

**DISCRETE LAX PAIRS, REDUCTIONS AND
HIERARCHIES**

by

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Abstract

The term ‘Lax pair’ refers to linear systems (of various types) that are related to nonlinear equations through a compatibility condition. If a nonlinear equation possesses a Lax pair, then the Lax pair may be used to gather information about the behaviour of the solutions to the nonlinear equation. Conserved quantities, asymptotics and even explicit solutions to the nonlinear equation, amongst other information, can be calculated using a Lax pair. Importantly, the existence of a Lax pair is a signature of integrability of the associated nonlinear equation.

While Lax pairs were originally devised in the context of continuous equations, Lax pairs for discrete integrable systems have risen to prominence over the last three decades or so and this thesis focuses entirely on discrete equations. Famous continuous systems such as the Korteweg de Vries equation and the Painlevé equations all have integrable discrete analogues, which retrieve the original systems in the continuous limit. Links between the different types of integrable systems are well known, such as reductions from partial difference equations to ordinary difference equations. Infinite hierarchies of integrable equations can be constructed where each equation is related to adjacent members of the hierarchy and the order of the equations can be increased arbitrarily.

After a literature review, the original material in this thesis is instigated by a completeness study that finds all possible Lax pairs of a certain type, including one for the lattice modified Korteweg de Vries equation. The lattice modified Korteweg de Vries equation is subsequently reduced to several q -discrete Painlevé equations, and the reductions are used to form Lax pairs for those equations. The series of reductions suggests the presence of a hierarchy of equations, where each equation is obtained by applying a recursion relation to an earlier member of the hierarchy, this is confirmed using expansions within the Lax pairs for the q -Painlevé equa-

tions. Lastly, some explorations are included into fake Lax pairs, as well as sets of equivalent nonlinear equations with similar Lax pairs.

To clarify the original contribution made by the author: the completeness study of chapter 3 is based on an arXiv publication [1] of which MH is the sole author. Chapter 4, on reductions, is based on a published collaboration [2] between MH, J. Hietarinta, N. Joshi and F. Nijhoff, to which MH is the primary contributor. The chapter on Hierarchies, chapter 5, is based on a publication [3] authored solely by MH. Chapter 6 includes unpublished new material by MH.

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Originality Declaration

The work in this thesis was performed between March 2004 and August 2008 in the School of Mathematics and Statistics at the University of Sydney. The work presented was conducted by myself unless stated otherwise. None of this material has been presented previously for the purpose of obtaining any other degree.

Michael C. Hay

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Chapter 1

Introduction

This thesis is devoted to the study of nonlinear integrable discrete equations through their associated linear problems called Lax pairs. This broad topic is investigated from three angles, the first aspect concerns partial difference equations (P Δ Es), their Lax pairs and how those Lax pairs are found. The second concerns reductions from P Δ Es to ordinary difference equations (O Δ Es), which are the discrete analogues of the Painlevé equations, and subsequently finding Lax pairs for those O Δ Es. The third aspect is the construction of hierarchies of O Δ Es starting from expansions within their Lax pairs. Finally, we note some results about Lax pairs that might lead to future investigations.

Around the end of the nineteenth century, Paul Painlevé (1863-1933) and his colleagues set about classifying a large class of second-order ordinary differential equations (ODEs). The equations studied were of the form $y''(t) = F(y', y, t)$ where F is rational in y' , algebraic in y and analytic in t , and they were classified on the basis of whether their solutions were single-valued around all movable singularities [4]. Those ODEs whose solutions were found to behave in this manner were said to possess the Painlevé property.

Fifty canonical classes of equations that possess the Painlevé property were found

by Painlevé [5], his student Gambier [6], and Fuchs [7]. Of the fifty, six equations were found to define new transcendental functions, while the remainder could be solved in terms of these six or other special functions. These six equations were thus called the Painlevé equations and are listed here:

$$\begin{aligned}
\text{P}_I: \quad & y''(t) = 6y^2 + t, \\
\text{P}_{II}: \quad & y''(t) = 2y^3 + ty + \alpha, \\
\text{P}_{III}: \quad & y''(t) = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \\
\text{P}_{IV}: \quad & y''(t) = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \\
\text{P}_V: \quad & y''(t) = \left(\frac{1}{2y} + \frac{1}{y-1} \right) (y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2} \left(\alpha + \frac{\beta}{y} \right) + \frac{\gamma y}{t} \\
& \quad + \frac{\delta y(y+1)}{y-1}, \\
\text{P}_{VI}: \quad & y''(t) = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) (y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{t-y} \right) y' \\
& \quad + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left(\alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2} \right),
\end{aligned}$$

where α , β , γ and δ are constant parameters and primes denote differentiation with respect to t . Ince [8] provides derivations of the list of fifty canonical classes that were identified by the Painlevé school. ODEs may be tested for the possession of the Painlevé property with the aid of the Ablowitz-Ramani-Segur algorithm. This provides a necessary, but not sufficient, test which may be regarded as a generalization of Frobenius analysis to nonlinear equations [9].

Integrable nonlinear equations arise as the compatibility conditions of associated pairs of linear differential and/or difference equations often called Lax pairs [10, 11]. Essential to Lax pairs associated with ODEs are the monodromy data, explained briefly in the next few sentences. From some fixed point P on the Riemann sphere, analytic continuation of the solution Φ around any pole of the system produces a new solution Φ' . These solutions are related by the monodromy matrix M , associated

with the pole in question, such that

$$\Phi' = \Phi M.$$

The monodromy matrices define the monodromy data for the system. The compatibility of the type of Lax pairs that give rise to the Painlevé equations ensures that the monodromy data remains invariant through the process of analytic continuation and, as such, the Painlevé equations are often said to be the isomonodromy conditions for the associated linear problems, which in turn are sometimes called isomonodromy Lax pairs.

The monodromy data can be used to characterize the behaviour of the solutions of the associated nonlinear system. It is possible to use the associated linear system to solve the Painlevé equation for a large class of initial data, this process is known as the isomonodromy deformation method of solution [12]. Evidence has been put forward that suggests a relationship between the existence of an isomonodromy Lax pair for an equation with said equation's possession of the Painlevé property [9, 13].

Also arising as the compatibility condition of Lax pairs, nonlinear integrable partial differential equations (PDEs) are intimately related to the Painlevé equations. The method used to solve PDEs using Lax pairs is rather distinct from that used to solve ODEs, integrable nonlinear PDEs arise as isospectral conditions rather than isomonodromy conditions. These may be solved, for a large class of initial data, by the inverse scattering technique [14]. Well known nonlinear PDEs that have been studied in this context include the Korteweg-de Vries (KdV) [15] and modified Korteweg-de Vries (mKdV) [16] equations,

$$\text{KdV:} \quad y_t + 6yy_t + y_{xxx} = 0, \tag{1.1}$$

$$\text{mKdV:} \quad y_t - 6y^2y_t + y_{xxx} = 0, \tag{1.2}$$

where the subscripts denote partial differentiation. The Painlevé test has been extended to PDEs as a test for their integrability [17]. Ablowitz, Ramani and Segur

observed that similarity reductions of integrable PDEs possess the Painlevé property, possibly after a change of variables [18, 19].

A natural extension of the work of the Painlevé school is finding higher order or higher degree equations that possess the Painlevé property. Toward this goal, Chazy [20], Garnier [21] and Bureau [22] began a classification of third-order ODEs of the form $y'''(t) = F(y'', y', y, t)$ where F is rational in its variables. The second-order and second-degree ODEs of the form $(y''(t))^2 = F(y', y, t)$ have all been classified by Cosgrove and Scoufis [23]. They found exactly six canonical classes of second-degree equations that possess the Painlevé property which were denoted SD_I, \dots, SD_{VI} .

The bustling activity surrounding integrable PDEs and Painlevé equations is motivated by their appearance in mathematical models in diverse areas of physics. The KdV and mKdV equations describe solitary wave behaviour which arises in atmospheric dynamics [24], ocean dynamics [25] and nonlinear optics [26]. The Painlevé equations are integrable reductions of these soliton equations and therefore also influence the understanding of solitary waves [27, 14]. The generic solutions of the Painlevé equations are higher transcendental functions that cannot be expressed in terms of the classical special functions [8]. The Painlevé equations also arise as models in areas such as quantum gravity [28] and general relativity [29, 30, 31].

However, this thesis is concerned with discrete equations, both PΔEs and OΔEs, for which interest has grown rapidly since the early 1990's, and where there continues to be a great deal of activity today. Grammaticos, Ramani *et al* [32, 33] proposed a discrete version of the Painlevé property, called the singularity confinement property (see section 2.1.2). The property has been used to derive discrete versions of the Painlevé equations, so called because they have the Painlevé equations as continuum limits. For example, a general form of the q -discrete third Painlevé equation is

$$qP_{III}: \quad \bar{x}x = \frac{(x - k_1\kappa^l)(x - k_2\kappa^l)}{(1 - k_3\kappa^l x)(1 - x/k_3)}, \quad (1.3)$$

where x depends on the discrete variable l and we have use the notation $x(l+1) = \bar{x}$,

$x(l-1) = \underline{x}$, and where κ and k_i ($i = 1, 2, 3$) are constants.

The singularity confinement property has also been used to find integrable, non-autonomous PΔEs. One equation thus found that is of particular interest to the present thesis is the lattice modified Korteweg-de Vries equation (LMKdV)

$$\text{LMKdV: } \hat{\hat{x}} = x \frac{\bar{x} - r\hat{x}}{\hat{x} - r\bar{x}} \quad (1.4)$$

where we use the notation $\hat{x} = x(l, m+1)$ and $\bar{x} = x(l+1, m)$. Equation (1.4) is non-autonomous because r depends on the lattice variables l and m . The test for singularity confinement shows that the equation possesses this property when r satisfies

$$\hat{\hat{r}}r = \bar{r}\hat{r}.$$

This is solved by $r = \lambda(l)\mu(m)$, for arbitrary functions $\lambda(l)$ and $\mu(m)$.

The names given to equations (1.3) and (1.4) arise from their continuum limits, however, continuum limits are not unique. For example, there are at least two well known integrable discrete equations that tend to P_I, these and other discrete Painlevé equations may be found in [34]. Therefore, other forms of nomenclature would be preferable, but these historical names persist despite the discovery of other mechanisms that are capable of identifying the equations [35, 36].

An important method of finding and classifying nonlinear integrable OΔEs was developed by Sakai [35]. Sakai's methodology is based on the geometry of rational surfaces, where each equation is defined by an affine Weyl group of Cremona transformations on a certain family of rational surfaces obtained from \mathbb{P}^2 by blow ups.

Another important result on classification, this time for PΔEs and by an entirely different approach to that used by Sakai, was found by Adler, Bobenko and Suris (ABS) in [36]. They began by equating integrability with three-dimensional consistency because the latter had previously been shown to ensure the existence of a Lax pair (and thus integrability) [37, 38], three-dimensional consistency is explained in

chapter 2. All three-dimensionally consistent equations on quadrilateral lattices that met certain symmetry assumptions were classified based on a discriminant derived from each equation. Although this represents a landmark result, and much work has been conducted in the area since, some authors have expressed concern over whether the assumptions used are too restrictive to capture all interesting equations on quad-graphs [39, 40, 41].

The motivation for studying the discrete Painlevé equations was initially due to their connection with their continuous counterparts, however nowadays many now believe that the discrete equations are actually more fundamental than the continuous ones. Each continuous differential equation has many discretizations. Connections between difference equations and continuous equations are made through continuum limit calculations. Conversely, many discrete equations may share the same continuum limit (for example dP_I and $\text{alt-}dP_I$ both tend to P_I). Furthermore, there exists integrable discrete equations that do not have any continuous counterparts. Moreover, discrete Painlevé equations have appeared independently in studies of two-dimensional quantum gravity [42, 43, 28] and orthogonal polynomial theory [43, 44].

It has been known for some time that the singularity confinement property alone is not sufficient for integrability [45], however many of the discrete Painlevé equations found by Grammaticos, Ramani *et al* are known to be integrable in the sense that each is a compatibility condition that ensure isomonodromy for an associated linear problem [46, 47]. The confinement property is equivalent to the well-posedness of the discrete equation for x , even through apparent singularities on the complex sphere of x -values in both forward and backward evolution in the lattice variables.

The outline for the remainder of this thesis runs as follows: in chapter 2, we continue the background material on integrable nonlinear PΔEs in more detail. Putting them into an historical setting beginning with the continuous KdV equation, we show where the PΔEs arise, why they are considered integrable and how we can

find reductions from them to integrable O Δ Es. A thorough review of the relevant literature on O Δ Es is also included. In the subsequent chapters, we add to the existing body of knowledge on discrete integrable systems. A completeness study is conducted to find all the possible 2×2 Lax pairs, with certain restrictions, in chapter 3. In chapter 4 we extend a previously known method of reducing P Δ Es to O Δ Es and show how the reductions can work on Lax pairs for the P Δ Es to produce Lax pairs for the O Δ Es. In this way, we produce the first known 2×2 Lax pair for qP_{III} , with multiple free parameters, amongst other results. The Lax pairs for the discrete Painlevé equations found in chapter 4 are generalized in chapter 5 and used to find two hierarchies of q -discrete nonlinear integrable equations. Naturally, many directions remain to be explored and major questions remain to be answered. Some of these new directions are investigated in chapter 6, however, each chapter ends with its own discussion outlining possible future research in the relevant areas.

Chapter 2

Background

This chapter summarizes part of the history of discrete Painlevé equations as well as integrable partial difference equations. Topics to be considered include: where these equations arise, why they are believed to be discrete versions of their continuous counterparts, and reductions from PΔEs to discrete Painlevé equations. The focus of this historical perspective lies with Lax pairs for the equations of interest, as Lax pairs are the principal topic of this thesis.

2.1 Integrable ordinary difference equations

2.1.1 Where they arose

As pointed out in [34], from a historical perspective, Laguerre [48] was the first to derive an integrable nonlinear ordinary difference equations. This instance was from the perspective of orthogonal polynomials and lead to a higher order equation. Many years later, using a similar method, Shohat [49] arrived at a second order difference equation, which was identified as dP_I decades later again. There was another unidentified occurrence of an integrable nonlinear difference equation in

Jimbo and Miwa's [50, 51, 52] papers on the continuous Painlevé equations, and they also surfaced in some field-theoretic papers. The direct link between these difference equations and the continuous Painlevé equations had to wait for a paper by Brézin and Karakov [53] where a field-theoretic model of 2-D gravity lead to precisely the same equation as was found by Shohat over 50 years earlier. This time, however, a continuous limit was found to be the first Painlevé equation and so the new equation was called the discrete first Painlevé equation or dP_I .

The discrete Painlevé equations have numerous physical applications particularly as a partition function in quantum gravity [54, 53, 43, 28, 55].

So, nonlinear integrable difference equations have cropped up in a variety of settings, even before they were identified as discrete versions of the Painlevé equations. Once they were identified as such there was greatly increased activity in their study. The names for the equations were generally obtained from their continuous limits, dP_I tends to P_I and so forth. The d in dP_I indicates the type of discretization involved with that equation. A d indicates that the discretization is of the most common, additive type, q indicates that the discretization is of the type where the independent variable takes on a discrete exponential form, $l \rightarrow ql$, and no prefix indicates that the equation is continuous. There is also the elliptic type of discretization, discovered by Sakai [35], however we do not refer to it often enough in this thesis to require its own label.

2.1.2 QRT map and singularity confinement to construct discrete Painlevé equations

Discrete Painlevé equations have been found using a number of different techniques. One widely utilized method of attack was to begin with a Quispel-Roberts-Thompson (QRT) form [56] and then apply some integrability predicting algorithm in order to zero in on the appropriate nonautonomous terms.

The QRT mapping can be separated into two categories: symmetric and asymmetric. The symmetric mapping generally has the form

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1}f_2(x_n)}{f_3(x_n) - x_{n-1}f_4(x_n)} \quad (2.1)$$

while the asymmetric form is

$$x_{n+1} = \frac{f_1(y_n) - x_n f_2(y_n)}{f_2(y_n) - x_n f_3(y_n)} \quad (2.2a)$$

$$y_{n+1} = \frac{g_1(x_{n+1}) - y_n g_2(x_{n+1})}{g_2(x_{n+1}) - y_n g_3(x_{n+1})} \quad (2.2b)$$

The functions f_i and g_i are given by the following process. Take matrices

$$A_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \delta_i & \epsilon_i & \zeta_i \\ \kappa_i & \lambda_i & \mu_i \end{pmatrix}$$

for $i = 0, 1$. If the matrices A_i are symmetric then we arrive at the symmetric QRT mapping (2.1), if not then we have the asymmetric mapping (2.2). The terms within these matrices become the parameters in the system. Put

$$\mathbf{X} = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}$$

Then the functions f_i and g_i are obtained by

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = (A_0 \mathbf{X}) \times (A_1 \mathbf{X})$$

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = (A_0^T \mathbf{X}) \times (A_1^T \mathbf{X})$$

Since the matrices A_i are 3×3 , it would appear as though there were up to eighteen free parameters in the asymmetric case and twelve in the symmetric case.

However, through various transformations, the effective number of free parameters is reduced to eight for the asymmetric case and five for the symmetric. The system is de-autonomized by assuming that the parameters can vary with the discrete independent variable n . So that only integrable equations are achieved, we pass the resulting nonautonomous mapping through one of a variety of integrability detecting algorithms to find the form of the nonautonomous terms. One of the most widely used techniques is singularity confinement.

Singularity confinement was conceived as a discrete version of the Painlevé property. The Painlevé property is possessed when the solutions to a differential equation contain no moveable singularities other than poles. The way singularity confinement works is to look at all the possible moveable singularities in turn and ensure that each one is a pole in the discrete sense, *i.e.* that the solution returns to a finite value within a finite number of iterations after the initial divergence.

The prototypical example of singularity confinement is dP_I . We show how dP_I may be derived by requiring that a de-autonomized difference equation of the form (2.1) possess the singularity confinement property [32]. Consider equation (2.1) with

$$f_1(x_n) = x_n^2 - ax_n - b, \quad f_2(x_n) = -x_n \quad \text{and} \quad f_3(x_n) = 0.$$

If a and b are now permitted to depend on n , equation (2.1) becomes

$$x_{n+1} + x_{n-1} = -x_n + a_n + \frac{b_n}{x_n}. \tag{2.3}$$

Equation (2.3) has a movable singularity at $x_n = 0$. For the singularity to be confined, a_n and b_n must be of a specific form, we find the admissible forms below. For initial data $x_{n-1} = \kappa$, $x_n = \epsilon$, it can be shown that x_{n-2} and x_{n-3} are ordinary

points and that

$$\begin{aligned}
x_{n+1} &= \frac{b_n}{\epsilon} + O(1), & x_{n+2} &= -\frac{b_n}{\epsilon} + O(1), \\
x_{n+3} &= a_{n+2} - a_{n+1} + O(\epsilon), \\
x_{n+4} &= \frac{b_n}{\epsilon} + O(1), & x_{n+5} &= -\frac{b_n}{\epsilon} + O(1), \\
x_{n+6} &= a_{n+5} - a_{n+4} + a_{n+2} - a_{n+1} + O(\epsilon), \\
x_{n+7} &= \frac{b_n}{\epsilon} + O(1), & x_{n+8} &= -\frac{b_n}{\epsilon} + O(1), \quad \dots
\end{aligned}$$

A first non trivial condition to yield singularity confinement or the isolated movable nature of the singularity is $a_{n+2} - a_{n+1} = 0$ or $a_n = C_1 = \text{constant}$. Including this condition in the iterations gives

$$x_{n+4} = -\frac{b_n(b_{n+3} - b_{n+2} - b_{n+1} + b_n)}{(b_{n+2} + b_{n+1} - b_n)\epsilon} + O(1).$$

A second condition for singularity confinement is therefore $b_{n+3} - b_{n+2} - b_{n+1} + b_n = 0$ or $b_n = C_2 + C_3(-1)^n + C_4n$. Equation (2.3) now satisfies the singularity confinement test and is of the form

$$x_{n+1} + x_n + x_{n-1} = C_1 + \frac{C_2 + C_3(-1)^n + C_4n}{x_n}. \quad (2.4)$$

The first condition for singularity confinement is clearly satisfied as equation (2.4) is linear in x_{n+1} and x_{n-1} . For $C_3 = 0$, equation (2.4) is dP_I.

It has been shown that the singularity confinement property is not sufficient to guarantee integrability of a particular mapping and, as such, a number of refinements have been added to the original theory including ‘keeping the memory’ and ‘non-proliferation of pre images’.

2.2 PΔEs

2.2.1 Historical setting: from continuous origins

Initially put forward as a nonlinear shallow water wave model over a century ago [57], the KdV equation

$$u_t + u_{xxx} - 6uu_x = 0 \quad (2.5)$$

is now one of the most famous objects in the field of integrable nonlinear systems. The first genus-2 solution of the KdV equation was published in 1897 by Baker in [58]. Work on the equation laid dormant for many decades until, in 1965, Zabusky and Kruskal published the results of their numerical experiments into the KdV equation which was found to be the continuum limit of the Fermi-Pasta-Ulam (FPU) chain with quadratic nonlinearity [59]. There it was found that solitary wave solutions, despite being nonlinear waves themselves, behaved similarly to linear waves which obey the principal of superposition. The term ‘soliton’ was initiated to convey the idea that such solitary waves behaved like particles.

Shortly thereafter, Muira *et al.* found a total of ten conserved quantities for the KdV equation and suggested that infinitely many might exist [60, 61]. While studying the KdV equation, Muira also examined a related entity called the modified KdV equation (MKdV)

$$v_t + 6v^2v_x + v_xxx = 0 \quad (2.6)$$

which is also found as a continuum limit of the FPU chain, this time with a cubic nonlinearity. Mirroring the KdV case, Miura found another set of conservation laws for the MKdV equation and found a connection between the two sets;

$$u = v_x + v^2 \quad (2.7)$$

where solutions to the KdV and MKdV equations are represented by u and v respectively. Moreover, setting

$$\mathcal{N}_1(u) = u_t - 6uu_x + u_{xxx} = 0 \quad (2.8)$$

and

$$\mathcal{N}_2(v) = v_t + 6v^2v_x + v_{xxx} = 0 \quad (2.9)$$

then it was shown that

$$\mathcal{N}_1(u) = \left(2v + \frac{\partial}{\partial x}\right)\mathcal{N}_2(v). \quad (2.10)$$

This Miura transformation illustrates the link between the KdV and MKdV equations.

2.2.2 Integrable partial difference equations

Although Toda constructed what is now a famous, integrable differential-difference equation [62], this was only discrete in one independent direction, and it was not related to Lax pairs at the time. From the Lax pair perspective, the recent history of integrable partial difference equations begins with work by Ablowitz and Ladik in 1976 [63]. Those authors formulated an integrable difference scheme that was, in part, motivated by a need to solve continuous PDEs numerically, but was also of theoretical interest. Starting with a Lax pair expansion based on the eigenvalue problem of Zakharov and Shabat [64], Ablowitz and Ladik derive differential-difference and P Δ E versions of the nonlinear Schrödinger equation. Solutions are also derived by discrete inverse scattering and similar results are given in their 1977 follow up paper [65].

Starting with an assumption about the form of a Lax pair

The method used by Ablowitz and Ladik in these early papers to obtain P Δ E Lax pairs is especially pertinent to later chapters of this thesis. The starting point was a discrete eigenvalue problem that can be written in matrix form as follows:

$$\begin{aligned} \bar{\theta} &= L\theta, \\ \hat{\theta} &= M\theta. \end{aligned}$$

Where Lax matrices L and M are given by

$$L = \begin{pmatrix} \nu & b \\ c & 1/\nu \end{pmatrix},$$

$$M = \begin{pmatrix} \alpha + 1 & \beta \\ \gamma & \delta + 1 \end{pmatrix},$$

where ν is the spectral variable, $b = b(l, m)$ and $c = c(l, m)$ are lattice terms, *i.e.* terms that depend only on the lattice variables l and m , and α, β, γ and δ are general terms that may depend on the spectral variable as well as the lattice variables. The compatibility condition for this Lax pair is $\bar{L}M = \widehat{M}L$, which leads to the following four equations:

$$\nu(\bar{\alpha} - \alpha) = \hat{b}\gamma - \bar{\beta}c, \quad (2.11a)$$

$$\frac{1}{\nu}(\bar{\delta} - \delta) = \hat{c}\beta - b\bar{\gamma}, \quad (2.11b)$$

$$\hat{b} - b = \frac{1}{\nu}\bar{\beta} - \nu\beta + \bar{\alpha}b - \delta\hat{b}, \quad (2.11c)$$

$$\hat{c} - c = \nu\bar{\gamma} - \frac{1}{\nu}\gamma + \bar{\delta}c - \alpha\hat{c}. \quad (2.11d)$$

At this point, directly after equation (2.14) in the original paper by Ablowitz and Ladik, assumptions were made, based on previous results with continuous systems [66], about the form of the general terms:

$$\alpha = \alpha_0 + \nu^2\alpha_2, \quad (2.12a)$$

$$\delta = \frac{1}{\nu^2}\delta_2 + \delta_0, \quad (2.12b)$$

$$\beta = \frac{1}{\nu}\beta_0 + \nu\beta_1, \quad (2.12c)$$

$$\gamma = \frac{1}{\nu}\gamma_0 + \nu\gamma_1, \quad (2.12d)$$

where the right hand sides are separated such that the spectral variable, ν , appears explicitly and all those terms with a subscript depend on the lattice variables, l and m , only.

Substituting (2.12) into (2.11) immediately shows that:

$$\alpha_2 = \mu_1, \quad (2.13a)$$

$$\delta_2 = \mu_2, \quad (2.13b)$$

$$\bar{\beta}_0 = \mu_2 \hat{b}, \quad (2.13c)$$

$$\beta_1 = \mu_1 b, \quad (2.13d)$$

$$\gamma_0 = \mu_2 c, \quad (2.13e)$$

$$\bar{\gamma}_1 = \mu_1 c, \quad (2.13f)$$

$$(2.13g)$$

where $\mu_i = \mu_i(m)$. Ablowitz and Ladik made also assume that $\mu_1 = -\mu_2 = k$, k being a constant associated with the continuum limit of the resulting nonlinear equation (below), and $b = \mp c^*$, where star denotes complex conjugation. With these values in place, it is not difficult to see that

$$\begin{aligned} \alpha &= k \left[\nu^2 - 1 \mp \sum_{j=-\infty}^l (c^*(j-1, m+1)c(j-2, m+1) - c^*(j, m)c(j-1, m)) \right], \\ \beta &= k(\mp \nu c^* \pm \frac{1}{\nu} \hat{c}^*), \\ \gamma &= k(\nu \hat{c} - \frac{1}{\nu} c), \\ \delta &= k \left[\frac{1}{\nu^2} - 1 \mp \sum_{j=-\infty}^l (c^*(j-2, m+1)c(j-1, m+1) - c^*(j-1, m)c(j, m)) \right], \end{aligned}$$

where $c = c(l, m)$ is expressed in terms of its arguments only when needed to clarify the summations. The resulting nonlinear equation associated with the Lax pair is

$$c - \hat{c} = \quad (2.14)$$

$$\begin{aligned} &k(\bar{c} - c - \hat{c} + \hat{\bar{c}}) \\ &\mp k \left[c \sum_{j=-\infty}^l (c^*(j-2, m+1)c(j-1, m+1) - c^*(j-1, m)c(j, m)) \right] \\ &\mp k \left[\hat{c} \sum_{j=-\infty}^l (c^*(j-1, m+1)c(j-2, m+1) - c^*(j, m)c(j-1, m)) \right]. \end{aligned} \quad (2.15)$$

This equation is a discrete version of the nonlinear Schrödinger equation.

These equations (2.15) involve infinite sums that can be viewed as discrete versions of integrals, in this sense they are more akin to integro-differential equations than partial differential equations such as the KdV. In any case, it should be clear that equation (2.15) is not the most general equation that could be derived from a Lax pair of this form. A number of assumptions were made in solving the compatibility condition, which effectively reduced the order and complexity of the final equation, refer to chapter 3.2 for general examples of similar calculations. A vast quantity of nonlinear equations have been derived in this manner, not only P Δ E's, but also PDEs, partial differential-difference equations, ODEs, O Δ E's and delay equations. It is a well trodden path that is extended in chapter 3.2.

The bilinear approach

Quite a different approach was instigated by Hirota in 1977-78 [67, 68, 69] in a series of five papers that explain how one can obtain integrable nonlinear partial difference equations through a bilinear approach. These results built on previous work on bilinear forms in the continuous realm, by the same author. To find integrable nonlinear P Δ E's, the corresponding continuous PDEs were put in to the bilinear form by a dependent variable transformation which was discretized. The discrete bilinear form was transformed back to a P Δ E by the associated dependent variable transformation, thereby producing a discrete analogue of the original nonlinear PDE. In the series of five papers, discrete analogues of the KdV, Toda and sine-Gordon equations were derived along with Bäcklund transformations and N -soliton solutions. Similar results for a collection of nonlinear PDEs including Burger's equation, that were previously known to be linearizable, were also presented.

Direct linearization of various nonlinear P Δ E's has been achieved via the use of linear integral equations having arbitrary measure and contour, based on the work

of Fokas and Ablowitz in the continuous case [70]. The first of such results for discrete equations was by Date, Jimbo and Miwa in 1982 who published five related papers [71, 72, 73, 74, 75] that uncovered many examples of PΔEs in the hope that a taxonomy of these equations could be clarified. Their equations are arranged into three families corresponding to Kadomstev-Petviashvili (KP), B-type Kadomstev-Petviashvili (BKP) and elliptic equations. Further results were obtained in a similar way by Nijhoff, Quispel *et al* in 1983 and 1984 [76, 56], Wiersma and Capel in 1987 [77]. Many other results of this nature were published, up to the mid 1990's, for a review of the topic see [78].

Consistency around a cube

Of late there has been a cacophony of activity surrounding the multidimensional consistency approach, often referred to as consistency around a cube (CAC). The idea is summarized, drawing largely from [40], in this section. The types of PΔEs to which the CAC approach has so far been applied can be written as

$$\hat{x} = Q(x, \bar{x}, \hat{x}; p_{lm}), \quad (2.16)$$

where $Q(x, \bar{x}, \hat{x}; p_{lm})$ is autonomous, first order in each of two discrete independent variables, and are linear in each argument (affine linear). The quantity p_{lm} lists the parameters relevant to the l - m -face. The PΔEs of interest are initially described over two discrete dimensions l and m , to these we add a third dimension, the n direction say, (see figure 2.1). Assume that the same equation holds along all planes in the basic cube formed by adding the third dimension, which is to say that we should also have the equations

$$\tilde{x} = Q(x, \bar{x}, \tilde{x}; p_{ln}), \quad (2.17)$$

$$\hat{\tilde{x}} = Q(x, \tilde{x}, \hat{\tilde{x}}; p_{mn}). \quad (2.18)$$

where $\tilde{x} = x(l, m, n + 1)$. Referring to figure 2.1, given the values of x, \bar{x}, \hat{x} and \tilde{x} as initial data, located at the full circles, the values of $\hat{\tilde{x}}, \tilde{\tilde{x}}$ and $\hat{\hat{\tilde{x}}}$, located at the open

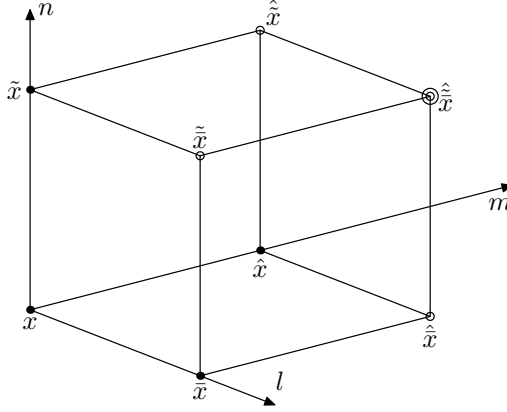


Figure 2.1: For CAC we add a third direction, n . Initial data lie at the full circles, uniquely determined data at the open circles, and CAC requires that there is no conflict at the double open circle $\hat{\tilde{x}}$.

circles, can all be determined by Q in the relevant variables. Hence, it is possible to calculate the value of $\hat{\tilde{x}}$ in three different ways, corresponding to the three adjacent faces of the cube. For the equation described by $Q(x, \bar{x}, \hat{x}; p_{lm})$ to be ‘consistent around a cube’, the same result must be attained for $\hat{\tilde{x}}$ in all three cases, or

$$\hat{\tilde{x}} = Q(\tilde{x}, \tilde{\tilde{x}}, \tilde{\hat{x}}; p_{lm}) = Q(\hat{x}, \hat{\tilde{x}}, \hat{\hat{x}}; p_{lm}) = Q(\bar{x}, \bar{\tilde{x}}, \bar{\hat{x}}; p_{mn})$$

In 2002, Nijhoff published [38] a Lax pair for the system now known as Q_4 (but was then known as either the Adler or lattice Krichever-Novikov system), where the Lax pair was constructed from the equation itself using its CAC property. In the same year, Bobenko and Suris published a paper about general integrable systems that exist on quad-graphs and possess the CAC property [37]. There, the same method was used to construct Lax pairs for three equations that are related to one another: the cross-ratio equation, the shifted cross-ratio system and the hyperbolic shifted ratio system. In the same paper, these equations were also shown to be related to Toda type systems on the lattice. As the CAC method of Lax pair construction is relevant to this thesis, we present the construction of a Lax pair for LMKdV below.

Begin with LMKdV

$$\hat{\tilde{x}} = x \frac{\bar{x} - r\hat{x}}{\hat{x} - r\bar{x}} \quad (2.19)$$

where $r(l, m) = \lambda(l)\mu(m)$, which has been shown to possess the CAC property [78].

As such, a Lax pair for LMKdV is constructed by allowing

$$\tilde{x} = \frac{\theta_2}{\theta_1}, \quad (2.20)$$

where θ_i are the elements of the two component vector which solves the linear systems that form the Lax pair

$$\begin{aligned} \bar{\theta} &= L\theta, \\ \hat{\theta} &= M\theta. \end{aligned} \quad (2.21)$$

Equation (2.21) can be written

$$\begin{pmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \end{pmatrix} = \begin{pmatrix} L_1\theta_1 + L_2\theta_2 \\ L_3\theta_1 + L_4\theta_2 \end{pmatrix},$$

and taking a ratio of the components yields

$$\frac{\bar{\theta}_2}{\bar{\theta}_1} = \tilde{x} = \frac{L_3\theta_1 + L_4\theta_2}{L_1\theta_1 + L_2\theta_2}. \quad (2.22)$$

As mentioned above, the same equation must exist on all faces of the cube, so LMKdV also arises as $\tilde{x} = q(x, \bar{x}, \tilde{x}; p_{ln})$ from equation (2.22). Comparison with equation (2.19) shows that one satisfactory choice for the undetermined components of the L matrix is

$$\begin{aligned} L_1 &= r\bar{x}/x \\ L_2 &= 1/x \\ L_3 &= \bar{x} \\ L_4 &= r \end{aligned}$$

where $r = \lambda(l)\nu(n)$ is the product of the non-autonomous parameter functions corresponding to the ln -plane where (2.22) exists. We therefore write the L matrix

$$L = D_1 \begin{pmatrix} \frac{\lambda\bar{x}}{x}\nu & \frac{1}{x} \\ \bar{x} & \lambda\nu \end{pmatrix} \quad (2.23)$$

where D_1 is a prefactor that does not effect equation (2.22) and is often taken to be the square root of the determinant of L .

Because the same equation lives on the mn -plane, a similar construction shows that the other matrix in the Lax pair is the same as L , with \bar{x} replaced by \hat{x} and $\lambda(l)$ replaced by $\mu(m)$. The parameter function $\nu(n)$ is regarded as the spectral variable in the Lax pair for the equation in the lm -plane.

$$M = D_2 \begin{pmatrix} \frac{\mu\hat{x}}{x}\nu & \frac{1}{x} \\ \hat{x} & \mu\nu \end{pmatrix} \quad (2.24)$$

It is easy to check that LMKdV arises from the compatibility condition of this Lax pair, $\widehat{L}M = \overline{M}L$, for many choices of D_1 and D_2 , including $D_1 \equiv D_2 \equiv 1$.

The CAC approach was famously used to derive and classify all one-field equations on quad-graphs under certain assumptions [36], where the most complex equation found was the Q_4 system mentioned earlier. This was a landmark paper, but some of its assumptions have been questioned and scrutinized since. In particular the tetrahedron assumption, which states that the expression for $\hat{\hat{x}}$ does not depend on x (see figure 2.1), has received attention [40, 79]. However, at least one of the integrable equations not possessing the tetrahedron property that have been so far, has been shown to be solvable by direct linearization [80].

2.2.3 Reductions

Reductions from partial differential equations (PDEs) to ordinary differential equations (ODEs) provide a natural way to gain further insight into either system, depending on the approach. On the one hand, there is a conjectured link between the integrability of a PDE and the possession of the Painlevé property by its reductions [19]. While on the other hand, where the reductions of some PDEs lead to Painlevé equations, the inverse scattering transform solutions to the former have been utilized to construct solutions of the latter [18].

For two-dimensional continuous systems, similarity reductions involve finding a special combination of both the independent variables such that the resulting equation only depends on one variable, *i.e.* the initial PDE is reduced to an ODE. The most famous example from the perspective of Painlevé equations was first given in [81] and reduces the modified KdV equation

$$u_t - 6u^2u_x + u_{xxx} = 0$$

to P_{II}

$$w'' = 2w^3 + zw + c$$

after using

$$z = \frac{x}{(3t)^{1/3}}, \quad w(z) = u(z)(3t)^{1/3}$$

and integrating once. The key lies in finding the appropriate combination of the two independent variables, x and t , which is done using symmetry arguments about the initial PDE.

Discrete symmetry reductions

The focus of this thesis is entirely on discrete equations, which presented their own problems, distinct from those of their continuous analogues. The study of reductions of partial difference equations to ordinary difference equations was conducted most vigorously by Nijhoff and collaborators in the 1990's. In these works [82, 83] the analogue of a similarity reduction in a discrete setting was given to be a pair of equations, termed the lattice equation and the similarity constraint, defined such that localized configurations of initial data can be iterated throughout the lattice leading to a global solution. The example of this type of reduction most pertinent here is that which reduces the LMKdV equation to a discrete form of the second Painlevé equation. The form of the LMKdV equation used is

$$\hat{x} = x \frac{\hat{x} - r\bar{x}}{\bar{x} - r\hat{x}} \tag{2.25}$$

where r is an autonomous parameter. If we consider which lattice points, (l, m) , are involved at some iteration of the LMKdV equation (2.25), we arrive at the plaquette of figure 2.2, where the axes are included to indicate directions only, the plaquette

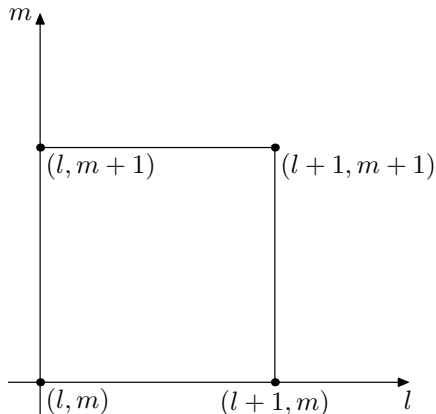


Figure 2.2: A square plaquette with four vertices.

drawn might lie anywhere in the (l, m) plane. Since LMKdV (2.25) is linear in each iteration of x , the value of x , at any vertex of the plaquette in figure 2.2, can be written in terms of the values of x at the other three vertices.

Turning to the similarity constraint

$$l \frac{\bar{x} - \underline{x}}{\bar{x} + \underline{x}} + m \frac{\hat{x} - \check{x}}{\hat{x} + \check{x}} = \frac{3k_1 - 1}{2} - k_2(-1)^{l+m}, \quad (2.26)$$

where x is the dependent variable and k_i are constants, we see that four vertices are involved which lie in the arrangement shown in figure 2.3.

Equations (2.25) and (2.26) are iterated to form the global solution, starting with a configuration of initial data of the form shown in figure 2.4. The value of x at some locations is found using the lattice equation (2.25), these are marked with stars \star , while the value of x at other locations is found using the similarity constraint (2.26), these points are marked with circles \circ . At some location there will be two expressions for x , one from each of the lattice equation and the similarity constraint, this point is marked with the symbol \otimes . The two expressions at the point marked

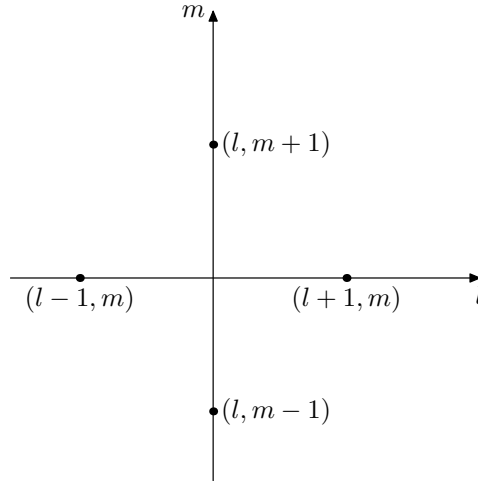


Figure 2.3: A cross-shaped plaquette with four vertices.

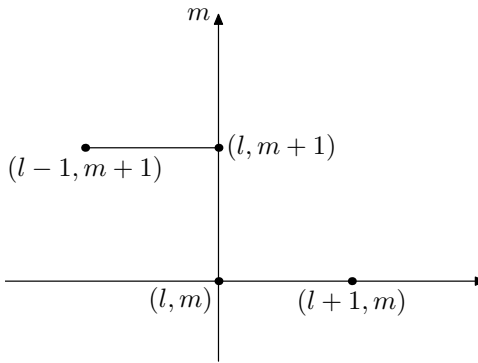


Figure 2.4: Initial data occupying four vertices in a stair configuration.

⊗ in figure 2.5 must be equivalent for the lattice equation and similarity constraint to be compatible. It has been put forward that this type of compatibility could be used as the definition of symmetry for discrete equations [84].

In some cases, compatible pairs of lattice equations and similarity constraints can be used to completely remove the dependence on one of the lattice variables and thereby reduce the lattice equations, that are PΔEs, to OΔEs. The example of LMKdV (2.25) with the similarity constraint (2.26) given above, reduces LMKdV to a rich OΔE that can be considered to be a discrete form of P_{II}, P_{III}, P_V or P_{VI}

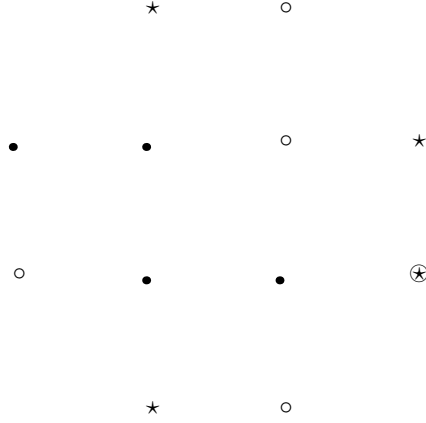


Figure 2.5: Part of the global solution generated by either the lattice equation (2.25) marked \circ , or the similarity constraint (2.26) marked \star , the position marked \otimes has two expressions, one from both the lattice equation and the similarity constraint. Full circles indicate initial data positions.

as follows [82, 85]: define a and b to represent the terms on the left hand side of equation (2.26).

$$a = \frac{\bar{x} - x}{\bar{x} + x} \tag{2.27}$$

$$b = \frac{\hat{x} - \hat{x}}{\hat{x} + \hat{x}}. \tag{2.28}$$

Also define u and v as

$$u = \frac{x}{\hat{x}} \tag{2.29a}$$

$$v = \frac{\bar{x}}{\hat{x}}. \tag{2.29b}$$

The key is to remove the dependence of a and b on either of the lattice variables l or m , we choose to remove m here. Notice that a is already without terms that have been shifted in m , we can write a in terms of u and v quite simply

$$a = \frac{v - u}{v + u}. \tag{2.30}$$

An expression for b that only involves l shifts can be obtained by taking equation (2.29) and adding its backward m -shifted counterpart to either side. Rearranging

yields

$$(u + r)\bar{b} + u = (v - r)b + v. \quad (2.31)$$

The similarity constraint (2.26) is used to write b as

$$b = \frac{1}{m} \left[k_1 - k_2(-1)^{l+m} - l \frac{v - \underline{u}}{v + \underline{u}} \right] \quad (2.32)$$

which can be substituted into equation (2.31) to obtain the sought after nonlinear OΔE

$$\begin{aligned} & (l + 1)(r + x)(1 + rx) \frac{\bar{x} - x + r(1 - x\bar{x})}{\bar{x} + x + r(1 + x\bar{x})} - n(1 - r^2)x \frac{x - \underline{x} + r(1 - x\underline{x})}{x + \underline{x} + r(1 + x\underline{x})} \\ & = k_1 r(1 + 2rx + x^2) + k_2(-1)^{l+m}(r + 2x + rx^2) - mr(1 - x^2) \end{aligned}$$

A type of reduction for nonautonomous partial difference equations

In [86] Grammaticos, Ramani and collaborators published a new method by which to construct discrete Painlevé equations starting with partial difference equations (PΔEs). An important point of difference between this and previous work on the subject was that, in the new setting, the PΔEs used were non-autonomous, where previous studies on discrete reductions had used only autonomous equations. For example, one PΔE studied was a non-autonomous version of the LMKdV (2.25) encountered in the previous section. This equation was made autonomous by allowing the parameters k and q to depend on the lattice variables l and m . The dependence is not arbitrary, a special dependence was found using singularity confinement to retain integrability of the equation. The form of LMKdV used in [86] was

$$\hat{\hat{x}} = x \frac{r\hat{x} - \bar{x}}{r\bar{x} - \hat{x}} \quad (2.33)$$

where the non-autonomous term r must be separable, *i.e.* $r(l, m) = \lambda(l)\mu(m)$, to ensure its integrability. The condition on r , derived by singularity confinement, is

$$\hat{\hat{r}} = \hat{r}\bar{r} \quad (2.34)$$

which plays an important role later on, and is clearly solved when r is separable.

Two reductions from LMKdV to q -discrete Painlevé equations were presented. The first involved a reduction of the form

$$\hat{x} = \bar{\bar{x}}. \quad (2.35)$$

Simply substituting equation (2.35) into (2.33) and introducing $y = \bar{\bar{x}}/\bar{x}$ results in the following nonlinear OΔE

$$\bar{y}\underline{y} = \frac{1 - ry}{y(r - y)}. \quad (2.36)$$

The form of the equation is tantalizingly similar to some known q -discrete Painlevé equations, however, it cannot be identified until the non-autonomous terms are known explicitly, which requires further analysis.

On the (l, m) lattice, this reduction is described by figure 2.6. The reduction

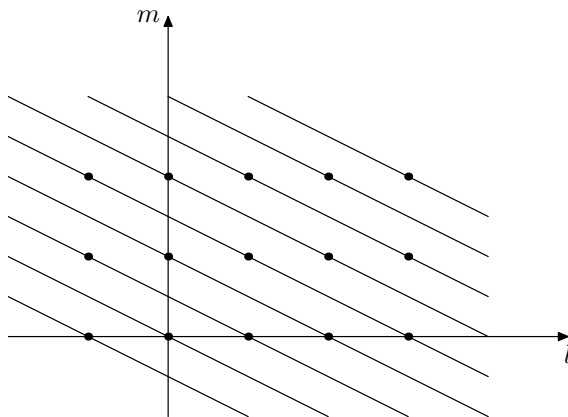


Figure 2.6: Lines joining lattice points of equal x value under the reduction $\hat{x} = \bar{\bar{x}}$.

used to achieve equation (2.35) was carried out on the plaquette at lattice points (l, m) , $(l + 1, m)$, $(l, m + 1)$ and $(l + 1, m + 1)$, and the reduced equation can be thought to exist along the points where m is fixed in the (l, m) lattice, *i.e.* along the l axis in figure 2.6. However, the same reduction should work anywhere in the (l, m) plane and we can find the explicit form of the non-autonomous terms in the reduced equation by considering the PΔE at a position moved up one step in m . A

'hatted' version of equation (2.33) is

$$\hat{\hat{x}} = \hat{x} \frac{\hat{r}\hat{\hat{x}} - \hat{\hat{x}}}{\hat{r}\hat{\hat{x}} - \hat{\hat{x}}}. \quad (2.37)$$

Noting that, under the present reduction, $\hat{\hat{x}} = \hat{\hat{x}} = \hat{x}$, equation (2.37) is reduced to

$$\bar{\bar{y}}\bar{y} = \frac{1 - \hat{r}\bar{y}}{\bar{y}(\hat{r} - \bar{y})}. \quad (2.38)$$

from where we may shift the whole equation down twice in the l direction to obtain

$$\bar{y}\underline{y} = \frac{1 - \hat{r}\underline{y}}{y(\hat{r} - \underline{y})}. \quad (2.39)$$

A comparison between equations (2.36) and (2.39) shows that the condition that r must satisfy is the same as the reduction on x

$$\hat{r} = \bar{r}. \quad (2.40)$$

The explicit form of r is gleaned when this condition is used in conjunction with equation (2.34)

$$r = k_1 k_2^l k_3^{(-1)^l} \quad (2.41)$$

where k_i are constants. With r of this form, equation (2.36) is identified with qP_{II} when $k_3 = 0$ [87], or with an asymmetric form of qth when $k_3 \neq 0$ [88].

The second reduction from LMKdV, reported in the same paper, used $\hat{x} = \bar{\bar{x}}$ and resulted in

$$\bar{w}\underline{w} = \frac{1 - rw}{r - w} \quad (2.42)$$

where $w = \bar{\bar{x}}/\bar{x}$ and $\log r = k_1 + k_2 l + k_3 j_3 l + k_4 j_3 2l$ with $j_3^3 = 1$. This equation was identified as qP_{II} when $k_3 = k_4 = 0$ [87], or as qfi in the generic case.

Grammaticos *et al.* also documented two reductions of this type from the lattice sine-Gordon equation (LSG) to q -discrete Painlevé equations, all working in the same way as described above, and using LSG as their starting point.

$$\hat{\hat{x}} = x \frac{1 + r\bar{x}\hat{\hat{x}}}{\bar{x}\hat{\hat{x}} + r}. \quad (2.43)$$

The following reductions hold:

- $\hat{x} = \bar{x}$: leads to $\bar{x}\underline{x} = \frac{1+rx}{r+x}$, where $r = k_1 k_2^l$, this is a special case of qP_{III} .
- $\hat{x} = \bar{\bar{x}}$: leads to $\bar{y}\underline{y} = \frac{y(1-ry)}{r-y}$, where $y = \bar{\bar{x}}/\bar{x}$ and $\log r = k_1 + k_2 l + k_3(-1)^l$, this equation is equivalent to (2.36) on taking the reciprocal of either even or odd iterates of y .

Other reductions of this kind are also possible, see chapter 4. The series of reductions presented in chapter 4 suggests the existence of hierarchies of integrable q -difference equations, this is confirmed in chapter 5.

2.3 Hierarchies

This section contains a summary of the literature that pertains to chapter 5, where the construction of two new hierarchies of q -discrete O Δ Es is explained, based on Lax pair expansions.

The first “higher order” integrable system of Painlevé type was published by Garnier in 1912 [21] and is now known as the Garnier system. These are systems arising from compatibility of linear systems with n singularities, where $n \geq 4$ (while the linear system for P_{VI} has three singularities). The resulting compatibility conditions are nonlinear PDEs with $n - 2$ independent variables, or ODEs with one independent variable and $n - 3$ parameters. They provide hierarchies of arbitrary order [21, 89]. Since then, the majority of results concerning higher order integrable equations have been on partial differential and/or difference equations.

Most results in the literature on hierarchies of discrete equations have concentrated on partial difference or differential-difference equations. The earlier work, since around the beginning of the 1980’s, concentrated on the Toda lattice hierarchy, as well as its variations such as the relativistic Toda Hierarchy [90, 91, 92, 93]. Other lattice hierarchies that have since been studied include a q -discrete version

of the KP hierarchy [94, 95, 96], and generalizations thereof such as the ‘universal character’ hierarchy [97], or the $\widehat{\mathfrak{gl}}_3$ hierarchy [98].

While these have led to some reductions to additive O Δ E s [89, 95, 98, 97], no previous attempts appear to have been made to find hierarchies of q- difference equations via reductions. In chapter 5, we find hierarchies of q-difference equations through the expansion of Lax pairs. Those expansions are motivated by two series of reductions from a single P Δ E, LMKdV, but those reductions play no part in the actual formation of the hierarchy.

Hierarchies via Lax pair expansions

The publications most relevant to the results presented in chapter 5 are [99, 100]. In those papers hierarchies of d -discrete Painlevé equations are derived from expansions within differential-discrete Lax pairs. For example, the dP $_{II}$ hierarchy derived in [99] begins with the following linear problem

$$\bar{\theta} = L\theta, \tag{2.44a}$$

$$\frac{\partial\theta}{\partial\nu} = N\theta, \tag{2.44b}$$

which is a differential-difference isomonodromy Lax pair with spectral variable ν . L and M take the forms

$$L = \begin{pmatrix} \nu & x \\ x & 1/\nu \end{pmatrix}, \tag{2.45a}$$

$$M = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}, \tag{2.45b}$$

where $x = x(l)$, l being the discrete, independent variable and A, B and C depend on l and ν . The compatibility condition for (2.44) is

$$\frac{\partial L}{\partial\nu} + LM = \bar{M}L \tag{2.46}$$

which gives three equations for A, B and C in terms of x and ν . B and C can be eliminated from the system, resulting in an expression for A

$$\begin{aligned} \nu \bar{x} \underline{x} + \bar{x} x [A(1 + \underline{x}^2) + \underline{A}(\underline{x}^2 - 1)] - x \underline{x} [\bar{A}(1 + \bar{x}^2) + \bar{\bar{A}}(\bar{x}^2 - 1)] \\ + \nu^2 \bar{x} \underline{x} (A - \bar{A}) - \frac{x}{\nu} (\bar{x} + \underline{x}) - \frac{\bar{x} \underline{x}}{\nu^2} (\bar{A} - A) + \frac{\bar{x} \underline{x}}{\nu^3} = 0. \end{aligned} \quad (2.47)$$

It is convenient to rewrite this in the following operator form

$$\nu - \frac{\bar{x}}{\nu} \left(\frac{1}{\bar{x}} + \frac{1}{x} \right) = \left[\left(\nu^2 + \frac{1}{\nu^2} \right) (\Delta^2 + \Delta) + \mathcal{J} \right] A, \quad (2.48)$$

where

$$\mathcal{J} = \bar{x} \left[\Delta(\Delta + 2) \left(\left(x - \frac{1}{x} \right) \Delta + 2x \right) - \frac{2}{x} \Delta \right]. \quad (2.49)$$

The hierarchy is brought about by allowing A to be an expansion in powers of ν , with coefficients that depend on l , to be determined from the compatibility condition. Recalling that every order of ν that arises in (2.47) forms a separate condition, equation (2.47) suggests that A be rational in ν . As such, we allow A to adopt the following form

$$A = \frac{a_1}{\nu} + \sum_{i=1}^m \left(\nu^{(2i-1)} + \frac{1}{\nu^{(2i+1)}} \right) a_{2i+1}, \quad (2.50)$$

where $a_i = a_i(l)$. Using this expansion in equation (2.48) results in a set of equations for the terms a_i , each equation multiplied by a distinct order of the spectral variable ν . In fact, there are twice as many distinct orders of ν as there are variables a_i , however, each equation repeats twice, resulting in precisely the correct number of conditions to solve for every a_i in the expansion, plus one extra condition that becomes the evolution equation. At the extreme orders, $\nu^{(-2m-3)}$ and $\nu^{(2m+1)}$, we find

$$(\Delta^2 + \Delta) a_{2m+1} = 0$$

which is solved for $a_{2m+1} \equiv k_{2m+1}$, with k_{2m+1} a constant. At the next orders in

from the extremes, $\nu^{(-2m-1)}$ and $\nu^{(2m-1)}$, we make the following calculation

$$\begin{aligned}
\mathcal{J}a_{2m+1} + (\Delta^2 + \Delta)a_{2m-1} &= \bar{x}\Delta(\Delta + 2)2xk_{2m+1} + (\Delta^2 + \Delta)a_{2m-1}, \\
&= 2k_{2m+1}\bar{x}(\bar{x} - x) + a_{2m-1} - a_{2m-1}, \\
&= 2k_{2m+1}\Delta(x\bar{x}) + \Delta a_{2m-1}, \\
&= 0,
\end{aligned}$$

which is solved for $a_{2m-1} = -2k_{2m+1}x\bar{x} + k_{m+1}$, where k_{m+1} is another constant. Moving successively from these extreme orders in ν toward the inner orders, we obtain

$$a_{2i-3} = -a_{2i+1} - (\Delta^2 + \Delta)^{-1}a_{2i-1} \quad \text{for } i = 3, \dots, m, \quad (2.51)$$

$$a_1 = l - a_5 - (\Delta^2 + \Delta)^{-1}a_3, \quad (2.52)$$

$$\bar{x} \left(\frac{1}{\bar{x}} + \frac{1}{x} \right) = -2(\Delta^2 + \Delta)a_3 - \mathcal{J}a_1. \quad (2.53)$$

Equation (2.51) is used iteratively to calculate the coefficients a_{2m-3} to a_3 one by one, equation (2.52) gives the value of a_1 and the final equation, (2.53), is the evolution equation of the system. Thus, each positive integer value of m yields a different equation in the hierarchy, $m = 0$ delivers a trivial equation, $m = 1$ brings about the well know (second order) dP_{II} equation:

$$\bar{x} + \underline{x} = \frac{(k_1 + k_2l)x + k_3 + k_4(-1)^l}{1 - x^2}$$

where k_i are constants, manipulated to retrieve the familiar form of dP_{II}. The next equation in the hierarchy is fourth order, obtained when $m = 2$

$$\begin{aligned}
2k_4(\bar{x}(1 - \bar{x}^2) - \underline{x}(1 - \underline{x}^2))(1 - x^2) - 2k_4x(\bar{x} + \underline{x})^2(1 - x^2) \\
+ 2k_3(\bar{x} + \underline{x})(1 - x^2) + x(2k_2 + 2l + 1) = k_1 + k_0(-1)^l
\end{aligned}$$

As m is increased, evolution equations are produced that are of order $2m$ and are of increasing complexity.

No general formula for the m th equation in the hierarchy is given in [99], one must calculate each member in turn. Although, it is not difficult to produce the

$m + 1$ st member from the m th because all of the coefficients a_i , $i > 3$, remain the same, except for a change of index $a_i \rightarrow a_{i+2}$. The new coefficients a_1 and a_3 are calculated using (2.51) and (2.52), then the new evolution equation is (2.53). The hierarchies produced in chapter 5 do not contain any equivalent of a_i that remain unchanged from one member of the hierarchy to the next. However, a general formula for every coefficient is given, as is the form of every equation in the hierarchy. It is interesting to note that, while the Lax pairs for successive equations in that hierarchy become increasingly complex with increasing order of the equations, the equations themselves retain the same simple form while their order increases.

Chapter 3

Completeness Study on Discrete 2×2 Lax Pairs

3.1 Introduction

Despite the existence of a Lax pair often being used as the definition of integrability for a given equation [101], there have been few studies that sought to find or categorize nonlinear equations that used Lax pairs as their starting point. Of those studies that did begin with Lax pairs, most chose a form of the Lax pair *a priori*, that is an assumption was made concerning the dependence of the linear systems on the spectral parameter, thus limiting the possible results.

Lax pairs can appear in many guises, the type that we are exclusively concerned with in this chapter consist of a pair of linear problems written:

$$\begin{aligned}\theta(l+1, m) &= L(l, m)\theta(l, m) \\ \theta(l, m+1) &= M(l, m)\theta(l, m).\end{aligned}\tag{3.1}$$

where $\theta(l, m)$ is a two-component vector and $L(l, m)$ and $M(l, m)$ are 2×2 matrices. These linear problems are described by L and M , which are referred to as the Lax

matrices. The easily derived compatibility condition on this Lax pair is

$$L(l, m + 1)M(l, m) = M(l + 1, m)L(l, m)$$

and it is through this compatibility condition that we arrive at the integrable non-linear equation associated with the Lax pair.

The present chapter is focussed wholly on Lax pairs that are 2×2 , where each entry of the Lax matrices contains only one separable term. The Lax pairs are otherwise general in that no assumptions are made as to the explicit dependence of any quantities within the Lax matrices on the lattice variables, l and m , or on the spectral variable n . By one separable term we mean that each entry contains a term that can be split into a product of two parts, one that depends on the lattice variables, and another that depends only on the spectral variable. For example, in the 11 entry of the L matrix, we write the separable term $a(l, m)A(n)$. Both a and A may contain multiple terms themselves, say $a = \sum_i a_i$, $A = \sum_j A_j$, however, all terms within a must multiply all those within A , the other entries their own similar term. So, we could have an L matrix

$$L = \begin{pmatrix} [a_1(l, m) + \dots a_M(l, m)][A_1(n) + \dots + A_N(n)] & b(l, m)B(n) \\ c(l, m)C(n) & d(l, m)D(n) \end{pmatrix}$$

where we have written the 11 entry in the expanded form and left the other entries abbreviated to save space, no other terms can be added into any entry. M must also contain just one separable term in each entry, although these terms are independent of those in L .

The reason for limiting the Lax pairs to those that are 2×2 with one separable term in each entry is two fold. Firstly, Lax pairs with more terms typically lead to equations of higher order, as can be seen from hierarchies of equations with Lax pairs [3, 99], therefore we constrain our study to the lower order equations by limiting the number of terms. Secondly, we limit the number of terms present in the compatibility condition and thus render it less complicated to examine all

of the combinations of terms that can arise there. A combination of terms in the compatibility condition defines a system of equations that we subsequently solve, in a manner that preserves its full generality, up to a point where a nonlinear evolution equation is apparent, or it has been shown that the system cannot be associated with a nonlinear equation. Testing all combinations of terms, we thereby survey the complete set of Lax pairs of the type described.

In fact, of all the potential Lax pairs identified by this method, only two lead to interesting evolution equations. These are higher order varieties of the lattice sine-Gordon (LSG) and the LMKdV equations, which can be found in section 3.1.1. The remaining systems are shown to be trivial, overdetermined or underdetermined. This finding adds impetus to the suggested connection between the singularity confinement method and the existence of a Lax pair made in [102].

As we do not make any assumptions about the explicit dependence on the spectral parameter, we show that a particular nonlinear equation may have many Lax pairs, all depending on the spectral parameter in different ways. The effect that this freedom has on the process of inverse scattering is, as yet, unclear.

This chapter is organized as follows: section 3.1.1 presents the major results, those being the higher order versions of LMKdV and LSG, as well as a statement of the completeness theorem. The method of identifying and analyzing the viable Lax pairs is laid out in section 3.2, where a representative list of all the Lax pairs identified can be found. Section 3.3 explains how the higher order LSG and LMKdV equations are derived from the general form of their Lax pairs and section 3.4 provides examples that describe why most Lax pairs found in section 3.2 lead to trivial systems. A discussion section rounds out the chapter.

3.1.1 Results

Note that all difference equations in the remainder of this chapter will use the notation

$$\bar{x} = x(l+1, m),$$

$$\hat{x} = x(l, m+1).$$

As one of the two main results of this chapter, we present two new integrable nonlinear partial difference equations. The first equation,

$$\text{LSG}_2: \quad \frac{\rho}{\sigma} \frac{\hat{x}}{x} + \lambda_1 \mu_1 \hat{x} \hat{y} = \frac{\sigma}{\rho} \frac{\bar{x}}{\bar{x}} + \frac{\lambda_2 \mu_2}{x \bar{y}} \quad (3.2a)$$

$$\frac{\sigma}{\rho} \frac{\hat{y}}{\hat{y}} + \frac{\lambda_2 \mu_2}{\hat{x} y} = \frac{\rho}{\sigma} \frac{\bar{y}}{y} + \lambda_1 \mu_1 \bar{x} \hat{y} \quad (3.2b)$$

is referred to as LSG_2 because it is second order in each of the lattice dimensions, and because setting $x = y$ returns the familiar LSG equation, (3.3) below, in a non-autonomous form. Here $x = x(l, m)$ and $y = y(l, m)$ are the dependent variables, $\lambda_i = \lambda_i(l)$ and $\mu_i = \mu_i(m)$ are functions of the lattice variables that play the same role as parameters in autonomous PDEs, as are $\rho = \lambda_3^{(-1)^m}$ and $\sigma = \mu_3^{(-1)^l}$.

$$x \hat{x} = \frac{\lambda_2 \mu_2 - \frac{\rho}{\sigma} \bar{x} \hat{x}}{\lambda_1 \mu_1 \bar{x} \hat{x} - \frac{\sigma}{\rho}} \quad (3.3)$$

Similarly, the equation

$$\text{LMKdV}_2: \quad \frac{\lambda_1}{\sigma} \frac{\hat{x}}{\hat{x}} + \frac{\mu_2}{\rho} \frac{y}{\hat{y}} = \lambda_2 \sigma \frac{y}{\bar{y}} + \rho \mu_1 \frac{\hat{x}}{\bar{x}} \quad (3.4a)$$

$$\rho \mu_1 \hat{x} \hat{y} + \lambda_2 \sigma x \hat{y} = \frac{\mu_2}{\rho} x \bar{y} + \frac{\lambda_1}{\sigma} \bar{x} \hat{y} \quad (3.4b)$$

where the terms are as for LSG_2 , is referred to as LMKdV_2 , again because setting $x = y$ brings about LMKdV as in equation (3.5) below. Note that the version of LMKdV so attained is of a more general form than the most common variety listed in chapter 2, equation (2.33).

$$\hat{x} = x \frac{\lambda_2 \sigma \hat{x} - \frac{\mu_2}{\rho} \bar{x}}{\frac{\lambda_1}{\sigma} \bar{x} - \mu_1 \rho \hat{x}} \quad (3.5)$$

The following definition is introduced to clarify Theorem 3.1 below:

Definition 1 A *separable* term is one that can be written as a product of quantities, one quantity that depends solely on the lattice variables and another that depends solely on the spectral variable.

Note, as described in the introduction to this chapter, each of these two quantities that make up a separable term can possibly be expressed as a sum, provided that all parts of the sum depending on the lattice variables multiply all those depending on the spectral variable.

The second main result is the following theorem:

Theorem 3.1 *The system of equations that arise via the compatibility condition of any 2×2 Lax pair (3.1) with one nonzero, separable term in each entry of each matrix is either trivial, underdetermined, overdetermined, or can be reduced to one of LSG_2 or $LMKdV_2$.*

The proof of Theorem 3.1 lies in considering all of the possible sets of equations that can arise from the compatibility condition of such Lax pairs, and solving those sets of equations in a way that retains their full freedom. This proof occupies the remainder of the chapter.

The Lax pairs associated with LSG_2 and $LMKdV_2$ respectively are listed below, these Lax pairs are derived in section 3.3. The Lax pair for LSG_2 is

$$L = \begin{pmatrix} F_1/\rho & F_2\lambda_1\bar{x} \\ F_2\lambda_2/x & F_1\rho\bar{y}/y \end{pmatrix} \quad (3.6a)$$

$$M = \begin{pmatrix} F_2\hat{x}/(\sigma x) & F_1/y \\ F_1\mu_1\hat{y} & F_2\sigma \end{pmatrix} \quad (3.6b)$$

While the Lax pair for LMKdV₂ is

$$L = \begin{pmatrix} F_1 \lambda_1 \bar{x}/x & F_2 \rho/y \\ F_2 \bar{y}/\rho & F_1 \lambda_2 \end{pmatrix} \quad (3.7a)$$

$$M = \begin{pmatrix} F_1 \mu_1 \hat{x}/x & F_2 \bar{y}/\sigma \\ F_2 \sigma \bar{y} & F_1 \mu_2 \end{pmatrix} \quad (3.7b)$$

where $F_i = F_i(n)$ are arbitrary functions of the spectral variable n with the condition that $F_1 \neq kF_2$, where k is a constant.

3.2 Method of identifying potential Lax pairs

From Lax pairs of the type we consider here, the compatibility condition produces a set of equations, each equation being due to one of the linearly independent spectral terms that arises in some entry. Studies that have searched for integrable systems by beginning with a Lax pair, whether an isospectral or isomonodromy Lax pair or otherwise, typically assume some dependence of the Lax pair on the spectral parameter, then solve the compatibility condition for the evolution equation [66, 63, 103, 46]. Most often a polynomial or rational dependence on the spectral variable is used [104], but any type of explicit dependence could be investigated, for example Weierstrass elliptic functions.

Example 1 Consider the following L and M matrices

$$L = \begin{pmatrix} a\wp' & 4b\wp \\ c & d\wp' \end{pmatrix}$$

$$M = \begin{pmatrix} \alpha(\wp^2 + 1) & \beta\wp' \\ \gamma\wp' & \delta\wp \end{pmatrix}$$

where \wp is the Weierstrass elliptic function in the spectral parameter n only, and all other quantities are functions of both the lattice variables l and m . From the

compatibility condition, $\widehat{L}M = \overline{M}L$, and noting that $\wp'^2 = \frac{1}{4}\wp^3 - g_2\wp - g_3$ where $g_i = \text{constant}$, we find the following equations in the 12 entry

$$\begin{aligned}
\boxed{12} \quad \wp^3 : \quad \hat{a}\beta &= b\bar{\alpha} + d\bar{\beta} \\
\wp^2 : \quad \hat{b}\delta &= 0 \\
\wp : \quad \hat{a}\beta &= d\bar{\beta} - \frac{b\bar{\alpha}}{4g_2} \\
\wp^0 : \quad \hat{a}\beta &= d\bar{\beta}
\end{aligned} \tag{3.8}$$

where we have separated out the equations coming from different orders of the spectral parameter.

Clearly, this choice of L and M does not yield an interesting evolution equation through their compatibility, this example was instead chosen because it illustrates an important point. Multiplying together two functions of the spectral parameter can produce numerous orders that may or may not be proportional to other spectral term products in the compatibility condition. From example 1, the spectral term multiplying \wp^2 did not ‘match up’ with other terms in the 12 entry at that order, which brought about the equation $\hat{b}\delta = 0$, forcing some term to be zero. However, the other spectral terms did turn out more meaningful equations, highlighting the need to choose the dependence on the spectral parameter carefully so that none of the resulting equations force any lattice terms to be zero.

The inclusion of a zero lattice term does not preclude the existence of an interesting evolution equation. However, we are essentially classifying Lax pairs by the number of terms in their entries and the class of Lax pairs presently under inspection contains one separable term in each entry of their 2×2 matrices. If any of those terms were forced to be zero then the resulting Lax pair would actually come under a different category in the present framework.

The most general form of a 2×2 Lax pair with exactly one separable term in each entry of the L and M matrices is:

$$L = \begin{pmatrix} aA & bB \\ cC & dD \end{pmatrix}$$

$$M = \begin{pmatrix} \alpha\Lambda & \beta\Xi \\ \gamma\Gamma & \delta\Delta \end{pmatrix}$$

Where lower cases represent lattice terms and upper cases represent spectral terms. The compatibility condition is $\widehat{L}M = \overline{M}L$, of which we initially concentrate on the 12 entry.

$$\hat{a}\beta A\Xi + \hat{b}\delta B\Delta = b\bar{\alpha}B\Lambda + d\bar{\beta}D\Xi \quad (3.9)$$

At this stage we are only concerned with the various linearly independent spectral terms that appear. These will determine the set of equations that come out of the compatibility condition, which are subsequently solved to find the corresponding evolution equation. That being the case, there are four quantities to contend with in this entry, $A\Xi$, $B\Delta$, $B\Lambda$ and $D\Xi$. Any of these four products can lead to multiple, linearly independent spectral terms, all of which must match up with at least one other spectral term from one of the three remaining products in this entry. If there exists some spectral term that does not match up with a spectral term from another product, then the lattice term that multiplies it will have to be zero, which is forbidden.

Let us label the terms in each product as follows $A\Xi = \sum_i F_{A\Xi_i}$, $B\Delta = \sum_i F_{B\Delta_i}$, etc. All the spectral terms that occur in the 12 entry of the compatibility condition can be sorted into four groups according to the lattice terms that they multiply.

$$\boxed{12} \quad \begin{array}{cc|cc} (\hat{a}\beta) & (\hat{b}\delta) & (b\bar{\alpha}) & (d\bar{\beta}) \\ F_{A\Xi_1} & F_{B\Delta_1} & F_{B\Lambda_1} & F_{D\Xi_1} \\ F_{A\Xi_2} & F_{B\Delta_2} & F_{B\Lambda_2} & F_{D\Xi_2} \\ \vdots & \vdots & \vdots & \vdots \end{array} \quad (3.10)$$

Where the line separating the four groups marks the position of the equals sign in the associated lattice term equations. Organizing the terms from example 1 in this way leads to

$$\boxed{12} \quad \begin{array}{cc|cc}
(\hat{a}\beta) & (\hat{b}\delta) & (b\bar{\alpha}) & (d\bar{\beta}) \\
\wp^3 & \wp^2 & \wp^3 & \wp^3 \\
\wp & & \wp & \wp \\
\wp^0 & & & \wp^0
\end{array}$$

The present study will utilize the word ‘group’ in the general English sense, our groups refer to collections of spectral terms that multiply the same lattice terms in an entry of the compatibility condition. We have already seen that all terms in all groups must be proportional to a term from at least one of the other three groups in this entry. Conversely, where there are spectral terms that are proportional to others from other groups, an equation relating the corresponding lattice terms will thus be defined. Still with the above example, that Lax pair has a dependence on the spectral parameter such that the groups multiplying $\hat{a}\beta$, $b\bar{\alpha}$ and $d\bar{\beta}$ all contain the spectral term \wp^3 , for which the corresponding equation in the lattice terms is $\hat{a}\beta = b\bar{\alpha} + d\bar{\beta}$.

At this stage, the number of possible sets of proportional spectral terms, and therefore the number of possible combinations of equations yielded by the compatibility condition, is unmanageably large. We require further considerations to bring the problem under control.

3.2.1 Links and equivalent equations at different orders

The word *proportional* is used in this chapter in the sense that two terms F_1 and F_2 are proportional if $F_1 = kF_2$ for some finite constant k .

Definition 2 A *link* is a set of proportional spectral terms in the same matrix entry of the compatibility condition. A set of two proportional terms is a single link, three

terms a double link, and four terms a triple link.

Naturally, the spectral terms that comprise a link must each reside in a different group of spectral terms within an entry. If there are two or more spectral terms that are proportional to one another within the same group, they are simply added together to make one term. Since each group of spectral terms multiplies the same lattice term, one may speak of either links between the groups of spectral terms or links between lattice terms, with the same meaning. The above definition captures the idea that the entries of the compatibility condition give rise to different lattice term equations at different orders in the spectral term, without appealing to powers of some basic function.

With the employment of proportional terms comes the possibility of constants of proportionality, which we begin to deal with here.

Fact 1 *If there exist two distinct single links between the same two groups of spectral terms, then the corresponding constants of proportionality must be equal.*

The proof of this fact is elementary: say that one single link is formed by allowing $F_{A\Xi_1} \propto F_{D\Xi_1}$, where neither of these terms is proportional to any other spectral term that arises in the 12 entry, and the other single link corresponds to $F_{A\Xi_2} \propto F_{D\Xi_2}$, where again these terms link with no others. By including some constants of proportionality, k_1 and k_2 respectively, we can write down the equations that correspond to these two single links, those being $\hat{a}\beta = k_1 d\bar{\beta}$ and $\hat{a}\beta = k_2 d\bar{\beta}$. Since both equations must hold, and none of the lattice terms can be zero, it is clear that we must have $k_1 = k_2$.

This simple fact proves to be rather important because it ensures that all links between the same two groups can be bundled together. Further, all the spectral terms, in some group, that correspond to those single links with one other group, can be treated as a single spectral term. So, where $F_{A\Xi_1}$ and $F_{D\Xi_1}$ formed one single

link and $F_{A\Xi_2}$ and $F_{D\Xi_2}$ formed another between the same two groups, we can lump together $F_{A\Xi_1} + F_{A\Xi_2} = G_1$ and be sure that it links with $F_{D\Xi_1} + F_{D\Xi_2} = kG_1$.

Still with the 12 entry, if there exist multiple double links between the same three lattice terms, $\hat{a}\beta$, $b\bar{\alpha}$ and $d\bar{\beta}$ say, then the lattice term equations that result from those links can be written

$$K \begin{pmatrix} \hat{a}\beta \\ b\bar{\alpha} \\ d\bar{\beta} \end{pmatrix} = 0 \quad (3.11)$$

Where K is a matrix of the constants of proportionality between the various spectral terms, normalized so that each entry of the first column of K is unity. Each row of K corresponds to a double link. If K is such that equation (3.11) is overdetermined or uniquely solvable, then any Lax pair possessing the corresponding links is inconsistent or contains a zero lattice term. Therefore, we need not consider more than two double links between the same three spectral terms, although there may be multiplicity within those two double links. By the same argument we can allow a maximum of three different triple links between the same four lattice terms in an entry of the compatibility condition.

3.2.2 Link symbolism

The abundance of Lax pairs that need to be checked necessitates the introduction of a shorthand, which will be based on their links. The off-diagonal entries both contain four groups of spectral terms, each of which multiplies a single product of lattice terms. The 12 entry contains the spectral term products $A\Xi$, $B\Delta$, $B\Lambda$, and $D\Xi$ associated with the lattice term products $\hat{a}\beta$, $\hat{b}\delta$, $b\bar{\alpha}$ and $d\bar{\beta}$ respectively. For the shorthand, we always set out the spectral terms in the same way on the page

$$\begin{array}{cc} A\Xi & B\Lambda \\ B\Delta & D\Xi \end{array}$$

Each link can be represented by lines between the quantities that are proportional to each other, and so we will use the symbols listed in table 3.1 to represent the combinations of links in the 12 entry, where $F_{A\Xi}$ is some term from the group of spectral terms formed by taking the product $A\Xi$, and other terms are similarly labeled.

Symbol	Links
\times	$F_{A\Xi} \propto F_{D\Xi}, F_{B\Delta} \propto F_{B\Lambda}$
$ $	$F_{A\Xi} \propto F_{B\Delta}, F_{B\Lambda} \propto F_{D\Xi}$
$—$	$F_{A\Xi} \propto F_{B\Lambda}, F_{B\Delta} \propto F_{D\Xi}$
\square	$F_{A\Xi} \propto F_{D\Xi} \propto F_{B\Delta} \propto F_{B\Lambda}$
$\nabla $	$F_{A\Xi} \propto F_{B\Delta} \propto F_{B\Lambda_i}, F_{B\Lambda_j} \propto F_{D\Xi}$
\vdots	\vdots

Table 3.1: Symbols used to represent the link combinations in the off-diagonal entries. Note that $F_{B\Lambda_i} \neq kF_{B\Lambda_j}$, k a constant

The 21 entry is similar to the 12 entry in that it possesses four distinct products of spectral terms. The same symbols listed in table 3.1 are used again for the 21 entry, with clear meaning given that the spectral products are set out as follows

$$\begin{array}{cc} C\Lambda & A\Gamma \\ D\Gamma & C\Delta \end{array}$$

However, the diagonal entries are slightly different because each diagonal entry contains a product that occurs twice: $A\Lambda$ occurs twice in the 11 entry and $D\Delta$ twice in 22. This automatically causes the associated lattice terms to be paired in their respective entries and, as such, $A\Lambda$ and $D\Delta$ need not be linked with another spectral term to prevent a zero lattice term. This being the case, there are really only three spectral term products to consider in both of the diagonal entries, one of which need not be linked to the other two, and our symbols reflect that. Positioning

the spectral term products as follows

$$A\Lambda \quad \begin{matrix} B\Gamma \\ C\Xi \end{matrix}$$

We symbolize the links as indicated in table 3.2, where the symbol ‘ \dashv ’ is used to represent the repeated spectral term products.

Symbol	Links
$\dashv\langle$	$F_{A\Lambda} \propto F_{B\Gamma} \propto F_{C\Xi}$
$\dashv\rangle$	$F_{A\Lambda}$ alone, $F_{B\Gamma} \propto F_{C\Xi}$
$\dashv\langle$	$F_{A\Lambda_i} \propto F_{B\Gamma}$ and $F_{A\Lambda_j} \propto F_{C\Xi}$ separately
\vdots	\vdots

Table 3.2: Symbols used to represent the link combinations in the diagonal entries of the compatibility condition. Note that $F_{A\Lambda_i} \neq kF_{A\Lambda_j}$, k a constant

3.2.3 Which Lax pairs need to be checked?

All link combinations that do not force any lattice terms to be zero are checked systematically. The procedure for doing this runs as follows:

1. Begin with the 12 entry, assume that only single links exist there, and list all single link combinations that produce different lattice term equations.
2. Construct the links found in the previous step by choosing proportional sets of spectral terms in the appropriate groups.
3. Move to the 21 entry, noting the spectral term constructions from the previous step, and identify all the viable link combinations in this entry.
4. Repeat the previous step in the diagonal entries.

After identifying all link combinations with single links in the 12 entry, we repeat the entire process assuming that double and possible single links exist there, and then repeat once more with triple, and possibly double and single links in the 12 entry. Finally, the corresponding set of lattice term equations for each Lax pair must be analyzed to find the resulting evolution equation. This analysis needs to be conducted in a manner that preserves the full freedom of the system, as described in section 3.3. Note that this final step of the procedure is not conducted in the present section, here we only identify the Lax pairs that need to be analyzed, we leave the analysis itself to sections 3.3 and 3.4.

Single links in the 12 entry

The four groups of spectral terms in the 12 entry are

$$\begin{array}{cc} A\Xi & B\Lambda \\ B\Delta & D\Xi \end{array}$$

Each group must be linked to another so, using single links, the group $A\Xi$ must be linked to at least one of the three other groups, $B\Lambda$, $B\Delta$, or $D\Xi$. For arguments sake, say that there exists a single link between $A\Xi$ and $B\Lambda$.

$$\begin{array}{cc} A\Xi & B\Lambda \\ \text{---} & \\ & \dots \\ B\Delta & D\Xi \end{array}$$

That leaves both groups $B\Delta$ and $D\Xi$ requiring links and, since we are only concerned with single links at the moment, these two groups of spectral terms can be linked to each other, or to one of $A\Xi$ or $B\Lambda$, in distinct, single links.

At this point, to make the number of Lax pairs that are to be scrutinized later more manageable, we segregate a certain class of Lax pairs with single links. It can be shown that if any group in either off-diagonal entry possesses single links between it and two other groups, the resulting Lax pair is associated with a trivial evolution equation (see Proposition 3.1 below).

Proposition 3.1 *If there exist two (or more) single links between some lattice term and two others in an off-diagonal entry, then the resulting evolution equation is trivial.*

Proof 3.1 *The proof of Proposition 3.1 lies in checking all the possible link combinations that meet the criterion. This is not written down here but can be done in a way that is analogous to the analysis of the other link types.*

Proposition 3.1 is used to exclude a class of Lax pairs from explicit analysis, we do not take it any further here. There is nothing inherently special about this class of Lax pair, it is only treated separately because it contains many cases that are similar to others studied below, so including these would unnecessarily lengthen the argument, see table 3.3 for examples of the types of links combinations excluded by Proposition 3.1.

$$\boxed{12} \quad \begin{array}{cc} A\Xi & B\Lambda \\ B\Delta & D\Xi \end{array} \quad \begin{array}{c} \times \\ \sqcap \\ \sphericalangle \\ \dots \end{array}$$

Table 3.3: Proposition 3.1 excludes Lax pairs that have two distinct single links between some entry and two others in 12 entry

Hence, without considering the link combinations excluded by Proposition 3.1 and given the first link between $A\Xi$ and $B\Lambda$, the only other single link that needs to be considered is between $B\Delta$ and $D\Xi$. A similar argument holds when we choose a single link between $A\Xi$ and $B\Delta$ or between $A\Xi$ and $D\Xi$, therefore, table 3.4 lists the only single link combinations in the 12 entry that require further analysis.

We shall proceed with the analysis under the assumption that single links between $A\Xi$ and $B\Lambda$, and between $B\Delta$ and $D\Xi$, although the other combinations can be dealt with in the same way. The specified links are constructed by choosing

$$\boxed{12} \quad \begin{array}{cc|cc} A\Xi & B\Lambda & = & || \times \\ B\Delta & D\Xi & & \end{array}$$

Table 3.4: Single link combinations in the 12 entry

spectral terms

$$A = \frac{1}{\Xi}(F_1 + \dots) \quad (3.12a)$$

$$D = \frac{1}{\Xi}(F_2 + \dots) \quad (3.12b)$$

$$\Lambda = \frac{1}{B}(F_1 + \dots) \quad (3.12c)$$

$$\Delta = \frac{1}{B}(F_2 + \dots) \quad (3.12d)$$

where $F_i = F_i(n)$ and $F_1 \neq kF_2$, k a constant. It is understood that while there is room for other spectral terms in the expression $A = \frac{1}{\Xi}(F_1 + \dots)$, there cannot be a term proportional to F_2/Ξ in A , as this would cause the single link corresponding to F_2 to become a double link. In this way the desired links are constructed, plus we have allowed for additional links should they be appropriate or required later. Readers may note the omission of any constants of proportionality in the above, however, the constants that could have been written at this point can all be absorbed into the lattice terms that they multiply.

Turning our attention to the 21 entry of the compatibility condition, $\hat{c}\alpha C\Lambda + \hat{d}\gamma D\Gamma = a\bar{\gamma}A\Gamma + c\bar{\delta}C\Delta$, the following spectral terms appear

$$\boxed{21} \quad \begin{array}{cc|cc} (\hat{c}\alpha) & (\hat{d}\gamma) & | & (a\bar{\gamma}) & (c\bar{\delta}) \\ \frac{C}{B}F_1 & \frac{\Gamma}{\Xi}F_2 & | & \frac{\Gamma}{\Xi}F_1 & \frac{C}{B}F_2 \\ \vdots & \vdots & | & \vdots & \vdots \end{array} \quad (3.13)$$

Since every spectral term must be proportional to another in the same entry, $\frac{C}{B}F_1$ must be proportional to $\frac{\Gamma}{\Xi}F_1$ or $\frac{\Gamma}{\Xi}F_2$. Clearly, $\frac{C}{B}F_1$ cannot be proportional to $\frac{C}{B}F_2$, nor can it link with some other term that we are yet to define, as this would introduce single links between some lattice term and two others, thus leading

to a trivial evolution equation by Proposition 3.1. Moreover, we can exclude the case with $\frac{C}{B}F_1 \propto \frac{\Gamma}{\Xi}F_2$ because this also requires the remaining spectral terms to be proportional to one another, *i.e.* $\frac{C}{B}F_2 \propto \frac{\Gamma}{\Xi}F_1$. These two conditions on the spectral terms imply that $F_1 \propto F_2$, contradicting a previous assumption. Hence, the links chosen in the 12 entry leave only one choice for the links in the 21 entry of the compatibility condition: $\frac{C}{B}F_1 \propto \frac{\Gamma}{\Xi}F_1$ and $\frac{C}{B}F_2 \propto \frac{\Gamma}{\Xi}F_2$, which can be written more succinctly as $C\Xi \propto B\Gamma$

$$\begin{array}{ccc} \boxed{12} & A\Xi & B\Lambda \\ & \text{---} & \\ & \text{---} & \\ & B\Delta & D\Xi \end{array} \Rightarrow \begin{array}{ccc} \boxed{21} & C\Lambda & A\Gamma \\ & \text{---} & \\ & \text{---} & \\ & D\Gamma & C\Delta \end{array}$$

The terms arising in the diagonal entries are listed below

$$\begin{array}{ccc} \boxed{11} & & | \\ & (\hat{a}\alpha - a\bar{\alpha}) & F_1^2/(B\Xi) & | & (c\bar{\beta} - \hat{b}\gamma) & B\Gamma \\ & & & & & | \\ & & & & & | \\ & & & & & | \\ & & & & & | \end{array} \tag{3.14}$$

$$\begin{array}{ccc} \boxed{22} & & | \\ & (\hat{d}\delta - d\bar{\delta}) & F_2^2/(B\Xi) & | & (b\bar{\gamma} - \hat{c}\beta) & B\Gamma \\ & & & & & | \\ & & & & & | \end{array}$$

Note that the terms in equation (3.14) are set out slightly differently to those in table 3.2 because we have already determined that $B\Gamma \propto C\Xi$.

The spectral terms in the diagonal entries, in this case, do not necessarily have to link with others because they are multiplied by more than one lattice term, as indicated in equation (3.14) where the lattice terms appear in parentheses to the left of the spectral terms they multiply. A lone spectral term in the diagonal entries, given the links already constructed in the off-diagonal entries, will not bring about a zero lattice term. Consequently, there exist three possible link combinations for the diagonal entries: $B\Gamma \propto F_1^2/(B\Xi)$, $B\Gamma \propto F_2^2/(B\Xi)$ or $B\Gamma$ is proportional to

neither $F_2^2/(B\Xi)$ nor $F_1^2/(B\Xi)$. All three choices lead to trivial evolution equations and we shall continue the analysis under the assumption that $B\Gamma \propto F_1^2/(B\Xi)$. A link combination has now been chosen in each of the entries of the compatibility condition, these are shown in table 3.5

$\boxed{12}$	$\boxed{21}$	$\boxed{11}$	$\boxed{22}$
$A\Xi \quad B\Lambda$	$C\Lambda \quad A\Gamma$	$B\Gamma$ $A\Lambda$	$B\Gamma$ $D\Delta$
$B\Delta \quad D\Xi$	$D\Gamma \quad C\Delta$	$C\Xi$	$C\Xi$
--- ---	--- ---	$\text{---} <$	$\text{---} <$

Table 3.5: An example of the links that define a Lax pair

Gauge transformations can be used to remove the dependence on some of the spectral terms and, as such, we expect some redundancy. The links constructed above are achieved by setting the values of the spectral terms to

$$\begin{aligned}
 A &= F_1, & \Lambda &= F_1 \\
 B &= 1, & \Xi &= 1 \\
 C &= F_1^2, & \Gamma &= F_1^2 \\
 D &= F_2, & \Delta &= F_2
 \end{aligned}$$

where F_1 and F_2 are any functions of the spectral variable n , such that $F_1 \neq kF_2$, $k = \text{constant}$. Note that one of F_1 or F_2 may itself be a constant. No constants of proportionality are required as these can be absorbed into lattice terms in this case. The same links are reproduced by any suite of spectral terms that satisfies the

following conditions, brought about by the links described above

$$\begin{aligned}
A &= F_1/\Xi, & \Lambda &= F_1/B \\
C &= F_1^2/\Xi, & \Gamma &= F_1^2/B \\
D &= F_2/\Xi, & \Delta &= F_2/B
\end{aligned}$$

The resulting set of equations that are produced by the compatibility condition for this Lax pair are written in (3.15), although they can be read from the links given in table 3.5.

$$\begin{aligned}
\hat{a}\alpha - a\bar{\alpha} &= c\bar{\beta} - \hat{b}\gamma \\
\hat{d}\delta - d\bar{\delta} &= 0 \\
b\bar{\gamma} - \hat{c}\beta &= 0 \\
\hat{a}\beta &= b\bar{\alpha} \\
\hat{b}\delta &= d\bar{\beta} \\
\hat{c}\alpha &= a\bar{\gamma} \\
\hat{d}\gamma &= c\bar{\delta}
\end{aligned} \tag{3.15}$$

This rounds the description of the Lax pairs with only single links in the off-diagonal entries, in practice one would continue by analyzing equations (3.15) to find the associated evolution equation, which in this case is trivial (for more on the analysis systems of equations leading to a trivial evolution equation see section 3.4.1). On the possibility of including extra spectral terms to augment the links used here, see equation (3.12), we note that fewer terms allow greater freedom and that any additional links could only lead to a more constrained system, one that certainly could not sustain an interesting evolution equation considering that the less constrained example here leads to a trivial result. Also, the alternative links that cause there to be one equation in the 22 entry and two equations in the 11 entry of the compatibility condition, see after equation (3.14), lead to the same evolution equations found here as the Lax pair is symmetric in that sense.

Double links in the 12 entry

Here link combinations that consist of double and possibly single links in the 12 entry are investigated. There are more possibilities in this class than when only considering single links, however the number is reduced by noting that some sets of link combinations are equivalent from the perspective of the lattice term equations. For example the following pair of link combinations in the 12 entry are clearly equivalent when considered as proportionality statements.

$$\boxed{12} \quad \begin{array}{cc} A\Xi & B\Lambda \\ B\Delta & D\Xi \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \leftrightarrow \begin{array}{c} \diagdown \\ \diagup \end{array}$$

$$\begin{array}{l} A\Xi + k_1 B\Delta = k_2 B\Lambda \\ B\Delta = k_3 D\Xi \end{array} \leftrightarrow \begin{array}{l} A\Xi = k_4 B\Lambda + k_5 D\Xi \\ B\Delta = k_3 D\Xi \end{array} \quad (3.16)$$

Also, all link combinations that possess two double links in the 12 entry are nearly equivalent, enough to consider them all together, what is meant by ‘nearly equivalent’ is clarified in the next two sentences. The equivalence is because the lattice term equations corresponding to any two double links in this entry can be manipulated so that they are the same as those from any other combination of two double links. The difference that may arise comes from the diagonal entries, where the particular pair of double links chosen in the 12 entry can affect the variety of link combinations possible. However, the difference is not sufficient to alter the overall outcome that these systems are overdetermined. The complete list of double link combinations in the 12 entry that need to be analyzed is shown in table 3.6

$$\boxed{12} \quad \begin{array}{cc} A\Xi & B\Lambda \\ B\Delta & D\Xi \end{array} \quad \begin{array}{cccccccc} \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \end{array}$$

Table 3.6: All double link combinations in the 12 entry that require analysis

To exemplify the method of construction of Lax pairs with double links in the 12 entry, we choose link combination in table 3.6 where there is a double link between $A\Xi$, $B\Delta$ and $B\Lambda$ and a single link between $B\Delta$ and $D\Xi$. Spectral terms that generate this choice of links are given in equation (3.17).

$$\begin{aligned} A &= F_1/\Xi, & \Lambda &= F_1/B \\ \Delta &= (F_1 + kF_2)/B, & D &= F_2/\Xi \end{aligned} \quad (3.17)$$

where our usual nomenclature applies, *i.e.* $F_i = F_i(n)$, $k = \text{constant}$.

Using the expressions found in the 12 entry, the terms that arise in the 21 entry are F_1C/B , $F_2\Gamma/\Xi$, $F_1\Gamma/\Xi$ and F_2C/B , where F_1C/B arises twice. It is convenient to rearrange the terms found in the 21 entry into columns of terms that cannot be linked, this is done in equation (3.18).

$$\begin{array}{cc} F_1C/B & F_1\Gamma/\Xi \\ F_2\Gamma/\Xi & (F_1 + F_2)C/B \end{array} \Rightarrow \begin{array}{cc} F_1\Gamma/\Xi & F_1C/B \text{---} \\ F_2\Gamma/\Xi & F_2C/B \end{array} \quad (3.18)$$

As in section 3.2.2, the ‘---’ is used to symbolize a repeated spectral term product that does not necessarily have to be linked. $F_2\Gamma/\Xi$ must link with F_2C/B because linking with F_1C/B would lead to the contradictory $F_1 \propto F_2$, since that would also necessitate a link between F_2C/B and $F_1\Gamma/\Xi$. That leaves two possibilities: $B\Gamma \propto C\Xi$, or we can split $F_2C/B \propto (F_1 + F_2)\Gamma/\Xi$, noting that the multiplicity of the term F_1C/B means that it need not link to another in this entry. The second possibility is neglected, though, as it gives rise to a zero lattice term in one of the diagonal entries to be considered below. Thus, by a process of elimination, the links selected in the 12 entry leave only one choice for the 21 entry, which is shown in equation (3.19)

$$\begin{array}{ccc} \boxed{12} & A\Xi & B\Lambda \\ & \diagdown & \\ & B\Delta & D\Xi \end{array} \Rightarrow \begin{array}{ccc} \boxed{21} & C\Lambda & A\Gamma \\ & \diagdown & \\ & D\Gamma & C\Delta \end{array} \quad (3.19)$$

Move now to the 11 entry where the relevant spectral term products are $A\Lambda \propto F_1^2/(B\Xi)$, and $B\Gamma \propto C\Xi$, see equation (3.20). We choose to link these two spectral

terms to get one equation in the 11 entry and find that there are necessarily two equations in the 22 entry. Note that the opposite case where there are two equations in the 11 entry and one in the 22 entry can also exist by splitting up $B\Gamma$ into two terms, however this will lead to the same evolution equation.

$$\begin{array}{ccc|c}
 \boxed{11} & & & | \\
 (\hat{a}\alpha - a\bar{\alpha}) & F_1^2/(B\Xi) & & | (c\bar{\beta} - \hat{b}\gamma) B\Gamma \\
 & & & | \\
 & & & | \\
 \boxed{22} & F_1F_2/(B\Xi) & & | \\
 (\hat{d}\delta - d\bar{\delta}) & & & | (b\bar{\gamma} - \hat{c}\beta) B\Gamma \\
 & F_2^2/(B\Xi) & & |
 \end{array} \tag{3.20}$$

Using our symbolism, the links that define this Lax pair are shown in table 3.7.

$\boxed{12}$	$\boxed{21}$	$\boxed{11}$	$\boxed{22}$
$A\Xi \quad B\Lambda$	$C\Lambda \quad A\Gamma$	$B\Gamma$	$B\Gamma$
$B\Delta \quad D\Xi$	$D\Gamma \quad C\Delta$	$A\Lambda$	$D\Delta$
$C\Xi$	$C\Xi$	$C\Xi$	$C\Xi$
$\underline{\nearrow}$	$\underline{\searrow}$	\dashleftarrow	\dashrightarrow

Table 3.7: Links for a Lax pair with double links in the off-diagonal entries

The spectral term relations that must be satisfied to bring about the links in

table 3.7 are given in (3.21).

$$\begin{aligned}
A &= F_1, & \Lambda &= F_1 \\
B &= 1, & \Xi &= 1 \\
C &= F_1^2, & \Gamma &= F_1^2 \\
D &= F_2, & \Delta &= F_1 + kF_2
\end{aligned} \tag{3.21}$$

where F_1 and F_2 are any functions of the spectral variable n , such that $F_1 \neq kF_2$, $k = \text{constant}$. Notice that a constant appears in the expression for Δ in equation (3.21), since one of the constants of proportionality in this expression cannot be absorbed into the multiplying lattice term, δ .

The lattice term equations that arise via the compatibility condition from this Lax pair are shown in equation (3.22).

$$\begin{aligned}
\hat{a}\alpha - a\bar{\alpha} &= c\bar{\beta} - \hat{b}\gamma \\
\hat{d}\delta - d\bar{\delta} &= 0 \\
b\bar{\gamma} - \hat{c}\beta &= 0 \\
\hat{a}\beta + \hat{b}\delta &= b\bar{\alpha} \\
\hat{b}\delta &= kd\bar{\beta} \\
\hat{c}\alpha &= d\bar{\gamma} + c\bar{\delta} \\
\hat{a}\gamma &= kc\bar{\delta}
\end{aligned} \tag{3.22}$$

We thus conclude the description of the formation of Lax pairs with at least one double link in the 12 entry. Naturally there are other Lax pairs of this type but they are formed in a similar manner to that described here. A description of the analysis of the systems of equations produced by the compatibility of these Lax pairs is left until sections 3.3 and 3.4, although no Lax pairs of this type lead to unconstrained, nonlinear evolution equations.

A triple link in the 12 entry

Lastly, link combinations including triple, and possibly double and single links in the 12 entry must be considered. It is not difficult to see that the only possibility that needs to be investigated is that with one triple link between all four lattice terms in the 12 entry, any additional links in this entry constrain the problem too heavily and lead to trivial evolution equations only.

A triple link in the 12 entry of the compatibility condition can be formed by setting the spectral terms to those in equation (3.23) below.

$$\begin{array}{lcl}
 A = F_1/\Xi, & \Lambda = F_1/B & \Rightarrow \begin{array}{cc} \boxed{12} & A\Xi & B\Lambda \\ & & \square \\ & B\Delta & D\Xi \end{array} \\
 \Delta = F_1/B, & D = F_1/\Xi &
 \end{array} \tag{3.23}$$

Given the values in (3.23), the resulting spectral terms that appear in the 21 entry are as shown in equation (3.24).

$$\begin{array}{lcl}
 \boxed{21} & C\Lambda & A\Gamma & \Rightarrow & F_1C/B & F_1\Gamma/\Xi \\
 & D\Gamma & C\Delta & & F_1\Gamma/\Xi & F_1C/B
 \end{array} \tag{3.24}$$

We therefore have two possibilities in the 21 entry depending on whether or not $B\Gamma \propto C\Xi$. If $B\Gamma$ is proportional $C\Xi$ then there is another triple link in the 21 entry, otherwise there is a pair of single links. The latter case yields the links given in item 4 of table 3.8 and leads to a trivial evolution equation, while the former yields Lax pairs that include one for the LMKdV₂ system.

We continue the analysis here assuming $B\Gamma \propto C\Xi$, in this case the spectral terms

	$\boxed{12}$	$\boxed{21}$	$\boxed{11}$	$\boxed{22}$	Evolution Eqn
	$A\Xi \quad B\Lambda$ $B\Delta \quad D\Xi$	$C\Lambda \quad A\Gamma$ $D\Gamma \quad C\Delta$	$A\Lambda \quad B\Gamma$ $C\Xi$	$B\Gamma$ $D\Delta \quad C\Xi$	
1	\times	\times	\dashleftarrow	\dashleftarrow	LSG ₂ (3.2)
2	\square	\square	\dashrightarrow	\dashrightarrow	LMKdV ₂ (3.4)
3	\times	\times	\dashrightarrow	\dashrightarrow	Trivial
4	\times	\square	\rightleftharpoons	\rightleftharpoons	Trivial
5	\times	\times	\rightleftharpoons	\rightleftharpoons	Zero term
6	—	—	\dashleftarrow	\dashrightarrow	Trivial
7	$ $	$ $	\dashleftarrow	\dashleftarrow	Underdetermined
8	∇	∇	\rightleftharpoons	\dashleftarrow	Trivial
9	∇	∇	\dashleftarrow	\rightleftharpoons	Trivial
10	\boxtimes	\boxtimes	\rightleftharpoons	\dashleftarrow	Overdetermined
11	∇	∇	\rightleftharpoons	\dashleftarrow	Trivial
12	\boxtimes	\boxtimes	\rightleftharpoons	\dashleftarrow	Overdetermined
13	∇	∇	\rightleftharpoons	\dashleftarrow	Overdetermined
14	\boxtimes	\boxtimes	\rightleftharpoons	\dashleftarrow	Overdetermined
15	\square	\square	\dashleftarrow	\dashleftarrow	Special case

Table 3.8: List of link combinations used to construct possible Lax pairs

3.3 Derivation of higher order LSG and LMKdV systems

While fourteen potentially viable types of Lax pairs were identified in section 3.2, only two types lead to non-trivial, well determined evolution equations. The two systems thus found are LMKdV₂ and LSG₂ and this section describes the derivation of these systems from the general form of their Lax pairs. It is important to note that no freedom in the lattice terms is lost through this process, all values that the lattice

terms take are dictated by the sets of equations effectuated by the compatibility condition. As such we conclude that the systems so derived (or equivalent systems) are the most general ones that can be associated with their Lax pairs.

In fact, such calculations have been performed many times before and yet neither LSG_2 or $LMKdV_2$ appear to have been published, despite coming from Lax pairs with simple forms that have certainly already been considered previously [102, 105]. Hence, it is necessary to outline the method used to derive LSG_2 and $LMKdV_2$ in detail.

3.3.1 LSG_2

For LSG_2 the Lax pair used as a starting point is

$$L = \begin{pmatrix} F_1 a & F_2 b \\ F_2 c & F_1 d \end{pmatrix} \quad (3.26a)$$

$$M = \begin{pmatrix} F_2 \alpha & F_1 \beta \\ F_1 \gamma & F_2 \delta \end{pmatrix} \quad (3.26b)$$

Where $a, b, c, d, \alpha, \beta, \gamma$ and δ are all functions of both the lattice variables l and m , referred to as lattice terms, and F_1 and F_2 depend on the spectral variable only, we refer to these terms as spectral terms.

It is possible to remove some of the lattice terms using a gauge transformation. While this would reduce the complexity of the system, it is not clear at this point which of the lattice terms would best be removed. Experience shows that natural transformations present themselves in the course of solving the compatibility condition and so we shall wait until later to remove some lattice terms, keeping in mind that we expect some freedom to disappear from each of the Lax matrices L and M .

The compatibility condition for (3.26), $\widehat{L}M = \overline{M}L$, leads to the following system

of difference equations in the lattice terms

$$\hat{a}\alpha + \hat{b}\gamma = a\bar{\alpha} + c\bar{\beta} \quad (3.27a)$$

$$\hat{d}\delta + \hat{c}\beta = d\bar{\delta} + b\bar{\gamma} \quad (3.27b)$$

$$\hat{a}\beta = d\bar{\beta} \quad (3.27c)$$

$$\hat{b}\delta = b\bar{\alpha} \quad (3.27d)$$

$$\hat{c}\alpha = c\bar{\delta} \quad (3.27e)$$

$$\hat{d}\gamma = a\bar{\gamma} \quad (3.27f)$$

Some of equations (3.27) are linear and some nonlinear. The linear equations can be solved easily when in the form

$$k_1\hat{\phi} - k_2\phi = k_3\bar{\psi} - k_4\psi \quad (3.28)$$

where ϕ and ψ are lattice terms and k_i constants. Equation (3.28) implies

$$\begin{aligned} \psi &= k_1\hat{v} - k_2v + \mu\left(\frac{k_4}{k_3}\right)^l \\ \phi &= k_3\bar{v} - k_4v + \lambda\left(\frac{k_2}{k_1}\right)^m \end{aligned}$$

where we have introduced the new lattice term $v = v(l, m)$, and $\lambda = \lambda(l)$ and $\mu = \mu(m)$ are constants of integration. The same fact also applies in a multiplicative sense, in particular

$$\frac{\hat{\phi}}{\phi} = \frac{\bar{\psi}}{\psi} \quad \Rightarrow \quad \phi = \lambda\frac{\bar{v}}{v}, \quad \psi = \mu\frac{\hat{v}}{v}$$

We now proceed to solve the system (3.27). Equations (3.27c) to (3.27f) are linear and may be solved in pairs as follows. Multiply equations (3.27c) by (3.27f) to find $\hat{a}\hat{d}/(ad) = \bar{\beta}\bar{\gamma}/(\beta\gamma)$, which is integrated for

$$a = \lambda_0^2\bar{v}_5/(v_5d)$$

$$\beta = \mu_5\hat{v}_5/(v_5\gamma)$$

where we have introduced $\lambda_0 = \lambda_0(l)$, $\mu_5 = \mu_5(m)$ and $v_5 = v_5(l, m)$. Now use equation (3.27c) again to find $\bar{v}_5\bar{\gamma}/(v_5\gamma) = \frac{\hat{d}}{\lambda_0} \frac{d}{\lambda_0}$ and integrate for

$$d = \lambda_0 \rho \bar{v}_2 / v_2$$

$$\gamma = \mu_2 \hat{v}_2 v_2 / v_5$$

where $\rho = \lambda_3(l)^{(-1)^m}$. Substitute these values back into the expressions for a and β , and replace $v_5/v_2 \mapsto v_1$, and $\mu_5/\mu_2 \mapsto \mu_1$ resulting in

$$a = \frac{\lambda_0 \bar{v}_1}{\rho v_1} \quad (3.29a)$$

$$d = \lambda_0 \rho \frac{\bar{v}_2}{v_2} \quad (3.29b)$$

$$\beta = \mu_1 \frac{\hat{v}_1}{v_2} \quad (3.29c)$$

$$\gamma = \mu_2 \frac{\hat{v}_2}{v_1} \quad (3.29d)$$

Perform similar calculations on equations (3.27d) and (3.27e) to find

$$b = \lambda_1 \frac{\bar{v}_3}{v_4} \quad (3.30a)$$

$$c = \lambda_2 \frac{\bar{v}_4}{v_3} \quad (3.30b)$$

$$\alpha = \frac{\mu_0 \hat{v}_3}{\sigma v_3} \quad (3.30c)$$

$$\delta = \mu_0 \sigma \frac{\hat{v}_4}{v_4} \quad (3.30d)$$

Where $\sigma = \mu_3(m)^{(-1)^l}$.

When equations (3.29) and (3.30) are substituted into (3.27a) and (3.27b) we find the following equations respectively

$$\lambda_0 \rho \frac{\mu_0}{\sigma} \frac{\hat{v}_1 \hat{v}_3}{\hat{v}_1 v_3} + \lambda_1 \mu_1 \frac{\hat{v}_2 \hat{v}_3}{v_1 \hat{v}_4} = \frac{\lambda_0}{\rho} \mu_0 \sigma \frac{\bar{v}_1 \hat{v}_3}{v_1 \bar{v}_3} + \lambda_2 \mu_2 \frac{\hat{v}_1 \bar{v}_4}{\bar{v}_2 v_3} \quad (3.31a)$$

$$\frac{\lambda_0}{\rho} \mu_0 \sigma \frac{\hat{v}_2 \hat{v}_4}{\hat{v}_2 v_4} + \lambda_2 \mu_2 \frac{\hat{v}_4 \hat{v}_1}{\hat{v}_3 v_2} = \lambda_0 \rho \frac{\mu_0}{\sigma} \frac{\hat{v}_2 \hat{v}_4}{v_2 \bar{v}_4} + \lambda_1 \mu_1 \frac{\hat{v}_2 \bar{v}_3}{\bar{v}_1 v_4} \quad (3.31b)$$

Multiplying (3.31a) by v_1/\hat{v}_1 and (3.31b) by v_4/\hat{v}_4 indicates that certain variables always appear in combination. As such, we set $v_1, v_2 \equiv 1$ without loss of generality,

and rename $v_3 = x$ and $v_4 = y$. This is the manifest reduction in freedom that was expected from the perspective of gauge transformations. The parameter functions similarly appear in ratios, hence we set $\lambda_0 \equiv 1$ and $\mu_0 \equiv 1$ without loss of generality. Making the substitutions we arrive at a pair of nonlinear partial difference equations in x and y , with arbitrary non-autonomous terms $\lambda_i(l)$ and $\mu_i(m)$, which together form LSG₂, the evolution equation associated with the Lax pair.

$$\frac{\rho}{\sigma} \frac{\hat{x}}{x} + \lambda_1 \mu_1 \frac{\hat{x}\hat{y}}{x\bar{y}} = \frac{\sigma}{\rho} \frac{\hat{x}}{\bar{x}} + \frac{\lambda_2 \mu_2}{x\bar{y}} \quad (3.32a)$$

$$\frac{\sigma}{\rho} \frac{\hat{y}}{\hat{y}} + \frac{\lambda_2 \mu_2}{\hat{x}y} = \frac{\rho}{\sigma} \frac{\bar{y}}{y} + \lambda_1 \mu_1 \frac{\bar{x}\hat{y}}{x\bar{y}} \quad (3.32b)$$

This pair of equations can be thought of as a higher order lattice sine-Gordon system because the lattice sine-Gordon equation (LSG) is retrieved by setting $y = x$ in either expression

$$\text{LSG} : \quad \hat{x}x \left(\frac{\sigma}{\rho} - \lambda_1 \mu_1 \bar{x}\hat{x} \right) = \frac{\rho}{\sigma} \bar{x}\hat{x} - \lambda_2 \mu_2$$

The Lax pair (3.6) for LSG₂ is obtained by substituting the calculated values of the lattice terms back into (3.26).

3.3.2 LMKdV₂

The general form of the Lax pair for LMKdV₂ is similar to that for LSG₂, the only difference being that here both matrices exhibit the same dependence on the spectral variable, while the dependence was antisymmetric in the previous case.

$$L = \begin{pmatrix} F_1 a & F_2 b \\ F_2 c & F_1 d \end{pmatrix} \quad (3.33)$$

$$M = \begin{pmatrix} F_1 \alpha & F_2 \beta \\ F_2 \gamma & F_1 \delta \end{pmatrix} \quad (3.34)$$

As above, $a, b, c, d, \alpha, \beta, \gamma$ and δ are all functions of the both the lattice variables l and m , F_1 and F_2 depend on the spectral variable only.

The compatibility condition leads to six equations coming from the different orders of the spectral variable in each entry.

$$\hat{a}\alpha = a\bar{\alpha} \quad (3.35a)$$

$$\hat{b}\gamma = c\bar{\beta} \quad (3.35b)$$

$$\hat{d}\delta = d\bar{\delta} \quad (3.35c)$$

$$\hat{c}\beta = b\bar{\gamma} \quad (3.35d)$$

$$\hat{a}\beta + \hat{b}\delta = b\bar{\alpha} + d\bar{\beta} \quad (3.35e)$$

$$\hat{c}\alpha + \hat{d}\gamma = a\bar{\gamma} + c\bar{\delta} \quad (3.35f)$$

Equations (3.35a) and (3.35c) are integrated immediately, while (3.35b) and (3.35d) are multiplied together and dealt with by a similar method to that used for the LSG₂ system of section 3.3.1 leading to the following results

$$a = \lambda_1 \bar{v}_1 / v_1 \quad (3.36a)$$

$$b = \lambda_0 \rho \bar{v}_3 / v_4 \quad (3.36b)$$

$$c = \lambda_0 \bar{v}_4 / (\rho v_3) \quad (3.36c)$$

$$d = \lambda_2 \bar{v}_2 / v_2 \quad (3.36d)$$

$$\alpha = \mu_1 \hat{v}_1 / v_1 \quad (3.36e)$$

$$\beta = \mu_0 \hat{v}_3 / (\sigma v_4) \quad (3.36f)$$

$$\gamma = \mu_0 \sigma \hat{v}_4 / v_3 \quad (3.36g)$$

$$\delta = \mu_2 \hat{v}_2 / v_2 \quad (3.36h)$$

Where $v_i = v_i(l, m)$, $\lambda_i = \lambda_i(l)$, $\mu_i = \mu_i(m)$, $\rho = \lambda_3^{(-1)^m}$ and $\sigma = \mu_3^{(-1)^l}$. Substituting these values into the two remaining equations, (3.35e) and (3.35f), shows that terms consistently appear in ratios, as they did with the LSG₂ system. We choose to set $v_2 \equiv v_3 \equiv 1$, $\lambda_0 \equiv \mu_0 \equiv 1$, $v_1 = x$ and $v_4 = y$ and arrive at LMKdV₂:

$$\frac{\lambda_1 \hat{x}}{\sigma \hat{x}} + \frac{\mu_2 y}{\rho \hat{y}} = \lambda_2 \sigma \frac{y}{\hat{y}} + \rho \mu_1 \frac{\hat{x}}{\bar{x}} \quad (3.37)$$

$$\rho \mu_1 \hat{x} \hat{y} + \lambda_2 \sigma x \hat{y} = \frac{\mu_2}{\rho} x \bar{y} + \frac{\lambda_1}{\sigma} \bar{x} \hat{y} \quad (3.38)$$

3.4 How most link combinations lead to bad evolution equations

Here we explain how most of the potentially viable Lax pairs fail to produce interesting evolution equations. The topic is split into four parts dealing with Lax pairs that lead to trivial, over-determined, under-determined evolution equations and the special case of item 15 in table 3.8.

3.4.1 Lax pairs that yield only trivial evolution equations

This section is pertinent to items 3, 4, 6, 8, 9 and 11 in table 3.8, those Lax pairs whose compatibility conditions can be solved to a point where only linear equations remain, or the equations can be reduced to first order equation in one lattice direction only.

The simplest route to triviality is to have a Lax pair with a set of equations that are all linear, as per item 3 in table 3.8. The compatibility condition leads to a set of eight linear equations in the eight initial lattice terms.

$$\begin{aligned}
 \hat{a}\alpha &= a\bar{\alpha}, & \hat{b}\gamma &= c\bar{\beta} \\
 \hat{d}\delta &= d\bar{\delta}, & \hat{c}\beta &= n\bar{\gamma} \\
 \hat{a}\beta &= d\bar{\beta}, & \hat{b}\delta &= b\bar{\alpha} \\
 \hat{d}\gamma &= a\bar{\gamma}, & \hat{c}\alpha &= c\bar{\delta}
 \end{aligned}
 \tag{3.39}$$

Equations (3.39) can be solved easily using techniques described in section 3.3, with the result of a simple, linear evolution equation. However, it is not necessary to conduct such analysis on this system since all equations (3.39) are linear and they can not be expected to produce a nonlinear evolution equation. For this reason, other examples of link combinations that produce only linear equations have been omitted from table 3.8.

Item number 8 from table 3.8 is an example of a Lax pair that leads to a trivial evolution equation in a more complex way. The equations that come out of its compatibility condition are

$$\hat{a}\alpha = a\bar{\alpha} \quad (3.40a)$$

$$\hat{b}\gamma = c\bar{\beta} \quad (3.40b)$$

$$\hat{d}\delta + \hat{c}\beta = d\bar{\delta} + b\bar{\gamma} \quad (3.40c)$$

$$\hat{a}\beta + \hat{b}\delta = b\bar{\alpha} \quad (3.40d)$$

$$kb\bar{\alpha} = -d\bar{\beta} \quad (3.40e)$$

$$\hat{c}\alpha = a\bar{\gamma} + c\bar{\delta} \quad (3.40f)$$

$$k\hat{c}\alpha = -\hat{d}\gamma \quad (3.40g)$$

Where k is a constant. This system of equations is solved as follows: integrate (3.40a) for $a = \lambda\bar{v}/v$, $\alpha = \mu\hat{v}/v$, introducing $\lambda = \lambda(l)$, $\mu = \mu(m)$ and $v = v(l, m)$. Multiply equations (3.40b) and (3.40g), then divide the product by (3.40e) and use the values calculated for a and α to find

$$\frac{\hat{b}\hat{c}\hat{v}}{bcv} = \frac{\hat{d}\hat{v}}{dv}. \quad (3.41)$$

Equation (3.41) is integrated for b , which is substituted into equations (3.40e) and (3.40b) yielding the following values

$$\begin{aligned} b &= \lambda_2 \frac{d\bar{v}}{cv} \\ \bar{\beta} &= -k\mu\lambda_2 \frac{\hat{v}}{cv} \\ \gamma &= -k\mu \frac{\hat{c}\hat{v}}{\hat{d}v} \end{aligned}$$

There are now three equations left to solve, (3.40c), (3.40d) and (3.40f), where it is found that c and v always appear as a product which suggests we introduce $u = cv$. Equation (3.40c) is then used to write

$$\bar{\delta} = \mu \frac{\hat{u}}{u} + k\mu\lambda \frac{\hat{u}}{u\hat{d}}$$

Finally, we introduce $x = \underline{u}d/u$ and achieve the final two equations in x

$$\lambda_2(\hat{x} - x) = k(\underline{\lambda}_2\lambda - \underline{\lambda}\lambda_2) \quad (3.42a)$$

$$\hat{x} - x + k(\underline{\lambda} - \lambda_2) = k\frac{x}{\hat{x}}(\mu\lambda - \lambda_2) \quad (3.42b)$$

Equations (3.42) is an overdetermined set of two equations in the one variable, x , which cannot hope to yield an interesting evolution equation. This is because (3.42a) can be used to remove the dependence of x on m (m is the independent variable of the ‘ \wedge ’ direction), leaving x with a first order dependence on l in (3.42b) at best.

3.4.2 Overdetermined Lax pairs

Here items 10 and 12 through 14 in table 3.8 are dealt with. Such Lax pairs have compatibility conditions that boil down to more equations than there are free lattice terms. These are not necessarily trivial, as the solution to one equation may solve another as well, that is it may be possible to make one or more equations redundant. Some systems of this type, where any hope of supporting an interesting evolution equation has been quashed, have already been considered in section 3.4.1. The remaining over-determined systems are considered here, however we do not attempt to resolve the issue of whether these systems support interesting evolution equations, we simply list them as being overdetermined.

The most interesting instances of this type of system arise from Lax pairs with two double links in the off-diagonal entries, those represented by item 14 from table 3.8. All such systems, with minimum constraint on the lattice terms in the diagonal entries of the compatibility condition, lead to the same evolution equations. An example of a Lax pair with two double links in the off-diagonal entries is

$$L = \begin{pmatrix} F_1a & b \\ (F_1^2 + k_4F_1F_2)c & F_2d \end{pmatrix} \quad (3.43a)$$

$$M = \begin{pmatrix} (F_1 + k_2F_2)\alpha & \beta \\ (F_1^2 + k_3F_1F_2)\gamma & (F_1 + k_1F_2)\delta \end{pmatrix} \quad (3.43b)$$

where lower case letters except k_i are lattice terms, k_i are constants of proportionality and F_i are spectral terms with F_1 not proportional to F_2 . This particular Lax pair is especially interesting because the solution of its compatibility condition involves integrating both additive and multiplicative linear difference equations, as described near equation (3.28) in section 3.3. The final evolution equations achieved are relatively complicated and nonlinear, however there are two equations for one variable and it remains to be seen whether they can be reconciled. The following outlines how to solve the compatibility condition.

In the compatibility condition, $\widehat{L}M = \overline{M}L$, the 11 entry dictates $k_2 = k_3 = k_4$. Redefine $F_2 \mapsto k_2 F_2$ to remove k_2 from everywhere except the 22 entry of L where we set $k_0 = 1/K_2$ and thus arrive at the following equations from the various orders of the compatibility condition

$$\hat{a}\alpha + \hat{b}\gamma = a\bar{\alpha} + c\bar{\beta} \quad (3.44a)$$

$$\hat{c}\bar{\beta} = b\bar{\gamma} \quad (3.44b)$$

$$\hat{d}\delta = d\bar{\delta} \quad (3.44c)$$

$$\hat{a}\beta + \hat{b}\delta = b\bar{\alpha} \quad (3.44d)$$

$$k_1\hat{b}\delta = b\bar{\alpha} + k_0d\bar{\beta} \quad (3.44e)$$

$$\hat{c}\alpha = a\bar{\gamma} + c\bar{\delta} \quad (3.44f)$$

$$\hat{c}\alpha + k_0\hat{d}\gamma = k_1c\bar{\delta} \quad (3.44g)$$

Integrate (3.44c) for $d = \lambda\bar{v}/v$, $\delta = \mu\hat{v}/v$, introducing $v = v(l, m)$, $\lambda = \lambda(l)$ and $\mu = \mu(m)$, and use equation (3.44b) to see that $\beta = b\bar{\gamma}/\hat{c}$. Now define $s = c/\bar{v}$, $t = \gamma/\hat{v}$ and $u = bv$ so that

$$\beta = b\bar{t}/\hat{s} \quad (3.45a)$$

$$(3.44g) \Rightarrow \alpha = \frac{1}{\hat{s}}(k_1\mu s - k_0\lambda t) \quad (3.45b)$$

$$(3.44f) \Rightarrow a = \frac{1}{\bar{t}}((k_1 - 1)\mu s - k_0\lambda t) \quad (3.45c)$$

Substituting these (3.45a) values into equation (3.44e) and rearranging leads to

$$k_1 \left(\frac{\widehat{u\bar{s}}}{\lambda\bar{\lambda}} \right) - k_1 \left(\frac{u\bar{s}}{\lambda\bar{\lambda}} \right) = k_0 \left(\frac{\overline{u\bar{t}}}{\lambda\bar{\mu}} \right) - k_0 \left(\frac{u\bar{t}}{\lambda\bar{\mu}} \right) \quad (3.46)$$

which is an additive linear difference equation that can be integrated to find

$$\frac{u\bar{s}}{\lambda\bar{\lambda}} = \bar{w} - w + \lambda_2 \quad (3.47a)$$

$$\frac{u\bar{t}}{\lambda\bar{\mu}} = \hat{w} - w + \mu_2 \quad (3.47b)$$

The two equations that remain, equations (3.44a) and (3.44d) respectively, are written in terms of s , t and u

$$\frac{\hat{s}}{\hat{t}} \left((k-1)\hat{\mu} - k_0\lambda\frac{\hat{t}}{\hat{s}} \right) \left(k_1\mu\frac{s}{t} - k_0\lambda \right) + \hat{u}\hat{s} = \left((k-1)\mu\frac{s}{t} - k_0\lambda \right) \left(k_1\mu - k_0\bar{\lambda}\frac{\bar{t}}{\bar{s}} \right) + \bar{u}\bar{t}\frac{\bar{s}}{t} \quad (3.48a)$$

$$\frac{\hat{s}}{\hat{t}} \left((k-1)\hat{\mu} - k_0\lambda\frac{\hat{t}}{\hat{s}} \right) + \mu\frac{\hat{u}\hat{s}}{u\bar{t}} = k_1\mu\frac{\bar{s}}{\bar{t}} - k_0\bar{\lambda} \quad (3.48b)$$

Make a further change of variables $x = u\bar{t}$, $y = u\bar{s}$, to completely remove u from equations (3.48) and (3.47). Equation (3.48b) becomes

$$\mu\frac{\hat{y}}{\bar{x}} + \frac{\hat{y}}{\hat{x}} \left((k_1-1)\hat{\mu} - k_0\lambda\frac{\hat{x}}{\hat{y}} \right) = k_1\mu\frac{\bar{y}}{\bar{x}} - k_0\bar{\lambda} \quad (3.49)$$

while equation (3.48a) becomes

$$\frac{\hat{y}}{\hat{x}} \left((k_1-1)\hat{\mu} - k_0\lambda\frac{\hat{x}}{\hat{y}} \right) \left(k_1\mu\frac{y}{x} - k_0\lambda \right) + \hat{y} = \left((k_1-1)\mu\frac{y}{x} - k_0\lambda \right) \left(k_1\mu - k_0\bar{\lambda}\frac{\bar{x}}{\bar{y}} \right) + \bar{x}\frac{y}{x} \quad (3.50)$$

where equation (3.47) shows that x and y can both be written in terms of w according to

$$\bar{x} = \frac{\lambda\mu}{k_0}(\hat{w} - w + \mu_2), \quad (3.51a)$$

$$\bar{y} = \frac{\lambda\bar{\lambda}}{k_1}(\bar{w} - w - \lambda_2). \quad (3.51b)$$

Hence, there are two complicated, nonlinear equations for the one variable w , (3.49) and (3.50) and the system is therefore overdetermined. However, these two equations may or may not be reconcilable, one way that they might be reconcilable is if one equation was shown to be a compatible similarity constraint for the other, in the sense of [82, 78].

3.4.3 Underdetermined Lax pairs

There are Lax pairs whose compatibility condition can be solved completely, while still leaving at least one lattice term free. In these cases there is no genuine evolution equation, although the freedom inherent in the system could cause it to appear as though there was. In fact, any evolution equation, trivial, integrable or even chaotic, could be falsely represented by such a Lax pair. One such case is item 7 on table 3.8 that has as its compatibility condition

$$\hat{a}\alpha + \hat{b}\gamma = a\bar{\alpha} + c\bar{\beta} \quad (3.52a)$$

$$\hat{d}\delta + \hat{c}\beta = d\bar{\delta} + b\bar{\gamma} \quad (3.52b)$$

$$\hat{a}\beta = -\hat{b}\delta \quad (3.52c)$$

$$b\bar{\alpha} = -d\bar{\beta} \quad (3.52d)$$

$$\hat{c}\alpha = -\hat{d}\gamma \quad (3.52e)$$

$$a\bar{\gamma} = -c\bar{\delta} \quad (3.52f)$$

Here is a roadmap to the solution: multiply equation (3.52c) by (3.52f) and divide by (3.52d) and (3.52e) to find an expression that can be integrated for

$$d = \lambda_1 \bar{v} c b / (v a)$$

$$\gamma = \mu_1 \hat{v} \alpha \delta / (v \beta)$$

Substituting these values for d and γ into the ratio of equations (3.52d) and (3.52f) shows that $v = \lambda_1 \mu_1 \hat{v}$, which indicates that v must be separable into a product such as $v = \lambda_2(l) \mu_2(m)$, where $\lambda_1 = \lambda_2 / \bar{\lambda}_2$ and $\mu_1 = \mu_2 / \hat{\mu}_2$. With the above values included and no further integration required, equation (3.52c) is used to find

δ while (3.52d) offers c . In summary:

$$d = -b\bar{\alpha}/\bar{\beta} \quad (3.53a)$$

$$\gamma = \pm\hat{a}\alpha/\hat{b} \quad (3.53b)$$

$$\delta = -\hat{a}\beta/\hat{b} \quad (3.53c)$$

$$c = \pm a\bar{\alpha}/\bar{\beta} \quad (3.53d)$$

The key feature is that, with all negative quantities in (3.53), equations (3.52a) and (3.52b) are automatically satisfied, furnishing us with no more constraints on the remaining lattice terms a , b , α and β . Therefore, equations (3.53) are the only conditions that must be met in order to satisfy consistency, but these are not uniquely determined and there is freedom enough to write any equation at all into this set. The undeniable conclusion is that any Lax pair of this type is false as one would expect that the information to be gleaned about the solution to any evolution equation associated with this Lax pair must be as vague as equation (3.53). Note further that the level of freedom left in the system is exactly that which can be removed by gauge transformations, so, in essence, this system's compatibility is simply determined by the values of the parameter functions and no evolution equation exists.

When c and γ in equations (3.53) are positive the situation is slightly more complicated because equations (3.52a) and (3.52b) are not automatically satisfied. However, the qualitative outcome is the same and the system remains underdetermined.

3.4.4 A special case

Item 15 from table 3.8 is a special case that sees all lattice terms linked in all four entries of the compatibility condition. Thus, the compatibility condition supplies only nonlinear equations, none of which can be explicitly integrated, and so no parameter

functions present themselves as constants of integration. This is an unusual situation but it is not necessary to solve the system because of the following arguments regarding the spectral dependence.

To form the links that define this Lax pair we require the following conditions, or an equivalent set, on the spectral terms.

$$\begin{aligned} A &= B\Gamma/\Lambda & \Xi &= \Lambda^2/\Gamma \\ C &= B\Gamma^2/\Lambda^2 & \Delta &= \Lambda \\ D &= B\Gamma/\Lambda \end{aligned} \tag{3.54}$$

Equation (3.55), below, displays a Lax pair of general form that possesses the links in question

$$L = BF_1 \begin{pmatrix} a & b/F_1 \\ cF_1 & d \end{pmatrix}, \tag{3.55a}$$

$$M = \Lambda \begin{pmatrix} \alpha & \beta/F_1 \\ \gamma F_1 & \delta \end{pmatrix}, \tag{3.55b}$$

where $F_1(n) = \Gamma/\Lambda$. The prefactors in equation (3.55) are irrelevant because they cancel in the compatibility condition. This leaves the Lax pair with the same dependence on just one spectral term in both the L and M matrices. As such, a gauge transformation can be used to completely remove the dependence on the spectral variable from the linear problem, implying that equation (3.55) is actually not a Lax pair at all.

3.5 Discussion

All 2×2 Lax pairs, with one separable term in the four entries of each matrix, have been considered through the various combinations of terms possible in their compatibility conditions. It has been shown that the only non-trivial evolution equations that can be supported are the higher order generalizations of the LMKdV

and LSG equations, with the possible exception of over-determined systems that may yet be consistent, see section 3.4.2.

There is an important question in where the present work sits in relation to studies in multidimensional consistency, or consistency around a cube (CAC), that provide a method of searching for integrable partial difference equations where a Lax pair can be derived as a bi-product of the procedure [38, 37]. The present work adds to multidimensional consistency studies by removing the restriction that the Lax matrices must be symmetric, a consequence of using the same Q equation in all three directions of the cube [40], and also leads to equations with more non-autonomous terms than multidimensional consistency studies have so far allowed. CAC studies do lead to non-autonomous equations, by allowing the parameters belonging to each side of a face to vary only with one lattice variable, but they have not used any non-autonomous terms that are arbitrary in one lattice variable and have a specific dependence on another, like ρ and σ in equation (3.37). Non-autonomy is vital to reductions of the type used in [86] and [2], that lead to nonlinear ordinary difference equations and Lax pairs for them. In addition, while it has been shown that (CAC) ensures the existence of a Lax pair, the converse is not necessarily true, which in itself indicates a need for the present completeness study.

Future studies using the techniques explained here should investigate Lax pairs with non-separable terms or with more terms in each matrix entry, and possibly aim for a general algorithm for dealing with an arbitrary number of terms in each entry of the Lax matrices. Other types, such as differential difference Lax pairs, should be considered, as should purely continuous Lax pairs where there is already a vast body of knowledge with which to compare results. Further, this technique is easily adaptable to isomonodromy Lax pairs.

There is some conflict about whether the parameters of the Q_4 equation, in the ABS scheme, must lie on elliptic curves or not [36, 106, 107]. The Lax pair for Q_4 , found via multidimensional consistency, is for a version of the equation where the

autonomous parameters are restricted to elliptic curves [38]. An exploration into the links present in that Lax pair might resolve the issue by providing a Lax pair for Q_4 with non-autonomous, and possibly free, parameters.

Chapter 4

Reductions and Lax pairs for the reduced equations

4.1 Introduction

Nonlinear evolution equations occur frequently in physical modeling and applied mathematics. Nonlinear integrable lattices provide a natural discrete extension of classically integrable systems. More recently, there has been great interest in nonlinear ordinary difference equations. We consider reductions from lattice equations to ordinary difference equations which constitute a natural link between the two classes of equations. Our main perspective will lie in the construction of Lax pairs for difference equations.

Most studies [108, 78, 82, 109, 110, 111] of reductions of lattice equations focus on equations in which all parameters are independent of lattice variables. For example, the lattice modified Korteweg-de Vries equation [78]

$$\text{LMKdV : } x_{l+1,m+1} = x_{l,m} \frac{(p x_{l+1,m} - q x_{l,m+1})}{(p x_{l,m+1} - q x_{l+1,m})}$$

contains lattice parameters p , q which are considered to be independent of the

lattice variables l, m . In [86], a new type of reduction from the lattice equations to ordinary difference equations was introduced by starting with non-autonomous lattice equations. In this approach, the lattice parameters p, q were considered to be functions of l, m , under the condition that the lattice equation satisfied the singularity confinement property. Note also that higher order versions of the non-autonomous LMKdV and lattice sine-Gordon (LSG), with similar constraints on the non-autonomous terms, were found in chapter 3 by an entirely different approach. In [86], such non-autonomous forms of well-known lattice equations were shown to reduce to q -discrete Painlevé equations, including qP_{II} , qP_{III} and qP_V .

The q -discrete Painlevé equations are of fundamental interest in the theory of integrable systems and random matrix theory. We note that the full generic form of qP_{III} was first obtained in [33]. Its natural generalization is a q -discrete sixth Painlevé equation (qP_{VI}) first obtained by [112]. The integrability of such equations lies in the fact that they can be solved through an associated linear problem called a Lax pair. For qP_{III} the Lax pair was obtained by [47], with a notable feature that the linear problem is a matrix problem involving matrices of size 4×4 . On the other hand, the Lax pairs of lattice equations, such as the LMKdV [108, 78], and many discrete Painlevé equations, such as qP_{VI} are typically matrix problems of size 2×2 . In [113], a 2×2 Lax pair was given for a special case of qP_{III} .

In this chapter, we present two types of results. First, we show that an extension of the reduction method given by [86] is possible and, by using the extension, deduce a sequence of discrete Painlevé equations as reductions of lattice equations. Second, we give a Lax pair of the non-autonomous LMKdV and show that it gives rise to 2×2 matrix Lax pairs under the reductions to q -Painlevé equations. In obtaining the latter, a key observation was needed that arises from the multi-dimensional embedding of lattice equations in a self-consistent way in three directions. The resulting theory [89, 38, 37, 40] is often referred to as “consistency around a cube”.

The chapter is organized as follows. In section 4.2, we recall the Lax pair

of LMKdV and generalize it to provide a non-autonomous Lax pair for the non-autonomous version of LMKdV. We also show that this Lax pair and the generalized LMKdV form a multi-dimensional system that satisfies the self-consistency property. In section 4.3, we consider the reductions of the non-autonomous LMKdV to ordinary difference equations and provide extensions of previously considered reductions. In section 4.4, we show that 2×2 Lax pairs for the reductions can be found by applying the idea of self-consistency and reductions to the Lax pair of the LMKdV. We end the chapter with a conclusion where we also point out some open problems.

4.2 Lax Pair and Self-Consistency of the Non-autonomous LMKdV

While a linear problem, or Lax pair, associated with the LMKdV has been known for a long time [108, 78], it appears that linear problems associated with the non-autonomous version of the LMKdV have not been written down before. We provide an explicit Lax pair for the non-autonomous version of the LMKdV in the first subsection below.

Furthermore, while the theory of multi-dimensional extensions of lattice equations has been explored fairly widely, the theory has not been applied explicitly to non-autonomous lattice equations. We provide such an application to the non-autonomous LMKdV in the second sub-section below.

4.2.1 Lax Pair of the Non-autonomous LMKdV

A Lax Pair for the LMKdV is a linear problem of the form

$$\begin{aligned}\theta(l+1, m) &= L(l, m)\theta(l, m), \\ \theta(l, m+1) &= M(l, m)\theta(l, m).\end{aligned}\tag{4.1}$$

whose compatibility condition, namely, $L(l, m+1)M(l, m) = M(l+1, m)L(l, m)$, is the LMKdV.

Hereafter we adopt the notation $\bar{\theta} = \theta(l+1, m)$ and $\hat{\theta} = \theta(l, m+1)$. (We have used l in place of the more traditional n here because it is notationally more appropriate that the L matrix should create a shift in l and M in m . Later, in §3, we will see that a third Lax matrix, N arises whose associated shifts will be in n .) Now set

$$L = \begin{pmatrix} \bar{x}/x & -\lambda/(\nu x) \\ -\lambda\bar{x}/\nu & 1 \end{pmatrix},\tag{4.2a}$$

$$M = \begin{pmatrix} \hat{x}/x & -\mu/(\nu x) \\ -\mu\hat{x}/\nu & 1 \end{pmatrix}.\tag{4.2b}$$

where ν is a spectral variable, μ is a function of m alone, and λ is a function of l alone.

Compatibility occurs when $\hat{L}M = \bar{M}L$. In this equation, it is straightforward to check that the diagonal entries yield identities and the off-diagonal entries each contain the lattice mKdV equation in the following way. The top-right entry yields

$$\frac{\mu\hat{x}}{\hat{x}x} + \frac{\lambda}{\hat{x}} = \frac{\lambda\hat{x}}{\bar{x}x} + \frac{\mu}{\bar{x}} \Rightarrow \hat{x}(\mu\bar{x} - \lambda\hat{x}) = x(\mu\hat{x} - \lambda\bar{x}).$$

Similarly, the bottom left entry yields the same equation. Thus we arrive at the following form of the LMKdV equation

$$\hat{x} = x \frac{\bar{x} - r\hat{x}}{\hat{x} - r\bar{x}}\tag{4.3}$$

where we have introduced

$$r(l, m) = \frac{\mu(m)}{\lambda(l)}. \quad (4.4)$$

This form of the LMKdV equation is identical to the one used in [86], except for a factor of (-1) which is inconsequential. Indeed, we can achieve the equation used there exactly if we premultiply each of L and M by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. We use the slightly different form here simply because it allows for a more symmetric Lax Pair. In [86] it is noted that r must separate as in (4.4) because it has to satisfy

$$\hat{r}r = \hat{r}\bar{r} \quad (4.5)$$

for the singularity confinement property to be satisfied. We note that integrability conditions for lattice equations have also been studied recently by [102].

4.2.2 Consistency Around a Cube

In this subsection, we show that the Lax Pair, (5.32), given in the previous subsection, is multi-dimensionally consistent with the lattice equation LMKdV.

In this point of view, the lattice variables l, m provide a two-dimensional slice of a three-dimensional space in which the third direction, coordinatized by n say, can be thought of as providing the spectral direction for the Lax pair. The shifts $l \mapsto l + 1, m \mapsto m + 1, n \mapsto n + 1$ describe a fundamental cube in this multi-dimensional space. The term “consistent around a cube” arises from the fact that the iteration of the map on any face of the fundamental cube provides a corner value that is the same as that provided by iteration on an intersecting face.

Define $\tilde{x} = x(l, m, n + 1)$, such that $\tilde{x} = u/t$, where u and t are the components of the eigenfunction $\theta(l, m, n)$, i.e.,

$$\theta = \begin{pmatrix} t \\ u \end{pmatrix}$$

where θ satisfies the linear system (5.1).

Since $\bar{\theta} = L\theta$,

$$\begin{aligned}\tilde{x} &= \bar{u}/\bar{t} \\ &= \frac{u - \lambda\bar{x}t/\nu}{\bar{x}t/x - \lambda u/(\nu x)} \\ \tilde{x} &= x \frac{\tilde{x} - \rho\bar{x}}{\bar{x} - \rho\tilde{x}}\end{aligned}\tag{4.6}$$

where we have allowed ν to depend on n and replaced λ/ν by $\rho(l, n)$. Since M takes on the same form as L , we can clearly find an equivalent expression in the m and n directions. And, because (4.6) is the LMKdV equation again, we conclude that the Lax Pair is multi-dimensionally consistent with the LMKdV equation.

Essentially we have done the reverse of the usual operation, ordinarily one begins with a system that is consistent around a cube and then constructs its Lax Pair (see [40] or [89, 38]). However, here we began with the Lax Pair and showed that it is multi-dimensionally consistent with the LMKdV equation.

4.3 Reductions to Ordinary Difference Equations

In this section, we consider reductions from the partial difference equation (4.3) to a sequence of ordinary difference equations. These include qP_{II} , a three-parameter version of qP_{III} , a special case of qP_{V} , and, moreover, some higher-order difference equations. We present the results in a series of subsections.

Let $\hat{x} = f(\mathbf{x})$ where \mathbf{x} represents x and its iterates, $\bar{x}, \bar{\bar{x}}\dots$. Thus, (4.3) becomes

$$\overline{f(\mathbf{x})} = x \frac{\bar{x} - r f(\mathbf{x})}{f(\mathbf{x}) - r \bar{x}}.\tag{4.7}$$

For $f(\mathbf{x})$ to be valid, it must produce the same reduced equation when we begin with the mKdV equation iterated up once in m . That is

$$\hat{\hat{x}} = \hat{x} \frac{\hat{\hat{x}} - \hat{r}\hat{\hat{x}}}{\hat{\hat{x}} - \hat{r}\hat{x}}\tag{4.8}$$

must lead to the same reduction, with possible conditions placed on r . We note that $\hat{x} = f(\mathbf{x})$ so $\hat{\hat{x}} = f(\hat{\mathbf{x}}) = f(f(\mathbf{x}))$ and so (4.8) is equivalent to

$$f(f(\bar{\mathbf{x}})) = f(\mathbf{x}) \frac{f(\bar{\mathbf{x}}) - \hat{r}f(f(\mathbf{x}))}{f(f(\mathbf{x})) - \hat{r}f(\bar{\mathbf{x}})}$$

4.3.1 $f(\mathbf{x}) = \bar{x}^\xi$

The first reduction we consider is $f(\mathbf{x}) = \bar{x}^\xi$, ξ constant, in (4.7) so that

$$\bar{\bar{x}}^\xi = x \frac{\bar{x}^{1-\xi} - r}{1 - r\bar{x}^{1-\xi}} \quad (4.9)$$

and we use the same $f(\mathbf{x})$ in (4.8) to get

$$\bar{\bar{x}}^\xi = x \left[\frac{\bar{x}^{\xi(1-\xi)} - \hat{r}}{1 - \hat{r}\bar{x}^{\xi(1-\xi)}} \right]^{1/\xi} \quad (4.10)$$

The two expressions for $\bar{\bar{x}}^\xi$, (4.9) and (4.10), agree if $\xi = 1$ but this leads to a linear equation. Another solution is $\xi = -1$ and $\hat{r} = \bar{r}$. The latter condition on r dictates through (4.5) that $r = k_1 k_2^l$, where k_i are constant, so that the final form of the reduced equation is

$$\bar{\bar{x}}x = \frac{k_1 k_2^l \bar{x}^2 - 1}{k_1 k_2^l - \bar{x}^2} \quad (4.11)$$

which is a special case of a q -discrete Painlevé III equation (qP_{III}) found by [33].

This equation was already obtained in [86] as a reduction of the lattice sine-Gordon equation (LSG). The advantage of the reduction presented above is that it comes with a Lax Pair (see §4.4.2). As a point of interest we mention that the LMKdV can be transformed to the LSG by allowing $\hat{x} \rightarrow 1/\hat{x}$.

4.3.2 $f(\mathbf{x}) = \bar{\bar{x}}^\xi$

Now consider $f(\mathbf{x}) = \bar{\bar{x}}^\xi$ the same analysis as above shows that, again, $\xi = 1$ or $\xi = -1$ will lead to valid reductions. When $\xi = 1$, we must set $\log r = k_1 l + k_2 + k_3(-1)^l$

and, after introducing $y = \bar{\bar{x}}/\bar{x}$, we are left with

$$\bar{y}\underline{y} = \frac{1 - ry}{y(y - r)}. \quad (4.12)$$

The same equation as (4.12) was found in [86] where the equation was identified as either qP_{II} or qP_{III} , depending on whether $k_3 = 0$.

Now take the case when $\xi = -1$, this time (4.7) becomes

$$\bar{\bar{x}}x = \frac{\bar{\bar{x}}\bar{x}\bar{x}x}{\bar{\bar{x}}\bar{x}} = \frac{r\bar{x}\bar{x} - 1}{r - \bar{\bar{x}}\bar{x}}$$

whereupon setting $y = \bar{\bar{x}}\bar{x}$ we find

$$\bar{y}\underline{y} = y \frac{ry - 1}{r - y}. \quad (4.13)$$

To find the required form of the parameter functions we must compare this to (4.8) with the same y substituted

$$\bar{\bar{y}}\underline{y} = \bar{y} \frac{1 - \hat{r}\bar{y}}{\bar{y} - \hat{r}}$$

Clearly, the equivalence between these two mappings is satisfied by taking r as for the case when $\xi = +1$. Equation (4.13) is actually equivalent to a special case of (4.12) and was also derived in [86] but from the lattice Sine-Gordon equation rather than the lmKdV.

4.3.3 $f(\mathbf{x}) = \bar{\bar{x}}$

We let $f = \bar{\bar{x}}$ and, on substituting $w = \bar{\bar{x}}/\bar{x}$, we have

$$\bar{w}\underline{w} = \frac{1 - rw}{w - r}. \quad (4.14)$$

Here $\log r = k_1 l + k_2 + k_3 j^l + k_4 j^{2l}$, k_i constants and $j^3 = 1$. This equation was shown to be a qP_{II} when $k_3 = k_4 = 0$ [87] or a qP_V in the general case [88].

4.3.4 $f(\mathbf{x}) = 1/\bar{\bar{x}}$

Specifying $f = 1/\bar{\bar{x}}$ leads to what appears to be an irreducible, fourth order, integrable difference equation, namely

$$\bar{\bar{x}} = \frac{\bar{x} - r}{1 - r\bar{x}} \quad (4.15)$$

where again $\log r = k_1 l + k_2 + k_3 j^l + k_4 j^{2l}$ and $j^3 = 1$.

4.3.5 $f(\mathbf{x}) = x_{l+4}$

The reductions of orders higher than third all lead to equations that are not reducible to a second order form. The next reduction to consider is $f = x_{l+4}$ which becomes

$$\bar{\bar{y}} = \frac{1 - r\bar{y}}{\bar{y} - r} \quad (4.16)$$

with $y = \bar{\bar{x}}/\bar{x}$, and $\log r = k_1 l + k_2 + k_3(-1)^l + k_4 \cos(\frac{l\pi}{2}) + k_5 \sin(\frac{l\pi}{2})$, $k_i = \text{constants}$.

4.3.6 $f(\mathbf{x}) = 1/x_{l+4}$

Using $f = 1/x_{l+4}$ allows the reduction of the lmKdV to

$$\bar{\bar{y}} = \bar{y} \frac{r\bar{y} - y}{-ry - \bar{y}} \quad (4.17)$$

where r is the same as in the previous example but $y = \bar{\bar{x}}$.

Arbitrarily high order equations can be generated in this manner.

4.4 Lax Pairs for the Reduced Equations

In this section, we deduce Lax pairs for the q -discrete Painlevé equations derived in the previous section by applying the observations obtained from the multi-dimensional

self-consistency of the LMKdV system and its Lax pair. Since the reductions leading to qP_{II} , qP_{III} and qP_V differ, we give the details of each separately in three subsections.

So far the linear system is given by (5.1) but now we wish to include the third direction, n , that we introduced in section 4.2.2. n will come into L via ν which plays the role of the spectral variable, i.e., we allow $\nu = \nu(n)$. This gives rise to the reduced equations through compatibility between the l and n directions. We write $L(l, m, \nu(n)) = L(l, m, n)$, $M(l, m, \nu(n)) = M(l, m, n)$ and introduce a matrix $N(l, m, \nu(n)) = N(l, m, n)$ such that

$$\theta(l+1, m, n) = L(l, m, \nu(n))\theta(l, m, n) \quad (4.18a)$$

$$\theta(l, m, n+1) = N(l, m, \nu(n))\theta(l, m, n) \quad (4.18b)$$

where L in the first equation is the same as in Equation (4.2a). We indicate a shift in n by $a(l, m, \nu(n+1)) =: a(l, m, n+1) =: \tilde{a}$. Now the compatibility condition of the above two equations is $\tilde{L}N = \tilde{N}L$.

We label the components of θ , as in §4.2.2, by

$$\theta = \begin{pmatrix} t \\ u \end{pmatrix} \quad (4.19)$$

and let $\tilde{x} = u/t$. We are now in a position to find the form of N through the reduction.

4.4.1 Lax Pair for qP_{II}

First consider the reduction $\hat{x} = \bar{\bar{x}}$ which reduces the LMKdV equation to qP_{II} . Recall that $\bar{v} = Lv$ and $\hat{v} = Mv$ so on the one hand,

$$\hat{\theta} = \begin{pmatrix} \hat{t} \\ \hat{u} \end{pmatrix} = \hat{t} \begin{pmatrix} 1 \\ \hat{x} \end{pmatrix} = \hat{t} \begin{pmatrix} 1 \\ \bar{\bar{x}} \end{pmatrix}$$

and on the other hand,

$$\bar{\theta} = \bar{t} \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix} = \frac{\bar{t}}{\bar{t}} \hat{v}$$

But $\hat{\theta} = (\hat{N})^{-1}M\theta$ and $\bar{\theta} = (\bar{N})^{-1}\bar{L}L\theta$. Thus

$$\bar{t}\bar{N} = \bar{L}LM^{-1}\hat{t}\hat{N} \quad (4.20)$$

We now use (5.2b) as a guide and try a general N that has the same form as $\bar{L}LM^{-1}$, where by the same form we mean that it contains the same powers of ν .

$$\bar{L}LM^{-1} = \frac{1}{\nu^2 - \mu^2} \begin{pmatrix} \nu^2 + \lambda\bar{\lambda}x/\bar{x} - \mu\lambda\bar{x}/\bar{x} - \mu\bar{\lambda}x/\bar{x} & \nu(\mu/x - \lambda\bar{x}/\bar{x}x - \bar{\lambda}/\bar{x}) + \frac{1}{\nu}\mu\bar{\lambda}\lambda/\bar{x} \\ \nu(\mu x - \bar{\lambda}\bar{x} - \lambda\bar{x}x/\bar{x}) + \frac{1}{\nu}\mu\bar{\lambda}\lambda\bar{x} & \nu^2 + \bar{\lambda}\lambda\bar{x}/x - \mu\bar{\lambda}\bar{x}/x - \mu\lambda\bar{x}/\bar{x} \end{pmatrix}$$

Since the prefactor cancels in the compatibility condition (4.22), we take N to be

$$N = \begin{pmatrix} \nu^2 a_2 + a_0 & \nu b_1 + b_0/\nu \\ \nu c_1 + c_0/\nu & \nu^2 d_2 + d_0 \end{pmatrix} \quad (4.21)$$

where the coefficients a_i, b_i, c_i and d_i are functions of l only. Before we continue, we will assume $\nu(n) = q^n$ and keep ν as the spectral variable in L and N . As we are seeking a Lax Pair with coefficient matrices N, L that depend on a spectral parameter ν , x should be independent of ν . Hence we take $\tilde{x} = x$ in the following.

Now the coefficients of the various powers of ν in N are determined by the compatibility condition

$$\tilde{L}N = \bar{N}L \quad (4.22)$$

which is the compatibility condition of the system (4.18). Going through the calculations in detail would be somewhat tedious so only a guide will be given here. The compatibility condition gives a total of ten equations, three in each of the diagonal entries and two in the off-diagonal entries. The equations in the diagonal entries at

order ν^2 and ν^{-2} are solved in a straightforward manner, yielding

$$\begin{aligned} a_2 &= \text{constant} \\ d_2 &= \text{constant} \\ b_0 &= \frac{k_1}{x\sigma} \\ c_0 &= \frac{k_1 x}{\sigma} \end{aligned}$$

where k_1 is a constant and $\sigma = q^l$. The remaining six equations read as follows:

$$\bar{a}_0 - a_0 = \lambda(x\bar{b}_1 - c_1/q\bar{x}) \quad (4.24a)$$

$$\bar{d}_0 - d_0 = \lambda(\bar{c}_1/x - \bar{x}b_1/q) \quad (4.24b)$$

$$\bar{b}_1 x - b_1 \bar{x} = \lambda(a_2 - d_2/q) \quad (4.24c)$$

$$\bar{c}_1/x - c_1/\bar{x} = \lambda(d_2 - a_2/q) \quad (4.24d)$$

$$\bar{a}_0 - d_0/q = \frac{k_1}{\lambda\sigma} \left(\frac{x}{q\bar{x}} - \frac{\bar{x}}{x} \right) \quad (4.24e)$$

$$\bar{d}_0 - a_0/q = \frac{k_1}{\lambda\sigma} \left(\frac{\bar{x}}{qx} - \frac{x}{\bar{x}} \right) \quad (4.24f)$$

To solve these, use (4.24c) to replace $\bar{b}_1 x$ in (4.24a), and the resulting expression to replace \bar{a}_0 in (4.24e). Now do the same with (4.24d) and (4.24b) in (4.24f), then solve these two equations for a_0 and d_0 to find

$$a_0 = -\frac{k_1 \bar{x}}{\lambda\sigma x} - \lambda b_1 \bar{x} - \lambda^2 a_2 \quad (4.25a)$$

$$d_0 = -\frac{k_1 x}{\lambda\sigma \bar{x}} - \frac{\lambda c_1}{\bar{x} - \lambda^2 d_2} \quad (4.25b)$$

One can now use (4.25a) and (4.25b) in equations (4.24a) and (4.24b), replace \bar{b}_1 and \bar{c}_1 via (4.24c) and (4.24d), then solve the remainder for b_1 and c_1 . All this reduces to

$$a_0 = \frac{a_2 \lambda \bar{\lambda} x}{\bar{x}} - \frac{k_1 \bar{x}}{\sigma} \left(\frac{1}{\bar{\lambda} \bar{x}} + \frac{1}{\lambda x} \right) \quad (4.26a)$$

$$d_0 = \frac{d_2 \lambda \bar{\lambda} \bar{x}}{x} - \frac{k_1}{\sigma \bar{x}} \left(\frac{\bar{x}}{\bar{\lambda}} + \frac{x}{\lambda} \right) \quad (4.26b)$$

$$b_1 = \frac{k_1}{\lambda \bar{\lambda} \sigma \bar{x}} - \frac{a_2}{\bar{x}} \left(\lambda + \bar{\lambda} \frac{x}{\bar{x}} \right) \quad (4.26c)$$

$$c_1 = \frac{k_1 \bar{x}}{\lambda \bar{\lambda} \sigma} - d_2 \bar{x} \left(\lambda + \bar{\lambda} \frac{\bar{x}}{x} \right) \quad (4.26d)$$

Finally, these calculated values should be substituted back into equations (4.24a)–(4.24f). On doing this and making the substitution $y = \bar{x}/\bar{x}$, we find that one of the two following forms of qP_{II} arises in each case

$$\frac{y\bar{y}}{y} = \frac{1}{q} \frac{k_1 \lambda y - a_2 \lambda \bar{\lambda} \bar{\lambda}^2 q \sigma}{k_1 \bar{\lambda} - d_2 \lambda^2 \bar{\lambda} \bar{\lambda} \sigma y} \quad (4.27)$$

$$\frac{y\bar{y}}{y} = \frac{1}{k_1 \lambda - d_2 \lambda \bar{\lambda} \bar{\lambda}^2 q \sigma y} \frac{k_1 \bar{\lambda} q y - a_2 \lambda^2 \bar{\lambda} \bar{\lambda} \sigma}{\lambda} \quad (4.28)$$

Equations (4.27) and (4.28) can be reconciled by setting $\log \lambda = k_2 + k_3(-1)^l + \frac{1}{2} \log q$, with constant k_2 and k_3 , which gives the same form of qP_{II} that was given earlier (and found in [86]) by a reduction from the LMKdV equation. Hence we have calculated a Lax Pair for qP_{II} , which explicitly takes the form

$$\theta(l+1, n) = L(l, \nu(n))\theta(l, n) \quad (4.29a)$$

$$\theta(l, n+1) = N(l, \nu(n))\theta(l, n) \quad (4.29b)$$

where

$$L = \begin{pmatrix} \frac{\bar{x}}{x} & -\frac{\lambda}{\nu x} \\ -\frac{\lambda \bar{x}}{\nu} & 1 \end{pmatrix}, \quad (4.30)$$

$$N = \begin{pmatrix} a_2 \nu^2 + \frac{a_2 \lambda \bar{\lambda} x}{\bar{x}} - \frac{k_1 \bar{x}}{\sigma} \left(\frac{1}{\lambda \bar{x}} + \frac{1}{\lambda x} \right) & \nu \left[\frac{k_1}{\lambda \lambda \sigma \bar{x}} - \frac{a_2}{x} \left(\lambda + \bar{\lambda} \frac{x}{\bar{x}} \right) \right] + \frac{k_1}{\nu \sigma x} \\ \nu \left[\frac{k_1 \bar{x}}{\lambda \lambda \sigma} - d_2 \bar{x} \left(\lambda + \bar{\lambda} \frac{\bar{x}}{x} \right) \right] + \frac{k_1 x}{\nu \sigma} & d_2 \nu^2 + \frac{d_2 \lambda \bar{\lambda} \bar{x}}{x} - \frac{k_1}{\sigma \bar{x}} \left(\frac{\bar{x}}{\lambda} + \frac{x}{\lambda} \right) \end{pmatrix} \quad (4.31)$$

4.4.2 Lax Pair for qP_{III}

The next reduction to be considered is that taking LMKdV to qP_{III} , i.e., $\hat{x} = 1/\bar{x}$. The reciprocal in the latter reduction introduces a difference in the method used to find the corresponding Lax Pair. We now have

$$\hat{\theta} = \hat{t} \begin{pmatrix} 1 \\ \hat{x} \end{pmatrix} = \hat{t} \begin{pmatrix} 1 \\ 1/\bar{x} \end{pmatrix} = \hat{t}/\bar{x} \begin{pmatrix} \bar{x} \\ 1 \end{pmatrix}$$

and

$$\bar{\theta} = \bar{t} \begin{pmatrix} 1 \\ \bar{x} \end{pmatrix} = \frac{\bar{x} \bar{t}}{\bar{t}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{v}$$

So this time the suggested form of N is the same as $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} LM^{-1}$ and, as such, we choose

$$N = \begin{pmatrix} a/\nu & b_0 + b_2/\nu^2 \\ c_0 + c_2/\nu^2 & d/\nu \end{pmatrix} \quad (4.32)$$

In this case the compatibility condition contains eight equations that are solved in a similar way to before, and these lead to

$$a = -\lambda k_1 x \bar{x} - \frac{k_2 \bar{x}}{\lambda \sigma x} \quad (4.33a)$$

$$b_0 = k_1 x \quad (4.33b)$$

$$b_2 = \frac{k_2}{\sigma x} \quad (4.33c)$$

$$c_0 = \frac{k_3}{x} \quad (4.33d)$$

$$c_2 = \frac{k_2 x}{\sigma} \quad (4.33e)$$

$$d = -\frac{\lambda k_3}{x \bar{x}} - \frac{k_2 x}{\lambda \sigma \bar{x}} \quad (4.33f)$$

where k_2, k_1 and k_3 are all constants, $\lambda = \delta q^{-l}$ where $\delta = \text{constant}$ and $\sigma = q^l$.

The final form of the qP_{III} equation obtained from the compatibility conditions is

$$x \bar{x} = \frac{\mu_1 q^l \bar{x}^2 + \mu_2}{\mu_3 q^l + \bar{x}^2} \quad (4.34)$$

which is a non-autonomous equation with three free parameters μ_i . The Lax Pair is of the form

$$\theta(l+1, n) = L(l, \nu(n)) \theta(l, n) \quad (4.35a)$$

$$\theta(l, n+1) = N(l, \nu(n)) \theta(l, n) \quad (4.35b)$$

where L as before is given by

$$L = \begin{pmatrix} \frac{\bar{x}}{x} & -\frac{\lambda}{\nu x} \\ -\frac{\lambda \bar{x}}{\nu} & 1 \end{pmatrix}, \quad (4.36)$$

and

$$N = \begin{pmatrix} -\frac{1}{\nu}(\lambda k_1 x \bar{x} + \frac{k_2 \bar{x}}{\lambda \sigma x}) & k_1 x + \frac{k_2}{\nu^2 \sigma x} \\ \frac{k_3}{x} + \frac{k_2 x}{\nu^2 \sigma} & -\frac{1}{\nu}(\frac{\lambda k_3}{x \bar{x}} + \frac{k_2 x}{\lambda \sigma \bar{x}}) \end{pmatrix} \quad (4.37)$$

4.4.3 Lax Pair for qP_V

Lastly, a Lax Pair for qP_V (see equation (4.14)) is presented. Following the previous analysis, we begin with N of the same form as $\overline{\overline{LLM}}^{-1}$ or

$$N = \begin{pmatrix} a_1 \nu^2 + a_0 + a_2 / \nu^2 & b_1 \nu + b_0 / \nu \\ c_1 \nu + c_0 / \nu & d_1 \nu^2 + d_0 + d_2 / \nu^2 \end{pmatrix} \quad (4.38)$$

After similar processes to those used earlier, we arrive at the following, where $\log(T_2) = K_1 + K_2(-1)^l$, K_i constant, is a function of period two, and $\sigma = q^l$ as before.

$$\begin{aligned} a_0 &= \frac{T_2 \bar{x}}{\sigma \bar{x}} \left(\frac{\bar{x}}{\lambda \lambda x} + \frac{\bar{x} \bar{x}}{\lambda \bar{\lambda} x \bar{x}} + \frac{1}{\bar{\lambda} \bar{\lambda}} \right) + a_1 \left(\lambda \bar{\lambda} \frac{x}{\bar{x}} + \lambda \bar{\lambda} \frac{x \bar{x}}{\bar{x} \bar{x}} + \bar{\lambda} \bar{\lambda} \frac{\bar{x}}{x} \right) \\ a_1 &= \text{constant} \\ a_2 &= \frac{T_2}{\sigma} \\ d_0 &= \frac{\bar{T}_2 \bar{x}}{\sigma \bar{x}} \left(\frac{x}{\lambda \lambda \bar{x}} + \frac{x \bar{x}}{\lambda \bar{\lambda} \bar{x} \bar{x}} + \frac{1}{\lambda \bar{\lambda}} \right) + d_1 \left(\lambda \bar{\lambda} \frac{\bar{x}}{x} + \lambda \bar{\lambda} \frac{\bar{x} \bar{x}}{x \bar{x}} + \bar{\lambda} \bar{\lambda} \frac{\bar{x}}{x} \right) \\ d_1 &= \text{constant} \\ d_2 &= \frac{\bar{T}_2}{\sigma} \\ b_0 &= -\frac{T_2 \bar{x}}{\sigma x} \left(\frac{1}{\bar{\lambda} \bar{x}} + \frac{1}{\bar{\lambda} \bar{x}} + \frac{x}{\lambda \bar{x} \bar{x}} \right) - a_1 \frac{\lambda \bar{\lambda} \bar{x}}{\bar{x}} \\ b_1 &= -\frac{T_2}{\lambda \bar{\lambda} \bar{\lambda} \sigma \bar{x}} - \frac{a_1}{\bar{x}} \left(\lambda \frac{\bar{x}}{x} + \bar{\lambda} \frac{x \bar{x}}{\bar{x} \bar{x}} + \bar{\lambda} \frac{x}{\bar{x}} \right) \\ c_0 &= -\frac{\bar{T}_2 x}{\sigma \bar{x}} \left(\frac{\bar{x}}{\lambda} + \frac{\bar{x}}{\bar{\lambda}} + \frac{\bar{x} \bar{x}}{\lambda x} \right) - d_1 \lambda \bar{\lambda} \bar{x} \\ c_1 &= -\frac{\bar{T}_2 \bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \sigma} - d_1 \bar{x} \left(\lambda \frac{\bar{x}}{\bar{x}} + \bar{\lambda} \frac{\bar{x} \bar{x}}{x \bar{x}} + \bar{\lambda} \frac{\bar{x}}{x} \right) \end{aligned}$$

We also find that $\log \lambda = k_1 + k_2 j_3^l + k_3 j_3^l - ql/3$, $j_3^3 = 1$ and the form of the

resulting evolution equation is

$$\underline{y\bar{y}} = \frac{q\bar{T}_2\bar{\bar{\lambda}}y + a_1\lambda^2\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\lambda}}\nu}{T_2\lambda + qd_1\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\lambda}}\nu y} \quad (4.39)$$

where we have made the substitution $y = \bar{\bar{x}}/\bar{x}$.

In this case, the Lax pair takes the form

$$\theta(l+1, n) = L(l, \nu(n))\theta(l, n) \quad (4.40a)$$

$$\theta(l, n+1) = N(l, \nu(n))\theta(l, n) \quad (4.40b)$$

where L as before is given by

$$L = \begin{pmatrix} \frac{\bar{x}}{x} & -\frac{\lambda}{\nu x} \\ -\frac{\lambda\bar{x}}{\nu} & 1 \end{pmatrix}, \quad (4.41)$$

and we write N as

$$N = N_2\nu^2 + N_1\nu + N_0 + \frac{N_{-1}}{\nu} + \frac{N_{-2}}{\nu^2} \quad (4.42)$$

where

$$\begin{aligned} N_2 &= \begin{pmatrix} a_1 & 0 \\ 0 & d_1 \end{pmatrix} \\ N_1 &= \begin{pmatrix} 0 & -\frac{T_2}{\lambda\bar{\lambda}\sigma\bar{x}} - \frac{a_1}{\bar{x}}(\lambda\frac{\bar{x}}{x} + \bar{\lambda}\frac{x\bar{x}}{\bar{x}\bar{x}} + \bar{\bar{\lambda}}\frac{x}{\bar{x}}) \\ -\frac{\bar{T}_2\bar{x}}{\lambda\bar{\lambda}\sigma} - d_1\bar{x}(\lambda\frac{\bar{x}}{x} + \bar{\lambda}\frac{x\bar{x}}{\bar{x}\bar{x}} + \bar{\bar{\lambda}}\frac{x}{\bar{x}}) & 0 \end{pmatrix} \\ N_0 &= \begin{pmatrix} \frac{T_2\bar{x}}{\sigma\bar{x}}(\frac{\bar{x}}{\lambda\lambda x} + \frac{x\bar{x}}{\lambda\lambda x\bar{x}} + \frac{1}{\lambda\bar{\lambda}}) + a_1(\lambda\bar{\lambda}\frac{x}{\bar{x}} + \lambda\bar{\bar{\lambda}}\frac{x\bar{x}}{\bar{x}\bar{x}} + \bar{\bar{\lambda}}\bar{\bar{\lambda}}\frac{x}{\bar{x}}) & \\ & 0 \end{pmatrix} \\ N_{-1} &= \begin{pmatrix} 0 & \frac{T_2\bar{x}}{\sigma\bar{x}}(\frac{x}{\lambda\bar{\lambda}} + \frac{x\bar{x}}{\lambda\bar{\lambda}x\bar{x}} + \frac{1}{\lambda\bar{\lambda}}) + d_1(\lambda\bar{\lambda}\frac{\bar{x}}{x} + \lambda\bar{\bar{\lambda}}\frac{x\bar{x}}{\bar{x}\bar{x}} + \bar{\bar{\lambda}}\bar{\bar{\lambda}}\frac{\bar{x}}{x}) \\ -\frac{T_2\bar{x}}{\sigma\bar{x}}(\frac{\bar{x}}{\lambda} + \frac{\bar{x}}{\lambda} + \frac{x\bar{x}}{\lambda x}) - d_1\lambda\bar{\bar{\lambda}}\bar{\bar{\lambda}}\bar{x} & 0 \end{pmatrix} \end{aligned}$$

and

$$N_{-2} = \begin{pmatrix} \frac{T_2}{\sigma} & 0 \\ 0 & \frac{\bar{T}_2}{\sigma} \end{pmatrix}.$$

4.5 Discussion

In this chapter we have presented a new Lax Pair for a lattice, non-autonomous, modified Korteweg de Vries equation and shown that it forms a consistent multi-dimensional system when considered together with its Lax pair. We also gave reductions of this non-autonomous LMKdV to q -difference Painlevé equations and found the Lax Pairs corresponding to those reduced equations. A notable feature of these results is that they provide 2×2 Lax pairs for the first time for these versions of qP_{II} , qP_{III} , and qP_V .

It is worth noting that only one simple form of reduction was investigated here. It remains to be seen whether other types of reductions, that is other forms of $f(\mathbf{x})$ in (4.7), can be used with the LMKdV or other lattice equations.

We note that, so far, there appears not to be a direct method for reducing the lattice Lax pair L, M to the ordinary difference equation's Lax pair L, N . There is a jump in our process of finding N after reduction. The main obstacle is that it is not known whether equations of the form (5.2b) can be solved to find N directly. Instead, we have chosen to use the form of the equation to motivate the dependence of N on the spectral parameter ν and then used the compatibility conditions to deduce the entries of N .

A feature of the Lax pairs we deduce is that they share the same L . We note here that the calculation of a series of Lax Pairs is also possible for other reductions, including the higher-order difference equations found in §2. These would also share the same L matrix. This is analogous to the case of integrable differential-equation hierarchies and suggests the existence of a hierarchy for each of the reductions we have studied here. An open problem is to find reductions of lattice equations to infinite hierarchies of q -difference equations along with their Lax pairs. It would be eventually interesting to find reductions from lattice equations to the q -Garnier hierarchy constructed by Sakai in [35].

It is striking to note also that very little information is known about the generic solutions of q -Painlevé equations. To our knowledge, Birkhoff's theory of linear q -difference equations has not been applied to deduce information about the solutions of q -Painlevé equations. The question of the global properties of solutions remains completely open.

Chapter 5

Hierarchies of equations

5.1 Introduction

We have seen in chapter 4 that reductions constitute a natural connection between lattice equations and ordinary difference equations. The LMKdV equation is de-autonomized by allowing p and r to depend on l and m and there are known reductions from the non-autonomous LMKdV equation to q -discrete forms of the second, third and fifth Painlevé equations, denoted qP_{II} , qP_{III} , and qP_V respectively [86, 2]. Different types of these reductions are possible, one of the simplest of which is to set $x_{l,m+1} = x_{l+d,m}$ for some positive integer d . In fact the reductions that take the LMKdV equations to qP_{II} and qP_V are of this type with $d = 2$ and $d = 3$ respectively. There apparently exist an infinite series of such reductions that result in equations of arbitrary order, which suggests the existence of a hierarchy of equations. In a recent paper [2] we established a connection between these non-autonomous reductions and reductions of a Lax pair for the LMKdV equation itself. In this way, Lax pairs for non-autonomous versions of qP_{II} , qP_{III} , and qP_V with multiple free parameters were discovered.

Here we find Lax pairs for higher order equations and we thereby lay the ground-

work for a hierarchy of equations. Two hierarchies are shown to exist, at the base of one lies qP_{II} and qP_V , while qP_{III} lies at the base of the other.

While there is a relatively large body of literature focussing on continuous Painlevé hierarchies [89, 114, 115] and some results concerning hierarchies of d -discrete nonlinear equations [99, 89], the problem of q -discrete hierarchies has been more elusive. Although a hierarchy of integrable nonlinear q -difference equations was found in [98], we believe that the results presented in this work are first example of such a hierarchy found by expansions of Lax pairs.

The chapter is organized as follows. In section 5.2, we derive the formulas used to calculate quantities that exactly describe an equation in one of the hierarchies. These are given in terms of the same quantities describing the equation at a lower order and so we thereby obtain a recursive method of finding the hierarchies. We go further to purport a general formula that yields those quantities for any member of either hierarchy, and thus any equation in the hierarchies with its Lax pair. In section 5.3, we clarify the application of the formulas found in section 5.2 and use them to confirm a known result. In section 5.4, we derive new equations and Lax pairs. We conclude the chapter with a discussion where we also point out some open problems.

5.2 How to Construct the hierarchy

In this section we will establish the existence of two hierarchies of nonlinear, integrable, ordinary q -difference equations that are each obtainable from the LMKdV equation via a reduction. In section 5.2.1, the procedure for constructing the hierarchies will be derived in relation to the first hierarchy, which corresponds to reductions of the type $x_{l,m+1} = x_{l+d,m}$ for some integer d . Subsequently, in section 5.2.2 we shall outline an analogous process that leads to the second hierarchy corresponding to reductions of the type $x_{l,m+1} = 1/x_{l+d,m}$.

We establish the existence of the hierarchies by developing formulas that construct a member of the hierarchy from the preceding, lower order member. However, rather than iterating the equation or terms in the Lax pair directly, as has been the procedure used for some other systems [99], we will derive formulas for iterating a set of coefficients, introduced in equation (5.14), that describe the Lax pairs for the equations in the hierarchy.

5.2.1 Hierarchy corresponding to reductions of the type $x_{l,m+1} =$

$$x_{l+d,m}$$

Begin with the linear problem

$$\begin{aligned}\theta(l+1, n) &= L(l, n)\theta(l, n), \\ \theta(l, n+1) &= N(l, n)\theta(l, n).\end{aligned}\tag{5.1}$$

whose compatibility condition is $L(l, n+1)N(l, n) = N(l+1, n)L(l, n)$. Hereafter we adopt the notation $\bar{\theta} = \theta(l+1, n)$ and $\tilde{\theta} = \theta(l, n+1)$. Now set

$$L = \begin{pmatrix} \bar{x}/x & -\nu/(\lambda x) \\ -\nu\bar{x}/\lambda & 1 \end{pmatrix},\tag{5.2a}$$

$$N = \begin{pmatrix} a_0 + a_2\nu^2 + \dots + a_{2\rho}\nu^{2\rho} & b_1\nu + b_3\nu^3 + \dots + b_{2\rho\pm 1}\nu^{2\rho\pm 1} \\ c_1\nu + c_3\nu^3 + \dots + c_{2\rho\pm 1}\nu^{2\rho\pm 1} & d_0 + d_2\nu^2 + \dots + d_{2\rho}\nu^{2\rho} \end{pmatrix}\tag{5.2b}$$

where k is associated with the spectral variable n such that $\nu = \nu_0 q^n$, and x , λ and all of a_i, b_i, c_i, d_i are functions of l alone. The diagonal entries of the N matrix in the Lax pair contain only even powers of ν , including a term constant in ν , up to $\nu^{2\rho}$ where ρ is a positive integer. The off-diagonals of N contain only odd powers of ν up to $\nu^{2\rho\pm 1}$ depending on which part of the hierarchy we are considering.

Compatibility occurs when $\tilde{L}N = \bar{N}L$. It is not difficult to show that the

compatibility condition can be written as follows

$$\bar{a}_i = a_i + \frac{1}{\lambda}(x\bar{b}_{i-1} - q\frac{c_{i-1}}{\bar{x}}), \quad i \text{ even} \quad (5.3a)$$

$$x\bar{b}_i = \bar{x}b_i + \frac{1}{\lambda}(\bar{a}_{i-1} - qd_{i-1}), \quad i \text{ odd} \quad (5.3b)$$

$$\frac{\bar{c}_i}{x} = \frac{c_i}{\bar{x}} + \frac{1}{\lambda}(\bar{d}_{i-1} - qa_{i-1}), \quad i \text{ odd} \quad (5.3c)$$

$$\bar{d}_i = d_i + \frac{1}{\lambda}(\frac{\bar{c}_{i-1}}{x} - q\bar{x}b_{i-1}), \quad i \text{ even} \quad (5.3d)$$

corresponding to entries (1, 1), (1, 2), (2, 1) and (2, 2) respectively. Some equivalences may be found between equations (5.3) if, at this point, we introduce the quantities

$$A_i = \begin{cases} a_i, & i \text{ even} \\ \bar{x}b_i, & i \text{ odd} \end{cases} \quad (5.4)$$

$$D_i = \begin{cases} d_i, & i \text{ even} \\ c_i/\bar{x}, & i \text{ odd} \end{cases} \quad (5.5)$$

so that equations (5.3) become

$$\frac{x}{\bar{x}}\bar{A}_{i-1} = \lambda(\bar{A}_i - A_i) + qD_{i-1}, \quad i \text{ even} \quad (5.6a)$$

$$\bar{A}_{i-1} = \lambda(\frac{x}{\bar{x}}\bar{A}_i - A_i) + qD_{i-1}, \quad i \text{ odd} \quad (5.6b)$$

$$\bar{D}_{i-1} = \lambda(\frac{\bar{x}}{x}\bar{D}_i - D_i) + qA_{i-1}, \quad i \text{ odd} \quad (5.6c)$$

$$\frac{\bar{x}}{x}\bar{D}_{i-1} = \lambda(\bar{D}_i - D_i) + qA_{i-1}, \quad i \text{ even} \quad (5.6d)$$

In equations (5.6) we may substitute

$$X_i = \left(\frac{\bar{x}}{x}\right)^{\frac{1-(-1)^i}{2}} = \begin{cases} \bar{x}/x, & i \text{ odd} \\ 1, & i \text{ even} \end{cases}$$

so that either equation (5.6a) or (5.6b), with i even or odd respectively, will become

$$\frac{\bar{A}_{i-1}}{X_{i-1}} = qD_{i-1} + \lambda\left(\frac{\bar{A}_i}{X_i} - A_i\right), \quad \forall i \quad (5.7)$$

and similarly equations (5.6c) and (5.6d), with i odd or even respectively, become

$$\bar{D}_{i-1}X_{i-1} = qA_{i-1} + \lambda(\bar{D}_iX_i - D_i), \quad \forall i \quad (5.8)$$

By repeated use of equations (5.7) and (5.8) respectively, we arrive at the following

$$\bar{A}_i = X_i[qD_i - \sum_{j=i+1}^m \lambda^{j-i}(A_j - qD_j)] \quad (5.9a)$$

$$\bar{D}_i = \frac{1}{X_i}[qA_i - \sum_{j=i+1}^m \lambda^{j-i}(D_j - qA_j)] \quad (5.9b)$$

Where m is equal to the greatest degree of the polynomials in k located in the entries of the N matrix (5.2b), *i.e.* m is either 2ρ or $2\rho + 1$. Now add $q/X_i \times (5.9a)$ to $X_i \times (5.9b)$ so that

$$\frac{q}{X_i}\bar{A}_i + X_i\bar{D}_i = q^2D_i + qA_i + \sum_{j=i+1}^m (q^2 - 1)\lambda^{j-i}D_j \quad (5.10)$$

However, at $i = 0$, (5.6) shows that both A_0 and D_0 are constant, meaning that (5.10) at $i = 0$ becomes

$$-D_0 = \sum_{j=1}^m \lambda^j D_j$$

which we can rearrange so that

$$-D_1 = D_0/\lambda + \sum_{j=2}^m \lambda^{j-1} D_j \quad (5.11)$$

Similarly $q(5.9a) + (5.9b)$ gives us an expression for A_1

$$-A_1 = A_0/\lambda + \sum_{j=2}^m \lambda^{j-1} A_j \quad (5.12)$$

These expressions for A_1 and D_1 , (5.11) and (5.12), can be substituted back into (5.7) and (5.8) to find expressions for A_2 and D_2 in terms of A_i, D_i with $i > 2$. We can continue this process to successively calculate all the terms in the Lax pair, A_i and D_i , thus resolving the Lax pair for a particular value of m . However, in the interest of establishing the existence of a hierarchy of equations, we will proceed to

derive a formula for calculating successive iterates from previous ones. To do this we first rewrite (5.9a) as

$$-\bar{A}_i + qX_i D_i + X_i \sum_{j=i+1}^m \lambda^{j-i} (qD_j - A_j) = 0 \quad (5.13)$$

We aim to calculate each of the quantities in the Lax pair, A_i, D_i , in terms of the remaining quantities, $A_j, D_j \forall j > i$ and A_0, D_0 . In general any A_i of interest might be found in terms of all A_j and $D_j, \forall j > i$. However, when the calculations are performed, it is observed that terms A_i only depend terms A_j (not D_j) so it is conjectured that we can write any A_i or D_i as

$$-A_i = \alpha_0^i A_0 + \sum_{j=i+1}^m \alpha_j^i A_j \quad (5.14a)$$

$$-D_i = \delta_0^i D_0 + \sum_{j=i+1}^m \delta_j^i D_j \quad (5.14b)$$

Where we have introduced a series of coefficients α_j^i and δ_j^i that need to be found. Substitute the expansion (5.14a) into (5.9a), noting that $A_0 = \text{constant}$, to get

$$\bar{\alpha}_0^i A_0 + \sum_{j=i+1}^m \bar{\alpha}_j^i \bar{A}_j + qX_i D_i + X_i \sum_{j=i+1}^m \lambda^{j-i} (qD_j - A_j) = 0$$

which, upon exploiting (5.9a) again to replace \bar{A}_j , becomes

$$\begin{aligned} \bar{\alpha}_0^i A_0 - qX_i \delta_0^i D_0 + \sum_{j=i+1}^m \bar{\alpha}_j^i X_j \sum_{k=j+1}^m \lambda^{k-j} (qD_k - A_k) \\ + \sum_{j=i+1}^m \{D_j q[X_i(\lambda^{j-i} - \delta_j^i) + \bar{\alpha}_j^i X_j] - \lambda^{j-i} A_j X_i\} = 0 \end{aligned}$$

Now we may rearrange the double sum to arrive at the following

$$\begin{aligned} \bar{\alpha}_0^i A_0 - qX_i \delta_0^i D_0 + D_{i+1} q[X_i(\lambda - \delta_{i+1}^i) + \bar{\alpha}_{i+1}^i X_{i+1}] - \lambda A_{i+1} X_i \\ + \sum_{j=i+2}^m \{D_j q[X_i(\lambda^{j-i} - \delta_j^i) + \bar{\alpha}_j^i X_j + \sum_{k=i+1}^{j-1} \lambda^{j-k} X_k \bar{\alpha}_k^i] \\ - A_j(\lambda^{j-i} X_i + \sum_{k=i+1}^{j-1} \lambda^{j-k} X_k \bar{\alpha}_k^i)\} = 0 \quad (5.15) \end{aligned}$$

A repeat of this process beginning with (5.9b) brings us to

$$\begin{aligned} \bar{\delta}_0^i D_0 - q \frac{\alpha_0^i}{X_i} A_0 + A_{i+1} q \left[\frac{1}{X_i} (\lambda - \alpha_{i+1}^i) + \frac{\bar{\delta}_{i+1}^i}{X_{i+1}} \right] - \frac{\lambda D_{i+1}}{X_i} \\ + \sum_{j=i+2}^m \left\{ A_j q \left[\frac{1}{X_i} (\lambda^{j-i} - \alpha_j^i) + \frac{\bar{\delta}_j^i}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \bar{\delta}_k^i}{X_k} \right] \right. \\ \left. - D_j \left(\frac{\lambda^{j-i}}{X_i} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \bar{\delta}_k^i}{X_k} \right) \right\} = 0 \quad (5.16) \end{aligned}$$

To calculate the next sets of coefficients α_j^{i+1} and $\bar{\delta}_j^{i+1}$, $j = i + 2, \dots, m$, we must combine equations (5.15) and (5.16) in the correct way. We claim that we can eliminate the quantities D_i by adding $q(5.16) + \bar{\delta}_0^i / (\delta_0^i X_i)$ (5.15), which tallies to

$$\begin{aligned} A_0 \left(\frac{\bar{\alpha}_0^i \bar{\delta}_0^i}{X_i \delta_0^i} - q^2 \frac{\alpha_0^i}{X_i} \right) + A_{i+1} \left\{ q^2 \left[\frac{1}{X_i} (\lambda - \alpha_{i+1}^i) + \frac{\bar{\delta}_{i+1}^i}{X_{i+1}} \right] - \frac{\lambda \bar{\delta}_{i+1}^i}{\delta_0^i} \right\} \\ + \sum_{j=i+2}^m A_j \left\{ q^2 \left[\frac{1}{X_i} (\lambda^{j-i} - \alpha_j^i) + \frac{\bar{\delta}_j^i}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \bar{\delta}_k^i}{X_k} \right] - \left(\frac{\lambda^{j-i} \bar{\delta}_0^i}{\delta_0^i} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} X_k \bar{\delta}_0^i \bar{\alpha}_k^j}{X_i \delta_0^i} \right) \right\} = 0 \quad (5.17) \end{aligned}$$

It is not obvious that every D_i should be canceled out in the above sum but this occurs in every calculation performed to date and we conjecture that it is always the case. From here we can make $-A_{i+1}$ the subject and so find the sought after coefficients

$$\begin{aligned} -A_{i+1} = A_0 \frac{\bar{\alpha}_0^i \bar{\delta}_0^i - q^2 \alpha_0^i \delta_0^i}{q^2 [\delta_0^i (\lambda - \alpha_0^i) + \frac{X_i}{X_{i+1}} \delta_0^i \bar{\delta}_{i+1}^i] - \lambda X_i \bar{\delta}_0^i} \\ + \sum_{j=i+2}^m A_j \left\{ \frac{q^2 [\delta_0^i (\lambda^{j-i} - \alpha_j^i) + \frac{\delta_0^i \bar{\delta}_j^i}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \delta_0^i \bar{\delta}_k^i X_i}{X_k}] - (\lambda^{j-i} \bar{\delta}_0^i X_i + \sum_{k=i+1}^{j-1} \lambda^{j-k} \bar{\delta}_0^i \bar{\alpha}_k^j)}{q^2 [\delta_0^i (\lambda - \alpha_0^i) + \frac{X_i}{X_{i+1}} \delta_0^i \bar{\delta}_{i+1}^i] - \lambda X_i \bar{\delta}_0^i} \right\} \quad (5.18) \end{aligned}$$

Comparing (5.18) with (5.14a) shows that

$$\alpha_0^{i+1} = \frac{\bar{\alpha}_0^i \bar{\delta}_0^i - q^2 \alpha_0^i \delta_0^i}{q^2 \delta_0^i [\lambda - \alpha_{i+1}^i + \frac{X_i}{X_{i+1}} \bar{\delta}_{i+1}^i] - \lambda X_i \bar{\delta}_0^i} \quad (5.19a)$$

$$\alpha_j^{i+1} = \frac{1}{G_i(\alpha, \delta, \mathbf{X})} \left\{ q^2 \delta_0^i [\lambda^{j-i} - \alpha_j^i + \frac{\bar{\delta}_j^i X_i}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \bar{\delta}_k^i X_i}{X_k}] - \bar{\delta}_0^i (\lambda^{j-i} X_i + \sum_{k=i+1}^{j-1} \lambda^{j-k} \bar{\alpha}_k^i X_k) \right\} \quad (5.19b)$$

Where G_i in (5.19b) is the same as the denominator in (5.19a). In fact we shall say

$$\alpha_0^{i+1} = \frac{H_0^i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})}{G_i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})} \quad (5.20a)$$

$$\alpha_j^{i+1} = \frac{H_j^i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})}{G_i(\boldsymbol{\alpha}, \boldsymbol{\delta}, \mathbf{X})} \quad (5.20b)$$

The H and G quantities in (5.20) are defined by comparison with (5.19) where we have introduced the bold face notation $\boldsymbol{\alpha}$ to signify all $\alpha, \bar{\alpha}$, etc, with any superscripts and subscripts.

Naturally, we must also repeat the operations to find an expression for the other coefficients δ_j^i , this begins with adding $q(5.15) + \bar{\alpha}_0^i X_i / \alpha_0^i$ (5.16), and leads to

$$\delta_0^{i+1} = \frac{\bar{\delta}_0^i \bar{\alpha}_0^i - q^2 \delta_0^i \alpha_0^i}{q^2 \alpha_0^i [\lambda - \delta_{i+1}^i + \frac{X_{i+1}}{X_i} \bar{\alpha}_{i+1}^i] - \frac{\lambda \bar{\alpha}_0^i}{X_i}} \quad (5.21a)$$

$$\delta_j^{i+1} = \frac{1}{G_i(\boldsymbol{\delta}, \boldsymbol{\alpha}, 1/\mathbf{X})} \left\{ q^2 \alpha_0^i [\lambda^{j-i} - \delta_j^i + \frac{\bar{\alpha}_j^i X_j}{X_i} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \bar{\alpha}_k^i X_k}{X_i}] - \bar{\alpha}_0^i (\lambda^{j-i} / X_i + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \bar{\delta}_k^i}{X_k}) \right\} \quad (5.21b)$$

Notice that the quantities H and G from (5.20) arise again in equations (5.21) but this time as

$$\delta_0^{i+1} = \frac{H_0^i(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}})}{G_i(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}})} \quad (5.22a)$$

$$\delta_j^{i+1} = \frac{H_j^i(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}})}{G_i(\boldsymbol{\delta}, \boldsymbol{\alpha}, \frac{1}{\mathbf{X}})} \quad (5.22b)$$

Importantly, since $\alpha_j^1 = \delta_j^1$, (5.20) and (5.22) indicate that $\alpha_j^i(\boldsymbol{\lambda}, \mathbf{x}) = \delta_j^i(\boldsymbol{\lambda}, \frac{1}{\mathbf{x}})$ for $i > 1$. Hence, we only need to calculate the coefficients α_j^i in practice as δ_j^i follow from these results.

Because these coefficients exactly describe a Lax pair and an associated nonlinear equation, we have shown that this system does indeed constitute a hierarchy by constructing a general operation that takes the members at any level of the hierarchy to the next level. To find a particular Lax pair in the hierarchy, we truncate the series at some point A_m, D_m say, and use the coefficients to calculate each of the terms A_i, D_i that appear in the N matrix of the Lax pair, the L matrix is always the

same. The equation associated with any Lax pair can be found via the compatibility condition. We may also continue to find higher order members of the hierarchy by subsequently reinstating some of the terms A_ι, D_ι with $\iota > m$ and calculating the coefficients needed to describe those terms, α_j^i and δ_j^i , through equations (5.19) and (5.21).

5.2.2 Hierarchy corresponding to reductions of the type $x_{l,m+1} = 1/x_{l+d,m}$

The formulas for the coefficients that were found in the preceding section corresponded to equations that can be obtained from the LMKdV equation via the reduction $\hat{x} = x_{l+d}$. However, in [2] it was shown that reductions of the type $\hat{x} = 1/x_{l+d}$ can also be used and that these reductions lead to q -discrete Painlevé equations as well. The hierarchy of equations that springs from this type of reduction has Lax pairs that are very similar to the ones used in section 5.2.1 and fit easily into the present framework. The main difference between the two sets of Lax pairs can be described in terms of the spectral parameter k . The Lax pairs in section 5.2.1 all have the following form:

$$N = \begin{pmatrix} a_0 + a_2k^2 + \dots + a_{2\rho}k^{2\rho} & b_1k + b_3k^3 + \dots + b_{2\rho\pm 1}k^{2\rho\pm 1} \\ c_1k + c_3k^3 + \dots + c_{2\rho\pm 1}k^{2\rho\pm 1} & d_0 + d_2k^2 + \dots + d_{2\rho}k^{2\rho} \end{pmatrix}$$

Observe that the terms that contain the lowest power of k , that is terms that are constant in k , appear in the diagonal entries of N . Also note that the other half of the Lax pair, the L matrix from equation (5.2a) remains the same for both types of reduction.

Moving now to the hierarchy associated with reductions of the type $\hat{x} = 1/x_{l+d}$, we find that the associated Lax pairs have a form similar to the former case, except here the lowest powers of k appear in the off diagonal entries. This can be achieved

simply by removing the constant terms from the diagonal entries.

$$N = \begin{pmatrix} a_2 k^2 + \dots + a_{2\rho} k^{2\rho} & b_1 k + b_3 k^3 + \dots + b_{2\rho\pm 1} k^{2\rho\pm 1} \\ c_1 k + c_3 k^3 + \dots + c_{2\rho\pm 1} k^{2\rho\pm 1} & d_2 k^2 + \dots + d_{2\rho} k^{2\rho} \end{pmatrix}$$

We can find the hierarchy that arises from this case in a congruent manner to the last with only minor alterations. The differences here arise because A_1 is now the first term in the series but it is not a constant, as was A_0 in the previous case. Instead $A_1 = \beta_1 x \bar{x}$, $\beta_1 = \text{constant}$, which introduces additional factors of $X_1 = \bar{x}/x$ into the equations after (5.14a). Following the same procedure as in section 5.2.1, it is not difficult to show that in this case the formulas analogous to (5.19) are

$$\alpha_1^{i+1} = \frac{\bar{\alpha}_1^i \bar{\delta}_1^i X_i^2 - q^2 \alpha_1^i \delta_1^i}{q^2 \delta_1^i [\lambda - \alpha_{i+1}^i + \frac{X_i}{X_{i+1}} \bar{\delta}_{i+1}^i] - \lambda X_1 X_i \bar{\delta}_1^i} \quad (5.23a)$$

$$\alpha_j^{i+1} = \frac{1}{G_i} \left\{ q^2 \delta_1^i [\lambda^{j-i} - \alpha_j^i + \frac{\bar{\delta}_j^i X_i}{X_j} + \sum_{k=i+1}^{j-1} \frac{\lambda^{j-k} \bar{\delta}_k^i X_i}{X_k}] - \bar{\delta}_1^i (\lambda^{j-i} X_1 X_i + \sum_{k=i+1}^{j-1} \lambda^{j-k} \bar{\alpha}_k^i X_1 X_k) \right\} \quad (5.23b)$$

Where G_i in (5.23b) is the same as the denominator in (5.23a). We will present the first few equations in this hierarchy in section 5.4.2

5.2.3 General Coefficients

It is conjectured that all the coefficients for the hierarchy of equations that arise from reductions of the type $x_{l,m+1} = x_{l+d,m}$ are given by the following equations

$$\alpha_j^k = \sum_{i_1=0}^{j-k} \sum_{i_2=0}^{j-k-I_1} \dots \sum_{i_{k-1}=0}^{j-k-I_{k-2}} \left[\frac{\prod_{h=1, h \text{ odd}}^{k-2} X_{I_h - I_{k-1}}^h}{\prod_{g=0, g \text{ even}}^{k-2} X_{I_g - I_{k-1}}^g} \right]^{(-1)^k} \prod_{f=0}^{k-1} \lambda^{i_f} \quad (5.24a)$$

$$\alpha_0^k = \left(\prod_{h=0}^{k-1} \lambda \right)^{-1} \left(\prod_{g=0}^{k-2} X_{k+g+1}^g \right)^{-1} \quad (5.24b)$$

Where $I_k = \sum_{s=1}^k i_s$, $i_0 = j - k - I_{k-1}$,

$$X_i = \left(\frac{\bar{x}}{x} \right)^{\frac{1-(-1)^i}{2}} = \begin{cases} \bar{x}/x, & i \text{ odd} \\ 1, & i \text{ even} \end{cases}$$

and we use the notation $\overset{f}{\lambda} = \lambda_{l+f}$.

These formulas can be used to find any coefficient of interest, which vastly decreases the number of calculations required to find an equation of any order in the hierarchy.

We obtain similar results for the coefficients for the hierarchy corresponding to reductions of the type $x_{l,m+1} = 1/x_{l+d,m}$.

$$\alpha_{j+1}^{k+1} = \sum_{i_1=0}^{j-k} \sum_{i_2=0}^{j-k-I_1} \cdots \sum_{i_{k-1}=0}^{j-k-I_{k-2}} \left[\frac{\prod_{g=0, g \text{ even}}^{k-2} \overset{g}{X}_{I_g-I_{k-1}}}{\prod_{h=1, h \text{ odd}}^{k-2} \overset{h}{X}_{I_h-I_{k-1}}} \right]^{(-1)^k} \prod_{f=0}^{k-1} \overset{f}{\lambda}^{i_f} \quad (5.25a)$$

$$\alpha_1^{k+1} = \left(\prod_{h=0}^{k-1} \overset{h}{\lambda} \right)^{-1} \prod_{g=0}^{k-2} \overset{g}{X}_{k+g+1} \quad (5.25b)$$

These coefficients α_{j+1}^{k+1} are equal to δ_j^k from the first hierarchy.

5.3 A Known Example

In this section we will implement the formulas (5.24) to explicitly find a known example. The procedure runs as follows:

- First, we decide how many terms we will keep in the Lax pair, i.e. we decide which A_i , and D_i will be nonzero, up to $i = m$ say.
- Second, use equation (5.24) to calculate all the coefficients α_j^i up to α_m^{m-1} . It is necessary to calculate every α_j^i with $j \leq m$ and $i \leq m-1$ in order to specify the Lax pair.
- Third, calculate the terms A_i and D_i from equations (5.14a) and (5.14b), noting that any δ_j^i is equal to α_j^i with $X_i, \overline{X}_i, \overline{\overline{X}}_i, \dots$ replaced with $1/X_i, 1/\overline{X}_i, 1/\overline{\overline{X}}_i, \dots$. We may then find the corresponding nonlinear equation using the compatibility conditions (5.7) and (5.8).

For our example we shall retain only those terms A_i, D_i with $0 \leq i \leq 3$, which causes there to be two terms in each entry of the N matrix of the Lax pair (see (5.2b)). A Lax pair of this form was already presented in [2] where it was shown to correspond to qP_{II} , we expect the same to occur here.

The next step is to calculate the coefficients α_j^i, δ_j^i up to $i = 2$ and $j = 3$. We begin with the coefficients α_j^1 and δ_j^1 , for which inspection of equations (5.11) and (5.12) indicates

$$\alpha_j^1 = \delta_j^1 = \lambda^{j-1} \quad (5.26)$$

Directly from (5.24) we find

$$\alpha_0^2 = -1/(\lambda\bar{\lambda}X_1) \quad (5.27a)$$

$$\alpha_3^2 = \lambda + \bar{\lambda}/X_1 \quad (5.27b)$$

We have now calculated all the coefficients needed for the present example but we will list the next four as well, for future reference.

$$\alpha_4^2 = \lambda^2 + \lambda\bar{\lambda}/X_1 + \bar{\lambda}^2 \quad (5.28a)$$

$$\alpha_5^2 = \lambda^3 + \lambda^2\bar{\lambda}/X_1 + \lambda\bar{\lambda}^2 + \bar{\lambda}^3/X_1 \quad (5.28b)$$

$$\alpha_6^2 = \lambda^4 + \lambda^3\bar{\lambda}/X_1 + \lambda^2\bar{\lambda}^2 + \lambda\bar{\lambda}^3/X_1 + \bar{\lambda}^4 \quad (5.28c)$$

$$\alpha_7^2 = \lambda^5 + \lambda^4\bar{\lambda}/X_1 + \lambda^3\bar{\lambda}^2 + \lambda^2\bar{\lambda}^3/X_1 + \lambda^4\bar{\lambda} + \bar{\lambda}^5/X_1 \quad (5.28d)$$

At this point we use the coefficients to calculate the values of the nonzero terms in the N matrix. Since A_0 and A_3 are at the ends of the sequence, we can calculate their values directly from (5.7), using the appropriate values of i in that equation. Trivially, these are found to be $A_0 = a_0 = \text{constant}$ and $A_3 = \bar{x}b_3 = T_2\sigma\bar{x}/x$ where T_2 is an arbitrary period two function of l and $\sigma = q^l$. The lower case a_0 and b_3 are the original variables in the N matrix, see (5.2b). Using (5.14a)

$$\begin{aligned} -A_2 &= \alpha_0^2 A_0 + \alpha_3^2 A_3 \\ &= -\frac{a_0 x}{\lambda\bar{\lambda}\bar{x}} + \left(\lambda + \frac{\bar{\lambda}x}{\bar{x}}\right) \frac{T_2\sigma\bar{x}}{x} \end{aligned} \quad (5.29)$$

We then use the coefficients from the previous step to calculate A_1

$$\begin{aligned} -A_1 &= \alpha_0^1 A_0 + \alpha_2^1 A_2 + \alpha_3^1 A_3 \\ &= a_0 \left(\frac{1}{\lambda} + \frac{x}{\lambda \bar{x}} \right) - \frac{\lambda \bar{\lambda} T_2 \sigma \bar{x}}{\bar{x}} \end{aligned} \quad (5.30)$$

Finally, we can obtain the related equation by substituting these values into (5.7) at $i = 3$ whence we recover qP_{II} as expected. The form of the equation is

$$\bar{y} \underline{y} = \frac{1 - T_2 r y}{y(\gamma y - \bar{T}_2 r)} \quad (5.31)$$

where $\log r = \gamma_0 + \gamma_1(-1)^l - ql/2$, $\gamma, \gamma_i = \text{constant}$, and $y = \bar{x}/x$. Actually this version of qP_{II} contains more parameters than those found in [86, 2] as γ and T_2 , which are described after the Lax pair below, were not present in those papers. The corresponding Lax pair is

$$\begin{aligned} L &= \begin{pmatrix} \bar{x}/x & -\nu/(\lambda x) \\ -\nu \bar{x}/\lambda & 1 \end{pmatrix}, \\ N &= \begin{pmatrix} a_0 + \nu^2 \frac{a_0 x}{\lambda \bar{\lambda} \bar{x}} - (\lambda + \frac{\bar{\lambda} x}{\bar{x}}) \frac{T_2 \sigma \bar{x}}{x} \nu^2 & \nu \left(-\frac{a_0}{\bar{x}} \left(\frac{1}{\lambda} + \frac{x}{\lambda \bar{x}} \right) + \frac{\lambda \bar{\lambda} T_2 \sigma}{\bar{x}} \right) + \nu^3 \frac{T_2 \sigma}{x} \\ \nu \left(-d_0 \bar{x} \left(\frac{1}{\lambda} + \frac{\bar{x}}{\lambda x} \right) + \lambda \bar{\lambda} \bar{T}_2 \sigma \bar{x} \right) + \bar{T}_2 \sigma x \nu^3 & d_0 + \nu^2 \frac{d_0 \bar{x}}{\lambda \bar{\lambda} x} - \nu^2 \left(\lambda + \frac{\bar{\lambda} \bar{x}}{x} \right) \frac{\bar{T}_2 \sigma x}{\bar{x}} \end{pmatrix} \end{aligned}$$

The terms in the N matrix are related to those in (5.31) by $\gamma = d_0/a_0$, T_2 is an arbitrary, period-two function of l and $r = \lambda \bar{\lambda} \bar{\lambda} \sigma/a_0$ where $\sigma = q^l$. The spectral parameter is n and it enters the Lax pair via $\nu = \nu_0 q^n$.

5.4 Higher Order Equations

Now that the formulas for finding all the equations in the hierarchy have been derived and their use explained, we will write down some higher order equations and their associated Lax pairs. Section 5.4.1 will deal with equations obtained from the LMKdV equations via reductions of the type $x_{l,m+1} = x_{l+d,m}$ and 5.4.2 will deal with the type $x_{l,m+1} = 1/x_{l+d,m}$.

5.4.1 Equations Corresponding to Reductions of the Type

$$x_{l,m+1} = x_{l+d,m}$$

This subsection pertains to higher order equations that can be obtained from the LMKDV equation by using a reduction of the type $\hat{x} = x_{l+d}$ where d is some positive integer. The coefficients, α_j^3 , that will be required for all of the equations presented in this section are calculated using (5.24) and are listed below.

$$\begin{aligned}\alpha_0^3 &= 1/(\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{X}_1) \\ \alpha_4^3 &= \lambda + \bar{\lambda}X_1 + \frac{\bar{\bar{\lambda}}X_1}{\bar{X}_1} \\ \alpha_5^3 &= \lambda^2 + \lambda\bar{\lambda}X_1 + \frac{\bar{\bar{\lambda}}\bar{\lambda}}{\bar{X}_1} + \frac{\lambda\bar{\bar{\lambda}}X_1}{\bar{X}_1} + \bar{\lambda}^2 + \bar{\bar{\lambda}}^2 \\ \alpha_6^3 &= \lambda^3 + \lambda^2\bar{\lambda}X_1 + \frac{\lambda^2\bar{\bar{\lambda}}X_1}{\bar{X}_1} + \lambda\bar{\lambda}^2 + \frac{\lambda\bar{\bar{\lambda}}\bar{\lambda}}{\bar{X}_1} \\ &\quad \lambda\bar{\bar{\lambda}}^2 + \bar{\lambda}^3X_1 + \frac{\bar{\bar{\lambda}}^2\bar{\lambda}X_1}{\bar{X}_1} + \bar{\lambda}\bar{\bar{\lambda}}^2X_1 + \frac{\bar{\bar{\lambda}}^3X_1}{\bar{X}_1}\end{aligned}$$

The Lax pair and associated equation that is achieved by truncating the series A_i at A_4 was derived in [2]. The equation is qP_V, which can be obtained from the LMKdV equation via the reduction $\hat{x} = \bar{\bar{x}}$ [86, 2].

$$\text{qP}_V : \quad \underline{w}\bar{w} = \frac{1 + T_2rw}{\bar{T}_2r + \gamma w} \quad (5.34)$$

Where $w = \bar{\bar{x}}/\bar{x}$, $\gamma = \text{constant}$, T_2 is an arbitrary period-two function and $\log r = \gamma_0 + \gamma_1j_3^l + \gamma_2j_3^{2l} - ql/3$, with $\gamma_\iota = \text{constant}$. Noting that $X_1 = \bar{\bar{x}}/x$, we can use the coefficients α_j^i to find the Lax pair for this qP_V equation through (5.14a). The Lax pair lies below and the relationships between the terms in the Lax pair and those

in the equation (5.34) follow.

$$\begin{aligned}
L &= \begin{pmatrix} \bar{x}/x & -\nu/(\lambda x) \\ -\nu\bar{x}/\lambda & 1 \end{pmatrix} \\
N_{11} &= a_0 + \nu^2 a_0 \left(\frac{x}{\lambda\bar{\lambda}\bar{x}} + \frac{\bar{x}}{\bar{\lambda}\bar{\lambda}\bar{x}} + \frac{x\bar{x}}{\lambda\bar{\lambda}\bar{x}\bar{x}} \right) \\
&\quad + \bar{T}_2 \sigma \left(\frac{\lambda\bar{\lambda}\bar{x}}{x} + \frac{\lambda\bar{\lambda}\bar{x}\bar{x}}{x\bar{x}} + \frac{\bar{\lambda}\bar{\lambda}\bar{x}}{\bar{x}} \right) + \nu^4 \bar{T}_2 \sigma \\
N_{12} &= -\nu a_0 \left(\frac{1}{\lambda\bar{x}} + \frac{x}{\bar{\lambda}\bar{x}\bar{x}} + \frac{x}{\bar{\lambda}\bar{x}\bar{x}} \right) - \nu \bar{T}_2 \sigma \frac{\lambda\bar{\lambda}\bar{\lambda}}{\bar{x}} - \frac{\nu^3 a_0}{\lambda\bar{\lambda}\bar{\lambda}\bar{x}} + \bar{T}_2 \sigma \left(\frac{\lambda}{\bar{x}} + \frac{\bar{\lambda}\bar{x}}{x\bar{x}} + \frac{\bar{\lambda}\bar{x}}{x\bar{x}} \right) \\
N_{21} &= -\nu d_0 \left(\frac{\bar{x}}{\lambda} + \frac{\bar{x}\bar{x}}{\bar{\lambda}x} + \frac{\bar{x}\bar{x}}{\bar{\lambda}x} \right) - \nu T_2 \sigma \lambda\bar{\lambda}\bar{\lambda}\bar{x} - \frac{\nu^3 d_0 \bar{x}}{\lambda\bar{\lambda}\bar{\lambda}} + T_2 \sigma \left(\lambda\bar{x} + \frac{\bar{\lambda}x\bar{x}}{\bar{x}} + \frac{\bar{\lambda}x\bar{x}}{\bar{x}} \right) \\
N_{22} &= d_0 + \nu^2 d_0 \left(\frac{\bar{x}}{\lambda\bar{\lambda}x} + \frac{\bar{x}}{\bar{\lambda}\bar{\lambda}\bar{x}} + \frac{\bar{x}\bar{x}}{\lambda\bar{\lambda}x\bar{x}} \right) \\
&\quad + T_2 \sigma \left(\frac{\lambda\bar{\lambda}x}{\bar{x}} + \frac{\lambda\bar{\lambda}x\bar{x}}{x\bar{x}} + \frac{\bar{\lambda}\bar{\lambda}\bar{x}}{\bar{x}} \right) + \nu^4 T_2 \sigma
\end{aligned} \tag{5.35}$$

Where $\lambda = \lambda(l)$, $\nu = \nu_0 q^n$, n being the spectral parameter, and $\sigma = q^l$. The compatibility condition for this Lax pair produces a series of equations that are either identities or one of two slightly different copies of qP_V . These two copies of qP_V are equal if $q\bar{\lambda} = \lambda$, there are no other restrictions on the parameters. To get from the form of qP_V that comes directly from the Lax pair to the form as listed in (5.34), we set $r = \lambda\bar{\lambda}\bar{\lambda}\bar{\lambda}\sigma/a_0$, $\gamma = d_0/a_0$ and T_2 remains as is. This type of condition on λ is common to every equation that has been calculated by the author. Indeed it is expected that, when considering a Lax pair with N matrix truncated at A_j , λ must satisfy $q\lambda(l+j-1) = \lambda(l)$ and $r = \frac{\sigma}{a_0} \prod_{h=0}^j \lambda(l+h)$.

We also point out that if one were only interested in this Lax pair and equation, the coefficients α_j^3 with $j > 4$ would be superfluous, they are written above because they are required to find subsequent Lax pairs and equations listed below.

Continuing to the next level in the hierarchy, we obtain the coefficients:

$$\begin{aligned}\alpha_0^4 &= -1/(\lambda\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{\bar{\lambda}}}X_1\bar{X}_1) \\ \alpha_5^4 &= \lambda + \frac{\bar{\lambda}}{X_1} + \frac{\bar{\bar{\lambda}}\bar{X}_1}{X_1} + \frac{\bar{\bar{\bar{\lambda}}}\bar{X}_1}{X_1\bar{\bar{\bar{X}}}_1} \\ \alpha_6^4 &= \lambda^2 + \frac{\lambda\bar{\lambda}}{X_1} + \frac{\lambda\bar{\bar{\lambda}}\bar{X}_1}{X_1} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{X}_1}{X_1\bar{\bar{\bar{X}}}_1} + \bar{\lambda}^2 + \lambda\bar{\bar{\lambda}}\bar{X}_1 + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{X}_1}{\bar{\bar{\bar{X}}}_1} + \bar{\lambda}^2 + \frac{\bar{\bar{\bar{\lambda}}}\bar{\lambda}}{\bar{\bar{\bar{X}}}_1} + \bar{\bar{\bar{\lambda}}}^2\end{aligned}$$

Ceasing at A_5 yields a new Lax pair for a fourth order equation also written in [2].

The associated equation, given below, is a reduction of the LMKDV equation under $\hat{x} = \bar{x}$. The notation $\bar{x} = x_{l+4}$ is used instead of $\bar{\bar{\bar{x}}}$ because too many bars become difficult to read.

$$\bar{\bar{\bar{y}}}\bar{\bar{\bar{y}}}\bar{\bar{\bar{y}}}\bar{\bar{\bar{y}}} = \frac{1 - T_2 r \bar{\bar{\bar{y}}}\bar{\bar{\bar{y}}}\bar{\bar{\bar{y}}}}{\gamma \bar{\bar{\bar{y}}}\bar{\bar{\bar{y}}} - \bar{\bar{\bar{T}}}_2 r}$$

where $y = \bar{x}/\bar{\bar{x}}$, $\log r = \gamma_0 + \gamma_1 i^l + \gamma_2 (-1)^l + \gamma_3 (-i)^l - ql/4$ and $\gamma_i = \text{constant}$.

The relationship between r and quantities in the Lax pair for this equation is $r = \lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\sigma/a_0$ and $q\bar{\lambda} = \lambda$ to ensure compatibility. The L matrix in the Lax pair, as always, is as in (5.35) and components of the N matrix in the Lax pair are

$$\begin{aligned}N_{11} &= a_0 + \nu^2 a_0 \left(\frac{x}{\lambda\bar{\lambda}\bar{\bar{x}}} + \frac{x\bar{x}}{\lambda\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}} + \frac{x\bar{\bar{x}}}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}}} + \frac{\bar{x}}{\lambda\bar{\bar{\bar{x}}}} + \frac{\bar{\bar{x}}}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}}} + \frac{\bar{\bar{\bar{x}}}}{\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}}} \right) \\ &\quad - \nu^2 \bar{\bar{\bar{T}}}_2 \sigma \left(\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}} \frac{\bar{\bar{\bar{x}}}}{x} + \lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}} \frac{\bar{\bar{\bar{\bar{x}}}}}{x\bar{x}} + \lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}} \frac{\bar{\bar{\bar{\bar{x}}}}}{x\bar{x}} + \lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}} \frac{\bar{\bar{\bar{x}}}}{x} \right) \\ &\quad - \frac{\nu^4 a_0 x}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}} - \nu^4 \bar{\bar{\bar{T}}}_2 \sigma \left(\frac{\lambda\bar{x}}{x} + \frac{\bar{\lambda}\bar{x}}{\bar{\bar{x}}} + \frac{\bar{\bar{\lambda}}\bar{x}}{\bar{\bar{\bar{x}}}} + \frac{\bar{\bar{\bar{\lambda}}}\bar{x}}{\bar{\bar{\bar{\bar{x}}}}} \right) \\ N_{12} &= -\nu a_0 \left(\frac{1}{\lambda\bar{x}} + \frac{x}{\bar{\lambda}\bar{\bar{x}}} + \frac{x}{\bar{\bar{\lambda}}\bar{\bar{\bar{x}}}} + \frac{x}{\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}}} \right) + \nu \bar{\bar{\bar{T}}}_2 \sigma \lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\frac{1}{\bar{\bar{\bar{x}}}} \\ &\quad - \nu^3 a_0 \left(\frac{1}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\lambda}}}} + \frac{\bar{x}}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}\bar{\bar{\bar{\bar{x}}}}} + \frac{\bar{x}}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}\bar{\bar{\bar{\bar{x}}}}} + \frac{x}{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{\bar{x}}}\bar{\bar{\bar{\bar{x}}}}} \right) \\ &\quad + \nu^3 \bar{\bar{\bar{T}}}_2 \sigma \left(\frac{\lambda\bar{\bar{\lambda}}}{\bar{\bar{x}}} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}}{\bar{\bar{x}}\bar{\bar{x}}} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}}{\bar{\bar{x}}\bar{\bar{x}}\bar{\bar{x}}} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}}{x\bar{\bar{x}}} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}\bar{\bar{\bar{x}}}}{x\bar{\bar{x}}\bar{\bar{x}}\bar{\bar{x}}} + \frac{\lambda\bar{\bar{\bar{\lambda}}}\bar{\bar{\bar{x}}}}{x\bar{\bar{x}}\bar{\bar{x}}} \right) + \frac{\nu^5 \bar{\bar{\bar{T}}}_2 \sigma}{x}\end{aligned}$$

$$\begin{aligned}
N_{21} = & -\nu d_0 \left(\frac{\bar{x}}{\lambda} + \frac{\bar{x}\bar{x}}{\lambda x} + \frac{\bar{x}\bar{x}}{\bar{\lambda}x} + \frac{\bar{x}\bar{x}}{\bar{\lambda}x} \right) + \nu T_2 \sigma \lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \bar{x}^4 \\
& -\nu^3 d_0 \left(\frac{\bar{x}}{\lambda \bar{\lambda} \bar{\lambda}} + \frac{\bar{x}\bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \bar{x}} + \frac{\bar{x}\bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \bar{x}} + \frac{\bar{x}\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{\lambda} x} \right) \\
& + \nu^3 T_2 \sigma \left(\lambda \bar{\lambda} \bar{x} + \frac{\lambda \bar{\lambda} \bar{x} \bar{x}}{\bar{x}} + \frac{\lambda \bar{\lambda} \bar{x} \bar{x}}{\bar{x}} + \frac{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}}{\bar{x}} + \frac{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}}{\bar{x}\bar{x}} + \frac{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}}{\bar{x}} \right) + \nu^5 T_2 \sigma x
\end{aligned}$$

$$\begin{aligned}
N_{22} = & d_0 + \nu^2 d_0 \left(\frac{\bar{x}}{\lambda \bar{\lambda} x} + \frac{\bar{x}\bar{x}}{\lambda \bar{\lambda} x \bar{x}} + \frac{\bar{x}\bar{x}}{\lambda \bar{\lambda} x \bar{x}} + \frac{\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{x}} + \frac{\bar{x}\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}} + \frac{\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{x}} \right) \\
& -\nu^2 T_2 \sigma \left(\lambda \bar{\lambda} \bar{\lambda} \frac{x}{\bar{x}} + \lambda \bar{\lambda} \bar{\lambda} \frac{x\bar{x}}{\bar{x}\bar{x}} + \lambda \bar{\lambda} \bar{\lambda} \frac{x\bar{x}}{\bar{x}\bar{x}} + \lambda \bar{\lambda} \bar{\lambda} \frac{x}{\bar{x}} \right) \\
& \frac{\nu^4 d_0 \bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} x} - \nu^4 T_2 \sigma \left(\frac{\lambda x}{\bar{x}} + \frac{\bar{\lambda} \bar{x}}{\bar{x}} + \frac{\bar{\lambda} \bar{x}}{\bar{x}} + \frac{\bar{\lambda} \bar{x}}{\bar{x}} \right)
\end{aligned}$$

We will list one more higher order equation. The next set of coefficients that we require are

$$\begin{aligned}
\alpha_5^5 &= 1/(\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \bar{\lambda} \bar{X}_1 \bar{X}_1) \\
\alpha_6^5 &= \lambda + \bar{\lambda} X_1 + \frac{\bar{\lambda} X_1}{\bar{X}_1} + \frac{\bar{\lambda} X_1 \bar{X}_1}{\bar{X}_1} + \frac{\lambda X_1 \bar{X}_1}{\bar{X}_1 \bar{X}_1}
\end{aligned} \tag{5.37a}$$

Where $X_1 = \bar{x}/x$ as usual. These coefficients lead to a Lax pair with the L matrix as before (see (5.35)) and the N matrix below.

$$\begin{aligned}
N_{11} = & a_0 + \nu^2 a_0 \left(\frac{x\bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} x} + \frac{x}{\lambda \bar{\lambda} \bar{x}} + \frac{\bar{x}}{\bar{\lambda} \bar{\lambda} x} + \frac{\bar{x}\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{\lambda} x} + \frac{\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{x}} + \frac{x\bar{x}}{\lambda \bar{\lambda} \bar{x} \bar{x}} + \frac{\bar{x}}{\bar{\lambda} \bar{\lambda} x} \right) \\
& + \frac{x\bar{x}}{\lambda \bar{\lambda} \bar{x} \bar{x}} + \frac{\bar{x}\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}} + \frac{\bar{x}}{\bar{\lambda} \bar{\lambda} x} + \nu^2 \bar{T}_2 \sigma \left(\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \frac{x}{\bar{x}} + \bar{\lambda} \bar{\lambda} \bar{\lambda} \bar{\lambda} \frac{x}{\bar{x}} + \lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \frac{x\bar{x}}{x} \right) \\
& + \lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \frac{\bar{x}\bar{x}}{x} + \lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \frac{\bar{x}\bar{x}}{x} + \nu^4 a_0 \left(\frac{x}{\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} x} + \frac{\bar{x}}{\bar{\lambda} \bar{\lambda} \bar{\lambda} \bar{\lambda} x} + \frac{x\bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \bar{x} \bar{x}} \right) \\
& + \frac{x\bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} x} + \frac{x\bar{x}}{\lambda \bar{\lambda} \bar{\lambda} \bar{\lambda} \bar{x} \bar{x}} + \nu^4 \bar{T}_2 \sigma \left(\frac{\lambda \bar{\lambda} \bar{x} \bar{x}}{x} + \frac{\bar{\lambda} \bar{\lambda} \bar{x}}{\bar{x}} + \frac{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}}{\bar{x}\bar{x}} + \frac{\lambda \bar{\lambda} \bar{x}}{x} \right) \\
& + \frac{\lambda \bar{\lambda} \bar{x}}{\bar{x}} + \frac{\lambda \bar{\lambda} \bar{x} \bar{x}}{x} + \frac{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}}{\bar{x}\bar{x}} + \frac{\lambda \bar{\lambda} \bar{x} \bar{x}}{x\bar{x}} + \frac{\bar{\lambda} \bar{\lambda} \bar{x}}{x} + \frac{\bar{\lambda} \bar{\lambda} \bar{x} \bar{x}}{\bar{x}\bar{x}} + \nu^6 \bar{T}_2 \sigma
\end{aligned}$$

$$\begin{aligned}
N_{22} = & d_0 + \nu^2 d_0 \left(\frac{\overset{4}{x}\overset{5}{x}}{\lambda\lambda x\bar{x}} + \frac{\bar{x}}{\lambda\bar{\lambda}x} + \frac{x}{\bar{\lambda}\bar{\lambda}\bar{x}} + \frac{\overset{4}{x}\overset{5}{x}}{\bar{\lambda}\bar{\lambda}\bar{x}\bar{x}} + \frac{\bar{x}}{\bar{\lambda}\bar{\lambda}\bar{x}} + \frac{\overset{\bar{x}}{x}\overset{4}{x}}{\lambda\lambda x\bar{x}} + \frac{x}{\bar{\lambda}\bar{\lambda}\bar{x}} \right) \\
& + \frac{\bar{x}\bar{x}}{\lambda\bar{\lambda}x\bar{x}} + \frac{\overset{\bar{x}}{x}\overset{4}{x}}{\bar{\lambda}\bar{\lambda}\bar{x}\bar{x}} + \frac{x}{\bar{\lambda}\bar{\lambda}\bar{x}} + \nu^2 T_2 \sigma \left(\lambda\bar{\lambda}\bar{\lambda}\bar{\lambda} \frac{x}{x} + \bar{\lambda}\bar{\lambda}\bar{\lambda}\bar{\lambda} \frac{\overset{5}{x}}{\bar{x}} + \lambda\bar{\lambda}\bar{\lambda}\bar{\lambda} \frac{\overset{5}{x}}{\bar{x}\bar{x}} \right) \\
& + \lambda\bar{\lambda}\bar{\lambda}\bar{\lambda} \frac{\overset{5}{x}}{\bar{x}\bar{x}} + \lambda\bar{\lambda}\bar{\lambda}\bar{\lambda} \frac{\overset{5}{x}}{\bar{x}\bar{x}} + \nu^4 d_0 \left(\frac{x}{\lambda\bar{\lambda}\bar{\lambda}\bar{\lambda}x} + \frac{x}{\bar{\lambda}\bar{\lambda}\bar{\lambda}\bar{\lambda}\bar{x}} + \frac{\bar{x}\bar{x}}{\lambda\bar{\lambda}\bar{\lambda}\bar{\lambda}x\bar{x}} + \frac{\overset{4}{x}\overset{5}{x}}{\lambda\bar{\lambda}\bar{\lambda}\bar{\lambda}x\bar{x}} \right) \\
& + \frac{\bar{x}\bar{x}}{\lambda\bar{\lambda}\bar{\lambda}\bar{\lambda}x\bar{x}} + \nu^4 T_2 \sigma \left(\frac{\lambda\lambda x\bar{x}}{\bar{x}\bar{x}} + \frac{\bar{\lambda}\bar{\lambda}\bar{x}}{x} + \frac{\bar{\lambda}\bar{\lambda}\bar{x}\bar{x}}{\bar{x}\bar{x}} + \frac{\lambda\bar{\lambda}x}{\bar{x}} + \frac{\lambda\bar{\lambda}\bar{x}}{\bar{x}} + \frac{\lambda\bar{\lambda}x\bar{x}}{\bar{x}\bar{x}} \right) \\
& + \frac{\bar{\lambda}\bar{\lambda}\bar{x}\bar{x}}{\bar{x}\bar{x}} + \frac{\lambda\bar{\lambda}x\bar{x}}{\bar{x}\bar{x}} + \frac{\bar{\lambda}\bar{\lambda}\bar{x}}{\bar{x}} + \frac{\bar{\lambda}\bar{\lambda}\bar{x}\bar{x}}{\bar{x}\bar{x}} + \nu^6 T_2 \sigma
\end{aligned}$$

Incredibly this cumbersome Lax pair has as its compatibility condition the following, rather simple, fourth order equation

$$\frac{\bar{\bar{w}}\bar{\bar{w}}}{w} = \frac{1}{w} \frac{1 + T_2 r \bar{w} w}{\gamma \bar{w} \underline{w} + T_2 r} \quad (5.38)$$

where $w = \frac{x}{\bar{x}}$ and $\log r = -ql/5 + k_0 + k_1 j_5^l + k_2 j_5^{2l} + k_3 j_5^{l^3} + k_4 j_5^{4l}$, with $k_i = \text{constant}$, $j_5 = 1^{1/5}$, and T_2 is an arbitrary, period-two function of l .

5.4.2 Equations corresponding to reductions of the type

$$x_{l,m+1} = 1/x_{l+d,m}$$

Here we will write down some equations, with their Lax pairs, from the hierarchy that arises from the LMKdV equation via reductions of the type $x_{l,m+1} = 1/x_{l+d,m}$, $d = \text{constant}$. The procedure used to obtain these results is just the same as that explained in section 5.3. However, as outlined in section 5.2.2, now $A_1 = k_1 x\bar{x}$, $k_1 = \text{constant}$, is an end point of the series of terms in the Lax pairs, and the odd and even powers of ν have been redistributed. We will not list the coefficients used in finding the results presented here because they are easily obtained from those used in section 5.4.1. To find a_l^k as required with the present hierarchy, use α_{j-1}^{k-1} from section 5.4.1 and replace $X_i \rightarrow 1/\bar{X}_i$.

The first non-trivial equation in this part of the hierarchy is qP_{III} :

$$x\bar{x} = \frac{1 + \bar{T}_2 r \bar{x}^2}{\gamma \bar{x}^2 + T_2 r} \quad (5.39)$$

which was found with the following Lax pair in [2], except here the equation has two extra free parameters coming from the T_2 term which is an arbitrary, period-two function of l , and $\gamma = \text{constant}$. The Lax pair has L as in (5.35) and

$$N = \begin{pmatrix} \nu^2 \left(\frac{k_1 x \bar{x}}{\lambda} + \lambda \bar{T}_2 \sigma \frac{\bar{x}}{x} \right) & \nu k_1 x + \nu^3 \lambda \bar{T}_2 \frac{\sigma}{x} \\ \nu \frac{k_2}{x} + T_2 \sigma x & \nu^2 \left(\frac{k_2}{\lambda x \bar{x}} + \lambda T_2 \sigma \frac{x}{\bar{x}} \right) \end{pmatrix} \quad (5.40)$$

and $\lambda = q\bar{\lambda}$ for compatibility so $r = \lambda\bar{\lambda}\sigma/\beta_2 = \gamma_0 q^{-l}$ with $\gamma_0 = \text{constant}$.

The next equation in the hierarchy is an alternative qP_{II} . After setting $y = \bar{x}\bar{x}$

$$\bar{y}y = y \frac{1 - \bar{T}_2 r y}{\gamma y - T_2 r} \quad (5.41)$$

The N matrix of the Lax pair for this equation is

$$\begin{aligned} N_{11} &= \nu^2 \left[k_1 \bar{x} \left(\frac{x}{\lambda} + \frac{\bar{x}}{\lambda} \right) - \lambda \bar{\lambda} \bar{T}_2 \sigma \frac{\bar{x}}{x} \right] + \nu^4 \bar{T}_2 \sigma \\ N_{12} &= \nu k_1 x + \nu^3 \left[k_1 \frac{\bar{x}\bar{x}}{\lambda\lambda} - \left(\lambda + \bar{\lambda} \frac{\bar{x}}{x} \right) \bar{T}_2 \sigma \right] \\ N_{21} &= \nu k_2 x + \nu^3 \left[\frac{k_2}{\lambda \bar{\lambda} \bar{x} \bar{x}} - \left(\lambda + \bar{\lambda} \frac{x}{\bar{x}} \right) \bar{T}_2 \sigma \right] \\ N_{22} &= \nu^2 \left[\frac{k_1}{\bar{x}} \left(\frac{1}{\lambda x} + \frac{1}{\bar{\lambda} \bar{x}} \right) - \lambda \bar{\lambda} \bar{T}_2 \sigma \frac{x}{\bar{x}} \right] + \nu^4 \bar{T}_2 \sigma \end{aligned}$$

where k_i are constant and, to ensure compatibility, $\lambda = q\bar{\lambda}$. We set $r = \lambda\bar{\lambda}\bar{\lambda}\sigma/k_2$ causing $\log r = k_0 + k_3(-1)^l - ql/2$, since $\sigma = q^l$.

The final Lax pair that will be presented from this part of the hierarchy is for the fourth order equation

$$x\bar{x}^4 = \frac{1 + \bar{T}_2 r \bar{x}\bar{x}}{\gamma \bar{x}\bar{x} + T_2 r} \quad (5.42)$$

Where $\gamma = k_1/k_2 = \text{constant}$ and $r = \lambda\bar{\lambda}\bar{\lambda}\bar{\lambda}\sigma/k_2$. The Lax pair for this equation

has the same L matrix again (5.35) and the components of the N matrix are

$$\begin{aligned}
N_{11} &= -\nu^2[k_1(\frac{x\bar{x}}{\lambda} + \frac{\bar{x}\bar{x}}{\bar{\lambda}} + \frac{\bar{\bar{x}}\bar{\bar{x}}}{\bar{\bar{\lambda}}}) + \lambda\bar{\lambda}\bar{\bar{\lambda}}T_2\sigma\frac{\bar{\bar{x}}}{x}] - \nu^4[k_1\frac{x\bar{x}}{\lambda\bar{\lambda}\bar{\bar{\lambda}}} + (\lambda\frac{\bar{x}}{x} + \bar{\lambda}\frac{\bar{x}}{\bar{x}} + \bar{\bar{\lambda}}\frac{\bar{\bar{x}}}{\bar{\bar{x}}})T_2\sigma] \\
N_{12} &= \nu k_1 x + \nu^3[k_1(\frac{\bar{x}}{\lambda\bar{\lambda}} + \frac{x\bar{x}}{\bar{\lambda}\bar{\lambda}\bar{x}} + \frac{\bar{\bar{x}}\bar{\bar{x}}}{\lambda\bar{\lambda}\bar{x}}) + (\frac{\lambda\bar{\lambda}}{\bar{x}} + \frac{\lambda\bar{\bar{\lambda}}\bar{\bar{x}}}{\bar{x}\bar{x}} + \frac{\bar{\lambda}\bar{\bar{\lambda}}\bar{\bar{x}}}{x\bar{x}})T_2\sigma] + \nu^5 T_2 \frac{\sigma}{x} \\
N_{21} &= \nu \frac{k_2}{x} + \nu^3[k_2(\frac{1}{\lambda\bar{\lambda}\bar{x}} + \frac{\bar{x}}{\bar{\lambda}\bar{\lambda}\bar{x}\bar{x}} + \frac{\bar{x}}{\lambda\bar{\lambda}\bar{x}\bar{x}}) + (\lambda\bar{\lambda}\bar{x} + \frac{\lambda\bar{\bar{\lambda}}\bar{x}\bar{x}}{\bar{x}} + \frac{\bar{\lambda}\bar{\bar{\lambda}}x\bar{x}}{\bar{x}})T_2\sigma] + \nu^5 T_2 \sigma x \\
N_{22} &= -\nu^2[k_2(\frac{1}{\lambda x\bar{x}} + \frac{1}{\bar{\lambda}\bar{x}\bar{x}} + \frac{1}{\bar{\bar{\lambda}}\bar{x}\bar{x}}) + \lambda\bar{\lambda}\bar{\bar{\lambda}}T_2\sigma\frac{x}{\bar{x}}] - \nu^4[\frac{k_2}{\lambda\bar{\lambda}\bar{\bar{\lambda}}x\bar{x}} + (\frac{\lambda x}{\bar{x}} + \frac{\bar{\lambda}\bar{x}}{\bar{x}} + \frac{\bar{\bar{\lambda}}\bar{\bar{x}}}{\bar{x}})T_2\sigma]
\end{aligned}$$

With this member of the hierarchy we require $\lambda = q\bar{\lambda}$ for compatibility, which causes $\log r = k_0 + k_3 j_3^l + k_4 j_3^{2l} - ql/3$, where k_i are constant.

5.5 Discussion

In this chapter we have presented two new hierarchies of nonlinear q -difference equations, one of which includes qP_{II} and qP_V , the other of which includes qP_{III} in addition to higher order equations. The relationship between the equations in each hierarchy was found using a series of Lax pairs and, as such, a Lax pair accompanies each equation in the hierarchy. All of the resulting equations are non-autonomous and contain multiple free parameters while each Lax pair is 2×2 .

Even though these Lax pairs increase in complexity at each level of the hierarchy, the equations retain the same simple structure while increasing in order and the number of free parameters. The persistence of a simple structure in the equations may facilitate the discovery of special solutions applicable to all members of the hierarchy.

We must point out that some key features of the method used to establish the hierarchy have not been proven in generality. We simply conjecture their validity based on agreement with results.

We note that these hierarchies have their roots in reductions from the lattice

modified KdV equation, it remains to be seen whether similar results lie behind other partial difference equations. It would eventually be interesting to find reductions from lattice equations to the q -Garnier hierarchy constructed by Sakai in [35].

At this point there is still a significant deficiency in knowledge about the generic solutions of q -Painlevé equations. The author is unaware of any instances where Birkhoff's theory of linear q -difference equations has been applied to deduce information about the solutions of q -Painlevé equations. The question of the global properties of solutions remains completely open.

Chapter 6

Some future directions: true and false Lax pairs and equivalent classes of equations

In this final chapter we explore how false Lax pairs can arise and how they can be identified by investigating their general forms. We also demonstrate an equivalence within some sets of nonlinear equations, by examining parameterizations thereof.

6.1 False Lax pairs

The existence of false Lax pairs is well known [116, 117, 118]. False Lax pairs are those whose compatibility condition appears to be an interesting nonlinear equation, but are nevertheless unable to provide any information about the solutions of that nonlinear equation. For example, take the following linear system:

$$\begin{aligned}\theta(l+1, n) &= L\theta(l, n), \\ \theta(l, n+1) &= N\theta(l, n).\end{aligned}$$

Where the Lax matrices take the form

$$L = \begin{pmatrix} \nu[(1+k)(1-\bar{x}) - \frac{\bar{x}(1-\bar{x})}{x(1-x)}] & \sigma^{-1}(1-x)(1-\bar{x}) \\ \sigma^{-1} & \nu\bar{x}/x \end{pmatrix}, \quad (6.2a)$$

$$N = \begin{pmatrix} \nu\sigma & 1-x \\ 1/(1-x) & \nu k\sigma \end{pmatrix}, \quad (6.2b)$$

where n plays the role of the spectral variable and $\nu = q^n$, $\sigma = q^l$, k is a constant parameter, $x = x(l)$ and $\bar{x} = x(l+1)$. It is easy to show that the compatibility condition

$$L(l, n+1)N(l, n) = N(l+1, n)L(l, n) \quad (6.3)$$

leads to the following equation, without any conditions on the parameter k

$$\bar{x} = kx(1-x) \quad (6.4)$$

Of course, (6.4) is the logistic equation, possibly the most famous and well studied chaotic equation known. It is believed that no chaos can exist in any equation that is solvable through a Lax pair. Therefore, the association between equation (6.4) and the linear systems in (6.2) is suspected to be fake in some way and warrants further investigation. The question of what might render this association fake is a difficult one that has had attention in the continuous domain [119] but not so much attention in the discrete domain. To explain the problem with this Lax pair, we consider a similar linear problem with general entries in the L and M matrices.

$$L = \begin{pmatrix} \nu a & b \\ c & \nu d \end{pmatrix}, \quad (6.5a)$$

$$N = \begin{pmatrix} \nu\alpha & \beta \\ \gamma & \nu\delta \end{pmatrix} \quad (6.5b)$$

where ν depends on the spectral variable and all other quantities are functions of l alone. The compatibility condition (6.3) then yields the following system of

equations.

$$qa\alpha = a\bar{\alpha} \quad (6.6a)$$

$$b\gamma = c\bar{\beta} \quad (6.6b)$$

$$qd\delta = d\bar{\delta} \quad (6.6c)$$

$$c\beta = b\bar{\gamma} \quad (6.6d)$$

$$qa\beta + b\delta = b\bar{\alpha} + d\bar{\beta} \quad (6.6e)$$

$$qd\gamma + c\alpha = a\bar{\gamma} + c\bar{\delta} \quad (6.6f)$$

Clearly, equations (6.6a) to (6.6d) yield

$$\alpha = \alpha_0\sigma$$

$$\delta = \delta_0\sigma$$

$$\gamma = \gamma_0/\beta$$

$$b = \frac{c}{\gamma_0}\beta\bar{\beta}$$

where $\alpha_0, \beta_0, \gamma_0 = \text{constant}$ and $\sigma = q^l$. Substituting these values in equations (6.6e) and (6.6f) brings us to the final two equations.

$$qd\gamma_0/\beta + \alpha_0\sigma c = \gamma_0 a/\bar{\beta} + q\delta_0 c\sigma \quad (6.8a)$$

$$qd\bar{\beta} + \frac{\alpha_0}{\gamma_0}\sigma c\beta\bar{\beta} = \frac{q\delta_0}{\gamma_0}c\beta\bar{\beta}\sigma + a\beta \quad (6.8b)$$

add (6.8a) and (6.8b) and solve the result for a to get

$$a = \frac{\sigma}{\gamma_0}(\alpha_0 + \delta_0)c\bar{\beta} - d\bar{\beta}/\beta \quad (6.9)$$

Finally we can put this expression for a back into either (6.8a) or (6.8b) to get the equation that is the compatibility condition for the linear system we are analysing.

$$\frac{\delta_0}{\gamma_0}\sigma c\beta = d \quad (6.10)$$

There are no conditions on, or relationships between, the quantities in (6.10) that come naturally out of the linear system itself, meaning that the system is underdetermined. Anything we add from this point is artificial and completely arbitrary.

To obtain the logistic equation, and corresponding false Lax pair, we set

$$\begin{aligned}\gamma_0 &= 1 \\ \delta_0 &= \lambda \\ c &= \sigma^{-1} \\ d &= \bar{x}/x \\ \beta &= 1 - x\end{aligned}$$

All the other quantities in the Lax pair are given by (6.6).

The key feature in identifying this as a false Lax pair is that there is too much freedom in the system. No information is given about the evolution of the quantities in (6.10) and so we cannot gain any information about the solution of the logistic equation, nor any other equation, from this linear system. This is proof of the falsehood of the Lax pair considered because the alternative, that this is a real Lax pair, implies every difference equation is integrable, which is certainly absurd.

There are examples of this type of false Lax pairs that appear in the literature, claimed to be real. For example, Papageorgiou, Nijhoff, Grammaticos and Ramani [47] presented the following Lax pair for qP_I

$$L = \begin{pmatrix} 0 & 0 & \underline{x} & q\underline{x} \\ 0 & \frac{x}{\sigma(x+1)} & 0 & 0 \\ 0 & 0 & 1/x & q/x \\ \nu & 0 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & \sigma/x & 0 \\ 0 & 0 & \underline{x} & q\underline{x} \\ \nu x & 0 & 1 & q \\ 0 & \nu \frac{\sigma}{q\underline{x}} & 0 & 0 \end{pmatrix}$$

This Lax pair has dP_I as its compatibility condition, which is $\bar{x}\underline{x} = q\sigma^2(x+1)/x^2$. However, beginning with the general Lax matrices of this form, we see that the

compatibility condition is a much simpler equation, this fact is illustrated below.

$$L = \begin{pmatrix} 0 & 0 & a & b \\ 0 & c & 0 & 0 \\ 0 & 0 & d & f \\ \nu & 0 & 0 & 0 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & 0 & \alpha & 0 \\ 0 & 0 & \beta & \gamma \\ \nu\delta & 0 & 1 & q \\ 0 & \nu\xi & 0 & 0 \end{pmatrix}$$

Where $\nu = \nu(n)$ and n is the spectral parameter, q is a constant and everything else is a function of l only. Imposing compatibility (6.3) on this linear system leads to the following conditions.

$$a\beta = b\bar{\alpha} \tag{6.11a}$$

$$a\gamma = c\bar{\alpha} \tag{6.11b}$$

$$\delta = \bar{\gamma} \tag{6.11c}$$

$$b\bar{\beta} = 1 \tag{6.11d}$$

$$c\bar{\beta} = q \tag{6.11e}$$

$$b\delta = q \tag{6.11f}$$

$$c\xi = a\bar{\delta} \tag{6.11g}$$

$$c = qb \tag{6.11h}$$

$$\bar{\xi} = q\alpha \tag{6.11i}$$

This is a simple set of equations to solve, we only write everything out to show that there are no shortcuts taken and that the solution we obtain is the only one available. There are numerous ways to proceed, all leading to the same outcome, we choose to save equations (6.11a), (6.11b) and (6.11g) until last, solving all the

other equations for the relevant quantities in terms of a , α and b . This yields

$$\begin{aligned} \xi &= q\underline{\alpha} & c &= qb \\ \delta &= q/b & \beta &= 1/\underline{b} \\ \gamma &= q/\underline{b} \end{aligned} \tag{6.12}$$

With these values (6.11a) and (6.11b) are identical so we are left with just two equations

$$a = \bar{\alpha}\underline{b}b \tag{6.13}$$

$$a = q\underline{\alpha}b\bar{b} \tag{6.14}$$

which can be reconciled by setting $\alpha = \sqrt{\sigma}b$ whereupon each of (6.13) and (6.14) become

$$a = \sqrt{\bar{\alpha}\underline{b}b\bar{b}} \tag{6.15}$$

again, there are no further conditions on a or b that come out of the compatibility of the two linear systems. In [47] these quantities were set to $a = \frac{x}{\sqrt{\sigma(x+1)}}$, $b = 1/x$ so that (6.15) yields qP_I , but one could also set $a = \lambda\sqrt{\bar{\sigma}\underline{b}b^2(1-b)}$ to find yet another false Lax pair for the logistic equation. Any other equation can also be represented by this fake Lax pair using the appropriate choices of a and b .

We remark that these two linear systems do in fact form a true Lax pair, albeit for a trivial equation. This can be seen from by canceling a in (6.13) and (6.14) which, after substituting $y = \alpha/\beta$, yields

$$\bar{y} = q\underline{y}$$

This is the only evolution equation that arises naturally from the system itself. It is therefore concluded that the only equations for which solutions can be reconstructed from the monodromy data, is this simple equation and its transformations and reductions.

One further example, this time of a false Lax pair for a PΔE, was worked through in section 3.4.3. The Lax pair identified there takes the form

$$L = \begin{pmatrix} a F_1 & b \\ c F_1 F_2 & d F_2 \end{pmatrix}, \quad (6.16a)$$

$$M = \begin{pmatrix} \alpha F_2 & \beta \\ \gamma F_1 F_2 & \delta F_1 \end{pmatrix}. \quad (6.16b)$$

Where lower case letters depend on the independent lattice variables l and m , and the upper case F_i depend on the spectral variable such that $F_1 \neq kF_2$, k constant. In section 3.4.3 it was explained that the evolution equation for the above Lax pair is simply the following, underdetermined set of equations:

$$d = -b\bar{\alpha}/\bar{\beta}, \quad (6.17a)$$

$$\gamma = -\hat{a}\alpha/\hat{b}, \quad (6.17b)$$

$$\delta = -\hat{a}\beta/\hat{b}, \quad (6.17c)$$

$$c = -a\bar{\alpha}/\bar{\beta}. \quad (6.17d)$$

This underdetermined set of equations does not properly define a nonlinear system, which is why the Lax pair is false, but could mistakenly be imbued with meaning under certain choices of variables. For example, setting

$$\alpha = -\frac{\beta}{\underline{b}}f(\underline{d}),$$

while keeping d as the dependent variable and allowing equations (6.17) to define γ , δ and c , leads to the following Lax pair

$$L = \begin{pmatrix} a F_1 & b \\ \frac{a}{\underline{b}}f(\underline{d}) F_1 F_2 & d F_2 \end{pmatrix},$$

$$M = \begin{pmatrix} -\frac{\beta}{\underline{b}}f(\underline{d}) F_2 & \beta \\ \frac{\hat{a}\beta}{\hat{b}\underline{b}}f(\underline{d}) F_1 F_2 & -\frac{\hat{a}}{\hat{b}}\beta F_1 \end{pmatrix}.$$

It is easy to check that this Lax pair has as its evolution equation:

$$d = f(d),$$

where $f(d)$ is an arbitrary function of d and any iterations thereof.

6.2 Equivalent evolution equations

In this section we describe how some sets of well known nonlinear integrable PDEs can actually be thought of as one equivalent system.

6.2.1 Common partial difference equations

The motivation for this section lies with the following conundrum: Lax pairs with the following spectral dependence

$$L = \begin{pmatrix} a & b \\ \nu c & d \end{pmatrix}, \quad (6.18a)$$

$$M = \begin{pmatrix} \alpha & \beta \\ \nu \gamma & \delta \end{pmatrix}, \quad (6.18b)$$

where ν is the spectral variable and all other terms are lattice terms, fall into the class that have LMKdV₂ as their evolution equation. However, in [37], Lax pairs with exactly this spectral dependence were shown to be associated with the cross ratio equation,

$$\frac{(x - \bar{x})(\hat{x} - \hat{\hat{x}})}{(\bar{x} - \hat{\hat{x}})(x - \hat{x})} = \frac{\lambda_1^2}{\mu_1^2}, \quad (6.19)$$

where $\lambda_i = \lambda_i(l)$ and $\mu_i = \mu_i(m)$ are arbitrary throughout this chapter. This suggests a transformation between LMKdV and the cross ratio equation that the Lax pair itself should be able to elucidate (possibly the known Miura transformation [78]). During a study of the Lax pair for the cross ratio equation, it was noticed that

the set equations from the compatibility condition of the general Lax pair in (6.18) would yield either the cross ratio equation or LMKdV, depending on the order in which the equations were solved. How can one set of equations, being solved in a general way without any arbitrary conditions being imposed, yield two different evolution equations?

The answer is that neither the LMKdV equation or the cross ratio equation has solved the compatibility condition fully, each of these evolution equations can be parameterized one more time. To see this, rewrite (6.19) as

$$\lambda_1(x - \bar{x})\lambda_1(\hat{x} - \hat{x}) = \mu_1(x - \hat{x})\mu_1(\bar{x} - \hat{x}),$$

which is parameterized for

$$\bar{y}y = \frac{\rho}{\lambda_1}(\bar{x} - x), \quad (6.20a)$$

$$\hat{y}y = \frac{\sigma}{\mu_1}(\hat{x} - x), \quad (6.20b)$$

where $\log \rho = (-1)^m \log \lambda(l)$ and $\log \sigma = (-1)^l \log \mu(m)$.

On the other hand, LMKdV often takes the form

$$\hat{y} = y \frac{\hat{y} - \frac{\mu_2}{\lambda_2} \bar{y}}{\bar{y} - \frac{\mu_2}{\lambda_2} \hat{y}},$$

which can be rewritten for

$$\frac{\hat{y}\bar{y}}{\mu_2} - \frac{\hat{y}y}{\mu_2} = \frac{\hat{y}\hat{y}}{\lambda_2} - \frac{\bar{y}y}{\lambda_2}.$$

In this form, it is clear that the LMKdV equation supports the following parametrization

$$\bar{y}y = \lambda_2(\bar{x} - x + \lambda_3), \quad (6.21a)$$

$$\hat{y}y = \mu_2(\hat{x} - x + \mu_3), \quad (6.21b)$$

Comparing equations (6.20) and (6.21) shows immediately that LMKdV and the cross ratio equation are in fact equivalent. The superficial differences arise through

the terms ρ and σ in (6.20) and λ_3 and μ_3 in (6.21), however, these terms are all introduced during the process of parametrization and can be set to unity or zero respectively, thus bringing about the equivalence. In fact, the common forms of LMKdV and the cross ratio equation do not hold all of the possible ‘parameter functions’ (the arbitrary functions ρ , σ , λ_1 and μ_1 that play the role of parameters), when all of these are included both equations can be parameterized to obtain

$$\bar{y}y = \rho_4\lambda_4(\bar{x} - x + \lambda_5), \quad (6.22a)$$

$$\hat{y}y = \sigma_4\mu_4(\hat{x} - x + \mu_5). \quad (6.22b)$$

Equation (6.22) is a integrable, coupled pair of first order difference equations that encompasses generalized forms of both LMKdV and the cross ratio equation.

6.2.2 Higher order versions

The full pair of coupled equations referred to as LMKdV₂ is,

$$\frac{\lambda_1 \hat{\hat{x}}}{\sigma \hat{x}} + \frac{\mu_2 y}{\rho \hat{y}} = \lambda_2 \sigma \frac{y}{\bar{y}} + \rho \mu_1 \frac{\hat{\hat{x}}}{\bar{x}}, \quad (6.23a)$$

$$\rho \mu_1 \frac{\hat{\hat{x}}}{x} + \lambda_2 \sigma \frac{\hat{y}}{\bar{y}} = \frac{\mu_2 \bar{y}}{\rho \hat{y}} + \frac{\lambda_1 \bar{x}}{\sigma x}. \quad (6.23b)$$

Where it is understood that the terms in (6.23a) are not necessarily related to terms using the same notation as in the previous section, although the naming system is consistent with $\lambda_i = \lambda_i(l)$, $\mu_i = \mu_i(m)$, $\log \rho = (-1)^m \lambda_3(l)$ and $\log \sigma = (-1)^l \mu_3(m)$. This pair of second order nonlinear equations describes the general dynamics that arise from the compatibility of the two linear systems (6.18). A further parametrization is achieved by adding the two equations in (6.23a) together and finding exact differences, although to bring these about we must set ρ and σ to

unity. It is not difficult to verify that the following parametrization holds:

$$\lambda_1 \frac{\bar{x}}{x} = \bar{v} - w + \lambda_3 - r_1, \quad (6.24a)$$

$$\mu_1 \frac{\hat{x}}{x} = \hat{v} - w + \mu_3 - s_1, \quad (6.24b)$$

$$\lambda_2 \frac{\bar{y}}{y} = \bar{w} - v + \lambda_3 + r_1, \quad (6.24c)$$

$$\mu_2 \frac{\hat{y}}{y} = \hat{w} - v + \mu_3 + s_1, \quad (6.24d)$$

where v and w are introduced terms dependent on both lattice variables l and m , $r_1 = (-1)^m \lambda_4(l)$ and $s_1 = (-1)^l \mu_v(m)$.

By eliminating x and y from equations (6.24), a pair of coupled, cross ratio type equations in v and w can be extracted.

$$(\hat{w} - \hat{v} + \lambda_3 - r_1)(\hat{w} - v + \mu_3 + s_1) = (\hat{w} - \bar{v} + \mu_3 - s_1)(\bar{w} - v + \lambda_3 + r_1),$$

$$(\hat{v} - \hat{w} + \lambda_3 + r_1)(\hat{v} - w + \mu_3 - s_1) = (\hat{v} - \bar{w} + \mu_3 + s_1)(\bar{v} - w + \lambda_3 - r_1).$$

6.3 Discussion

In this chapter, we explored different ways in which false Lax pairs can occur and how to find equivalences between equations by considering their parameterizations. In [38] it was shown (see equation (2.6) in that paper) that the Adler system, also known as Q_4 , can be written in the form

$$f(x)f(\hat{x}) + g(x)g(\bar{x}) = k, \quad (6.25)$$

where k is a constant. It will be interesting to see what equations Q_4 is equivalent to.

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