 . Then subject to (5.82), the defining equation (4.21) is

$$\mathring{A}_n + \hat{A}_n + 2^{n+1} (g_{n+1}, 0)^T = \mathcal{O}(z^{-1}, \hat{U}^2),$$

where \mathring{A}_n is given by (5.23) and

$$\hat{A}_n = \mathring{A}'_n \hat{U}, \quad (5.83)$$

where $'$ denotes the Fréchet derivative with respect to \hat{U} . On noting (5.31) and $\hat{U} \ll 1$ it follows that the asymptotic behaviour of \hat{U} is found by

solving the ODE

$$\widehat{\mathbf{A}}_n + \mathcal{O}(z^{-1}) = 0. \quad (5.84)$$

Let \mathbf{Y} be a vector of $2n$ components

$$y_{2i-1} = \widehat{U}_{i-1}, \quad y_{2i} = \widehat{V}_{i-1}, \quad i = 1, \dots, n.$$

Then the system of two n th-order equations (5.84) can be rewritten as a system of $2n$ first-order equations

$$\partial_z \mathbf{Y} = \mathbf{K} \mathbf{Y}, \quad (5.85)$$

where \mathbf{K} is a matrix of asymptotically expanded coefficients

$$\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{\infty} \frac{1}{z^i} \mathbf{K}_i.$$

The above construction of the first-order system (5.85) is familiar from the proof of Proposition 5.21, where we showed that in the system (5.33) the following holds: $\partial_z \mathbf{F}_0|_{\mathbf{Y}=0} = \mathbf{J}$. Given (5.24) and (5.83), it is clear that $\widehat{\mathbf{A}}_n = \widetilde{\mathbf{A}}_n|_{\widehat{\mathbf{U}}=\widehat{\mathbf{U}}}$ and hence by the same reasoning as previously made we find $\mathbf{K}_0 = \mathbf{J}$. Thus we can rewrite the system (5.85) as

$$\partial_z \mathbf{Y} = \mathbf{K} \mathbf{Y}, \quad \mathbf{K} = \mathbf{J} + \sum_{i=1}^{\infty} \frac{1}{z^i} \mathbf{K}_i. \quad (5.86)$$

For diagonalisable \mathbf{J} , a fundamental matrix solution of (5.86) is given in Theorem 2.24 by (2.25) where each matrix \mathbf{H} , \mathbf{D} , and \mathbf{Q} is $2n$ -by- $2n$ in this case. Extracting the asymptotic behaviour of $\widehat{\mathbf{U}}$ from \mathbf{Y} given by (2.25) we obtain

$$\widehat{\mathbf{U}} \sim \sum_{i=1}^{2n} \mathbf{C}_i \exp(\lambda_i z). \quad (5.87)$$

where each \mathbf{C}_i is a vector of two arbitrary constants and λ_i are the eigenvalues of \mathbf{J} . This argument is familiar from Chapter 2.

The asymptotic behaviour found for $\widehat{\mathbf{U}}$ (5.87) must be consistent with the original assumption that $\widehat{\mathbf{U}} \ll 1$. This is only the case if $\Re(\lambda_i z) < 0$ or $\mathbf{C}_i = 0$, for each i in (5.87). The sectors defined by those z for which $\Re(\lambda_i z) < 0$ are $S_{i,l}$ (5.81). In each sector $S_{i,l}$ the corresponding constant \mathbf{C}_i is free. If two sectors overlap, then in the sector of overlap two constants are free, and so on. The orientation of the sectors $S_{i,l}$ (5.81) is determined

by φ_i and hence by the eigenvalues of J . \square

For for Case A, we showed in Proposition 5.65 that the eigenvalues of J are distinct, and thus the assumptions of Proposition 5.80 are immediately satisfied. Consider Case C.

Proposition 5.88. Case C: J is diagonalisable.

Proof. In this case, n is odd and the eigenvalues of J satisfy the characteristic equation (5.78) where $p(\lambda)$ is defined by (5.77). The proof of the proposition is in three parts. We first show that

1. $\lambda = 2i\sqrt{c_5}$ and $\lambda = -2i\sqrt{c_5}$ are not roots of $p(\lambda)$; and,
2. $p(\lambda)$ has no repeated roots,

from which we conclude that J has two distinct eigenvalues $\pm 2i\sqrt{c_5}$, and $(n-1)$ eigenvalues of multiplicity two, which correspond to the simple roots of $p(\lambda)$. We then show that

3. for each repeated eigenvalue of J there exist two linearly independent eigenvectors.

To prove 1, substitute the definition of $e_{n,2i+2}$ (5.58) into equation (5.77) to yield

$$p(\lambda) = (2\sqrt{c_5})^{n-1} \left(\frac{n}{2}\right)! \sum_{j=0}^{(n-1)/2} \frac{1}{(j + \frac{1}{2})! \left(\frac{(n-1)}{2} - j\right)!} \left(\frac{\lambda^2}{4c_5}\right)^j,$$

and rewrite this polynomial as follows:

$$p(\lambda) = \frac{2^n (\sqrt{c_5})^{n-1} \Gamma\left(\frac{n+1}{2} + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right)} F(a, b; c; z), \quad (5.89)$$

where $F(a, b; c; z)$ is a standard hypergeometric series with arguments

$$a = 1, \quad b = -\frac{(n-1)}{2}, \quad c = \frac{3}{2}, \quad z = -\frac{\lambda^2}{4c_5}, \quad (5.90)$$

Note that since n is odd, b is a negative integer and the series is indeed terminating. Assume that $\lambda = \pm 2i\sqrt{c_5}$ is a root of $p(\lambda)$. Then (5.89) and (5.90) imply $F(a, b; c; 1) = 0$. But

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

(see for example [3]) which is clearly non-zero. Thus our assumption is incorrect and $\lambda = \pm 2i\sqrt{c_5}$ is a not root of $p(\lambda)$.

To prove 2, we use the polynomial Euclidean algorithm to calculate $\gcd(p, p')$ where $'$ denotes differentiation with respect to λ . We find

1. $p = \beta_1 \lambda p' + r_1$,
2. $p' = \beta_2 \lambda r_1 + r_2$,
3. $r_1 = \beta_3 \lambda r_2 + r_3$,

where r_1 and r_2 are polynomials in λ given by

$$r_1 = \sum_{j=0}^{(n-3)/2} (1 - 2j\beta_1) g_{n,j} \lambda^{2j},$$

$$r_2 = \sum_{j=1}^{(n-3)/2} (2j g_{n,j} - \beta_2 (1 - 2(j-1)\beta_1) g_{n,j-1}) \lambda^{2j-1},$$

where $g_{n,j} = c_5^{1+2i-n} e_{n,2j+1}$ and $\beta_1, \beta_2, \beta_3$, and r_3 are the following constants:

$$\beta_1 = \frac{1}{n-1}, \quad \beta_2 = \frac{(n-1)^2}{4n}, \quad \beta_3 = -\frac{n}{n-1}, \quad r_3 = g_{n,0}.$$

This implies $\gcd(p, p') = g_{n,0}$. Since $g_{n,0}$ is a non-zero constant, p and p' have no common factors and hence p has no repeated roots.

To prove 3, we note that the repeated eigenvalues of J correspond to roots of $p(\lambda)$. Let λ_0 denote a root of $p(\lambda)$ so that

$$p(\lambda_0) = 0. \tag{5.91}$$

Let w and \widehat{w} denote vectors with $2n$ components

$$w_{2j-1} = \lambda_0^{j-n} = \widehat{w}_j, \quad w_{2j} = 0 = \widehat{w}_{2j-1}, \tag{5.92}$$

for $j = 1, \dots, n$. By (5.92), w and \widehat{w} are clearly linearly independent. Let $\mathcal{M} = J - \lambda \mathcal{I}$. For $\lambda = \lambda_0$ denote the product of \mathcal{M} with w and \widehat{w} as follows:

$$\mathcal{M} w = (m_1, \dots, m_{2n})^T, \tag{5.93a}$$

$$\mathcal{M} \widehat{w} = (\widehat{m}_1, \dots, \widehat{m}_{2n})^T. \tag{5.93b}$$

Recall that J is given by (5.27). Thus by (5.27) and (5.92), the components of (5.93) are

$$m_j = -\lambda_0 (\lambda_0^{j-n}) + \lambda_0^{j+1-n} = 0,$$

$$\widehat{m}_j = 0,$$

for $j = 1, \dots, (2n - 2)$ and the remaining components are given by

$$m_{2n-1} = \lambda_0^{1-n} \rho_1(\lambda_0), \quad m_{2n} = \lambda_0^{1-n} \rho_4(\lambda_0), \quad (5.94)$$

$$\widehat{m}_{2n-1} = \lambda_0^{1-n} \rho_3(\lambda_0), \quad \widehat{m}_{2n} = \lambda_0^{1-n} \rho_2(\lambda_0), \quad (5.95)$$

where ρ_i is given in (5.61). By (5.76), each $\rho_i(\lambda)$ is equal to $c_i p(\lambda)$ for constant c_i . Thus by (5.91) each term in (5.94) and (5.95) is zero and hence $\mathcal{M} w = \mathcal{M} \widehat{w} = 0$. \square

Remark 5.96. Proposition 5.88 shows that the assumptions of Proposition 5.80 are fulfilled for Case C and therefore the sectors of validity of the asymptotic descriptions are given by (5.81). We note that these sectors are implicitly defined in terms of the eigenvalues of J , which arise as the solution of equation (5.78) in this case.

Conjecture 5.97. Case B: J is diagonalisable.

5.5 UNIQUE SOLUTIONS

Denote the one-parameter solutions defined in Theorem 5.79 as follows:

$$(u_{t,A,j}, v_{t,A,j}) \Big|_{\substack{|x| \rightarrow \infty \\ x \in \sigma_{1,k}}} \sim (u_f, v_f) \Big|_{u_0 = u_{A,j}}.$$

Theorem 5.98. Given $x_0 \neq 0$, the true solutions $(u_{t,A,j}, v_{t,A,j})$ are unique in the sectors

$$\Sigma_{k,j} = \left\{ x \in \mathbb{C} \mid |x| > |x_0|, \frac{k\pi}{(n+2)} < \arg(x - x_0) < \frac{(k + \beta_{k,j})\pi}{(n+2)} \right\},$$

where $k = 0, \dots, (2n + 3)$. For each k , j takes n distinct values j_1, \dots, j_n , from $\{1, \dots, (n + 1)\}$, and $\beta_{k,j_l} = n + 1 + l$. If $j \in \{j_1, \dots, j_n\}$ for both k and $k + 1$ then $\beta_{k,j} = \beta_{k+1,j} - 1$.

Proof. The result follows from propositions 5.65 (above) and 5.99 (below). \square

By Proposition 5.80, the families of solutions asymptotic to (5.18) in sectors (5.81) contain at least one parameter C_i hidden beyond all orders. We now ask in which sectors of the plane there exists a unique solution asymptotic to (5.18).

Proposition 5.99. If J is diagonalisable then there exist unique solutions

of (4.21) which are asymptotic to (5.18) in sectors

$$T_{i,l} = S_{i,l} \cup S_{i+1,l}, \quad (5.100)$$

where $S_{i,l}$ and $S_{i+1,l}$ are given by (5.81) for each i and l .

Proof. For some choice of i and l , define the following two sectors:

$$\widehat{S}_{i,l,\pm\epsilon} = \{z \in \mathbb{C} \mid \varphi_i \pm \epsilon + 2l\pi < \arg z < \varphi_i \pm \epsilon + (2l+1)\pi\},$$

which are related to $S_{i,l}$ (5.81) by angular rotation by $\pm\epsilon$, $\epsilon \ll 1$. Let U_1 and U_2 be two solutions whose asymptotic behaviour is known to be (5.18) in the respective sectors $\widehat{S}_{i,l,+\epsilon}$ and $\widehat{S}_{i+1,l,-\epsilon}$. Define the overlap of these two sectors as

$$\widetilde{S}_{i,l,\epsilon} = \widehat{S}_{i,l,+\epsilon} \cap \widehat{S}_{i+1,l,-\epsilon}.$$

We have two pieces of information we can exploit. Firstly, the two solutions have the same asymptotic behaviour for $z \in \widetilde{S}_{i,l,\epsilon}$. Thus,

$$W := U_1 - U_2 \underset{\substack{|z| \rightarrow \infty \\ z \in \widetilde{S}_{i,l,\epsilon}}}{=} o(z^{-m}), \quad (5.101)$$

for all positive integers m . Secondly, since U_1 and U_2 are both solutions of (4.21), their difference W must satisfy the system of ODEs

$$A_n|_{U=U_1} - A_n|_{U=U_2} = \mathcal{O}(z^{-1}). \quad (5.102)$$

Rearrange the first equation in (5.101) for U_1 to obtain $U_1 = U_2 + W$, and substitute this expression into equation (5.102) to yield

$$(A_n|_{U=U_2})' W = \mathcal{O}(z^{-1}, W^2), \quad (5.103)$$

where $'$ denotes the Fréchet derivative with respect to U_2 . This kind of system is familiar from our earlier proofs. In particular, if we replace U_2 with its asymptotic behaviour (5.18) then (5.103) is identical to equation (5.84) with $\widehat{U} \mapsto W$. In equation (5.103) we can ignore the contribution of higher-degree W terms since $W = o(z^{-m})$ for all $m \in \mathbb{N}$. Thus the asymptotic behaviour of W can be derived from those terms in (5.103) which are linear in W , or equivalently from a system of the form (5.86), which is solved by (2.25) where each matrix H , D , and Q is $2n$ -by- $2n$ in this case.

Extracting the asymptotic behaviour of W from Y given by (2.25), we

obtain

$$W \sim \sum_{r=1}^{2n} \exp(\lambda_r z) C_r, \quad z \in \tilde{S}_{i,l,\epsilon}, \quad (5.104)$$

where λ_r are the eigenvalues of J and C_r are free vector constants. The two expressions for the asymptotic behaviour of W , (5.101) and (5.104), must be consistent. Since $\varphi_i = \pi/2 - \phi_i$ and $\phi_i = \arg \lambda_i$, it is clear that for each r in (5.104) there exists some z in $\tilde{S}_{i,l,\epsilon}$ for which $\Re(\lambda_r z) > 0$. Thus (5.101) and (5.104) are consistent only if each C_r in (5.104) is set to zero. This gives $U_1 = U_2$ in $\tilde{S}_{i,l,\epsilon}$. The sector of validity for the asymptotic behaviour can be analytically continued into the extended sector $S_{i,l} \cup S_{i+1,l}$. This argument is familiar from Chapter 2. \square

Remark 5.105. Given Proposition 5.88, the assumption of Proposition 5.99 is fulfilled for Case c. Thus we can construct sectors of validity (5.100) in which there is a unique solution for Case c. We note that these sectors are implicitly defined in terms of the eigenvalues of J , which arise as the solution of equation (5.78) in this case.

5.6 DISCUSSION

This chapter provided a comprehensive structure for the analysis of solutions with algebraic asymptotics. Our methods can be applied to any member of the Jimbo-Miwa hierarchy, and indeed to any hierarchy which is defined recursively by two n th-order non-linear ODEs. We found three classes of special asymptotic behaviours of JM $P_{\text{II}}^{(n)}$ and proved that there exist true solutions with these behaviours in sectors of the complex x -plane of a specified angular opening. We explicitly determined the sectors of validity (that is, orientation as well as size) for families of solutions, and unique solutions, in one of these cases. The solutions cannot be characterised via the naming convention of Boutroux as discussed for the fourth-order equation in §2.5, as this requires an explicit expression for the sectors of validity in *all* cases.

It is interesting to note that the formal solution denoted Case B does not appear for the classical equation, and that Case c is valid only for the odd members of the hierarchy. Thus it is only for JM $P_{\text{II}}^{(3)}$ that all three asymptotic behaviours given in Theorem 5.1 are initially valid - these behaviours are written explicitly in the discussion in §2.6. Let us revisit the following claim that we made in that discussion: that the algebraic asymptotic expansions for $u(x)$ in the system of equations JM $P_{\text{II}}^{(2m-1)}$

are comparable to those of FN $P_{\text{II}}^{(m)}$, given in [33]. Theorem 5.1 shows that JM $P_{\text{II}}^{(2m-1)}$ admits three possible algebraic expansions of the form

$$u(x) = u_0 \left(1 + \mathcal{O}(x^{-\kappa(1-\epsilon)})\right), \quad v(x) = v_0 \left(1 + \mathcal{O}(x^{-\kappa(1-\epsilon)})\right),$$

for $\kappa = (2m + 1)/(2m)$ and $\epsilon > 0$ where

$$\begin{aligned} u_0 &\sim x^{1/(2m)}, & v_0 &\sim x^{-(2m-1)/(2m)}, \\ u_0 &\sim x^{1/(2m)}, & v_0 &\sim x^{1/m}, \\ u_0 &\sim x^{-1}, & v_0 &\sim x^{1/m}. \end{aligned}$$

Let $y(x)$ denote the solution of FN $P_{\text{II}}^{(m)}$. Then in [33], it was shown that $y(x)$ admits two possible expansions of the form

$$y(x) = y_0 \left(1 + \mathcal{O}(x^{-\kappa(1-\epsilon)})\right),$$

where $y_0 \sim x^{1/(2m)}$ and $y_0 \sim x^{-1}$, and our claim therefore holds. Thus in terms of the algebraic asymptotics, for $m \geq 1$ there is a relationship between the odd $(2m - 1)$ members of the Jimbo-Miwa hierarchy and all members m of the Flaschka-Newell hierarchy, rather than a direct correspondence between equations at the same level of the two hierarchies.

We now consider the general asymptotic behaviours admitted by equations in the Jimbo-Miwa hierarchy.

6.1 ABSTRACT

We conclude our asymptotic study of the Jimbo-Miwa hierarchy by considering the most general asymptotic behaviour admitted by $\text{JM P}_{\text{II}}^{(n)}$. We find that the general asymptotic behaviour of the solutions at each level of the hierarchy are described to leading-order by two related genus- n hyperelliptic functions, and we construct these functions explicitly in this chapter. This result provides a natural extension of the classical elliptic function (genus-1) solutions for P_{II} and the genus-2 hyperelliptic function solutions derived for $\text{JM P}_{\text{II}}^{(2)}$ in Chapter 3.

6.2 LEADING-ORDER ASYMPTOTICS

Proposition 6.1. Let $\gamma_1(z), \dots, \gamma_n(z)$ be defined by the inversion of the hyperelliptic integrals

$$\sum_{j=1}^n \int_{\infty}^{\gamma_j} \frac{s^k ds}{\sqrt{P(s)}} = c_k, \quad k = 0, \dots, n-2, \quad (6.2a)$$

$$\sum_{j=1}^n \int_{\infty}^{\gamma_j} \frac{s^{n-1} ds}{\sqrt{P(s)}} = z + c_{n-1}, \quad (6.2b)$$

where c_0, \dots, c_{n-1} are n arbitrary constants of integration, and P is the $(2n+2)$ th-degree polynomial

$$P(s) = \sum_{i=0}^{2n+2} p_i s^i, \quad p_{2n+1} = 0, \quad (6.3)$$

where n of the coefficients p_i are arbitrary. In the limit as $|x| \rightarrow \infty$ the general asymptotic behaviour of $\text{JM P}_{\text{II}}^{(n)}$ is given in terms of Boutroux coordinates (4.20) where $U(z)$ and $V(z)$ denote the functions

$$U = -2 \sum_{i=1}^n \gamma_i + \mathcal{O}(z^{-1}), \quad (6.4a)$$

$$V = -\sum_{i=1}^n (\gamma_i)_z - 2 \left(\sum_{i=1}^n \gamma_i \right)^2 + 2 \sum_{\substack{i,j=1 \\ i \neq j}}^n \gamma_i \gamma_j + \mathcal{O}(z^{-1}). \quad (6.4b)$$

Proof. The solutions are constructed from the leading-order equation (4.54a) following Drach's method presented in §3.2.2. Consider equation (4.54a). Given \mathcal{F} (4.54b) and $\tilde{\mathcal{H}}_n$ (4.54c) are both polynomial in μ it follows that \tilde{P} is polynomial in μ . By balancing the highest degree terms in (4.54a) we find that \tilde{P} is a $(2n+2)$ th-degree polynomial, and hence $\tilde{P} = P$ (6.3). Thus equation (4.54a) is rewritten as

$$2\tilde{\mathcal{H}}_n(\tilde{\mathcal{H}}_n)_{zz} - (\tilde{\mathcal{H}}_n)_z^2 - 4\mathcal{F}(\tilde{\mathcal{H}}_n)^2 + P(\mu) = 0. \quad (6.5)$$

Rewrite $\tilde{\mathcal{H}}_n$ (4.54c) in factorised form as follows:

$$\tilde{\mathcal{H}}_n = \prod_{i=1}^n (\mu - \mathcal{G}_i), \quad (6.6)$$

for n functions $\mathcal{G}_i(z)$. Evaluate equation (6.5) at points where $\tilde{\mathcal{H}}_n$ (6.6) vanishes, that is $\mu = \mathcal{G}_j$, to obtain a system of n ODEs for \mathcal{G}_j ,

$$(\mathcal{G}_j)_z^2 \prod_{\substack{i=1 \\ i \neq j}}^n (\mathcal{G}_j - \mathcal{G}_i)^2 = P(\mathcal{G}_j), \quad j = 1, \dots, n. \quad (6.7)$$

Take the square root of (6.7) and rewrite as follows:

$$\frac{(\mathcal{G}_j)_z}{\sqrt{P(\mathcal{G}_j)}} = \frac{1}{\mathbf{P}_j}, \quad \mathbf{P}_j = \prod_{\substack{i=1 \\ i \neq j}}^n (\mathcal{G}_j - \mathcal{G}_i). \quad (6.8)$$

Let \mathbf{S}_k denote the sum

$$\mathbf{S}_k = \sum_{j=1}^n \frac{\mathcal{G}_j^k}{\mathbf{P}_j}, \quad k = 0, \dots, (n-1), \quad (6.9)$$

then \mathbf{S}_k is given by

$$\mathbf{S}_k = 0, \quad k = 0, \dots, (n-2), \quad (6.10a)$$

$$\mathbf{S}_{n-1} = 1. \quad (6.10b)$$

The identities (6.10a) and (6.10b) follow from a formula for the inverse of the Vandermonde matrix [44, 61]. Substitute (6.9) into (6.10a) and

(6.10b), and replace $1/P_j$ using (6.8) to yield

$$\sum_{j=1}^n \frac{\mathcal{G}_j^k(\mathcal{G}_j)_z}{\sqrt{P(\mathcal{G}_j)}} = 0, \quad k = 0, \dots, (n-2), \quad (6.11a)$$

$$\sum_{j=1}^n \frac{\mathcal{G}_j^{n-1}(\mathcal{G}_j)_z}{\sqrt{P(\mathcal{G}_j)}} = 1. \quad (6.11b)$$

Integrate (6.11a) and (6.11b) with respect to z to give

$$\sum_{j=1}^n \int_{\infty}^{\mathcal{G}_j} \frac{s^k ds}{\sqrt{P(s)}} = c_k, \quad k = 0, \dots, n-2, \quad (6.12a)$$

$$\sum_{j=1}^n \int_{\infty}^{\mathcal{G}_j} \frac{s^{n-1} ds}{\sqrt{P(s)}} = z + c_{n-1}, \quad (6.12b)$$

for integration constants c_0, \dots, c_{n-1} . On comparison of (6.2) and (6.12) we conclude that

$$\mathcal{G}_j = \gamma_j + \mathcal{O}(z^{-1}), \quad j = 1, \dots, n, \quad (6.13)$$

where the correction terms arise because the equation (6.5), from which (6.12) is derived, is correct only to leading-order.

Substitute (6.13) into (6.6) to yield

$$\tilde{\mathcal{H}}_n = \prod_{i=1}^n (\mu - \gamma_i), \quad (6.14)$$

up to $\mathcal{O}(z^{-1})$. The two definitions of $\tilde{\mathcal{H}}_n$, (4.54c) and (6.14), must be consistent. Equate (4.54c) and (6.14) and balance powers of μ to give the leading-order relations

$$\mu^0: \quad 2^{-1}A_{0,1} = -\sum_{i=1}^n \gamma_i, \quad \mu^1: \quad 2^{-2}A_{1,1} = \sum_{\substack{i,j=1 \\ i \neq j}}^n \gamma_i \gamma_j. \quad (6.15)$$

Recall that $A_{0,1}$ and $A_{1,1}$ are given by

$$A_{0,1} = U, \quad A_{1,1} = -U_z + U^2 + 2V, \quad (6.16)$$

then substitute (6.16) into (6.15) and rearrange to give (6.4).

There are presently $(3n+2)$ constants in our description of the solution: n integration constants c_k and $(2n+2)$ coefficients p_i . Since the governing system of equations JM $P_{\text{II}}^{(n)}$ is $(2n)$ th-order we require that $(n+2)$ of the coefficients p_i are prescribed. This leaves n arbitrary coefficients, as

stated in the proposition. \square

Proposition 6.1 shows that the leading-order solutions of JM $P_{\text{II}}^{(n)}$ are given by two hyperelliptic functions of genus- n . The symmetric combination of the functions γ_i and the derivatives $(\gamma_i)_z$ in the expressions for U and V (6.4) ensure that these functions are meromorphic. Such globally meromorphic functions, which are constructed from functions that arise from the inversion of hyperelliptic integrals, are hyperelliptic functions. The genus g of these functions is calculated from the degree d of the polynomial (6.3) with $g = (d - 2)/2$.

Remark 6.17. While the two definitions of $\tilde{\mathcal{H}}_n$ (4.54c) and (6.6) have been shown to be consistent at powers of μ^1 and μ^0 in the proof of Proposition 6.1, at higher powers of μ we find the following relations:

$$2^{-(k+1)}A_{k,1} = (-1)^{k+1} \sum \gamma_{i_1} \cdots \gamma_{i_{k+1}}, \quad k = 2, \dots, (n-1),$$

where the sum is over multiple indices i_1, \dots, i_k where i_l runs from 1 to n and $i_l \neq i_m$ for all $l, m \in \{1, \dots, k\}$. In addition, although we have considered equation (6.5) at each zero of $\tilde{\mathcal{H}}_n$ in the proof of Proposition 6.1, we have not considered the consistency of this equation when \mathcal{F} (4.54b), $\tilde{\mathcal{H}}_n$ (4.54c), and P (6.3) are substituted. These concerns are resolved provided the $(n+2)$ conditions on the coefficients of the polynomial P are imposed.

Conjecture 6.18. The $(n+2)$ conditions on the coefficients of P are

$$\begin{aligned} p_{2n+2} &= 4, \\ p_i &= 0, \quad i = (n+2), \dots, 2n, \\ p_{n+1} &= 4g_{n+1}, \\ p_n &= 0. \end{aligned}$$

Remark 6.19. Given the conditions in Conjecture 6.18, there are n arbitrary coefficients in the polynomial P (6.3).

Conjecture 6.20. Let (M_i, N_i) denote the integrating factors

$$M_i = (A_{i-1,2})_z, \quad N_i = (A_{i-1,1})_z, \quad i = 1, \dots, n. \quad (6.21)$$

Then the n arbitrary coefficients of P define the following leading-order first integrals for JM $P_{\text{II}}^{(n)}$:

$$E_{n,i} = \int M_i (A_{n,1} + 2^n g_{n+1}) + N_i A_{n,2} dz, \quad i = 1, \dots, n, \quad (6.22)$$

where each (6.22) is written locally and is related to p_i by

$$p_i = 2^{i+2-2n} \mathbf{E}_{n,n-i}, \quad i = 1, \dots, n-1, \quad (6.23a)$$

$$p_0 = 2^{2-2n} \left(\mathbf{E}_{n,n} + (2^{n-1} g_{n+1})^2 \right). \quad (6.23b)$$

Remark 6.24. Drach [18] explicitly identified this split between the highest and lowest coefficients of P . It is easily understood by returning to the leading-order compatibility condition (4.53). This equation is of degree $(n+2)$ in μ and defines $\text{JM P}_{\text{II}}^{(n)}$. Recall that in order to obtain (4.54a) from (4.53), we use the integrating factor $2\mathcal{H}$ - which is an n th-degree polynomial. Hence the n additional degrees of freedom must define first integrals. In fact, the first integrals (6.22) are equivalent to equations (6.11), given (6.4), (6.13) and (6.23).

The precise expressions for p_i given in conjectures 6.18 and 6.20 can be obtained by substituting \mathcal{F} (4.54b), $\tilde{\mathcal{H}}_n$ (4.54c), and P (6.3) into equation (6.5) and balancing powers of μ - the coefficients p_{2n+2} and p_{2n} are easily derived in this way. However given (6.5) is non-linear in $\tilde{\mathcal{H}}_n$, and given the definition of $\tilde{\mathcal{H}}_n$ in terms of $\mathbf{A}_{i,1}$ (4.54c), it follows that for $i = (n+2), \dots, (2n-1)$, at any power μ^i the expression for p_i is an algebraic function of $\mathbf{A}_{0,1}, \dots, \mathbf{A}_{2n+1-i,1}$, and derivatives of these terms. Similarly, for $i = 0, \dots, (n+1)$, at μ^i the expression for p_i is an algebraic function of $\mathbf{A}_{0,1}, \dots, \mathbf{A}_{n-1,1}$, and derivatives of these terms. To simplify such expressions, note that the defining recurrence relation for $\mathbf{A}_i = (\mathbf{A}_{i,1}, \mathbf{A}_{i,2})^T$ (4.38) can be rewritten in terms of $\mathbf{A}_{i,1}$ as follows:

$$(\mathbf{A}_{i,1})_z = (U_z + 2U\partial_z) \mathbf{A}_{i-1,1} + D\mathbf{A}_{i-2,1}, \quad (6.25)$$

where D is the differential operator

$$D = \partial_z^2 - \frac{1}{2} (2U_z + U^2 + 4V)_z - (2U_z + U^2 + 4V) \partial_z.$$

However given the relation (6.25) is only second-order, the expressions for p_i , which are given by a function of possibly all of $\mathbf{A}_{0,1}, \dots, \mathbf{A}_{n-1,1}$, and their derivatives, are not easy to simplify in general.

The definition of the first integrals (6.22) and integrating factors (6.21) is familiar from our study of the fourth-order case in Chapter 3. They also appear in the definition of the terms \mathbf{A}_n . Equations (4.38) and (4.39)

imply that the second component of \mathbf{A}_{n+1} is given by

$$\begin{aligned} A_{n+1,2} &= d_0 A_{n,1} + \tilde{d}_0 A_{n,2} - \int (A_{0,2})_z A_{n,1} + (A_{0,1})_z A_{n,2} dz, \\ &= d_0 A_{n,1} + \tilde{d}_0 A_{n,2} + 2^n g_{n+1} A_{0,2} - E_{n,1}, \end{aligned}$$

where $d_0 = 2V$ and $\tilde{d}_0 = U + \partial_z$, and $E_{n,1}$ is given by (6.22). In addition, the first-order recurrence relation (4.38) implies a sequence of recurrence relations

$$A_{n+1} = R^{i+1} A_{n-i}, \quad i = 0, \dots, n.$$

For $i = 1$ and $i = 2$ we find, respectively,

$$\begin{aligned} A_{n+1,2} &= d_1 A_{n-1,1} + \tilde{d}_1 A_{n-1,2} + 2^n g_{n+1} A_{1,2} - E_{n,2}, \\ A_{n+1,2} &= d_2 A_{n-2,1} + \tilde{d}_2 A_{n-2,2} + 2^n g_{n+1} A_{2,2} - E_{n,3} + T, \end{aligned}$$

where $T = 2A_{0,2}(2^n g_{n+1} A_{0,2} - E_{n,1})$, and

$$\begin{aligned} d_1 &= 2(V_z + 2UV), \\ \tilde{d}_1 &= (-U_z + U^2 + 2V) + 2V + 2U\partial_z + \partial_z^2, \\ d_2 &= 2(V_{zz} + 3UV_z + 3V^2 + 3U^2V) + 2V^2 + V_z\partial_z + 2V\partial_z^2, \\ \tilde{d}_2 &= (U_{zz} - 3UU_z + 6UV + U^3) + 2(V_z + 2UV) + 2UV \\ &\quad + (3U^2 + 4V)\partial_z + 3U\partial_z^2 + \partial_z^3. \end{aligned}$$

Therefore if we express $A_{n+1,2}$ in terms of $A_{n-i,1}$ and $A_{n-i,2}$ for $i = 0, \dots, (n-1)$, then the term $E_{n,i+1}$ also appears in these expressions.

6.3 DISCUSSION

In this chapter we constructed the general leading-order behaviour of each equation $\text{JM } P_{\text{II}}^{(n)}$ in terms of two related genus- n hyperelliptic functions. The results presented here are consistent with the solutions derived in Chapter 3 for $\text{JM } P_{\text{II}}^{(2)}$. Moreover, they show that the elliptic function behaviour of the classical second Painlevé transcendent admits a natural extension to all of the higher-order equations in the Jimbo-Miwa hierarchy. Together with the results of Chapter 5, which describe the special asymptotic behaviours admitted by $\text{JM } P_{\text{II}}^{(n)}$, we have obtained a detailed picture of the asymptotic structure of the hierarchy. We complete our study of the Jimbo-Miwa hierarchy with an investigation of special solutions which arise for particular values of the parameter α_n .

SPECIAL SOLUTIONS

7.1 ABSTRACT

In §2.6 we identified a special solution of the sixth-order Jimbo-Miwa second Painlevé equation JM $P_{\text{II}}^{(3)}$ given in terms of the classical first Painlevé equation P_{I} . This solution was found by considering a degeneration of the leading-order algebraic asymptotics. In this chapter we derive special solutions for the entire Jimbo-Miwa hierarchy by considering a degeneration of the asymptotics derived in Chapter 5. We prove that for each $m \geq 1$ special solutions of JM $P_{\text{II}}^{(2m+1)}$ are expressed in terms of solutions of equations in the first Painlevé hierarchy $P_{\text{I}}^{(m)}$. Special solutions given in terms of hyper-Airy functions are also presented.

7.2 PRELIMINARIES

Before presenting our results, we define the first Painlevé hierarchy and the hyper-Airy equations.

For each $m \geq 1$ the first Painlevé hierarchy is defined by a $(2m)$ th-order scalar ordinary differential equation

$$P_{\text{I}}^{(m)} : \quad \mathbf{d}_m[w] + 2^{2m-1}g_{2m+1}x = 0, \quad (7.1)$$

where \mathbf{d}_m is determined by the Lenard recursion relation

$$\partial_x \mathbf{d}_{m+1}[w] = (\partial_x^3 - 8w\partial_x - 4w_x) \mathbf{d}_m[w], \quad \mathbf{d}_0[w] = -w. \quad (7.2)$$

The P_{I} hierarchy is derived from a reduction of the KdV hierarchy through the mKdV hierarchy in [40]. The presentation above is given in [60], except for the coefficient of x which we have chosen for convenience. Equation (7.2) implies

$$\mathbf{d}_1[w] = -w_{xx} + 6w^2, \quad (7.3)$$

$$\mathbf{d}_2[w] = -w_{4x} + 20w w_{xx} + 10w_x^2 - 40w^3, \quad (7.4)$$

such that (7.1) gives

$$\begin{aligned} P_I^{(1)} : \quad w_{xx} &= 6w^2 + 2g_3x, \\ P_I^{(2)} : \quad w_{4x} &= 20ww_{xx} + 10w_x^2 - 40w^3 + 8g_5x. \end{aligned}$$

The classical Airy equation is the second-order linear ordinary differential equation $\phi_{xx} + x\phi = 0$. For $n \geq 1$ define a hierarchy of Airy equations, or hyper-Airy equations (see [5]), as

$$A^{(n)} : \quad (-1)^{n+1}\phi_{n+1} + 2^n g_{n+1}x\phi = 0, \quad \phi_{n+1} = \partial_x^{n+1}\phi, \quad (7.5)$$

where the coefficient of x and the alternating sign of the highest derivative are chosen for convenience. Set $n = 1$ in (7.5) to recover the classical equation with a scaled coefficient of x .

7.3 RESULTS

Recall the three leading-order asymptotic behaviours (5.4), (5.5) and (5.6), derived for JM $P_{II}^{(n)}$ in Chapter 5. For special values of α_n some components of the leading-order terms are equal to zero: if $\alpha_n = -\frac{1}{2}g_{n+1}$ then $u_{C,j} = 0$ by (5.6a), and if $\alpha_n = 0$ then $v_{A,j} = 0$ by (5.4b). Here we ask: what are the solutions of JM $P_{II}^{(n)}$ corresponding to these two leading-order degeneracies.

Proposition 7.6. For $m \geq 1$ there exist special solutions of JM $P_{II}^{(2m+1)}$ given by

$$u = 0, \quad v = -2w, \quad \alpha_n = -\frac{1}{2}g_{n+1}, \quad (7.7)$$

where $w(x)$ solves $P_I^{(m)}$.

Proof. Let $n = 2m + 1$. Substitute $\alpha_n = -\frac{1}{2}g_{n+1}$ and $u = 0$ into (1.12) to give

$$\mathbf{a}_n + 2^{n-1}g_{n+1}(2x, 1)^T = 0, \quad (7.8)$$

where \mathbf{a}_n is defined recursively by

$$\mathbf{a}_{n+1} = r\mathbf{a}_n, \quad \mathbf{a}_0 = (0, v)^T, \quad (7.9)$$

and the operator r (1.14) takes the form

$$r = \begin{pmatrix} -\partial_x & 2 \\ 2v - \partial_x^{-1}v_x & \partial_x \end{pmatrix}. \quad (7.10)$$

Equations (7.9) and (7.10) imply the second-order recurrence relation

$$\mathbf{a}_{n+2} = r^2 \mathbf{a}_n, \quad (7.11a)$$

$$r^2 = \begin{pmatrix} \partial_x^2 + 2(2v - \partial_x^{-1}v_x) & 0 \\ r_{21} & \partial_x^2 + 2(2v - \partial_x^{-1}v_x) \end{pmatrix}, \quad (7.11b)$$

where r_{21} is given by

$$r_{21} = \partial_x (2v - \partial_x^{-1}v_x) - (2v - \partial_x^{-1}v_x) \partial_x.$$

Let us show by induction (on m) that the components of \mathbf{a}_n satisfy

$$\mathbf{a}_n = (\mathbf{a}_{n,1}, \mathbf{a}_{n,2})^T, \quad \mathbf{a}_{n,2} = \frac{1}{2} \partial_x \mathbf{a}_{n,1}. \quad (7.12)$$

The $m = 1$ case is trivial to check. Suppose (7.12) is valid for some $m \geq 1$ then the $(m + 1)$ result follows immediately from (7.11).

Given (7.12), the system (7.8) is equivalent to the system

$$\mathbf{a}_{n,1} + 2^n g_{n+1} x = 0, \quad (7.13a)$$

where $\mathbf{a}_{n,1}$ is defined recursively by

$$\mathbf{a}_{n+2,1} = r_{1,1} \mathbf{a}_{n,1}, \quad \mathbf{a}_{3,1} = 2v_{xx} + 6v^2, \quad (7.13b)$$

and $r_{1,1}$ is given by

$$r_{1,1} = \partial_x^2 + 2(2v - \partial_x^{-1}v_x). \quad (7.13c)$$

Set $v = -2w$ and $\mathbf{a}_{n,1} = \frac{1}{4} \mathbf{d}_m[w]$ where $n = 2m + 1$. Then (7.13) is equivalent to the definition of $P_I^{(m)}$ given by (7.1) and (7.2). \square

Proposition 7.14. For $n \geq 1$ special solutions of JM $P_{II}^{(n)}$ are given by

$$u = -\varphi, \quad v = 0, \quad \alpha_n = 0, \quad (7.15)$$

where $\varphi(x) = \phi_x/\phi$ and $\phi(x)$ solves $A^{(n)}$.

Proof. Substitute (7.15) into (1.12) to give

$$\mathbf{a}_n + 2^n g_{n+1}(x, 0)^T = 0, \quad (7.16)$$

where \mathbf{a}_n is defined recursively by

$$\mathbf{a}_{n+1} = r\mathbf{a}_n, \quad \mathbf{a}_0 = (-\varphi, 0)^T, \quad (7.17)$$

and the operator r (1.14) takes the form

$$r = \begin{pmatrix} \partial_x - \varphi & 2 \\ 0 & -\varphi + \partial_x^{-1}\varphi_x \end{pmatrix}. \quad (7.18)$$

Let us show by induction that the components of \mathbf{a}_n satisfy

$$\mathbf{a}_n = (\mathbf{a}_{n,1}, \mathbf{a}_{n,2})^T, \quad \mathbf{a}_{n,1} = (-1)^{n+1} \frac{\phi_{n+1}}{\phi}, \quad \mathbf{a}_{n,2} = 0. \quad (7.19)$$

The $n = 1$ case is trivial to check. Assume (7.19) holds for some $n \geq 1$ then by (7.17) and (7.18) the claim is immediately true for $(n + 1)$. Given (7.19) the system (7.16) is equivalent to (7.5). \square

A related solution is easily derived by substituting $\alpha_n = -g_{n+1}$ and $v = u_x$ into (1.12). For $n = 1$ we find

$$\text{JM P}_{\text{II}}^{(1)} : \quad \begin{aligned} u_x + u^2 + 2xg_2 &= 0, \\ \partial_x(u_x + u^2 + 2xg_2) &= 0, \end{aligned}$$

and for $n = 2$ we have

$$\text{JM P}_{\text{II}}^{(2)} : \quad \begin{aligned} u_{xx} + 3uu_x + u^3 + 4xg_3 &= 0, \\ \partial_x(u_{xx} + 3uu_x + u^3 + 4xg_3) &= 0. \end{aligned}$$

Via the same procedure as in the proof above, these equations can be linearised and the solution generalised for the entire hierarchy of equations. This gives another special solution

$$u = \Phi, \quad v = u_x, \quad \alpha_n = -g_{n+1}, \quad (7.20)$$

where $\Phi(x) = \Phi_x/\Phi$ and $\Phi(x)$ solves an alternative hyper-Airy equation

$$\Phi_{n+1} + 2^n g_{n+1} x \Phi = 0, \quad \Phi_{n+1} = \partial_x^{n+1} \Phi.$$

7.4 DISCUSSION

The special solutions of JM $P_{\text{II}}^{(n)}$ presented in this chapter complement the well known special solutions of P_{II} . Possible future work could involve generating a sequence of solutions for different values of the parameter α_n via Bäcklund transformations applied to these seed solutions. This work would be analogous to existing studies of the classical Painlevé equations. Bäcklund transformations for the fourth-order equation JM $P_{\text{II}}^{(2)}$ are given in [58], where the solutions (7.15) and (7.20) (for $n = 2$) are constructed by considering the conditions under which the given transformations break down.

The solutions (7.7), which are expressed in terms of equations in the first Painlevé hierarchy, are particularly interesting. Potentially these special solutions are more appropriately considered as *coalescence limits* of higher-order Painlevé equations, as in [56]. In this case, the reduction of the linear problem, from JM $P_{\text{II}}^{(2m+1)}$ to $P_{\text{I}}^{(m)}$, may fit into a scheme analogous to the one presented in [52] for the classical Painlevé equations.

This concludes our study of JM $P_{\text{II}}^{(n)}$. In the last part of this thesis we derive exact solutions of a particular class of reaction-diffusion equations which are given in terms of classical Painlevé transcendents.

Part III

APPLICATION

NEW SOLUTIONS OF VARIABLE COEFFICIENT REACTION-DIFFUSION EQUATIONS

8.1 ABSTRACT

In this chapter we return to the classical Painlevé equations and show how they arise in the study of reaction-diffusion equations. We consider reaction-diffusion equations that include a spatio-temporal dependence in the source terms and show that solutions are given in terms of the classical Painlevé transcendents. We consider reaction-diffusion equations with cubic and quadratic source terms. A feature of our analysis is that the coefficient functions are also solutions of differential equations, including the Painlevé equations. Special cases arise with elliptic functions as solutions. Additional solutions given in terms of equations that are not integrable are also considered. Solutions are constructed using a Lie symmetry approach.

8.2 INTRODUCTION

In this chapter we study solutions of the following two reaction-diffusion equations:

$$u_t = u_{xx} + q_0(x, t)u - u^2, \quad (8.1)$$

$$u_t = u_{xx} + q_1(x, t)u + q_2(x, t)u^2 - u^3. \quad (8.2)$$

The coefficient functions $q_i(x, t)$, $i = 0, 1, 2$, are not specified explicitly at the outset, but are chosen to satisfy particular equations which enable exact solution of equations (8.1) and (8.2). The motivating equations for this work are the standard reaction-diffusion equations with quadratic and cubic source terms. These equations are ubiquitous in biological and physical systems (see [10, 49, 53], for example) and take the following form: Fisher's equation

$$u_t = u_{xx} + u(1 - u),$$

and the Nagumo and Huxley-type equations respectively,

$$u_t = u_{xx} + u(1-u)(u-a), \quad u_t = u_{xx} + u^2(1-u),$$

where a is constant. By generalising the Fisher and Nagumo equations to include variable coefficients, we arrive at equations (8.1) and (8.2). Equations of Huxley-type with spatial heterogeneity included in the source terms are considered in [9], and will not be treated in this study.

Our main observation concerning equation (8.1) is the following: letting

$$u(x, t) = \eta(x, t) + \frac{1}{2}q(x, t), \quad (8.3)$$

in equation (8.1) with $q_0 \equiv q$ yields

$$\eta_t + \frac{1}{2}q_t = (\eta_{xx} - \eta^2) + \frac{1}{2}(q_{xx} + \frac{1}{2}q^2), \quad (8.4)$$

where we note that the η and q terms are additively separated. Thus, if we require that q satisfies the equation

$$q_t = q_{xx} + \frac{1}{2}q^2 + k(x, t), \quad (8.5)$$

for an arbitrary function k , then (8.4) implies that η must satisfy

$$\eta_t = \eta_{xx} - \eta^2 - \frac{1}{2}k(x, t). \quad (8.6)$$

Equations (8.5) and (8.6) can be solved independently, and then the solutions to (8.1) can be reconstructed via (8.3).

In §8.3.1 we carry out symmetry analysis of equation (8.1) to justify this observation. We find that equation (8.1) has no non-trivial classical symmetries unless q satisfies equation (8.5). Furthermore, the form of the transformation (8.3) is explicitly identified in the symmetry analysis. We thus find that symmetry analysis of equation (8.1) leads directly to symmetry analysis of equations (8.5) and (8.6). These equations are reduced to ordinary differential equations by exploiting their invariance under found symmetries. In §8.3.2 an analogous approach is followed for equation (8.2).

We find that the reduced ordinary differential equations corresponding to (8.1) and (8.2) are explicitly linked with equations of the respective forms

$$y_{\lambda\lambda} = 6y^2 + f(\lambda), \quad (8.7)$$

$$y_{\lambda\lambda} = 2y^3 + g(\lambda)y + h(\lambda), \quad (8.8)$$

where $y = y(\lambda)$. It is well known that these equations are integrable, provided the functions f and g are linear, and h is constant. In §8.4.1 and §8.4.2, we restrict our attention to these cases, and express exact solutions to the reaction-diffusion equations (8.1) and (8.2) in terms of solutions of associated integrable ordinary differential equations of the form (8.7) and (8.8).

The first and second Painlevé equations are examples of integrable equations of the form (8.7) and (8.8), respectively. For $y = y(\lambda)$ and α constant the equations are written as

$$P_I : \quad y_{\lambda\lambda} = 6y^2 + \lambda, \quad (8.9)$$

$$P_{II} : \quad y_{\lambda\lambda} = 2y^3 + \lambda y + \alpha. \quad (8.10)$$

We provide solutions to equations (8.1) and (8.2) in terms of these classical Painlevé transcendents. The recognition of the connection between the Painlevé equations (8.9) and (8.10) and variable-coefficient reaction-diffusion equations of the form (8.1) and (8.2) is believed to be new.

Solutions to equations (8.1) and (8.2) are also given in terms of Weierstrass and Jacobi elliptic functions. These functions solve the autonomous forms of equations (8.7) and (8.8) respectively. In [2], travelling wave solutions to Fisher's equation are given in terms of Weierstrass elliptic functions. These solutions are obtained by demanding that the reduced ordinary differential equation passes the Painlevé test (see [39]). This restricts the wave speed to a particular value. The solutions that we construct for the generalised Fisher type equation (8.1) include this as a special case. Other investigations of reaction-diffusion equations which have found Jacobi elliptic function solutions are: [9], where a spatial dependence is included in a Huxley-type equation, and [11], where reaction-diffusion equations with non-linear diffusion are considered. In both cases the solutions are constructed using non-classical symmetry methods. The simpler classical symmetry approach followed in this study produces not only elliptic function solutions, but also Painlevé transcendent solutions. The coefficient functions $q_i(x, t)$ afford a degree of freedom to equations (8.1) and (8.2) which makes them amenable to study.

8.3 SYMMETRY ANALYSIS

In this section we use symmetry analysis to systematically construct solutions to the partial differential equations (8.1) and (8.2). We follow the classical approach, which determines the forms of $X(x, t, u)$, $T(x, t, u)$,

and $U(x, t, u)$ for which the equation of interest is invariant under the point transformations

$$\begin{aligned}x_1 &= x + \epsilon X(x, t, u) + \mathcal{O}(\epsilon^2), \\t_1 &= t + \epsilon T(x, t, u) + \mathcal{O}(\epsilon^2), \\u_1 &= u + \epsilon U(x, t, u) + \mathcal{O}(\epsilon^2),\end{aligned}$$

for $\epsilon \ll 1$. These point transformations form a group with generator

$$\Gamma = X(x, t, u)\partial_x + T(x, t, u)\partial_t + U(x, t, u)\partial_u, \quad (8.11)$$

and the invariants of the group, denoted F , are solutions of the equation $\Gamma F = 0$. These invariants are used to construct similarity solutions of equations (8.1) and (8.2) in terms of solutions to ordinary differential equations (see [6]).

8.3.1 Quadratic case

Let $q_0 \equiv q$. Equation (8.1) has no non-trivial classical symmetries if q is free. Fix q as a solution to equation (8.5) where k satisfies

$$(c_1 + \frac{1}{2}c_3x)k_x + (c_2 + c_3t)k_t + 2c_3k = 0, \quad (8.12)$$

and c_1, c_2, c_3 are arbitrary parameters. Equation (8.1) is invariant under a three parameter Lie group of point symmetries with group operator (8.11) where

$$\begin{aligned}X(x) &= c_1 + \frac{1}{2}c_3x, & T(t) &= c_2 + c_3t, \\U(x, t, u) &= \frac{1}{2} \left\{ (c_1 + \frac{1}{2}c_3x)q_x + (c_2 + c_3t)q_t \right\} - c_3 \left(u - \frac{1}{2}q \right).\end{aligned} \quad (8.13)$$

If k satisfies (8.12) with $c_i \mapsto c_{i+3}$

$$(c_4 + \frac{1}{2}c_6x)k_x + (c_5 + c_6t)k_t + 2c_6k = 0, \quad (8.14)$$

then equation (8.5) is also invariant under a three parameter Lie group of point symmetries. In this case $u \mapsto q$ and $U \mapsto Q$ in equation (8.11), and

$$X(x) = c_4 + \frac{1}{2}c_6x, \quad T(t) = c_5 + c_6t, \quad Q(q) = -c_6q. \quad (8.15)$$

X and T are identical in (8.13) and (8.15), thus each invariant of the independent variables (x, t) holds for both equations (8.1) and (8.5).

Table 2: Reductions of the independent variables

Case	Reduction	c_1, c_4	c_2, c_5	c_3, c_6	Invariant
(1)	Stationary	0	1	0	x
(2)	Travelling Wave	c	1	0	$z = x - ct$
(3)	Scaling Reduction	0	0	1	$\zeta = x/t^{1/2}$

These invariants are constructed from linear combinations of the group operators Γ_i associated with each constant c_i in (8.13) or (8.15). We explore three standard combinations, as detailed in Table 2, where c is an arbitrary constant.

By (8.13), the solution to equation (8.1) is given by (8.3) where

$$\eta(x, t) = \frac{\tilde{\eta}(*)}{c_2 + c_3 t}, \tag{8.16}$$

and $\tilde{\eta}(*)$ is a function of the respective invariant for each case given in Table 2. Similarly by (8.15), the solution to equation (8.5) is

$$q(x, t) = \frac{\tilde{q}(*)}{c_5 + c_6 t}, \tag{8.17}$$

where $\tilde{q}(*)$ is a function of each respective invariant. Let Case (ij) denote Case (i) applied to (8.16), and Case (j) applied to (8.17), where $i, j \in \{1, 2, 3\}$. For these cases, Table 3 provides the explicit form of (8.16) and (8.17), the equations satisfied by the reduced functions $\tilde{\eta}(*)$ and $\tilde{q}(*)$, and the corresponding reductions of k , the solution of (8.12) or (8.14). A_i and B_i denote equations which are given in Table 4.

Table 3: Reductions of $\eta(x, t)$, $q(x, t)$, and $k(x, t)$

Case	$\eta(x, t)$	$\tilde{\eta}(*)$	$k(x, t)$	Case	$q(x, t)$	$\tilde{q}(*)$	$k(x, t)$
(1j):	$\tilde{\eta}(x)$	A_1	$\tilde{k}_j(x)$	(i1):	$\tilde{q}(x)$	B_1	$\tilde{k}_i(x)$
(2j):	$\tilde{\eta}(z)$	A_2	$\tilde{k}_j(z)$	(i2):	$\tilde{q}(z)$	B_2	$\tilde{k}_i(z)$
(3j):	$\frac{1}{t}\tilde{\eta}(\zeta)$	A_3	$\frac{1}{t^2}\tilde{k}_j(\zeta)$	(i3):	$\frac{1}{t}\tilde{q}(\zeta)$	B_3	$\frac{1}{t^2}\tilde{k}_i(\zeta)$

Remark 8.18. Note that each equation A_i in Table 4 is a reduction of (8.6), and each B_i is a reduction of (8.5).

The reduced form of the solution (8.3) is constructed from the individual reduced forms for η and q given in Table 3. In each case, the corresponding form of k must be consistent. Table 5 details any restric-

Table 4: $\tilde{\eta}(\ast)$ and $\tilde{q}(\ast)$ equations

A ₁ :	$\tilde{\eta}_{xx} - \tilde{\eta}^2 = \frac{1}{2}\tilde{k}_j(x)$
A ₂ :	$\tilde{\eta}_{zz} + c\tilde{\eta}_z - \tilde{\eta}^2 = \frac{1}{2}\tilde{k}_j(z)$
A ₃ :	$\tilde{\eta}_{\zeta\zeta} + \frac{1}{2}\zeta\tilde{\eta}_\zeta - \tilde{\eta}^2 + \tilde{\eta} = \frac{1}{2}\tilde{k}_j(\zeta)$
B ₁ :	$\tilde{q}_{xx} + \frac{1}{2}\tilde{q}^2 + \tilde{k}_i(x) = 0$
B ₂ :	$\tilde{q}_{zz} + c\tilde{q}_z + \frac{1}{2}\tilde{q}^2 + \tilde{k}_i(z) = 0$
B ₃ :	$\tilde{q}_{\zeta\zeta} + \frac{1}{2}\zeta\tilde{q}_\zeta + \frac{1}{2}\tilde{q}^2 + \tilde{q} + \tilde{k}_i(\zeta) = 0$

tions that are required for consistency. In this table, a , b , d , and e are arbitrary constants.

Table 5: Restrictions on $\tilde{k}(\ast)$

Case	$\tilde{k}_j(\ast)$	$\tilde{k}_i(\ast)$	Case	$\tilde{k}_j(\ast)$	$\tilde{k}_i(\ast)$	Case	$\tilde{k}_j(\ast)$	$\tilde{k}_i(\ast)$
(11):	free	free	(21):	a	a	(31):	$\frac{b}{\zeta^4}$	$\frac{b}{x^4}$
(12):	d	d	(22):	free	free	(32):	0	0
(13):	$\frac{e}{x^4}$	$\frac{e}{\zeta^4}$	(23):	0	0	(33):	free	free

Remark 8.19. In cases (23) and (32) we have set $\tilde{k}_2 = \tilde{k}_3 = 0$. Other functional forms satisfying $\tilde{k}_i(z) = \tilde{k}_j(\zeta)/t^2$ do not give rise to exact solutions in §8.4.1.

Example 8.20. Case (3j): When $c_1 = c_2 = 0$ and $c_3 = 1$, solving (8.12) gives

$$k(x, t) = \tilde{k}_j(\zeta)/t^2, \quad \zeta = x/t^{1/2}, \quad (8.21)$$

with $\tilde{k}_j(\zeta)$ an arbitrary function. Hence we obtain the (partial) scaling reduction

$$u(x, t) = \frac{1}{t}\tilde{\eta}(\zeta) + \frac{1}{2}q(x, t), \quad \zeta = x/t^{1/2}, \quad (8.22)$$

where $\eta(\zeta)$ satisfies equation A₃, and $q(x, t)$ satisfies equation (8.5).

Example 8.23. Case (i1): When $c_4 = c_6 = 0$, and $c_5 = 1$, solving (8.14) gives

$$k(x, t) = \tilde{k}_i(x), \quad (8.24)$$

with $\tilde{k}_i(x)$ an arbitrary function. Hence we obtain the stationary reduction

$$q(x, t) = \tilde{q}(x), \quad (8.25)$$

where $q(x)$ satisfies equation B₁.

Example 8.26. Case (31): Equations (8.22) and (8.25) imply that

$$u(x, t) = \frac{1}{t}\tilde{\eta}(\zeta) + \frac{1}{2}\tilde{q}(x).$$

Equations (8.21) and (8.24) imply that

$$\tilde{k}_1(\zeta) = b/\zeta^4, \quad \tilde{k}_3(x) = b/x^4,$$

where b is an arbitrary constant.

8.3.2 Cubic case

Let $q_2 \equiv q$. Equation (8.2) has no non-trivial classical symmetries if q and q_1 are free. Fix q as a solution to the following equation:

$$q_t = q_{xx} - \frac{1}{9}q^3 + l(x, t)q + k(x, t), \quad (8.27)$$

and fix q_1 as a function of q as follows:

$$q_1(x, t) = l(x, t) - \frac{1}{3}q(x, t)^2,$$

where k and l satisfy

$$\begin{aligned} (c_1 + \frac{1}{2}c_3x)k_x + (c_2 + c_3t)k_t + \frac{3}{2}c_3k &= 0, \\ (c_1 + \frac{1}{2}c_3x)l_x + (c_2 + c_3t)l_t + c_3l &= 0, \end{aligned} \quad (8.28)$$

and c_1, c_2, c_3 are arbitrary parameters. Equation (8.2) is invariant under the three parameter Lie group of point symmetries with group operator (8.11) where

$$\begin{aligned} X(x) &= c_1 + \frac{1}{2}c_3x, & T(t) &= c_2 + c_3t, \\ U(x, t, u) &= \frac{1}{3} \left\{ (c_1 + \frac{1}{2}c_3x)q_x + (c_2 + c_3t) \right\} - \frac{1}{2}c_3 \left(u - \frac{1}{3}q \right). \end{aligned} \quad (8.29)$$

If k and l satisfy equations (8.28) with $c_i \mapsto c_{i+3}$

$$\begin{aligned} (c_4 + \frac{1}{2}c_6x)k_x + (c_5 + c_6t)k_t + \frac{3}{2}c_6k &= 0, \\ (c_4 + \frac{1}{2}c_6x)l_x + (c_5 + c_6t)l_t + c_6l &= 0, \end{aligned}$$

Table 6: Reductions of $\eta(x, t)$, $q(x, t)$, $l(x, t)$, and $k(x, t)$

Case	$\eta(x, t)$	$\tilde{\eta}(\ast)$	$l(x, t)$	$k(x, t)$
(1j):	$\tilde{\eta}(x)$	C_1	$\tilde{l}_j(x)$	$\tilde{k}_j(x)$
(2j):	$\tilde{\eta}(z)$	C_2	$\tilde{l}_j(z)$	$\tilde{k}_j(z)$
(3j):	$\frac{1}{t^{1/2}}\tilde{\eta}(\zeta)$	C_3	$\frac{1}{t}\tilde{l}_j(\zeta)$	$\frac{1}{t^{3/2}}\tilde{k}_j(\zeta)$
(i1):	$\tilde{q}(x)$	D_1	$\tilde{l}_i(x)$	$\tilde{k}_i(x)$
(i2):	$\tilde{q}(z)$	D_2	$\tilde{l}_i(z)$	$\tilde{k}_i(z)$
(i3):	$\frac{1}{t^{1/2}}\tilde{q}(\zeta)$	D_3	$\frac{1}{t}\tilde{l}_i(\zeta)$	$\frac{1}{t^{3/2}}\tilde{k}_i(\zeta)$

Table 7: $\tilde{\eta}(\ast)$ and $\tilde{q}(\ast)$ equations

C_1 :	$\tilde{\eta}_{xx} - \tilde{\eta}^3 + \tilde{l}_j(x)\tilde{\eta} = \frac{1}{3}\tilde{k}_j(x)$
C_2 :	$\tilde{\eta}_{zz} + c\tilde{\eta}_z - \tilde{\eta}^3 + \tilde{l}_j(z)\tilde{\eta} = \frac{1}{3}\tilde{k}_j(z)$
C_3 :	$\tilde{\eta}_{\zeta\zeta} + \frac{1}{2}\zeta\tilde{\eta}_{\zeta} - \tilde{\eta}^3 + \frac{1}{2}(2\tilde{l}_j(\zeta) + 1)\tilde{\eta} = \frac{1}{3}\tilde{k}_j(\zeta)$
D_1 :	$\tilde{q}_{xx} - \frac{1}{9}\tilde{q}^3 + \tilde{l}_i(x)\tilde{q} + \tilde{k}_i(x) = 0$
D_2 :	$\tilde{q}_{zz} + c\tilde{q}_z - \frac{1}{9}\tilde{q}^3 + \tilde{l}_i(z)\tilde{q} + \tilde{k}_i(z) = 0$
D_3 :	$\tilde{q}_{\zeta\zeta} + \frac{1}{2}\zeta\tilde{q}_{\zeta} - \frac{1}{9}\tilde{q}^3 + \frac{1}{2}(2\tilde{l}_i(\zeta) + 1)\tilde{q} + \tilde{k}_i(\zeta) = 0$

then equation (8.27) is also invariant under a three parameter Lie group of point symmetries. In this case $u \mapsto q$ and $U \mapsto Q$ in equation (8.11), and

$$X(x) = c_4 + \frac{1}{2}c_6x, \quad T(t) = c_5 + c_6t, \quad Q(q) = -\frac{1}{2}c_6q. \quad (8.30)$$

By (8.29), the solution to equation (8.2) is given by

$$u(x, t) = \eta(x, t) + \frac{1}{3}q(x, t), \quad \eta(x, t) = \frac{\tilde{\eta}(\ast)}{(c_2 + c_3t)^{1/2}},$$

where $\tilde{\eta}(\ast)$ is a function of the respective invariant for each case given in Table 2. Similarly by (8.30), the solution to equation (8.27) is

$$q(x, t) = \frac{\tilde{q}(\ast)}{(c_5 + c_6t)^{1/2}},$$

where $\tilde{q}(\ast)$ is a function of each respective invariant. From here we proceed as in the quadratic case. Our results are summarised in Tables 6 - 8, where a, b, d, e and $\hat{a}, \hat{b}, \hat{d}, \hat{e}$ are arbitrary constants.

Table 8: Restrictions on $\tilde{k}(\ast)$ and $\tilde{l}(\ast)$

Case	$\tilde{k}_j(\ast)$	$\tilde{k}_i(\ast)$	Case	$\tilde{k}_j(\ast)$	$\tilde{k}_i(\ast)$	Case	$\tilde{k}_j(\ast)$	$\tilde{k}_i(\ast)$
(11):	free	free	(21):	a	a	(31):	$\frac{b}{\zeta^3}$	$\frac{b}{x^3}$
(12):	d	d	(22):	free	free	(32):	0	0
(13):	$\frac{e}{x^3}$	$\frac{e}{\zeta^3}$	(23):	0	0	(33):	free	free

Case	$\tilde{l}_j(\ast)$	$\tilde{l}_i(\ast)$	Case	$\tilde{l}_j(\ast)$	$\tilde{l}_i(\ast)$	Case	$\tilde{l}_j(\ast)$	$\tilde{l}_i(\ast)$
(11):	free	free	(21):	\hat{a}	\hat{a}	(31):	$\frac{\hat{b}}{\zeta^2}$	$\frac{\hat{b}}{x^2}$
(12):	\hat{d}	\hat{d}	(22):	free	free	(32):	0	0
(13):	$\frac{\hat{e}}{x^2}$	$\frac{\hat{e}}{\zeta^2}$	(23):	0	0	(33):	free	free

8.4 EXACT SOLUTIONS

8.4.1 Quadratic case

Table 3 summarises all possible solutions by reduction of (8.1) to the equations in Table 4. We now investigate when these equations can be solved in terms of a first Painlevé transcendent or a Weierstrass elliptic function.

Transformations

Let \ast denote $x, z,$ or ζ , let $\tilde{k}_i, \tilde{k}_j \equiv \tilde{k}$, and set

$$\tilde{\eta}(\ast) = 6 \left(\frac{d}{d\ast}\gamma(\ast)\right)^2 y(\lambda) - \delta(\ast), \tag{8.31a}$$

$$\tilde{q}(\ast) = -2 \left\{ 6 \left(\frac{d}{d\ast}\gamma(\ast)\right)^2 \hat{y}(\lambda) - \delta(\ast) \right\}, \tag{8.31b}$$

$$\tilde{k}(\ast) = 12 \left(\frac{d}{d\ast}\gamma(\ast)\right)^4 f(\lambda) - \beta(\ast), \tag{8.31c}$$

where $\lambda = \gamma(\ast)$ and $\delta(\ast), \beta(\ast),$ and $\gamma(\ast)$ are given in Table 9 where

$$p_2(\zeta) = 3\zeta^2 - 40, \quad p_4(\zeta) = \zeta^4 + 40\zeta^2 - 200.$$

Then, each equation in Table 4 transforms to (8.7), or (8.7) with $y \mapsto \hat{y}$.

Solutions

Recall the restrictions placed on \tilde{k} in Table 5, and the transformation between \tilde{k} and f (8.31c). In order to solve equation (8.7) we additionally require that f be linear. This is possible in five cases, and exact solutions for these cases are given in Table 10. In Table 10, P_I denotes a solution of

Table 9: Terms in the transformations (8.31)

Equations	*	$\delta(*)$	$\beta(*)$	$\gamma(*)$
A ₁ B ₁	x	0	0	x
A ₂ B ₂	z	$3c^2/25$	$18c^4/625$	$\exp(-cz/5)$
A ₃ B ₃	ζ	$p_2(\zeta)/100$	$9p_4(\zeta)/5000$	$\sqrt{5\pi}\operatorname{erf}(\zeta/(2\sqrt{5}))$

Table 10: Exact solutions of transformed equations

Case	Solution $y(\lambda)$	Solution $\hat{y}(\lambda)$	λ
(11)	P _I or $\wp(\lambda; g_2, g_3)$	P _I or $\wp(\lambda; g_2, \hat{g}_3)$	x
(12)	$\wp(x; b, g_3)$	$\wp(\lambda; 0, \hat{g}_3)$	$\gamma(z)$
(21)	$\wp(\lambda; 0, g_3)$	$\wp(x; b, \hat{g}_3)$	$\gamma(z)$
(22)	P _I or $\wp(\lambda; g_2, g_3)$	P _I or $\wp(\lambda; g_2, \hat{g}_3)$	$\gamma(z)$
(33)	P _I or $\wp(\lambda; g_2, g_3)$	P _I or $\wp(\lambda; g_2, \hat{g}_3)$	$\gamma(\zeta)$

the first Painlevé equation (8.9), and $\wp(\lambda; g_2, g_3)$ denotes the Weierstrass elliptic function which satisfies

$$\wp_\lambda^2 = 4\wp^3 - g_2\wp - g_3.$$

In Table 10, g_2, g_3 , and \hat{g}_3 are free constants, and $b = 3c^4/625$.

Remark 8.32. In each Case (ii), the solution to (8.1) can be written concisely as a difference of first Painlevé transcendents or elliptic functions, as follows:

$$(11) \quad u(x, t) = 6 \{y(x) - \hat{y}(x)\},$$

$$(22) \quad u(x, t) = 6 \left(\frac{d}{dz}\gamma(z)\right)^2 \{y(\lambda) - \hat{y}(\lambda)\}, \quad \lambda = \gamma(z),$$

$$(33) \quad u(x, t) = \frac{6}{t} \left(\frac{d}{d\zeta}\gamma(\zeta)\right)^2 \{y(\lambda) - \hat{y}(\lambda)\}, \quad \lambda = \gamma(\zeta),$$

where $\gamma(*)$ is given in Table 9, and z, ζ are given in Table 2. These solutions are constructed from the reductions given in Table 3 and the transformations (8.31). The corresponding form of $q(x, t)$ is found in the same way.

Table 11: Terms in the transformations (8.33) and (8.34)

Equations	*	$\beta(*)$	$\gamma(*)$
C ₁ D ₁	x	0	x
C ₂ D ₂	z	$2c^2/9$	$-3 \exp(-cz/3)/c$
C ₃ D ₃	ζ	$(\zeta^2 - 6)/18$	$\sqrt{3\pi} \operatorname{erf}(\zeta/(2\sqrt{3}))$

8.4.2 Cubic case

Table 6 summarises all possible solutions by reduction of (8.2) to the equations in Table 7. We now investigate when these equations can be solved in terms of a second Painlevé transcendent or a Jacobi elliptic function.

Transformations

Let $*$ denote x , z , or ζ , let $\tilde{k}_i, \tilde{k}_j \equiv \tilde{k}$, $\tilde{l}_i, \tilde{l}_j \equiv \tilde{l}$, and set

$$\tilde{\eta}(*) = \sqrt{2} \left(\frac{d}{d*} \gamma(*) \right) y(\lambda), \quad (8.33a)$$

$$\tilde{q}(*) = 3\sqrt{2} \left(\frac{d}{d*} \gamma(*) \right) \hat{y}(\lambda), \quad (8.33b)$$

$$\tilde{l}(*) = - \left(\frac{d}{d*} \gamma(*) \right)^2 g(\lambda) + \beta(*), \quad (8.34a)$$

$$\tilde{k}(*) = 3\sqrt{2} \left(\frac{d}{d*} \gamma(*) \right)^3 h_{\pm}(\lambda), \quad (8.34b)$$

where $\lambda = \gamma(*)$, and $\beta(*)$ are given in Table 11. Then each equation in Table 7 transforms to (8.8) with $h_{\pm} = h$, or (8.8) with $y \mapsto \hat{y}$ and $h_{\pm} = -h$.

Solutions

Recall the restrictions placed on \tilde{k} and \tilde{l} in Table 8, and the transformations between \tilde{l} and g , and \tilde{k} and h_{\pm} (8.34). In order to solve equation (8.8) we additionally require that g be linear and h_{\pm} constant. This is possible in five cases, and exact solutions for these cases are given in Table 12. In this table we denote the solution to P_{II} (8.10) as $y(\lambda) = P_{\text{II}}(\lambda; \alpha)$, and the solution to the elliptic equation (8.8) with $g(\lambda) = b$ and $h(\lambda) = d$ for arbitrary constants b, d , as $y(\lambda) = \text{JE}(\lambda; b, d)$. The exact expression of the solution in terms of the standard Jacobi elliptic functions depends on the value of the constants b and d (see [3]).

Table 12: Exact solutions of transformed equations

Case	Solution $y(\lambda)$	Solution $\hat{y}(\lambda)$	λ
(11)	$P_{II}(\lambda; \alpha)$ or $JE(\lambda; b, d)$	$P_{II}(\lambda; -\alpha)$ or $JE(\lambda; b, -d)$	x
(12)	$JE(x; -2c^2/9, 0)$	$JE(\lambda; 0, 0)$	$\gamma(z)$
(21)	$JE(\lambda; 0, 0)$	$JE(x; -2c^2/9, 0)$	$\gamma(z)$
(22)	$P_{II}(\lambda; \alpha)$ or $JE(\lambda; b, d)$	$P_{II}(\lambda; \alpha)$ or $JE(\lambda; b, -d)$	$\gamma(z)$
(33)	$P_{II}(\lambda; \alpha)$ or $JE(\lambda; b, d)$	$P_{II}(\lambda; \alpha)$ or $JE(\lambda; b, -d)$	$\gamma(\zeta)$

8.5 DISCUSSION

In this chapter we have found five exact solutions to each of the variable-coefficient reaction-diffusion equations (8.1) and (8.2). These were constructed by forcing the reduced ordinary differential equations associated with (8.1) and (8.2) to be integrable. The symmetry analysis also revealed solutions given in terms of non-integrable ordinary differential equations, cases (13), (23), (31) and (32), which remain to be explored. The exact solutions we have found are given in terms of Painlevé transcendents or elliptic functions. The initial conditions for the Painlevé transcendents, and the parameter for P_{II} , have not been specified here. Similarly, the invariants g_2 and g_3 in the Weierstrass elliptic function solutions, and the constants b and d describing the Jacobi elliptic function solutions, have been left arbitrary in many cases. Based on this work, the specific dynamics of these systems can be investigated for any choice of parameters that is of interest.

While precise applications of this work are yet to be explored, it is hoped that these results can be realised in terms that are biologically or physically significant. One area of particular relevance is ecology, where spatial and/or temporal environmental and biotic features of different species need to be incorporated into reaction-diffusion models [10].

An interesting question arises as to whether there is a connection between higher-order Painlevé equations and systems of reaction-diffusion equations. A starting point for extension of these results is the fourth-order second Painlevé equation $JM P_{II}^{(2)}$ (1.11), studied in Part i, which can be considered as a stationary reduction of the following system of

reaction-diffusion equations:

$$\begin{aligned}u_t + u_{xx} - 3uu_x + u^3 + 6uv + 4xg_3 &= 0, \\v_t + v_{xx} + 3uv_x + 3v^2 + 3u^2v &= 4\alpha_2.\end{aligned}$$

The pole free solutions derived in Chapter 2 could be of use in describing physically meaningful solutions of this system in particular domains.

Part IV

CONCLUSION

CONCLUDING REMARKS

In this thesis we have obtained a detailed description of the asymptotic behaviour of the Jimbo-Miwa second Painlevé hierarchy in the limit as the independent variable approaches infinity. The study was motivated by Boutroux's classical investigation of special tronquée solutions and general elliptic asymptotics admitted by the second Painlevé equation, and by recent interest in higher-order Painlevé equations and hierarchies.

Our most substantial results were derived for the fourth-order Jimbo-Miwa second Painlevé equation, examined in Part [i](#). In Chapter [2](#) we demonstrated the existence of new types of higher-tronquée solutions of this equation. These solutions were shown to be asymptotic to two types of algebraic formal power series, and we provided explicit details on the size and orientation of sectors in the complex plane where such asymptotic descriptions are valid. We found two-, one- and zero-parameter tronquée solutions.

In Chapter [3](#) we constructed the general, four-parameter, asymptotic behaviour of the fourth-order Jimbo-Miwa second Painlevé equation. We showed that the general asymptotic behaviour is described to leading-order by two related genus-2 hyperelliptic functions. Together, these higher-tronquée solutions and hyperelliptic function asymptotics show that the asymptotic behaviour of the classical second Painlevé equation admits a natural extension to higher-order equations in the Jimbo-Miwa hierarchy.

In addition to the leading-order study of the general asymptotic behaviour of the fourth-order equation, we investigated two energy-type parameters in the hyperelliptic functions. These parameters were shown to satisfy particular difference equations over a period of the leading-order hyperelliptic functions and, using an averaging method, they were also shown to be bounded. Such results provide a foundation for future extension of the locally-valid hyperelliptic asymptotics to a wider domain of validity. A related problem, also for future study, is to establish a connection between the cases of degeneracy of the hyperelliptic function behaviours and the special rays in the complex plane identified in the analysis of the tronquée solutions.

In Part [ii](#) we studied the asymptotic behaviour of the entire Jimbo-Miwa hierarchy. The structure of each in equation in the hierarchy was

expounded in Chapter 4, and special and general asymptotic behaviours of these equations were considered in Chapters 5 and 6, respectively. We derived three classes of special asymptotic behaviours and proved that there exist true solutions with these behaviours in particular sectors of the complex plane. For one class of algebraic behaviours, at each level n of the Jimbo-Miwa hierarchy we found n -, $(n - 1)$ -, \dots , one- and zero-parameter solutions. In addition to these special asymptotics, we constructed the general, $2n$ -parameter, leading-order behaviour of the equations in the hierarchy in terms of two related genus- n hyperelliptic functions. Concluding Part ii, in Chapter 7 we derived exact special solutions for the higher-order second Painlevé equations for particular values of the parameter.

In our investigation of the Jimbo-Miwa second Painlevé hierarchy we highlighted a number of connections with other higher-order Painlevé equations. Interestingly, we encountered different behaviours in the algebraic asymptotics and exact special solutions at even and odd levels of the hierarchy. For the odd members of the hierarchy, the exact solutions were expressed in terms of equations in the first Painlevé hierarchy, and for all members, solutions were given in terms of hyper-Airy functions. In particular, we found that the special solutions at odd levels $(2m + 1)$ of the Jimbo-Miwa second Painlevé hierarchy are given in terms of equations at each level m of the first Painlevé hierarchy. We also showed that there is a relationship between the types of algebraic asymptotic behaviours admitted at odd levels $(2m - 1)$ of the Jimbo-Miwa hierarchy, and at each level m of the Flaschka-Newell hierarchy, rather than a direct correspondence between the equations at the same level of the two hierarchies. For further study, the relationships between different higher-order equations should be analysed via a more rigorous approach through a transformation of variables or coalescence limits.

In Part iii we identified a connection between variable-coefficient reaction diffusion equations and classical Painlevé equations. In Chapter 8 we found five exact solutions to two classes of reaction-diffusion equations, with quadratic or cubic source terms, and these solutions were expressed by first or second Painlevé transcendents. Future work could include an investigation of a related connection between higher-order Painlevé equations and systems of variable coefficient reaction-diffusion equations.

BIBLIOGRAPHY

- [1] M. J. Ablowitz and H. Segur. Exact linearization of a Painlevé transcendent. *Phys. Rev. Lett.*, 38(20):1103–1106, 1977. (Cited on pages 3 and 9.)
- [2] M. J. Ablowitz and A. Zeppetella. Explicit solutions of Fisher’s equation for a special wave speed. *Bull. Math. Biol.*, 41(6):835–840, 1979. (Cited on page 99.)
- [3] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964. (Cited on pages 75, 79, and 107.)
- [4] H. F. Baker. *Abelian functions*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995. Abel’s theorem and the allied theory of theta functions, Reprint of the 1897 original. (Cited on page 33.)
- [5] C. M. Bender and S. A. Orszag. *Advanced mathematical methods for scientists and engineers. I*. Springer-Verlag, New York, 1999. Asymptotic methods and perturbation theory, Reprint of the 1978 original. (Cited on pages 8 and 92.)
- [6] G. W. Bluman and S. C. Anco. *Symmetry and integration methods for differential equations*, volume 154 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 2002. (Cited on page 100.)
- [7] P. Boutroux. Recherches sur les transcendentes de M. Painlevé et l’étude asymptotique des équations différentielles du second ordre. *Ann. Sci. École Norm. Sup. (3)*, 30:255–375, 1913. (Cited on page 1.)
- [8] P. Boutroux. Recherches sur les transcendentes de M. Painlevé et l’étude asymptotique des équations différentielles du second ordre (suite). *Ann. Sci. École Norm. Sup. (3)*, 31:99–159, 1914. (Cited on page 1.)
- [9] B. H. Bradshaw-Hajek, M. P. Edwards, P. Broadbridge, and G. H. Williams. Nonclassical symmetry solutions for reaction-diffusion

- equations with explicit spatial dependence. *Nonlinear Anal.*, 67(9): 2541–2552, 2007. (Cited on pages 98 and 99.)
- [10] R. S. Cantrell and C. Cosner. *Spatial ecology via reaction-diffusion equations*. Wiley Series in Mathematical and Computational Biology. John Wiley & Sons Ltd., Chichester, 2003. (Cited on pages 97 and 108.)
- [11] P. A. Clarkson and E. L. Mansfield. Symmetry reductions and exact solutions of a class of nonlinear heat equations. *Phys. D*, 70(3): 250–288, 1994. (Cited on page 99.)
- [12] P. A. Clarkson and E. L. Mansfield. The second Painlevé equation, its hierarchy and associated special polynomials. *Nonlinearity*, 16(3):R1–R26, 2003. (Cited on page 5.)
- [13] P. A. Clarkson, N. Joshi, and A. Pickering. Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach. *Inverse Problems*, 15(1):175–187, 1999. (Cited on page 5.)
- [14] P. A. Clarkson, N. Joshi, M. Mazzocco, F. W. Nijhoff, and M. Noumi. Preface: One hundred years of PVI, the Fuchs-Painlevé equation. *J. Phys. A*, 39(39):v–xii, 2006. With a reprint of Richard Fuchs’s paper “Sur quelques équations différentielles linéaires du second ordre” [C. R. Acad. Sci. Paris 141 (1905), 555–558]. (Cited on page 9.)
- [15] C. M. Cosgrove. Higher-order Painlevé equations in the polynomial class. I. Bureau symbol P2. *Stud. Appl. Math.*, 104(1):1–65, 2000. (Cited on pages 5 and 33.)
- [16] C. M. Cosgrove. Higher-order Painlevé equations in the polynomial class. II. Bureau symbol P1. *Stud. Appl. Math.*, 116(4):321–413, 2006. (Cited on page 5.)
- [17] M. V. Demina and N. A. Kudryashov. Special polynomials and rational solutions of the hierarchy of the second Painlevé equation. *Teoret. Mat. Fiz.*, 153(1):58–67, 2007. (Cited on page 5.)
- [18] J. Drach. Sur l’intégration par quadratures de l’équation $d^2y/dx^2 = [\phi(x) + h]y$. *Comptes Rendus de l’Académie des Sciences*, 168(7): 337–340, 1919. reprinted in D. V. Chudnovsky and G. V. Chudnovsky, eds., *Classical and Quantum Models and Arithmetic Problems*, Lecture Notes in Pure and Applied Mathematics 92:455–458, Marcel Dekker, New York, 1984. (Cited on pages 6, 7, 29, and 89.)

- [19] K. Driver and P. Duren. Zeros of the hypergeometric polynomials $F(-n, b; 2b; z)$. *Indag. Math. (N.S.)*, 11(1):43–51, 2000. (Cited on page 75.)
- [20] B. Dubrovin, T. Grava, and C. Klein. On universality of critical behaviour in the focusing nonlinear Schrödinger equation, elliptic umbilic catastrophe and the *tritonquée* solution to the Painlevé-I equation. *arXiv:0704.0501*, To appear in *Journal of Nonlinear Science*, 2007. (Cited on page 8.)
- [21] H. Flaschka and A. C. Newell. Monodromy- and spectrum-preserving deformations. I. *Comm. Math. Phys.*, 76(1):65–116, 1980. (Cited on page 4.)
- [22] A. S. Fokas, A. R. Its, A. A. Kapaev, and V. Y. Novokshenov. *Painlevé transcendents*, volume 128 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006. The Riemann-Hilbert approach. (Cited on page 2.)
- [23] P. R. Gordoa, N. Joshi, and A. Pickering. On a generalized $2 + 1$ dispersive water wave hierarchy. *Publ. Res. Inst. Math. Sci.*, 37(3):327–347, 2001. (Cited on pages 4 and 7.)
- [24] P. R. Gordoa, N. Joshi, and A. Pickering. Second and fourth Painlevé hierarchies and Jimbo-Miwa linear problems. *J. Math. Phys.*, 47(7):073504, 16, 2006. (Cited on page 4.)
- [25] S. P. Hastings and J. B. McLeod. A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation. *Arch. Rational Mech. Anal.*, 73(1):31–51, 1980. (Cited on page 3.)
- [26] E. L. Ince. *Ordinary Differential Equations*. Dover Publications, New York, 1944. (Cited on page 6.)
- [27] A. R. Its and A. A. Kapaev. Quasi-linear Stokes phenomenon for the second Painlevé transcendent. *Nonlinearity*, 16(1):363–386, 2003. (Cited on page 17.)
- [28] M. Jimbo and T. Miwa. Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II. *Phys. D*, 2(3):407–448, 1981. (Cited on page 4.)
- [29] N. Joshi. The second Painlevé hierarchy and the stationary KdV hierarchy. *Publ. Res. Inst. Math. Sci.*, 40(3):1039–1061, 2004. (Cited on pages 4, 6, and 29.)

- [30] N. Joshi. The second Painlevé equation in the large-parameter limit. I. Local asymptotic analysis. *Stud. Appl. Math.*, 102(4):345–373, 1999. (Cited on page 27.)
- [31] N. Joshi and M. D. Kruskal. The Painlevé connection problem: an asymptotic approach. I. *Stud. Appl. Math.*, 86(4):315–376, 1992. (Cited on pages 2, 13, and 46.)
- [32] N. Joshi and M. D. Kruskal. An asymptotic approach to the connection problem for the first and the second Painlevé equations. *Phys. Lett. A*, 130(3):129–137, 1988. (Cited on page 2.)
- [33] N. Joshi and M. Mazzocco. Existence and uniqueness of tri-tronquée solutions of the second Painlevé hierarchy. *Nonlinearity*, 16(2):427–439, 2003. (Cited on pages 8, 23, and 84.)
- [34] N. Joshi, A. V. Kitaev, and P. A. Treharne. On the linearization of the First and Second Painlevé equations. *arXiv:0806.0271v1*, 2008. (Cited on page 4.)
- [35] T. Kawai and Y. Takei. WKB analysis of higher order Painlevé equations with a large parameter—local reduction of 0-parameter solutions for Painlevé hierarchies (P_J) ($J = \text{I, II-1 or II-2}$). *Adv. Math.*, 203(2):636–672, 2006. (Cited on page 5.)
- [36] A. V. Kitaev. Caustics in 1 + 1 integrable systems. *J. Math. Phys.*, 35(6):2934–2954, 1994. (Cited on page 6.)
- [37] A. V. Kitaev. Elliptic asymptotics of the first and second Painlevé transcendents. *Uspekhi Mat. Nauk*, 49(1(295)):77–140, 1994. (Cited on page 2.)
- [38] M. Kruskal. Asymptotic theory of Hamiltonian and other systems with all solutions nearly periodic. *J. Mathematical Phys.*, 3:806–828, 1962. (Cited on page 42.)
- [39] M. D. Kruskal, N. Joshi, and R. Halburd. Analytic and asymptotic methods for nonlinear singularity analysis: a review and extensions of tests for the Painlevé property. In *Integrability of nonlinear systems (Pondicherry, 1996)*, volume 495 of *Lecture Notes in Phys.*, pages 171–205. Springer, Berlin, 1997. (Cited on page 99.)
- [40] N. A. Kudryashov. The first and second Painlevé equations of higher order and some relations between them. *Phys. Lett. A*, 224(6):353–360, 1997. (Cited on page 91.)

- [41] R. C. Littlewood. Hyperelliptic asymptotics of Painlevé-type equations. *Nonlinearity*, 12(6):1629–1641, 1999. (Cited on pages 6 and 33.)
- [42] J. J. Mahony. Validity of averaging methods for certain systems with periodic solutions. *Proc. Roy. Soc. London Ser. A*, 330:349–371, 1972. (Cited on page 42.)
- [43] A. I. Markushevich. *Introduction to the classical theory of abelian functions*, volume 96 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992. Translated from the 1979 Russian original by G. Blüher. (Cited on page 33.)
- [44] S. Matsutani. Hyperelliptic solutions of KdV and KP equations: re-evaluation of Baker’s study on hyperelliptic sigma functions. *J. Phys. A*, 34(22):4721–4732, 2001. (Cited on page 86.)
- [45] M. Mazzocco and M. Y. Mo. The Hamiltonian structure of the second Painlevé hierarchy. *Nonlinearity*, 20(12):2845–2882, 2007. (Cited on page 5.)
- [46] B. M. McCoy, C. A. Tracy, and T. T. Wu. Painlevé functions of the third kind. *J. Mathematical Phys.*, 18(5):1058–1092, 1977. (Cited on page 3.)
- [47] B. M. McCoy, C. A. Tracy, and T. T. Wu. Connection between the KdV equation and the two-dimensional Ising model. *Phys. Lett. A*, 61(5):283–284, 1977. (Cited on page 3.)
- [48] G. Moore. Geometry of the string equations. *Comm. Math. Phys.*, 133(2):261–304, 1990. (Cited on page 8.)
- [49] J. D. Murray. *Mathematical biology. I*, volume 17 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, third edition, 2002. (Cited on page 97.)
- [50] Y. Nishikawa. Toward the exact WKB analysis of the P_{II} hierarchy. *Stud. Appl. Math.*, 119(1):1–71, 2007. (Cited on page 5.)
- [51] M. Noumi. *Painlevé equations through symmetry*, volume 223 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2004. Translated from the 2000 Japanese original by the author. (Cited on page 8.)
- [52] Y. Ohyama and S. Okumura. A coalescent diagram of the Painlevé equations from the viewpoint of isomonodromic deformations. *J. Phys. A*, 39(39):12129–12151, 2006. (Cited on page 95.)

- [53] A. Okubo and S. A. Levin. *Diffusion and ecological problems: modern perspectives*, volume 14 of *Interdisciplinary Applied Mathematics*. Springer-Verlag, New York, second edition, 2001. (Cited on page 97.)
- [54] P. Painlevé. Mémoire sur les équations différentielles dont l'intégrale générale est uniforme. *Bull. Soc. Math. France*, 28:201–261, 1900. (Cited on page 1.)
- [55] P. Painlevé. Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme. *Acta Math.*, 25(1):1–85, 1902. (Cited on page 1.)
- [56] A. Pickering. Coalescence limits for higher order Painlevé equations. *Phys. Lett. A*, 301(3-4):275–280, 2002. (Cited on page 95.)
- [57] J. N. Ridley. An elementary proof on location of zeros. *Amer. Math. Monthly*, 107(4):357–360, 2000. (Cited on page 75.)
- [58] A. H. Sakka. Schlesinger transformations for the second members of PII and PIV hierarchies. *J. Phys. A*, 40(27):7687–7697, 2007. (Cited on page 95.)
- [59] H. Segur and A. Finkel. An analytical model of periodic waves in shallow water. *Stud. Appl. Math.*, 73(3):183–220, 1985. (Cited on page 49.)
- [60] S. Shimomura. Poles and α -points of meromorphic solutions of the first Painlevé hierarchy. *Publ. Res. Inst. Math. Sci.*, 40(2):471–485, 2004. (Cited on page 91.)
- [61] L. R. Turner. Inverse of the Vandermonde matrix with applications. *NASA Technical Note*, NASA-TN-D-3547, 1966. (Cited on page 86.)
- [62] W. Wasow. *Asymptotic expansions for ordinary differential equations*. Dover Publications Inc., New York, 1987. Reprint of the 1976 edition. (Cited on pages 14 and 17.)
- [63] G. B. Whitham. *Linear and nonlinear waves*. Wiley-Interscience [John Wiley & Sons], New York, 1974. Pure and Applied Mathematics. (Cited on page 42.)