

A COMPARATIVE STUDY
OF AMERICAN OPTION VALUATION
AND COMPUTATION

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Abstract

For many practitioners and market participants, the valuation of financial derivatives is considered of very high importance as its uses range from a risk management tool, to a speculative investment strategy or capital enhancement. A developing market requires efficient but accurate methods for valuing financial derivatives such as American options.

A closed form analytical solution for American options has been very difficult to obtain due to the different boundary conditions imposed on the valuation problem. Following the method of solving the American option as a free boundary problem in the spirit of the “no-arbitrage” pricing framework of Black-Scholes, the option price and hedging parameters can be represented as an integral equation consisting of the European option value and an early exercise value dependent upon the optimal free boundary.

Such methods exist in the literature and along with risk-neutral pricing methods have been implemented in practice. Yet existing methods are accurate but inefficient, or accuracy has been compensated for computational speed. A new numerical approach to the valuation of American options by cubic splines is proposed which is proven to be accurate and efficient when compared to existing option pricing methods. Further comparison is made to the behaviour of the American option’s early exercise boundary with other pricing models.

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