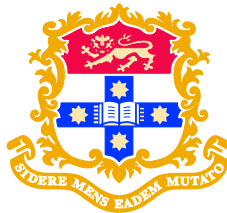


On Monoids Related to Braid Groups and Transformation Semigroups

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