Real Estate Leases and Real Options

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Abstract

This thesis builds on the real estate lease model of Grenadier which consists of the Black Scholes PDE and an upper reflecting boundary condition. Extending the method of images of Buchen, a new technique was developed to solve this class of problems. Problems that previously required difficult integration can now be solved with algebra and simple integrals. In addition, the compound option in this framework is solved using this new technique. To the best of our knowledge the solution of the compound problem has not been published. An interesting symmetry between this class of problems and the lookback option was also discovered and described in this thesis.

The extension of the method of images to include problems with the reflecting boundary condition in the context of real estate leases was presented at the *Financial Integrity Research Network* Doctoral Tutorials at the University of Technology, Sydney, in 2006. The presentation was awarded the "FIRN Best Paper Award". This paper has been submitted to the Journal of Financial Mathematics for publication.

The solution to the compound problem in the context of the upward-only market review option is the subject of the next paper.

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Chapter 1

Introduction

1.1 Motivation

The thesis was motivated by a simple question. Given a lease on an office space what is the value of the upward only market review clause added by the landlord? Indeed this is a problem faced by many tenants of commercial office space as this specific clause is becoming a standard feature of this type of lease contract.

1.2 Black Scholes and Real Estate

Lease contracts are common in our economy with a large percentage of both commercial and residential real estate being leased. These leases can be quite complex and contain clauses that change the value of the lease. These clauses cover all aspects of the lease from the permitted usage of the property to rules that govern future changes in the rent. Such clauses have the capacity to change the value of the lease, yet few are routinely valued, which is partly due to a lack of available pricing techniques [1]. It transpires that these clauses contain embedded options in the financial mathematical sense.

Initial models to value real estate leases focused on taxation implications and opportunity cost. Early work, at the same time as the Black Scholes option pricing model was being introduced, priced the real estate lease by calculating the economic rationale for leasing. Meyers [38] considered the opportunity cost of a lease versus the cost of borrowing to purchase the property. Their formula used the present value of cash flows associated with the cost of leasing, cost of borrowing and property ownership, and taxation implication.

The Black Scholes option pricing techniques were later applied to the lease problem. McConnell [18] priced leases and lease options in the Black Scholes framework. Whereas Meyers used a simple mathematical model but included many practical considerations such as taxes, McConnell assumed a perfectly efficient market and concentrated on the mathematical model. This allowed him to price the lease and several lease options such as the purchase option and renewal option. He went further to consider lease options with American features. However one of his assumptions was that the return followed a log normal process. He did not impose an upper boundary condition on the return from the lease.

Buetow [14] also followed an option theoretic approach and assumed the underlying stochastic process was the rent and the consumer price index (CPI). However, the model proposed by Buetow was that the rent and CPI followed mean reverting processes. This, therefore, had more similarity with interest rate models. Buetow solved his model numerically.

One of the more recent approaches to real estate leases, and in particular the upward only market review option, was by Ambrose [4]. This approach deviated from the option theoretic approach and although some results from Grenadier were used, the upward only market review option was priced by solving an infinite system of algebraic equations.

The notion of an option in a lease is not obvious and occurs at two levels. In the model for real estate leases used in this thesis, the basic valuation of the lease involves a simple call option which will be described later in detail. In addition to this option there are also options within the lease that add or remove value from the lease and so favour either the lessor or lessee. These embedded options are quite common in leases. It is these embedded options that will be referred to as *lease options* and it is the pricing of these lease options that are interesting and provide the motivation for this work.

Lease options are usually not valued using modern pricing techniques in financial mathematics. A method to place a value on these lease options would be of great value to both the lessor and lessee and may assist in the negotiation of leases. Current lease negotiations rely on comparing the value of leases of similar real estate. This is used in the absence of any other guide to valuation. Other lease options form part of the standard leases yet there is no justification as to why this should be, apart from historical practice in the industry. Rowland [27] agrees that lease negotiations are facilitated by valuing these lease options and states "business negotiations are more effective when each issue is expressed as a monetary equivalent because this encourages the parties to trade concessions to reach mutually beneficial agreement".

The lease option that has provided much of the motivation for this research is the upward-only market review option. The upward-only market review option can be a feature of the commercial real estate lease and provides protection to the lessor against upward movement in the rent. To place a value on this option would be of great help to the lessee and could be used as a bargaining tool to obtain other considerations in the lease.

Indeed the upward-only market review option has been quite controversial. In the UK in the early 1990s legislation was proposed to make this option illegal. The Department of Environment issued a consultation paper seeking comment on this proposed legislation [5].

The problem with an options theoretic approach to valuing leases and lease options is that there is no visible underlying process, unlike the share market where share price movements are monitored continuously during trading. Rent and real estate values are observed infrequently. Estimating the price of real estate is difficult as there is both spatial and temporal averaging. This smoothing causes an under estimate of the volatility or risk in property investment returns. Attempts have been made to correct for this smoothing [12], whereas other authors have acknowledged that there is an inherent problem with underestimation of volatility and they have resorted to estimating the implied volatility [19].

The class of contracts where the underlying stochastic process can not be directly observed, as in the case of real estate leases, are called *real options*. Indeed there are many real option problems as many decisions in business do not have an underlying observable process such as a share price. Examples include business valuation and the decision to close a factory. However, many of these real options can be priced in the Black Scholes framework.

The Black-Scholes model was first published in 1973 with a closed form solution for the price of a European call and put option [10]. This model made several assumptions [37]. Firstly, the market is assumed to be frictionless and continuous with no transaction costs or taxes and short selling is allowed without penalty. Secondly, the underlying asset follows geometric Brownian motion described by the sde

$$dS = (\alpha S - q(S,t))dt + \sigma S dW$$

where S is the price of the underlying asset, q is the dividend, α is the expected return,

σ is the volatility and W is a standard Weiner process. Thirdly, there exists a default free bond with a constant interest rate, r. Fourthly, that investors prefer more to less and make decisions accordingly and all investors agree on the volatility, σ, but not necessarily on the expected return, α. Finally, the option price is a twice continuously differentiable function of the asset price, default free bond price and time. Merton [36] showed that the last assumption is actually a consequence of the model and need not be assumed ab initio.

The Black-Scholes hedging argument involves a portfolio constructed by purchasing the option and short selling the underlying asset. This is done in such a ratio of the option to asset as to render the portfolio riskless. This technique came to be known as *delta hedging*, and is still very much in use today as a risk management tool. Of course, this hedge has to be maintained dynamically by constantly adjusting the ratio of the option to the underlying asset. One of the most extreme examples of delta hedging was the story of Long Term Capital Management where it was used successfully for many years [25]. LTCM eventually failed spectacularly in 1999, and an account of this as well as possible explanations has been the subject of many articles and books.

The result of this analysis is the celebrated Black-Scholes pde

$$\begin{aligned} \mathcal{L}V &= 0 \qquad (1.1) \\ \mathcal{L} &= \frac{\partial}{\partial t} - r + (r-q)x\frac{\partial}{\partial x} + \tfrac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} \end{aligned}$$

where $V = V(x,t)$ is the option price, x is the price of the underlying asset and t is time. Different options are defined by adding different boundary conditions. Solving the Black-Scholes PDE with these boundary conditions provides the price of the option and this price is guaranteed to be arbitrage free.

In this research the PDE approach to option pricing will be followed. However, following 1973 the Black-Scholes model was expanded providing a second approach that involved the discounted expectation method. This method shows that the price of an option is the expected value of the payoff discounted at the risk free interest rate. The expectation is taken with respect to the risk neutral measure. Indeed this approach can be much simpler than solving the Black Scholes pde, and in this thesis has been used to price some of the simple options such as binary options that are used to build more complex options. The expectation approach can be found in Cox and Ross [15], Harrison and Kreps [16] and Harrison and Pliska [17]. These two approaches can be demonstrated to be equivalent using the Feynman-Kac formula [2]. A summary of these approaches can be found in Buchen [33].

The use of the Black-Scholes option pricing equation in real option theory is controversial. Critics of real option theory argue that the Black-Scholes model was derived with the assumption that the underlying stochastic process is continuously traded. Since the underlying process for a real option is generally not traded critics argue that the Black-Scholes model is not appropriate. Merton [37] argues that a "hypothetical financial intermediary" can create derivative securities. These derivative securities can be used with the underlying non traded asset to replicate the hedging portfolio in a real option strategy. Another portfolio of traded assets can be created to approximate this hedging portfolio. Using arbitrage arguments the equilibrium price of the derivative security is the same as if the underlying asset was traded continuously. Merton concludes that the Black-Scholes formula can be applied even when the underlying asset is not traded continuously.

The underlying stochastic process that would be desirable to monitor would be the Equilibrium Instantaneous Lease Rate described by Grenadier [40], which he denotes as $P(t)$. In this paper he considered an oligopoly of property developers who compete to supply new commercial office space. When this competitive oligopoly is in (Nash) equilibrium there exists an upper reflecting barrier on $P(t)$.

The instantaneous lease rate is assumed to evolve to clear the market and the market inverse demand function is of constant elasticity form

$$P(t) = D(t)Q(t)^{-1/\gamma}$$

where $D(t)$ is the multiplicative demand shock, $Q(t)$ is the cumulative supply from the oligopoly of property developers, $\gamma > 1/n$, n is the number of property developers in the oligopoly. The demand shock $D(t)$ is assumed to follow geometric Brownian motion.

Grenadier argues that in the case of this (Nash) equilibrium argument if the equilibrium instantaneous lease rate increases above the barrier level then the property developers act to increase the supply $Q(t)$. The Nash equilibrium used by Grenadier will not be explored further in this work. It will be simply assumed the instantaneous lease rate and the barrier exist. In addition the term structure of rent which is analogous to the term structure of interest rates will not be considered in detail.

Every lease contract specifies a lease payment that is usually paid monthly or weekly in advance. This is the base nominal rent and should not be confused with the equilibrium instantaneous lease rate. However, there is a term structure of the nominal rent which for short term leases is higher than for long term leases. In addition, as part of the lease negotiations, changes in the nominal rent are traded for lease options. For example a potential lessee may accept a higher nominal rent in the lease in exchange for a rent free period at the start of the lease. Allowing for the time value of money the total rent paid may be the same so that the value of the leases are the same, however the lessee benefits from less expenses at the start of the lease when the business is new and revenue is low.

The base nominal rent quoted in a lease contract is still the most prominent feature of the document and is what is usually known as the rent. It is really the regular lease payment. This nominal rent is not to be confused with the Grenadier's equilibrium instantaneous lease rate. The nominal rent that appears on the lease contract is then a function of the value of the rent process at the time of negotiations of the lease and the result of the negotiations of the lease options. The "rent" used in this work, if not qualified, will refer to the equilibrium instantaneous lease rate which is the underlying stochastic process.

In summary it is assumed that the instantaneous rent rate is the underlying random process, which we will simply refer to as *rent* in the rest of this work. The rent is then considered to follow geometric Brownian motion. This means that the Black Scholes option pricing methodology can be used.

The unusual feature of this model is the addition of a reflecting boundary condition. This boundary condition is uncommon in financial mathematics and mathematical finance. It adds considerable complexity to the pricing problem. In fact we know of no other financial contract with this feature. Although this model is used in this work, no justification, beyond the brief discussion above, will be provided regarding the appropriateness of this model. There are many models that could be used, such as a mean reverting process. However, validating the model against real data is not in the scope of this thesis.

Data from the Sydney commercial real estate market provided by the Property Council of Australia does give some insight into the suitability of such a model. Even within the relatively short period from 1990 to 2004 which the data covers, the nature of the market clearly changes. That is, the problem is non-stationary so even if the model is appropriate at a particular point in time it may not be appropriate if the market conditions change. This will be discussed briefly in chapter 8.

Residential real estate will not be considered as there are subjective factors that seem to dominate this market. Factors such as proximity to water, social status of neighbours and the history of the suburb have greater influence than for commercial real estate.

This thesis is primarily concerned with pricing lease options in the framework described by Grenadier. Simply stated the primary goal is to solve the Black-Scholes partial differential equation with the addition of a reflecting boundary condition for different payoffs. In order to do this the method of images of Buchen was extended for this reflecting boundary condition. Using this methodology to solve this class of problems the compound problem was also solved. A compound option is one where the payoff is also an option. This work also demonstrated a symmetry between the lookback option and the reflecting boundary. In fact the equivalence of the lookback option and the reflecting barrier was found first and the extension of the method of images followed from this.

One of the fundamental concepts used in this research is the notion of *Static Replication*. The basis for this is the linearity of the Black-Scholes operator. If the price of an option satisfies the Black-Scholes PDE and has a payoff at expiry as below. $V(x,t)$ is defined by

$$\left.\begin{array}{rcll} \mathcal{L}V &=& 0 & 0<t<T \\ V(x,T) &=& \sum_{i=1}^{n} f_i(x) & \end{array}\right\} \tag{1.2}$$

then the option price is

$$V(x,t) = \sum_{i=1}^{n} V_i(x,t) \tag{1.3}$$

where each component $V_i(x,t)$ is defined by

$$\left.\begin{aligned} \mathcal{L}V_i &= 0 \qquad 0<t<T \\ V_i(x,T) &= f_i(x) \end{aligned}\right\} \tag{1.4}$$

Although this is quite a simple idea it is quite powerful when combined with the method of images and the Equivalent European Payoff by Buchen. The method of images of Buchen will be described in chapter 4. This technique is used to solved the Black Scholes pde where there a boundary condition such as an absorbing barrier. The technique provides an alternative but equivalent description of the problem where the boundary condition is removed and the payoff is changed so that the solution is in the form of a portfolio of European options. Hence given a problems of the form

$$\begin{aligned} \mathcal{L}V &= 0 \qquad \text{in} \qquad t>0,\ \ x<b \\ V(x,T) &= \text{“payoff”} \\ & \quad\ \text{“Boundary Condition at } x=b\text{”} \end{aligned}$$

the method of images provides the alternate description

$$\begin{aligned} \mathcal{L}V &= 0 \qquad \text{in} \qquad t>0,\ \ x>0 \\ V(x,T) &= \text{“equivalent payoff”} \end{aligned}$$

Note that the domain of the x changes.

1.3 The Structure of this thesis

The basic model will be defined and then solved using the method of images. This technique is significantly simpler than that used by Grenadier. He uses the mathematics of first pass probability which involves difficult and complex integration.

Once the method is demonstrated it will be used to solve several lease options. This includes the purchase option and the market review option which are the most interesting.

The next section shows the simple transformation that takes this problem into a lookback option. The lookback option solution of Buchen and Konstandatos [34] is given and is shown to be equivalent to the solution found by the more direct approach.

The Sydney CBD commercial office space data from the Property Council of Australia is considered. Macroeconomic factors that have influence on the commercial property market are discussed.

Finally an example from the Sydney CBD office market is used to illustrate how the implied volatility can be obtained and then used to price the market review option.

1.4 Original work in this thesis

There are three original contributions to financial mathematics presented in this thesis.

The first original work is a new method to solve an existing problem. The method of images was applied to the lease type problem. The equivalent European payoff for this class of problems has not been described before and has greatly simplified the solution. Further, the derivation of the equivalent European payoff, for this class of problems, from first principles and then using the log-volution theory of Buchen is presented.

The second original work is the application of this new theory to the compound problem in this framework. In particular, the market review option was priced and the analytical solution presented.

The third original work is the discovery of a symmetry between the lease type problems and the lookback option. Although this is not the main result of the thesis it was the first breakthrough made that led to the other original, and more significant, contributions.

Chapter 2

Real Options

2.1 Real Estate Demand and Supply

Since the publication of the Black Scholes model of option pricing [10] there has been an increase in activity not only in the financial markets, in particular the derivative securities markets, but also in financial mathematics. The first public options exchange was the Chicago Board of Exchange which began trading in 1973. By 1975 traders in this exchange were using the Black Scholes model to price options and to hedge their positions [37]. Thirty years on this is still the dominant model in option markets world wide. However, the origins of this work predated Black and Scholes by seven decades with the mathematics of Louis Bachelier and Kiyoshi Itô.

The Black Scholes partial differential equation is essentially a model that embodies the notions of arbitrage free markets and stock prices that follow geometric Brownian motion. We will state the Black Scholes partial differential equation here without proof as this is readily available in most financial mathematics textbooks [28].

Let S_t be the price of the underlying share with constant drift μ and volatility σ. That is

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where dW_t is a Weiner process. This describes geometric Brownian motion having constant drift and constant volatility. The Black Scholes PDE which describes the price of a derivative security, $V(s,t)$, whose underlying share price is described by S_t is

$$V_t = rV - rsV_s - \tfrac{1}{2}\sigma^2 s^2 V_{ss}; \qquad s > 0, \;\; t > T \tag{2.1}$$

subject to an expiry condition $V(s,T) = F(s)$ for some function F and conditioned on $s = S_t$. The subscript refers to the partial derivative with respect to the subscript and $F(s)$ is called the payoff function.

The idea of no arbitrage means that if an opportunity occurred in the market where a risk free profit could be made then enough traders would exploit this opportunity and market forces of demand and supply would rapidly correct this aberration. Geometric Brownian motion is a natural choice as it is the maximum entropy distribution where the only assumption is that the underlying process is non negative and only the mean and variance of the log-return is given. From the mathematical point of view this model is essentially a Gaussian model and so there are numerous symmetry properties that can be exploited to provide analytical solutions. Current work in the field of option pricing is looking at more generalised stochastic processes, however few, if any, analytical solutions exist.

It is well known that the Gaussian distribution is a poor model for the stochastic process that drives the log-return of the stock price. In particular the kurtosis, or thickness of the tails of the distribution of the stock price changes do not fit the Gaussian distribution. The reality is that large changes in the stock price occur more frequently than described by a Gaussian with constant volatility. The result is that if the volatility is calculated from the model and applied to current market prices then the implied volatility is not constant and is higher for large price changes. This is the so called volatility smile.

Despite this poor fit of the model to market data where the underlying process is monitored almost continuously the model is being used for markets where this is not the case. If all that is known about a stochastic process is that it is non negative then this is a good starting point. If the underlying process is not traded then a derivative or option based on this process is called a real option. Real option theory has also recently flourished as options are ubiquitous. For example, consider a business that wants to build a new factory. The factory can be built at any time and the fact that the business can either build now or wait until there is a change in the market, is itself an option. There is a value that can be placed on the option to wait. The real estate market also has an abundance of real options. In this work only those related to real estate leases will be considered.

In the real estate market there is a value that can be placed on the deferral of usage of a property. This is a central concept in trying to place a value on usage of a property. As will be described in more detail later the value of a lease is related to the owner being denied usage of a property for a period of time and is irrespective of the actual usage of the property during that period. This constitutes an option. In real estate leases there are a number of conditions or options that restrict the use of the property and hence change the value of a lease. These also constitute real options. Examples of these include the purchase option, the market review option, the rent collar and cap and the renewal option.

The purchase option gives the lessee the right to purchase the property at the end of the lease for a pre-determined price. In the simplest case the price is set at the beginning of the lease. In more complex cases the purchase price can be dependent on an economic parameter such as the rate of inflation or the exchange rate. The market review option gives the lessor the right to re value the lease at a fixed time during the term of the lease and is related to the change in the underlying rent. The amount that the lessor can change the lease value at the market review can be limited and these are described as caps and collars.

Leases and the terms and conditions in the lease are currently negotiated without any valuation of the embedded options in the lease. To a large extent lease negotiators use experience and common practices as a guide [1].

The underlying value of real estate is also needed before it is possible to describe the model that will be used to price leases and lease options. To some extent in this model the absolute value of the property is not needed but instead its relative change in value is used. Although the value of a property is known when a sale occurs, this is an infrequent event in the history of a property. To value a property one could look at recent sales of similar properties. However unlike shares, every property is different. Even an identical building must have a different location. Valuation based on this then suffers from temporal and spatial smoothing. This is the traditional method of valuation of real estate and does not take into account the value of the property from it's utility.

The other approach to valuing a property is by considering the return from the property. This has the converse problem of not considering the intrinsic value of a property. However, in the example of commercial office space this approach is more appropriate as an office space that cannot attract rent has no value. This approach is used in the form of a long term equilibrium value which will be described later.

The return on a rental property is in the form of a lease or rent. However, the lease usually has a term measured in years and the payment is usually in monthly installments. Hence measuring quantities such as instantaneous return and volatility directly from lease data is impossible.

One of the goals of this work is to try to place a value on these lease options and provide, at least, a starting point for negotiations and comparison of leases. As the data from which this will be derived is sparse compared to the high frequency data that comes out of the stock market, many parameters will have to be estimated from existing and historical market data.

The term 'real estate' encompasses a wide range of properties. It would be difficult to attempt to model all real estate. To narrow down the field only commercial real estate in major cities is considered. This has a number of advantages. Commercial property has a well defined classification and detailed information is kept for each class and each city. Regular valuations are performed on commercial properties and accurate information about supply and return are known. Commercial property within a class is to some extent homogeneous. Certainly there is evidence to suggest that the market views these properties as homogeneous. The specific details of this will be addressed in section 8.1.

Much of the literature on lease options has a foundation in the work by Dixit and Pyndick [3]. Consider a market that consists of many identical properties. Let this idealised market also contain many similar businesses that lease property. The number of these businesses and their activity determines the level of demand for property. Each business will then have one or more leases on property.

When a lease is negotiated its value will be a function of the value of rent at the time of negotiation. To understand this boundary condition the demand function has to be considered. What follows is an outline of the justification for the existence of an upper barrier and the reflecting boundary condition as presented by Dixit and Pindyck [3]. Again it should be stated that rent is the equilibrium instantaneous lease rate described by Grenadier and its value is modeled by geometric Brownian motion.

Consider again, the inverse demand function which is of constant elasticity form so

$$X_t = D_t Q_t^{-1/\gamma} \tag{2.2}$$

where X_t is the rent, D_t is the instantaneous demand and Q_t is the total supply and γ is a positive constant [40].

From this model of rent, if demand is constant then an increase in supply will result in a decrease in rent. For a constant supply, rent is proportional to demand. This model would appear to be intuitive.

As the economy grows one would expect demand for commercial property to increase. From the demand function this would cause an increase in rent. However, there must exist a level of rent where it is financially viable to build more property. If property supply increases then, again from the above demand function, rent must decrease.

In this market model we assume that there are no restrictions to entry so that a property developer and builder could instantaneously increase supply. Let the value of rent above which it is financially viable to build new property be b. When demand increases and rent increases to this value, there is an instantaneous increase in supply which causes the rent to fall below b. This is a "reflecting" boundary condition. It places a ceiling on the rent.

Let us represent the rent process X, as a random variable that follows geometric Brownian motion.

$$dX = \mu X_t dt + \sigma X_t dW_t \tag{2.3}$$

Here μ is the drift, σ is the volatility and W_t is a standard Weiner process.

Given any function of the rent process and time, $V(X_t, t)$, the reflecting boundary condition is represented by the Neumann boundary condition [3].

$$\frac{\partial V}{\partial x} = 0 \qquad \text{when} \qquad X_t = b. \tag{2.4}$$

How this reflecting barrier is calculated or derived from the market is addressed later. Grenadier [40] uses the Nash equilibrium for an oligopoly to find this value.

The first comment about the equation is that the drift term is μ and not the risk free rate r. This is expected for any asset except riskless securities such as Australian Treasury bonds. As will be demonstrated the same hedging argument as used in the Black Scholes equation makes the real drift irrelevant.

2.2 Rent and the PDE

One of the fundamental assumptions is the existence of a long term equilibrium value of real estate. Whether such an equilibrium exists is contentious, but this argument is beyond the scope of this work. It is a fundamental assumption of Dixit and Pyndick [3] and this will be followed here. What follows is a hypothetical portfolio derived to find the equilibrium value of real estate. In the process we will also derive the fundamental diffusion equation that governs derivative securities driven by the rent process.

So consider a property with value $H(X, t)$ at time t. Consider being able to sell or buy a unit of rent. That is, a rent unit is a contract whose value at time t is X_t. This rent unit has to pay a dividend or a return of q say. Hence the instantaneous return on a unit of rent is qX_t. This has an analogy with a unit in a listed property trust.

Consider a portfolio that is long one property and short n units of rent. Call the value of this portfolio ϕ. Then

$$\phi(t) = H(X_t, t) - nX_t$$

In the next time increment the portfolio receives rent of X_t but has to pay nqX_t. That is

$$\phi(t + dt) = H(X_{t+dt}, t + dt) - nX_{t+dt} + X_t dt - nqX_t dt$$

Formally this gives, conditional on $X_t = x$

$$d\phi = dH(x, t) - ndx + xdt - nqxdt$$

Using Itô's lemma

$$\begin{aligned} d\phi &= \left(H_t + \mu x H_x + \tfrac{1}{2}\sigma^2 x^2 H_{xx}\right) dt + \sigma x H_x dW \\ &- n\mu x dt - n\sigma x dW \\ &+ xdt - nqxdt \end{aligned}$$

The value of the portfolio, ϕ, has a deterministic component and a stochastic component. The stochastic component is $\sigma x(H_x - n)dW$. Setting $n = H_x$ removes the stochastic component and the return on the portfolio is then risk free. The value of the portfolio is deterministic and must equal the risk free rate of return. If the portfolio returned either more or less than the risk free rate of return it can be shown that an arbitrage profit can be made.

For instance assume that the portfolio with $n = H_x$ returned $\mu > r$ where r is the risk free rate. Construct a portfolio that is long the property, short H_x units of rent and the difference is financed by borrowing at the risk free rate of return, r. The portfolio will increase in value at a rate of μ and the cost of financing the property will be at a rate of r hence a nett profit is made.

Hence

$$\begin{aligned} d\phi &= \left(H_t - qxH_x + \tfrac{1}{2}\sigma^2 x^2 H_{xx} + x\right) dt \\ &= r\phi dt \\ &= r(H - xH_x)dt \end{aligned}$$

This leads to the pde

$$H_t - rH + (r-q)xH_x + \tfrac{1}{2}\sigma^2 x^2 H_{xx} + x = 0 \tag{2.5}$$

This is similar to the Black Scholes equation with continuous dividend. The only difference is the extra term, x, for rent.

The interpretation of q is difficult in the above derivation as a unit of rent as a tradable asset does not exist. A practical interpretation comes from Patel and Sing [19]. They interpret q as the rental yield on the property. In appraisal literature it is estimated as the difference between the "equated yield that measures the required rate of return on comparable assets" and "the expected rental growth" [19]. In Patel and Sing in their work on estimating real estate volatility they use the Investment Chronicle Hillier Parker Index as a proxy for this rental yield. Its justification however can also come from appealing to the real options argument of Merton [37]. This is explained by Dixit and Pindyck [3] by analogy with a financial American call option. If V is the price of the underlying stock for this call option then q is the dividend rate on this stock. The total return on the stock is the sum of the dividend and the capital growth of the stock. If the stock does not pay a dividend then the call option will be held to maturity. If the stock does pay a dividend then the call option can be exercised as the opportunity cost of not exercising the call option early is the dividend stream that could be obtained by exercising the option and buying the stock. In terms of real estate q would be the opportunity cost of delaying construction of a building or purchasing real estate instead of just keeping the option to invest open.

2.2.1 Long Term Equilibrium

Dixit and Pyndick assert that there is an equilibrium in the real estate market. This is the steady state solution of equation 2.5 where

$$\frac{\partial H}{\partial t} = 0\,, \quad \text{i.e.} \quad H = H(x) \quad .$$

To find this equilibrium the following ode is solved

$$\begin{aligned} \tfrac{1}{2}\sigma x^2 H_{xx} + (r-q)xH_x - rH + x &= 0 \quad \text{in} \quad 0 < x < b \\ H_x &= 0 \quad \text{at} \quad x = b \\ H &= 0 \quad \text{at} \quad x = 0 \end{aligned}$$

This has solution

$$H(x) = Bx^{\beta} + x/q$$

where β is the positive root of

$$\tfrac{1}{2}\sigma^2\beta(\beta-1)+(r-q)\beta-r=0$$

and B is a constant. The negative root can be excluded as this would imply that value increases as rent goes to zero. This clearly cannot occur in any feasible real estate market.

At the upper limit $x=b$ from the boundary condition

$$H_x(b)=\beta B b^{\beta-1}+1/q=0$$

hence

$$B=-b^{1-\beta}/(\beta q)\,.$$

The solution has an economic interpretation. The term x/q is the value of the property from the continuous cash flows to perpetuity. It represents the discounted present value of rent payment discounted by the rate q. This correlates well with the Patel and Sing interpretation of q [19]. The term Bx^β is the correction for the upper limit on the rent x. Also note that B is negative as expected. One can see that the dividend factor q, even though it does not have an obvious interpretation, is critical.

The equilibrium property value can be re written in a more intuitive form

$$H(x)=\frac{b}{q}\left[\frac{x}{b}-\frac{1}{\beta}\left(\frac{x}{b}\right)^\beta\right] \tag{2.6}$$

This will be useful later.

2.3 Chapter Summary

The background of this thesis is not simple and relies on the work of a few pioneers in real options. Apart from introducing very briefly the Black Scholes pde this chapter introduced two very important concepts developed by Dixit and Pindyck. The first is inverse demand function and how this function justifies the existence of an upper reflecting barrier on rent in the real estate market. The second is the concept of the long term equilibrium value of real estate.

These three concepts are the basis for the next chapter which describes the Grenadier model for real estate leases.

Chapter 3

The Lease Framework

Smith [6] described a lease in the following way. When a tenant receives a lease they in effect receive the property for their use for a specified period of time. The tenant pays rent to the owner and at the end of the period of use, hands the property back to the owner. In the contingent claims analysis this is equivalent to the lessor giving the property to the the lessee. The lessee gives to the lessor a call option with zero strike and expiry being the end of the lease. At the expiry the lessor exercises the option and regains the property. The value of the lease is then the difference between the value of the property and the value of the call option.

The call option in this case is European. The lease payments can be calculated as an annuity. There is an assumption that the lessee does not default on this lease. Adding default risk increases the complexity of the problems.For now it is assumed that the payments will be riskless.

Complementary to lessee default risk is early termination of the lease by the lessor. This can be modelled as early exercise of the zero strike call option. This adds an American type feature. Again, this will not be considered in this work.

It is important to consider the lease payment schedule, although this is not developed further in this thesis. The lease payment is what is commonly understood by the term "rent". The value of the first payments, being the nominal rent, as appears in the lease contract is described in this lease payment schedule. Grenadier develops this further and uses it to find the term structure of rent.

For a risk free tenant the lease payment schedule is just a type of annuity payment where the present value of the installments is the lease value. This is described by Grenadier as follows. Let r be the risk free interest rate for the purpose of discounting. Let L be the lease value at the start of the lease. Let T be the termination time of the lease. Let N be the number of installments. Then the payment is

$$\text{“rent” payment} = L\frac{1-e^{-rT/N}}{1-e^{-rT}} .$$

In general in this thesis the term "rent" will refer to the underlying stochastic process rather than the colloquial usage being the lease payments or the nominal rent that appears in the lease contract. The distinction between these two concepts is important and should not be confused.

The lease value L is then written as

$$L(x,t;T) = H(x) - C(x,t) \quad \text{conditional on} \quad X_t = x \tag{3.1}$$

Now $H(x)$ is the equilibrium value of the property at time t and T is the time at expiry of the lease. $C(x,t)$ is the value of the zero-strike call option that expires at time $t=T$.

The derivation for the PDE for the value of the option is similar to the derivation for the property value.

Consider a portfolio that is long one option and short n units of rent. let the value of this portfolio be ϕ. Then

$$\phi(t) = C(X,t) - nX \, .$$

In the next time increment the portfolio receives rent of X_t. Unlike a portfolio that contains a property no rent is paid. That is

$$\phi(t+dt) = C(X, t+dt) - nX_{t+dt} - nqX_t dt \, .$$

Formally this gives, conditional on $X_t = x$

$$d\phi = dC(x,t) - ndx - nqxdt$$

Using Itô's lemma

$$\begin{aligned} d\phi &= \left(C_t + \mu x C_x + \tfrac{1}{2}\sigma^2 x^2 C_{xx}\right) dt + \sigma x C_x dW \\ &- n\mu x dt - n\sigma x dW - nqxdt \end{aligned}$$

This can be made risk free by letting $n = C_x$. If the portfolio is risk free, in order to avoid arbitrage, it can only earn the risk free rate of return. Hence

$$\begin{aligned} d\phi &= \left(C_t - qxC_x + \tfrac{1}{2}\sigma^2 x^2 C_{xx}\right) dt \\ &= r\phi dt \\ &= rCdt - rxC_x dt \end{aligned}$$

Equating the coefficients of dt gives following pde

$$\tfrac{1}{2}\sigma^2 x^2 C_{xx} + (r-q)xC_x + C_t - rC = 0 \quad \text{for} \quad t < T, \;\; x < b \tag{3.2}$$

with the boundary conditions

$$\left.\begin{array}{rcl} C(0,t) & = & 0 \\ C(x,T) & = & H(x) \\ C_x(b,t) & = & 0 \end{array}\right\} \tag{3.3}$$

This is the Black Scholes equation and is similar to equation 2.5 in Chapter 2.

The first boundary condition, at $x = 0$, is an absorbing boundary. If the property receives no rent then it has no value. The second condition, at $t = T$, is the payoff at expiry. The last condition, at $x = b$, is a Neumann boundary condition and embodies the reflecting barrier at $X = b$, as mentioned previously. This Neumann boundary condition is rarely seen in financial mathematics.

3.1 Chapter Summary

This short chapter had the single purpose of introducing the Grenadier model for real estate leases in its most compact form. That is, the call option in this model can be described by the Black Scholes pde, an absorbing boundary at zero rent, a reflecting upper barrier and the long term equilibrium value of real estate as the payoff.

This model with an arbitrary payoff defines the class of problem that is of interest and whose general solution is presented in this thesis. The path to this general solution takes us to the Method of Imaging of Buchen which becomes the topic of the next chapter.

Chapter 4

The Method of Images

4.1 The Equivalent European Payoff

In this chapter we describe the method of images of Buchen [31]. This is similar to the method of images well known in thermodynamics and electrostatics. The difference is that the image solution is non trivial for the Black Scholes PDE.

Given a problem in a restricted domain a more general problem can be found in the complete unconstrained domain that is easier to solve. The domain can be extended by adding the virtual domain to the original restricted or 'physical' domain. In our problem the physical domain is $0 < x < b$ where x is the rent and b is the natural upper boundary of the rent. The virtual domain is $x > b$. Together the physical and virtual domains give the complete unconstrained domain, $x > 0$. The solution found in the virtual domain is the image solution. The solution can then be expressed in terms of the solution to this more general problem and it's image.

In option pricing theory the method of images was first applied to pricing barrier options by Buchen [31]. For a knock-out barrier option there is an absorbing boundary condition.

That is the boundary condition can be represented as

$$V(b,t) = 0 \qquad \text{for} \ \ t > 0$$

When the price of the underlying asset reaches the boundary before expiry then the option value drops immediately to zero.

The barrier option was first priced by Merton [35] using the Black-Scholes PDE. Rich and Rubenstein used discounted expectations under the risk neutral measure [8] [22] [23], however, calculating the probability densities at the boundary involved solving complex integrals.

The method of images applied to the problem represented a return to the PDE approach. However, the technique greatly simplifies the calculations by exploiting symmetries of the system. The technique requires only algebra and at most simple integrals which are trivial in comparison with previous solutions.

In this thesis the boundary condition needed for the lease model is a reflecting boundary and although the final form of the solution is more complex than for the absorbing boundary the integration is still quite simple and has an economic interpretation.

Practically speaking the method of images of Buchen allows the option pricing problem to be re formulated without the boundary condition by changing the payoff. Without the boundary condition, the problem becomes a European type option problem. The new payoff is then referred to as the equivalent European payoff. The Black Scholes pde is, of course, unchanged.

4.2 Absorbing Barriers

To illustrate the method of images with an absorbing boundary condition the down-and-out barrier option will be used. This satisfies the Black-Scholes pde with the usual payoff at expiry but with the addition of the absorbing boundary condition. The lease problem is an example of a Black-Scholes PDE with the usual payoff at expiry but with a reflecting boundary. This will be the subject of later chapters and is the solution to this problem, using the method of images is one of the main results of this thesis.

For any function $V(x,t)$ which is continuous in x and t, the image, denoted as $\overset{*}{V}(x,t)$, relative to the Black-Scholes operator and the barrier b is

$$\overset{*}{V}(x,t) = \mathcal{I}[V(x,t)] = \left(\frac{b}{x}\right)^{\alpha} V\left(\frac{b^2}{x}, t\right); \qquad \alpha = \frac{2(r-q)}{\sigma^2} - 1 \tag{4.1}$$

$\mathcal{I}$ denotes the Black Scholes image operator. The image solution has a number of important properties. These include

1. $\overset{**}{V}(x,t) = \mathcal{I}[\overset{*}{V}(x,t)] = \mathcal{I}^2 V(x,t) = V(x,t)$
2. $V = \overset{*}{V}$ when $x = b$
3. If $V(x,t)$ solves the BS-pde with payoff $V(x,T) = F(x)$, then $\overset{*}{V}(x,t)$ solves the BS-pde with payoff $\overset{*}{V}(x,T) = \overset{*}{F}(x)$.

The idea behind an out barrier option is that if the asset price reaches the barrier during the life of the option then the value of the option drops to zero. For an in barrier option the option does not have value until it reaches the barrier. The up (and down) refers to the whether the asset price starts below (or above) the barrier.

Hence there are four basic barrier options

- down-and-out, V_{do}
- up-and-out, V_{uo}

- down-and-in, V_{di}
- up-and-in, V_{ui}

Let $\mathcal{L}$ as usual, denote the Black Scholes differential operator. The down-and-out barrier option is defined by

$$\begin{aligned}
\mathcal{L}V_{do}(x,t) &= 0 \qquad x > b, \;\; t < T \\
V_{do}(x,T) &= f(x) \\
V_{do}(b,t) &= 0
\end{aligned}$$

The up-and-out barrier is defined by

$$\begin{aligned}
\mathcal{L}V_{uo}(x,t) &= 0 \qquad x < b, \;\; t < T \\
V_{uo}(x,T) &= f(x) \\
V_{uo}(b,t) &= 0
\end{aligned}$$

The up-and-in barrier is defined by

$$\begin{aligned}
\mathcal{L}V_{ui}(x,t) &= 0 \qquad x < b, \;\; t < T \\
V_{ui}(x,T) &= 0 \\
V_{ui}(b,t) &= V_0(b,t)
\end{aligned}$$

The down-and-in barrier is defined by

$$\begin{aligned}
\mathcal{L}V_{di}(x,t) &= 0 \qquad x > b, \;\; t < T \\
V_{di}(x,T) &= 0 \\
V_{di}(b,t) &= V_0(b,t)
\end{aligned}$$

For the 'in' barrier options the value at the boundary is the price of the standard option, $V_o(x,t)$, defined by

$$\begin{aligned} \mathcal{L}V_0(x,t) &= 0 \qquad x > 0, \;\; t > T \\ V_0(x,T) &= f(x) \end{aligned}$$

and evaluated at the boundary, $x = b$.

It is clear from the above prescription that for 'up' barrier options the physical domain is $x < b$ and for down barrier options the domain is $x > b$. There are also symmetries that exist for these four options that can be used to price these barrier options without having to solve each barrier option directly.

Consider the following 'up' binary option $V_b(x,t)$ defined by

$$\begin{aligned} \mathcal{L}V_b(x,t) &= 0 \qquad x > 0, \;\; t < T \\ V_b(x,T) &= f(x)\mathbb{I}(x > b) \end{aligned}$$

It transpires that the four standard barrier options can be described entirely in terms of the standard option, $V_0(x,t)$, the 'up' binary option $V_b(x,t)$ and their respective images. These two option, being simple European style options, are quite straight forward to price using standard option pricing techniques and their solutions are well known.

It should be noted at this point that in the above prescriptions of the barrier options that for 'in' barrier options the option expires worthless if the barrier is not reached during the life of the option and for the 'out' type barrier options the option becomes worthless if the barrier is reached during the life of the option. There are also barrier options where the is a cash rebate if an 'out' barrier hits the barrier or an 'in' barrier does not. However, these will not be considered here for simplicity.

The first symmetry is found by inspection of the prescriptions for the down-and-out and the down-and-in barrier options. These two options cover the domain $x > b$ and have the same payoff at expiry. The standard option $V_0(x,t)$ has the same payoff at expiry. Hence it follows immediately that in the domain $x > b$

$$V_{do}(x,t) + V_{di}(x,t) = V_0(x,t) \qquad x > b$$

Likewise the same arguments can be used for up barrier options to give

$$V_{uo}(x,t) + V_{ui}(x,t) = V_0(x,t) \qquad x < b$$

These two symmetry relations are known as the in-out parity.

To find the remaining symmetry relations consider the image of the down-and-in barrier option. By the properties of the image the domain is $x < b$ and the boundary condition at $x = b$ is unchanged. This is exactly the prescription for the up-and-in barrier. Hence

$$\overset{*}{V}_{di}(x,t) = V_{ui}(x,t) \qquad \text{and} \qquad \overset{*}{V}_{ui}(x,t) = V_{di}(x,t)$$

This is known as the up-down parity.

Theorem. *Equivalent payoff for absorbing BV problems*

The equivalent European payoff for an absorbing BV problem for the Black-Scholes pde in $x > b$, with expiry T payoff $F(x)$ is given by

$$V^{\text{eq}}(x,T) = F(x)\mathbb{I}(x{>}b) - \overset{*}{F}(x)\mathbb{I}(x{<}b) \tag{4.2}$$

The last term above follows from the observation, using Eq(4.1)

$$\mathcal{I}[F(x)\mathbb{I}(x>b)] = \overset{*}{F}(x)\mathbb{I}(x<b)$$

This theorem allows us to solve absorbing BV problems by instead solving a related terminal value problem, *i.e.* a European option without the presence of the barrier at $x = b$, but with a modified payoff function.

This is an important theorem and the proof can be found in the appendix. It can be immediately applied to the down-and-out barrier option to show that the equivalent European payoff is

$$V_{do}^{\mathrm{eq}}(x,T) = f(x)\mathbb{I}(x > b) - \overset{*}{f}(x)\mathbb{I}(x < b)$$

From this it follows that

$$V_{do}(x,t) = V_b(x,t) - \overset{*}{V}_b(x,t)$$

Using the symmetry relations the remaining barrier options follow by simple algebra

$$\begin{aligned} V_{di}(x,t) &= V_0 - (V_b - \overset{*}{V}_b) \\ V_{ui}(x,t) &= \overset{*}{V}_0 + (V_b - \overset{*}{V}_b) \\ V_{uo}(x,t) &= (V_0 - \overset{*}{V}_0) - (V_b - \overset{*}{V}_b). \end{aligned}$$

4.3 Chapter Summary

The Method of Images of Buchen was described here and the general solution to the class of problems in the Black Scholes framework with an absorbing boundary condition was presented. However, the class of problems of interest in this thesis has a reflecting boundary condition. The extension of the method of images to solve the class of problems with the reflecting boundary condition will be presented in the next chapter.

Chapter 5

Reflecting Barriers

The lease framework of Grenadier has an upper reflecting boundary condition. This is quite a different problem to the absorbing barrier type options. In the reflecting boundary problems there are multiple times that the value of the underlying random process can reach the boundary. Grenadier solves this using first pass probabilities which involves “grueling integration” [39].

Extending the method of images of Buchen allows us to re formulate the problem by removing the boundary condition and introducing a new payoff which as before, is the equivalent European payoff. This equivalent European payoff can be quite complicated but the integration is greatly simplified and the problem is easily broken down to simpler components using the linearity of the Black Scholes pde.

For standard barrier options, the solutions could be expressed in terms of a linear combination of binary options. In the lease problem the solution can be expressed in terms of turbo binary options. The turbo option is a European option where the payoff is a power of the underlying asset multiplied by an indicator function. It is further defined and solved in appendix B.

The first step is to derive the equivalent European payoff for these reflecting boundary condition problems. What follows is a direct method which involves a transformation to the heat equation, solving the heat equation problem using Laplace Transforms and then transforming back to the original Black-Scholes variables. The theory of Log-volutions of Buchen, however, provides a more elegant derivation of the same result. The log-volution theory [32] is summarised in appendix C.

The problem we wish to solve if the following:

$$
\begin{aligned}
\mathcal{L}V(x,t) &= 0 \qquad \text{in} \qquad x < b, \;\; t > 0 \\
V(x,T) &= f(x) \\
V_x(b,t) &= 0
\end{aligned}
$$

where $\mathcal{L}$ is the Black Scholes operator, T is the time of expiry, $f(x)$ is any payoff function and b is the barrier level.

5.1 Solution from First Principles

Theorem *Equivalent payoff for reflecting boundary value problems*

The equivalent European payoff for a reflecting boundary value problem for the Black-Scholes pde in $x < b$, with expiry T payoff $F(x)$ is given by

$$V^{\mathrm{eq}}(x,T) = f(x)\mathbb{I}(x<b) + \overset{*}{f}(x)\mathbb{I}(x>b) + \alpha\mathbb{I}(x>b)\int_b^x \overset{*}{f}(y)\frac{dy}{y} \tag{5.1}$$

where $\alpha = 2(r-q)/\sigma^2 - 1$.

Proof: The proof has six steps.

- Transformation to the heat equation
- Laplace Transform in time
- Solve the transformed ordinary differential equation (ODE)
- Inverse Laplace transform
- Convert back to original variables
- Evaluate the solution at $t = T$

Transformation to the heat equation

The Black Scholes PDE with the reflecting boundary

$$\begin{aligned}
V_t &= rV - (r-q)xV_x - \tfrac{1}{2}\sigma^2 x^2 V_{xx} \qquad \text{in} \qquad x < b,\ t \le T \\
V(x,T) &= f(x) \\
V_x(b,t) &= 0 \\
V(0,t) &= 0
\end{aligned}$$

can be transformed to the Heat Equation using

$$\tau = T - t, \qquad \xi = \log b/x, \qquad V = e^{\frac{1}{2}\alpha\xi - \beta\tau} u(\xi,\tau)$$

This gives the mixed boundary value problem

$$\begin{aligned}
u_t &= \tfrac{1}{2}\sigma^2 u_{\xi\xi} \qquad \text{in } \xi > 0,\ \tau > 0 \\
u(\xi,0) &= e^{-\frac{1}{2}\alpha\xi} f(be^{-\xi}) \ = \ h(\xi) \\
u_\xi + \tfrac{1}{2}\alpha u &= 0 \qquad \text{when } \xi = 0;\ \ u \to 0 \ \text{ as } \xi \to \infty
\end{aligned}$$

where $\alpha = 2(r-q)/\sigma^2 - 1$ and $\beta = r + 1/8\alpha^2\sigma^2$.

Laplace Transform

Take the Laplace Transform in τ. Let $\hat{u}(\xi, s) = \mathrm{LT}[u(\xi, \tau)]$ then the problem transforms to

$$\begin{aligned} s\hat{u}(\xi, s) - h(\xi) &= \tfrac{1}{2}\sigma^2 \hat{u}_{\xi\xi}(\xi, s); \qquad \xi > 0 \\ \hat{u}_\xi + \tfrac{1}{2}\alpha\hat{u} &= 0 \qquad \xi = 0; \end{aligned}$$

which can be rewritten as an ODE

$$\begin{aligned} \hat{u}_{\xi\xi} - k^2\hat{u} &= -\frac{2}{\sigma^2}h(\xi) \qquad \xi > 0 \\ \hat{u}_\xi + \tfrac{1}{2}\alpha\hat{u} &= 0 \qquad \xi = 0 \end{aligned}$$

where $k = \frac{\sqrt{2s}}{\sigma}$ and $\hat{u} \to 0$ as $\xi \to 0$. This has the solution

$$\hat{u}(\xi, s) = Ae^{-k\xi} + Be^{k\xi} + \frac{1}{\sigma^2 k}\int_0^\infty e^{-k|\xi-\eta|}h(\eta)d\eta$$

for some constants A and B. However, $B = 0$ since $\hat{u} \to 0$ as $\xi \to \infty$.

Next determine A from the boundary condition $\hat{u}_\xi + \frac{1}{2}\alpha\hat{u} = 0$ at $\xi = 0$. This gives

$$A = \left[1 + \frac{\alpha}{k - \frac{1}{2}\alpha}\right]\frac{1}{\sigma^2 k}\int_0^\infty e^{-k\eta}h(\eta)d\eta$$

The solution of the ODE can be written as

$$\begin{aligned} \hat{u}(\xi, s) &= \frac{1}{\sigma^2 k}\int_{-\infty}^\infty e^{-k|\xi-\eta|}h(\eta)\mathbb{I}(\eta > 0)d\eta \\ &\quad + \left[1 + \frac{\alpha}{k - \frac{1}{2}\alpha}\right]\frac{1}{\sigma^2 k}\int_{-\infty}^\infty e^{-k(\xi+\eta)}h(\eta)\mathbb{I}(\eta > 0)d\eta \end{aligned}$$

Inverse Laplace transform of the solution

There are two results that are useful to simplify the calculation. The first involves the Green's function and is as follows (see Abramowitz and Stegun [20])

$$\begin{aligned}\text{ILT}\left\{\frac{1}{\sigma^2 k}e^{-k|\xi-\eta|}\right\} &= G(\xi-\eta,\tau)\\ &= \frac{e^{-(\xi-\eta)^2/2\sigma^2\tau}}{\sqrt{2\pi\sigma^2\tau}}\end{aligned}$$

The second result is used to give an alternative form of the solution and is as follows.

$$\frac{e^{-k\xi}}{k-\frac{1}{2}\alpha} = e^{-\frac{1}{2}\alpha\xi}\int_{\xi}^{\infty} e^{(-k+\frac{1}{2}\alpha)\nu}d\nu$$

The solution can therefore be written as

$$\begin{aligned}\hat{u}(\xi,s) &= \frac{1}{\sigma^2 k}\int_{-\infty}^{\infty} e^{-k|\xi-\eta|}h(\eta)\mathbb{I}(\eta>0)d\eta\\ &+\frac{1}{\sigma^2 k}\int_{-\infty}^{\infty} e^{-k(\xi+\eta)}h(\eta)\mathbb{I}(\eta>0)d\eta\\ &+\frac{\alpha}{\sigma^2 k}\int_{\xi}^{\infty} e^{\frac{1}{2}\alpha(\nu-\xi)}\int_{-\infty}^{\infty} e^{-k(\nu+\eta)}h(\eta)\mathbb{I}(\eta>0)\,d\eta\,d\nu\end{aligned}$$

Apply the inverse Laplace transform to get:

$$\begin{aligned}u(\xi,\tau) &= \int_{-\infty}^{\infty} G(\xi-\eta)h^+(\eta)d\eta\\ &+\int_{-\infty}^{\infty} G(\xi+\eta)h^+(\eta)d\eta\\ &+\alpha\int_{\xi}^{\infty} e^{\frac{1}{2}\alpha(\nu-\xi)}d\nu\int_{-\infty}^{\infty} G(\nu+\eta)h^+(\eta)d\eta\end{aligned}$$

where $h^+(\eta) = h(\eta)\mathbb{I}(\eta > 0)$. This form of the solution is particularly useful as it known that

$$\bar{u}(\xi,\tau) = \int_{-\infty}^{\infty} G(\xi-\eta)h^+(\eta)d\eta$$

is the solution of the initial value problem

$$\begin{aligned}\bar{u}_\tau &= \tfrac{1}{2}\sigma^2\bar{u}_{\xi\xi} \qquad \xi \in \Re, \qquad \tau > 0\\ \bar{u}(\xi,0) &= h(\xi)\mathbb{I}(\xi > 0) \;=\; h^+(\xi)\end{aligned}$$

Also since G is an even function, that is $G(-\xi,\tau) = G(\xi,\tau)$, then the image of $\bar{u}$ is

$$\bar{u}(-\xi,\tau) = \int_{-\infty}^{\infty} G(\xi+\eta)h^+(\eta)d\eta$$

The solution can be written in terms of $\bar{u}$ and it's image.

$$u(\xi,\tau) = \bar{u}(\xi,\tau) + \bar{u}(-\xi,\tau) + \alpha\int_{\xi}^{\infty}\bar{u}(-\nu,\tau)e^{\frac{1}{2}\alpha(\nu-\xi)}d\nu \tag{5.2}$$

Before converting back to the original variables it is instructive to check if this solution is correct. Firstly the integral

$$\begin{aligned}\int_{\xi}^{\infty}\bar{u}(-\nu,\tau)e^{\frac{1}{2}\alpha(\nu-\xi)}d\nu &= \int_{-\infty}^{\infty}\bar{u}(-\nu,\tau)e^{\frac{1}{2}\alpha(\nu-\xi)}d\nu\mathbb{I}(\nu > \xi)\\ &= \bar{u}(-\xi,\tau)\star\phi(\xi) \qquad \text{i.e. a convolution}\end{aligned}$$

with $\phi(\xi) = e^{-\frac{1}{2}\alpha(\nu-\xi)}\mathbb{I}(\xi < 0)$. But $\bar{u}(-\xi,\tau)$ satisfies the heat equation and so does the convolution. By linearity $u(\xi,\tau)$ also satisfies the heat equation.

Secondly the initial condition at $\tau = 0$ yields

$$\begin{aligned}u(\xi,0) &= h(\xi)\mathbb{I}(\xi > 0) + h(\xi)\mathbb{I}(\xi < 0)\\ &\quad +\alpha\int_{\xi}^{\infty}h(-\nu)e^{\frac{1}{2}\alpha(\nu-\xi)}d\nu\mathbb{I}(\xi < 0)\\ &= h(\xi) \qquad \text{when } \xi > 0\end{aligned}$$

Finally the boundary condition at $\xi = 0$ is

$$\begin{aligned} u_0' + \tfrac{1}{2}\alpha u_0 &= \bar{u}_0' - \bar{u}_0' - \alpha\bar{u}_0 - \tfrac{1}{2}\alpha^2 \int_0^\infty \bar{u}(-\nu,\tau)e^{\frac{1}{2}\alpha\nu}d\nu \\ &\quad + \tfrac{1}{2}\alpha[\bar{u}_0 + \bar{u}_0 + \tfrac{1}{2}\alpha^2 \int_0^\infty \bar{u}(-\nu,\tau)e^{\frac{1}{2}\alpha\nu}d\nu] \\ &= 0 \qquad \text{as required.} \end{aligned}$$

Conversion back to the Black Scholes variables

Let $\xi = \log b/x$, $t = T - \tau$ and

$$\begin{aligned} V(x,t) &= e^{\frac{1}{2}\alpha\xi - \beta\tau} u(\xi,\tau) \\ V_b(x,t) &= e^{\frac{1}{2}\alpha\xi - \beta\tau} \bar{u}(\xi,\tau) \end{aligned}$$

Now $V_b(x,t)$ solves

$$\begin{aligned} \mathcal{L}V_b(x,t) &= 0 \qquad \text{in } t < T,\ x > 0 \\ V_b(x,T) &= f(x)\mathbb{I}(x<b) \end{aligned}$$

Further, its image with respect to b is

$$\begin{aligned} \overset{*}{V}_b(x,t) &= (b/x)^\alpha V_b(b^2/x,t) \\ &= (b/x)^\alpha (x/b)^{\frac{1}{2}\alpha} e^{-\beta\tau} u(\log x/b,\tau) \\ &= (b/x)^{\frac{1}{2}\alpha} e^{-\beta\tau} u(-\log b/x,\tau) \\ &= e^{\frac{1}{2}\alpha\xi - \beta\tau} \bar{u}(-\xi,\tau) \end{aligned}$$

Multiplying equation 5.2 by $e^{\frac{1}{2}\alpha\xi - \beta\tau}$ gives

$$V(x,t) = V_b(x,t) + \overset{*}{V}_b(x,t) + \alpha \int_\xi^\infty \overset{*}{V}_b(be^{-\nu},t)d\nu$$

The last term can be simplified by using $\xi = \log b/x$ and $y = be^{-\nu}$ to change variables giving

$$V(x,t) = V_b(x,t) + \overset{*}{V}_b(x,t) + \alpha \int_0^x \overset{*}{V}_b(y,t)\frac{dy}{y}$$

The Equivalent Payoff

To find the equivalent European payoff simply evaluate at $t = T$ giving

$$\begin{aligned} V^{\text{eq}}(x,T) &= f(x)\mathbb{I}(x<b) + \overset{*}{f}(x)\mathbb{I}(x>b) + \alpha \int_0^x \overset{*}{f}(y)\mathbb{I}(y>b)\frac{dy}{y} \\ &= f(x)\mathbb{I}(x<b) + \overset{*}{f}(x)\mathbb{I}(x>b) + \alpha \int_b^x \overset{*}{f}(y)\frac{dy}{y}\mathbb{I}(x>b) \end{aligned}$$

This completes the proof.

5.2 Solution using Log-volutions

The log-volution as described by Buchen (appendix C) is defined for arbitrary functions $h(x)$ and $g(x)$ to be

$$h(x) \star g(x) = \int_0^\infty h(y) g\left(\frac{x}{y}\right)\frac{dy}{y}$$

Let $D = x\frac{d}{dx}$. Let $\mathcal{I}_b$ represent the image operator with respect to the barrier b.

Theorem

The solution to the Black-Scholes option pricing problem in $x < b$ and $t > 0$,

$$\begin{aligned} \mathcal{L}V(x,t) &= 0 \\ V(x,T) &= f(x) \\ DV &= 0 \qquad \text{when } \; x = b \end{aligned}$$

where D is a linear differential operator, is given by

$$V(x,t) = V_b(x,t) - (D^{-1}.\mathcal{I}_b.D)\{V_b(x,t)\}$$

Here $V_b(x,t)$ is the solution to the initial value problem

$$\begin{aligned}
\mathcal{L}V_b(x,t) &= 0 \qquad \text{in } x > 0,\ t > 0 \\
V_b(x,T) &= f(x)\mathbb{I}(x < b)
\end{aligned}$$

Now take the Mellin transform of

$$DV = DV_b - \mathcal{I}_b DV_b$$

to give

$$-s\hat{V}(s,t) = -s\hat{V}_b(s,t) - b^{2s-\alpha}[(s-\alpha)\hat{V}_b(\alpha - s, t)]$$

Hence

$$\hat{V}(s,t) = \hat{V}_b(s,t) + b^{2s-\alpha}\left[1 - \frac{\alpha}{s}\right]\hat{V}_b(\alpha - s, t)$$

Take the inverse Mellin transform to give

$$\begin{aligned}
V(x,t) &= V_b(x,t) + \overset{*}{V}_b(x,t) + \alpha\mathbb{I}(x > 1) \star \overset{*}{V}_b(x,t) \\
&= V_b(x,t) + \overset{*}{V}_b(x,t) + \alpha\int_0^x \overset{*}{V}_b(y,t)\frac{dy}{y}
\end{aligned}$$

To find the equivalent European payoff set $t = T$ to give

$$V(x,T) = f(x)\mathbb{I}(x < b) + \overset{*}{f}(x)\mathbb{I}(x > b) + \alpha\mathbb{I}(x > b)\int_b^x \overset{*}{f}(y)\frac{dy}{y}$$

5.3 Examples with simple payoffs

In order to give a deeper understanding of the method of images for reflecting barriers we consider several simple options with a reflecting barrier. The pricing of these simple options using this technique requires simple integration only and algebra. These simple options form the building blocks for the more complex options priced later in this thesis.

5.3.1 $f(x) = p\mathbb{I}(x > h)$

Consider the following problem

$$\begin{aligned}
\mathcal{L}V(x,t) &= 0 \qquad \text{in } \; x < b, \;\; t < T \\
V(x,T) &= p\mathbb{I}(x > h) \qquad \text{for } \; h < b, \;\; p \in \Re^+ \\
V_x(b,t) &= 0
\end{aligned}$$

Using equation 5.1 to find the payoff for an equivalent European option

$$\begin{aligned}
V^{\text{eq}}(x,T) &= p\mathbb{I}(x > h)\mathbb{I}(x < b) + (b/x)^\alpha p\mathbb{I}(x < k)(x > b) \qquad \text{where } \; k = b^2/h \\
&\quad + \alpha\mathbb{I}(x > b)\int_b^x p(b/y)^\alpha \mathbb{I}(y < k)dy/y \\
&= p\mathbb{I}(h < x < b) + \overset{*}{p}\mathbb{I}(b < x < k) \\
&\quad + \alpha\mathbb{I}(x > b)\int_b^\xi pb^\alpha y^{-\alpha-1}dy \qquad \text{where } \; \xi = \min(x,k) \\
&= p\mathbb{I}(h < x < b) + \overset{*}{p}\mathbb{I}(b < x < k) \\
&\quad - pb^\alpha[y^{-\alpha}]_b^\xi \mathbb{I}(x > b) \\
&= p\mathbb{I}(x > h) - p(b/k)^\alpha \mathbb{I}(x > k)
\end{aligned}$$

The solution can be written down by inspection in terms of turbo binaries

$$V(x,t) = pP_h^+(x,\tau,0) - p(h/b)^\alpha P_k^+(x,\tau,0) \tag{5.3}$$

where $\tau = T - t$.

For completeness this can be written explicitly as

$$V(x,t) = pe^{-r\tau}\mathcal{N}(d_1) - p(h/b)^{\alpha}e^{-r\tau}\mathcal{N}(d_2)$$

and

$$\begin{aligned} d_1 &= \frac{\log x/h + (r - q - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\ d_2 &= \frac{\log x/k + (r - q - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

Note that the final form is the price absent the boundary and the second terms is the correction for it's presence.

5.3.2 $f(x) = (x/b)^{\beta}$

Consider the following problem

$$\begin{aligned} \mathcal{L}V(x,t) &= 0 \qquad \text{in } x < b, \ \ t < T \\ V(x,T) &= (x/b)^{\beta} \\ V_x(b,t) &= 0 \end{aligned}$$

Using equation 5.1 to find the payoff for an equivalent European option

$$\begin{aligned}
V^{\text{eq}}(x,T) &= \left(\frac{x}{b}\right)^{\beta}\mathbb{I}(x<b)+\left(\frac{b}{x}\right)^{\alpha}\left(\frac{b}{x}\right)^{\beta}\mathbb{I}(x>b) \\
&\quad +\alpha\int_b^x\left(\frac{b}{y}^{\alpha}\right)\left(\frac{b}{y}\right)^{\beta}\frac{dy}{y}\mathbb{I}(x>b) \\
&= \left(\frac{x}{b}\right)^{\beta}\mathbb{I}(x<b)+\left(\frac{b}{x}\right)^{\alpha+\beta}\mathbb{I}(x>b) \\
&\quad +\frac{\alpha}{\alpha+\beta}\left[1-\left(\frac{b}{x}\right)^{\alpha+\beta}\right]\mathbb{I}(x>b) \\
&= \left(\frac{x}{b}\right)^{\beta}\mathbb{I}(x<b)+\frac{\alpha}{\alpha+\beta}\mathbb{I}(x>b)+\frac{\beta}{\alpha+\beta}\left(\frac{b}{x}\right)^{\alpha+\beta}\mathbb{I}(x>b)
\end{aligned}$$

The solution can be expressed in terms of turbo binaries as follows

$$V(x,t)=P_b^-(x,\tau;\beta)+\frac{\alpha}{\alpha+\beta}B_b^+(x,\tau)+\frac{\beta}{\alpha+\beta}\overset{*}{P}{}_b^-(x,\tau;\beta) \tag{5.4}$$

This can be written without the image by using the result in appendix B.

$$V(x,t)=P_b^-(x,\tau;\beta)+\frac{\alpha}{\alpha+\beta}B_b^+(x,\tau)+\frac{\beta}{\alpha+\beta}P_b^+(x,\tau;-\alpha-\beta)$$

5.3.3 $f(x)=(x/b)^{\beta}\mathbb{I}(x<h)$; $h<b$

Consider the following problem

$$\begin{aligned}
\mathcal{L}V(x,t) &= 0 \qquad \text{in } x<b,\ t<T \\
V(x,T) &= (x/b)^{\beta}\mathbb{I}(x<h) \\
V_x(b,t) &= 0
\end{aligned}$$

Using equation 5.1 to find the payoff for an equivalent European option

$$\begin{aligned}
V^{\mathrm{eq}}(x,T) &= (x/b)^{\beta}\mathbb{I}(x<h)\mathbb{I}(x<b)+(b/x)^{\alpha+\beta}\mathbb{I}(x>k)\mathbb{I}(x>b)\\
&\quad+\alpha\mathbb{I}(x>b)\int_b^x (b/y)^{\alpha+\beta}\mathbb{I}(y>k)\frac{dy}{y} \qquad \text{where } \ k=b^2/h\\
&= (x/b)^{\beta}\mathbb{I}(x<h)\mathbb{I}(x<b)+(b/x)^{\alpha+\beta}\mathbb{I}(x>k)\mathbb{I}(x>b)\\
&\quad+\alpha\mathbb{I}(x>b)\int_k^x (b/y)^{\alpha+\beta}\frac{dy}{y}\mathbb{I}(x>k)\\
&= (x/b)^{\beta}\mathbb{I}(x<h)+(b/x)^{\alpha+\beta}\mathbb{I}(x>k)\\
&\quad+\frac{\alpha}{\alpha+\beta}((h/b)^{\alpha+\beta}-(b/x)^{\alpha+\beta})\mathbb{I}(x>k)
\end{aligned}$$

It is noted that $k>b$. This can be simplified to give

$$\left.\begin{aligned}
V^{\mathrm{eq}}(x,T)=(x/b)^{\beta}\mathbb{I}(x<h)+\tfrac{\beta}{\alpha+\beta}[(x/b)^{\beta}\mathbb{I}(x<h)]^{*}\\
+\tfrac{\alpha}{\alpha+\beta}(h/b)^{\alpha+\beta}\mathbb{I}(x>k)
\end{aligned}\right\} \tag{5.5}$$

The solution can be expressed in terms of turbo binaries as follows

$$\left.\begin{aligned}
V(x,\tau)=(h/b)^{\beta}P_h^{-}(x,\tau;\beta)+\tfrac{\beta}{\alpha+\beta}(h/b)^{\beta}\overset{*}{P}{}_h^{-}(x,\tau;\beta)\\
+\tfrac{\alpha}{\alpha+\beta}(h/b)^{\alpha+\beta}P_k^{+}(x,\tau;0)
\end{aligned}\right\} \tag{5.6}$$

5.3.4 $f(x)=(x/b)^{\beta}\mathbb{I}(x>h)$; $h<b$

Consider the following problem

$$\begin{aligned}
\mathcal{L}V(x,t) &= 0 \qquad \text{in } \ x<b,\ \ t<T\\
V(x,T) &= (x/b)^{\beta}\mathbb{I}(x>h) \qquad \text{for } \ h<b\\
V_x(b,t) &= 0
\end{aligned}$$

Using equation 5.1 to find the payoff for an equivalent European option

$$\begin{aligned}
V^{\text{eq}}(x,T) &= (x/b)^{\beta}\mathbb{I}(x>h)\mathbb{I}(x<b) + (b/x)^{\alpha+\beta}\mathbb{I}(x<k)\mathbb{I}(x>b) \\
&\quad + \alpha\mathbb{I}(x>b)\int_b^x (b/y)^{\alpha+\beta}\mathbb{I}(y<k)\frac{dy}{y} \qquad \text{where } k = b^2/h \\
&= (x/b)^{\beta}(\mathbb{I}(x>h) - \mathbb{I}(x>b)) + (b/x)^{\alpha+\beta}((\mathbb{I}(x>b) - \mathbb{I}(x>k)) \\
&\quad + \alpha\mathbb{I}(x>b)\left\{\int_b^k \frac{b^{\alpha+\beta}}{y^{\alpha+\beta+1}}dy\mathbb{I}(x>k) + \int_b^x \frac{b^{\alpha+\beta}}{y^{\alpha+\beta+1}}dy\mathbb{I}(x<k)\right\}
\end{aligned}$$

It is noted that $k > b$. The integral is evaluated

$$b^{\alpha+\beta}\int_b^x y^{-\alpha-\beta-1}dy = \frac{1}{\alpha+\beta}\left[1-(b/x)^{\alpha+\beta}\right]$$

The equivalent European payoff can be simplified to give

$$\begin{aligned}
V^{\text{eq}}(x,T) &= (x/b)^{\beta}(\mathbb{I}(x>h) - \mathbb{I}(x>b)) + \frac{\alpha}{\alpha+\beta}\mathbb{I}(x>b) \\
&\quad - \frac{\alpha}{\alpha+\beta}(b/k)^{\alpha+\beta}\mathbb{I}(x>k) \\
&\quad + \frac{\beta}{\alpha+\beta}(b/x)^{\alpha+\beta}(\mathbb{I}(x>b) - \mathbb{I}(x>k))
\end{aligned}$$

The solution can be expressed in terms of turbo binaries as follows

$$\left.\begin{aligned}
V(x,\tau) &= (h/b)^{\beta}P_h^{+}(x,\tau;\beta) - P_b^{+}(x,\tau;\beta) \\
&\quad + \tfrac{\alpha}{\alpha+\beta}\left(P_b^{+}(x,\tau;0) - (b/k)^{\alpha+\beta}P_k^{+}(x,\tau;0)\right) \\
&\quad + \tfrac{\beta}{\alpha+\beta}\left(\overset{*}{P}{}_b^{-}(x,\tau;\beta) - (h/b)^{\beta}\overset{*}{P}{}_h^{-}(x,\tau;\beta)\right)
\end{aligned}\right\} \tag{5.7}$$

The solution to these problems require only simple integration.

5.3.5 Two period payoff

This is an important example as it is used to price the upward-only market review option.

$(X_2/b)^\beta \mathbb{I}(X_1 > h)$ is a short hand description of the payoff of a compound option. This option pays $(x/b)^\beta$ at time $t = T_2$ if the underlying rent process exceeded a threshold, h, at time $t = T_1$. However it is more intuitive to consider this option as one that at time $t = T_1$ pays another option if the rent exceeds a threshold. The option that is the payoff of the first option expires at time $t = T_2$. In this case the payoff of this option is simple. It is this approach that will be taken to solve this problem.

More precisely the compound option can be described by

$$\left.\begin{array}{rcl} \mathcal{L}V & = & 0 \qquad \text{in} \qquad t < T_1 \; ; \; x < b \\ V(X_1, T_1) & = & Q(X_1, \tau; \beta)\mathbb{I}(X_1 > h) \; ; \;\; h < b \; , \;\; \tau = T_2 - T_1 \\ \frac{\partial V}{\partial x} & = & 0 \qquad \text{at} \;\; x = b \end{array}\right\} \tag{5.8}$$

where $Q(x, \tau; \beta)$ solves the first-order turbo problem

$$\left.\begin{array}{rcl} \mathcal{L}Q & = & 0 \qquad \text{in} \qquad \tau_2 = T_2 - t, \;\; t > T_1 \; ; \; x < b \\ Q(X_2, 0; \beta) & = & (X_2/b)^\beta \qquad \text{at} \;\; t = T_2 \\ \frac{\partial Q}{\partial x} & = & 0 \qquad \text{at} \;\; x = b \end{array}\right\} \tag{5.9}$$

Consider the option $Q(x, \tau, \beta)$. This has already been solved in subsection 5.3.2. Hence $Q(x, \tau, \beta)$ can be expressed as follows using turbo binary options

$$Q(x, \tau_2, \beta) = P_b^-(x, \tau_2; \beta) + \frac{\beta}{\alpha + \beta} \overset{*}{P}{}_b^-(x, \tau_2; \beta) + \frac{\alpha}{\alpha + \beta} B_b^+(x, \tau_2) \tag{5.10}$$

Next consider the second order problem defined in equation 5.8. The method of images and the equivalent payoff at time T_1 can be applied again.

$$\begin{aligned}
V_\beta^{\text{eq}}(X_1,T_1) &= Q(X_1,\tau_2,\beta)\mathbb{I}(h<X_1<b)+[Q(X_1,\tau_2,\beta)\mathbb{I}(h<X_1<b)]^* \\
&+ \alpha\left(\int_b^{X_1}\overset{*}{Q}(y,\tau_2,\beta)\mathbb{I}(\frac{b^2}{y}>h)\frac{dy}{y}\right)\mathbb{I}(X_1>b) \\
&= V^{'}+V^{''}+V^{'''} \qquad \text{say}
\end{aligned}$$

Now the components can be evaluated

$$\begin{aligned}
V^{'}(X_1,T_1) &= Q(X_1,\tau,\beta)\mathbb{I}(h<X_1<b) \qquad \tau=T_2-T_1 \\
&= Q(X_1,\tau,\beta)[\mathbb{I}(X_1>h)-\mathbb{I}(X_1>b)] \\
V^{''}(X_1,T_1) &= \overset{*}{Q}(X_1,\tau,\beta)\mathbb{I}(b<X_1<k) \qquad k=b^2/h \\
&= \overset{*}{Q}(X_1,\tau_2,\beta)[\mathbb{I}(X_1<k)-\mathbb{I}(X_1<b)] \\
V^{'''}(X_1,T_1) &= \alpha\int_b^{\xi}\overset{*}{Q}(y,\tau_2,\beta)\frac{dy}{y}\mathbb{I}(X_1>b) \qquad \text{where } \xi=\min(X_1,k)
\end{aligned}$$

The third component can be evaluated using the following lemma.

Lemma

$$x\frac{d}{dx}\left[\Phi(x)-\overset{*}{\Phi}(x)\right]=\alpha\overset{*}{\Phi}(x)$$

where $\Phi(x)$ is the equivalent European payoff for a general reflecting boundary value problem.

Proof. Let $f(x)$ be an arbitrary function. Define $\Phi(x)$ as follows

$$\Phi(x)=f(x)\mathbb{I}(x<b)+\overset{*}{f}(x)\mathbb{I}(x>b)+\alpha\int_b^x\overset{*}{f}(\xi)\frac{d\xi}{\xi}\mathbb{I}(x>b)$$

Then

$$\Phi(x) - \overset{*}{\Phi}(x) = \alpha \int_b^x \overset{*}{f}(\xi)\frac{d\xi}{\xi}\mathbb{I}(x > b) - \left(\alpha \int_b^x \overset{*}{f}(\xi)\frac{d\xi}{\xi}\right)^* \mathbb{I}(x < b)$$

Define the differential operator $D = x\frac{d}{dx}$ and using the definition of the image $\overset{*}{V}(x) = (b/x)^\alpha V(b^2/x)$, we get

$$\begin{aligned}
D\overset{*}{V}(x) &= x\frac{d}{dx}\left(\frac{b}{x}\right)^\alpha V\left(\frac{b^2}{x}\right) \\
&= x\left(-\alpha\frac{b}{x^{\alpha+1}}V\left(\frac{b^2}{x}\right) + \left(\frac{b}{x}\right)^\alpha\left(-\left(\frac{b}{x}\right)^2 V'\left(\frac{b^2}{x}\right)\right)\right) \\
&= -\alpha\left(\frac{b}{x}\right)^\alpha V\left(\frac{b^2}{x}\right) - \left[xV'(x)\right]^* \\
&= -\alpha\overset{*}{V}(x) - (DV(x))^*
\end{aligned}$$

Hence

$$\begin{aligned}
D[\Phi(x) - \overset{*}{\Phi}(x)] &= \alpha(\overset{*}{f}(x)\mathbb{I}(x > b) + x\int_b^x \overset{*}{f}(\xi)\frac{d\xi}{\xi}\delta(x-b)) \\
&\quad - \Big\{-(\alpha^2\int_b^x \overset{*}{f}(\xi)\frac{d\xi}{\xi}\mathbb{I}(x > b))^* \\
&\quad -\alpha(\overset{*}{f}(x)\mathbb{I}(x > b) + x\int_b^x \overset{*}{f}(\xi)\frac{d\xi}{\xi}\delta(x-b))\Big\} \\
&= \alpha\left[\overset{*}{f}(x)\mathbb{I}(x > b) + f(x)\mathbb{I}(x < b) + \alpha\int_b^x \overset{*}{f}(\xi)\frac{d\xi}{\xi}\mathbb{I}(x > b)\right] \\
&= \alpha\overset{*}{Q}(x)
\end{aligned}$$

using the result $f(x)\delta(x-a) = 0$ if $f(a) = 0$. From this it follows by integration that

$$\alpha\int_b^x \overset{*}{\Phi}(\xi)\frac{d\xi}{\xi} = \Phi(x) - \overset{*}{\Phi}(x)$$

Combining the preceding results and the above lemma,

$$
\begin{aligned}
V^{\mathrm{eq}}(X_1, T_1) \;&=\; Q(X_1, \tau, \beta)\mathbb{I}(h < X_1 < b) \\
&+\overset{*}{Q}(X_1, \tau, \beta)\mathbb{I}(b < X_1 < k) \\
&+[Q(X_1, \tau, \beta) - \overset{*}{Q}(X_1, \tau, \beta)]\mathbb{I}(X_1 > b)\mathbb{I}(X_1 < k) \\
&+[Q(k, \tau, \beta) - \overset{*}{Q}(k, \tau, \beta)]\mathbb{I}(X_1 > b)\mathbb{I}(X_1 > k)
\end{aligned}
$$

which simplifies

$$
\left.
\begin{aligned}
V^{\mathrm{eq}}(X_1, T_1) \;&=\; Q(X_1, \tau, \beta)\mathbb{I}(h < X_1 < k) \\
&+[Q(k, \tau; \beta) - \overset{*}{Q}(k, \tau; \beta)]\mathbb{I}(X_1 > k)
\end{aligned}
\right\} \tag{5.11}
$$

Using equation 5.10 and 5.11 and writing $\mathbb{I}(h < x < k) = \mathbb{I}(x > h) - \mathbb{I}(x > k)$

$$
\begin{aligned}
V^{\mathrm{eq}}(X_1, T_1) \;&=\; [P_b^-(X_1, \tau_2; \beta) + \frac{\beta}{\alpha+\beta}\overset{*}{P}{}_b^-(X_1, \tau_2; \beta) + \frac{\alpha}{\alpha+\beta}B_b^+(X_1, \tau_2)]\mathbb{I}(x > h) \\
&-[P_b^-(X_1, \tau_2; \beta) + \frac{\beta}{\alpha+\beta}\overset{*}{P}{}_b^-(X_1, \tau_2; \beta) + \frac{\alpha}{\alpha+\beta}B_b^+(X_1, \tau_2)]\mathbb{I}(X_1 > k) \\
&+\frac{\alpha}{\alpha+\beta}[P_b^-(k, \tau_2; \beta) - \overset{*}{P}{}_b^-(k, \tau_2; \beta) + (B_b^+(k, \tau_2) - \overset{*}{B}{}_b^+(k, \tau_2))]\mathbb{I}(X_1 > k)
\end{aligned}
$$

The present value is then, by inspection, given by:

$$
\left.
\begin{aligned}
V(x, t) \;&=\; [P_{hb}^{+-}(x, \tau_1, \tau_2; \beta) + \tfrac{\beta}{\alpha+\beta}\overset{*}{P}{}_{kb}^{--}(x, \tau_1, \tau_2; \beta) \\
&\qquad +\tfrac{\alpha}{\alpha+\beta}B_{hb}^{++}(x, \tau_1, \tau_2)] \\
&-[P_{kb}^{+-}(x, \tau_1, \tau_2; \beta) + \tfrac{\beta}{\alpha+\beta}\overset{*}{P}{}_{hb}^{--}(x, \tau_1, \tau_2; \beta) \\
&\qquad +\tfrac{\alpha}{\alpha+\beta}B_{kb}^{++}(x, \tau_1, \tau_2)] \\
&+\tfrac{\alpha}{\alpha+\beta}[P_b^-(k, \tau_2; \beta) - \overset{*}{P}{}_b^-(k, \tau_2; \beta) \\
&\qquad +(B_b^+(k, \tau_2) - \overset{*}{B}{}_b^+(k, \tau_2))]B_k^+(x, \tau_1)
\end{aligned}
\right\} \tag{5.12}
$$

5.4 Solving the Lease Problem

For the lease problem defined by Eq(3.1), Eq(3.3), Eq(3.2) the payoff is $H(x)$ given by equation (2.6). Define $F_a(x) = (x/b)^a$. Then

$$H(x) = \frac{b}{q}\left[F_1(x) - \frac{1}{\beta}F_\beta(x)\right]$$

By linearity, if $C_a(x,t)$ solves the reflecting boundary value problem with payoff $F_a(x)$, then

$$C(x,t) = \frac{b}{q}\left[C_1(x,t) - \frac{1}{\beta}C_\beta(x,t)\right] \tag{5.13}$$

solves the Black-Scholes partial differential equation payoff $H(x)$.

The equivalent payoff for the reflecting boundary value problem with payoff $F_a(x)$, using equation (5.1) and the image $\mathcal{I}[(x/b)^a] = (b/x)^{a+\alpha}$ has already been solved in subsection 5.3.2 and hence using Eq(5.4) $C_a(x,t)$ is given by

$$C_a(x,t) = P_b^-(x,t;a) + \frac{a}{a+\alpha}\overset{*}{P}{}_b^-(x,t;a) + \frac{\alpha}{a+\alpha}P_b^+(x,t;0) \tag{5.14}$$

Substituting into equation (5.13), we obtain

$$\begin{aligned} C(x,t;T) \quad = \quad & \frac{b}{q}\left[P_b^-(x,t;1) + \frac{1}{\alpha+1}\overset{*}{P}{}_b^-(x,t;1) + \frac{\alpha}{\alpha+1}P_b^+(x,t;0)\right. \\ & \left. -\frac{1}{\beta}P_b^-(x,t;\beta) - \frac{1}{\alpha+\beta}\overset{*}{P}{}_b^-(x,t;\beta) - \frac{\alpha}{\beta(\alpha+\beta)}P_b^+(x,t;0)\right] \end{aligned} \tag{5.15}$$

The call option is seen to be a portfolio of turbo binaries (and their images), all with the same exercise price b, but with different power indices $(1,\beta,0)$. Note further that although the two expressions in $P_b^+(x,t;0)$ can be combined, we leave them as above, in order to facilitate comparison with Grenadier's result. One final point to observe is that the images in (5.15) can be evaluated using (4.1) and leads to the identity

Lemma

$$\overset{*}{P}{}_b^s(x,t;c) = P_b^{-s}(x,t;-c-\alpha) \tag{5.16}$$

Proof

Consider the European payoff for $P_b^{-s}(x,t;-c-\alpha)$

$$\begin{aligned} P_b^{-s}(x,T;-c-\alpha) &= (x/b)^{-c-\alpha}\mathbb{I}(-sx > -sb) \\ &= \mathcal{I}(x/b)^c\mathbb{I}(sx > sb) \\ &= P_b^s(x,T;c) \end{aligned}$$

Thus the image of a turbo binary is itself a turbo binary but of opposite type.

Assume the following default parameters $r = 0.05$, $q = 0.07$, $\sigma = 0.15$, $x = 100$, $\tau = 5$ and $b = 150$. In figure 5.1 the plots are obtained for terms ranging from 5 to 20 years. In figure 5.2 the plots are obtained for rent values from 20 to 120. The code used to generate the figures can be found in the appendix. The programs were written in C and using the Gnu Scientific Library (www.gnu.org/software/gsl).

While our graph of Lease versus Time looks quite different to Grenadier [40] figure 1, the two are in fact consistent. In Grenadier's term structure of lease rates, he includes an interest rate factor, $\frac{r}{1-\exp(-r.T)}$, which accounts fully for the difference. Indeed, while our lease curves (fig 5.2) are all monotone increasing with time, this factor permits humped shaped and monotone decreasing term structures as well .

It should be noted in figure (5.1) that the slope of the curve vanishes at the reflecting boundary, $x = b$, as expected.

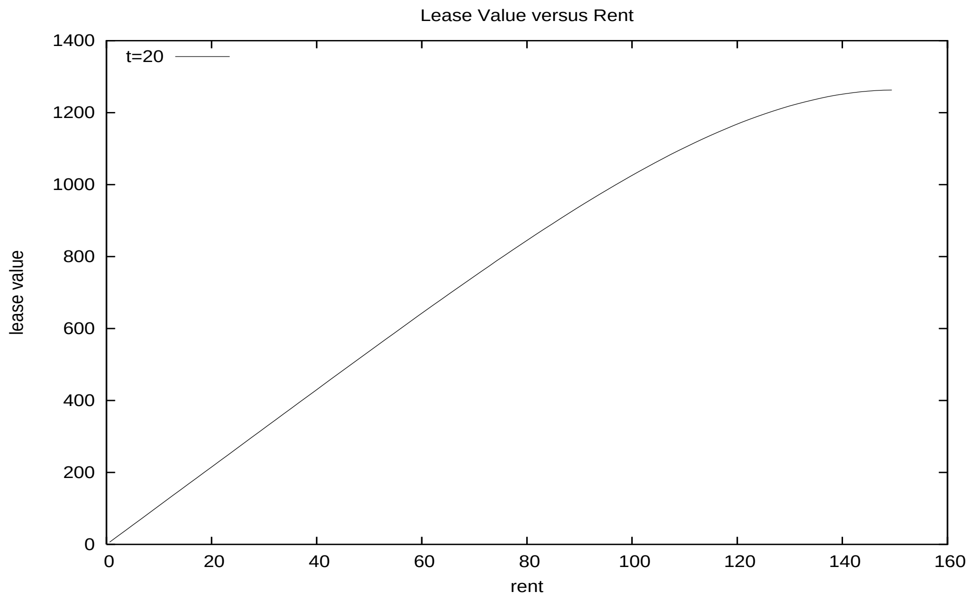

Figure 5.1: Lease versus Rent

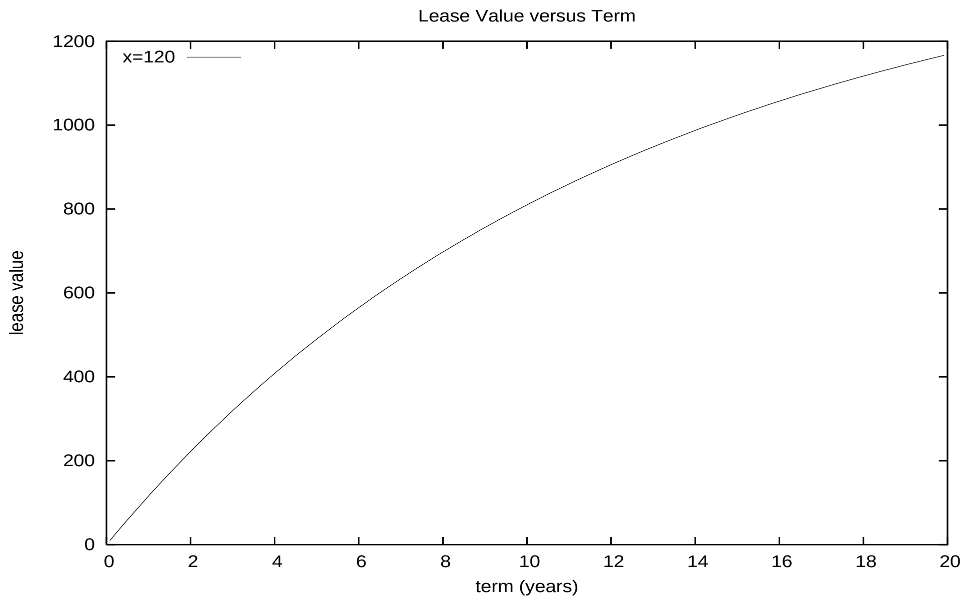

Figure 5.2: Lease versus Term

We can also demonstrate the analytical equivalence of our pricing formula (5.15) with his equation [40](29) and [40](30), provided we correct a typographical error in his equation [40](30): the factor $[1-h(w)]$ should read $[1+h(w)]$. Grenadiers solution is provided for comparison with Eq(5.15)

$$C(P,0;T) = \frac{\nu_n e^{-rT}}{r-\alpha}[f(P,T,-1) - f(P,T,-\beta)/\beta]$$

where

$$\begin{aligned}
f(P,T,\omega) &= g_1(P,T,\omega)g_3(P,T,\omega) + [1-h(\omega)]g_2(P,T,\omega)g_4(P,T,\omega) \\
&\quad -h(\omega)g_2(P,T,-2\mu/\sigma^2) \\
g_1(P,T,\omega) &= N\left[\frac{(\omega\sigma^2+\mu)T + \log(\nu_n/P)}{\sigma\sqrt{T}}\right] \\
g_2(P,T,\omega) &= 1 - g_1(P,T,-\omega-2\mu/\sigma^2) \\
g_3(P,T,\omega) &= \left(\frac{P}{\nu_n}\right)^{-\omega}\exp[(\omega^2\sigma^2/2+\omega\mu)T] \\
g_4(P,T,\omega) &= g_3(P,T,-\omega-2\mu/\sigma^2) \\
h(\omega) &= \frac{-2\mu}{\omega\sigma^2+2\mu} \\
\mu &= \sigma^2/2-\alpha
\end{aligned}$$

In the above equation the rent process is denoted as P, the barrier as ν_n, the dividend is $r-\alpha$. Note that Grenadier uses α in a different context than in this thesis.

5.5 Chapter Summary

This is the central chapter of this thesis. The method of images was extended to solve the class of problems in the Black Scholes framework with a reflecting boundary condition. The general solution in the form of the equivalent European payoff derived from first principles. An alternate, and more elegant, derivation using the theory of log-volutions of Buchen was also provided.

The technique was illustrated using several different simple payoffs. The solutions of these simple examples provide the building blocks for the solutions to the more complicated lease options presented in the next chapter. The chapter concludes by pricing the call option in the basic lease.

Chapter 6

Lease Options

Most leases are not simple and contain clauses that relate to the usage of the property and the lease payments to the owner. Many of these are included in standard lease contracts. In practice these extra clauses are negotiated prior to drawing up the lease and takes the form of a document called the "heads of agreement". It is in this document that additional lease options are added. Currently these options are added and removed from the lease without an attempt to value them using modern financial mathematics.

Some clauses relate to the lessee such as allowed usage of the property, hours of trading, noise restrictions and keeping of pets. These clauses do embody options, which are difficult to price. Other clauses such as changes in the lease value during the lease term or rent reviews also embody options that can be valued. These are the lease options that will be priced in this chapter.

These lease options add or subtract value from the lease and so benefit either the lessee or the lessor. During the lease negotiation process, value judgments regarding these options are made by both sides. For instance a prospective tenant may agree to a rent review half way through the term of the lease in exchange for a rent free period at the start of the lease. Some negotiations involve the term of the lease and a renewal of the lease. Being able to place a value on these options would benefit both parties. Indeed even being able to compare the relative values of two options would be of benefit.

In the preceding chapters the basic model framework has been established. The method of images has been extended to the case of options with a reflecting barrier and the equivalent European payoff was found to considerably simplify solving this class of problem. In this chapter lease options will be described and priced in this real options framework. The results from the preceding chapters will be used with the principle of static replication, described in chapter 1 Eq(1.2) to (1.4).

Of the lease options to be presented the most interesting options are the market review option which has a number of variations including the upward-only, the cap and the collar. The cap places an upper limit on the increase in rent that can occur at the market review and the collar places a lower limit on the potential decrease in rent. The upward-only market review is simply a market review with a collar of zero. That is, the rent can go up but not down.

The purchase option has been priced by Grenadier [40] using more traditional techniques and our solution corresponds to his, which verifies the method of images at least for simple cases. The market review option is a compound or multiperiod option and no analytical solution has yet been published to our knowledge. In this chapter the analytical solution for this option will be presented in detail.

Some of the options presented here are virtually trivial, such as the forward lease, yet can be priced simply using the method of images. Other options introduce American features with a floating boundary condition and have not been considered in this work. That does not imply that they are not important and indeed they may be some of the most important lease options currently used, yet remain unpriced. The list of possible options are endless, like the vast over the counter options of the derivative market.

6.1 Pre-leasing

Pre-leasing is not an option but rather a forward lease contract. It appears in the market review option as a component of the payoff which will be shown later.

In the case of the lessee it guards against a rise in the cost of leasing if the property is not required immediately. For the lessor it ensures that there will be a cash flow in the future which may be a requirement by a financier providing the capital for a new development.

The forward lease is defined by

$$\left.\begin{aligned} L^f(x,t,T_1,T_2) &= L(x,t,T_2) - L(x,t,T_1) \\ &= C(x,t,T_1) - C(x,t,T_2) \end{aligned}\right\} \tag{6.1}$$

for $t < T_1 < T_2$ and $x = X_t$ using Eq(3.1). Since we have already priced $C(x,t,T)$ in Eq(5.15) we immediately get the forward lease value.

6.2 Renewal and Cancellation Options

At the end of a term of a lease the renewal option allows the lessee to enter into another predetermined term. It is of benefit to the lessee as the cost of relocation can be substantial.

Most leases have renewal options which allow a lessee to extend their lease for a predetermined term. For instance a lease can have an initial term of 5 years with two renewal options for 5 years each. Hence the lessee can effectively lease the premises for 5, 10 or

15 years. This lease could also be considered to be a lease with a 15 year term and two cancellation options which can be exercised at year 5 and year 10 of the lease.

Assuming relocation costs are small, this option is of clear benefit to the lessee. In fact the value of the renewal option to the lessee is the same as the value of the market review option to the lessor. If the market value of rent is lower than the rent specified in the option the lessee simply moves to new premises. If the market value of rent is higher than the rent specified in the renewal option then the lessee benefits by exercising this option. As a result this option is often balanced with a market review of the rent at the time of exercise of the renewal option.

The cancellation option allows the lessee to exit the lease at any time. This option will not be priced in this thesis as it has an American feature. It can be modelled using binomial trees.

6.3 Sale Lease Back Option

This is not really an option but is mentioned as it is complementary to the lease purchase option considered in section 6.4. This transaction can be used to raise finance for a business [40]. If a building owner needs finance for a business and cannot borrow the money an alternative is to sell the property at above market rate and then lease the property from the new owners at above market rent.

Let X be the current rent then the equilibrium value of the property is $H(X)$ as defined in Eq (2.6). Let the above market sale price of the property be S then the rent that must be charged, $\hat{X}$, should satisfy the following equation

$$S - H(X) = L(\hat{X}, t; T) - L(X, t; T) \tag{6.2}$$

where $L(x, t; T)$ is the lease value defined in Eqs (3.1) and (5.15) and T is the time of expiry of the lease. Since we have value L we can in principle use Eq (6.2) to find $\hat{X}$.

6.4 Purchase Option

Consider a lease where at the end of the lease the lessee has the option to purchase the property for K. The problem in this framework can be stated as

$$\begin{aligned} \mathcal{L}V &= 0 \quad \text{in} \quad 0 < x < b,\ t < T \\ V(x, T) &= (H(x) - K)^{+} \\ \frac{\partial V}{\partial x}(b, t) &= 0 \qquad \text{at} \qquad x = b \end{aligned}$$

where $H(x)$ is defined in Eq (2.6).

Consider the condition $H(x) > K$ which can formally be written

$$\left(\frac{x}{b}\right)^{\beta} - \beta\frac{x}{b} + \beta\frac{q}{b}K < 0$$

Consider

$$g(x) = (x/b)^{\beta} - \beta(x/b) + \beta\frac{q}{b}K < 0$$

in the interval $0 < x < b$.

The value of a perpetual annuity that has return b and yield q is

$$\int_0^{\infty} be^{-qt}dt = b/q$$

hence this option will only be accepted if $K < b/q$. That is, if the property is considered to be a perpetual annuity its maximum value is b/q so an option to purchase the property must have a strike price less than this maximum value.

Now $g(0) = \beta q K/b > 0$. We want to find $\xi \in [0, b]$ such that $g(\xi) = 0$ hence we require that $g(b) < 0$.

$$g(b) < 0 \qquad \Rightarrow \qquad \beta > \frac{1}{1 - Kq/b}$$

This is satisfied as $K < b/q$ and further $\beta > 1$.

$$\frac{dg}{dx} = \beta((x/b)^{\beta-1} - 1)$$

so in the interval $[0, b]$, $dg/dx < 0$. That is $g(x)$ is monotonic decreasing and hence ξ is unique.

We can then write

$$\mathbb{I}(H(x) > K) = \mathbb{I}(x > \xi)$$

where ξ satisfies $g(\xi) = 0$ The payoff is therefore

$$\left(\frac{b}{q}\left(\frac{x}{b} - \frac{1}{\beta}\left(\frac{x}{b}\right)^{\beta}\right) - K\right)\mathbb{I}(x > \xi)$$

So the components have the form

$$\left(\frac{x}{b}\right)^{n}\mathbb{I}(x > \xi) \qquad \text{for} \qquad n = 0, 1, \beta$$

The solution using Eq (5.7) is

$$\left.\begin{array}{rcl} V^n(x,\tau) & = & (\xi/b)^n P_\xi^+(x,\tau;n) - P_b^+(x,\tau;n) \\ & & +\frac{\alpha}{\alpha+n}(P_b^+(x,\tau;0) - (b/k)^{\alpha+n} P_k^+(x,\tau;0) \\ & & +\frac{n}{\alpha+n}(\overset{*}{P}{}_b^-(x,\tau;n) - (\xi/b)^n \overset{*}{P}{}_\xi^-(x,\tau;n) \end{array}\right\} \tag{6.3}$$

where $k = b^2/\xi$.

Finally the solution can be written down in the form

$$V(x,t) = \frac{b}{q}\left(V^1 - \frac{1}{\beta}V^\beta\right) - KV^0.$$

After careful comparison this was found to correspond to Grenadier's solution [40].

6.5 Market Review

A market review is a common clause in a lease. It breaks up the term of the lease so that the first part of the lease has a different value to the second part. That is, before the end of the term of a lease, the rent paid by the tenant changes based on the current rent at the time of the review. As the rent can increase or decrease this can benefit either the lessee or the lessor. As a result there are a variety of conditions that can be placed on the market review to limit the loss or gain to the lessee or lessor. This is usually in the form of a cap or a collar. A cap limits the increase in rent at the time of the market review and the collar is the limit on the decrease in the rent at the time of the market review. A special case of a collar is the upward only market review which is also known as the market review with a ratchet clause. That is, at the time of the market review, the rent is increased if the current rent is greater than the rent at the beginning of the lease but if the current rent has decreased then the rental rate stays the same. It is obviously of benefit only to the lessor and protects against a decrease in rent in long term leases. A tenant might agree to this as the alternative may be a shorter term lease and the prospect of having to relocate and negotiate a new lease. It may be easier to just accept that rent will increase but not have to relocate a business.

The problem of valuing the upward only market review has been addressed by researchers such as Ward and Ambrose. Some researchers as will be discussed below have used some elements of the Grenadier model. However, none have considered the problem in the Grenadier framework with a reflecting barrier on the rent process.

Ward [5] values the upward only market review and compares it with the rent review where there could be either an upward or downward revision. He uses the simple and then multi period binomial model before generalising to the Black Scholes model. The assumption is that the rent can be modelled by geometric Brownian motion. Rowland [27] implements the binomial option pricing approach and uses the simplifying assumption that in equilibrium the return in rent is the same as the cost of borrowing.

Ambrose [4] prices the upward only market review by modelling the service flow as a random process. He prices the lease using Grenadier's model [39] and formulates an infinite series of non linear algebraic equations. This series has an exact solution. This approach differs quite significantly from the model presented in this paper as it does not constrain rental growth and assumes an infinite time horizon.

6.5.1 The simple market review option

Consider the simplest market review option without a cap or a collar. At the review date the rent is set to the current rental rate at the time of the review. This means that the rent can increase or decrease. Further there is no limit to this rental rate change. This can be described by

$$\left.\begin{array}{rcl} \mathcal{L}V & = & 0 \qquad \text{in } \ x < b, \ \ t < T_1 \\ V(X_1, T_1) & = & L(X_1, T_1, T_2) - L(X_0, T_0, T_2) + L(X_0, T_0, T_1) \\ \frac{\partial V}{\partial x}(x, t) & = & 0 \qquad \text{at } \ x = b \end{array}\right\} \tag{6.4}$$

The payoff here is the difference between the value of the lease at time T_1 for the term (T_1, T_2) and the forward lease for the same term but priced at time T_0. That is, the forward lease is the component of the payoff $K = L(X_0, T_0, T_2) - L(X_0, T_0, T_1)$. To price this option consider the payoff

$$\begin{aligned} V(X_1, T_1) &= L(X_1, T_1, T_2) - L(X_0, T_0, T_2) + L(X_0, T_0, T_1) \\ &= L(X_1, T_1, T_2) - K \qquad K \text{ is a constant for } t > T_0 \\ &= H(X_1) - C(X_1, T_1, T_2) - K \\ &= \frac{b}{q}\left(\frac{X_1}{b} - \frac{1}{\beta}\left(\frac{X_1}{b}\right)^{\beta}\right) - K \\ &\quad -\frac{b}{q}\left(P_b^-(X_1, \tau, 1) + \frac{1}{\alpha+1}\overset{*}{P}{}_b^-(X_1, \tau, 1) + \frac{\alpha}{\alpha+1}B_b^+(X_1, \tau_2)\right. \\ &\quad \left. - \frac{1}{\beta}P_b^-(X_1, \tau, \beta) - \frac{1}{\alpha+\beta}\overset{*}{P}{}_b^-(X_1, \tau, \beta) - \frac{\alpha}{\beta(\alpha+\beta)}B_b^+(X_1, \tau)\right) \\ &= H(X_1) - K \\ &\quad -\frac{b}{q}\left(Q(X_1, \tau, 1) - \frac{1}{\beta}Q(X_1, \tau, \beta)\right) \\ &= W_1 - W_2 - W_3, \qquad \text{say,} \end{aligned}$$

where

$$\begin{aligned} W_1 &= H(X_1) \\ W_2 &= K \\ W_3 &= \frac{b}{q}\left(Q(X_1, \tau, 1) - \frac{1}{\beta}Q(X_1, \tau, \beta)\right) \end{aligned}$$

Here $K = L(X_0, T_0, T_2) - L(X_0, T_0, T_1)$ is the forward lease and can be evaluated on $L(x, t, T)$ is known. For W_1 consider the option problem

$$\begin{aligned} \mathcal{L}U_\beta &= 0 \qquad x < b, \;\; t < T_2 \\ U_\beta(X_1, T_1) &= \left(\frac{X_1}{b}\right)^\beta \\ \frac{\partial U_\beta}{\partial x}(x, t) &= 0 \qquad \text{at } \; x = b \end{aligned}$$

using Eq(5.4) the solution is

$$U_\beta(x,t) = P_b^-(x, \tau_1; \beta) + \frac{\alpha}{\alpha+\beta} B_b^+(x, \tau_1) + \frac{\beta}{\alpha+\beta} \overset{*}{P}{}_b^-(x, \tau_1; \beta)$$

for $\tau_1 = T_1 - t$. Hence

$$W_1 = \frac{b}{q}\left(U_1(x,t) - \frac{1}{\beta} U_\beta(x,t)\right).$$

The evaluation of W_2 is trivial and is given by the present value of the K, viz

$$W_2(x,t) = K e^{-r\tau_1}$$

W_3 appears to be a compound option however the result simplifies. Consider the following prescription

$$\left.\begin{aligned} \mathcal{L}V_\beta &= 0 \qquad x < b, \;\; t < T_1 \\ V_\beta(X_1, T_1) &= Q(X_1, \tau; \beta) \\ V_x(x,t) &= 0 \qquad \text{at } \; x = b \end{aligned}\right\} \tag{6.5}$$

where Q is defined by the following

$$\begin{aligned} \mathcal{L}Q &= 0 \qquad x < b, \;\; t < T_2 \\ Q(X_2, 0; \beta) &= (X_2/b)^\beta \qquad \text{at } \; t = T_2 \\ Q_x(x, \tau_2; \beta) &= 0 \qquad \text{at } \; x = b \end{aligned}$$

using Eq(5.4) the solution is

$$Q(x,\tau_2;\beta) = P_b^-(x,\tau_2;\beta) + \frac{\alpha}{\alpha+\beta} B_b^+(x,\tau_2) + \frac{\beta}{\alpha+\beta} P_b^+(x,\tau_2;-\alpha-\beta)$$

for $\tau_2 = T_2 - t$. Solving the second order problem Eq(6.5) using the method of images, the equivalent European payoff at $t = T_1$ is

$$\begin{aligned}
V_\beta^{eq} &= Q(X_1,\tau 2;\beta)\mathbb{I}(x<b) + [Q(X_1,\tau 2;\beta)\mathbb{I}(x<b)]^* \\
&\quad +\alpha \int_b^x \overset{*}{Q}(y,\tau_2;\beta)\frac{dy}{y}\mathbb{I}(x>b) \\
&= Q(X_1,\tau_2;\beta)\mathbb{I}(x<b) + [Q(X_1,\tau 2;\beta)\mathbb{I}(x<b)]^* \\
&\quad [Q(X_1,\tau_2,\beta) - \overset{*}{Q}(x_1,\tau_2;\beta)]\mathbb{I}(x>b) \\
&= Q(X_1,\tau_2;\beta)
\end{aligned}$$

Hence

$$V_\beta(x,t) = e^{-r\tau_1} Q(x,\tau_2;\beta)$$

So

$$W_3 = e^{-r\tau_1}\frac{b}{q}\left(Q(x,\tau_2,1) - \frac{1}{\beta}Q(x,\tau_2;\beta)\right)$$

Hence the total solution of the market review option without a cap or collar is

$$\begin{aligned}
V(x,t) &= \frac{b}{q}\left(Q(x,\tau_1;1) - \frac{1}{\beta}Q(x,\tau_1;\beta)\right) - e^{-r\tau_1}K \\
&\quad -\frac{be^{-r\tau_1}}{q}\left(Q(x,\tau_2;1) - \frac{1}{\beta}Q(x,\tau_2;\beta)\right).
\end{aligned}$$

6.5.2 Market review with limits

Let the cap (rent ceiling) be denoted as X_c and the collar (rent floor) by X_f. If the rent at the beginning of the lease is X_0 then $X_f < X_0 < X_c$. If the market review occurs at time T_1 then the payoff at the time of the review for the cap is

$$\begin{aligned} V(X_1, T_1) &= [L(X_1, T_1, T_2) - L(X_0, T_0, T_2) + L(X_0, T_0, T_1)]\mathbb{I}(X_1 < X_c) \\ &\quad + [L(X_c, T_1, T_2) - L(X_0, T_0, T_2) + L(X_0, T_0, T_1)]\mathbb{I}(X_1 > X_c) \end{aligned}$$

The payoff for at time T_1 for the collar

$$\begin{aligned} V(X_1, T_1) &= [L(X_1, T_1, T_2) - L(X_0, T_0, T_2) + L(X_0, T_0, T_1)]\mathbb{I}(X_f < X_1) \\ &\quad + [L(X_f, T_1, T_2) - L(X_0, T_0, T_2) + L(X_0, T_0, T_1)]\mathbb{I}(X_1 < X_f) \end{aligned}$$

The general cases for the cap and collar will not be presented here as it is an algebraic exercise. That is, it is just the sum of the market review with a collar and the market review with a cap. To illustrate the technique the upward only market review option will be presented. It is really just a market review with a collar however the collar level is just the rent at the start of the lease. That is $X_f = X_0$. The upward only market review option is described by the following

$$\begin{aligned} \mathcal{L}V &= 0 \qquad x < b, \;\; t < T_2 \\ V(X_1, T_1) &= [L(X_1, T_1, T_2) - L(X_0, T_0, T_2) + L(X_0, T_0, T_1)]\mathbb{I}(X_1 > X_0) \\ V_x(x, t) &= 0 \qquad \text{at} \;\; x = b \end{aligned}$$

First some notation. Let $L(X_i, T_i, T_j)$ for $T_i < T_j$ represent the value of a lease from time T_i to time T_j. X_i is the rent process at time T_i. The "ratchet clause" introduces the indicator function $\mathbb{I}(X_1 > h)$ and $h = X_0$ is the rent at T_0, the beginning of the lease.

The value of the upward only market review at time T_1 is therefore

$$(L(X_1, T_1, T_2) - L(h, T_0, T_2) + L(h, T_0, T_1))\mathbb{I}(X_1 > h)$$

$L(h, T_0, T_2) - L(h, T_0, T_1)$ is the forward lease as determined at time T_0. For the period (T_1, T_2), $L(X_1, T_1, T_2)$ is the lease determined at time T_1 for the same period.

The lease value $L(X, t, t+\tau)$ is dependent on the rent X, and the term τ, and independent of t. The lease value $L(X, t, t+\tau)$ is a monotonic increasing function of τ with positive concavity for a fixed value of the rent X. It follows that

$$L(X, T_0 + t, T_0 + t + \tau) > L(X, T_0, T_0 + t + \tau) - L(X, T_0, T_0 + t)$$

Further, the lease value $L(X, t, t+\tau)$ is monotonic increasing in X given a fixed term, that is, if τ is a constant. Hence

$$L(X_1, t, t+\tau) > L(h, t, t+\tau)$$

for $X_1 > h$.

It follows that

$$L(X_1, T_1, T_2) > L(h, T_0, T_2) - L(h, T_0, T_1) \qquad \text{for } X_1 > h$$

From the Eq (3.1) the lease value is

$$L(X_1, T_1, T_2) = H(X_1) - C(X_1, T_1, T_2)$$

and at time T_1 the value of $H(X_1)$ is known.

The payoff at time T_1 can be written as

$$(H(X_1) - C(X_1, T_1, T_2) - K)\mathbb{I}(X_1 > h) \tag{6.6}$$

where $K = L(h, T_0, T_2) - L(h, T_0, T_1)$ is a known constant after time T_0.

The payoff Eq (6.6) can be decomposed into two types of binary options with payoffs given by:

- $X_1^{\beta}\mathbb{I}(X_1 > h)$ paid at time T_1 and
- $X_2^{\beta}\mathbb{I}(X_1 > h)$ paid at time T_2

Further the solution can be expressed in the form of a linear combination of first and second order turbo binary options which are denoted as $P_h^s(x, \tau_i, \beta)$ and $P_{h_1 h_2}^{s_1 s_2}(x, \tau_1, \tau_2, \beta)$ where $\tau_i = T_i - t$. A discussion of turbo binary options can be found in the appendix.

The solution of the simple lease problem of $L(x, t, T)$ has already been derived is given by Eqs (3.1) and (5.15). The value can be expressed in terms of the turbo binaries as follows.

$$L(x, t, T) = H(x) - C(x, t; T) \tag{6.7}$$

where

$$\begin{aligned} C(x,t,T) \quad = \quad & \frac{b}{q}\Big[P_b^-(x,\tau;1) + \tfrac{1}{\alpha+1}\overset{*}{P}{}_b^-(x,\tau;1) + \tfrac{\alpha}{\alpha+1}B_b^+(x,\tau) \\ & -\tfrac{1}{\beta}P_b^-(x,\tau;\beta) - \tfrac{1}{\alpha+\beta}\overset{*}{P}{}_b^-(x,\tau;\beta) - \tfrac{\alpha}{\beta(\alpha+\beta)}B_b^+(x,\tau)\Big] \end{aligned} \tag{6.8}$$

Here $*$ represents the image relative to $x = b$ in the Black-Scholes framework and is

$$\overset{*}{f}(x) = (b/x)^{\alpha} f(b^2/x)$$

for any function $f(x)$.

The payoff (6.6) has three components which we will denote as follows

$$W = W_1 - W_2 - W_3$$

where

$$\left.\begin{array}{rcl} W_1(X_1,T_1) & = & H(X_1)\mathbb{I}(X_1 > h) \\ W_2(X_1,T_1) & = & C(X_1,T_1,T_2)\mathbb{I}(X_1 > h) \\ W_3(X_1,T_1) & = & K\mathbb{I}(X_1 > h)] \end{array}\right\} \tag{6.9}$$

The third component, W_3, has solution expressed in terms of turbo binaries as follows

$$W_3(x,t) = KB_h^+(x,\tau_1) - K(b/h)^\alpha P_{b^2/h}^+(x,\tau_1;0)$$

See Eq (5.3) in section 5.3.1 The first component of the payoff at time T_1, W_1, is

$$W_1(X_1,T_1) = \frac{b}{q}\left[\frac{X_1}{b} - \frac{1}{\beta}\left(\frac{X_1}{b}\right)^\beta\right]\mathbb{I}(X_1 > h)$$

The sub-option U_β with payoff $(X_1/b)^\beta\mathbb{I}(X_1 > h)$ has solution

$$\begin{aligned} U_\beta \;=\; & (h/b)^\beta P_h^+(x,\tau_1;\beta) - P_b^+(x,\tau_1;\beta) \\ & + \frac{\alpha}{\alpha+\beta}\left(B_b^+(x,\tau_1) - (h/b)^{\alpha+\beta}B_b^+(x,\tau_1)\right) \\ & + \frac{\beta}{\alpha+\beta}\left(\overset{*}{P}{}_b^-(x,\tau_1;\beta) - (h/b)^\beta \overset{*}{P}{}_h^-(x,\tau_1;\beta)\right) \end{aligned} \tag{6.10}$$

from equation (5.7).

So we can write

$$W_1(x,t) = \frac{b}{q}\left[U_1(x,t) - \frac{1}{\beta}U_\beta(x,t)\right]$$

The second component $W_2(X_1,T_1) = C(X_1,T_1,T_2)\mathbb{I}(X_1 > h)$ requires the second order turbo binary and is equivalent to a payoff at time T_2 of

$$\frac{b}{q}\left(\frac{X_2}{b}-\frac{1}{\beta}\left(\frac{X_2}{b}\right)^{\beta}\right)\mathbb{I}(X_1>h)$$

This is a compound option. The basic component is of the form

$$\left(\frac{X_2}{b}\right)^{\beta}\mathbb{I}(X_1>h)$$

where $\beta > 0$.

The solution of this has already been evaluated in section 5.3.5, Eq (5.12) and is

$$\begin{aligned}
V_\beta(x,t) &= [P^{+-}_{hb}(x,\tau_1,\tau_2;\beta)+\frac{\beta}{\alpha+\beta}\overset{*}{P}{}^{--}_{kb}(x,\tau_1,\tau_2;\beta)+\frac{\alpha}{\alpha+\beta}B^{++}_{hb}(x,\tau_1,\tau_2)]\\
&- [P^{+-}_{kb}(x,\tau_1,\tau_2;\beta)+\frac{\beta}{\alpha+\beta}\overset{*}{P}{}^{--}_{hb}(x,\tau_1,\tau_2;\beta)+\frac{\alpha}{\alpha+\beta}B^{++}_{kb}(x,\tau_1,\tau_2)]\\
&+ \frac{\alpha}{\alpha+\beta}[P^{-}_{b}(k,\tau_2;\beta)-\overset{*}{P}{}^{-}_{b}(k,\tau_2;\beta)+(B^{+}_{b}(k,\tau_2)-\overset{*}{B}{}^{+}_{b}(k,\tau_2))]B^{+}_{k}(x,\tau_1)
\end{aligned}$$

Finally we can write

$$W_2(x,t)=(b/q)(V_1(x,t)-(1/\beta)V_\beta(x,t))$$

The solution to the upward-only market review option is given by

$$W(x,t)=W_1(x,t)-W_2(x,t)-W_3(x,t)$$

Comment

This solution is, perhaps, the most significant in this thesis. It answers the original question by providing an explicit solution to the upward-only market review problem. To our knowledge, this analytical solution has not previously been published. The solution,

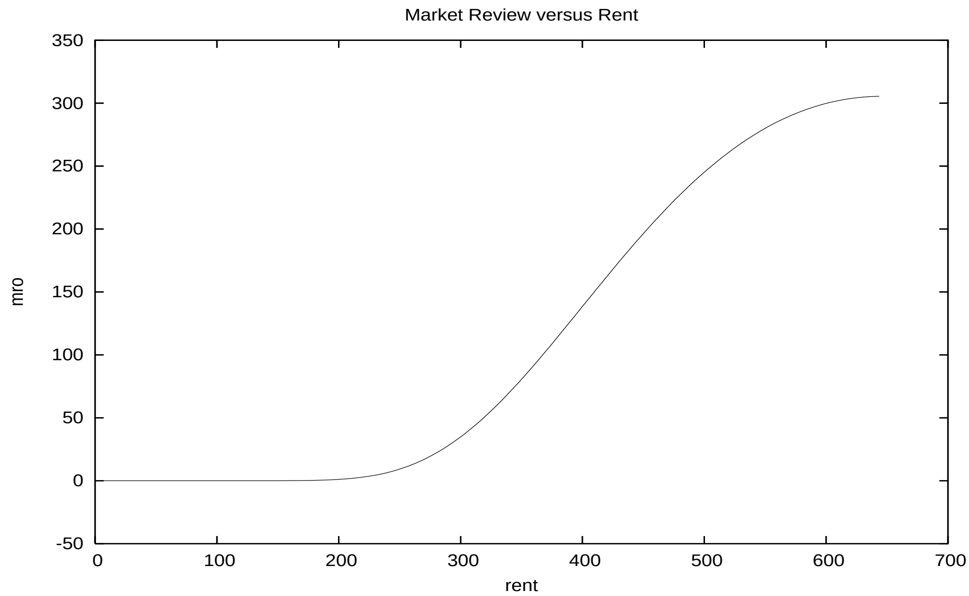

Figure 6.1: Market Review Option versus Rent

although complex, simply involves a portfolio of first order and second order turbo binary options, and required only simple integration.

As an example consider the following. Assume the risk free rate, r, is 8%, q is 7%, the volatility, σ, is 15%, the barrier is \$650, the rent at the start of the lease is \$380, the lease expires in five years and the market review occurs in three years. Figure 6.1 is a plot of the value of the market review option for current rent ranging from zero to the barrier. The behaviour is as expected for this option. At low current rent the value of the option approaches zero and as the current rent approaches the barrier the gradient of the curve approaches zero.

6.6 Ground Lease

This is a type of lease where the land is leased for a long term. The lessee builds on the land and adds value to the land. The lessee can then use the buildings or lease it to other

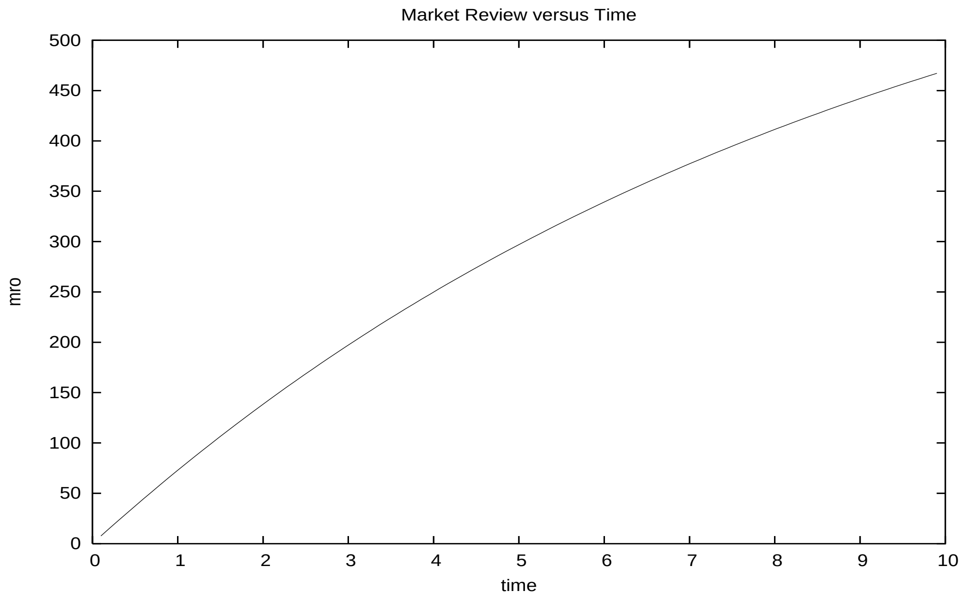

Figure 6.2: Market Review Option versus $\tau_2 - \tau_1$

businesses. At the end of the ground lease the ownership of the buildings often goes to the land owner.

Grenadier [40] models this by considering the infinite lease where the land is essentially being sold for a flow of lease payments. The flow of payments can be considered as an annuity and it's value must equal the long term equilibrium value of the land. The land is developed when the underlying rent process first hits the barrier. Recall that the barrier level is the level at which it is profitable to develop more buildings. The ground lease and hence the value of the land is then tied to the lease payments that come from the development of then land.

This is not same type of lease option seen in the rest of this chapter however it introduces the concept of development as an option. This idea is very important and developed extensively by Dixit and Pindyck [3].

6.7 Chapter Summary

In this chapter the solution to the original motivation of this thesis was presented. That is, several lease options are priced including the upward-only market review option. To the best of our knowledge the solution to the upward-only market review option within this framework has not been published. It transpires that these complicated options can be represented as portfolios of the simple options presented in chapter 5.

Chapter 7

Leases and Lookbacks

7.1 Motivation

A lookback option is an option whose payoff at expiry depends on the value of a state variable of a predefined monitoring period. This is an exotic option that is described as path dependent. In the simplest case this monitoring period is the time from the activation or purchase of the option to expiry. The typical state variables for this monitoring period is a supremum or infimum of the value of the underlying stock price. The payoff can be the difference between the maximum price and a fixed price. This is the fixed strike call option.

Let the underlying stochastic process be Y_t. This new notation is being introduced to distinguish this stochastic process from the related stochastic process denoted by X_t in the first few chapters. The simple relationship between these two process will be described shortly. Let the monitoring period be $[0, T]$, then the maximum over $[0, t]$ with $t \leq T$ is defined as

$$Z_t = \max_{0 \leq s \leq t} \{Y_s\}$$

The payoff at expiry of a fixed strike call is then

$$V(Y,T) = (Z_T - k)^+$$

Compare this with an ordinary vanilla call option where the payoff at expiry is

$$V(Y,T) = (Y_T - k)^+$$

The fixed strike lookback call option clearly has a greater value than the vanilla European call.

In pricing these lookback options it is quite surprising to find that they satisfy the Black-Scholes PDE and the state variable can be treated merely as a parameter. The pricing methodology then exploits this feature and in particular exploits the observation that if $V(y,z,t)$ is a solution of the BS-PDE then so is

$$\bar{V}(y,z,t) = \mathcal{P}_z V(y,z,t)$$

Here y is the underlying stochastic process and z is the state variable. The convention used here is if t is the current time then $y = Y_t$ and $z = Z_t$. $\mathcal{P}_z$ is any linear operator in z such as a differential or integral operator.

In lookback theory the linear operator for these maximum type problems is $\mathcal{P}_z = \partial/\partial z$ and the boundary condition that is needed is

$$V_z = 0 \qquad \text{at } \; y = z.$$

The argument for this vanishing partial derivative can be found in Wilmott 1995 [29]. Buchen, however, provides a more intuitive heuristic argument [33] which will be presented here.

Consider a portfolio containing one long lookback option and h short shares of stock. The value of this portfolio whose value is denoted as P is

$$P(y, z, t) = V(y, z, t) - hy$$

Using Itô's lemma the change in the value of this portfolio after the time step dt is

$$dP = dV - hdy$$

where

$$dV = V_y dx + V_z dz + (V_t + \tfrac{1}{2}\sigma^2 y^2 V_{yy})dt$$

Consider the term $V_z dz$. Here z is the current maximum at time t. z does not change unless the stock price y cross the maximum value in the time interval $(t, t + dt)$. Hence dz is zero most of the time. Using arbitrage, it can be argued that dV cannot have finite jumps, so if $V_z dz$ is zero before the stock price reaches the maximum z, then it must be zero after as well. Since dz is non zero only when $y = z$ at some point in $(t, t + dt)$ the partial derivative V_z must vanish at $y = z$.

Since $V_z dz$ is the only term that contains the state variable z explicitly, the hedging argument leads to the classic Black-Scholes PDE and the only place that the state variable occurs is in the boundary condition [34]. The arbitrage free price of a maximum type lookback option is then given by the solution to

$$\left.\begin{aligned} \mathcal{L}V(y, z, t) &= 0 \text{ in } y < z, \;\; 0 < t < T \\ V(y, z, T) &= g(y, z) \\ \tfrac{\partial V}{\partial z} &= 0 \text{ at } y = z \end{aligned}\right\} \tag{7.1}$$

This is similar to the reflecting boundary value problem except the boundary condition has a partial derivative with respect to the state variable rather than the underlying stochastic variable.

One way to solve Eq (7.1) was first noted by Buchen in a private communication [30] and follows. It was later exploited by Konstandatos in his PhD thesis [26].

Define $U(y,z,t) = \frac{\partial}{\partial z}V(y,z,t)$. Then U satisfies

$$\left.\begin{array}{rcl} \mathcal{L}U &=& 0 \text{ in } y < z, \;\; t < T \\ U(y,z,T) &=& g_z(y,z) \\ U(y,z,t) &=& 0 \text{ at } y = z \end{array}\right\} \tag{7.2}$$

which exactly defines an up and out barrier option with payoff $g_z(y,z)$ and barrier $y = z$. Using the method of images as defined in chapter 4 this can be solved by finding the equivalent European payoff which is

$$U^{\text{eq}}(y,z,T) = g_z(y,z)\mathbb{I}(x < z) - \overset{*}{g}_z(y,z)\mathbb{I}(y > z)$$

Integrating with respect to the state variable z gives

$$\begin{aligned} V^{\text{eq}}(y,z,T) &= V^{\text{eq}}(y,\infty,T) \\ &\quad - \int_z^\infty g_\xi(y,\xi)\mathbb{I}(y < \xi) - \overset{*}{g}_\xi(y,\xi)\mathbb{I}(y > \xi)d\xi \\ &= g(y,\infty,T) - \int_{\max(y,z)}^\infty g_\xi(y,\xi)d\xi \\ &\quad + \int_z^y \overset{*}{g}_\xi(y,\xi)d\xi\mathbb{I}(y > z) \end{aligned}$$

Simplifying yields

$$\left.\begin{array}{rcl} V^{\mathrm{eq}}(y,z,T) & = & g(y,z)\mathbb{I}(y<z)+g(y,y)\mathbb{I}(y>z) \\ & & +\int_z^y \overset{*}{g}_\xi(y,\xi)d\xi\,\mathbb{I}(x>z) \end{array}\right\} \tag{7.3}$$

This is a very important formula as it gives the equivalent European payoff for a maximum type look back option. That is, this is the equivalent payoff in terms of European options where the payoff depends on the maximum value that the underlying stochastic process achieved during the monitoring period.

Let us now consider the original leasing problem presented in chapter 3. The value of the lease at time t is

$$L(x,t;T)=H(x)-C(x,t;T)$$

Of interest is the option $C(x,t;T)$ which will now be reconsidered.

The problem can be solved by first converting it into an equivalent lookback option. Define a new random variable $Y_t=y$ such that

$$dy=\mu ydt+\sigma ydW_t$$

in the domain $0<t\leq T$ and $y>0$. Y_t has the same drift and volatility as X_t and is driven by the same Weiner process.

Define a supremum process

$$Z_t=\sup_{0\leq s\leq t}\{Y_s\}$$

This is essentially the maximum value that Y_t achieved since time $t=0$. However, the argument is still valid for any monitoring period that extends up to the current time.

Make the change of variable

$$x = b\frac{y}{z}$$

The PDE for the lease option can be expressed in terms of y and z rather than x.

$$\tfrac{1}{2}\sigma^2 y^2 V_{yy} + (r-q)yV_y - rV + V_t = 0 \quad \text{in} \quad 0 < y < z\,, t < T$$

$$\begin{aligned} V(0,z,t) &= 0 \\ V(y,z,T) &= H(b\frac{y}{z}) = G(y,z) \\ V_y(y,z,t) &= 0 \quad \text{at } y = z \end{aligned}$$

Note that the Black Scholes operator $\mathcal{L}$ is invariant under scaling of x to y. Now consider the last boundary condition $V_y(y,z) = 0$ at $y = z$. Clearly $V(y,z,t)$ is homogeneous of degree zero in y and z. This is because the original problem was described in terms of the rent process x. In the new formulation x is substituted for an expression that is homogeneous of degree zero in y and z.

Euler's Theorem states that if a function, V, is homogeneous function of degree n in y and z then

$$yV_y + zV_z = nV$$

By Euler's equation and noting that the change of variable results in a function of degree zero in y and z then $yV_y + zV_z = 0$. It follows that if $yV_y = 0$ at $y = z$ then $zV_z = 0$ at $y = z$ hence $V_z(y,z,t) = 0$ at $y = z$. This new boundary condition changes the problem from a reflecting boundary value problem (or lease problem) into an equivalent lookback problem.

The option is now given by

$$\tfrac{1}{2}\sigma^2 y^2 V_{xx} + (r-q)yV_y - rV + V_t = 0 \quad \text{in} \quad 0 < y < z\,, t < T$$

$$\begin{aligned}
V(0,z,t) &= 0; \quad \text{where } y = 0 && (7.4)\\
V(y,z,T) &= \frac{b}{q}\left[-\frac{1}{\beta}\left(\frac{y}{z}\right)^{\beta} + \frac{y}{z}\right] = g(y,z) && (7.5)\\
V_z(y,z,t) &= 0 \quad \text{at } y = z && (7.6)
\end{aligned}$$

This can be solved by the method of Buchen and Konstandatos [34]. The equivalent payoff is (see Eq (7.3))

$$\begin{aligned}
V^{eq}(y,z,T) &= g(y,z)\mathbb{I}(y<z)\\
&+ \left(g^*(y,y) + \int_z^y (g_\xi(y,\xi))^* d\xi\right)\mathbb{I}(y>z)
\end{aligned}$$

Here the superscript * is the image with respect to the barrier z

$$f^*(y,z) = (z/y)^{\alpha} f(z^2/y, z)$$

and

$$\alpha = 2(r-q)/\sigma^2 - 1$$

Define time remaining to expiry as $\tau = T - t$.

Each of the components of the payoff shown in Eq (7.5) is a simple power function. The solution is then the sum of simple turbo options. The solution to this lookback option is identical to Grenadier's solution, which was re-derived in Eq(5.15) of this thesis.

The power option payoff is in the form of

$$W^\gamma(y,z,T) = (y/z)^\gamma \quad \text{for} \quad \gamma = \beta, 1$$

The equivalent pay off is

$$\begin{aligned} W^\gamma_{eq}(y,z,T) &= z^{-\gamma} y^\gamma \mathbb{I}(y<z) \\ &\quad + \left(\frac{\alpha}{\alpha+\gamma}\right) \mathbb{I}(y>z) \\ &\quad + \frac{\gamma z^{\alpha+\gamma}}{\alpha+\gamma} y^{-\alpha-\gamma} \mathbb{I}(y>z) \end{aligned}$$

This has the present-value for all $t < T$

$$\begin{aligned} W^\gamma(y,z,t) &= z^{-\gamma} e^{-r\tau+\gamma(r-q-\frac{1}{2}\sigma^2)\tau+\frac{1}{2}\gamma^2\sigma^2\tau} x^\gamma \mathcal{N}(-d_1) \\ &\quad + \left(\frac{\alpha}{\alpha+\gamma}\right) e^{-r\tau}\mathcal{N}(d_2) \\ &\quad + \frac{\gamma z^{\alpha+\gamma}}{\alpha+\gamma} e^{-r\tau-(\alpha+\gamma)(r-q-\frac{1}{2}\sigma^2)\tau+\frac{1}{2}(\alpha+\gamma)^2\sigma^2\tau} y^{-\alpha-\gamma}\mathcal{N}(d_3) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log(y/z) + (r-q-\frac{1}{2}\sigma^2)\tau + \gamma\sigma^2\tau}{\sigma\sqrt{\tau}} \\ d_2 &= \frac{\log(y/z) + (r-q-\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\ d_3 &= \frac{\log(y/z) + (r-q-\frac{1}{2}\sigma^2)\tau - (\alpha+\gamma)\sigma^2\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

The complete solution is then given by

$$V(y,z,t) = \frac{b}{q}\left(W^1(y,z,t) - \frac{1}{\beta}W^\beta(y,z,t)\right).$$

Converting back to the original variable provides the same solution as obtained using the equivalent payoff for the reflecting barrier.

7.2 Symmetry Relationship

The transformation from the Black Scholes PDE with a Neumann Boundary condition to a lookback option although simple demonstrates a quite surprising symmetry. To demonstrate this the equivalence of the two solutions has to be established. The direct solution of the Black Scholes PDE with a Neumann boundary condition has been obtained by Buchen and will be presented here. This is based on the idea of the log-volution also developed by Buchen and appears in the appendix C.

Introduce a change of variable Let $y = b\frac{y}{z}$ then $x < b$ iff $y < z$. Then the problems becomes

$$\begin{aligned}
\mathcal{L}U(y,z,t) &= 0 \qquad \text{in} \qquad y < z,\ t > 0 \\
U(y,z,0) &= f(by/z) = F(y,z) \\
U_z(y,z,t) &= 0 \qquad \text{when} \qquad y = z
\end{aligned}$$

Note that $zU_z = -xV_x(x,t) = 0$ when $x = b$ or $y = z$. This is a lookback option. From Buchen and Konstandatos [34] this has equivalent payoff of

$$U_{\text{eq}}(y,z,0) = F(y,z)\mathbb{I}(y,z) + F(y,y)\mathbb{I}(y > z) + \int_z^y (F')^*(y,\xi)d\xi \tag{7.7}$$

where $F' \equiv \frac{\partial F}{\partial \xi}$.

Both approaches must be equivalent. Next the equivalence of equation (5.1) and equation (7.7) will be shown.

First consider the payoff function

$$
\begin{aligned}
F(x,\xi) &= f(by/z) \\
F'(y,\xi) &= -by/z^2 f'(by/z) \\
(F')^*(y,\xi) &= -(\xi y)^\alpha \frac{b}{\xi^2}\left(\frac{\xi^2}{y}\right) f'\left(\frac{b\xi}{y}\right) = \frac{b\xi^\alpha}{y^{\alpha+1}} f'\left(\frac{b\xi}{y}\right)
\end{aligned}
$$

Consider the integral in equation (7.7).

$$
\begin{aligned}
I &= \int_z^y (F')^*(y,\xi)d\xi \\
&= -\frac{b}{y^{\alpha+1}}\int_z^y \xi^\alpha f'\left(\frac{b\xi}{y}\right) d\xi \\
&= -\frac{1}{y^\alpha}\int_z^y \xi^\alpha \frac{d}{d\xi} f\left(\frac{b\xi}{y}\right) d\xi \\
&= -\frac{1}{y^\alpha}\left\{\left[\xi^\alpha f\left(\frac{b\xi}{y}\right)\right]_z^y - \alpha\int_z^y \xi^{\alpha-1} f\left(\frac{b\xi}{y}\right) d\xi\right\} \\
&= -f(b) + \left(\frac{z}{y}\right)^\alpha f\left(\frac{bz}{y}\right) + \alpha\int_z^y \left(\frac{\xi}{y}\right)^\alpha f\left(\frac{b\xi}{y}\right)\frac{d\xi}{\xi}
\end{aligned}
$$

Hence the equivalent payoff is

$$
\begin{aligned}
U_{\text{eq}} &= f\left(\frac{by}{z}\right)\mathbb{I}(y<z) + f(b)\mathbb{I}(y>z) \\
&- f(b)\mathbb{I}(y>z) + \left(\frac{z}{y}\right)^\alpha f\left(\frac{bz}{y}\right)\mathbb{I}(y>z) \\
&+ \alpha\mathbb{I}(y>z)\int_z^y \left(\frac{z}{y}\right)^\alpha f\left(\frac{bz}{y}\right)\frac{d\xi}{\xi} \\
&= f\left(\frac{by}{z}\right)\mathbb{I}(y<z) + f^*\left(\frac{by}{z}\right)\mathbb{I}(y>z) \\
&+ \alpha\mathbb{I}(y>z)\int_z^y f^*\left(\frac{by}{z}\right)\frac{d\xi}{\xi}
\end{aligned}
$$

Next use the change of variable $y/z = x/b$ and $\xi = by/z$ giving

$$
U_{\text{eq}} = f(x)\mathbb{I}(x<b) + \left[f^*(x) + \alpha\int_b^x f^*(\eta)\frac{d\eta}{\eta}\right]\mathbb{I}(x>b) \tag{7.8}
$$

Equation (7.8) is the same as equation (5.1) which concludes the demonstration.

In the Black Scholes framework there is an equivalence between the reflecting barrier and the lookback process. Hence options with a reflecting barrier which is described by a Neumann boundary condition can be solved by the lookback methodology of Buchen and Konstandatos [34].

7.3 Purchase Option Using Lookbacks

The purchase option can also be considered in the lookback framework. The solution is more complicated than using the reflecting boundary framework, however, will be presented here in full to illustrate the equivalence of the methods.
The problem in the lookback framework can be stated as

$$
\begin{aligned}
\mathcal{L}V &= 0 \quad \text{in} \quad 0 < y < z, \ t < T \\
V(y,z,T,K) &= (H(y,z) - K)^+ \\
\frac{\partial V}{\partial z} &= 0 \qquad \text{at} \qquad y = z
\end{aligned}
$$

where $H(x)$ is defined in chapter 3 and $H(y,z)$ is defined as

$$
H(y,z) = \frac{b}{q}\left(\frac{y}{z} - \frac{1}{\beta}\left(\frac{y}{z}\right)^{\beta}\right)
$$

Consider the condition $(H(y,z) > K)$ which can be written

$$
\left(\frac{y}{z}\right)^{\beta} - \beta\frac{y}{z} + \beta\frac{q}{b}K < 0
$$

Consider

$$Y(\eta) = \eta^\beta - \beta\eta + \beta\frac{q}{b}K = 0$$

in the interval $0 < \eta < 1$. Using similar arguments as in section 6.4 there exists a unique η such that

$$\mathbb{I}(H(y,z) > K) = \mathbb{I}(y/z > \eta) = \mathbb{I}(y > \eta z)\,.$$

The payoff is therefore

$$\left(\frac{b}{q}\left(\frac{y}{z} - \frac{1}{\beta}\left(\frac{y}{z}\right)^\beta\right) - K\right)\mathbb{I}(y > \eta z) \qquad \text{with} \qquad 0 < \eta < 1.$$

So the components have the form

$$A\left(\frac{y}{z}\right)^n \mathbb{I}(y > \eta z) \qquad \text{for} \qquad n = 0, 1, \beta$$

Consider

$$V = \left(\frac{y}{z}\right)^\beta \mathbb{I}(y > \eta z)$$

The equivalent lookback payoff is (see Eq (7.7))

$$\begin{aligned} V^{\text{eq}} &= \left(\frac{y}{z}\right)^\beta \mathbb{I}(y > \eta z)\mathbb{I}(y < z) + \mathbb{I}(y > z) \\ &\quad + \int_z^y \left[\frac{d}{d\xi}\left\{\left(\frac{y}{\xi}\right)^\beta \mathbb{I}(y > \eta\xi)\right\}\right]^* d\xi \mathbb{I}(y > z) \end{aligned}$$

The first term can be simplified to

$$\mathbb{I}(y > \eta z)\mathbb{I}(y < z) = \mathbb{I}(y < z) - \mathbb{I}(y < \eta z) \qquad \text{since} \qquad \eta < 1$$

Now consider the integral term.

$$
\begin{aligned}
I &= \int_z^y \left[\frac{d}{d\xi} \left\{ \left(\frac{y}{\xi}\right)^\beta \mathbb{I}(y > \eta\xi) \right\} \right]^* d\xi \mathbb{I}(y > z) \\
&= \int_z^y \left[-\beta \frac{\xi^{\alpha+\beta-1}}{y^{\alpha+\beta}} \mathbb{I}(\xi > \eta y) - \left(\frac{\xi}{y}\right)^{\alpha+\beta} \delta(\xi - \eta y) \right] d\xi \; \mathbb{I}(y > z)
\end{aligned}
$$

There are two cases. If $z < \eta y < y$ then

$$
I = \left(\int_{\eta y}^y -\beta \frac{\xi^{\alpha+\beta-1}}{y^{\alpha+\beta}} d\xi - \eta^{\alpha+\beta} \right) \mathbb{I}(z < \eta y) \mathbb{I}(y > z)
$$

If $\eta y < z < y$ then

$$
I = \int_z^y -\beta \frac{\xi^{\alpha+\beta-1}}{y^{\alpha+\beta}} d\xi \quad \mathbb{I}(z > \eta y) \mathbb{I}(y > z)
$$

Hence

$$
\begin{aligned}
I &= \left(\int_{\eta y}^y -\beta \frac{\xi^{\alpha+\beta-1}}{y^{\alpha+\beta}} d\xi - \eta^{\alpha+\beta} \right) \mathbb{I}(z < \eta y) \mathbb{I}(y > z) \\
&\quad + \int_z^y -\beta \frac{\xi^{\alpha+\beta-1}}{y^{\alpha+\beta}} d\xi \quad \mathbb{I}(z > \eta y) \mathbb{I}(x > z)
\end{aligned}
$$

This can be evaluated by making use of the following identities

$$
\begin{aligned}
\mathbb{I}(z < \eta y) \mathbb{I}(y > z) &= \mathbb{I}(y > z/\eta) \\
\mathbb{I}(z > \eta y) \mathbb{I}(y > z) &= \mathbb{I}(y > z) - \mathbb{I}(y > z/\eta)
\end{aligned}
$$

So the integral is evaluated to give

$$
\begin{aligned}
I &= -\frac{\alpha}{\alpha+\beta} \eta^{\alpha+\beta} \mathbb{I}(y > z/\eta) \\
&\quad -\frac{\beta}{\alpha+\beta} \mathbb{I}(y > z) \\
&\quad +\frac{\beta}{\alpha+\beta} \left(\frac{z}{y}\right)^{\alpha+\beta} \left(\mathbb{I}(y > z) - \mathbb{I}(y > z/\eta) \right)
\end{aligned}
$$

The equivalent payoff becomes

$$\left.\begin{aligned} V^{\text{eq}} &= \left(\tfrac{y}{z}\right)^{\beta} \mathbb{I}(y > \eta z) + \tfrac{\alpha}{\alpha+\beta}\mathbb{I}(y > z) \\ &\quad -\tfrac{\alpha}{\alpha+\beta}\eta^{\alpha+\beta}\mathbb{I}(y > z/\eta) \\ &\quad +\tfrac{\beta}{\alpha+\beta}\left(\tfrac{z}{y}\right)^{\alpha+\beta}\left(\mathbb{I}(y > z) - \mathbb{I}(y > z/\eta)\right) \end{aligned}\right\} \tag{7.9}$$

The option price for this payoff can be found using standard techniques [24]

$$\begin{aligned} V^{\beta} &= \left(\frac{y}{z}\right)^{\beta} e^{-r\tau+\beta(r-q-\frac{1}{2}\sigma^2)\tau+\frac{1}{2}\beta^2\sigma^2\tau}\left(N(-d_1') - N(-d_1')\right) \\ &\quad +\frac{\alpha}{\alpha+\beta}e^{-r\tau}N(d_2) \\ &\quad -\frac{\alpha}{\alpha+\beta}\eta^{\alpha+\beta}N(d_2') \\ &\quad +\frac{\beta}{\alpha+\beta}\left(\frac{z}{y}\right)^{\alpha+\beta} e^{-r\tau-(\alpha+\beta)(r-q-\frac{1}{2}\sigma^2)\tau+\frac{1}{2}\sigma^2(\alpha+\beta)^2\tau}(N(d_3) - N(d_3')) \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\log(y/z) + (r-q-\frac{1}{2}\sigma^2) + \beta\sigma^2\tau}{\sigma\sqrt{\tau}} \\ d_1' &= \frac{\log(y/\eta z) + (r-q-\frac{1}{2}\sigma^2) + \beta\sigma^2\tau}{\sigma\sqrt{\tau}} \\ d_2 &= \frac{\log(y/z) + (r-q-\frac{1}{2}\sigma^2)}{\sigma\sqrt{\tau}} \\ d_2' &= \frac{\log(\eta y/z) + (r-q-\frac{1}{2}\sigma^2)}{\sigma\sqrt{\tau}} \\ d_3 &= \frac{\log(y/z) + (r-q-\frac{1}{2}\sigma^2) - (\alpha+\beta)\sigma^2\tau}{\sigma\sqrt{\tau}} \\ d_3' &= \frac{\log(y/\eta z) + (r-q-\frac{1}{2}\sigma^2) - (\alpha+\beta)\sigma^2\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

The last term of the payoff is a special case. This is the case where $n = 0$. So the payoff is

$$V_T^0 = \mathbb{I}(y > \eta z)$$

The equivalent payoff is then

$$\begin{aligned}
V^{\text{eq}} &= \mathbb{I}(y > \eta z)\mathbb{I}(y < z) + \mathbb{I}(y > z) \\
&\quad + \int_z^y \left(\frac{\partial}{\partial \xi} \mathbb{I}(y > \eta z) \right)^* d\xi \mathbb{I}(y > z) \\
&= \mathbb{I}(y > \eta z)\mathbb{I}(y < z) + \mathbb{I}(y > z) \\
&\quad + \int_z^y - \left(\frac{\xi}{y} \right)^\alpha \delta(\xi - z/\eta) d\xi
\end{aligned}$$

Only the case $\mathbb{I}(x > z/\eta)$ needs to be considered. So

$$V^{\text{eq}} = \mathbb{I}(y > \eta z) - \eta^\alpha \mathbb{I}(y > z/\eta)$$

Again the option price can be solved using standard techniques [24]

$$V^0 = e^{-r\tau} N(d_1'') - \eta^\alpha N(d_2')$$

where

$$d_1'' = \frac{\log(y/\eta z) + (r - q - \frac{1}{2}\sigma^2)}{\sigma\sqrt{\tau}}$$

Finally the solution can be written down in the form

$$V(y, z, t) = \frac{b}{q} \left(V^1 - \frac{1}{\beta} V^\beta \right) - K V^0$$

This full solution can be compared to the solution found in the reflecting boundary framework. However once the equivalent European payoff is found the remainder of the solution uses standard option pricing techniques in the Black Scholes framework. Hence to show that this is the same as the answer obtained using the equivalent payoff for the problem formulated using the reflecting barrier we simply need to consider the equivalent payoff Eq (7.9). Using the transformation to take the lookback option back to the variables used in the formulation using the reflecting barrier

$$\begin{aligned} \frac{y}{z} &= \frac{x}{b} \\ \eta &= \frac{\xi}{b} \end{aligned}$$

The payoff is equivalent to Eq (5.5).

7.4 A lookback at the Market Review Options

It has already been demonstrated in section 7.2 that the lookback problem is equivalent to the problem with a reflecting boundary condition. This was demonstrated by showing that the equivalent payoff of one problem could be transformed into the equivalent payoff of the other.

In section 7.3 the purchase option was valued in the lookback framework and again the solution was demonstrated to be equivalent to the solution found in the Grenadier framework with the reflecting boundary condition. In this chapter an outline of the solution to the market review option will be presented. The integration required to demonstrate that the solution is equivalent to the solution presented in section 6.5 is quite onerous and lengthy but relatively straightforward so will not be presented in this thesis. The solution to the market review option in the lookback framework will be outlined in this section to illustrate the approach and the difficulties that arise.

Hence, if a problem is difficult to solve in the lookback framework the transformation to the reflecting boundary framework may provide a simpler solution.
First some notation.

Let $L(Y_i, Z_i, T_i, T_j)$ represent the value of a lease from time T_i to time T_j. Y_s is the short term rent at time s. Z_s is the maximum value X reached up to time s.

Now the value of this upward only market review at time t is

$$(L(X_1, Z_1, T_1, T_2) - L(X_t, Z_t, t, T_2) + L(X_t, Z_t, t, T_1))^+$$

Firstly, $L(Y_t, Z_t, t, T_2) - L(Y_t, Z_t, t, T_1)$ is the forward lease and is determined at time t. $L(Y_1, Z_1, T_1, T_2)$ is the lease determined at time T_1 and expires at time T_2. From the previous discussion on the lease value this is

$$L(Y_1, Z_1, T_1, T_2) = H(Y_1, Z_1) - C(Y_1, Z_1, T_1, T_2)$$

At time T_1 the value of $H(Y_1, Z_1)$ is known.

Using the definition of the short term rent in terms of the demand function y and its running maximum z the rent is given by $b\frac{y}{z}$. As b is a constant the short term rent at time $T_0 < t$ is given by $\eta = \frac{Y_0}{Z_0}$. Hence at time T_1 the conditional is

$$\mathbb{I}\left(\frac{Y_1}{Z_1} > \eta\right)$$

As before define Y_i be the value of the random variable y at time T_i. Then the payoff becomes

$$(H(Y_1, Z_1) - C(Y_1, Z_1, T_1, T_2) - K)\mathbb{I}\left(\frac{X_1}{Z_1} > \eta\right)$$

where $K = L(Y_t, Z_t, t, T_2) - L(Y_t, Z_t, t, T_1)$.

There are three components

1. $W_1(Y_1, Z_1, T_1) = H(Y_1, Z_1)\mathbb{I}\left(\frac{X_1}{Z_1} > \eta\right)$

2. $W_2(Y_1, Z_1, T_1) = C(Y_1, Z_1, T_1, T_2)\mathbb{I}\left(\frac{X_1}{Z_1} > \eta\right)$

3. $W_3(Y_1, Z_1, T_1) = K\mathbb{I}\left(\frac{X_1}{Z_1} > \eta\right)$

K can be evaluated at time t and hence is a constant at time T_1.

Consider W_1. The basic element of this payoff is

$$\hat{V}(Y_1, Z_1, T_1; \gamma) = \left(\frac{Y_1}{Z_1}\right)^{\gamma} \mathbb{I}\left(\frac{X_1}{Z_1} > \eta\right)$$

since

$$H(Y_1, Z_1) = \frac{b}{q}\left(\frac{X_1}{Z_1} - \frac{1}{\beta}\left(\frac{Y_1}{Z_1}\right)^{\beta}\right)$$

This is the payoff of a simple lookback option that was solved in chapter 7.3. Setting $\gamma = 0, 1, \beta$ provides the solution to two components of the solution

$$W_1(y, z, t) = b/q(\hat{V}(y, z, t; 1) - \hat{V}(y, z, t; \beta)/\beta) \quad (7.10)$$

$$W_3(y, z, t) = K\,\hat{V}(y, z, t, 0) \quad (7.11)$$

The $W_2(Y_1, Z_1, T_1)$ component of the payoff contains an option

$$C(Y_1, Z_1, T_1, T_2)\mathbb{I}\left(\frac{Y_1}{Z_1} > \eta\right)$$

and can be described as follows.

$$\begin{aligned}\mathcal{L}C &= 0 \ \text{in}\, t < T_2,\ y < z \\ C(Y_2, Z_2, T_2) &= H(Y_2, Z_2) \\ \partial C/\partial z &= 0 \ \text{for}\ y = z\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}W_2 &= 0 \quad \text{in} \quad t < T_1,\ y < z \\ W_2(Y_1, Z_1, T_1) &= C(Y_1, Z_1, T_1, T_2)\mathbb{I}\left(\frac{Y_1}{Z_1} > \eta\right) \\ \partial W_2/\partial z &= 0 \ \text{at}\ y = z\end{aligned}$$

7.4.1 Solving the Compound Turbo Lookback

The following section is the solution to the problem of the form

$$\left.\begin{aligned}\mathcal{L}Q &= 0 \quad \text{in} \quad t < T_2,\ y < z \\ Q(Y_2, Z_2, T_2) &= \left(\tfrac{Y}{Z}\right)^{\beta} \\ \tfrac{\partial Q}{\partial z} &= 0 \quad \text{at} \quad y = z\end{aligned}\right\} \tag{7.12}$$

and

$$\left.\begin{aligned}\mathcal{L}V &= 0 \quad \text{in} \quad t < T_1,\ y < z \\ V(Y_1, Z_1, T_2) &= Q(Y_1, Z_1, T_1)\mathbb{I}\left(\tfrac{Y_1}{Z_1} > \eta\right) \\ \tfrac{\partial V}{\partial z} &= 0 \quad \text{at} \quad y = z\end{aligned}\right\}$$

This is really just another simple example of a compound lookback option but one that is very important in the market review option. The first step is to start at the payoff at expiry and evaluate the lookback option at time T_1.

The equivalent lookback payoff for Eq (7.12) is

$$Q_{\text{eq}} = \left(\frac{y}{z}\right)^{\beta} \mathbb{I}(y < z) + \mathbb{I}(x > z) \left[1 + \int_z^y \left(\frac{\partial}{\partial \xi}\left(\frac{y}{z}\right)^{\beta}\right)^* d\xi\right]$$

Now

$$\left(\frac{\partial}{\partial \xi}\left(\frac{y}{\xi}\right)^{\beta}\right)^* = -\beta \left(\frac{y}{\xi}\right)^{\alpha+\beta} \frac{1}{\xi}$$

so

$$\begin{aligned} Q_{\text{eq}} &= \left(\frac{y}{z}\right)^{\beta} \mathbb{I}(y < z) + \mathbb{I}(y > z) \left[1 - \frac{\beta}{\alpha+\beta}\left(1 - \left[\left(\frac{y}{z}\right)^{\beta}\right]^*\right)\right] \\ &= \left(\frac{y}{z}\right)^{\beta} \mathbb{I}(y < z) + \mathbb{I}(y > z) \left[\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\left[\left(\frac{y}{z}\right)^{\beta}\right]^*\right] \end{aligned}$$

The solution can be written down by inspection to be

$$Q(y, z, t) = P_z^-(y, \tau; \beta) + \frac{\alpha}{\alpha+\beta} B_z^+(y, \tau) + \frac{\beta}{\alpha+\beta} \overset{*}{P}{}_z^-(y, \tau; \beta) \tag{7.13}$$

where $t < T_1$ and $\tau = T_2 - t$. The explicit expression for the turbo binaries can be found in an appendix.

The next step is to evaluate the lookback option with payoff

$$V = Q(y, z, t)\mathbb{I}(y > \eta z_1); \quad t < T_1$$

The equivalent European formula 7.3 for the lookback can be applied again, however will have terms of the form

$$\int_z^y \left(\frac{\partial}{\partial \xi} \mathcal{N}(d(y, \xi))^*\right) d\xi$$

where $d(y, \xi)$ is of the form

$$d(y, \xi) \sim \frac{\log y/\xi}{\sigma\sqrt{\tau}} + \text{constant}$$

This integration is possible but complicated and will not be presented here.

The symmetry between the lookback option and the reflecting boundary condition clearly is of benefit in this problem. This symmetry can be utilised to choose which framework the problem is best solved in.

7.5 Chapter Summary

There is a symmetry between the lookback option and the reflection boundary condition in the Black Scholes framework. In fact there is a simple transformation that takes one problem to the other. This symmetry was presented by showing how the equivalent European payoff for the lookback problem can be transformed to the equivalent European payoff for the reflecting boundary value problem. To illustrate this, the purchase option is then priced in the lookback framework and shown to be equivalent to the solution the equivalent reflecting boundary problem. The difficulty of solving the compound problem in the lookback framework is illustrated with the upward-only market review option.

Hence, a lookback option may be more easily priced by transforming to the equivalent problem in the Black Scholes framework with a reflecting boundary and a lease type problem may be more easily solved by transforming to the equivalent lookback option.

Chapter 8

Application

8.1 Sydney CBD commercial office

The Grenadier model [40] has attracted considerable interest in the real estate leasing literature. However the relevance of such a model to Australia should be questioned. It is fair to say that there have been considerable changes in the Sydney commercial office over the last two decades. Although it is not in the scope of this work to determine if the Grenadier model can be applied to the Sydney CBD commercial office market the example in this chapter, of the practical application of the technique developed in this research does make this assumption. However, some insights can be obtained from inspecting the data from the Property Council of Australia (PCA).

The graphs presented here are from data contained in the direct property indices published by the PCA. The responsibility of collecting data from the office market has since been passed to Investment Property Databank (IPD). These PCA/IPD direct property indices are the benchmark series for Australian property performance [13]. It looks at 470 commercial properties which had a value of over $45 billion in June 2005.

From an international perspective the commercial property market in Australia is the 12th largest in the world [13] and in 2005 had an estimated value of $332 billion [7]. Long term trends in the property market has been extensively studied in the UK property market. Since 1990 lease lengths have fallen significantly. Prior to the 1990s the average lease term exceeded 20 years. In the first half of the 1990s, with the exception of retail warehouses, the average lease term fell to below 15 years. For older properties the average lease term was about 10 years [4]. The reasons for this has been outlined by Baum [1] and includes the effects of globalisation of business, changes in business practices that require more flexibility, accounting regulations, investment and lending criteria and flows of foreign investment.

There is also pressure for lenders and investors to accept standard institutional lease terms and Baum [1] states that "The lack of sophisticated techniques to price diverse lease terms and the reliance on comparative valuation methods increases the risk to landlords of accepting non standard terms". This implies that there is a high degree of correlation in the behaviour of lessors and lessees. For example in the UK, the rent review was introduced into leases in the 1960's and the standard review period was 5 years [4]. This review period has not changed despite the shortening of the lease terms, and in the current market the average lease would have one or two market reviews. Another factor is the increased use of the break provision linked to review dates. In the office sector in the UK the use of the market review fell from 75% in 1990 to 30% in 1998. Hence the influence of the market review on underlying rent has decreased and there is an increasing need to accurately price the market review option.

Compulsory superannuation was introduced in 1992 and resulted in a sharp increase in total assets in superannuation fund. In 2005 total assets in superannuation funds increase to $741 billion from $238 billion in 1995 [11]. In Australia these assets are expected to grow by 11% per annum from 2005 to 2008 and reach $1 trillion by 2010. This strong inflow of funds has been seen as a strong driver in the commercial property and listed property trusts as 95% of superannuation funds have a specific allocation to these two property asset classes.

West [41] used data of the Australian commercial, retail and industrial market from various sources including the Property Council of Australia. Her analysis using the generalised autoregressive conditional heteroskedasticity in mean model (GARCH-M) confirms that direct property returns are highly correlated with demand fundamentals in the economic cycle and in particular that macroeconomic factors of inflation, employment and interest rates are important in the commercial office returns. Interest rates, not surprisingly, were a significant risk factor across all types of property portfolios.

The total CBD commercial office market in Sydney was considered and graphed from the data provided by the Property Council of Australia for the period from 1986 to 2005. One of the events that dominated this period was the stock market crash that occurred in October 1987 and the recession that resulted in the early 1990's.

There appear to be two distinct regions in these graphs. The period prior to 1997 in figure 8.1 shows large changes in the inflation adjusted rent whereas in the period that followed the inflation adjusted rent appears more constrained. For the Grenadier model to apply to this post 1997 period an upper barrier on rent would be desirable. However this data can only suggest that one may exist.

The yield curve in figure 8.2 also appears to be divided into two region. Prior to 1995 there was a fall in yield before recovering to around 7%. This coincides with the recession of the early 1990's. Post 1995 the yield appears remarkably stable. In the Grenadier model the yield denoted by q is taken to be a constant and this data appears to support this

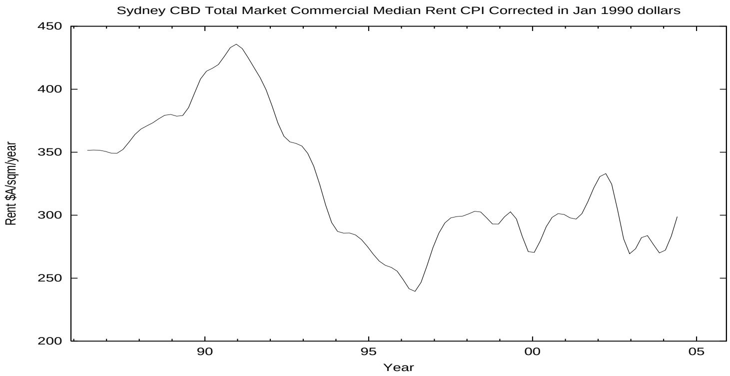

Figure 8.1: Commercial real estate rent from the Property Council of Australia

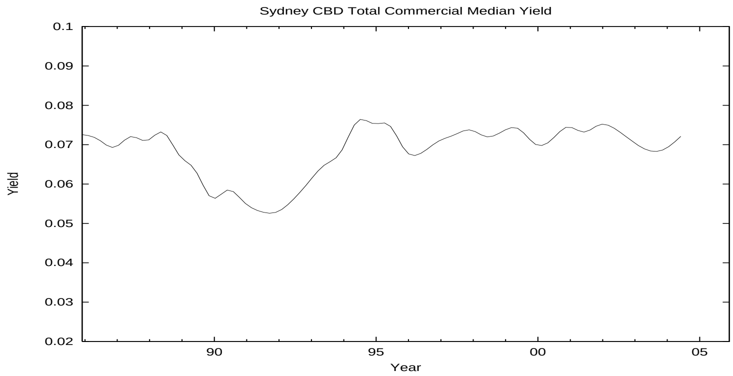

Figure 8.2: Commercial real estate yield from the Property Council of Australia

assumption, at least after 1995.

Finally figure 8.3 is a graph of supply and occupancy. Prior to 1997 the two graphs diverge whereas after 1997 the supply appears to be the supremum of the occupancy. If occupancy were to be a proxy for demand then this again would support the Grenadier model.

The data only covers a short period of time and in no way is considered proof of the applicability of the Grenadier model to the Sydney commercial office market. However it is felt that the data is suggestive and consideration should be given to further research to determine if this model is suitable for the purpose of lease option pricing in this market.

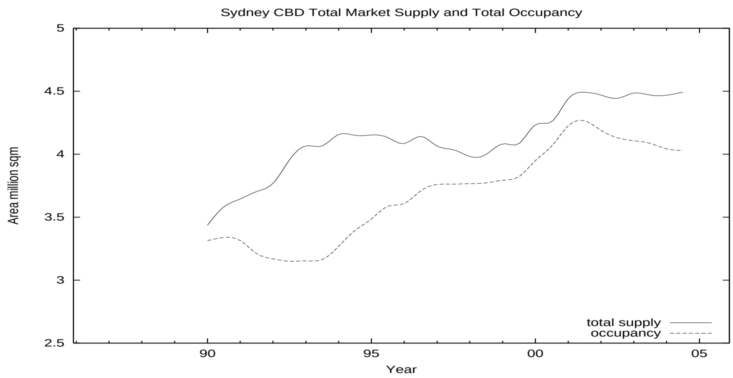

Figure 8.3: Commercial office total supply and occupancy from the Property Council of Australia

8.2 Implied Volatility-Barrier

In the previous chapter it was suggested that commercial office space in Sydney changed after 1997 partly due to superannuation. Indeed pension funds around the world from the USA to Norway are large investors and influence every asset class including commercial office. However it is not in the scope of this thesis to analyse the factors driving the commercial office market. We will simply apply the model to an example of commercial office space in the Sydney central business district to demonstrate how this theory can be used in a practical and achievable example.

The example I will use is a small office space in the central business district of Sydney. The property is on level 7 of a 7 floor building in Pitt Street near Martin Place. The property has been leased to lawyers who are sole practitioners and over the last several years has had a vacancy rate of 20%.

The total floor area of the office is 33 square metres. The 2007 land value as determined by the NSW Office of State Revenue for the purpose of land tax was \$83,520 or \$2,530 per square metre of office space. The rent was \$13,900 per year or \$420 per square metre per year. The outgoings is estimated at \$5,000 per year or \$152 per square meter per year.

The first comment to make is that the net return is \$8,900. As the original purchase price was \$135,000 in 2001 and recent sales in the same building for similar office spaces values the property at \$165,000 or \$5,000 per square metre. Hence the increase in value has been 3.4% compounded. This ignores inflation. The return from the property is

$$\begin{aligned} \text{rental return} &= \text{net rent / property value} * \text{occupancy rate} \\ &= 8900/16500 * 0.8 \\ &= 4.3\% \end{aligned}$$

Hence the total return from the property as an investment is 7.7%. This is consistent with the yield from a "property investment" in Sydney of 7% according to the figure from the property council of Australia and presented in the last chapter.

The barrier level commercial office space can be estimated in the spirit of the Grenadier argument. If the rent increases to such a level that the value of office space increases beyond the cost of building office space, then more developers will come into the market in order to build more office space. It transpires that the cost of building office space has already been estimated. The cost estimates are in fact published every year and used by architects to estimate the cost of construction [21]. The cost of building a prestige office building that is 7 to 20 floors of concrete frame construction with a membrane roof, aluminum windows, air conditioning and lift is \$2,780 per square metre. Hence the total cost of building per square metre an identical building including land cost is \$5310.

Now that we have the upper reflecting barrier for construction the next step is to find the barrier level for rent and the volatility. Unfortunately neither of these are known. However it is quite a simple exercise to write a program to calculate the model implied barrier level for different volatilities. The model for the equilibrium value of the property is

$$H(x) = \frac{b}{q}\left(\frac{x}{b} - \frac{1}{\beta}\left(\frac{x}{b}\right)^{\beta}\right)$$

This is similar to Patel and Singh who also used the Dixit and Pindyck model except did not have an upper reflecting barrier to provide a boundary condition so they appealed to smooth pasting for the boundary condition. Using data from real estate in the UK they calculated the implied volatility. For the UK market this implied volatility ranged from 5 to 30%.

Figure 8.4 is a table of the implied volatility and corresponding implied barrier level for the above example. It assumes that the risk free rate, r, is 8% and the rent at the beginning of the lease, h, is \$380.

Using the numbers for the example above for the current rent level of \$400, figure 8.4 shows that below 5% volatility the properties were under valued. Above 20% the properties are overvalued and there is no effective ceiling on rent.

These values of model determined volatility and upper reflecting barrier can then be used to determine the value of lease options such as the market review option. Consider the range of volatility between 10% and 20% in the previous table. Assume that the current rent is \$400, the rent at the beginning of the lease is \$380 and the market review occurs in year 3 of a 5 year lease. Using the closed form solution of the market review option we can calculate the value.

Depending on where one estimates the rent volatility and upper reflecting barrier the value of the market review option can be calculated. In this example if a volatility of 15% and a barrier level of \$650 is thought to be reasonable then the market review option is worth \$138. With the current rent of \$400 one would need a rent free period of at least 4 months to compensate for agreeing to a market review option in the lease. In fact looking at the range of values of the market review option from a volatility of 10% to 20%, which corresponds to an implied barrier of \$492 to \$930, the market review option value is reasonably

σ	barrier	σ	barrier
0.010	410.416021	0.160	694.256812
0.020	403.015008	0.170	743.622763
0.030	401.749674	0.180	798.897588
0.040	405.211438	0.190	860.809450
0.050	412.379677	0.200	930.217823
0.060	422.687378	0.210	1008.10751
0.070	435.854020	0.220	1095.63046
0.080	451.778135	0.230	1194.11169
0.090	470.465693	0.240	1305.09107
0.100	492.006222	0.250	1430.35322
0.110	516.560874	0.260	1571.99309
0.120	544.344516	0.270	1732.43393
0.130	575.631702	0.280	1914.51079
0.140	610.768609	0.290	2121.55413
0.150	650.155130	0.300	2357.45545

Figure 8.4: This table gives the values of the volatility, σ, and the barrier level for the example above. Below a volatility of 5% the property is under priced and above 20% there is no practical barrier.

σ	barrier	MRO
0.10	492.01	119.32
0.11	516.56	122.82
0.12	544.34	126.65
0.13	575.63	130.59
0.14	610.77	134.54
0.15	650.16	138.45
0.16	694.26	142.31
0.17	743.62	146.12
0.18	798.90	149.87
0.19	860.81	153.57
0.20	930.22	157.22

Figure 8.5: This table provides the value of the market review option for the range of feasible volatilities and barrier levels

insensitive to these parameters.

8.3 Chapter Summary

The motivation of this thesis was to price the upward-only market review option. In this chapter a real example is taken from the Sydney CBD office market. Using the equation for the long term equilibrium value of real estate the relation between the implied rent volatility and the upper reflecting barrier was calculated. Taking a typical range of these values the market review option was priced.

Chapter 9

Beyond Leases

Leases are an important financial instrument in the economy and central to many business processes. Of interest in this work is the real estate lease and in particular leases on commercial real estate. Real estate leases usually contain options that govern changes in rent over time and extensions on the term of the lease. An important option is the market review option which is a compound or multi-period option. To be able to price this option would be of value to both the lessor and lessee and would help in the lease negotiation.

To this lease problem, Grenadier [40] applied a model which in essence is the Black Scholes option pricing framework with the addition of a reflecting barrier on the underlying driving process. In this case the driving process is the rent. Mathematically the Grenadier model is the Black Scholes PDE in a domain bounded above and with a Neumann boundary condition.

In this thesis a technique to solve this class of problems has been developed based on the concepts of the image solution and the equivalent payoff of Buchen. The equivalent European payoff for this problem was derived. A surprising and interesting result that followed from this was the equivalence between this class of problems and the lookback option. This symmetry was demonstrated by proving that the equivalent European payoff for these two problems can be shown to be consistent.

This new technique was applied to the basic lease problem and the purchase option and the solution to both of these were shown to correspond to the published results. The market review option was then priced hence demonstrating the technique could be used to solve the compound problem.

A brief discussion of the Sydney commercial office market outlined the multiple factors that influence this market and suggested that the Grenadier model may be applicable to the Sydney commercial office market. Using a real example the model was used to back out the relation between the barrier and the implied rent volatility and this was used to price the market review option which was the original problem that motivated this thesis.

9.1 Extensions

Further research into the Sydney commercial real estate market would be useful to determine if the model is in fact appropriate. One of the main challenges would be to determine if an upper reflecting barrier on the rent process existed. Further it would be useful if this barrier could be estimated as this would increase the reliability of the model derived implied volatility. Alternatively if a reliable estimate of the rent volatility could be obtained from the market or from historical data then the model derived barrier could be obtained.

This model potentially has widespread applicability and could be extended to non-commercial real estate and leases on commodities such as farm equipment and cars.

Bibliography

[1] Baum A. Pricing the options inherent in leased commercial property: a uk case study. *Department of Real Estate and Planning, University of Reading Business School, UK*, 2003.

[2] Friedman A. *Stochastic Differential Equations and Applications.* Academic Press, New York, 1975.

[3] Dixit A.K. and Pindyck R.S. *Investment under Uncertainty.* Princeton University Press, 1994.

[4] Ambrose B.W., Hendershott P.H., and Klosek M.N. Pricing upward-only adjusting leases. *Journal of Real Estate Finance and Economics*, 25(1):33–49, 2002.

[5] Ward C. and French N. The valuation of upwards-only rent review: an option pricing model. *Journal of Property Valuation and Investment*, 15(2):171–182, 1996.

[6] Smith C.W. Application of option pricing analysis. *in: J.L. Bicksler, ed., Handbook of Financial Economics(North-Holland, Amsterdam)*, 1979.

[7] Higgins D. Modelling the australian property investment universe: a preliminary study. *Pacific Rim Property Research Journal*, pages 268–281, 2005.

[8] Rich D. The mathematical foundations of barrier option pricing theory. *Advances in Futures and Options Research*, 7:267–311, 1994.

[9] Kreyszig E. *Advanced Engineering Mathematics*. Wiley, 6 edition, 1988.

[10] Black F. and Scholes M.S. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–54, 1973.

[11] Newell G. The significance of property in industry based superannuation funds in australia. *Pacific Rim Real Estate Society Conference*, January 2006.

[12] Newell G. and MacFarlane J. Risk estimation and appraisal-smoothing in uk property returns. *Journal of Property Research*, 13:1–12, 1996.

[13] Newell G. and MacFarlane J. The accuracy of property forecasting in australia. *Pacific Rim Real Estate Society Conference*, January 2006.

[14] Buetow G.W. and Albert J.D. The pricing of embedded options in real estate lease contracts. *Journal of Real Estate Research*, 15(3), 1998.

[15] Cox J. and Ross S. The valuation of options for alternative stochastic processes. *Journal of Financial Economics*, 3(1/2):145–166, 1976.

[16] Harrison J. and Kreps D.M. Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory*, 20:381–408, 1979.

[17] Harrison J. and Pliska S. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Applications*, 11:215–260, 1981.

[18] McConnell J.J. and Schallheim J.S. Valuation of asset leasing contracts. *Journal of Financial Economics*, 12(2):237–261, August 1983.

[19] Patel K. and Sing T.F. Implied volatility in the u.k. commercial property market: Empirical evidence based on transaction data. *Journal of Real Estate Finance and Economics*, 20(1):5–24, 2000.

[20] Abramowitz M. and Stegun I.A. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* Dover, New York, ninth dover printing, tenth gpo printing edition, 1964.

[21] Bartlett M. *Cordell Commercial and Industrial Building Cost Guide*, volume 38:01. Reed Construction Data, 2007.

[22] Harrison M. *Brownian motion and stochastic flow systems.* John Wiley and Sons, New York, 1985.

[23] Rubenstein M. and Reiner E. Breaking down the barriers. *Risk Magazine*, 4(8):28–35, 1991.

[24] Skipper M. and Buchen P.W. A general formula for pricing mulit-asset, multi-period exotics. *Working Paper, University of Sydney*, pages 1–21, 2002.

[25] Dunbar N. *Inventing Money.* Wiley, 2001.

[26] Konstandatos O. A new framework for pricing barrier and lookback options. *PhD Thesis. University of Sydney*, 2003.

[27] Rowland P. Pricing lease covenants: Turning theory into practice. *International Real Estate Society Conference, Kuala Lumpur*, 1999.

[28] Wilmott P. *Derivatives The Theory and Practice of Financial Engineering.* Wiley, West Sussex, university edition, 1998.

[29] Wilmott P., Howison S., and Dewynne J. *The Mathematics of Financial Derivatives.* Cambridge University Press, 1995.

[30] Buchen P.W. Lookback option. *Private Communication - University of Sydney*, 2000.

[31] Buchen P.W. Image options and the road to barriers. *Risk Magazine*, 14(9):127–130, 2001.

[32] Buchen P.W. Log-volutions. *Working Paper - University of Sydney*, 2005.

[33] Buchen P.W. *Stocks, Derivatives and Exotics.* Lecture Notes: University of Sydney, 2005.

[34] Buchen P.W. and Konstandatos O. A new method of pricing lookback options. *Mathematical Finance*, 15(2):245–259, 2004.

[35] Merton R. The theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.

[36] Merton R.C. On the pricing of contingent claims and the modigliani-miller theorem. *Journal of Financial Economics*, 5(3):241–249, 1977.

[37] Merton R.C. Application of option-pricing theory: Twenty-five years later. *The American Economic Review*, 88(3):323–349, 1998.

[38] Myer S.C., Dill D.A., and Bautista A.J. Valuation of financial lease contracts. *Journal of Finance*, 31(3), 1976.

[39] Grenadier S.R. Valuing lease contracts: A real options approach. *The Journal of Financial Economics*, 38:297–331, 1995.

[40] Grenadier S.R. An equilibrium analysis of real estate leases. *The Journal of Business*, 78(4):1173(41 pages), 2005.

[41] West T. and Worthington A. Macroeconomic risk factors in australian commercial real estate, listed property trust and property sector stock returns: A comparative analysis using garch-m. *Journal of Financial Management of Property and Construction*, 11(2), August 2006.

Appendix A

Standard Results

A.1 The Normal Distribution

The normal distribution is also known as the Gauss distribution. This is a continuous distribution with a density of

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad \sigma > 0$$

A random variable with this distribution has mean μ and variance of σ^2. σ is also known as the standard deviation. The shorthand for this is $Z \sim N(\mu, \sigma)$ which means that the random variable Z is normally distributed with mean μ and variance σ^2 [9].

The symbol $\mathcal{N}$ is also used to denote the cummulative probability density function. Hence

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}\xi^2} d\xi$$

Hence if Z is a normally distributed random variable with mean μ and variance σ^2 then the probability that $Z > x$ is $\mathcal{N}\left(\frac{x-\mu}{\sigma}\right)$.

The log-normal distribution is defined using the normal distribution. If Z is a Gaussian $N(\mu, \sigma)$ then the random variable $Y = \exp(Z)$ is said to be log-normal and the probability density function is

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right); \qquad x > 0$$

A.2 Gaussian Shift Theorem

The Gaussian Shift theorem captures the change of numeraire. It exploits the high degree of symmetry of the Gaussian. Let $X \sim N(0, 1)$, a be any constant and $F(x)$ any measurable function with finite expectation. Then

$$\mathbb{E}\{e^{aX} F(X)\} = e^{\frac{1}{2}a^2} \mathbb{E}\{F(X + a)\}$$

A.3 Itô's Lemma

Define the underlying gBm as

$$dX_t = \mu X_t dt + \sigma X_t dW_t \quad \text{which is} \quad \mathcal{C}_{2,1}$$

Let F be a stochastic function dependent on t and X_t, *i.e.* $F(X_t, t)$ then

$$dF = (F_t + \mu F_x + \tfrac{1}{2}\sigma^2 F_{xx})dt + \sigma F_x dW_t$$

A.4 Image Solutions from Buchen (2001)

Let $V(x, t)$ be a continuous function in x and t. The image of $V(x, t)$ denoted as $\overset{*}{V}(x, t) = \mathcal{I}V(x, t)$ relative to the Black-Scholes operator $\mathcal{L}$ and the barrier $x = b$ satisfies the following properties [31]

- $\mathcal{I}\overset{*}{V}(x,t) = V(x,t)$

- $\mathcal{L}V(x,t) = 0$ implies $\mathcal{L}\overset{*}{V}(x,t) = 0$

- $V = \overset{*}{V}$ at the barrier, $x = b$

- If $x > b$ (respectively $x < b$) is the active domain of $V(x,t)$ then $x < b$ (respectively $x > b$) is the active domain of $\overset{*}{V}(x,t)$

As a consequence of these properties if the payoff is $f(x)\mathbb{I}(x > b)$ then the image of the payoff is $\overset{*}{V}(x,t)\mathbb{I}(x < b)$.

Theorem. *The Black Scholes Image*

For the Black-Scholes PDE

$$\overset{*}{V}(x,t) = (b/x)^{\alpha} V(b^2/x,t)$$

where $\alpha = 2r/\sigma^2 - 1$.

Proof. Let $V(x,t)$ be a solution of the Black-Scholes PDE in $t < T$. Using the following transformation

$$\begin{aligned} y &= \log x \\ \tau &= T - t \end{aligned}$$

we can write

$$V(x,t) = e^{-\frac{1}{2}\alpha y - \beta\tau} U(y,\tau)$$

where $\beta = (r + \frac{1}{2}\sigma^2)^2/2\sigma^2$ and $U(y, \tau)$ satisfies the heat equation

$$U_\tau = \tfrac{1}{2}\sigma^2 U_{yy}$$

The image solution for the heat equation relative to the barrier $c = \log b$ is simply

$$\overset{*}{U}(y, \tau) = U(2c - y, \tau)$$

Hence it follows from transformation to the original variables

$$\begin{aligned}
\overset{*}{V}(x, t) &= e^{-\frac{1}{2}\alpha y - \beta\tau} U(2c - y, \tau) \\
&= e^{\alpha(2c-y)} V(e^{2c-y}, T - \tau) \\
&= (b/x)^\alpha V(b^2/x, t)
\end{aligned}$$

Theorem. *Equivalent payoff for absorbing BV problems*

The absorbing boundary value problem defined by the benchmark down and out barrier option is

$$\begin{aligned}
\mathcal{L}V(x, t) &= 0 \qquad x > b \;\; t < T \\
V(x, T) &= f(x) \\
V(b, t) &= 0
\end{aligned}$$

has an equivalent form, without the boundary condition at $x = b$, when restricted to the domain $x > b$ of

$$\begin{aligned}
\mathcal{L}V_{\text{eq}}(x, t) &= 0 \qquad t < T \\
V_{\text{eq}}(x, T) &= f(x)\mathbb{I}(x > b) - [f(x)\mathbb{I}(x > b)]^*
\end{aligned}$$

Proof. To show this result the boundary and initial conditions have to be shown to be equivalent. Firstly, the equivalent payoff can be written

$$\begin{aligned} V_{\text{eq}}(x,T) &= f(x)\mathbb{I}(x>b) - \overset{*}{f}(x)\mathbb{I}(x<b) \\ &= \begin{cases} f(x) & \text{for} \quad x>b \\ -\overset{*}{f}(x) & \text{for} \quad x<b \end{cases} \end{aligned}$$

Hence in the domain $x > b$ the initial condition is satisfied as

$$V(x,T) = V_{\text{eq}}(x,T) = f(x) \qquad \text{in} \;\; x > b$$

Secondly, at $x = b$ from the properties of the image we have $V_b(b) = \overset{*}{V}_b(b)$. Given

$$V(x,t) = V_b(x,t) - \overset{*}{V}_b(x,t)$$

we obtain $V(b,t) = 0$ as required.

Appendix B

Binary Options

Binary options [33] are defined by

$$
\begin{aligned}
\mathcal{L}V_\xi^s(x,t) &= 0 \qquad \text{in } \ t > 0; \\
V_\xi^s(x,T) &= f(x)\mathbb{I}(sx > s\xi)
\end{aligned}
\tag{B.1}
$$

where $f(x)$ is an arbitrary function of x, ξ is a non negative exercise price and s is the '+' for an up binary and '-' for a down binary. $\mathcal{L}$ is the Black-Scholes operator. Hence the indicator function in the payoff determines whether the payoff is $f(x)$ or nothing.

B.1 Simple Binary Options

The asset binary, A_ξ^s, is defined by $f(x) = x$ and the bond binary, B_ξ^s, is defined by $f(x) = 1$. The asset binary pays the asset if the condition defined by the indicator function is satisfied. The bond binary pays a fixed cash amount if the condition defined by the indicator function is satified. Both of these are simple options and can be solved using standard techniques to give

$$A^s_\xi(x,t) = x\mathcal{N}(sd_\xi) \tag{B.2}$$

$$B^s_\xi(x,t) = e^{-r\tau}\mathcal{N}(sd'_\xi) \tag{B.3}$$

where

$$d_\xi, d'_\xi = [\log x/\xi + (r \pm \tfrac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}$$

B.2 First Order Turbo Binary Options

Turbo binary options are defined by

$$\begin{aligned} \mathcal{L}P^s_k(x,t;\beta) &= 0 \qquad \text{in } \ t > 0; \\ P^s_k(x,T;\beta) &= (x/k)^\beta \mathbb{I}(sx > sk) \end{aligned} \tag{B.4}$$

where β is a real number. The case of $\beta = 1$ is an asset binary. The case of $\beta = 0$ is a bond binary.

To illustrate the technique this can be solved by expressing the option as a discounted expectation and making use of the Gaussian Shift Theorem (see Appendix A.2).

$$P^s_k(x,\tau;\beta) = e^{-r\tau}\mathbb{E}\{(\frac{X_T}{k})^\beta \mathbb{I}(sX_T > sk)|X_t = x\} \tag{B.5}$$

where

$$X_t = xe^{(r-q-\frac{1}{2}\sigma^2)\tau+\sigma\sqrt{\tau}Z}$$

and $\tau = T - t$ and $Z \sim \mathcal{N}(0,1)$. Therefore

$$P^s_k = e^{-r\tau}(\frac{x}{k})^\beta e^{\beta(r-q-\frac{1}{2}\sigma^2)\tau}\mathbb{E}\{e^{\beta\sigma\sqrt{\tau}Z}\mathbb{I}(sZ > -sd'_k)\}$$

where

$$[d_k, d'_k] = [\log x/k + (r - q \pm \tfrac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}$$

Using Gaussian Shift Theorem

$$\begin{aligned}
P^s_k(x, t; \beta) &= e^{-r\tau}(\frac{x}{k})^\beta e^{\beta(r-q-\frac{1}{2}\sigma^2)\tau + \frac{1}{2}\sigma^2\beta^2\tau}\mathbb{E}\{\mathbb{I}(sZ + \beta\sigma\sqrt{\tau} > -sd'_k)\} \\
&= (\frac{x}{k})^\beta e^{-(r-\beta(r-q-\frac{1}{2}\sigma^2))\tau + \frac{1}{2}\sigma^2\beta^2\tau}\mathbb{E}\{\mathbb{I}(sZ > s(d'_k + \beta\sigma\sqrt{\tau}))\} \\
&= (\frac{x}{k})^\beta e^{-(r-\beta(r-q-\frac{1}{2}\sigma^2))\tau + \frac{1}{2}\sigma^2\beta^2\tau}\mathcal{N}[s(d'_k + \beta\sigma\sqrt{\tau})] \qquad \text{(B.6)}
\end{aligned}$$

B.2.1 A useful identity

$$\overset{*}{P}{}^s_b(x, \tau; c) = P^{-s}_b(x, \tau; -\alpha - c)$$

Proof

Consider the European payoff for the turbo binary option $P^{-s}_b(x, \tau; -\alpha - c)$

$$\begin{aligned}
P^{-s}_b(x, 0; -\alpha - c) &= (x/b)^{-\alpha-c}\mathbb{I}(-sx > -sb) \\
&= [(x/b)^c\mathbb{I}(sx > sb)]^* \\
&= \overset{*}{P}{}^s_b(x, 0; c)
\end{aligned}$$

B.3 Second Order Turbo Binary Options

The second order Turbo Binary option covers two contiguous time periods. The time intervals are $[0, T_1]$ and $(T_1, T_2]$. The payoff of x^β is paid at time $t = T_2$ only if at $t = T_1$, $s_1 x > s_1 k_1$ and at $t = T_2$, $s_2 x > s_2 k_2$.

The option is then defined by

$$
\begin{aligned}
\mathcal{L}P^{s_1,s_2}_{k_1,k_2}(x,t;\beta) &= 0 \\
P^{s_1,s_2}_{k_1,k_2}(x,T_2;\beta) &= \left(\frac{x_2}{k_2}\right)^{\beta} \mathbb{I}(s_1 x_1 > s_1 k_1)\mathbb{I}(s_2 x_2 > s_2 k_2)
\end{aligned} \tag{B.7}
$$

This can be solved using standard techniques to give

$$
P^{s_1,s_2}_{k_1,k_2} = \left(\frac{x}{k_2}\right)^{\beta} e^{-r\tau_2 + (r-q-\frac{1}{2}\sigma^2)\tau_2 + \frac{1}{2}\beta^2\sigma^2\tau_2} \mathcal{N}[s_1 d_1, s_2 d_2; s_1 s_2 \sqrt{\tau_1/\tau_2}] \tag{B.8}
$$

where

$$
\begin{aligned}
d_1 &= \frac{\log x/k_1 + (r-q-\frac{1}{2}\sigma^2)\tau_1 + \beta\sigma^2\tau_1}{\sigma\sqrt{\tau_1}} \\
d_2 &= \frac{\log x/k_2 + (r-q-\frac{1}{2}\sigma^2)\tau_2 + \beta\sigma^2\tau_2}{\sigma\sqrt{\tau_2}}
\end{aligned}
$$

and $\tau_i = T_i - t$ and $T_2 > T_1 > t$.

Appendix C

Log-volution

In this appendix the notion of the the logvolution by Buchen is presented. This is a natural way to describe the BS PDE.

C.1 Definition

Define the logvolution of functions $f(x)$ and $g(x)$ as

$$f(y) \star g(y) = \int_0^\infty f(\xi) g\left(\frac{y}{\xi}\right) \frac{d\xi}{\xi}$$

Let $y' = \log y$; $\xi' = \log \xi$; $F(y') = f(e^{y'})$; $G(y) = g(e^{y'})$. Then

$$f(y) \star g(y) = \int_{-\infty}^\infty F(\xi') G(y' - \xi') d\xi' = F(y') * G(y')$$

The last term is the convolution defined in the usual way. Define $D = x\frac{d}{dx}$ and $\Delta(x) = \delta(x-1)$ is the Dirac delta function.

Properties

1. Linearity: $f \star (\alpha g + \beta h) = \alpha(f \star g) + \beta(f \star h)$ for any scalarsα, β
2. Commutation: $f \star g = g \star f$
3. Associativity: $f \star (g \star h) = (f \star g) \star h$
4. Scaling: $h(x) = f(x) \star g(x) \rightarrow h(kx) = f(kx) \star g(x) = f(x) \star g(kx)$ for all $k > 0$
5. Power law: $x^k[f(x) \star g(x)] = x^k f(x) \star g(x)$ for any real k
6. Inversion: $h(x) = f(x) \star g(x) \rightarrow h(x^{-1}) = f(x^{-1}) \star g(x^{-1})$
7. Derivative: $D[f \star g] = Df \star g = f \star Dg$
8. Identity Element: $f(x) \star \Delta(x) = f(x)$
9. Log-volutional inverse: $f^{-1}(x) \star f(x) = \Delta(x)$

C.2 Mellin Transform

The Mellin or Scale transform is an integral transform and is useful in the theory of logvolutions to solve the Black Scholes PDE. For any function $f(x)$ on $x > 0$ the Mellin transform $F(s)$ is defined as

$$F(s) = T_s[f(x)] = \int_0^\infty f(s) x^{s-1} dx$$

Here the scale parameter s may be complex valued. $F(s)$ can be thought of as the *scale spectrum* and $|F(s)|^2$ is the *power spectrum.* There is a well known inversion formula

$$f(x) = T_x^{-1}[F(s)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) x^{-s} ds$$

Properties

1. Linearity: $T[\alpha f(x) + \beta g(x)] = \alpha F(s) + \beta G(s)$
2. x-scaling: $T[f(kx)] = k^{-s}F(s)$ for all $k > 0$
3. s-scaling: $T[f(x^k)] = |k|^{-1}F(s/k)$ for any real k
4. x-inversion: $T[f(1/x)] = F(-s)$
5. s-shift: $T[x^k f(x)] = F(s + k)$ for any real k
6. Derivative: $T[(-D)^n f(x)] = s^n F(s)$ where $D = x\frac{d}{dx}$
7. Log-volution: $T[f(x) \star g(x)] = F(s)G(s)$

Some Scale Transforms

1. Delta Function $T[\Delta(x)] = 1$
2. $T[x^\alpha \mathbb{I}(x < k)] = k^{s+\alpha}(s + \alpha)^{-1}$ with $\mathcal{R}(s + \alpha) > 0$
3. $T[x^\alpha \mathbb{I}(x > k)] = -k^{s+\alpha}(s + \alpha)^{-1}$ with $\mathcal{R}(s + \alpha) < 0$
4. $T[a^{-1}\phi((\log x)/a)] = e^{\frac{1}{2}a^2s^2}$ where $\phi(y) = e^{-\frac{1}{2}y^2}/\sqrt{2\pi}$ is the density of the standard normal $\mathcal{N}(0, 1)$ variate

C.3 The Black-Scholes PDE using Logvolutions

The Black Scholes PDE can be expressed in terms of the differential operator $D = x\frac{d}{dx}$ as

$$V_t = \tfrac{1}{2}\sigma^2 D^2 V + (r - q - \tfrac{1}{2}\sigma^2)DV - rV = Q(-D)V(x, t)$$

where $Q(s) = \frac{1}{2}\sigma^2 s^2 - (r - q - \frac{1}{2}\sigma^2)s - r$.

For the general case of the homogeneous BS-PDE for $V(x,t)$

$$\mathcal{L}V(x,t) = V_t - Q(-D) = 0; \qquad V(x,0) = f(x)$$

take the scale transform in x and $\hat{V}(s,t) = T_s[V(x,t)]$ to give

$$\hat{V}_t - Q(s)\hat{V} = 0; \qquad \hat{V}(s,0) = \hat{f}(s)$$

This is a first order linear ODE which is solved using standard techniques to give

$$\hat{V}(s,t) = \hat{f}(s)e^{Q(s)t}$$

Define the funcion $G(x,t) = T^{-1}[e^{Q(s)t}]$. This is the Greens function for the BS-PDE and using the scale transforms described above is

$$G(x,t) = \frac{e^{-rt}}{\sigma\sqrt{t}}\phi\left(\frac{\log x + (r - q - \frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right)$$

Hence the solution is

$$V(x,t) = f(x) \star G(x,t)$$

C.4 The Black Sholes Image using Logvolutions

Consider the Black Sholes polynomial defined above and rewrite it as follows

$$Q(s) = \frac{1}{2}\sigma^2(s - \frac{1}{2}\alpha)^2 - c \qquad \text{where } \alpha = 2(r-q)/\sigma^2 - 1$$

where c is a constant independent of s. It follows that there is a symmetry of this polynomial where $Q(s) = Q(\alpha - s)$. Recall that the scale transform of the Greens function for contains this polynomial. Hence we have

$$\hat{G}(s,t) = e^{Q(s)t} = e^{Q(\alpha-s)t} = \hat{G}(\alpha - s, t)$$

Taking the inverse transform gives

$$G(x,t) = x^{-\alpha} G(x^{-1}, t)$$

Defintion: Let $\mathcal{I}_b$ be the Black Scholes image operator with respect to $x = b > 0$ then

$$\mathcal{I}_b\{f(x)\} = (b/x)^{\alpha} f(b^2/x)$$

for any function $f(x)$ on $x > 0$.

C.5 Solving the Mixed Boundary Value problem

Theorem: The solution to the Black Scholes pricing problem

$$\begin{aligned} \mathcal{L}V &= 0 \qquad \text{in } \; x > 0 \;\; t > 0 \\ V(x,T) &= f(x) \\ \mathcal{B}V &= 0 \;\text{ when } \; x = b \end{aligned} \tag{C.1}$$

where $\mathcal{B}V = \mathcal{B}(D)V$ is a linear differential operator, is

$$V(x,t) = V_b(x,t) - (\mathcal{B}^{-1}.\mathcal{I}_b.\mathcal{B})\{V_b(x,t)\} \tag{C.2}$$

where $V_b(x,t)$ is the solution of

$$\begin{aligned} \mathcal{L}V_b(x,t) &= 0 \qquad \text{in } x>0 \;\; t>0 \\ V_b(x,T) &= f(x)\mathbb{I}(x<b) \end{aligned} \tag{C.3}$$

Proof:

- Since $\mathcal{L}U = 0$ then $\mathcal{L}(DU) = 0$ for some function U. It follows that $\mathcal{L}(\mathcal{B}U) = 0$ and also $\mathcal{L}(\mathcal{B}^{-1}U) = 0$. By definition of an image we also have $\mathcal{L}(\mathcal{I}_bU) = 0$. It follows that since $\mathcal{L}V_b = 0$ then $\mathcal{L}V = \mathcal{L}V_b - \mathcal{L}(\mathcal{B}^{-1}.\mathcal{I}_b.\mathcal{B})\{V_b(x,t)\} = 0$.

- Consider $\mathcal{B}V = \mathcal{B}V_b - \mathcal{I}_b.\mathcal{B}.V_b$. By definition of the image for any function U, $U = \mathcal{I}_bU$ at the boundary $x = b$. Hence $\mathcal{B}V = 0$ at $x = b$.

- $V(x,0) = f(x)\mathbb{I}(x<b) - g(x)\mathbb{I}(x>b)$ for some function $g(x)$. Since $\mathcal{I}_b(V_b) = 0$ for $x<b$ by linearity of the operator $\mathcal{B}$, it is also true that $\mathcal{B}V = 0$ and $\mathcal{B}^{-1}.\mathcal{I}_b.\mathcal{B}V_b = 0$ for $x<b$. Hence $V(x,0) = f(x)$ for $x<b$

Example

Consider the mixed boundary condition $\mathcal{B}V = DV - \beta V = 0$ at $x = b$. Take the scale transform of

$$\mathcal{B}V = \mathcal{B}V_b - \mathcal{I}_b\{\mathcal{B}V_b\}$$

to give

$$-(s+\beta)\hat{V}(s,t) = -(s+\beta)\hat{V}_b(s,t) - b^{2s-\alpha}[(\alpha-s+\beta)\hat{V}_b(\alpha-s,t)]$$

Hence

$$\hat{V}(s,t) = \hat{V}_b(s,t) + b^{2s-\alpha}\left[1 - \frac{\alpha+2\beta}{s+\beta}\right]\hat{V}_b(\alpha-s,t)$$

Take the inverse scale transform to give

$$\begin{aligned}
V(x,t) &= V_b(x,t) + \overset{*}{V}_b(x,t) + (\alpha+2\beta)x^\beta \mathbb{I}(x>1) \star \overset{*}{V}_b(x,t) \\
&= V_b(x,t) + \overset{*}{V}_b(x,t) + (\alpha+2\beta)\int_0^\infty \xi^\beta \mathbb{I}(\xi<x)\overset{*}{V}_b(\xi,t)\frac{d\xi}{\xi} \\
&= V_b(x,t) + \overset{*}{V}_b(x,t) + (\alpha+2\beta)\int_0^x \xi^{\beta-1}\overset{*}{V}_b(\xi,t)d\xi
\end{aligned}$$

Appendix D

Programs

The graphs of the option prices were generated from programs written in the C programming language and using the Gnu Scientific Library (GSL). A representative selection of these programs are presented here to illustrate the implementation of these formulae.

D.1 leaseVSrent.c

This program is used to generate the graphs of lease versus rent.

```
/*********************************************************************
*   Lease Value Versus Rent
*
*   compile using gcc -lm -lgsl -lgslcblas leaseVSrent.c
*
*   uses the Gnu Scientific Library
*
**********************************************************************/
#include <stdio.h>
#include <math.h>
```

```
#include <stdlib.h>
#include <gsl/gsl_sf_erf.h>

/* GLOBAL VARIABLES */
double r,q,si,T,m,alpha,beta,b;

/* Normal Distribution Function */
double NDF1(double x)
{
    return (1 + gsl_sf_erf(x/sqrt(2)))/2.0;
}

double turboP(double s, double x, double c)
{
    double u, d;
    u = 0.5 * si*si * c*c  + (r-q- si*si/2)*c - r;
    d = ((log(x/b) + (r-q+(c-0.5)*si*si)*T))/si/sqrt(T);
    return pow((x/b),c)*exp(u*T)*NDF1(s*d);
}

double Wfun(double gamma, double x)
{
    return turboP(-1,x,gamma)
           + gamma*turboP(+1,x,-gamma-alpha)/(gamma+alpha)
           + alpha*turboP(+1,x,0)/(alpha+gamma);
}
```

```
double H(double x)
{
    return (b/q*( x/b - pow((x/b),beta)/beta));
}




main(int argc, char *argv[])
{
    double x0, x1, dx, x, v;
    int i, nx;

    r = 0.05; q = 0.07; si = 0.15; T = 20.0;
    b = 150.0;

    m = r-q-0.5*si*si;
    beta = (-m+sqrt(m*m+2*r*si*si))/(si*si);
    alpha = 2*(r-q)/(si*si) -1;
    nx = 250;

    x0 = 0; x1 = b;
    dx = (x1 - x0)/nx;
    for (i=0;i<nx;i++)
    {
        x = x0 + i*dx;
v = b/q*(Wfun(1.0,x) - Wfun(beta,x)/beta);
        printf("%6.4f %6.4f \n", x,  H(x)-v);
```

```
    }

    return(0);
}
```

D.2 leaseVSterm.c

This program is used to generate the graphs of lease versus term.

```
/*********************************************************************
*   Lease Value Versus Term
*
*   compile using gcc -lm -lgsl -lgslcblas leaseVSterm.c
*
*   uses the Gnu Scientific Library
*
**********************************************************************/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <gsl/gsl_sf_erf.h>

/* GLOBAL VARIABLES */
double r,q,si,x,m,alpha,beta,z,b,T;
```

```
/* Normal Distribution Function */
double NDF1(double xx)
{
    return (1 + gsl_sf_erf(xx/sqrt(2)))/2.0;
}

double turboP(double s, double x, double c)
{
    double u, d;
    u = 0.5 * si*si * c*c  + (r-q- si*si/2)*c - r;
    d = ((log(x/b) + (r-q+(c-0.5)*si*si)*T))/si/sqrt(T);
    return pow((x/b),c)*exp(u*T)*NDF1(s*d);
}

double Wfun(double gamma, double x)
{
    return turboP(-1,x,gamma)
           + gamma*turboP(+1,x,-gamma-alpha)/(gamma+alpha)
           + alpha*turboP(+1,x,0)/(alpha+gamma);
}

double H(double x)
{
    return (b/q*( x/b - pow((x/b),beta)/beta));
}
```

```
main(int argc, char *argv[])
{
    double t0, t1, dt, v;
    int i, nt;
    double Hx;

    r = 0.05; q = 0.07; si = 0.15; x = 20.0;
    b = 150.0;

    m = r-q-0.5*si*si;
    beta = (-m+sqrt(m*m+2*r*si*si))/(si*si);
    alpha = 2*(r-q)/(si*si) -1;
    nt = 250;

    t0 = 0.0; t1 = 20;
    dt = (t1 - t0)/nt;
    Hx = H(x);
    /* printf("debug line 1\n"); */
    for (i=0;i<nt;i++)
    {
        T = t0 + i*dt;
v = b/q*(Wfun(1,x) - Wfun(beta,x)/beta);
        printf("%6.4f %6.4f \n", T, Hx - v);
    }
    return(0);
}
```

D.3 iv.c

This program uses the long term equilibrium value of real estate and binary search to generate the implied volatility-barrier pairs.

```
/*******************************************************************
*   Market Implied Volatility
*
*   Input the ...
*
*   compile using gcc -lm -lgsl -lgslcblas iv.c
*
*   uses the Gnu Scientific Library
*
********************************************************************/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <gsl/gsl_sf_erf.h>

/* define the risk free rate, barrier, yield and current rent */
double r,q,x,Hb;

double beta(double sig)
{
    double temp;
```

```
    temp = r-q-sig*sig/2.0;
    return ((-temp + sqrt(temp*temp + 2*sig*sig*r))/sig/sig);
}

double H(double bet, double b)
{
    return ( b/q*(x/b - pow(x/b,bet)/bet) );
}

double find_b(double bet)
{
    double high, low, tol, mid;
    low = 0.0;
    high = x * 1000000000000.0;
    mid = (high + low)/2.0;
    tol = 0.01;

    if (H(bet,high) < H(bet,low)) {
        printf("Error in binary search\n");
        return(0);
    }

    while (high-low > tol)
    {
        if (H(bet,mid) > Hb) high = mid;
        else low = mid;
        mid = (high + low)/2.0;
```

```
        /*printf("high = %f mid= %f low = %f\n",high, mid, low);*/
    }

    return(high);

}

main(int argc, char *argv[])
{
    double sig,temp;
    int i,N;

    /* initialise the global variables */
    Hb = 5300;
    q = 0.07;
    r = 0.08;
    x = 420;

    /* initialise local variables */
    N = 100;

    for (i=0;i<N;i++)
    {
        sig = (double)i/N;
        temp = find_b(beta(sig));
        /* printf("sigma = %f  beta = %f  b = %f \n", sig, beta(sig), temp); */
        printf("%f %f\n",sig,temp);
```

```
    }

}
```

D.4 MROimpVOL.c

This program uses the output from iv.c to calculate the value of the market review option.

```
/*********************************************************************
*   Market Review Option Price using implied volatility and barrier
*
*   Input the rent, the barrier and the term
*
*   compile using gcc -lm -lgsl -lgslcblas mro_imp_vol.c
*
*   uses the Gnu Scientific Library
*
**********************************************************************/
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <gsl/gsl_sf_erf.h>

/* define global variables */

    double r; /* risk free interest rate */
    double q; /* property yield */
```

```
    double sig; /* volatility */
    double tau_1, tau_2; /* tau_i = T_i - t */
    double b; /* barrier level */
    double alpha; /* constant from image theory */

/* Normal Distribution Function */
double ndf1(double x)
{
    return (1 + gsl_sf_erf(x/sqrt(2)))/2.0;
}

double fbiv(double a, double b, double c)
{
    int i, j;
    long double A[4]
        = {0.3253030, 0.4211071, 0.1334425, 0.006374323};
    long double B[4]
        = {0.1337764, 0.6243247, 1.3425378, 2.2626645};
    long double k1[4][4], k2[4][4], k3[4][4], F[4][4], AF[4], AFA;
    double bet = sqrt(1.0 - c*c);
    double s = sqrt(2.0);
    long double aa = (long double)a/bet/s;
    long double bb = (long double)b/bet/s;
    long double pi = 3.14159265;

    for(i=0;i<4;i++) for(j=0;j<4;j++)
        k1[i][j] = aa*(2*B[i]-aa);
```

```
    for(i=0;i<4;i++) for(j=0;j<4;j++)
        k2[i][j] = bb*(2*B[j]-bb);
    for(i=0;i<4;i++) for(j=0;j<4;j++)
        k3[i][j] = 2*(long double)c*(B[i]-aa)*(B[j]-bb);
    for(i=0;i<4;i++) for(j=0;j<4;j++)
        F[i][j] = exp(k1[i][j]+k2[i][j]+k3[i][j]);
    for(j=0;j<4;j++)
    {
         AF[j] = 0;
         for(i=0;i<4;i++) AF[j] += A[i]*F[i][j];
    }
    AFA = 0;
    for(i=0;i<4;i++) AFA += AF[i]*A[i];

    /* Debug
    printf("a = %f b = %f c = %f\n",a,b,c);
    printf("fbiv = %e\n",beta/(double)pi*(double)AFA);
    */

    return(bet/(double)pi*(double)AFA);
}

double M(double a, double b, double c)
{
    double h;
    double abc = a*b*c;
```

```
    if(abc > 0)
    {
        printf("ERROR abc positive non zero\n");
        exit(0);
    }
    if(c>0)
    {
        if ((a<=0) & (b>=0)) h = ndf1(a) - fbiv(a,-b,-c);
        else if ((a>=0) & (b<=0)) h = ndf1(b) - fbiv(-a,b,-c);
    }
    else if(c<0)
    {
        if ((a<=0) & (b<=0)) h = fbiv(a,b,c);
        else if ((a>=0) & (b>=0)) h = ndf1(a) + ndf1(b) - 1
                                        + fbiv(-a,-b,c);
    }
    else
    {
        printf("ERROR in M\n");
        exit(0);
    }
    return(h);
}


/* Bivariate Normal Distribution */
double ndf2(double a, double b, double c)
```

```
{
    double h, abc, rho1, rho2, d, sa, sb, delta;
    abc = a*b*c;



    if(fabs(c)>1.0)
    {
        printf("ERROR CORRELATION COEFFICIENT OUT OF RANGE\n");
        exit(0);
    }
    else if(c==0.0) h = ndf1(a)*ndf1(b);
    else if(c==1.0) h = ndf1( (a<b)?a:b );
    else if(abc <= 0) h = M(a,b,c);
    else if(abc > 0)
    {
        sa = a/fabs(a);
        sb = b/fabs(b);
        d = sqrt(a*a + b*b - 2*abc);
        rho1 = sa*(a*c-b)/d;
        rho2 = sb*(b*c-a)/d;
        delta = (1.0-sa*sb)/4.0;
        h = M(a,0,rho1) + M(b,0,rho2) - delta;
    }
    else
    {
        printf("ERROR in ndf1\n");
        exit(0);
```

```
    }
    return(h);

}

/* The first order Turbo Binary  */
double P1(double s, double k, double x, double tau, double gamma)
{
    double d, discount;

    d = (log(x/k) + (r-q-sig*sig/2.0)*tau
                        + gamma*sig*sig*tau)/sig/sqrt(tau);
    discount  = exp( -(r-gamma*(r-q-sig*sig/2.0))*tau
                           + sig*sig*gamma*gamma*tau/2.0);
    return( pow(x/k,gamma)*discount*ndf1(s*d));
}

/* The image of the first order Turbo Binary */
double IP1(double s,double k,double x,double tau,double gamma)
{
    /*return( P1(-s,k,x,tau,-alpha-gamma));*/
    return( pow(b/x,alpha)*P1(s,k,b*b/x,tau,gamma));
}

/* The second order Turbo Binary - algorithm from Dr Peter Buchen */
double P2(double s1, double k1, double s2, double k2,
          double x, double tau1, double tau2, double gamma)
```

```
{
    double d1, d2, discount, temp;

    d1 = (log(x/k1) + (r-q-sig*sig/2.0)*tau1
                        + gamma*sig*sig*tau1)/sig/sqrt(tau1);
    d2 = (log(x/k2) + (r-q-sig*sig/2.0)*tau2
                        + gamma*sig*sig*tau2)/sig/sqrt(tau2);
    discount = exp( -r*tau2 + gamma*(r-q-sig*sig/2.0)*tau2
               + gamma*gamma*sig*sig*tau2/2.0  );
    temp = pow(x/k2,gamma)*discount
                    *ndf2(s1*d1,s2*d2,s1*s2*sqrt(tau1/tau2));
    return(temp);
}
/* The image of the second order Turbo Binary */
double IP2(double s1, double k1, double s2, double k2,
           double x, double tau1, double tau2, double gamma)
{
    /*return(P2(-s1,k1,-s2,k2,x,tau1,tau2,-alpha-gamma));*/
    return(pow(b/x,alpha)*P2(s1,k1,s2,k2,b*b/x,tau1,tau2,gamma));
}

double H(double x, double gamma)
{
    return (b/q*( x/b - pow((x/b),gamma)/gamma));
}

double lease(double x,double tau,double gamma)
```

```
{
    double j1, jb;

    jb = P1(-1,b,x,tau,gamma)
           + gamma*IP1(-1,b,x,tau,gamma)/(gamma+alpha)
           + alpha*P1(+1,b,x,tau,0)/(alpha+gamma);
    j1 = P1(-1,b,x,tau,1)
           + 1*IP1(-1,b,x,tau,1)/(1+alpha)
           + alpha*P1(+1,b,x,tau,0)/(alpha+1);
    return( H(x, gamma) - b/q*(j1 - jb/gamma));
}


main(int argc, char *argv[])
{

    /* initialise global variables */
    r = 0.08;
    q = 0.07;
    tau_1 = 3.0;
    tau_2 = 5.0;

    /* define variables   */
    double beta, temp;
    double h = 380.0; /* rent at beginning of the lease */
    double x = 400.0; /* current rent */
    double k;
```

```
  double K;

  double w, w1, w2, w3;
  double u1, ub;
  double v1,v11,v12,v13, vb, vb1, vb2, vb3;
  int i;
  FILE *fp;

  fp = fopen(argv[1], "r");
  if(fp==NULL) {printf("fp NULL\n"); exit(1);}
  printf("opening file %s as read only\n", argv[1]);
  i = fscanf(fp, "%lf %lf", &sig, &b);
  printf("Volatility & Barrier & MRO\n");



while(i==2){

  alpha = 2.0*(r-q)/sig/sig -1.0;

  temp = r-q-sig*sig/2.0;
  beta = (-temp + sqrt(temp*temp + 2.0*r*sig*sig))/sig/sig;

  k=b*b/h;

  u1 = pow(h/b,1)*P1(1,h,x,tau_1,1) - P1(1,b,x,tau_1,1)
       + alpha/(alpha+1)*(P1(1,b,x,tau_1,0)
```

```
     - pow(h/b,alpha+1)*P1(1,k,x,tau_1,0))
     + 1/(alpha+1)*(IP1(-1,b,x,tau_1,1)
     - pow(h/b,1)*IP1(-1,h,x,tau_1,1));

ub = pow(h/b,beta)*P1(1,h,x,tau_1,beta) - P1(1,b,x,tau_1,beta)
     + alpha/(alpha+beta)*(P1(1,b,x,tau_1,0)
     - pow(h/b,alpha+beta)*P1(1,k,x,tau_1,0))
     + beta/(alpha+beta)*(IP1(-1,b,x,tau_1,beta)
     - pow(h/b,beta)*IP1(-1,h,x,tau_1,beta));

w1 = b/q*(u1 - ub/beta);

v11 = P2(1,h,-1,b,x,tau_1,tau_2,1)
     + 1/(alpha+1)*IP2(-1,k,-1,b,x,tau_1,tau_2,1)
     + alpha/(alpha+1)*P2(1,h,1,b,x,tau_1,tau_2,0);
v12 = - P2(1,k,-1,b,x,tau_1,tau_2,1)
     - 1/(alpha+1)*IP2(-1,h,-1,b,x,tau_1,tau_2,1)
     - alpha/(alpha+1)*P2(1,k,1,b,x,tau_1,tau_2,0);
v13 = alpha/(alpha+1)*( P1(-1,b,k,tau_2,1.)
     -IP1(-1,b,k,tau_2,1)+P1(1,b,k,tau_2,0)
     -IP1(1,b,k,tau_2,0) )*P1(1,k,x,tau_1,0);
v1 = v11 + v12 + v13;

vb1 = P2(1,h,-1,b,x,tau_1,tau_2,beta)
     + beta/(alpha+beta)*IP2(-1,k,-1,b,x,tau_1,tau_2,beta)
     + alpha/(alpha+beta)*P2(1,h,1,b,x,tau_1,tau_2,0);
vb2 = -P2(1,k,-1,b,x,tau_1,tau_2,beta)
```

```
          - beta/(alpha+beta)*IP2(-1,h,-1,b,x,tau_1,tau_2,beta)
          - alpha/(alpha+beta)*P2(1,k,1,b,x,tau_1,tau_2,0);
    vb3 = alpha/(alpha+beta)*( P1(-1,b,k,tau_2,beta)
          -IP1(-1,b,k,tau_2,beta)+P1(1,b,k,tau_2,0)
          -IP1(1,b,k,tau_2,0) )*P1(1,k,x,tau_1,0);
    vb = vb1 + vb2 + vb3;

    w2 = b/q*(v1-vb/beta);

    K = lease(h,tau_2,beta) - lease(h,tau_1,beta);

    w3 = K*P1(1,h,x,tau_1,0) - K*pow(h/b,alpha)*P1(1,k,x,tau_1,0);

    w = w1 - w2 - w3;

    printf("%4.2f & %4.2f & %4.2f\n",sig,b,w);

    i = fscanf(fp, "%lf %lf", &sig, &b);
  }


}
```