Information-Theoretic Control of Multiple Sensor Platforms

Ben Grocholsky

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Department of Aerospace, Mechatronic and Mechanical Engineering

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Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of the University or other institute of higher learning, except where due acknowledgement has been made in the text.

Ben Grocholsky

March 28, 2002

Ben Grocholsky
The University of Sydney

Abstract

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This thesis is concerned with the development of a consistent, information-theoretic basis for understanding of coordination and cooperation decentralised multi-sensor multi-platform systems. Autonomous systems composed of multiple sensors and multiple platforms potentially have significant importance in applications such as defence, search and rescue, mining or intelligent manufacturing. However, the effective use of multiple autonomous systems requires that an understanding be developed of the mechanisms of coordination and cooperation between component systems in pursuit of a common goal. A fundamental, quantitative, understanding of coordination and cooperation between decentralised autonomous systems is the main goal of this thesis.

This thesis focuses on the problem of coordination and cooperation for teams of autonomous systems engaged in information gathering and data fusion tasks. While this is a subset of the general cooperative autonomous systems problem, it still encompasses a range of possible applications in picture compilation, navigation, searching and map building problems. The great advantage of restricting the domain of interest in this way is that an underlying mathematical model for coordination and cooperation can be based on the use of information-theoretic models of platform and sensor abilities. The information theoretic approach builds on the established principles and architecture previously developed for decentralised data fusion systems. In the decentralised control problem addressed in this thesis, each platform and sensor system is considered to be a distinct decision maker with an individual information-theoretic utility measure capturing both local objectives and the inter-dependencies among the decisions made by other members of the team. Together these information-theoretic utilities constitute the team objective.

The key contributions of this thesis lie in the quantification and study of cooperative control between sensors and platforms using information as a common utility measure. In particular,

- The problem of information gathering is formulated as an optimal control problem by identifying formal measures of information with utility or pay-off.
- An information-theoretic utility model of coupling and coordination between decentralised decision makers is elucidated. This is used to describe how the information gathering strategies of a team of autonomous systems are coupled.
- Static and dynamic information structures for team members are defined. It is shown that the use of static information structures can lead to efficient, although sub-optimal, decentralised control strategies for the team.
- Significant examples in decentralised control of a team of sensors are developed. These include the multi-vehicle multi-target bearings-only tracking problem, and the area coverage or exploration problem for multiple vehicles. These examples demonstrate the range of non-trivial problems to which the theory in this thesis can be employed.

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My greatest debt is to my parents, sister and grandparents, who have provided inspiration, encouragement and unwavering support throughout my life.

This work is dedicated to
the memory of my Opa
and his dream

Contents

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Nomenclature

$(\cdot)(k)$	$(\cdot)$ at discrete time k
$(\cdot)(k \mid l)$	$(\cdot)$ at time k given l
$(\cdot)(t)$	$(\cdot)$ at continuous time t
$(\cdot)^T$	Matrix transpose
$(\cdot)^{-1}$	Matrix inverse
$\mathcal{B}(\cdot)$	Best response function of $(\cdot)$
$\mid (\cdot) \mid$	Matrix determinant of $(\cdot)$
$\dot{(\cdot)}$	Derivative of $(\cdot)$ with respect to time
$\boldsymbol{i}(\cdot)$	Entropic information measure
$\mathrm{E}\{\cdot\}$	Expected value of $(\cdot)$
$\hat{(\cdot)}$	Estimate of $(\cdot)$
$\mathcal{I}(\cdot)$	Mutual information measure
$\mathbf{I}(\cdot)$	Observation Information
$\mathbf{i}(\cdot)$	Information update vector
$\nabla^2_{(\cdot)} a$	Hessian of $(\cdot)$ with respect to a
$\nabla_{(\cdot)} a$	Jacobian of $(\cdot)$ with respect to a
ψ	Vehicle heading
$\Rightarrow$	Implies
θ	Bearing to feature
$\bar{\mathbf{u}}$	Fixed actions of other decision makers

$\tilde{\mathbf{J}}(\cdot)$	Expected utility of $(\cdot)$ formed from partial information
$\mathbf{J}(\cdot)$	Expected utility of $(\cdot)$
$\boldsymbol{u}_i^k$	Action at k^{th} stage of iteration
$\boldsymbol{u}_i$	Action of i^{th} decision maker
$\boldsymbol{u}_i^*$	Action optimal with respect to stated conditions
$\mathcal{U}(\cdot)$	Utility function of $(\cdot)$
$\boldsymbol{\epsilon}$	Generalised error symbol
$\boldsymbol{\sigma}$	Standard deviation
$\boldsymbol{\sigma}^2$	Variance
$\mathbf{f}(\cdot)$	Non-linear state transition model
$\mathbf{h}(\cdot)$	Non-linear observation model
$\mathbf{Z}^k$	Set of observations up to time k
$\mathbf{F}$	State transition matrix
$\mathbf{H}$	Linear observation matrix
$\mathbf{Q}$	Process noise covariance
$\mathbf{R}$	Observation noise covariance
$\mathbf{u}$	Control input vector
$\mathbf{v}$	Observation noise
$\mathbf{w}$	Process noise
$\mathbf{x}$	State vector
$\mathbf{z}$	Observation vector
$\mathbf{Y}(\cdot \mid \cdot)$	Information matrix
$\hat{\mathbf{y}}(\cdot \mid \cdot)$	Information state vector
$\{(\cdot)\}$	Set whose elements are $(\cdot)$
$a \triangleq b$	a is defined as b
r	Range to feature
$P(\cdot)$	Probability distribution

Chapter 1

Introduction

1.1 Objective of the Thesis

This thesis addresses the development of a consistent information-theoretic framework for engineering decentralised multi-sensor multi-vehicle systems. A theoretical basis for the realisation of the practical implementation of cooperative multi-sensor teams is presented. The approach taken builds on the established principles and architecture developed for decentralised data fusion and sensor management. Each vehicle and sensor system is considered to be a distinct decision maker within a team. Each member has an individual utility measure that captures the inter-dependencies among team members. Together these utilities constitute the team objective. The team members are organised and communicate in a manner that jointly achieves the team goal.

The scope of this study is focused on the investigation of distributed sensing, a task fundamental to autonomous operation of robotic systems. This work addresses the question:

What allows a collective of distributed autonomous decision makers to work together as a team toward a common objective?

The problem of seeking, sensing, interpreting perceptual information and interacting

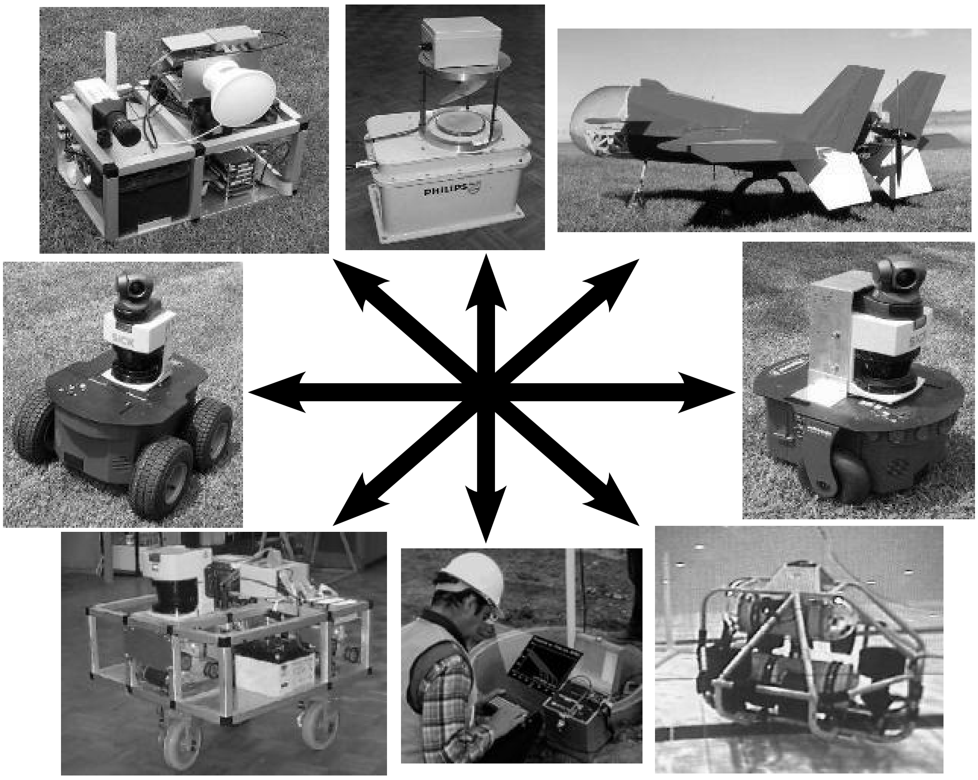


Figure 1.1: Examples of typical robotic vehicles and sensor sub-systems are illustrated. This thesis seeks to understand dependencies among system elements and develop a cooperative decentralised architecture that seamlessly achieves synergistic inter-operation of complementary system capabilities. Team decision making capability is added to the individual sub-systems, communication and negotiation realises their cooperative potential.

with other decision making elements in an inherently uncertain environment is a complex and, as yet, unsolved problem. A successful multi-robot system implementation would revolutionise approaches to a wide variety of practical tasks. Information gathering is the fundamental goal of tasks such as coastal surveillance, environmental monitoring, emergency search and rescue, relaying telecommunication signals and land-mine clearance. Sensor systems provide measurements used to search for, identify and localise features, thereby constructing and updating representations of the problem environment. Their use entails an imperative to optimise the allocation of resources and allow the synergistic inter-operation of all component elements in pursuit of this common goal, capitalising on their distinctive but complementary capabilities. Typical sensors and sensor platforms are shown in Figure 1.1.

Research in economics, social science, computer science, artificial intelligence and engineering has provided diverse approaches to this challenging problem. While interesting and significant progress has been made, this research has not adequately addressed the optimality of the collective system performance and the complexity of the solution process. To allow practical implementation, a consistent, coherent approach with quantifiable performance is essential. This work is motivated by the practical benefits of cooperative sensor teams and by the limitations of existing approaches.

1.2 An Information-Theoretic Approach

Decentralised decision making and control is a logical extension to information-theoretic decentralised data fusion methods. Once information is made available locally, in a decentralised form, and information based utility functions have been defined, it is then possible to implement a decentralised team decision process. Information-theoretic models offer a uniquely powerful method of mathematically describing large-scale systems. Decentralised methods allow information gathering and decision making systems to be described in a mathematically rigorous and modular manner. The global system can be considered as a system of interacting sub-systems, a concept labelled *Systems of Systems*. Resulting systems are analytic, predictive, modular and dynamically configurable.

The basis for the approach adopted in this thesis is provided by established methods in decentralised data fusion, team decision theory, information-theoretic utility and the best-response negotiation procedures. An amalgam of these ingredients offers a general approach to the decentralised control of active sensor teams.

1.3 Principal Contributions

The main contributions of this thesis are:

- Information gathering is formulated and solved as an optimal control problem. The utility associated with a planned sequence of control actions is determined *a priori* from the modelling of the vehicles, sensors and environment. The sensing task is converted into a numerical representation suitable for systematic optimisation.

- A consistent framework is developed for the design of multi-sensor cooperative teams. Information-theoretic utility measures and an established decentralised data fusion architecture are combined with the team decision problem formulation. This results in a decentralised cooperative control architecture for autonomous multi-sensor information gathering systems. Areas where engineering approximations can be applied to trade-off system performance against solution effort are identified.

- The utility of a team decision maker is considered from both an individual and a team perspective. This establishes the relationship between the individual and team optimal actions, and the complexity of possible cooperation. Cooperation is only beneficial when coupling in utility results in team optimal actions that differ from the individual actions determined in isolation. It is observed that coupled utility does not necessarily alter the individual actions. A situation is demonstrated where the action associated with absolute minimum individual utility is team optimal.

- A distinction is made between coordination and cooperation. Coordination is considered to occur when a mechanism coupling the actions of the system gives rise to an increase in the utility of the system. The cooperative solution is taken to be the negotiated equilibrium between sensor action plans. This distinction permits a range of practically useful coordinated solutions without the effort associated with seeking cooperation.

- The mechanisms that under-pin coordination and cooperation are investigated. These are identified as coupled team utility and communication of information. Communication of observed information couples future actions leading to coordination. Communication of expected observation information couples the individual decision making processes. Exchanging expected observation information allows decision makers to account for and influence each other leading to a cooperative solution.

- Scalable coordinated decision making is realised by addition of a local information seeking control layer to the established decentralised data fusion architecture. The decentralised data fusion network propagates observed information influencing the locally optimised sensing plans. A special case is obtained when the decisions are made without looking ahead in time. This requires extremely low solution effort and can be interpreted as 'surfing' the mutual information vector field.

- Decentralised cooperative sensing is achieved through anonymous negotiation based on propagation of expected observation information. Each decision maker updates their sensing plan using a *better-response* procedure and communicates the change in expected observation information. This negotiation cycle is repeated to determine the sensing actions that optimise the team objective.

- An endogeneous non-hierarchical node based cooperative sensor system architecture is proposed. This is an extension of the established decentralised data fusion node. The name 'endogeneous' emphasises that the functionality enabling team decision making is internal. Each node is augmented with an individual distributed decision making procedure and a negotiation communication manager. This architecture is the key to achieving transparent synergistic inter-operation among decision making elements of sensor teams.

1.4 Thesis Structure

Chapter 2 considers approaches to distributed multi-robot systems. Conventions used through the thesis are stated. These include definitions of coordination, cooperation and the characteristics of decentralised systems. The formulation of the team decision problem is presented and its connection to the Nash bargaining problem is established. An iterative procedure known as *better-response* negotiation is introduced as a means for determining Nash equilibria. Key elements in engineering decentralised decision making team members are identified as: modelling of the environment, sensors and vehicles; specification of communication structures; capturing team utility; parameterisation of actions and devising solution procedures.

Chapter 3 covers the problem of quantifying and fusing information in multi-sensor systems. Information is formally defined, in terms of uncertainty, by Fisher and Shannon measures. The Information filter is presented as a mechanism for scalable decentralised fusion of data from multiple sources. The manner in which information is lost and gained in the fusion process is discussed and quantified. Entropic and mutual information are determined to be appropriate expected utility measures for sensing actions. Common information among observations is identified as the source of coupling in team utility derived from entropic information. The decentralised data fusion process and information-theoretic utility structure are identified as forming a consistent basis for gathering, exchanging, evaluating and fusing information in the team decision problem. The approach is demonstrated through the analysis of a discrete sensor assignment problem.

Chapter 4 presents information gathering as an optimal control problem. Modelling of the environment, vehicles and sensors is combined with utility based on entropic information. This is applied to the determination of optimal *information seeking* trajectories for the case of a single bearings-only sensor platform localising a point feature. The implications of this example for active sensing tasks is explored and discussed. Consideration then turns to problems involving multiple sensor platforms. Attention is focused on the team utility structure and its role in cooperation. A proposed decomposition of the team utility is used to explore the influence of coupled utility on the optimal member decisions. This identifies relationships between the optimal individual and team solutions with implications for the complexity of the cooperative solution. A localisation example with two range-only sensors is used to illustrate these results. It is then demonstrated that the optimal team solution can be determined through a *better-response* negotiation procedure.

Chapter 5 explores communication and coupled utility among decision makers as fundamental mechanisms underlying coordination and cooperation. Propagation of observation information through the decentralised data fusion process leads to coordination by altering the prior information on which local decisions are based. The individual decision making processes become coupled when propagation of expected observation information is permitted. This enables determination of the cooperative team solution by negotiation. Coordinated and cooperative solutions are demonstrated through extension of the single vehicle bearings-only example from Chapter 4 to multiple sensor platforms and features. The applicability of this approach to other tasks is demonstrated through an area exploration problem. Finally, all the elements considered through this are brought together to form a general architecture for decentralised coordinated control of multi-sensor information gathering systems.

Chapter 6 presents the main conclusions and identifies a range of future research directions for the work described in this thesis.

Chapter 2

Distributed Systems

2.1 Introduction

This chapter introduces a framework for the design of decentralised multi-agent systems. The formulation of the team decision problem is presented and its connection to the Nash bargaining problem is established. Conventions used through the thesis are defined. These include the characteristics of decentralised systems in Section 2.2, the elements of the team problem in Section 2.3 and the distinction between coordination and cooperation in Section 2.4. Section 2.5 explores the relationship between solving team decision problems and methods in distributed optimisation. An iterative procedure known as *better-response* negotiation is introduced as a means for solving the team decision problem.

2.2 Decentralised System Architectures

A decentralised system consists of a network of agent nodes, each with its own processing facility, which together do not require any central fusion, control or communication facility. In a decentralised system, fusion and control occur locally at each node on the basis of local observations and the information communicated from neighbouring nodes. At no point is there a common place where fusion or global decisions are made. Decentralised systems should be distinguished from distributed systems which rely on some central facilities.

A decentralised system is characterised by three constraints [29]:

1. There is no single central decision centre; no one node should be central to the successful operation of the network.

2. There is no common communication facility; nodes cannot broadcast results and communication must be kept on a strictly node-to-node basis.

3. Nodes do not have any global knowledge of network topology; nodes should only know about connections in their own neighbourhood.

Figures 2.1 and 2.2 show two possible realisations of decentralised systems.

The constraints imposed provide a number of important characteristics for decentralised systems:

- Eliminating the central decision centre and any common communication facility ensures that the system is **scalable** as there are no limits imposed by centralised computational bottlenecks or lack of communication bandwidth.

- Ensuring that no node is central and that no global knowledge of the network topology is required for control means that the system can be made **survivable** to the on-line loss (or addition) of sensing nodes and to dynamic changes in the network structure.

- As all decision processes must take place locally at each site and no global knowledge of the network is required *a priori*, nodes can be constructed and programmed in a **modular** fashion.

The potential of distributed multi-robot systems can not be realised without adhering to a strict decentralised architecture.

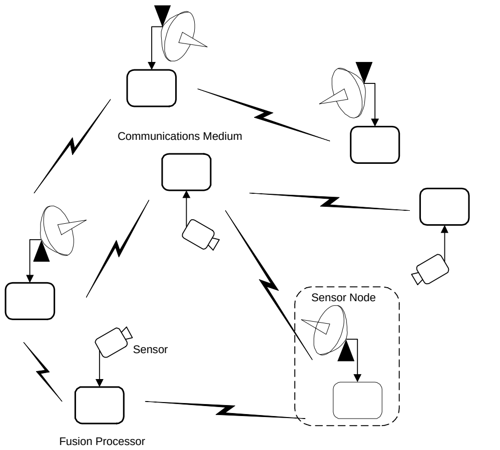


Figure 2.1: A decentralised data fusion system implemented with a point-to-point communication architecture.

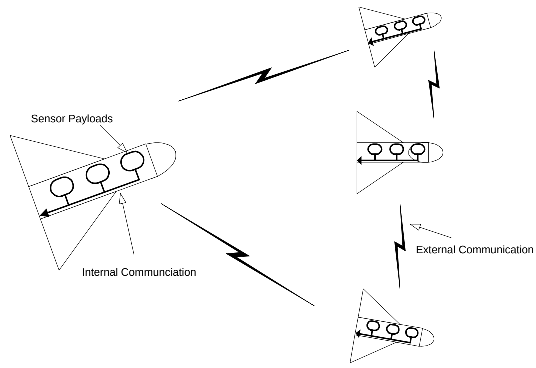


Figure 2.2: A decentralised data fusion system implemented with a hybrid, broadcast and point-to-point, communication architecture.

2.2.1 Decentralised Estimation, Decision and Control Algorithms

The ability to construct a decentralised system architecture clearly depends on whether it is possible to efficiently decentralise existing centralised data fusion and control algorithms. For many common data fusion and decision-theory algorithms, this turns out to be possible, and indeed many decentralised algorithms are, surprisingly, *more* efficient, in terms of both computation and communication, than conventional distributed, federated or hierarchical systems. In particular, the Kalman filter algorithm for target tracking and navigation applications[75], Bayesian methods, for identification and decision making[74, 46], and linear quadratic Gaussian (LQG) control algorithms [59] can all be efficiently decentralised.

Conventional decision and fusion algorithms employ the common notion of 'state' (position, velocity, identity, etc), together with associated probabilities and likelihoods, to assimilate data and generate control actions. Decentralised decision and fusion algorithms rely instead on the notion of *information*, formally defined through both Fisher and Shannon information measures, for continuous and discrete or continuous states respectively. The real advantage of an information measure is that it is straightforward to separate out what is new information from what is either prior knowledge or common information. Assimilation of information measures is additive. This means that any fusion or decision process is associative (it does matter what order it is done in) and thus can be decentralised without (too much) concern as to when information is communicated or assimilated. This is in stark contrast to conventional data fusion algorithms (such as the Kalman filter) where state fusion is *not* associative and so it matters when and how estimates are constructed.

2.2.2 An Example Decentralised System

Given the widely stated advantages of decentralised systems, it is surprising how few practical implementations exist. The OxNav project is an outstanding practical implementation of decentralised systems methodology. OxNav demonstrated fully decentralised and modular sensing, navigation and control for a single mobile robot. Current work presented in this thesis builds on this approach.

In the OxNav project a modular vehicle consisting of a number of standardised modular cages was designed (Figure 2.3). Each cage contained a specific part of the overall vehicle function; drive unit, sensor, power distribution, communication systems. Each cage contained a processor, power and communication facilities, and all local software to implement the required decentralised functions of that unit. There is no central unit or processor where information is combined or where control is coordinated. A wide range of different vehicle systems were constructed from a small number of standardised cages, without the need to change either hardware or software. The decentralised control system for the vehicle demonstrated that the design of local decentralised control algorithms for an individual driven wheel unit allows the control of vehicles with any number of and kinematic configuration of driven and steered wheels. The decentralised navigation system is also described. The system employs a number of modular tracking sonar units. Each unit employs a model of vehicle motion to track environment features to provide

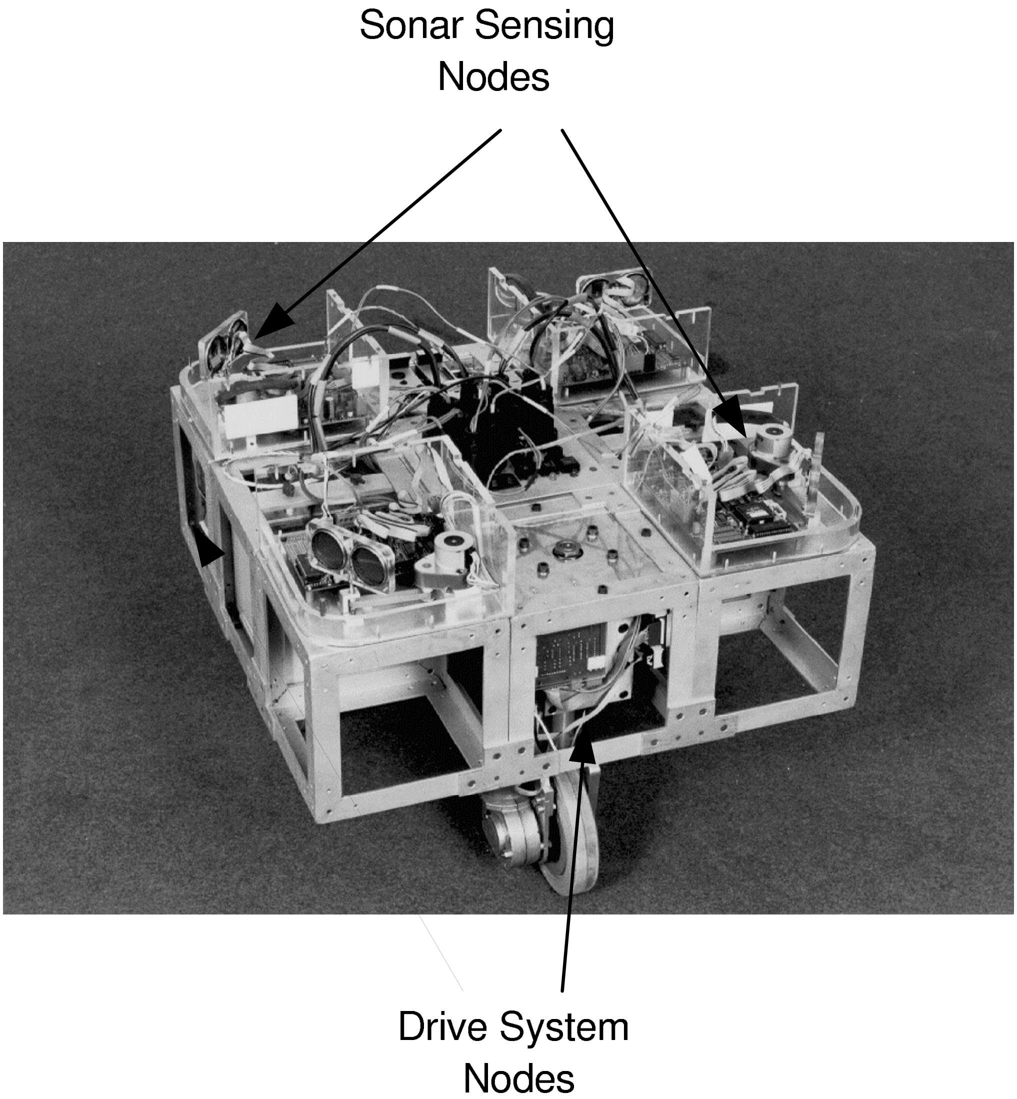


Figure 2.3: The OxNav Vehicle; a fully modular fully decentralised navigation and control system

independent estimates of vehicle location. The estimates are exchanged between sonar units to provide global navigation information. Vehicle guidance was achieved through exchange of information between vehicle drive units and sonar navigation sensors. The key demonstrations undertaken in this project were:

1. Modular hardware design by Burke [12, 13].

2. Scalable distributed and decentralised estimation and communication Berg [3, 4].

3. Distributed and decentralised trajectory tracking control by Mutambara [58, 57, 56, 55, 54, 59].

4. Information-theoretic sensor management for classification and estimation by Manyika [43, 45, 44, 46].

2.3 The Team Decision Problem

The formulation of the team decision problem presented here stems from the work of Marshak and Radner [48, 47, 73] in the study of decentralised decision making in economic systems. Team theory in the context of optimal control was developed by Ho *et. al.* [34, 35, 36, 33]. Application to control of robot sensing was explored by Durrant-Whyte and Hager [21, 30] and later Wen [99]. The formulation has recently been revisited and extended in the context of Distributed Artificial Intelligence by Pynadath and Tambe [71, 72].

A team consists of multiple decision makers. Each decision maker must make a decision that accounts for the decisions made by other members of the team. The key components of the team decision problem are:

1) The presence of *different but correlated* information for each decision maker regarding some underlying uncertainty;

2) The need for *coordinated* actions on the part of all decision makers in order to realise the *payoff*.

The team decision problem is to find optimal decision rules for each member so that the expected utility of the whole team is maximised. This cooperative situation falls under the general category of bargaining problems as defined by Nash [60].

2.3.1 Elements of the Team Problem

The team problem consists of five key elements:

1. **Uncertain World**

   A random vector $\mathbf{x} \equiv [\mathbf{x}_1, \ldots, \mathbf{x}_n] \in \mathbf{X} \subset \mathbb{R}^n$ with known probability distribution $P(\mathbf{x})$, represents all the uncertainties with bearing on the problem under consideration. This includes states, unknown initial conditions and uncertain parameters.

2. **Decision Variables**

   The means of actuation available to decision makers $\boldsymbol{u} \equiv [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m] \in \mathbf{U} \subset \mathbb{R}^s$.

3. **Information Structure**

   Relates observed and communicated information $\mathbf{z} \in \mathbf{Z} \subset \mathbb{R}^m$ to the *uncertain world* $\mathbf{x}$, $\boldsymbol{\eta} : \mathbf{X} \mapsto \mathbf{Z}$. The information structure allows for differing information between decision makers and determines which team member knows what at which time. The information structure is *static* if the information on which a decision is made is not affected by the actions of other decision makers $\mathbf{z} \equiv \boldsymbol{\eta}(\mathbf{x}) \in \mathbf{Z} \equiv [\boldsymbol{\eta}_1(\mathbf{x}), \ldots, \boldsymbol{\eta}_m(\mathbf{x})]$. In a *dynamic* information structure, the decision makers act on information influenced by the action of other team members $\mathbf{z} \equiv \boldsymbol{\eta}(\mathbf{x}, \boldsymbol{u}) \in \mathbf{Z} \equiv [\boldsymbol{\eta}_1(\mathbf{x}, \boldsymbol{u}), \ldots, \boldsymbol{\eta}_m(\mathbf{x}, \boldsymbol{u})]$.

4. **Utility Structure**

   Relates states and actions to payoff (or loss) $\mathcal{U} : \mathbf{U} \times \mathbf{X} \mapsto \mathbb{R}$.

5. **Decision Rule**

   A set of control laws or strategies that relate the information available to an individual to their control actions $\boldsymbol{\gamma} : \mathbf{Z} \mapsto \mathbf{U}$. $\boldsymbol{u}_i = \boldsymbol{\gamma}_i(\mathbf{z}_i),\ \boldsymbol{\gamma}_i \in \boldsymbol{\Gamma} \equiv [\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_m]$. This is a decentralised mechanism as the action is based on local information.

2.3.2 Modelling the Environment, Sensors and Vehicles

Designing decision making teams requires determining a suitable probabilistic model of the state, sensors and the environment. The approach taken throughout this work utilises *information filter* for decentralised data fusion. The information filter is the 'information' or 'inverse covariance' form of the extended Kalman Filter[25, 50]. The Kalman filter employs an explicit statistical model of how the parameter of interest $\mathbf{x}(t)$ evolves over time and an explicit model of how the observations $\mathbf{z}(t)$ are related to this parameter. Representation of the states, actions, observations and associated uncertainty is detailed here. A brief review of the information filter and its application to multi-sensor data fusion is conducted in Chapter 3. Detailed derivation of the information filter is presented in [43]. The implementation of a decentralised data fusion (DDF) architecture based on the information filter is summarised in Appendix B.1. The DDF architecture establishes an appropriate information structure for multi-sensor teams.

Representing the State of the World

Uncertain parameters of interest in the system are represented by a state vector $\mathbf{x}$. Uncertainty in the state $P(\mathbf{x})$ is parameterised by a Gaussian probability distribution

$$\begin{aligned} P(\mathbf{x}) &= \mathcal{N}(\overline{\mathbf{x}}, \mathbf{P}) \\ &= \frac{1}{\sqrt{(2\pi)^n |\, \mathbf{P} \,|}} \exp\left\{ -\frac{1}{2}(\mathbf{x}-\overline{\mathbf{x}})^T \mathbf{P}^{-1}(\mathbf{x}-\overline{\mathbf{x}}) \right\} \end{aligned} \tag{2.1}$$

Where $\overline{\mathbf{x}}$ is the mean and $\mathbf{P}$ the error covariance.

State Transition Model

The characteristics of the operating system and environment are described by a non-linear stochastic differential equations. Perturbation methods may be employed to linearise this system about a nominal trajectory $\mathbf{x}_n(t)$ and $\mathbf{u}_n(t)$ to yield a model linear in error [1],

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \boldsymbol{f}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)) \\ \delta\dot{\mathbf{x}}(t) &= \mathbf{F}(t)\delta\mathbf{x}(t) + \mathbf{B}(t)\delta\mathbf{u}(t) + \mathbf{G}(t)\mathbf{w}(t). \end{aligned} \tag{2.2}$$

Where,

$\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector of interest,

$\mathbf{u}(t) \in \mathbb{R}^s$ are the known control inputs,

$\boldsymbol{f} : \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^q \mapsto \mathbb{R}^n$ is a mapping of state and control input to state rates,

$\mathbf{w}(t) \in \mathbb{R}^q$ are random variables describing model and state evolution uncertainty,

$\mathbf{F}(t)$ is the $n \times n$ linearised time varying state matrix,

$\mathbf{B}(t)$ is the $n \times s$ linearised time varying input matrix, and

$\mathbf{G}(t)$ is the $n \times q$ linearised time varying noise matrix.

$$\mathbf{F}(t) = \left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}(t)=\mathbf{x}_n(t)\\ \mathbf{u}(t)=\mathbf{u}_n(t)}}, \ \mathbf{B}(t) = \left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{u}}\right|_{\substack{\mathbf{x}(t)=\mathbf{x}_n(t)\\ \mathbf{u}(t)=\mathbf{u}_n(t)}}, \ \mathbf{G}(t) = \left.\frac{\partial \boldsymbol{f}}{\partial \mathbf{v}}\right|_{\substack{\mathbf{x}(t)=\mathbf{x}_n(t)\\ \mathbf{u}(t)=\mathbf{u}_n(t)}} \tag{2.3}$$

$$\delta\mathbf{x}(t) \stackrel{\triangle}{=} \mathbf{x}(t) - \mathbf{x}_n(t), \quad \delta\mathbf{u}(t) \stackrel{\triangle}{=} \mathbf{u}(t) - \mathbf{u}_n(t) \tag{2.4}$$

The state transition noise $\mathbf{w}(t)$ is assumed to be a zero mean uncorrelated Gaussian process with covariance $\mathbf{Q}(t)$

$$\mathrm{E}\{\mathbf{w}(t)\} = 0, \quad \mathrm{E}\{\mathbf{w}(t)\mathbf{w}^T(\tau)\} = \mathbf{Q}(t)\delta(t-\tau). \tag{2.5}$$

Observation Model

An observation model relates sensed outputs of the system to the state. Again this can be perturbed to produce a linear error model,

$$\mathbf{z}(t) = \boldsymbol{h}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t))$$

[1] Stability of the error dynamics is not assured in linearised estimation problems without certain conditions. Maybeck [50] examines this in detail in the development of the Extended Kalman Filter.

$$\delta\mathbf{z}(t) \quad = \quad \mathbf{H}(t)\delta\mathbf{x}(t) + \mathbf{D}(t)\mathbf{w}(t). \tag{2.6}$$

Where,

$\mathbf{z}(t) \in \mathbb{R}^m$ is the observation vector at time t,

$\boldsymbol{h}(\cdots)$ is a mapping of state and control inputs to observations,

$\mathbf{v}(t) \in \mathbb{R}^r$ are random variables describing uncertainty in the model and observations,

$\mathbf{H}(t)$ is the $m \times n$ linearised time varying observation matrix, and

$\mathbf{D}(t)$ is the $m \times r$ linearised time varying observation noise matrix.

$$\mathbf{H}(t) = \left.\frac{\partial \boldsymbol{h}}{\partial \mathbf{x}}\right|_{\substack{\mathbf{x}(t)=\mathbf{x}_n(t) \\ \mathbf{u}(t)=\mathbf{u}_n(t)}}, \ \mathbf{D}(t) = \left.\frac{\partial \boldsymbol{h}}{\partial \mathbf{w}}\right|_{\substack{\mathbf{x}(t)=\mathbf{x}_n(t) \\ \mathbf{u}(t)=\mathbf{u}_n(t)}} \tag{2.7}$$

The observation noise represented by $\mathbf{v}(t)$ is a zero mean uncorrelated Gaussian process with covariance $\mathbf{R}(t)$. Further, the observation noise $\mathbf{v}(t)$ and the state transition noise $\mathbf{w}(t)$ are uncorrelated

$$\mathrm{E}\{\mathbf{v}(t)\} \ = 0, \quad \mathrm{E}\{\mathbf{v}(t)\mathbf{v}(\tau)\} \ = \mathbf{R}(t)\delta(t-\tau) \tag{2.8}$$

$$\mathrm{E}\{\mathbf{v}(t)\mathbf{w}^T(\tau)\} \ = 0 \quad \forall \ t, \tau. \tag{2.9}$$

2.3.3 Capturing Group and Individual Utility

Except in the case of perfect knowledge, a utility function $\mathcal{U}(\mathbf{x}, \mathbf{u})$ is not very useful because the true state of the world $\mathbf{x}$ is not known with precision. Hence, the true utility gain associated with the action $\mathbf{u}$ will not be known. Rather the probability distribution $P(\mathbf{x})$ summarises all the probabilistic information available about the state at the time of the decision. With this, one natural method of defining utility is expected utility (or

Bayes expected utility). For a continuous state variables is defined

$$\mathbf{J}(\mathbf{u}) \triangleq \mathrm{E}\{\mathcal{U}(\mathbf{x}, \mathbf{u})\} \;=\int_{-\infty}^{\infty} \mathcal{U}(\mathbf{x}, \mathbf{u}) P(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{2.10}$$

Clearly, Bayes expected utility weights the utility gained by the probability of occurrence (an average utility).

The team formulation associates an expected utility with each decision maker. There is no restriction on the differences or similarity among the individual team member expected utilities. The team utility functions jointly represent the global utility measure. Interdependencies among the individual utilities and actions require the joint optimisation of all actions.

2.3.4 Solving the Team Problem

For given strategies $\boldsymbol{\gamma}$, the utility $\mathcal{U}$ is a well defined function of the *state of the world* $\mathbf{x}$. Thus, the expectation of utility $\mathcal{U}$ with respect to $P(\mathbf{x})$ is well defined, and dependent on $\boldsymbol{\gamma}$. The team decision problem can now be stated as

$$\max_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}} \mathbf{J}(\boldsymbol{\gamma}) = \max_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}} \mathrm{E}_{\mathbf{x}}\{\mathcal{U}(\boldsymbol{u} = \boldsymbol{\gamma}(\boldsymbol{\eta}(\mathbf{x})), \mathbf{x})\} \;. \tag{2.11}$$

This is a deterministic functional optimisation problem. The general solution to this problem can, in principle, be obtained through the calculus of variations in the same manner as other optimal control problems (see Appendix A.3 for example). However, in practice an exact solution is often not tractable and recourse must be made to solution through approximate parameterisation.

As shown by Ho [33], the team problem of Equation 2.11 can be considered from the i^{th} decision makers point of view. Let $\bar{\boldsymbol{\gamma}}_i$ denote a fixed strategy for all other team members. Knowledge of this is determined by the specific information structure employed. Then the i^{th} decision makers problem is

$$\max_{\boldsymbol{\gamma}_i \in \boldsymbol{\Gamma}_i} \mathbf{J}(\boldsymbol{\gamma}_i, \bar{\boldsymbol{\gamma}}_i) = \max_{\boldsymbol{\gamma}_i \in \boldsymbol{\Gamma}_i} \mathrm{E}_{\mathbf{x}} \{\mathcal{U}(\boldsymbol{u}_i = \boldsymbol{\gamma}_i(\boldsymbol{\eta}_i(\mathbf{x})), \bar{\boldsymbol{\gamma}}_i, \mathbf{x})\} \ . \tag{2.12}$$

Since the information structure is fixed, $\mathbf{z}_i$ is a well defined random variable, $\mathrm{E}_{\mathbf{x}}$ can be replaced by $\mathrm{E}_{\mathbf{Z}_i}\ \mathrm{E}_{\mathbf{x}|\mathbf{Z}_i}$ where $\mathrm{E}_{\mathbf{x}|\mathbf{Z}_i}$ is the expectation conditional on $\mathbf{z}_i$. Determining the optimal $\boldsymbol{u}_i$ for $\mathbf{z}_i$ is equivalent to choosing $\boldsymbol{\gamma}_i$. Thus

$$\begin{aligned} \max_{\boldsymbol{\gamma}_i \in \boldsymbol{\Gamma}_i} \mathbf{J}(\boldsymbol{\gamma}_i, \bar{\boldsymbol{\gamma}}_i) &= \max_{\boldsymbol{\gamma}_i \in \boldsymbol{\Gamma}_i} \mathrm{E}_{\mathbf{Z}_i}\ \mathrm{E}_{\mathbf{x}|\mathbf{Z}_i} \{\mathcal{U}(\boldsymbol{\gamma}_i, \bar{\boldsymbol{\gamma}}_i, \mathbf{x})\} \\ &= \mathrm{E}_{\mathbf{Z}_i}\ \max_{\boldsymbol{u}_i \in \mathbf{U}_i} \mathrm{E}_{\mathbf{x}|\mathbf{Z}_i} \{\mathcal{U}(\boldsymbol{u}_i, \bar{\boldsymbol{\gamma}}_i, \mathbf{x})\} \end{aligned}$$

This provides the *person-by-person* form of the problem defined in Equation 2.11

$$\max_{\boldsymbol{u}_i \in \mathbf{U}_i} \mathbf{J}_i(\boldsymbol{u}_i, \mathbf{z}_i; \bar{\boldsymbol{\gamma}}_i) \equiv \max_{\boldsymbol{u}_i \in \mathbf{U}_i} \mathrm{E}_{\mathbf{x}|\mathbf{Z}_i} \{\mathcal{U}(\boldsymbol{u}_i, \bar{\boldsymbol{\gamma}}_i, \mathbf{x})\} \ , \quad \forall\ i. \tag{2.13}$$

Each decision makers optimisation problem is parameterised by the strategy of the other team members as well as its own decision variables. Thus, to solve Equation 2.13 in general will require an iterative loop as shown in Figure 2.4.

The *person-by-person optimality* condition or Nash equilibrium solution [61] is given by

$$\left.\begin{array}{ll} \text{Find } \boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_n^* \text{ such that} & \\ \mathbf{J}_i(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_n^*, \mathbf{z}_1, \mathbf{x}) \geq \mathbf{J}_i(\boldsymbol{u}_1, \boldsymbol{u}_2^*, \ldots, \boldsymbol{u}_n^*, \mathbf{z}_1, \mathbf{x}) & \forall \boldsymbol{u}_1 \in \mathbf{U}_1 \\ \vdots & \vdots \\ \mathbf{J}_i(\boldsymbol{u}_1^*, \ldots, \boldsymbol{u}_n^*, \mathbf{z}_n, \mathbf{x}) \geq \mathbf{J}_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{n-1}^*, \boldsymbol{u}_n, \mathbf{z}_n, \mathbf{x}) & \forall \boldsymbol{u}_n \in \mathbf{U}_n \end{array}\right\} \tag{2.14}$$

2.4 The Bargaining Problem and Nash Equilibrium

Nash's concept of equilibrium is remarkably simple and insightful. Its elegance has been lost in much of the work that stemmed from it. Nash revisited a classical economic

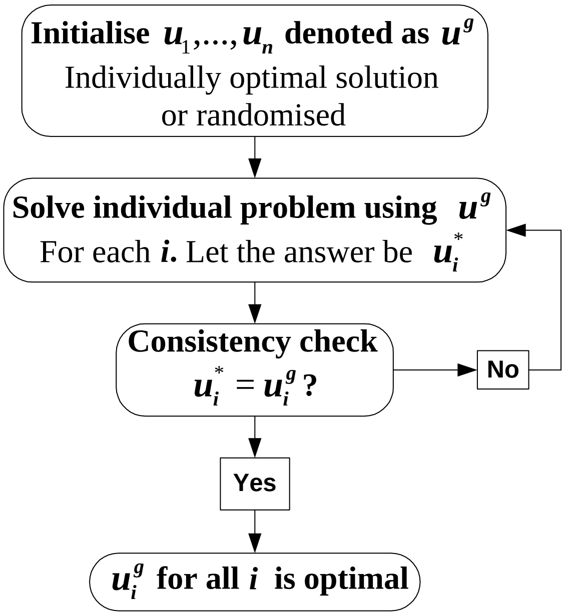


Figure 2.4: Conceptual team decision problem iterative solution procedure

problem. A situation where individuals take actions associated with a set of outcomes. Each individual desires to maximise their gain from a bargaining process. The individuals are able to accurately compare preferences. The exchange is cooperative in the sense that individuals are able to discuss and agree on a joint plan of action.

Nash argued axiomatically that a particular solution was the only rational solution to this problem [60]. The Nash equilibrium condition is stated in Equation 2.14. The individuals jointly maximise their rewards. At this condition no individual has an incentive to deviate. This situation is considered for two individuals in Figure 2.5.

Criticism of the Nash solution focuses on the issue of rationality. This may be arguable in certain contexts. In application to engineering cooperative robot teams there is no such concern. Engineered decision makers act rationally and make objective judgements on a common quantitative scale.

The approach of Nash is remarkably general. The theory of von Neumann and Morgenstern [98] allows a mechanism for decision makers to make “side payments” in a com-

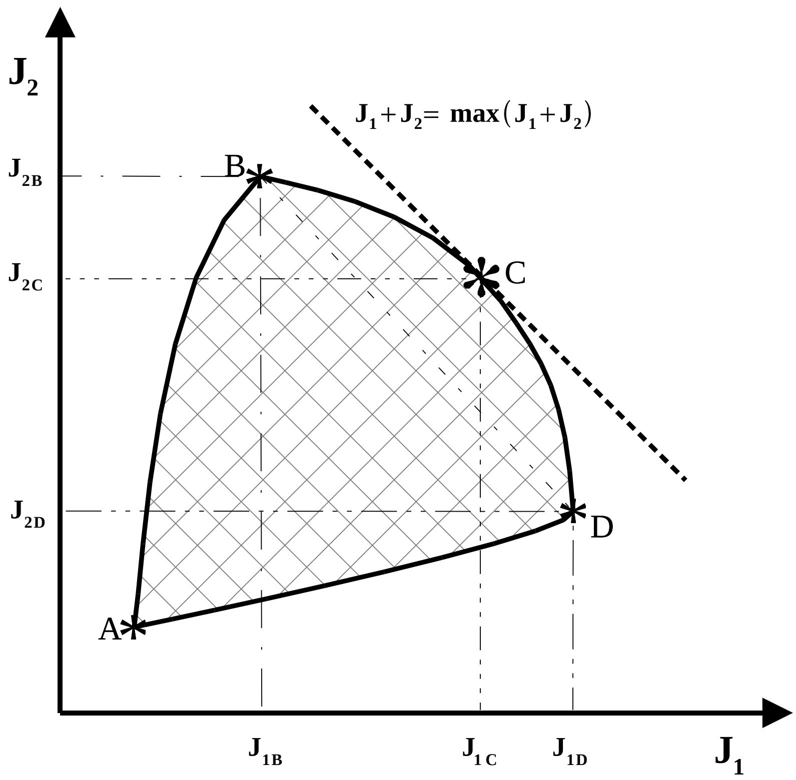


Figure 2.5: Two decision maker actions result in a set of possible utility outcomes. Surely the only rational decision is to maximise individual utility over the set of all possible group outcomes (point D for the 1st and point B for the 2nd decision maker). Careful consideration of the bargaining problem reveals an alternative situation. The outcome depends on the action of both individuals. The first decision maker only receives outcome $\mathbf{J}_{1D}$ if the second accepts $\mathbf{J}_{2D}$. Should both try to individually maximise, the outcome is a point within the set with lower than expected associated rewards. The arc of the hull BCD is Pareto efficient. Moving along this arc increases one individuals reward at the expense of the other. Point C is the Nash solution where individual rewards are jointly maximised. In dealing with rational bargainers, this is the only solution.

modity for which each individual has linear utility. It is possible to conceive other means by which decision makers my influence each others utility such as threats, bribes or altruism. However, as highlighted by Nash [62], these mechanisms simply affect the set of possible utility outcomes and equilibria. No special consideration is necessary as these mechanisms may be treated as any other activity that may take place in playing the game. Market mechanisms fall within the concept of the *Bargaining Problem*.

Market mechanisms have been promoted as vital to the successful operation of coordinated robotic systems [20]. Such reasoning is based on the assertion that profit motive ensures efficiency. This notion must be applied with caution. The Nash solution relies on jointly maximising individual rewards with respect to all involved decision makers. This is distinctly different to maximising individual reward without considering the likely actions of others. For a cooperative team it is vital that the utility measures used represent the true value for a task. It is not relevant or necessary for the utility structure to resemble a financial or commodity market. The market concept may prove to be an effective method when complex utility structures can be approximated by trading in an artificial commodity.

2.4.1 Coordination and Cooperation

The terms coordination and cooperation are used loosely throughout the robotics and distributed artificial intelligence literature. For consistence, definitions of cooperation and coordination are sought. A distinction is drawn between cooperation and coordination. Coordination between team members exists when one member's decision is influenced by the action of another decision maker. Definitions generally accepted within the English language [41] suggest coordination involves *"harmonious function for effective results"*. This is a broad definition. Implementations with wide variations in system performance could be considered coordinated. Cooperation among team members gives rise to joint action for common benefit. Cao *et al.* [14] define robot cooperation as *"a multiple-robot system displays cooperative behaviour if, due to some underlying mechanism, there is an increase in the total utility of the system"*. It is more than this. It must involve possessing and exchanging knowledge in some form.

Cooperation implies coordination. Cooperation is a form of coordination where the influence and benefit occurs in the present, rather than evolving over time. In order to realise this benefit cooperation must involve planning. The decision makers reach an equilibrium through negotiation. It would be to a decision makers detriment to depart from the cooperative solution. The convention used throughout this thesis is that the cooperative solution is the extremum of the range of coordinated solutions.

2.4.2 The Role of Communication

Communication is fundamental to coordination and cooperation. Communication allows decision makers to be obtain external knowledge required to plan and execute a task. Through communication, decision makers can influence and account for each others actions.

Stilwell [86, 85] proposes a powerful and sensible framework based on observer theory that establishes the communication requirements for a task. Within this, he differentiates between active and passive communication. Active in the sense that messages are explicitly passed between team members and passive where knowledge is implicitly inferred by observing the effects of team members. This is an important and practical distinction. The motivation to minimises active communication is provided by requirements for power conservation or stealth. These same motives apply to implicit communication: forces in 'box pushing' [14] that require energy to generate; and actuator acoustic noise[86] which may reveal knowledge of existence or location for example. In an adversarial situation, it is desirable to minimise knowledge implicitly communicated to an opponent. When communication is required, alternative means with differing associated costs must be considered.

A conjecture of some artificial intelligence researchers [24] is that sending signals or transmission of knowledge, only constitutes communication if it is intentional. This leads to misleading concepts such as "cooperation without communication" [14]. In this, decision makers are coordinated through observation of the unintentional effect of others on the environment. This would argue that they are in fact implicitly communicating through the environment. In this work, transmitting knowledge is considered communication, regardless of intent.

2.4.3 Levels of Coordination and Cooperation

To clarify the relation between coordination and cooperation, it is suggested that three categories exist:

1. **Cooperation or Negotiated Coordination** - Mutual agreement *(equilibrium)* is achieved through collective, coordinated and predictive planning and execution.

2. **Un-negotiated Coordination (Not Cooperation)** - An active mechanism gives rise to coordinated response but there is no mechanism for bargaining between decision makers. *"coordination without negotiation"*

3. **Passive Coordination (Not Cooperation)** - coupling through passive mechanism gives rise to coordinated response.

These conventions are used throughout this work.

2.5 Cooperative Solution by Negotiation

The iterative solution procedure to the team problem suggested in Figure 2.4 is known as *best-reply* iteration. This is strongly related to *fictitious play* concepts in game theory [76]. The formulation of the team decision problem results in a parameter optimisation. Optimisation is a large, diverse and active research area. What follows aims to establish the connection between conventional optimisation theory and the team decision problem. The global team problem Equation 2.11, is of the form

$$\mathbf{u}^{\star} = \arg\max_{\mathbf{u}\in\mathbf{U}} \mathbf{J}(\mathbf{u}), \quad \mathbf{u} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}^T. \tag{2.15}$$

General solution methods for this problem fall under the category of nonlinear programming (NLP). A common fixed-point iterative optimisation implementation is sequential quadratic programming (SQP) [26, 2]. Assuming the performance measure is twice differentiable, the unconstrained quadratic problem at each iteration stage can in principle be solved for the full parameter vector by Newton's method

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \left[\nabla^2_{\mathbf{u}} \mathbf{J}(\mathbf{u}^k)\right]^{-1} \nabla_{\mathbf{u}} \mathbf{J}(\mathbf{u}^k). \tag{2.16}$$

Newton's method is significant conceptually but in its exact form is only of limited practical use. This is primarily due to the expense of inverting the Hessian and complications when the Hessian is non-negative definite. This motivates the use of alternate schemes. Bertsekas [6] details algorithms suitable for parallel and distributed implementation. Two simple approaches are:

Generalised Jacobi Algorithm:

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \boldsymbol{\kappa} \left[\mathbf{D}(\mathbf{u}^k)\right]^{-1} \nabla_{\mathbf{u}} \mathbf{J}(\mathbf{u}^k), \tag{2.17}$$

where $\boldsymbol{\kappa}$ is a positive stepsize and $\mathbf{D}(\mathbf{u}^k)$ is a diagonal matrix i^{th} diagonal entry is $\nabla^2_{ii} \mathbf{J}(\mathbf{u}^k)$, assumed to be nonzero for each i. Note this is a generalisation of the Jacobi Over Relaxation scheme for solving linear equations.

Generalised Gauss-Seidel Algorithm:

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \boldsymbol{\kappa} \frac{\nabla_i \mathbf{J}(\mathbf{p}(i,k))}{\nabla^2_{ii} \mathbf{J}(\mathbf{p}(i,k))}, \quad i = 1, \ldots, n, \tag{2.18}$$

where $\mathbf{p}(i,k) = \left\{\mathbf{u}_1^{k+1}, \ldots, \mathbf{u}_{i-1}^{k+1}, \mathbf{u}_i^{k+1}, \ldots, \mathbf{u}_n^{k+1}\right\}$. Note this is a generalisation of the Successive Over Relaxation scheme for solving linear equations.

These algorithms, in effect, treat every parameter in the solution vector separately. Each parameter is updated individually accounting only for its own influence, rather than updating the entire solution vector simultaneously while accounting for all parameter interdependencies (as in Newton's method). In the Jacobi implementation updates are performed based on the solution from the last iteration stage. The Gauss-Seidel method sequentially updates the parameters using the most recent solutions.

Implemented in a single processing centre, these algorithms may simply offer a computational advantage over Newton's method. A significant additional benefit is achieved when it is realised that these algorithms can be implemented in a distributed fashion across a network of processors. Each processor node computes and communicates updates of its associated parameter. Calling each node a decision maker with a decision variable described by the parameter completes the analogy between distributed optimisation implementations and team decision problems.

Large-scale optimisation methods solve problems by decomposition into subproblems. Retaining coupling between sub-problems ensures the global result is achieved. Decomposition may be more efficient than global solution methods since it avoids directly trying to solve for the entire problem solution vector. Over a sequence of iterations, each sub-problem effectively 'learns' the coupling in a small region of the solution space, from the reaction of other sub-problems. Thus, the concept of learning equilibria in game theory [38]. Each robot in a multi-robot system is analogous to the subproblems in a numerical optimisation method. This analogy extends to more complex optimisation problems and solution methods. Further detailed analysis of distributed *block-iterative* algorithms is provided by Bertsekas [6].

2.5.1 Negotiation Through Better Response

Better-response negotiation is proposed as a method for finding pure Nash equilibria. The solution involves deterministic or stochastic [2] fixed point iterative schemes. Individual

[2] Stochastic methods are not discussed here. Probabilistic update methods are more computationally intensive and not decentralisable. Their advantage is provable global convergence.

decision makers act in manner that is the best response to the actions of other decision makers. A key feature of the Nash solution is that it is the *best response* to itself. Wilson [100] shows that these schemes are applicable if the problem has a finite number of Nash equilibria [3]. The form of the updated solution for the i^{th} decision maker is

$$\boldsymbol{u}_i^k = (1 - \boldsymbol{\kappa_k})\boldsymbol{u}_i^{k-1} + \boldsymbol{\kappa_k}\mathcal{B}_i(\bar{\mathbf{u}}), +\boldsymbol{\beta_k} \tag{2.19}$$

where k is the index in the iteration process. $\bar{\mathbf{u}} = \bar{\mathbf{u}}_1, \ldots, \bar{\mathbf{u}}_j, \ldots, \bar{\mathbf{u}}_n$, $j \neq i$ is the fixed action of the other decision makers. $\mathcal{B}(\bar{\mathbf{u}})$ is known as the *best response function*. $\boldsymbol{u}_i^\star = \mathcal{B}_i(\bar{\mathbf{u}})$ is the action that optimises the i^{th} decision makers expected utility with respect to the other team members actions. $\{\boldsymbol{\kappa_k} : 0 < \boldsymbol{\kappa_k} \leq 1\}$ is a step size that sets how far the solution moves towards the best response. $\boldsymbol{\beta_k}$ is an error that may be introduced to help escape unstable equilibria or weak local optimum solutions. Additionally, the order in which the decision makers update and communicate their actions determines $\bar{\mathbf{u}}$ on which the new decision is based. Generalising the terminology used in iterative solutions to linear equations, the scheme is referred to as *Jacobi* if the updates are synchronous, *Gauss-Seidel* if the updates are made sequentially using the latest available information and *Randomised-Gauss-Seidel* if the team members update their decisions according to a randomised order. Together, the schedules of $\boldsymbol{\kappa_k}$, $\boldsymbol{\beta_k}$ and the individual decision updates determine the region of contraction for a method and corresponding solution convergence rate. Two implementations are:

1. *Best Response.* $\boldsymbol{\kappa_k} = 1$, $\boldsymbol{\beta_k} = 0$ $\forall k$. The iteration update is the full best response.

2. *Deterministic Better Response.* $\boldsymbol{\kappa_k}$ and $\boldsymbol{\beta_k}$ are variables (possibly random) that are scheduled according to deterministic functions of the iteration number. $\boldsymbol{\kappa_k} \longrightarrow \mathbf{1}$ and $\boldsymbol{\beta_k} \longrightarrow \mathbf{0}$ as k increases.

[3] If not the problem is under-determined and the optimal actions are a function of each other.

The exact best response implementation suffers three weaknesses. Firstly, unstable non-optimal stationary points (saddle or singular points) may be fixed points of the sequential iteration. Secondly, the synchronous iteration may oscillate between stationary points or about symmetries between stationary solutions. Thirdly, full best response may only converge in a very small region about the true solution. These weaknesses can be overcome by relaxation and/or perturbation of the iteration process. Hence, adoption of the *better response* over *best response* implementation. Sequential updates break the oscillations that potentially occur in synchronous schemes. Randomising the update order breaks equilibria that arise from the sequential process and are not Nash solutions.

Given $\boldsymbol{\kappa_k}$ and $\boldsymbol{\beta_k}$ that provide a convergent sequence, convergence to solution may be undesirably slow. It is reasonable to expect that the utility function may be approximated by a polynomial in the region of the solution. This motivates the use of a *line search* procedure to accelerate solution convergence. A quadratic fit can be used to interpolate or extrapolate a solution update from three actions and their corresponding utilities. The procedure is illustrated in Figure 2.6 and the update is given by.

$$\boldsymbol{u}_i^{k+1} = \frac{1}{2}\frac{\boldsymbol{\alpha}_{12}\mathbf{J}(\boldsymbol{u}_i^k, \bar{\mathbf{u}}) + \boldsymbol{\alpha}_{20}\mathbf{J}(\boldsymbol{u}_i^{k-1}, \bar{\mathbf{u}}) + \boldsymbol{\alpha}_{01}\mathbf{J}(\boldsymbol{u}_i^{k-2}, \bar{\mathbf{u}})}{\boldsymbol{\mu}_{12}\mathbf{J}(\boldsymbol{u}_i^k, \bar{\mathbf{u}}) + \boldsymbol{\mu}_{20}\mathbf{J}(\boldsymbol{u}_i^{k-1}, \bar{\mathbf{u}}) + \boldsymbol{\mu}_{01}\mathbf{J}(\boldsymbol{u}_i^{k-2}, \bar{\mathbf{u}})} \tag{2.20}$$

where $\boldsymbol{\alpha}_{lm} = \boldsymbol{u}_i^{k-l} - \boldsymbol{u}_i^{k-m}$ and $\boldsymbol{\mu}_{lm} = (\boldsymbol{u}_i^{k-l})^2 - (\boldsymbol{u}_i^{k-m})^2$.

2.5.2 Solving Linear Equations as a Team Problem

A classical problem is used to highlight the connection between team decision problem and optimisation. Special cases in this problem emphasise the importance of considering the structure of utility in the solution process. Consider the system of n linear equations

$$\mathbf{A}\mathbf{u} = \mathbf{b}, \quad \mathbf{u} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\}^T. \tag{2.21}$$

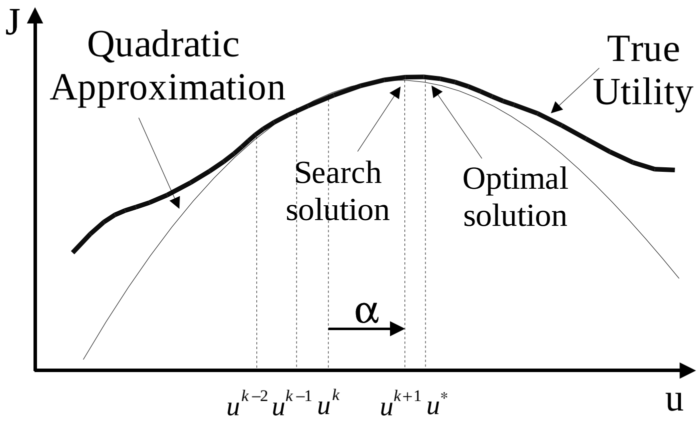


Figure 2.6: Quadratic line search procedure for accelerated convergence

The solution is equivalent to maximising the sum of squares utility measures

$$\begin{aligned}\mathbf{J}(\mathbf{u}) &= -(\mathbf{A}\mathbf{u}-\mathbf{b})^T(\mathbf{A}\mathbf{u}-\mathbf{b}) \\ &= -\sum_{i=1}^{n}(\mathbf{A}_i\mathbf{u}-\mathbf{b}_i)^2 \\ &= \sum_{i=1}^{n}\mathbf{J}_i(\boldsymbol{u}_i) \qquad (2.22)\end{aligned}$$

where $\mathbf{A}_i$ and $\mathbf{b}_i$ are the i^{th} row and element of $\mathbf{A}$ and $\mathbf{b}$. Two well known numerical solutions to this problem are Jacobi and Gauss-Seidel iteration. These can be expressed in terms of a decomposition of the matrix $\mathbf{A}$. Let $\mathbf{A} = \mathbf{D}+\mathbf{L}+\mathbf{U}$, where $\mathbf{D}$ are the diagonal elements of $\mathbf{A}$, $\mathbf{L}$ are the elements below the diagonal and $\mathbf{U}$ are the elements above the diagonal. These algorithms are:

$$\mathbf{Jacobi}: \quad \mathbf{u}^{k+1} = \mathbf{D}^{-1}\left[\mathbf{b}-(\mathbf{L}+\mathbf{U})\mathbf{u}^k\right] \qquad (2.23)$$

$$\mathbf{Gauss-Seidel}: \quad \mathbf{u}^{k+1} = (\mathbf{D}+\mathbf{L})^{-1}\left[\mathbf{b}-\mathbf{U}\mathbf{u}^k\right]. \qquad (2.24)$$

These are *best-response* procedures that update $\boldsymbol{u}_i^{k+1}$ by maximising $\mathbf{J}_i(\boldsymbol{u}_i^k, \bar{\mathbf{u}}^k)$ of Equation 2.22. This zeros the error in the ith equation with respect to the values of the other parameters. Not accounting for the effect $\boldsymbol{u}_i$ has on the error in all other equations. The Jacobi method updates simultaneously and the Gauss-Seidel method updates sequentially.

Substituting the Jacobian and Hessian of $\mathbf{J}(\mathbf{u})$

$$\nabla_{\mathbf{u}}\mathbf{J}(\mathbf{u}^k) = -2\mathbf{A}^T(\mathbf{A}\mathbf{u} - \mathbf{b}), \quad \nabla^2_{\mathbf{u}}\mathbf{J}(\mathbf{u}^k) = -2\mathbf{A}^T\mathbf{A}$$

into the generalised Jacobi and Gauss-Seidel methods of Equations 2.17 and 2.18 generates schemes that update each parameter accounting for their effect on errors in all equations. These are *best-response* procedures that update $\boldsymbol{u}_i^{k+1}$ by maximising $\mathbf{J}(\boldsymbol{u}_i^k, \bar{\mathbf{u}}^k)$ of Equation 2.22.

All of these approaches are *best-response* procedures derived from different utility structures. In the generalised approaches, each decision is made based on the same global utility measure. The other methods update decisions based on decomposed individual utilities. The implication for team decision making is that different utility structures can represent the problem but result in solution procedures with different convergence properties. Example solution trajectories are shown in Figure 2.7.

The structure of the matrix $\mathbf{A}$ determines the coupling among parameters in utility. Special cases provide insight to the complexity for different solution approaches. Consider:

$\mathbf{A}$ Diagonal:	$\Rightarrow$	All parameters are independent. No negotiation required. All approaches yield solution on first step.
$\mathbf{A}^T\mathbf{A}$ Diagonal:	$\Rightarrow$	The Hessian is diagonal. Generalised approaches yield solution on first step.
$\mathbf{A}$ Block Diagonal:	$\Rightarrow$	Coupling is within disjoint sub-sets of the parameter vector.
$\mathbf{A}$ Lower Triangular:	$\Rightarrow$	Coupling is hierarchical. A single pass of a sequential algorithm provides the solution
$\mathbf{A}$ Singular:	$\Rightarrow$	Infinite Nash equilibria. No Negotiated solution.

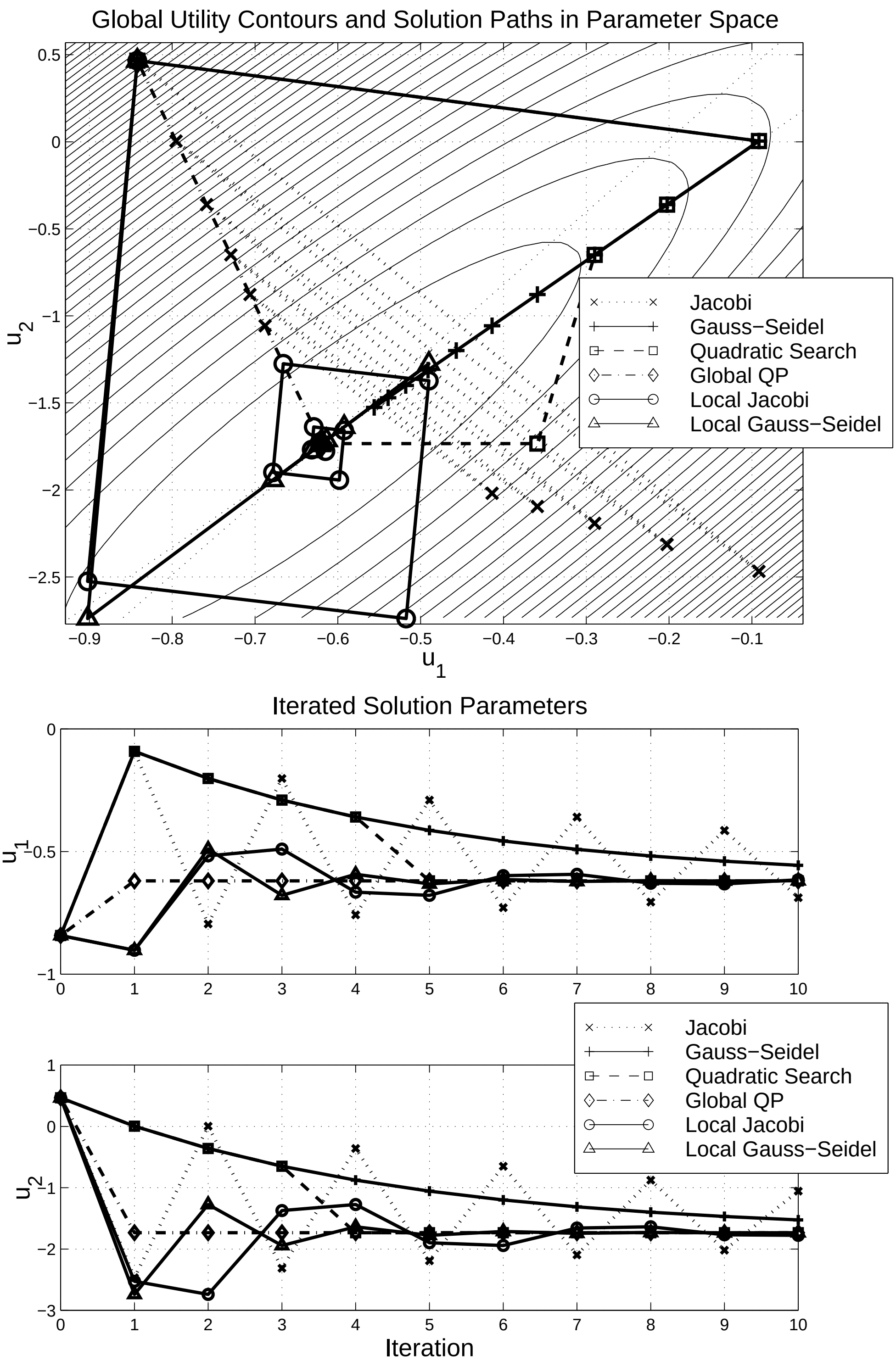


Figure 2.7: Solving a 2D system of linear equations as a team. Classical iterative solution methods are *best-response* procedures based on different utility and information structures. Algorithms that perform solution updates based on the effect on an individual equation are compared to those that consider the effect on all equations. On the 4th iteration, the second decision maker implements a quadratic line search. For this quadratic problem the line search immediately results in the exact solution.

2.6 Summary

This chapter defined a decentralised approach to multi-robot systems. The formulation of the team decision problem was presented. An overview is shown in Figure 2.8. The key elements in team decision making were identified as: modelling of the environment, sensors and vehicles; specification of communication structures; capturing team utility; parameterisation of actions and devising solution procedures. The remainder of this thesis develops this basis into a consistent approach to the control of cooperative sensor teams.

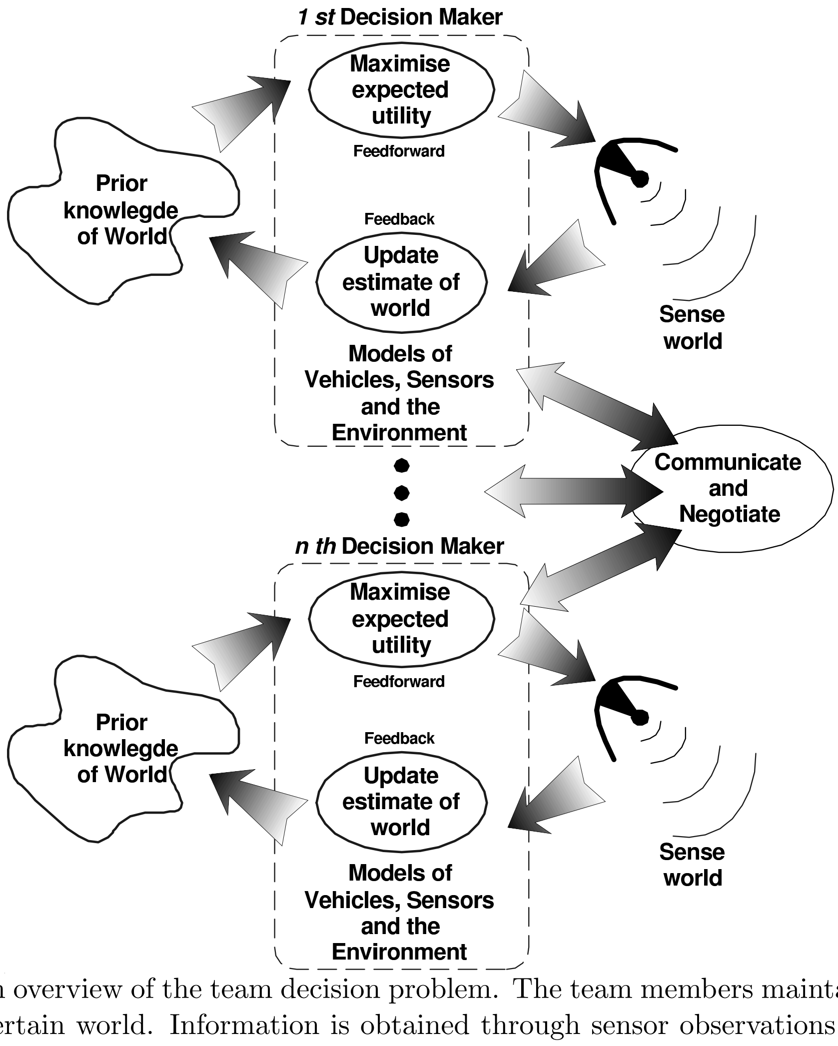


Figure 2.8: An overview of the team decision problem. The team members maintain information about an uncertain world. Information is obtained through sensor observations and communication. Probabilistic modelling of the environment, sensors and vehicles provides an means to capture and predict the expected utility for a planned sequence of actions. The decision makers jointly optimise their utilities through communication and negotiation.

Chapter 3

Measuring and Fusing Information

3.1 Introduction

Sensing tasks involve the gathering, exchange, evaluation and combination of information. Sensors are used to make observations of physical quantities with the objective of obtaining an estimate of some state of the world. Uncertainty lies at the heart of the sensing and estimation problem. Probabilistic methods can be used to combine measurement information with models of the sensors, vehicles and the environment. With a probabilistic model of information, estimates of an underlying state may be obtained in a coherent and principled manner. Given a probabilistic method for fusing sensor information and measures that quantify uncertainty, it is logical to ask: what sensing action should team members take to minimise the group uncertainty or, alternatively, to maximise group information?

Information in terms of uncertainty is formally defined in Section 3.2 through the Shannon and Fisher information measures. The 'information' or inverse covariance form of the estimation problem is introduced in Section 3.3. Section 3.4 extends the use of information measures to the general data fusion problem and illustrates the process of loss and gain of information. The issue of valuing and selecting a potential sensing action is addressed in Section 3.5. This includes consideration of actions made by individuals within a group of sensors. Finally in Section 3.6, these components are a applied to a practical multi-sensor decision problem. This sensor management task, aims to allocate limited sensing resources in order to maximise system information.

3.2 Information Measures

Estimation and control problems inherently deal with uncertainty of states, observations and actions. Uncertainty in these quantities is most usually described in terms of a probability distribution. It is essential to provide a measure of the informativeness of the probability distributions associated with the data fusion task. Two formal definitions of information are of particular practical use; the Shannon information (or entropy) and the Fisher information. Both of these measures evaluate the 'information' contained in a probability distribution in terms of its compactness. Both measures are a function of the distribution, rather than the underlying state or observation.

3.2.1 Entropic Information

The entropy or Shannon information $H(\mathbf{x})$ associated with a probability distribution $P(\mathbf{x})$, defined on a random variable $\mathbf{x}$, is defined as the expected value of minus the log-likelihood. For continuous-valued random variables this is given by (see [66] Chapter 15)

$$H(\mathbf{x}) \stackrel{\triangle}{=} -\mathrm{E}\{\log P(\mathbf{x})\} = -\int_{-\infty}^{\infty} P(\mathbf{x}) \log P(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{3.1}$$

The convention $H(\mathbf{x})$ is used to indicate entropy associated with the variable $\mathbf{x}$. The integral is taken over all values of $\mathbf{x}$ so $H_P(\cdot)$ is not strictly a function of $\mathbf{x}$ but rather the distribution $P(\cdot)$.

Entropic information $\boldsymbol{i}(\mathbf{x})$ is defined as the negative of entropy. Information is a maximum when entropy is a minimum

$$\boldsymbol{i}(\mathbf{x}) = -H(\mathbf{x}). \tag{3.2}$$

When $\mathbf{x}$ is continuous valued, the least informative distribution is uniform.

Entropy is the only reasonable definition of 'informativeness'. An excellent proof of this remarkable result (first shown by Shannon [80]) can be found in [15]. The implication of this is that entropy is a uniquely appropriate measure for evaluating information sources modelled by probabilistic descriptions.

Of particular interest in the following is the entropic information on an n-dimensional state $\mathbf{x}$ modelled by a Gaussian of mean $\overline{x}$ and covariance $\mathbf{P}$ (Equation 2.1)

$$\boldsymbol{i}(\mathbf{x}) = -H(\mathbf{x}) = -\frac{1}{2}\log[(2\pi e)^n |\ \mathbf{P}\ |] \tag{3.3}$$

as shown in Cover [17]. The entropy is proportional to the log of the determinant of the covariance. The determinant of a matrix is a volume measure; the entropy is a measure of the volume enclosed by the covariance matrix and consequently the compactness of the probability distribution.

3.2.2 Fisher Information

A second probabilistic information measure, widely used in estimation, is the Fisher Information. Unlike entropy, Fisher information is only defined on continuous distributions. The Fisher information $\mathcal{J}(x)$ is defined as the second derivative of the log-likelihood

$$\mathcal{J}(\mathbf{x}) = \frac{\mathrm{d}^2}{\mathrm{d}\mathbf{x}^2}\log P(\mathbf{x}). \tag{3.4}$$

For a vector $\mathbf{x}$, $\mathcal{J}(\mathbf{x})$ is a matrix referred to as the Fisher Information Matrix. The Fisher information describes the information contained in the distribution $P(\mathbf{x})$. It measures the surface of a bounding region containing the probability mass. Thus, like entropy, it measures the compactness of a density function. However, entropy is a scalar, volumetric measure. Fisher information is a matrix capturing the axes or area of the bounding surface.

Equation 3.4 can be used to determine the Fisher information for a Gaussian (normal) probability distribution $P(\mathbf{x}) = \mathcal{N}(\overline{\mathbf{x}}, \mathbf{P})$, as in Equation 2.1. Taking logs of this distribution and differentiating twice with respect to $\mathbf{x}$ gives $\mathcal{J}(\mathbf{x}) = \mathbf{P}^{-1}$. The Fisher information is simply the inverse covariance. For a Gaussian distribution, this shows the explicit relationship between the Fisher and entropic information measures through the determinant of $\mathbf{P}$

$$\begin{aligned} \boldsymbol{i}(\mathbf{x}) &= -\frac{1}{2}\log[(2\pi e)^n \mid \mathbf{P} \mid] \\ &= \frac{1}{2}\log[(2\pi e)^n \mid \mathcal{J}(\mathbf{x}) \mid]. \end{aligned} \tag{3.5}$$

3.3 Data Fusion

Data fusion seeks to combine information, about a state $\mathbf{x}$, from a variety of sources. The mechanism for combining prior information with observed and communicated information is Bayes theorem.

$$P(\mathbf{x} \mid \mathbf{z}) = \frac{P(\mathbf{z} \mid \mathbf{x})P(\mathbf{x})}{P(\mathbf{z})}. \tag{3.6}$$

The value of this theorem lies in the interpretation of the probability density functions $P(\mathbf{x} \mid \mathbf{z})$, $P(\mathbf{z} \mid \mathbf{x})$ and $P(\mathbf{x})$. Prior beliefs about the state of $\mathbf{x}$ are encoded in the form of relative likelihoods in the prior probability density $P(\mathbf{x})$. To obtain more information about the state $\mathbf{x}$ an observation $\mathbf{z}$ is made. The observations are modelled as a conditional probability density function $P(\mathbf{z} \mid \mathbf{x})$. This describes the likelihood of making observation

$\mathbf{z}$ for fixed state $\mathbf{x}$. The new likelihood associated with the state of the world $\mathbf{x}$ must now be computed from the prior information and the information obtained by observation. This is encoded in the posterior distribution $P(\mathbf{x} \mid \mathbf{z})$ which describes the likelihoods associated with $\mathbf{x}$ *given* the observation $\mathbf{z}$. The marginal distribution $P(\mathbf{z})$ serves to normalise the posterior. The value of Bayes theorem is now clear. It provides a direct means of combining observed information with prior beliefs of the state of the world.

Manyika and Durrant-Whyte have developed the 'Information Form' of the Extended Kalman Filter starting from Bayes theorem [46]. Estimates produced are equivalent to the conventional covariance formulation. However, the information formulation has many properties that make it highly suitable for decentralised multi-sensor data fusion. The derivation of the filter is omitted here. Decentralised Data Fusion (DDF) implementation details are summarised in Appendix B.1.

3.3.1 The Information Filter

The information form of the Kalman filter is obtained by replacing the representation of the state estimate $\hat{\mathbf{x}}$ and covariance $\mathbf{P}$ with the information state $\hat{\mathbf{y}}$ and Fisher information $\mathbf{Y}$. Notation $(i \mid j)$ is introduced to indicate a value at time i, conditional on observation information obtained up to time j. The information state and information matrix are defined as

$$\hat{\mathbf{y}}(i \mid j) \stackrel{\triangle}{=} \mathbf{P}^{-1}(i \mid j)\hat{\mathbf{x}}(i \mid j), \qquad \mathbf{Y}(i \mid j) \stackrel{\triangle}{=} \mathbf{P}^{-1}(i \mid j). \tag{3.7}$$

The state dynamics and observation processes are represented by discrete time versions of the models detailed in Section 2.3.2. In [43] it is shown by means of sufficient statistics (see [66]) that an observation $\mathbf{z}(k)$ at discrete time k, contributes $\mathbf{i}(k)$ to the information state $\hat{\mathbf{y}}$ and $\mathbf{I}(k)$ to the Fisher information $\mathbf{Y}$ where

$$\mathbf{i}(k) \stackrel{\triangle}{=} \mathbf{H}^T(k)\mathbf{R}^{-1}(k)\nu(k), \qquad \mathbf{I}(k) \stackrel{\triangle}{=} \mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{H}(k), \tag{3.8}$$

and where $\nu(k) = \mathbf{z}(k) - \boldsymbol{h}(\hat{\mathbf{x}}(k \mid k-1)$ is the observation innovation. The update stage of discrete time Kalman Filter reduces to

$$\hat{\mathbf{y}}(k \mid k) = \hat{\mathbf{y}}(k \mid k-1) + \mathbf{i}(k), \tag{3.9}$$

$$\mathbf{Y}(k \mid k) = \mathbf{Y}(k \mid k-1) + \mathbf{I}(k). \tag{3.10}$$

The probabilistic representation of prior information, observation likelihood and posterior information used in the information filter can be summarised as:

Prior:	$P(\mathbf{x} \mid \mathbf{Z}^{k-1})$	modelled by	$\hat{\mathbf{y}}(k \mid k-1)$ and $\mathbf{Y}(k \mid k-1)$
Likelihood:	$P(\mathbf{z}(k) \mid \mathbf{x})$	modelled by	$\mathbf{i}(k)$ and $\mathbf{I}(k)$
Posterior:	$P(\mathbf{x} \mid \mathbf{Z}^{k})$	modelled by	$\hat{\mathbf{y}}(k \mid k)$ and $\mathbf{Y}(k \mid k)$.

3.3.2 Multi-Sensor Information Fusion

Fisher information plays an important role in estimation problems involving multiple information sources. In conventional approaches to state estimation, it is difficult to capture the statistical relationships that exist between different estimates produced by different combinations of observations. Accounting for the cross-correlations between observation innovations results in a complex update stage in any multi-sensor Kalman filter implementation. This is most easily overcome by dealing directly with the likelihood functions of the observations. The Fisher information makes explicit the information in the likelihood function.

The contributions to the information state and information matrix made by observations relate directly to the underlying likelihood functions for the states, rather than the estimates themselves. Combined with the assumption that the sensor observations are conditionally independent, this leads to a remarkably simple observation fusion stage for the information filter. For N sensor information sources, the posterior information state and information matrix are obtained from

$$\hat{\mathbf{y}}(k \mid k) = \hat{\mathbf{y}}(k \mid k-1) + \sum_{i=1}^{N} \mathbf{i}_i(k), \tag{3.11}$$

$$\mathbf{Y}(k \mid k) = \mathbf{Y}(k \mid k-1) + \sum_{i=1}^{N} \mathbf{I}_i(k). \tag{3.12}$$

Where $\mathbf{I}_i(k)$ and $\mathbf{i}_i(k)$ are the information matrix and information state contributions of the sensors $i = 1, \ldots, N$. The posterior state estimate may be obtained from

$$\hat{\mathbf{x}}(k \mid k) = \mathbf{Y}^{-1}(k \mid k)\hat{\mathbf{y}}(k \mid k). \tag{3.13}$$

The simple additive nature of the update stage makes the Information filter highly attractive for multi-sensor, decentralised and distributed estimation. This feature is exploited in Decentralised Data Fusion (DDF) architecture developed and demonstrated by Durrant-Whyte *et al.* at the University of Oxford and University of Sydney.

3.4 Measuring Information in the Fusion Process

Entropy can be applied to quantitatively measure the information contained in the probability distributions involved in the fusion process. The entropic information gain associated with an observation is quantified by mutual information. The evolution of information measures in the information fusion process is illustrated through an example of estimating a scalar variable from uncertain observations.

3.4.1 Conditional Entropy

The entropic information measure can be extended to conditional entropy

$$H(\mathbf{x} \mid z) \triangleq \mathrm{E}\{-\log H(\mathbf{x} \mid z)\} \;= -\int_{-\infty}^{+\infty} P(\mathbf{x} \mid z) \log P(\mathbf{x} \mid z) \mathrm{d}\mathbf{x}. \tag{3.14}$$

This describes the information about $\mathbf{x}$ contained in $P(\cdot \mid \mathbf{z})$ following an observation z. $H(\mathbf{x} \mid \mathbf{z})$ is a function of $\mathbf{z}$ and as such depends on the value of the observation made.

The mean conditional entropy, $\overline{H}(\mathbf{x} \mid \mathbf{z})$, taken over all possible values of $\mathbf{z}$, is given by

$$\begin{aligned} \overline{H}(\mathbf{x} \mid \mathbf{z}) &\triangleq \mathrm{E}\{H(\mathbf{x} \mid \mathbf{z})\} \\ &= -\int_{-\infty}^{+\infty} P(z) \int_{-\infty}^{+\infty} P(\mathbf{x} \mid \mathbf{z}) \log P(\mathbf{x} \mid \mathbf{z}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{z} \\ &= -\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P(\mathbf{x}, \mathbf{z}) \log P(\mathbf{x} \mid \mathbf{z}) \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{z}. \end{aligned} \tag{3.15}$$

Note that $\overline{H}(\mathbf{x} \mid \mathbf{z})$ is not a function of either $\mathbf{x}$ or $\mathbf{z}$. It is a essentially a measure of the information that will be obtained (on the average) by making an observation before the value of the observation is known.

3.4.2 Mutual Information

With these definitions of entropy and conditional entropy, it is possible to write an 'information form' of Bayes theorem. Taking expectations of the logs of both sides of Equation 3.6 with respect to both the state $\mathbf{x}$ and the observation $\mathbf{z}$ gives

$$\overline{H}(\mathbf{x} \mid \mathbf{z}) = \overline{H}(\mathbf{z} \mid \mathbf{x}) + H(\mathbf{x}) - H(\mathbf{z}). \tag{3.16}$$

Simply, this describes the change in entropy or information following an observation from a sensor modelled by the likelihood $P(\mathbf{z} \mid \mathbf{x})$.

Being able to describe changes in entropy leads naturally to asking the important question: what is the most informative observation one can make? This question may be answered through the idea of mutual information.

The mutual information $\mathcal{I}(\mathbf{x}, \mathbf{z})$ obtained about a random variable $\mathbf{x}$ with respect to a second random variable $\mathbf{z}$ is now defined as

$$\begin{aligned} \mathcal{I}(\mathbf{x}, \mathbf{z}) &= -\mathrm{E}\{\log \frac{P(\mathbf{x}, \mathbf{z})}{P(\mathbf{x})P(\mathbf{z})}\} \\ &= -\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P(\mathbf{x}, \mathbf{z}) \log \frac{P(\mathbf{x}, \mathbf{z})}{P(\mathbf{x})P(\mathbf{z})} \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{z}. \end{aligned} \tag{3.17}$$

Mutual information is an *a priori* measure of the information to be gained through observation. The expectation is taken over $\mathbf{z}$ and $\mathbf{x}$, so the mutual information gives an average measure of the gain to be expected *before* making the observation.

As

$$\frac{P(\mathbf{x}, \mathbf{z})}{P(\mathbf{x})P(\mathbf{z})} = \frac{P(\mathbf{x} \mid \mathbf{z})}{P(\mathbf{x})}, \tag{3.18}$$

mutual information may be written in the alternative forms

$$\mathcal{I}(\mathbf{x}, \mathbf{z}) = -\mathrm{E}\{\log \frac{P(\mathbf{x} \mid \mathbf{z})}{P(\mathbf{x})}\} = -\mathrm{E}\{\log \frac{P(\mathbf{z} \mid \mathbf{x})}{P(\mathbf{z})}\}\ . \tag{3.19}$$

Mutual information is thus a function of the ratio of the density $P(\mathbf{x} \mid \mathbf{z})$ following an observation to the prior density $P(\mathbf{x})$. If $\mathbf{x}$ and $\mathbf{z}$ are independent, then $P(\mathbf{x} \mid \mathbf{z}) = P(\mathbf{x})$ and the expressions in Equation 3.18 become equal to one and (taking logs) the mutual information becomes equal to zero. This is logical; if knowledge of the state is independent of the observation, the information to be gained by taking an observation (the mutual information) is zero. Conversely, as $\mathbf{x}$ becomes more dependent on $\mathbf{z}$, then $P(\mathbf{x} \mid \mathbf{z})$ becomes more peaked or compact relative to the prior distribution $P(\mathbf{x})$ and so mutual information increases. Note that mutual information is always positive (it is not possible to lose information by taking observations). Equation 3.17 can be written in terms of the component entropies as

$$\begin{aligned}\mathcal{I}(\mathbf{x}, \mathbf{z}) &= H(\mathbf{x}) + H(\mathbf{z}) - H(\mathbf{x}, \mathbf{z}) \\ &= H(\mathbf{x}) - \overline{H}(\mathbf{x} \mid \mathbf{z}) \\ &= H(\mathbf{z}) - \overline{H}(\mathbf{z} \mid \mathbf{x}).\end{aligned} \tag{3.20}$$

Equation 3.20 measures the 'compression' of the probability mass caused by an observation. Mutual information provides an average measure of how much more information we would have about the random variable $\mathbf{x}$ if the value of $\mathbf{z}$ where known. Most importantly mutual information provides a *pre-experimental* measure of the usefulness of obtaining information (through observation) about the value of $\mathbf{z}$.

3.4.3 Information Evolution in Estimation

The evolution of information over time is of significant interest in sensing problems. Filtering approaches to estimation recursively calculate successive estimates of a state that evolves over time. This is implemented through periodic prediction and observation of that state. The entropic information and mutual information measures developed can be applied to quantify information gains and losses in the estimation process.

The information state and information matrix can be predicted forward in time through the stochastic process model (Equation 2.2). In continuous time the evolution of the Fisher Information can be described as the solution to the 'information form' of the Riccati equation

$$\dot{\mathbf{Y}}(t) = -\mathbf{F}(t)\mathbf{Y}(t) - \mathbf{F}^T(t)\mathbf{Y}(t) - \mathbf{Y}(t)\mathbf{G}(t)\mathbf{Q}(t)\mathbf{G}^T(t)\mathbf{Y}(t) + \sum_{i=1}^{N} \mathbf{H}_i^T(t)\mathbf{R}_i^{-1}(t)\mathbf{H}_i(t). \tag{3.21}$$

The Fisher information in the discrete time information filter is governed by the following prediction and updates stages [1] (see Appendix B.1)

[1]For a suitably small time increment Δt, the discrete time system matrices are approximated by $\mathbf{F}(k) = \exp\{\mathbf{F}(t)\Delta t\}$, $\mathbf{Q}(k) = \mathbf{G}^T(t)\mathbf{Q}(t)\mathbf{G}(t)\Delta t$ and $\mathbf{R}(k) = \frac{1}{\Delta t}\mathbf{R}(t)$

Prediction:

$$\mathbf{Y}(k \mid k-1) = \left[\mathbf{F}(k)\mathbf{Y}^{-1}(k \mid k-1)\mathbf{F}^T(k) + \mathbf{Q}(k)\right]^{-1} \tag{3.22}$$

Update:

$$\mathbf{Y}(k \mid k) = \mathbf{Y}(k \mid k-1) + \sum_{i=1}^{N} \mathbf{I}_i(k) \tag{3.23}$$

Equations 3.22, 3.23 and 3.21 indicate how the system dynamics, process noise and observation affect the information in the system. Since $\mathbf{Q}$ and $\mathbf{R}$ are positive semi-definite matrices, process noise cannot gain information and observation cannot lose information. It is most interesting to note that the system dynamics $\mathbf{F}$ can contribute either an information loss or gain over time.

The entropic information in this process at the prediction and updates stages is measured by

$$\textbf{Posterior Information}: \quad \boldsymbol{i}(k) = \frac{1}{2}\log[(2\pi e)^n \mid \mathbf{Y}(k \mid k) \mid], \tag{3.24}$$

$$\textbf{Prior Information}: \quad \boldsymbol{i}(k \mid k-1) = \frac{1}{2}\log[(2\pi e)^n \mid \mathbf{Y}(k \mid k-1) \mid]. \tag{3.25}$$

The entropic information change associated with transitions between probability distributions is the mutual information. There are three transitions of interest in the filter predict-update cycle illustrated in Figure 3.1: the information change in the prediction stage; the information gain associated with observation; and the overall dispersion or concentration of information for the combined stages. The corresponding mutual information measures are:

$$\textbf{Process Mutual Info.}: \quad \mathcal{I}(k, \mathbf{x}(k-1)) = \frac{1}{2}\log\left[\frac{\mid \mathbf{Y}(k \mid k-1) \mid}{\mid \mathbf{Y}(k-1 \mid k-1) \mid}\right] \tag{3.26}$$

$$\textbf{Observation Mutual Info.}: \quad \mathcal{I}(k, \mathbf{z}(k)) = \frac{1}{2}\log\left[\frac{\mid \mathbf{Y}(k \mid k) \mid}{\mid \mathbf{Y}(k \mid k-1) \mid}\right] \tag{3.27}$$

$$\textbf{Mutual Information}: \quad \mathcal{I}(k) = \frac{1}{2}\log\left[\frac{\mid \mathbf{Y}(k \mid k) \mid}{\mid \mathbf{Y}(k-1 \mid k-1) \mid}\right] \tag{3.28}$$

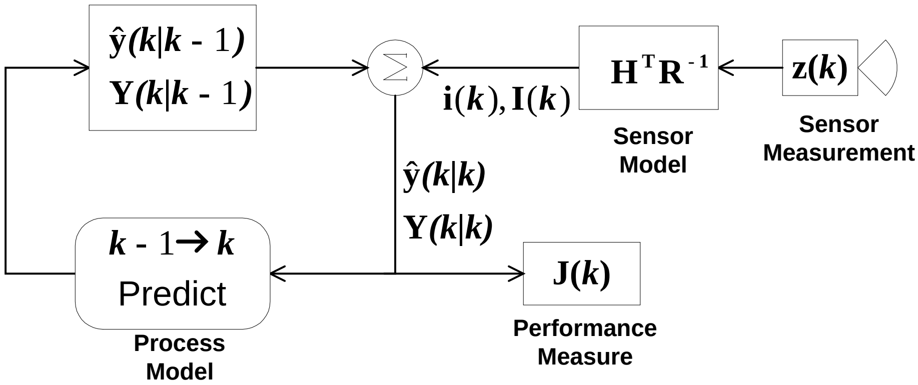


Figure 3.1: Information is lost and gained in the data fusion prediction and observation cycle. Entropic information and mutual information provide measures of the performance of the estimation process and the contributions made by sensor measurements.

The rate of change of entropic information can be determined from the Fisher information rate (Equation 3.21). Using matrix calculus identities from [42], the instantaneous rate of change of entropy, or mutual information rate is

$$\dot{\mathcal{I}}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log \mid \mathbf{Y}(t) \mid = \frac{1}{2}\mathrm{trace}\left(\mathbf{Y}^{-1}(t)\dot{\mathbf{Y}}(t)\right). \qquad (3.29)$$

This provides a measure analogous to mutual information in continuous time. It is not of practical use in determining mutual information over time as this can be found directly from the Fisher information. Equation 3.29 can be separated into contributions from the process model and observations

$$\dot{\mathcal{I}}(t, \mathbf{x}) = \frac{1}{2}\mathrm{trace}\left(\mathbf{Y}^{-1}\left(-\mathbf{FY} - \mathbf{F}^T\mathbf{Y} - \mathbf{YGQG}^T\mathbf{Y}\right)\right), \qquad (3.30)$$

$$\dot{\mathcal{I}}(t, \mathbf{z}) = \frac{1}{2}\mathrm{trace}\left(\mathbf{Y}^{-1}\sum_{i=1}^{N}\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{H}_i\right). \qquad (3.31)$$

3.4.4 Illustration of Information Measures

A simple process is presented to illustrate the temporal evolution of information in an estimator. A sensor is required to estimate the value of a scalar characteristic **x** through observation $\mathbf{z}$. The feature characteristic dynamics are assumed to be governed to a first order Gauss-Markov process

$$\dot{\mathbf{x}}(t) = -\boldsymbol{\beta}(\mathbf{x}(t) - \mathbf{w}(t)), \tag{3.32}$$

where [2]

$$\mathrm{E}\{\mathbf{w}(t)\} \ = \overline{\mathbf{w}}(t) = 0, \quad \mathrm{E}\{[\mathbf{w}(t) - \overline{\mathbf{w}}(\tau)]^2\} \ = \mathbf{Q}\delta(t-\tau).$$

The observation model is

$$\mathbf{z}(t) = \mathbf{x}(t) + \mathbf{v}(t) \tag{3.33}$$

where

$$\mathrm{E}\{\mathbf{v}(t)\} \ = \overline{\mathbf{v}}(t) = 0, \quad \mathrm{E}\{[\mathbf{v}(t) - \overline{\mathbf{v}}(\tau)]^2\} \ = \mathbf{R}(t)\delta(t-\tau), \quad \mathrm{E}\{\mathbf{w}(t)\mathbf{v}(\tau)\} \ = 0 \ \forall \ t, \tau.$$

The Fisher information $\mathbf{Y}(t)$ for this process is governed by the following scalar Riccati equation

$$\dot{\mathbf{Y}}(t) = 2\boldsymbol{\beta}\mathbf{Y}(t) - \boldsymbol{\beta}^2\mathbf{Q}\mathbf{Y}^2(t) + \mathbf{R}^{-1}(t).$$

For constant observation information $\mathbf{R}^{-1}$ and $\mathbf{Y}(0) \neq 0$, the analytic solution is

$$\mathbf{Y}(t) = \frac{1}{\boldsymbol{\beta}\mathbf{Q}}\left(\tanh\left(t\boldsymbol{\beta}\boldsymbol{\varsigma} + \operatorname{arctanh}\left(\frac{\mathbf{Y}(0)\boldsymbol{\beta}\mathbf{Q} - 1}{\boldsymbol{\varsigma}}\right)\right)\boldsymbol{\varsigma} + 1\right) \tag{3.34}$$

where $\boldsymbol{\varsigma} = \sqrt{\mathbf{R}^{-1}\mathbf{Q} + 1}$. The steady state solution to Equation 3.34 provides an upper bound on the information gathered through observation

$$\mathbf{Y}_{upper} = \lim_{t\to\infty}\mathbf{Y}(t) = \frac{\sqrt{\mathbf{R}^{-1}\mathbf{Q} + 1} + 1}{\mathbf{Q}\boldsymbol{\beta}}. \tag{3.35}$$

If observations are are stopped ($\mathbf{R}^{-1}(t) = 0$) at time $t = \tau$, the process loses information according to

$$\mathbf{Y}(t) = 2\left(\boldsymbol{\beta}\mathbf{Q} + \left(\frac{2}{\mathbf{Y}(\tau)} - \boldsymbol{\beta}\mathbf{Q}\right).e^{-2\boldsymbol{\beta}(t-\tau)}\right)^{-1}. \tag{3.36}$$

[2]The information measures used are applicable to uncertainty described by general probability distributions. Gaussian modelling significantly simplify the analysis.

The lower bound on information lost given by the steady state solution to Equation 3.36

$$\mathbf{Y}_{lower} = \lim_{t \to \infty} \mathbf{Y}(t) = \frac{2}{\beta \mathbf{Q}}. \tag{3.37}$$

The linearised discrete time solution is

$$\mathbf{Y}(k \mid k-1) = \mathbf{\Phi}(k)\mathbf{Y}(k-1 \mid k-1)$$

$$\mathbf{Y}(k \mid k) = \mathbf{Y}(k \mid k-1) + \mathbf{\Gamma}(k)\mathbf{R}^{-1}(t)$$

where

$$\mathbf{\Phi}(k) = e^{(2\boldsymbol{\beta} - \boldsymbol{\beta}^2 \mathbf{Q}\mathbf{Y}(k-1|k-1))\Delta T} \quad \text{and}$$

$$\mathbf{\Gamma}(k) = \frac{e^{(2\boldsymbol{\beta} - \boldsymbol{\beta}^2 \mathbf{Q}\mathbf{Y}(k-1|k-1))\Delta T} - 1}{2\boldsymbol{\beta} - \boldsymbol{\beta}^2 \mathbf{Q}\mathbf{Y}(k-1 \mid k-1)} \approx \Delta T. \quad [3]$$

Example Solution

An example solution for this process is shown in Figure 3.2. The parameters used are [4]

$$\mathbf{Q} = .1, \quad \boldsymbol{\beta} = .1, \quad \mathbf{R}^{-1} = 100, \quad \mathbf{Y}(0) = 150, \quad and \quad \Delta T = .5s.$$

Information gain and loss are shown over a 20 second time period. At $t = 8$ seconds observations are stopped.

The solution illustrates the 'information dynamics' that result as an output from the combined state dynamics, process noise and observations. Although the observation information is constant in the Fisher sense, the mutual information depends on the entropic information level. The information gain associated with observation is higher when in-

[3] $\mathbf{\Gamma}(k) \to \Delta T$ as the solution approaches steady state. Care should be applied in the discretisation to preserve the true contribution and DC gain of the observations.

[4] The parameters in this example are not given units. The situation corresponds to the generic task of estimating a arbitrary characteristic. The units are application specific.

formation is low. At steady state the information gain through observation equals the information loss through the process model.

The actual estimated value is deliberately not presented here. This is to emphasise a property of entropy and mutual information. The solution presented is the information that will be obtained over time, on average, by a system modelled by Equations 3.32 and 3.33. A particular observation innovation sequence may result in different *posterior* information. The value of the presented solution is that it provides a prediction of the expected information obtained, before the actual observations are made.

This example corresponds to a highly simplified situation. In general, the state dynamics, process noise and observation information are dependent on the state and control inputs, alluding to the possibility of using the control inputs to influence the evolution of information.

3.5 The Utility of Information

Natural measures of the information obtained through observation in a sensing task are provided by entropy and mutual information. In executing this task, a set of actions corresponding to different sensor configurations are available to the system. The system then must decide on actions that maximise some measure of utility. The goal is typically maximising information gain. It is therefore logical to establish a relationship between the utility of decisions and these information quantities.

Each decision maker has a an action $\mathbf{u} \in \mathbf{U}$ available to it. This is a general notion. The set $\mathbf{U}$ may be a continuous subspace $\mathbf{U} \subset \mathbb{R}^s$ or a set of m discrete s-tuples $\mathbf{U} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. In general, the effect of each possible action $\mathbf{u} \in \mathbf{U}$ is to induce a posterior probability distribution $P(\mathbf{x} \mid \mathbf{u})$ on the state $\mathbf{x}$. A utility function $\mathcal{U}(\mathbf{x}, \mathbf{u})$ places a value on each action.

Utility theory can be used to encode a preferential ordering over the actions available.

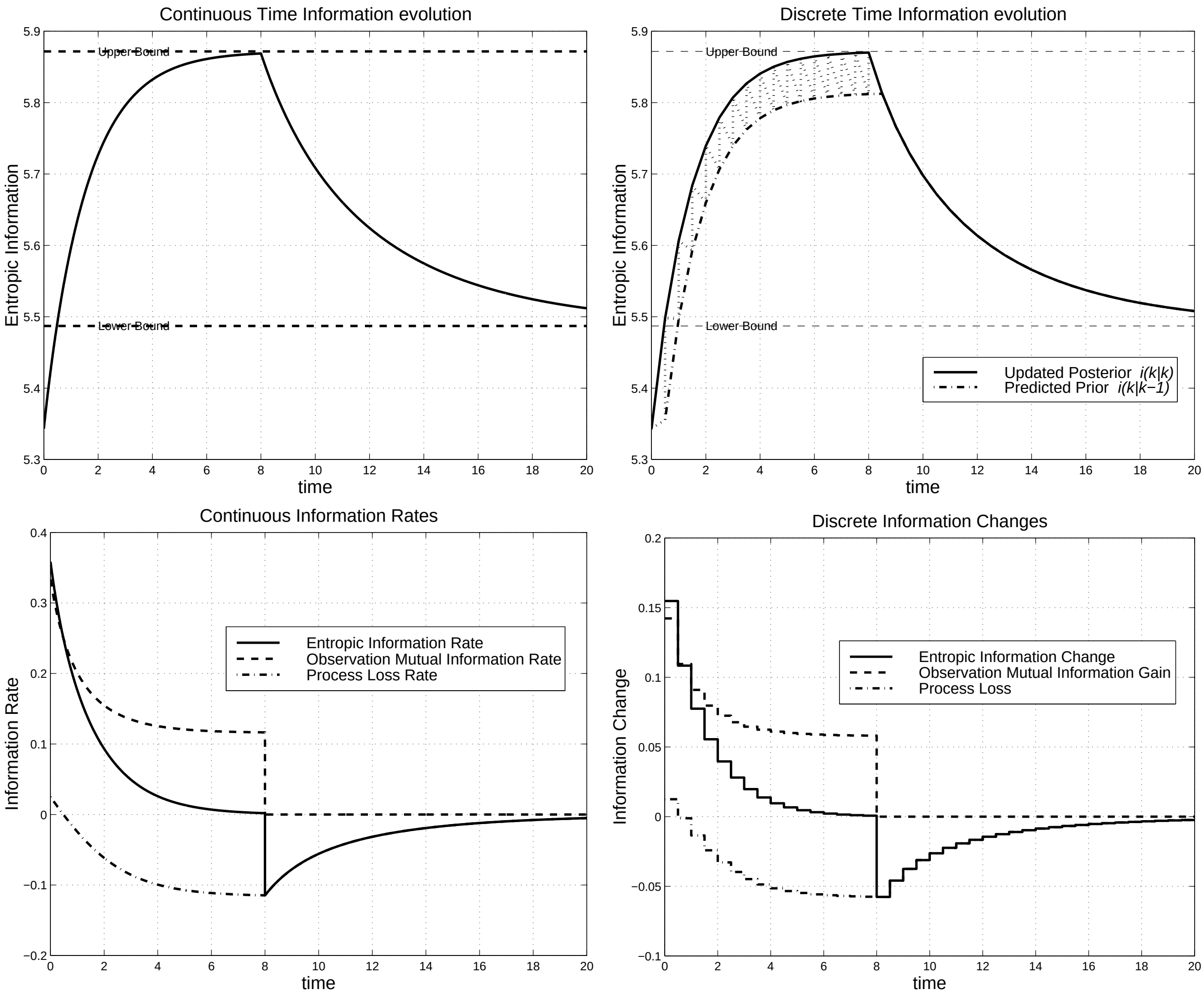


Figure 3.2: Illustration of information measures over continuous and discrete time. The solutions presented indicate the expected information gain for a system modelled by a first order Gauss-Markov process (Equations 3.32 and 3.33). The value of this is that it provides a prediction of the information that will be obtained (on average) from a sequence of observations, before the observations are made.

A usable utility measure is defined by the concept of Bayes expected utility (defined in Section 2.3.3). Decisions are made through the maximisation of this expected utility over all possible actions. The optimal action (or Bayes action) is defined through

$$\begin{aligned} \mathbf{u}^{\star} &= \arg\max_{\mathbf{u}\in\mathbf{U}} \mathbf{J}(\mathbf{u}) \\ &= \arg\max_{\mathbf{u}\in\mathbf{U}} \mathrm{E}\{\mathcal{U}(\mathbf{x},\mathbf{u})\}\ . \end{aligned} \tag{3.38}$$

It remains to determine appropriate utility functions for the sensing problem.

3.5.1 Entropy as Expected Utility

Consider the posterior density $P(\mathbf{x} \mid \mathbf{u})$ on a state of interest $\mathbf{x}$, given a sensing action $\mathbf{u}$. An appropriate definition of utility is now provided by the log-likelihood

$$\mathcal{U}(\mathbf{x}, \mathbf{u}) = \log P(\mathbf{x} \mid \mathbf{u}). \tag{3.39}$$

It was shown by Manyika [43] that the log-likelihood satisfies the utility or 'rationality' axioms guaranteeing a preference ordering. Proof that the axioms imply the existence of a utility function can be found in [5].

Taking expected values of Equation 3.39 gives

$$\begin{aligned} \mathbf{J}(\mathbf{u}) &= \mathrm{E}\{\mathcal{U}(\mathbf{x}, \mathbf{u})\} \\ &= \mathrm{E}\{\log P(\mathbf{x} \mid \mathbf{u})\} \\ &= \boldsymbol{i}(\mathbf{x} \mid \mathbf{u}). \end{aligned} \tag{3.40}$$

The efficacy of choosing log-likelihood as utility is now clear. The expected utility is the entropic information. The increase in expected utility for an action is the predicted mutual information or information gain. The Bayes action has the natural interpretation as the action that maximises mutual information or information gain. This allows the information-theoretic tools developed in Section 3.4 to be applied to decision problems involving sensing and communication. Most importantly, this enables the expected utility for an action to be determined *a priori* through the process probabilistic modelling, as indicated in Figure 3.3.

The means of constructing consistent utility functions attracts discord within the decision theory research community. However, in the engineering of autonomous decision

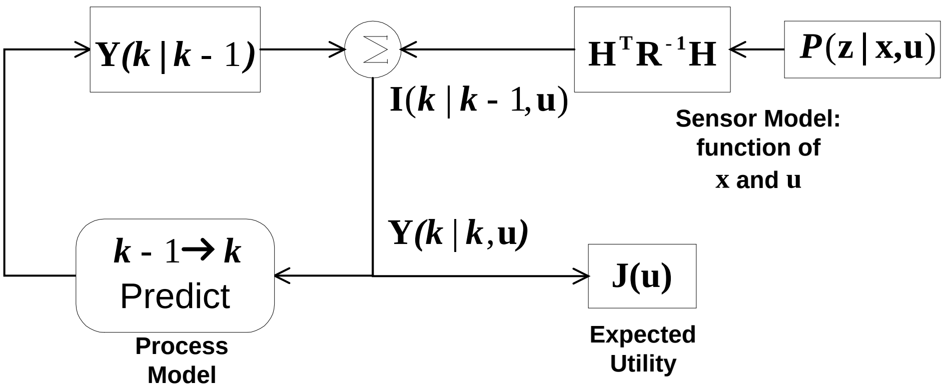


Figure 3.3: The expected utility for a planned sensing action is entropic information. The gain in expected utility is the mutual information or predicted information gain associated with the action. Probabilistic modelling of the system state and observation processes allow the expected reward for an action to be determined *a priori*.

makers, the primary importance is finding utility representations that truly capture the value of the actions. The sensing task presents a situation where this is relatively straight forward. Consistency in the modelling can be verified from the results of the actual measurement process.

3.5.2 Entropy as Team Expected Utility

The argument for entropic information as the expected utility for an individual sensing decision maker extends naturally to a team of decision makers. Consideration of multi-sensor information fusion in Section 3.3 reveals that the posterior Fisher information is simply the sum of the information in the prior and observation likelihoods. The relationship between Fisher information and entropy leads to the following entropic information group sensing expected utility function

$$
\begin{aligned}
\mathbf{J}(\mathbf{u}_1, \ldots, \mathbf{u}_n) &= \mathrm{E}\{\log P(\mathbf{x} \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_n\})\} \\
&= \boldsymbol{i}(\mathbf{x} \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}). \qquad (3.41)
\end{aligned}
$$

The Fisher information update following a group of observation is given by Equation 3.23. Entropic information is obtained from the Fisher information by Equation 3.5. This results in a utility measure for a sensor team in terms of the prior information and observation information corresponding to each action

$$\begin{aligned}
\mathbf{J}(\mathbf{u}_1, \ldots, \mathbf{u}_n, k) &= \boldsymbol{i}(\mathbf{x}(k) \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}) \\
&= \frac{1}{2} \log \left[(2\pi e)^n \left| \, \mathbf{Y}(k \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_n, k\}) \, \right| \right] \\
&= \frac{1}{2} \log \left[(2\pi e)^n \left| \, \mathbf{Y}(k \mid k-1) + \sum_{i=1}^{n} \mathbf{I}_i(k \mid \mathbf{u}_i) \, \right| \right]. \qquad (3.42)
\end{aligned}$$

There is no requirement for each decision maker to maintain a common state $\mathbf{x}$ or have identical prior information $\mathbf{Y}(k \mid k-1)$. Equation 3.42 can be applied to generate individual utility functions immediately usable in the team decision problem formulations of Sections 2.3.

The actions among decision makers are now coupled in utility through the common information resulting from all observations. The information common to the set of observations is the sum of mutual information $\mathcal{I}(\mathbf{u}_i, \mathbf{u}_j)$ between all pairs of actions. A consequence is that a decision maker's optimal action may change if it accounts for the information gathered by other team members. In turn, its actions influence the optimal decisions of other team members.

However, situations exist where the optimal actions are independent in utility; or equivalently incorporating the influence other decision makers changes utility but does not alter an optimal action. Figure 3.4 presents a visual interpretation of information as team utility. The entropy power inequality or Minkowski inequality [17] provides an upper bound on the information resulting from the combination of independent random sources. The combined information is less than the sum of the individual sources as there is information common to them. A consequence is that the team utility gain is always less than or equal to the sum of the information gains determined individually in ignorance of the other team members actions. This is stated in terms of mutual information as

$$\mathcal{I}(\mathbf{x}, \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}) \leq \sum_{i=1}^{n} \mathcal{I}(\mathbf{x}, \mathbf{u}_i). \tag{3.43}$$

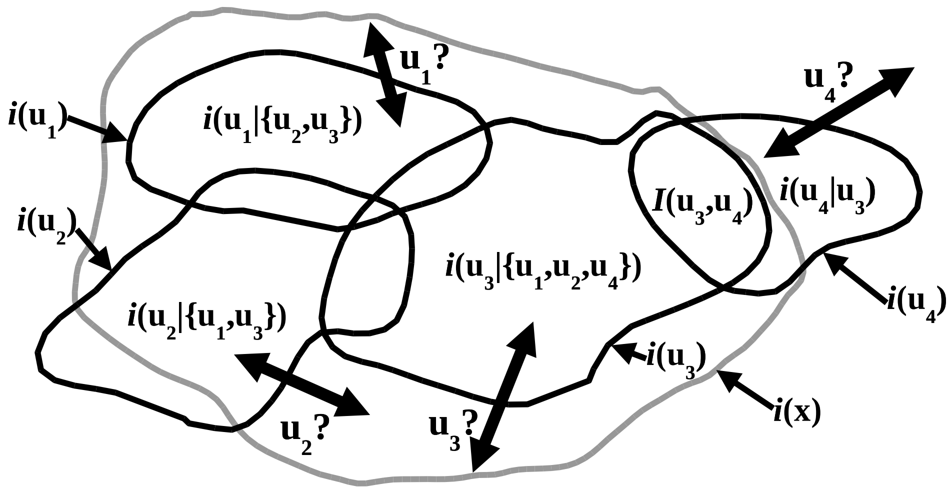


Figure 3.4: Visual interpretation of information as sensor team utility. Each information source has an associated entropic information $\boldsymbol{i}(\cdot)$, represented by the shapes in this diagram. The control actions $\mathbf{u}_i$ alter the shape and location of $\boldsymbol{i}(\mathbf{u}_i)$. The utility, as information gain $\mathcal{I}(\mathbf{x}, \{\mathbf{u}_1, \ldots, \mathbf{u}_n\})$, is the area of $\boldsymbol{i}(\mathbf{x})$ *(marked grey)* covered by the observation information $\boldsymbol{i}(\mathbf{u}_1), \ldots, \boldsymbol{i}(\mathbf{u}_n)$. The utility of actions are coupled through their common information. This 'information shape' analogy yields insight into the range of complexity for solution of the optimal decision problem. It is possible to think of situations where the best arrangement of shapes is rudimentary and others where many arrangements may require consideration. As the shapes change over time, the best arrangement may alter abruptly.

3.5.3 Alternate Information Related Measures

The previous section established entropy as an expected utility measure for sensing tasks. It is maintained that this is uniquely the most appropriate measure of the information in a probability distribution. It is possible and potentially suitable, however, to establish alternative measures. The reasons for this include: seeking acceptable simplified approximate solutions; overcoming adverse numerical conditioning; or desire for different 'risk taking' or 'risk averse' profiles (see [5]). The potential of the most likely candidates for expected utility in sensing tasks is now examined. This issue is revisited in Section 4.3.7 where the influence is illustrated by comparing trajectories for different measures applied to a single *bearings-only* sensor feature localisation problem.

Constructing an appropriate expected utility requires forming a scalar utility measure from the Fisher information. This scalar measure must combine and weight the elements or eigenvalues of the Fisher information or its inverse. Two likely alternatives to maximising entropic information are:

1. $\max \operatorname{trace}(\mathbf{Y})$ and
2. $\min \operatorname{trace}(\mathbf{Y}^{-1})$.

It is of use to recall the relation of these measures to the eigenvalues. This provides a geometric interpretation of alternate measures. Let $\{\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n\}$ be the eigenvalues of the Fisher information $\mathbf{Y}$, then

$$\begin{aligned} \mid \mathbf{Y} \mid &= \frac{1}{\mid \mathbf{Y}^{-1} \mid} = \prod_{i=1}^{n} \boldsymbol{\lambda}_i, \\ \operatorname{trace}(\mathbf{Y}) &= \sum_{i=1}^{n} \boldsymbol{\lambda}_i \text{ and} \\ \operatorname{trace}(\mathbf{Y}^{-1}) &= \sum_{i=1}^{n} \frac{1}{\boldsymbol{\lambda}_i}. \end{aligned} \tag{3.44}$$

A state vector is observable if the Fisher information is invertible. This requires a non-zero determinant of the Fisher information. This clearly relates entropic information to observability. If combined prior and observation Fisher information is singular for all possible actions, the team sensor suite is inadequate for the task. A serious flaw in using the trace of the Fisher information is that non-zero values can be obtained regardless of the invertibility Fisher information matrix.

Entropic information, as the log of the product of the eigenvalues of the Fisher information matrix, applies significant weight to decreasing the uncertainty in the states with lowest Fisher information. This is a highly desirable characteristic. Mutual information gain results in actions which seek to acquire knowledge about what is most uncertain.The trace of the covariance matrix, as the sum of the inverses of the eigenvalues also penalises low Fisher information. The trace of the Fisher information fails to distinguish between gains based on the value of the eigenvalues.

Insight into how these measures evolve over time is provided by the matrix calculus relations [42]

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mid \mathbf{Y}(t) \mid &= \mid \mathbf{Y}(t) \mid \mathrm{trace}\left(\mathbf{Y}^{-1}(t)\dot{\mathbf{Y}}(t)\right), \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\log \mid \mathbf{Y}(t) \mid &= \mathrm{trace}\left(\mathbf{Y}^{-1}(t)\dot{\mathbf{Y}}(t)\right), \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{trace}\left(\mathbf{Y}^{-1}(t)\right) &= -\mathrm{trace}\left(\mathbf{Y}^{-1}(t)\dot{\mathbf{Y}}(t)\mathbf{Y}^{-1}(t)\right) \text{ and} \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathrm{trace}\left(\mathbf{Y}(t)\right) &= \mathrm{trace}\left(\dot{\mathbf{Y}}(t)\right).
\end{aligned} \tag{3.45}$$

This reveals that the trace of the Fisher information does not weight instantaneous observation information by prior Fisher information. The trace of the covariance does, but in a manner different and more complicated than entropy.

The only practical criticism of entropy is in the case of poorly scaled Fisher information. A combination of extremely large and small eigenvalues of the Fisher information will give high entropic information. This is despite large uncertainty in components of the estimated state. In this case it could be desirable to operate on the eigenvalues directly.

Many other candidate measures for utility are possible including

- $\prod \mathrm{diag}\left(\mathbf{Y}^{-1}\right)$,
- $\min\max\mathrm{diag}\left(\mathbf{Y}^{-1}\right)$,
- $\max\min\mathrm{diag}\left(\mathbf{Y}\right)$ and
- $\max\min\mathrm{eig}\left(\mathbf{Y}\right)$.

All measures relate in some way to changes in the uncertainty. They vary in the different weighting attributed to prior information, process information loss and observation information gain. Differing decision rules with different characteristics and performance will result from the use of different measures.

3.5.4 Deriving Task Specific Utility From Entropy

Entropic information can be used to derive alternate utility measure suitable for specific tasks. One example would be a requirement to obtain a level of information that is lower than the achievable upper bound. In this situation the objective is not simply to maximise final information over a series of actions.

Consider the following extension to the Gauss-Markov process example form Section 3.4.4. An action $\mathbf{u}(t)$ is introduced that varies the sensor observation information according to

$$\mathbf{R}^{-1}(t) = \mathbf{R}^{-1}_{max}\mathbf{u}(t), \quad \mathbf{u}(t) \in [0,\ 1], \quad t \in [t_0,\ t_f].$$

The goal (utility) is to keep the total information above a certain threshold $\boldsymbol{i}_{Threshold}$ while simultaneously minimising the use of the sensor resource. An appropriate utility is [5].

$$\mathbf{J}(t_f) = \int_{t_0}^{t_f} [\boldsymbol{\omega}\boldsymbol{\delta}(t)(\boldsymbol{i}(t) - \boldsymbol{i}_{Threshold}) + (\boldsymbol{\omega} - 1)\mathbf{u}(t)]\, \mathrm{d}t \tag{3.46}$$

where, $\boldsymbol{\delta}(t) = \begin{cases} 1 & \text{if } \boldsymbol{i}(t) < \boldsymbol{i}_{Threshold} \\ 0 & \text{otherwise} \end{cases}$ and $\boldsymbol{\omega}$ is a weighting factor that is used to adjust the trade-off between the use of the sensor and information levels below the threshold. This is required as information measures alone do not establish a unique solution. Example action sequences which this utility measure could be used to optimise are shown in Figure 3.5.

This simple example indicates entropy can be used as a basis for other utility functions.

[5] Many functional forms are suitable. Since this form is linear, it is likely to result in a switching optimal action.

Entropy based utilities could be formed to capture the value of actions such as: activating or deactivating a sensor, adjusting a sensors operating mode, pointing a sensor in a particular direction, and communicating an observation to fellow decision makers.

3.6 Sensor Management

Sensor management concerns the optimal allocation of limited sensor resources to best maintain knowledge of inherently uncertain characteristics crucial to a task. This fundamentally involves making decisions associated with the evaluation, acquisition and exchange of information. As such, information-theoretic modelling and Bayesian decision theory offer an appropriate methodology for studying this problem. The tools required to model sensing and estimation tasks, place value on sensor actions, decide upon optimal actions, and fuse observation information from sensors have been presented in Sections 3.3, 3.4 and 3.5.

The sensor management problem is briefly explored here as an example of multi-sensor information fusion and team decision making. This is intended to demonstrate and illuminate concepts in the information-theoretic approach to decentralised data fusion and decision making. There is no attempt to comprehensively cover the issue of sensor management. The approach follows the work of Manyika [43, 46]. For a discussion of the problem intricacies and alternate approaches see McIntyrre [51] and Ng [64].

A decision making procedure is implemented on top of the decentralised data fusion (DDF) process. The actions considered are discrete. These include sensor to feature assignment, selecting the sensors operational mode, and activating communication channels. Modelling is applied to generate a dynamic probabilistic representation of the system. Information-theoretic methods are applied to form expected utilities for the different action combinations. A decision process seeks the action configuration that maximises expected utility. This solution structure is shown in Figure 3.6.

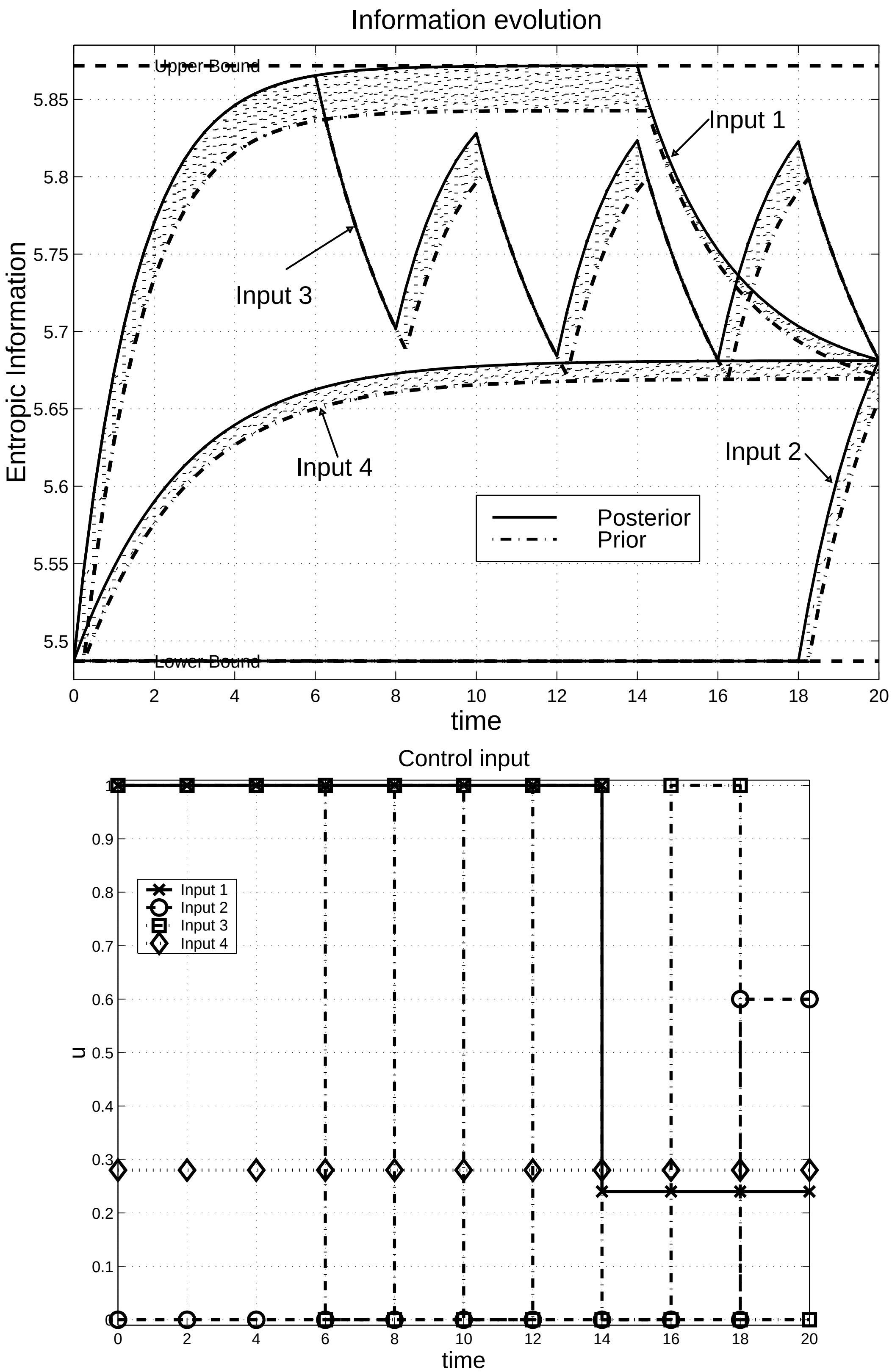


Figure 3.5: Entropy can be used as a basis for other utility functions. In this example it is desired to maintain entropic information above a threshold. Maximising entropic information does not capture the value of this task. Four action sequences are presented with equivalent mutual information gains. Equation 3.46 presents a utility function derived from entropy that assigns a value suitable for optimising this task.

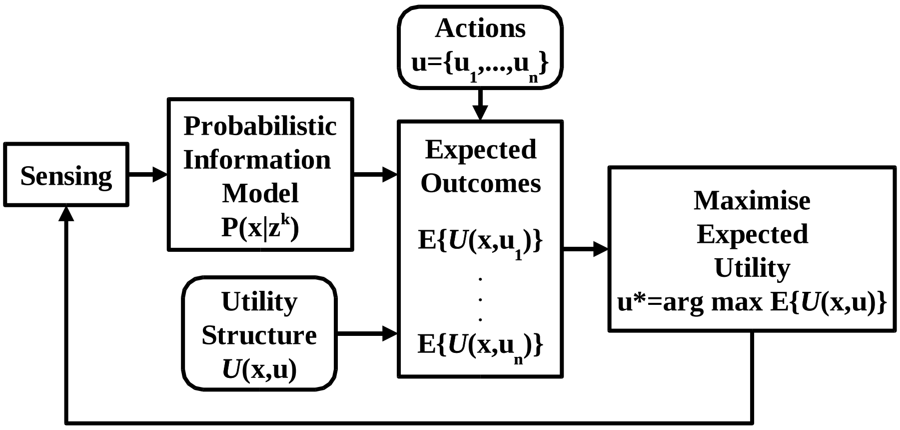


Figure 3.6: Structure of the sensor management problem

The following example is presented to illuminate the essential features of this approach.

3.6.1 Discrete Sensor Management for Target Tracking

Three tracking sensors at fixed locations $\mathbf{x}_{s,i} = [\boldsymbol{x}_{s,i}, \boldsymbol{y}_{s,i}]^T$ $i = 1, 2, 3$ make range and bearing observations of three targets with location and velocity $\mathbf{x}_j(k)$ $j = 1, 2, 3$ at discrete time k, where $t(k+1) - t(k) = \Delta T$. Each sensor is only capable of tracking a single target. The sensor's discrete control action is the target assignment. The control objective is to find the $1 \rightarrow 1$, $sensor \rightarrow target$ mapping that maximises the global entropic information of the target state estimates. The available sensor configurations are

$$\boldsymbol{a} = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \in \boldsymbol{A} = \left\{ \begin{array}{l} a_1 = (1 \rightarrow 1, 2 \rightarrow 2, 3 \rightarrow 3) \\ a_2 = (1 \rightarrow 1, 2 \rightarrow 3, 3 \rightarrow 2) \\ a_3 = (1 \rightarrow 2, 2 \rightarrow 1, 3 \rightarrow 3) \\ a_4 = (1 \rightarrow 2, 2 \rightarrow 3, 3 \rightarrow 1) \\ a_5 = (1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2) \\ a_6 = (1 \rightarrow 3, 2 \rightarrow 2, 3 \rightarrow 1) \end{array} \right\}.$$

3.6.2 Probabilistic Modelling

Process Model

Target dynamics are represented by the linear discrete time probabilistic model

$$\mathbf{x}_j(k+1) = \mathbf{F}\mathbf{x}_j(k) + \mathbf{w}_j(k)$$

$$\mathbf{x}_j(k) = \begin{bmatrix} \boldsymbol{x}_j(k) \\ \dot{\boldsymbol{x}}_j(k) \\ \boldsymbol{y}_j(k) \\ \dot{\boldsymbol{y}}_j(k) \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} 1 & \Delta T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \Delta T \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{w}_i(k)$ is taken to be a zero-mean $\mathrm{E}\{\mathbf{w}_j(k)\} = \mathbf{0}_{4\times 1}$ uncorrelated Gaussian sequence with variance,

$$\mathrm{E}\{\mathbf{w}_j(k)\mathbf{w}_j^T(k)\} = \mathbf{Q}_j = \begin{bmatrix} \frac{1}{3}\Delta T^3 & \frac{1}{2}\Delta T^2 & 0 & 0 \\ \frac{1}{2}\Delta T^2 & \Delta T & 0 & 0 \\ 0 & 0 & \frac{1}{3}\Delta T^3 & \frac{1}{2}\Delta T^2 \\ 0 & 0 & \frac{1}{2}\Delta T^2 & \Delta T \end{bmatrix} \boldsymbol{\sigma}_j^2.$$

Sensor Observation Model

The observation vector $\mathbf{z}_i(k) = [r(k), \theta(k)]^T$, is a non-linear function of the state of the target being observed.

$$\mathbf{z}_i(k) = \mathbf{h}(\mathbf{x}(k), \mathbf{v}_i(k))$$

$$\mathbf{h}_i(\mathbf{x}(k), \mathbf{v}_i(k)) = \begin{bmatrix} \mathbf{r}(k) \\ \theta(k) \end{bmatrix} = \begin{bmatrix} \sqrt{(x(k)-x_s)^2 + (y(k)-y_s)^2} \\ \arctan\left[\frac{x(k)-x_s}{y(k)-y_s}\right] \end{bmatrix} + \mathbf{v}_i(k)$$

where $\mathbf{v}_i(k)$ is taken to be a zero-mean $\mathrm{E}\{\mathbf{v}_i(k)\} = \mathbf{0}_{2\times 1}$ uncorrelated Gaussian sequence with variance,

$$\mathrm{E}\{\mathbf{v}_i(k)\mathbf{v}_i^T(k)\} = \mathbf{R}_i = \begin{bmatrix} \boldsymbol{\sigma}_{r,i}^2 & 0 \\ 0 & \boldsymbol{\sigma}_{\theta,i}^2. \end{bmatrix}$$

The Jacobian of the observation model with respect to target state is

$$\mathbf{H}(\mathbf{x}(k)) = \begin{bmatrix} \frac{x(k)-x_s}{\sqrt{(x(k)-x_s)^2+(y(k)-y_s)^2}} & \frac{y(k)-y_s}{\sqrt{(x(k)-x_s)^2+(y(k)-y_s)^2}} \\ \frac{y(k)-y_s}{(x(k)-x_s)^2+(y(k)-y_s)^2} & \frac{x_s-x(k)}{(x(k)-x_s)^2+(y(k)-y_s)^2} \end{bmatrix} = \begin{bmatrix} \sin\theta(k) & \cos\theta(k) \\ \frac{1}{r}\cos\theta(k) & -\frac{1}{r}\sin\theta(k) \end{bmatrix}.$$

The expected observation information for this sensor model is given by

$$\mathbf{I}(k) = \mathbf{H}^T(k)\mathbf{R}^{-1}\mathbf{H}(k) = \begin{bmatrix} \frac{\sin^2\theta(k)}{\sigma_r} + \frac{\cos^2\theta(k)}{r^2\sigma_\theta} & \frac{\sin\theta(k)\cos\theta(k)}{\sigma_r} - \frac{\sin\theta(k)\cos\theta(k)}{r^2\sigma_\theta} \\ \frac{\sin\theta(k)\cos\theta(k)}{\sigma_r} - \frac{\sin\theta(k)\cos\theta(k)}{r^2\sigma_\theta} & \frac{\cos^2\theta(k)}{\sigma_r} + \frac{\sin^2\theta(k)}{r^2\sigma_\theta} \end{bmatrix}.$$

Note, the determinant $\mid \mathbf{I}(k) \mid = \frac{1}{r^2\sigma_r\sigma_\theta}$. Hence, the observation information for this model is range dependent.

3.6.3 Decentralised Information Fusion

Each sensor runs an information filter with local knowledge of the global 12×1 information state and 12×12 block diagonal information matrix

$$\hat{\mathbf{y}}_i(k \mid k) = \begin{bmatrix} \hat{\mathbf{y}}_{i,1}(k \mid k) \\ \hat{\mathbf{y}}_{i,2}(k \mid k) \\ \hat{\mathbf{y}}_{i,3}(k \mid k) \end{bmatrix}, \quad \mathbf{Y}_i(k \mid k) = \begin{bmatrix} \mathbf{Y}_{i,1}(k \mid k) & 0 & 0 \\ 0 & \mathbf{Y}_{i,2}(k \mid k) & 0 \\ 0 & 0 & \mathbf{Y}_{i,3}(k \mid k) \end{bmatrix}.$$

The corresponding entropic information measure is

$$\boldsymbol{i}_i(k) = \frac{1}{2}\log\left[(2\pi e)^{12}\mid \mathbf{Y}_i(k \mid k) \mid\right] = \frac{1}{2}\sum_{j=1}^{3}\log\left[(2\pi e)^4\mid \mathbf{Y}_{i,j}(k \mid k) \mid\right].$$

The sensors now make observation $\mathbf{z}_i(k)$ of their allocated target $\mathbf{u}_i$ and determine the associated information state and matrix updates

$$\mathbf{i}_i(k) = \mathbf{H}_{i,\mathbf{u}_i}^T(k)\mathbf{R}_i^{-1}\mathbf{z}_i(k),$$

$$\mathbf{I}_i(k) = \mathbf{H}_{i,\mathbf{u}_i}^T(k)\mathbf{R}_i^{-1}\mathbf{H}_{i,\mathbf{u}_i}(k).$$

This information is propagated through the team information structure. The sensor nodes then update their local information state and information matrix by

$$\hat{\mathbf{y}}_{i,j}(k \mid k) = \hat{\mathbf{y}}_{i,j}(k \mid k-1) + \mathbf{i}_j(k), \quad j = 1, 2, 3$$

$$\mathbf{Y}_{i,j}(k \mid k) = \mathbf{Y}_{i,j}(k \mid k-1) + \mathbf{I}_j(k), \quad j = 1, 2, 3.$$

Each decision maker is made aware of the team information and can recover an individual estimate of the target states by

$$\hat{\mathbf{x}}_i(k \mid k) = \mathbf{Y}_i^{-1}(k \mid k)\hat{\mathbf{y}}_i(k \mid k).$$

3.6.4 Action Expected Utility

With its sensor model and predicted estimate of the target state, each sensor i constructs the expected observation information gain from observing target $j = 1, 2, 3$

$$\mathbf{I}_{i,j}(k) = \mathbf{H}_{i,j}^T(k)\mathbf{R}_i^{-1}\mathbf{H}_{i,j}(k).$$

From this the mutual information gain for sensor i observing each target j is

$$\mathcal{I}_{i,j}(k) = \frac{1}{2}\log\left[\frac{\mid \mathbf{Y}_{i,j}(k \mid k-1) + \mathbf{I}_{i,j}(k) \mid}{\mid \mathbf{Y}_{i,j}(k \mid k-1) \mid}\right].$$

The global utility for control action $\boldsymbol{a}_l = \{\mathbf{u}_{1,l}, \mathbf{u}_{2,l}, \mathbf{u}_{3,l}\} \in \boldsymbol{A}$ is

$$\mathbf{J}(\boldsymbol{a}_l, k) = \sum_{i=1}^{3} \mathcal{I}_{i,\boldsymbol{a}_l(i)}(k).$$

The individual mutual information measures can be added to form a team measure as the $1 \rightarrow 1$ sensor to target constraint makes the observations mutually exclusive. There is no information common to the observations. The optimal action is selected from

$$\boldsymbol{a}^\star(k) = \arg\max_{\boldsymbol{a}\in\boldsymbol{A}} \mathbf{J}(\boldsymbol{a}, k).$$

Note that this problem formulation results in a two dimensional *Linear Sum Assignment Problem* [6] where $\mathcal{I}_{i,j}(k)$ form the elements of the cost matrix and $\boldsymbol{a}$ corresponds to a permutation matrix [31, 6, 11].

3.6.5 Solution Process

The linear assignment problem is polynomial time solvable and has a extremely wide variety of solution approaches. Efficient combinatorial optimisation is a significant and ongoing research problem. Directly sorting all globally feasible solutions is rejected as the number of solutions is the factorial of the problem dimension. A review of solution methods is provided by the DIMACS Challenge [37] or Burkard [11]. The approaches are mainly simplex or primal-dual approaches. An instance of the primal-dual approach is the auction algorithm of Bertsekas [6, 7]. A very different and interesting approach by Haken [31], is the method of 'coupled selection equations'.

The approach taken here is a slight adjustment of the method used by Manyika [43]. Local action ranking can not be performed without considering the actions of others. An iterative best response procedure is used. At each stage, the decision makers choose and

[6] *asymmetric* if the number of targets is greater than the number of sensors

communicate their best action accounting for knowledge of $\mathcal{I}_{i,j}(k)$ communicated from other decision makers' previous preferences. The process is terminated once all sensors are assigned to a target and the assignment is the best response to itself.

3.6.6 Discussion of Solution Results

A variety of rigorous solution techniques can be applied to solve this problem in a decentralised manner. However, of most significant interest are the properties of the information-theoretic formulation and the characteristics the solution exhibits.

Results from a solution to this example problem are shown in Figure 3.7. Figure 3.7 (b) shows that the value of observing a target is range dependent. As the targets move the optimal group control action switches. This decision and communication structure has provided a coordinated solution to the global control objective. Local prior and communicated external information allow each node to arrive at a solution it believes is best for the group. This example indicates that without this communicated external information the resulting control action would differ. There are times during the solution where "greedy" allocation based on gain from individual sensor to target mutual information is not the best group decision. A powerful result is achieved when the communicated information is combined with the observation model. Then each sensor is aware of targets it can not see and can determine the utility associated with observing those targets. Therefore, the decision to switch target allocation is made based on this utility comparison without the sensors observing their future target. This action is referred to as sensor hand-off and cueing. This characteristic is not specified by the system designer. It arises when an optimal decision process is applied to the problem and utility formulation.

Significantly, it can be seen that the sensor nodes do not require any knowledge of the other sensors' location or characteristics. This simplicity is an important property of the decentralised information-theoretic approach. Everything required to select the optimal action is contained in the local $\hat{\mathbf{y}}_i(k \mid k)$, $\mathbf{Y}_i(k \mid k)$ and the communicated $\mathbf{I}_{i,j}(k)$. Individual decision makers influence each other through communicated expected observation information.

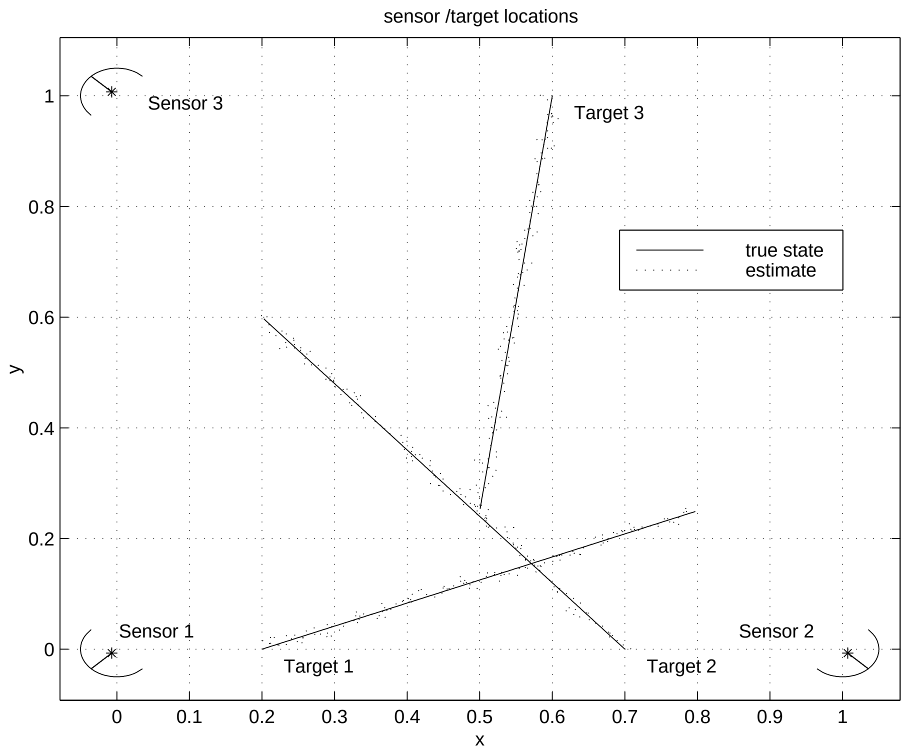
Note that in this case the local sensor knowledge is the true global information since all information is communicated between the sensor nodes. If all information is not available at each node this formulation is still valid. However, the local utility value associated with an action will differ between nodes.

The $1 \rightarrow 1$ sensor to target tracking problem results in a simple utility structure as there is no information mutual to the observations. This same approach can be applied to more complicated problems such as the beacon based navigation as demonstrated in the OxNav system [46, 43]. Although the assignment is one sensor to each beacon, the observation information is combined to localise the vehicle carrying the sensors. This results in common information among the observations. The value of each sensor assignment is dependent on the assignments of all other sensors. For this reason this problem is significantly more complex. The non-linear assignment problem resulting from the coupled utility is $\mathcal{NP}$-hard.

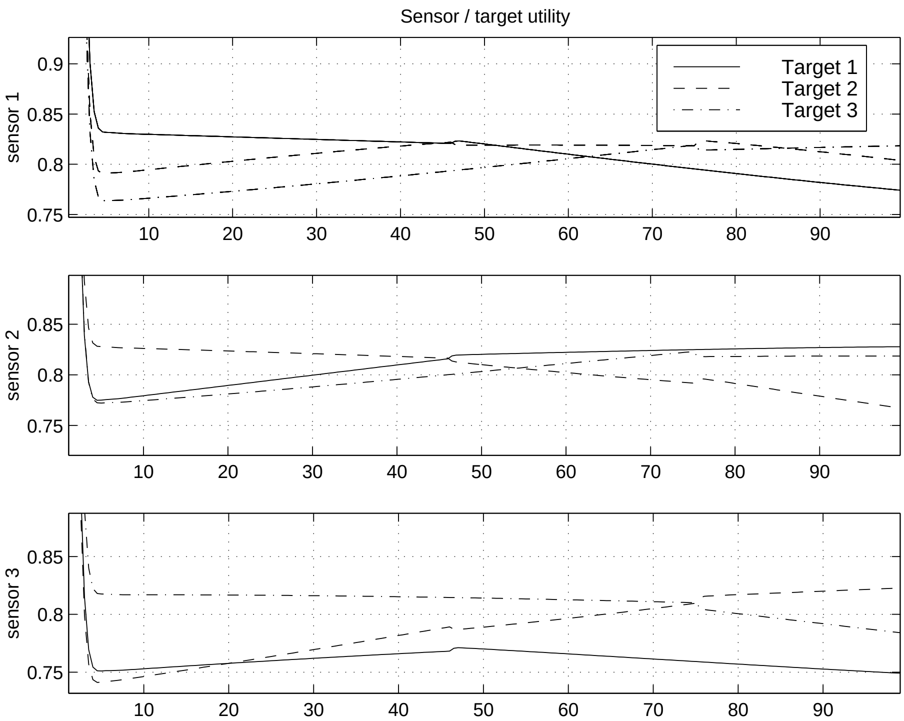
While highly simplified, the target and range-bearing sensor modelling used is representative of real world problems. The information-theoretic approach captures the value of sensor configurations. Only knowledge essential to evaluating action utility is communicated among decision makers. Irrelevant physical detail, such as the location, type and quality of the other sensors, is abstracted away. The dynamics of the information measures suggest an optimal system configuration that can be sought by decentralised decision making procedures.

3.7 Summary

This chapter formally defined information in terms of uncertainty, outlined a methodology for decentralised fusion of information from multiple sources and determined entropy and

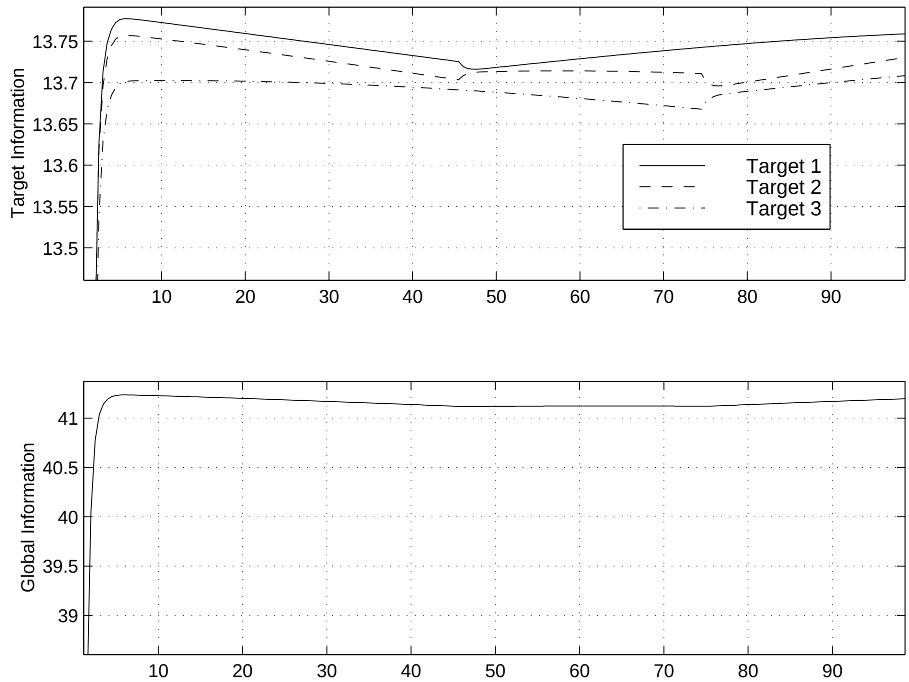


(a)

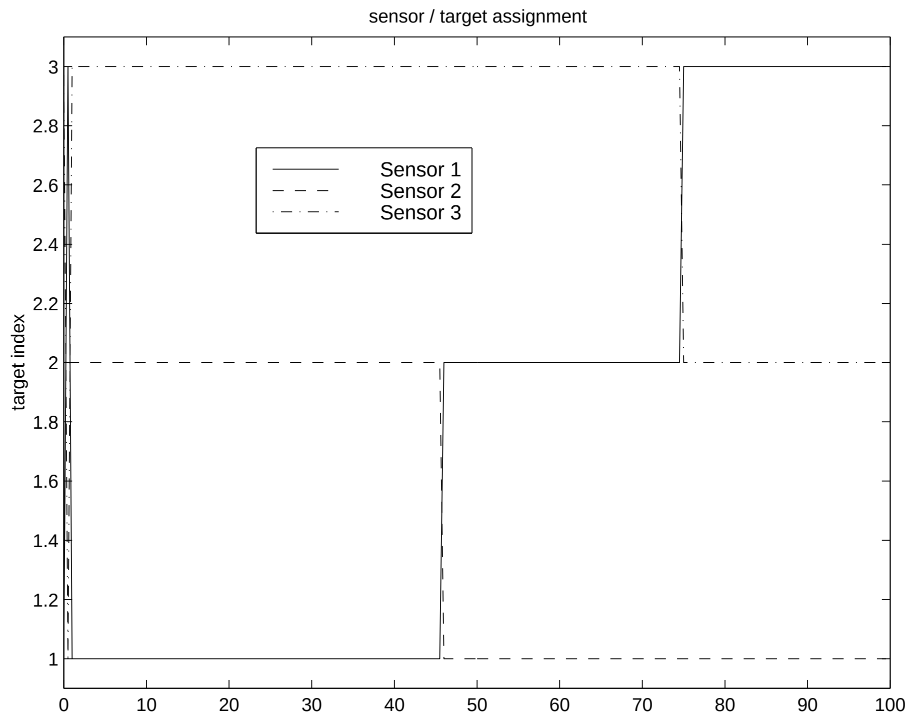


(b)

Figure 3.7: Information-theoretic approach to a discrete sensor management task. (a) Problem geography, (b) Information based Utility for each individual sensor $\rightarrow$ target assignment, (c) Entropic target information, (d) optimal group sensor $\rightarrow$ target assignment



(c)



(d)

Figure 5.8: (continued)

mutual information to be appropriate expected utility measures for sensing actions.

The problem of combining information form multiple sources was considered. The fusion process was observed to be simply the summation of the Fisher information of the source likelihood functions. The additive and associative properties of the information sources led to an efficient and scalable decentralised data fusion implementation based on the 'information' or 'inverse covariance' formulation of the extended Kalman filter.

Information-theoretic reasoning promoted entropic information and mutual information gain as natural expected utility measures for placing value on available sensing actions. Individual team member utility is coupled through information common to the team observations. Team optimal sensing actions can be determined through exchange of expected observation information.

Two complementary mechanisms were introduced: the decentralised data fusion architecture; and information-theoretic team expected utility. The resultant combination is a *consistent methodology* for gathering, exchanging, evaluating and fusing information. This framework is eminently compatible with the team decision making structure outlined in Chapter 2.

The viability and relative simplicity of this approach was demonstrated through the analysis of a discrete sensor assignment example. Attention is now turned to consideration of problems involving dynamic optimisation of continuous sensor and sensor platform trajectories.

Chapter 4

Control of Information Gathering Tasks

4.1 Introduction

This chapter investigates autonomous sensing tasks in the context of optimal control. Optimal control is a general dynamic optimisation problem where control actions are selected over time to optimise a performance index or utility function. Section 4.2 applies the information metrics developed in Chapter 3 to formulate utility measures suitable for information gathering tasks. Unlike the discrete sensor assignment problems considered earlier, this involves determining continuous trajectories for the sensors and sensor platforms in the system. These concepts are first applied to single vehicle single sensor situation in Section 4.3. Concepts and results are illustrated through an example of a single feature localised by a single bearings-only enabled vehicle. Subsequently, attention is focused on the far more complex multiple vehicle multiple sensor problem.

The multiple vehicle multiple sensor problem is posed as a team decision problem with information based utility measures. Many complex issues arise in this situation. Crucial to the multiple vehicle problem is coordination and cooperation among decision makers.

Coordination and cooperation is investigated by considering decomposition of the global utility structure in Section 4.4. Partial utility measures are formed by considering the local knowledge and influence of a decision maker. The usefulness and validity of these concepts is explored with regard to decentralised solution procedures. In Section 4.5 a simplified localisation problem is used to investigate these matters. The best response procedure from Section 2.5.1 is applied to determine the cooperative solution through decentralised negotiation.

The optimal control solution methods employed are well established and are detailed in Appendix A. The approach taken is to approximate the optimal solution through parameterisation of the control trajectories. Sequential quadratic programming is used to solve the resulting constrained parameter optimisation problem.

4.2 Information as a Control Objective

Chapter 3 introduced information based utility measures that capture the value of sensing actions. These were applied to determine optimal discrete sensor assignments for the sensor management problem. This concept is now extended to the problem of controlling continuous vehicle and sensor trajectories.

In the presence of uncertainty, the true value of a future action can not be determined exactly. Probabilistic dynamic models of the vehicle, sensors and environment allow *a priori* evaluation of the expected utility associated with a sequence of actions. Entropic information provides a natural utility measure representing the expected compactness of the posterior probability distribution conditional on the action sequence.

Typical real world sensors have spatially dependent observation information. The measurement uncertainty varies over a finite range with the relative distance to the object of interest. Control inputs, such as focal length or signal power level in electro-optic sensors, may be available to vary the measurement uncertainty. Dynamics and constraints

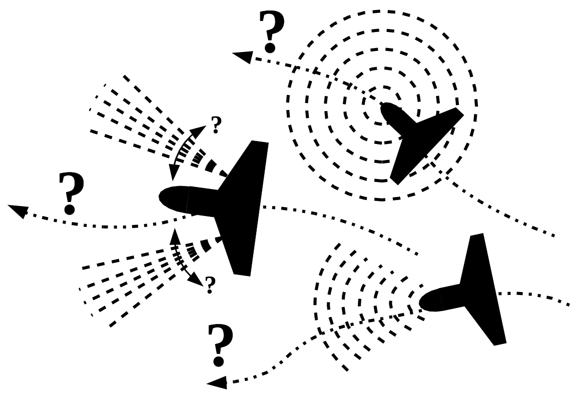


Figure 4.1: Mission objectives are cast in to utility functions that include information based measures. Utility is dependent on the control inputs to the vehicles and sensors. Generating vehicle and sensor trajectories is formulated as an optimal control problem.

may be associated with the allocation of the sensors to areas of interest. These sensors are carried on board vehicles that in turn have constrained dynamics and uncertainty associated with their state and controls. The sensors and vehicles operate in an uncertain and dynamic environment. Probabilistic models of the sensors, vehicles and environment can be combined to give an overall representation of the dynamic and control variables for the problem.

Objectives formed from information measures can be combined with other performance measures such as expended energy and risk. When combined with models of the environment, vehicles and sensors, the sensing requirements are mapped into a numerical description suitable for systematic optimisation. Given prior knowledge, the expected utility for a sequence of control actions can be evaluated. Differentiation of the models allows determination of the sensitivity of the utility measure to the control action. The best control action can be determined by an *extremum seeking* loop. This process is illustrated in Figure 4.2.

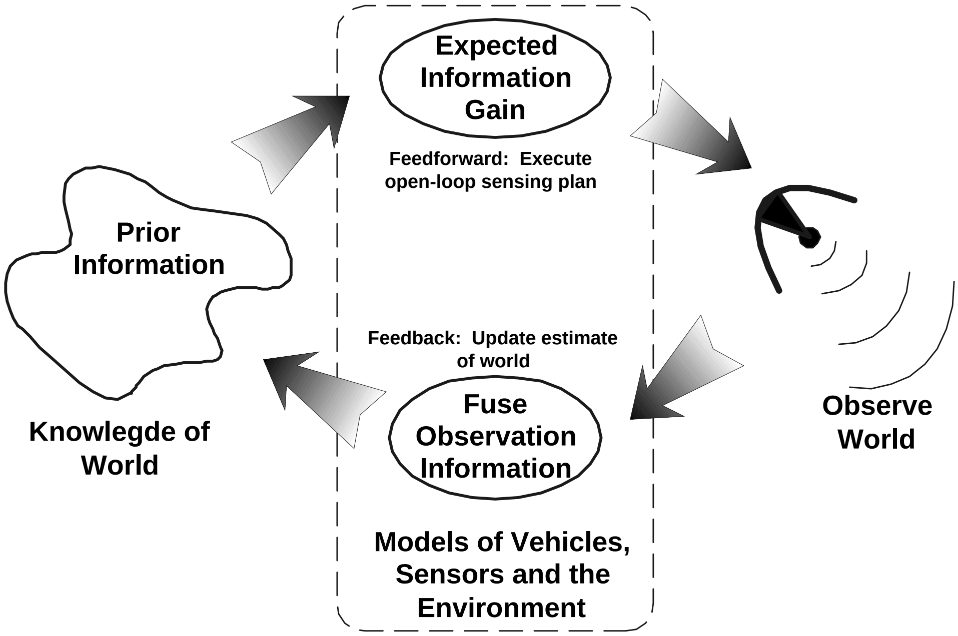


Figure 4.2: Information flow in active sensing. Models of vehicles sensors and the environment provide means to capture and predict *a priori* the expected utility of a sequence of actions. Fusing observation information updates the knowledge from which subsequent optimal actions are generated.

4.3 A Single Platform Example: Bearings-Only Feature Localisation

The use of information measures as a performance index in control problems is best illustrated through a motivational example. The bearings-only feature localisation problem is considered. This single vehicle problem has been studied widely by other researchers. Three studies by Oshman and Davidson [65], Tremois and Le Cadre [90] and Passerieux and Van Cappel [68] consider the problem from an optimal control perspective. The vehicle control action and trajectory is sought that minimises the determinant of the feature error covariance at a fixed terminal time t_f. This is equivalent to maximising final entropic information or the integral of mutual information gain over time. This example is revisited with the aim of illustrating the use of entropic information as a performance metric, the value of prior information, the effect of different information measures and the influence of varied optimisation time horizons. The geography of this problem is illustrated in Figure 4.3.

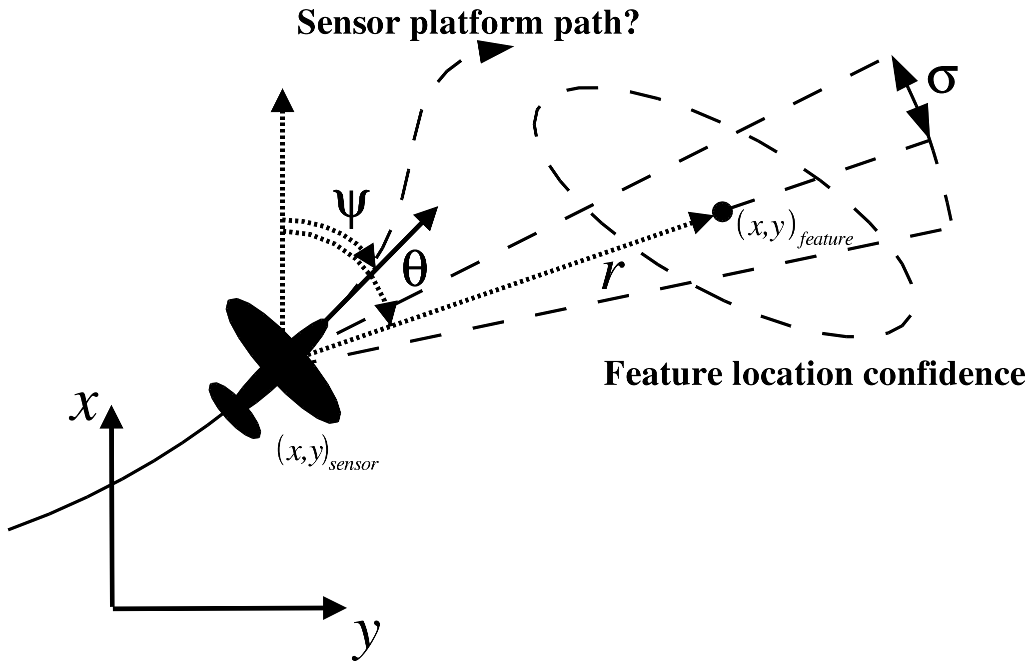


Figure 4.3: Bearings-only feature localisation problem

4.3.1 Modelling the Vehicle, Sensor and Environment

Central to this problem is the modelling of the vehicle, feature and sensor.

Sensor Platform Model

A sensor platform is moving in the xy plane with constant velocity V. The vehicle's location and heading at a time t is captured in the state $\mathbf{x}_s(t)$. The single control variable is the rate of change of platform heading. This is shown in Figure 4.4 and modelled by the following equations.

$$\mathbf{x}_s(t) = \begin{bmatrix} x(t) \\ y(t) \\ \psi(t) \end{bmatrix},\ \mathbf{x}_s(0) = \begin{bmatrix} x(0) \\ y(0) \\ \psi(0) \end{bmatrix},\ \mathbf{u}(t) = \dot{\psi}(t),\ \dot{\mathbf{x}}_s(t) = \begin{bmatrix} V\cos(\psi(t)) \\ V\sin(\psi(t)) \\ \mathbf{u}(t) \end{bmatrix} \tag{4.1}$$

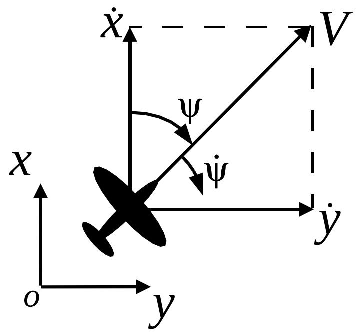


Figure 4.4: 2D sensor platform vehicle model

Feature Model

The feature is a stationary point on the xy plane. It is modelled by two Gaussian random constants representing its location $\mathbf{x}_f = [x_f, y_f]^T$ in the xy plane. The feature location is estimated by the conditional mean $\hat{\mathbf{x}}_f(t) = \mathrm{E}\{\mathbf{x}_f(t) \mid \mathbf{Z}^t\}$, where $\mathbf{Z}^t$ are the observations made up to time t. The feature location uncertainty is captured by the covariance of the two dimensional Gaussian distribution $\mathbf{P}_f(t) = \mathrm{E}\{(\mathbf{x}_f - \hat{\mathbf{x}}_f(t))^T(\mathbf{x}_f - \hat{\mathbf{x}}_f(t)) \mid \mathbf{Z}^t\}$. In the information filter this is represented by the inverse covariance $\mathbf{Y}(t) = \mathbf{P}_f^{-1}(t)$.

The feature state is not influenced by control input. Process noise is included in the model to allow for flexibility in the design of the estimator. It is valid to set this to zero but small non-zero values may improve numerical conditioning. The feature process model is given by

$$\dot{\mathbf{x}}_f(t) = \omega(t) \tag{4.2}$$

where $\omega(t)$ is represented by a zero mean Gaussian process with covariance $\mathbf{Q}(t)$ uncorrelated in time, $\mathrm{E}\{\omega(t)\} = 0$, $\mathrm{E}\{\omega(t)\omega^T(\tau)\} = \mathbf{Q}(t)\delta(t-\tau)$.

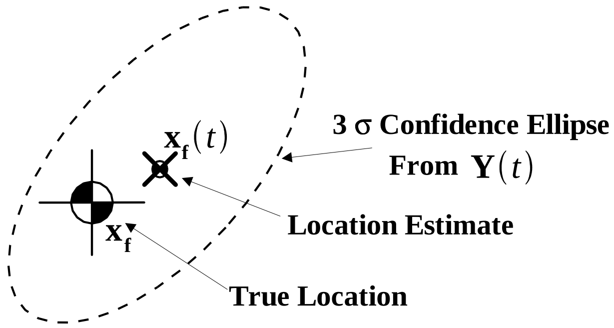


Figure 4.5: Feature representation for localisation

Sensor Model

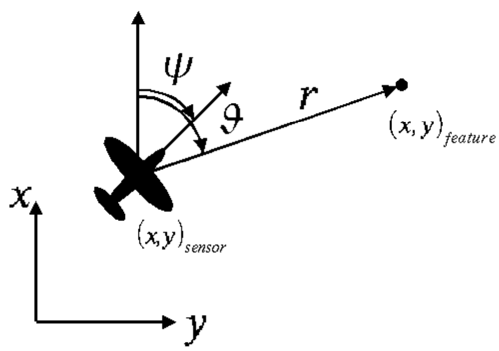


Figure 4.6: 2D range and/or bearing sensor model

The sensor platform carries a sensor making bearings-only observations of a point feature in the xy-plane. The observation is the bearing of the stationary feature $\mathbf{x}_f = [x_f, y_f]^T$ relative to the sensor platform location $\mathbf{x}_s(t)$. As indicated in Figure 4.6. The observation model equation is:

$$\begin{aligned}\mathbf{z}(t) &= \mathbf{h}(\mathbf{x}_f, \mathbf{x}_s) \\ \mathbf{h}(t) &= \theta(t) - \psi(t) + \nu(t)\end{aligned}$$

$$= \quad \arctan\left[\frac{y_f - y_s(t)}{x_f - x_s(t)}\right] - \psi(t) + \nu(t) \tag{4.3}$$

where $\nu(t)$ is a zero-mean scalar Gaussian processes with variance $\mathbf{R} = \sigma^2$,
$\mathrm{E}\{\nu(t)\} = 0$, $\mathrm{E}\{\nu(t)\nu^T(\tau)\} = \mathbf{R}\delta(t-\tau)$, $\mathrm{E}\{\nu(t)\omega^T(\tau)\} = 0 \ \forall \ t, \tau$
The Jacobian with respect to the feature state is:

$$\begin{aligned}
\mathbf{H}(t) &= \nabla_{\hat{\mathbf{x}}_f}\mathbf{h}(\mathbf{x}_f, \mathbf{x}_s) \\
&= \left[\frac{-(\hat{y}_f - y_s(t))}{(\hat{x}_f - x_s(t))^2 + (\hat{y}_f - y_s(t))^2}, \frac{\hat{x}_f - x_s(t)}{(\hat{x}_f - x_s(t))^2 + (\hat{y}_f - y_s(t))^2}\right] \\
&= \frac{1}{r(t)}\left[-\sin\hat{\theta}(t), \cos\hat{\theta}(t)\right]
\end{aligned}$$

Following the reasoning used in derivation of the linearised filter, the resulting observation information is:

$$\begin{aligned}
\mathbf{I}(t) &= \mathbf{H}^T(t)\mathbf{R}^{-1}\mathbf{H}(t) \\
&= \frac{1}{\sigma^2\hat{r}(t)^2}\begin{bmatrix} \sin^2\hat{\theta}(t) & -\sin\hat{\theta}(t)\cos\hat{\theta}(t) \\ -\sin\hat{\theta}(t)\cos\hat{\theta}(t) & \cos^2\hat{\theta}(t) \end{bmatrix}
\end{aligned} \tag{4.4}$$

4.3.2 System Equations

The system state consists of the current sensor platform location, feature location estimate, feature inverse error covariance and the performance index. In this case the feature state remains constant and is not included in the system equations for the optimisation process. The performance index is a function of the feature inverse error covariance at the optimisation horizon final time. Hence, there is no requirement to append an integral cost equation to the system equations. Since the feature information matrix is symmetric, only three of the four values need to be propagated. The evolution of the predicted feature Fisher information is given by

$$\dot{\mathbf{Y}}(t) = -\mathbf{Y}(t)\mathbf{Q}\mathbf{Y}(t) + \mathbf{I}(t, \hat{\mathbf{x}}_f(t), \mathbf{x}_s(t)). \tag{4.5}$$

Let

$$\mathbf{Y}(t) = \begin{bmatrix} \mathrm{Y}_x(t) & \mathrm{Y}_{xy}(t) \\ \mathrm{Y}_{xy}(t) & \mathrm{Y}_y(t) \end{bmatrix}, \quad \mathbf{I}(t) = \begin{bmatrix} \mathrm{I}_x(t) & \mathrm{I}_{xy}(t) \\ \mathrm{I}_{xy}(t) & \mathrm{I}_y(t) \end{bmatrix} \text{ and } \quad \mathbf{Q} = \begin{bmatrix} \mathrm{Q}_x & 0 \\ 0 & \mathrm{Q}_y \end{bmatrix},$$

the equations governing the evolution of the feature information are appended to the vehicles system dynamics. The stacked system equations become

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{\psi}(t) \\ \dot{\mathrm{Y}}_x(t) \\ \dot{\mathrm{Y}}_{xy}(t) \\ \dot{\mathrm{Y}}_y(t) \end{bmatrix} = \begin{bmatrix} V\cos(\psi(t)) \\ V\sin(\psi(t)) \\ \mathbf{u}(t) \\ -\mathrm{Y}_x^2(t)\mathrm{Q}_x - \mathrm{Y}_{xy}^2(t)\mathrm{Q}_y + \mathrm{I}_x(t) \\ -\mathrm{Y}_x(t)\mathrm{Q}_x\mathrm{Y}_{xy}(t) - \mathrm{Y}_{xy}(t)\mathrm{Q}_y\mathrm{Y}_y(t) + \mathrm{I}_{xy}(t) \\ -\mathrm{Y}_{xy}^2(t)\mathrm{Q}_x - \mathrm{Y}_y^2(t)\mathrm{Q}_y + \mathrm{I}_y(t) \end{bmatrix}. \tag{4.6}$$

With utility to maximise at terminal time t_f is

$$\mathbf{J}(t_f) = \mid \mathbf{Y}(t_f) \mid = \mathrm{Y}_x(t_f)\mathrm{Y}_y(t_f) - \mathrm{Y}_{xy}^2(t_f). \tag{4.7}$$

Note, in practice the estimation procedure is implemented in discrete time. Performing the optimisation in a continuous differential formulation allows use of ODE solvers more efficient than fixed interval recursive implementations. This is particularly justified when sensor sampling rates are significantly faster than the state dynamics.

4.3.3 Solution Procedure

The solution to this problem is approached using the control parameterisation scheme described in Appendix A. The control action, vehicle heading rate, is parameterised by a number of zero-order steps over the optimisation horizon. The resulting unconstrained nonlinear programming problem is solved using the SQP routine *fminunc* from the Matlab optimisation toolbox [8].

The solution procedure requires gradient information about the performance index with respect to the solution parameters. This is determined using the forward sensitivity analysis described in Appendix A. This procedure requires the Jacobians of the system equations and terminal cost with respect to state and control action. The Jacobians are

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \left[\begin{array}{cccccc} 0 & 0 & -\mathbf{V}\sin(\psi) & 0 & 0 & 0 \\ 0 & 0 & \mathbf{V}\cos(\psi) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{4(\hat{y}_f - y)^2(\hat{x}_f - x)}{\sigma^2 r^6} & \frac{4(\hat{y}_f - y)^3}{\sigma^2 r^6} - \frac{2(\hat{y}_f - y)}{\sigma^2 r^4} & 0 & -2\mathrm{Y}_x\mathrm{Q}_x & -2\mathrm{Y}_{xy}\mathrm{Q}_y & 0 \\ \frac{4(\hat{y}_f - y)(\hat{x}_f - x)^2}{\sigma^2 r^6} - \frac{(\hat{y}_f - y)}{\sigma^2 r^4} & \frac{4(\hat{y}_f - y)^2(\hat{x}_f - x)}{\sigma^2 r^6} - \frac{(\hat{x}_f - x)}{\sigma^2 r^4} & 0 & -\mathrm{Y}_{xy}\mathrm{Q}_x & -\mathrm{Y}_x\mathrm{Q}_x - \mathrm{Y}_y\mathrm{Q}_y & -\mathrm{Y}_{xy}\mathrm{Q}_y \\ \frac{4(\hat{x}_f - x)^3}{\sigma^2 r^6} - \frac{2(\hat{x}_f - x)}{\sigma^2 r^4} & \frac{4(\hat{y}_f - y)(\hat{x}_f - x)^2}{\sigma^2 r^6} & 0 & 0 & -2\mathrm{Y}_{xy}\mathrm{Q}_x & -2\mathrm{Y}_y\mathrm{Q}_y \end{array} \right], \tag{4.8}$$

where $r = \sqrt{(\hat{x}_f - x)^2 + (\hat{y}_f - y)^2}$, and

$$\frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \left[\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \end{array} \right]^T, \tag{4.9}$$

$$\frac{\partial \mathbf{J}}{\partial \mathbf{x}} = \left[\begin{array}{cccccc} 0 & 0 & 0 & -\mathrm{Y}_y & 2\mathrm{Y}_{xy} & -\mathrm{Y}_x \end{array} \right]. \tag{4.10}$$

To solve this problem initial conditions are required for the feature location estimate and inverse covariance. For the information form of the estimation process and entropic information metrics there is no problem with choosing zero initial information. The prior may be known from an alternate information source or estimation method. In the case that no initial information is available, the integral required to determine mutual information gain is indefinite. This can be addressed by choosing a suitable small non-zero initial value. The initial location estimate is required as the observation information prediction is a function of the relative range and bearing to the feature. This amounts to requiring an initial range estimate, bearing to the feature being provided by observations.

4.3.4 Solution

This seemingly simple problem belies significant complexity. The objective amounts to a complex trade-off between reducing range and maximising change in bearing. The solution trajectory and characteristics vary with initial range, sensor variance and prior information. This is reasonable given the form of the observation information Equation 4.4 and the control objective Equation 4.7. The observation information increases in inverse proportion to the range squared and the information gain is higher along bearings with lower prior information. A variety of solutions are presented to investigate these effects. In all cases the vehicle velocity is $1m/s$ and the bearing sensor makes observations at $16Hz$ with $\sigma = 2.5^o$.

Details of an example solution are shown in Figures 4.7 to 4.9. The control action is determined at $4Hz$ over a 1.5 second time horizon. The initial prior is constructed from a single bearing observation combined with a "guess" that the target range is 21 metres with large variance. The actual initial range is 14.1 metres. Figure 4.7 shows the optimal trajectory of sensor platform, the state estimate and the evolution of the feature

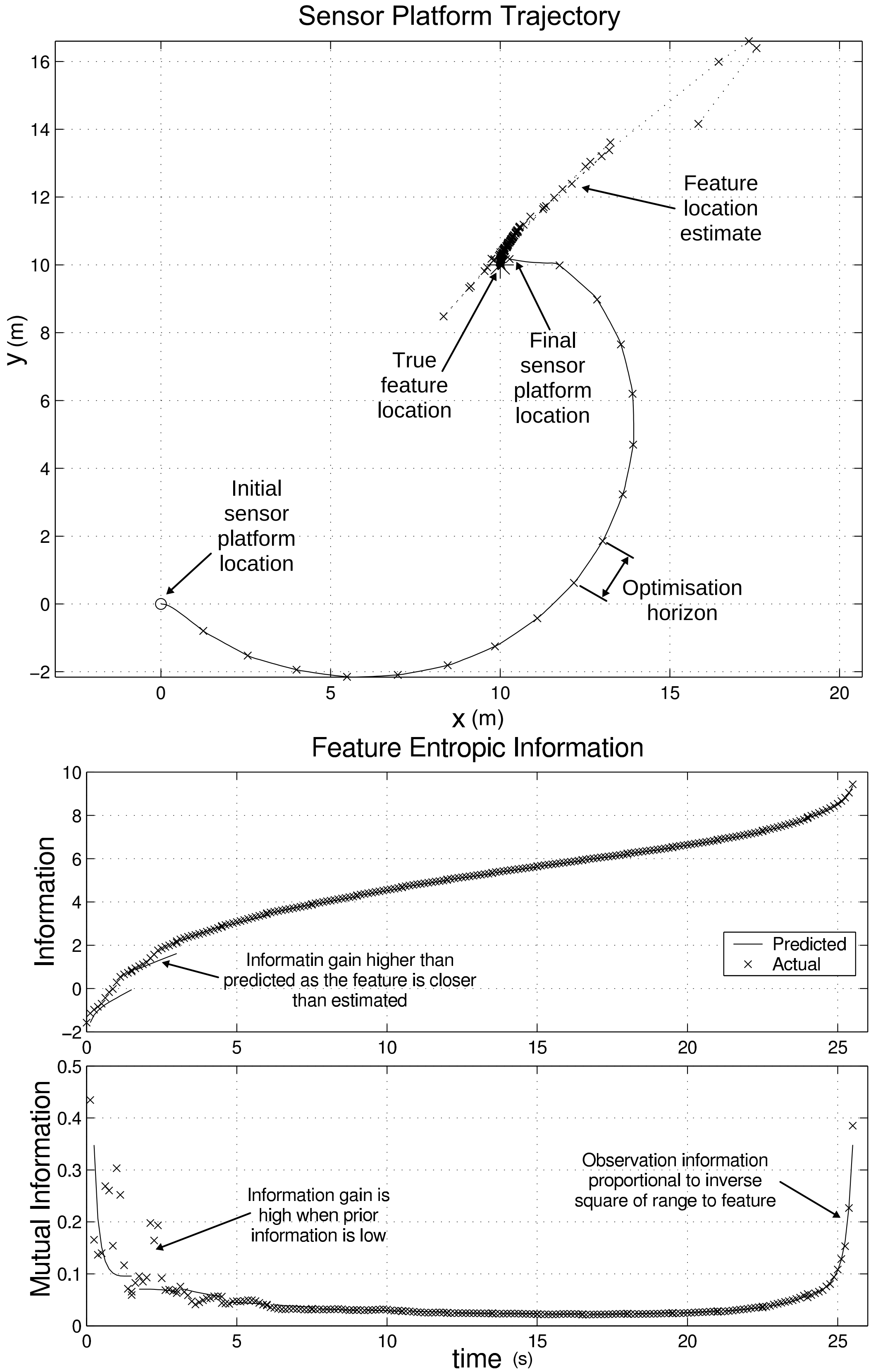


Figure 4.7: Trajectories of the sensor platform and location estimate along with the information evolution for the bearings-only localisation example. Bearings observations are made at $16Hz$. The open-loop control is calculated at $4Hz$ over a 1.5 second time horizon. The information gain associated with observations is initially high when there is large uncertainty and at the end of the sensing plan when the range to the feature is small increasing observation information.

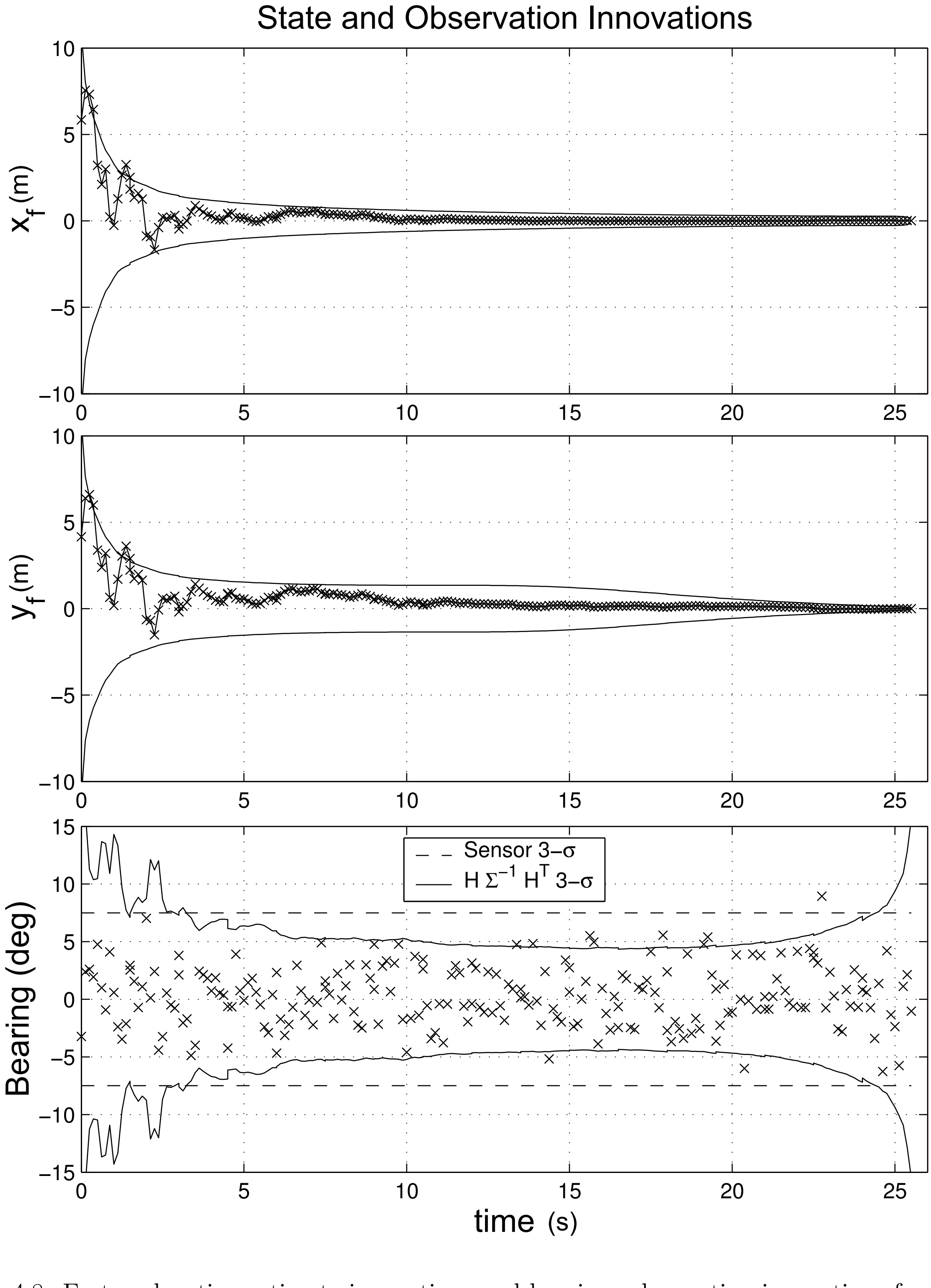


Figure 4.8: Feature location estimate innovations and bearings observation innovations for the localisation example. 3σ bounds are indicated corresponding to the state estimate covariance, the sensor observation variance and the observation innovation variance.

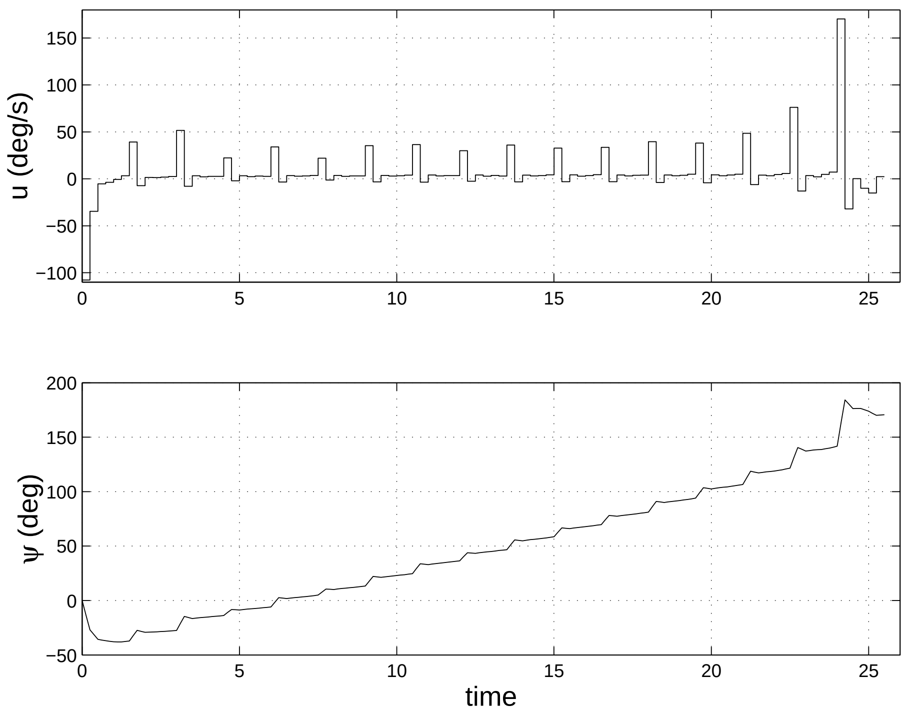


Figure 4.9: $4Hz$ parameterised control action and resulting vehicle heading.

information. Figure 4.8 shows the state and observation innovations with 3σ confidence levels. Figure 4.9 shows the parameterised control solution and vehicle heading.

As the feature location estimate approaches the true value, the predicted information evolution becomes more accurate. This is expected as the predicted information gain is a function of the state estimate conditional on prior observations. The control and estimation problems are indeed coupled. The open-loop sensing plan is based on the feature state estimate that is updated as observations are made.

4.3.5 Temporal Considerations

This feature localisation example highlights the temporal dependencies in controlling the sensing processes. To illustrate this solutions for varied time horizons are compared. The cases are listed in Table 4.1. In all cases the vehicle velocity is $1m/s$ and the bearing sensor makes observations at $16Hz$ with $\sigma = 2.5^o$.

Case	1	2	3	4	5	6	7	8	9
Final time $\mathbf{T}_f$	26	25	24	22.75	21.25	20	18.75	16.5	14
Time horizon $\mathbf{T}_h$	.5	1	2	3.25	4.25	5	6.25	8.25	14
Control parameters	2	2	4	4	4	5	4	7	7
Optimisation stages	52	25	12	7	5	4	3	2	1

Table 4.1: Details of parameterised solution cases used to investigate the influence of planning horizon duration. Final time, optimisation time horizon, the number of parameters used to represent the control action and the total number of optimisation stages are indicated.

To isolate the influence of varied time horizons the open-loop solutions are computed with the correct range to the feature. While unrealistic, this allows investigation of characteristics of the solution trajectories with varied time horizons. The resulting vehicle trajectories and expected information time series are shown in Figure 4.10. As the planning horizon increases, the optimal solutions to the localisation problem tend to initially reduce range to the feature, thus increasing the value of subsequent bearing changes.

As observations are made, the information and information state can be compared to their predicted values. This can be observed in figure 4.7. The initial rate of information gain is significantly greater than predicted. This is because the estimated range is far greater than the true value, revealing an inconsistency. As the optimisation horizon is increased, the ability to distinguish range inconsistencies is reduced over the initial portion of the solution. A long planning horizon, such as in trial 9, would not reveal whether the estimate on which it is based is significantly inaccurate.

Sensing problems inherently involve uncertain, linearised and potentially dynamic environments. Planning too far ahead is meaningless and incurs significant computational expense. Not looking ahead may fail to capture potential benefits and lead to undesirable variations in control. This implies a compromise is required in choosing the planning horizon and control parameterisation. With knowledge of the system dynamics and current state, a spectrally suitable action parameterisation can be made, optimised and updated.

Within a small region, perturbations about the open-loop trajectory can be used to

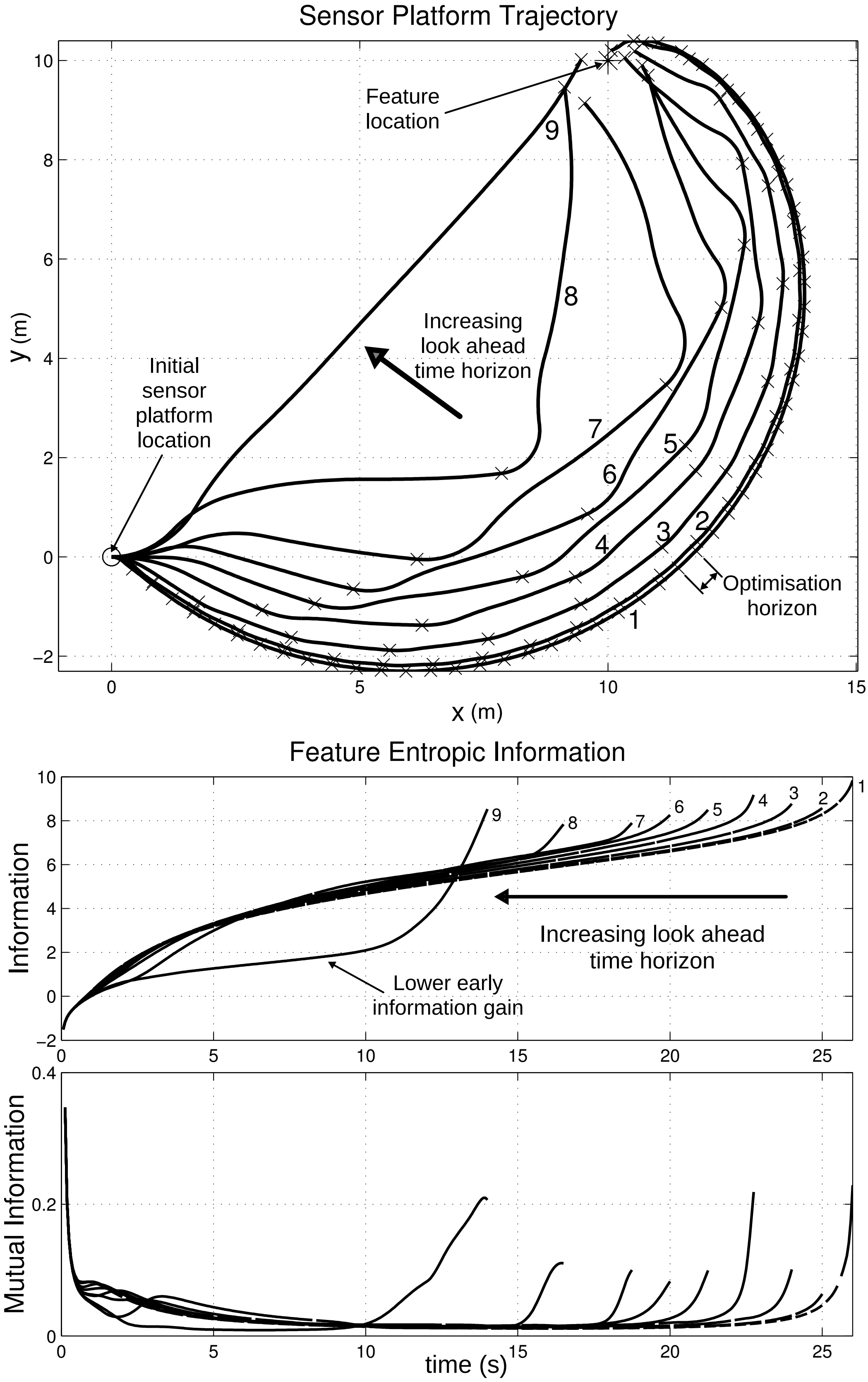


Figure 4.10: Solution trajectories investigating the influence of planning time horizon. As the optimisation time horizon increases, the solutions tend to initially reduce range to the feature, thus increasing the value of subsequent bearing changes.

obtain a feed-back solution. The scheme implemented in this example only updates the control solution at the end of each open-loop stage. An alternative would be to incorporate a perturbed feed-back rule as each observation is made.

4.3.6 Influence of Prior Information and Initial Range

To study the effects of feature prior information and initial range in bearings-only sensing, a number of solutions is presented with a fixed small optimisation horizon. Figure 4.11 shows fifteen solution trajectories with differing prior information. Figure 4.12 shows variations in the solution trajectories with initial range to the feature.

The first plot in each of these figures corresponds to the situation where there is no prior information concerning the range to the feature. The only information contained in the prior is the bearing to the feature. A closed form linearised solution to this situation with infinitesimally small look ahead is presented in [39]. In this case the initial action is to head perpendicular to the feature bearing. The heading rate is updated according to the law $\boldsymbol{u} = -2\dot{\theta}$. This results in a spiralling trajectory toward the feature. At small ranges to the feature, the linearisation on which this solution is based fails. The trajectories plotted agree with this solution. A small difference is introduced by the optimisation look ahead.

With alternate priors, the solution trajectories exhibit additional characteristics. The prior information alters the merit of reducing range over changing relative bearing. Consequently the optimal initial heading varies. The solution trajectories tend to move the sensor towards a location where observations provide the largest information gain. When the prior location estimate confidence is near circular there is little value in varying relative bearing. Thus, the solution reduces range exclusively. At relatively large ranges the value of bearing variations is also reduced. This effect is evident in Figure 4.12.

Intuitively, higher prior information allows development of more reliable longer term plans. A remarkable result is that the short term plan exhibits characteristics of the

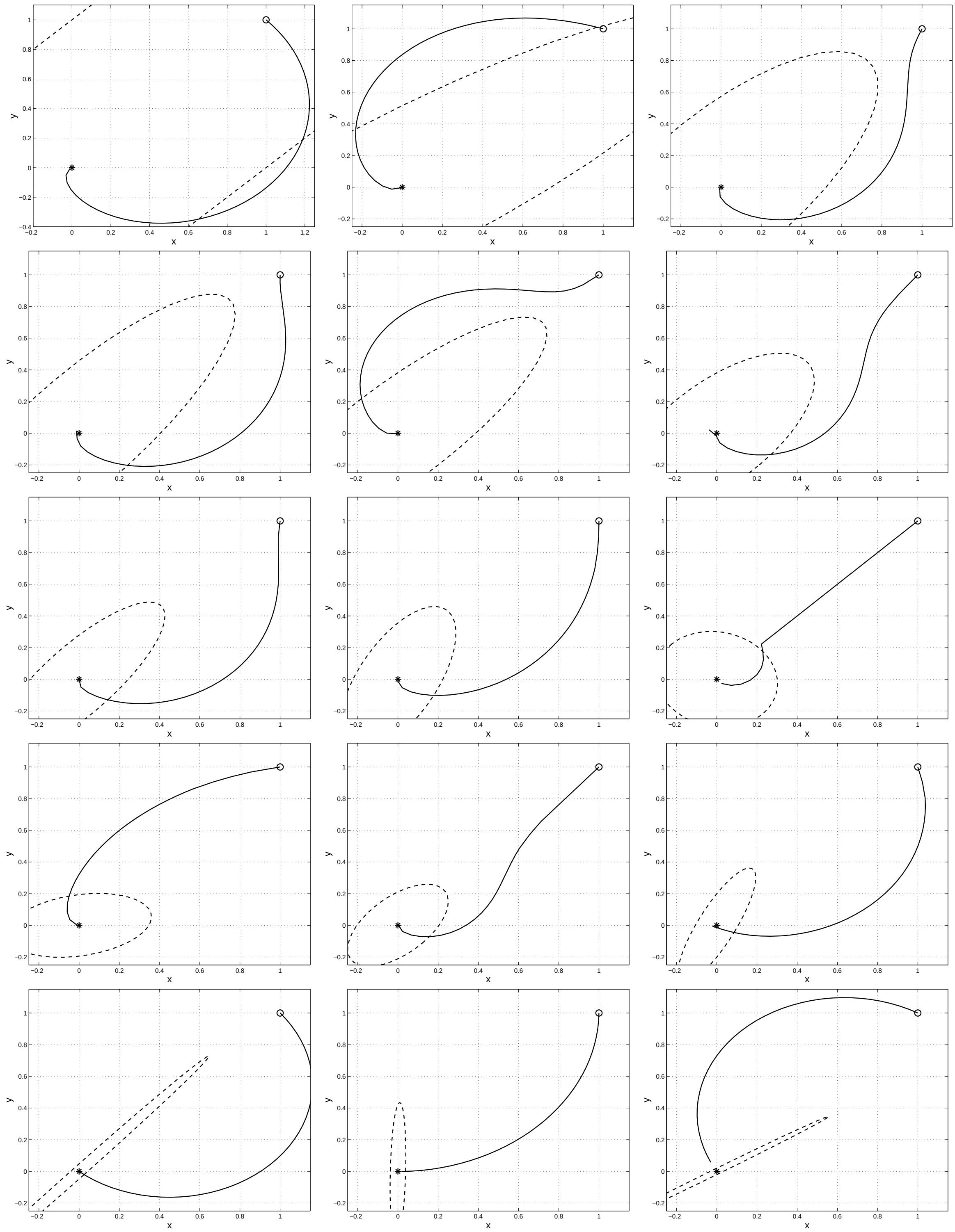


Figure 4.11: Fifteen trajectories showing the variation in solution characteristics with prior information. Prior information is indicated by the dashed confidence ellipse. The first plot corresponds to the situation where there is no initial range information. It is most interesting to observe that as the prior uncertainty is lower and more circular, the path to the feature is more direct.

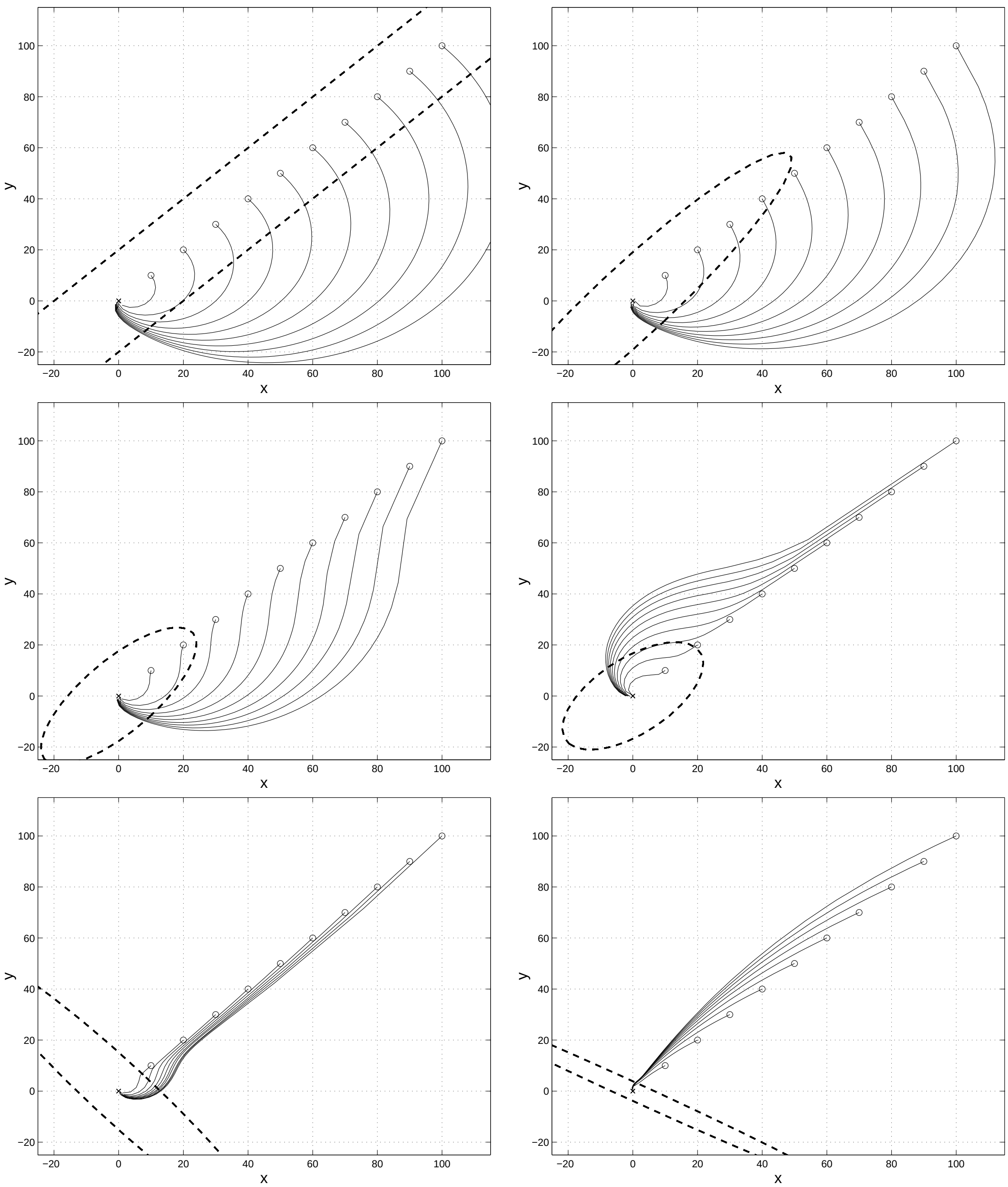


Figure 4.12: Influence of initial range to feature on sensing trajectory. Trajectories are shown for varied range with six different priors. Prior information is indicated by the dashed confidence ellipse. The control objective amounts to a tradeoff between reducing range and changing bearing. This highlights that this tradeoff is dependent on both range and prior information.

longer term plan as prior knowledge increases. This argues that short term look ahead provides an acceptable solution in addition to reduced solution effort.

4.3.7 Investigation of Alternate Utility Metrics

Section 3.5.3 proposed using metrics related to information other than entropy and mutual information. The effect, performance and applicability of these can be evaluated by applying them to this localisation example.

One possible utility measure is provided by the trace of the information matrix as suggested in [99]. For the bearings-only problem, maximising the trace of the information matrix minimises the distance to the feature, since $\mathrm{trace}\left(\mathbf{I}(k)\right) = \frac{1}{\sigma^2 r^2}$. The trace thus fails to capture the vital dependence on bearing in the sensor information or value prior information. The resulting trajectory, a straight line to the expected location of the feature, gathers the least information.

Figure 4.13 shows the variation in observer trajectories for three different cost functions with one step ahead optimisation:

1. $\max \mid \mathbf{Y}(k \mid k-1) \mid$
2. $\min \mathrm{trace}\left(\mathbf{Y}(k \mid k-1)^{-1}\right)$
3. $\max(\min \mathrm{eig}\left(\mathbf{Y}(k \mid k-1)\right))$

The differences in solutions are to be expected as the metrics weight contributions of the elements of the prior and observation information differently. To some extent, the most appropriate metric is application specific. Some applications may not wish to consider the probabilistic model as a whole. This could include optimising or bounding the uncertainty of a projection of the state estimate.

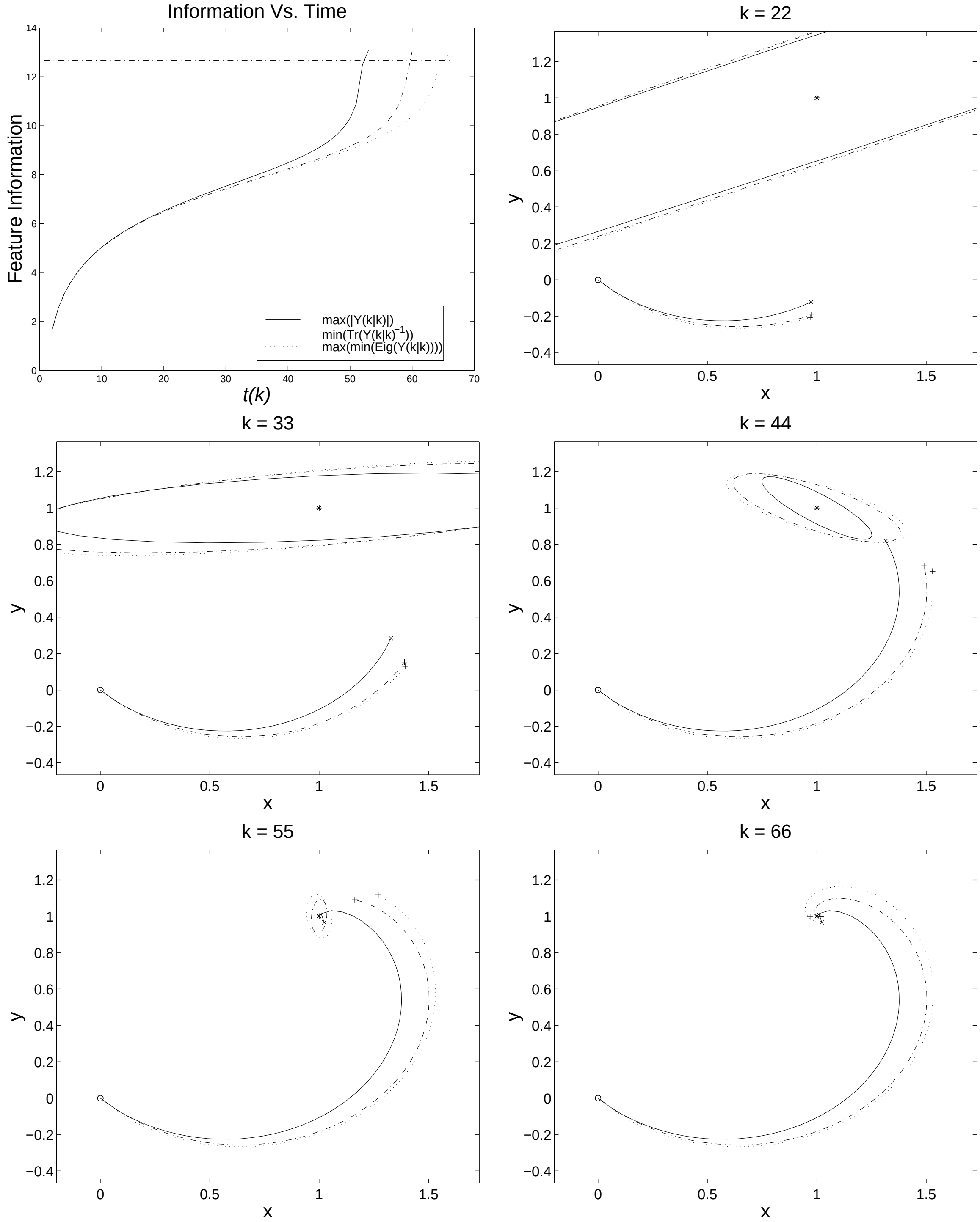


Figure 4.13: Comparison of information based utility measures. Trajectories are shown for three metrics: $\max \mid \mathbf{Y}(k \mid k-1) \mid$ (solid), $\min trace(\mathbf{Y}(k \mid k-1)^{-1})$ (dash dot) and $\max(\min eig(\mathbf{Y}(k \mid k-1)))$ (dotted). The etropic measure provides the best solution and has the lowest associated numerical effort.

4.3.8 Implications for Active Sensing

Active path control of a sensor can yield significant benefits. In the case of a bearing-only localisation problem, as described in this section, motion is essential to achieve accurate location estimates. By controlling or optimising the path taken by the sensor, substantial improvements in localisation performance may be obtained. Selection of a sensor motion based on entropic information maximisation results in a solution whose characteristics are intuitively appropriate and correct.

The example highlights the importance of control parameterisation and optimisation time horizon on solution performance. In a linearised, uncertain and unstructured world it is impossible to plan an entire task to completion. As uncertainty is reduced, the time over which predictions are reliable increases. Parameterisation of the control inputs admits a sub-optimal approximate solution. The time scales of system dynamics and performance measures indicate suitable bandwidth and horizon for control parameterisation. This helps to limit the computation required for an acceptable solution. The example demonstrates that this approach, illustrated in figure 4.14 , provides an effective solution to the active sensing problem.

4.4 Value in Multi-Vehicle Multi-Sensor Systems

In this section, the control of multi-vehicle multi-sensor systems is considered as a team decision problem in the form described in Section 2.3. Decentralised sensor systems are naturally considered in this framework. Value, in the sense of utility associated with a decision maker's action, and its dependence and influence on the actions of other members is central to the team decision problem. The underlying information in the sensor fusion problem forms a well defined utility structure. This section seeks to better understand the relationship between the value of a decision maker's actions, and the team decision problem solution characteristics.

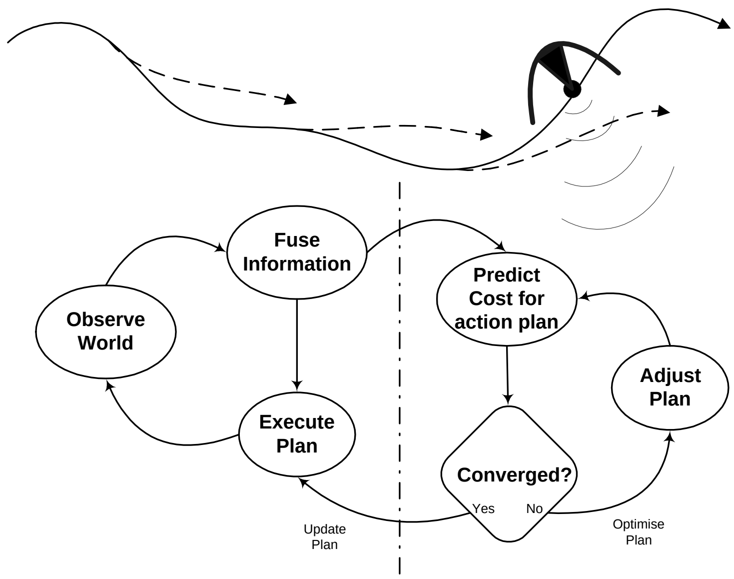


Figure 4.14: Predictive control for active sensing, based on expected utility with intermittent feedback through fusion of observed information

The team decision problem allows each decision maker to have different individual utility. There is no need for a global utility. The optimal solution is the actions that jointly maximise the team member's individual utility functions. While this idea is powerful and generic, it does not provide an intuitive view of the interrelation between members of decentralised decision making teams. More insight is provided by imposing additional structure in the form of a value.

The team sensing problem possesses inherent structure. Each team member estimates the state and maintains a measure of uncertainty about a list of objects in the environment. Decision makers may possess complementary sensory capability regarding a common state. The decision makers are coupled through this common information.

Value amongst decision makers is crucial to coordination and cooperation. Coupled utility may give rise to coupled, coordinated actions. Negotiation procedures allow *a priori* influence in utility among decision makers. This allows cooperation between team members.

Utility measures that do not reflect the true global value are of interest. This arises from the measures associated with the information available to or considered by an individual decision maker. Fundamental to cooperation is the effect on an indiviual's action caused by considering other decision makers. This section explores this issue by considering the the local decision problem from an individual and team perspective. Implications for the nature of cooperation and the solution complexity are established. This is illustrated through a simple two sensor feature localisation problem.

4.4.1 Local Partial utility and Global Utility

In addition to the individual's utility with in the team $\mathbf{J}_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$, it is of use to define the notion of a partial utility $\tilde{\mathbf{J}}_i(\boldsymbol{u}_i)$. This is a measure of the utility associated with an individual acting without knowledge or consideration of the other decision makers. This considers only accumulated prior knowledge and the information gathered locally. Extreme care must be applied in using the concept of partial utility. The partial utilities alone do not reflect the group utility. There is no reason ever to expect them to provide a conservative approximation to the group utility. They are of interest due to reduced complexity of the local solution.

Maximising local utility is often referred to as the "greedy" solution. This is a most inappropriate title as the result is almost certainly not the solution for the decision maker when the team is considered. By definition, the Nash Solution is one where no decision maker has an incentive to deviate. There is nothing "greedy" or "selfish" about acting without considering the influence of others. The "obtuse" solution may be a more apt title.

Partial utility measures do not account for the coupling between decisions and do not necessarily fully reflect the true global utility. It is important to make a distinction between coupled utility and coupled actions. If two decision makers' utilities are coupled, it does not imply their actions will change if the coupling were ignored. This idea leads to an important question: *When does maximising partial local utilities achieve the global* *optimal action?* This has significant implications for decentralised decision making. Identifying when individual actions based on partial local utility approximations are globally optimal provides a means for decomposing the team below the level at which actions are coupled in group utility.

4.4.2 Utility Structure Decomposition

Insight is sought into the structure of utility in a system of cooperative decision makers. While formulation of the team decision problem caters for each individual having a different utility function, its optimisation offers little insight into the decentralised decision making problem. An alternate structure is required that provides value measures to individual decision makers, while capturing the influence decision makers have on each other. To enable this, the notion of partial utility $\tilde{\mathbf{J}}_i(\boldsymbol{u}_i)$ is employed. An additional term $\mathbf{J}_c(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$, captures the coupling between decision makers. Partial local utility measures, when combined with the coupling function, are equivalent to the global utility.

A most desirable utility structure would be additive partial utility. This form is attractive due to the simple structure of its derivatives with respect to the decision variable. An additive global structure is

$$\mathbf{J}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) = \sum_{i=1}^{n} \tilde{\mathbf{J}}_i(\boldsymbol{u}_i) + \mathbf{J}_c(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) \tag{4.11}$$

with individual team member utility

$$\mathbf{J}_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) = \tilde{\mathbf{J}}_i(\boldsymbol{u}_i) + \frac{1}{n}\mathbf{J}_c(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n).$$

An alternative and more complicated form consists of multiplicative partial utilities,

$$\mathbf{J}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) = \mathbf{J}_c(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) \prod_{i=1}^{n} \tilde{\mathbf{J}}_i(\boldsymbol{u}_i). \tag{4.12}$$

Other functional forms, including combinations of 4.11 and 4.12 are of interest. Attention focuses on Equation 4.11 as the additive properties of information described in Chapter 3 suggest the use of this structure in sensing problems.

Potential situations exist where the utility coupling term $\mathbf{J}_c(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)$ may be further decomposed into a number of disjoint terms. This provides further simplification of the utility structure and required solution process. In the additive structure 4.11, it implies the global problem is composed of k independent sub-problems

$$\mathbf{J}_c(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) = \sum_{i=1}^{k} \mathbf{J}_c(\bar{\mathbf{u}}_i), \quad \bar{\mathbf{u}}_i \subset \mathbf{u}, \bar{\mathbf{u}}_i \cap \bar{\mathbf{u}}_j = 0, i \neq j.$$

Decision makers are members of independent coalitions within which utility is coupled.

4.4.3 Optimising Decomposed Utilities

A common approach in solving decision and control problems is to convert them into mathematical programming problems through parameterisation. For this reason it is of interest to recall a simple global parameter optimisation problem:

$$\mathbf{u}^{\star} = \arg\max_{\mathbf{u}} \mathbf{J}(\mathbf{u}), \quad \mathbf{u} = \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n\} \in \mathbf{U} \subset \mathbb{R}^n \tag{4.13}$$

The conditions for a local maximum are

$$\frac{\partial \mathbf{J}(\mathbf{u})}{\partial \boldsymbol{u}_i} = 0, \quad i = 1, \ldots, n \tag{4.14}$$

and that the $(n \times n)$ *Hessian* matrix be negative semidefinite

$$\frac{\partial^2 \mathbf{J}(\mathbf{u})}{\partial \mathbf{u}^2} \leq 0 \tag{4.15}$$

All points satisfying 4.14 are known as *stationary* points. The nature of these points may be determined from the eigenvalues of the *Hessian*:

1. *Maximum*: All eigenvalues are negative.

2. *Minimum*: All eigenvalues are positive.

3. *Saddle point*: Some eigenvalues are positive and some negative.

4. *Singular*: One or more eigenvalues are zero. Additional information is required to determine if such a point is an extremum.

The signs of determinants of the *principle minors* of the *Hessian* can be used to test for maxima rather than directly testing the sign of the eigenvalues. For two parameter systems $\mid \frac{\partial^2 \mathbf{J(u)}}{\partial \mathbf{u}^2} \mid > 0$ implies the eigenvalues are either both positive or both negative determined by the sign of $\frac{\partial^2 \mathbf{J(u)}}{\partial \boldsymbol{u}_1{}^2}$. $\mid \frac{\partial^2 \mathbf{J(u)}}{\partial \mathbf{u}^2} \mid < 0$ implies a positive and negative eigenvalue. $\mid \frac{\partial^2 \mathbf{J(u)}}{\partial \mathbf{u}^2} \mid = 0$ implies one or more zero eigenvalues, hence a *singular* stationary point with higher derivatives required to test for an extremum.

This analysis can be applied to the additive utility structure 4.11 to investigate the relationship between the individual and team optimal solutions.

4.4.4 Levels of Coordination and Cooperation

Section 2.4.1 discussed and attempted to define notions of coordination and cooperation. Coordination and cooperation can only occur between decision makers if their individual utility measures are coupled through their state and actions. This section explores the levels of coordination and cooperation that arise in application of additive partial utility structure 4.11 to the parameter optimisation problem 4.13. The objective is to address the following questions:

1. When and how are the decision makers decision processes coupled?

2. When are the actions that maximise partial utilities globally optimal?

3. Do non-maximal stationary points of partial utility ever become globally optimal?

4. How complex is finding the global optimal cooperative solution?

The conditions for global stationary points are obtained by differentiating 4.11.

$$\frac{\partial \mathbf{J}(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial \boldsymbol{u}_i} = \frac{\partial \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i} + \frac{\partial \mathbf{J}_c(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial \boldsymbol{u}_i} = 0, \quad i = 1,\ldots,n \tag{4.16}$$

The elements of the *Hessian* are

$$\frac{\partial^2 \mathbf{J}(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial {\boldsymbol{u}_i}^2} = \frac{\partial^2 \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial {\boldsymbol{u}_i}^2} + \frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial {\boldsymbol{u}_i}^2}, \quad i = 1,\ldots,n$$

$$\frac{\partial^2 \mathbf{J}(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial \boldsymbol{u}_i \partial \boldsymbol{u}_j} = \frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial \boldsymbol{u}_i \partial \boldsymbol{u}_j}, \quad i \neq j. \tag{4.17}$$

Equations 4.16 and 4.17 indicate how the coupling term influences the global topology and the locations of the global stationary points relative to the stationary points of the partial utility. From this the following situations emerge:

1. *Cooperation Not Required and Not Beneficial*

   No coupling in utility between actions,

   $$\frac{\partial \mathbf{J}_c(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial \boldsymbol{u}_i} = \frac{\partial \mathbf{J}_c(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial \boldsymbol{u}_j} = 0 \quad \forall \ \{\boldsymbol{u}_i, \boldsymbol{u}_j\} \in \mathbf{U} \subset \mathbb{R}$$

   implies $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ are not coupled and there is no benefit from cooperation. The actions are optimised independently.

2. *Elementary Cooperation Beneficial*

   Coupling between actions appears in the global utility (i.e. $\mathbf{J}_c(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n) \neq 0$) and the condition

   $$\left.\frac{\partial \mathbf{J}_c(\boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)}{\partial \boldsymbol{u}_i}\right|_{\boldsymbol{u}_i=\boldsymbol{u}_i^*} = 0, \text{ for any } \boldsymbol{u}_i^* \in \left\{\boldsymbol{u}_i : \frac{\partial \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i} = 0\right\} \text{ for } i = 1,\ldots,n$$

   is satisfied. This is a special case where some combinations of partial stationary points are global stationary points, hence, potential maxima. Each combination can be tested with three possible outcomes:

(i) The local maxima are the global maximum. In case (i), the individual utilities are coupled but the optimal actions are independent. The global solution is the set of actions maximising partial utility. This is of particular interest as it provides a means of decomposing problems to a level below that achieved through identifying coupling in utility.

(ii) The global maximum is a combination of the local maxima, minima, saddle and singular stationary points. The actions and utilities of the decision makers are coupled. Cooperation through some form of communication is required to identify and resolve this case. However, the global solution is immediately known from partial problems. With the possibility that a decision maker's best action is the worst obtained based on partial information. Realising this situation yields significant benefit to the solution process.

(iii) All combinations are global minima, saddle points or non-maximising singular points. Global maxima must exist and lie away from these points. The maximising solution is not locally convex in the region of the partial stationary points. *Full cooperation* between decision makers is required to reach the global maximum.

3. *Full Cooperation Required*

Coupling moves global maximum away from all partial stationary points

$$\frac{\partial \mathbf{J}_c(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n)}{\partial \boldsymbol{u}_i}\bigg|_{\boldsymbol{u}_i = \boldsymbol{u}_i^*} \neq 0 \ \forall \ \boldsymbol{u}_i^* \in \left\{ \boldsymbol{u}_i : \frac{\partial \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i} = 0 \right\} \text{ for } i = 1, \ldots, n.$$

Cooperation is required to obtain the global solution through some form of negotiation or bargaining. Solving the partial problems does not yield the global solution. The global solution may lie in a locally convex region near a combination of local fixed points. However, coupling may result in significant differences between topology of global and local utility. This can be determined from the curvature of the utility with respect to the actions at the partial stationary points; resulting in an indication of the effort and complexity required to determine the solution.

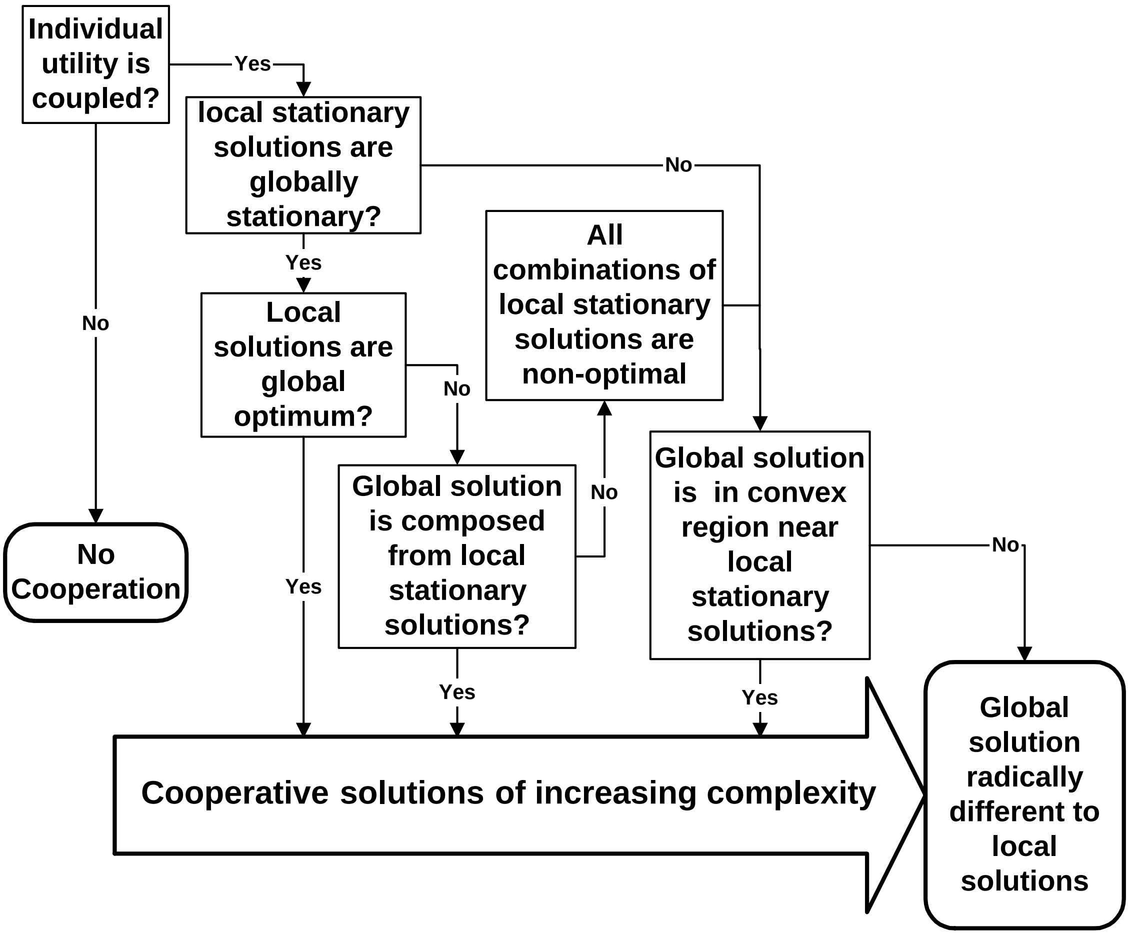


Figure 4.15: Levels of cooperation and solution complexity

From this reasoning, tests emerge that answer all of the questions posed at the beginning of this section. A simple example follows to illustrate the effect of coupling between decision makers with this utility structure.

4.5 A Multi-Platform Example: Feature Localisation With Two Range-Only Sensors

A simple example is presented to clarify and illuminate the issues regarding control of and value in multi-sensor teams. This example explores utility decomposition, partial and global solutions, and cooperation. It has significant implications for the problem of distributed versus global decision making.

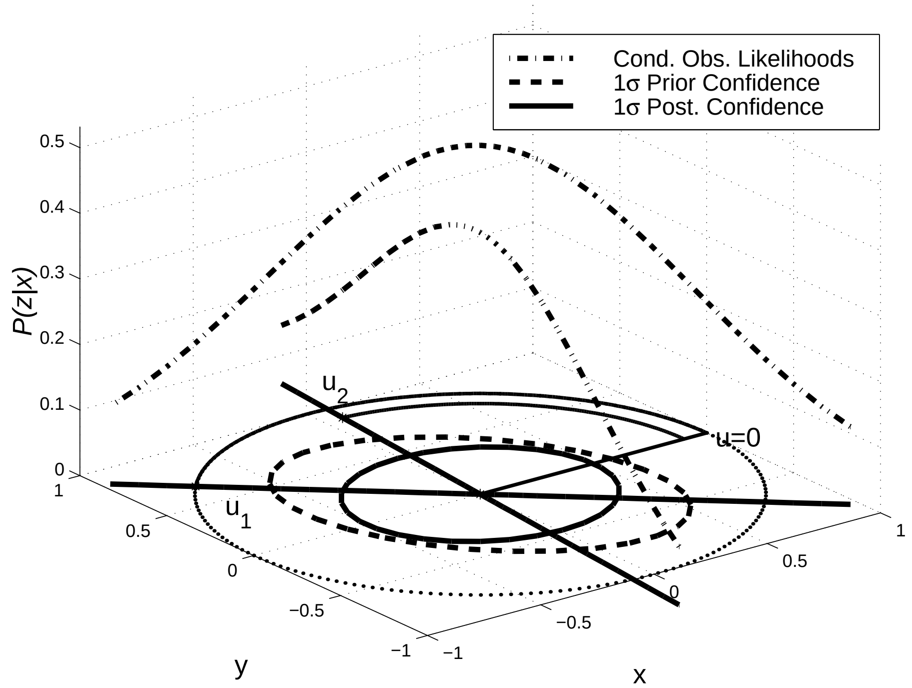


Figure 4.16: Single feature localisation with two range-only sensors. This illustrates how the control variables are mapped to utility. The actions are the orientation of the conditional observation likelihoods. The utility measure is the volume of the posterior distribution, when these observations are combined with prior information.

4.5.1 Formulation as Problems in Local and Global Utility

Two sensor platforms make range observations $\boldsymbol{z}_i,\ i = 1, 2$ of a features location $\mathbf{x}$ in the xy-plane. The control action $\boldsymbol{u}_i,\ i = 1, 2$ available to observers is their bearing relative to the expected location of the feature $\hat{\mathbf{x}}(k \mid k-1)$. The control objective is to maximise the expected entropic information at step k given prior inverse covariance $\mathbf{Y}(k \mid k-1)$ and the observation inverse covariance updates $\mathbf{I}_1(k \mid k-1, \boldsymbol{u}_1)$ and $\mathbf{I}_2(k \mid k-1, \boldsymbol{u}_2)$. The posterior inverse covariance is given by the update stage of the information filter

$$\mathbf{Y}(k \mid k) = \mathbf{Y}(k \mid k-1) + \mathbf{I}_1(k \mid k-1, \boldsymbol{u}_1) + \mathbf{I}_2(k \mid k-1, \boldsymbol{u}_2).$$

The posterior entropic information is

$$\boldsymbol{i}(k) = \frac{1}{2}\log\left((2\pi\boldsymbol{e})^2 \mid \mathbf{Y}(k \mid k) \mid\right)$$

$$\arg \max_{\{\boldsymbol{u}_1, \boldsymbol{u}_2\}} \boldsymbol{i}(k) \equiv \arg \max_{\{\boldsymbol{u}_1, \boldsymbol{u}_2\}} \mid \mathbf{Y}(k \mid k) \mid .$$

This problem is equivalent to maximising the utility function

$$\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) = \mid \mathbf{Y}(k \mid k-1) + \mathbf{I}_1(k \mid k-1, \boldsymbol{u}_1) + \mathbf{I}_2(k \mid k-1, \boldsymbol{u}_2) \mid . \tag{4.18}$$

The global utility in Equation 4.18 can be decomposed into the form [1]

$$\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) = \tilde{\mathbf{J}}_1(\boldsymbol{u}_1) + \tilde{\mathbf{J}}_2(\boldsymbol{u}_2) + \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2) \tag{4.19}$$

where

$$\begin{aligned} \tilde{\mathbf{J}}_1(\boldsymbol{u}_1) &= \mid \mathbf{Y}(k \mid k-1) + \mathbf{I}_1(k \mid k-1, \boldsymbol{u}_1) \mid \\ \tilde{\mathbf{J}}_2(\boldsymbol{u}_2) &= \mid \mathbf{Y}(k \mid k-1) + \mathbf{I}_2(k \mid k-1, \boldsymbol{u}_2) \mid \\ \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2) &= \mid \mathbf{I}_1(k \mid k-1, \boldsymbol{u}_1) + \mathbf{I}_2(k \mid k-1, \boldsymbol{u}_2) \mid - \mid \mathbf{Y}(k \mid k-1) \mid . \end{aligned} \tag{4.20}$$

The partial utilities and coupling term in Equation 4.20 have an interpretation in terms of information [2]. $\tilde{\mathbf{J}}_1(\boldsymbol{u}_1)$ and $\tilde{\mathbf{J}}_2(\boldsymbol{u}_2)$ are the information gains for the individual isolated sensing actions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. $\mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is the information common to the observations. Thus, the combined information gain is the sum of the individual gains less the common

[1] For any 2x2 matrices A, B and C, $\mid A+B+C \mid = \mid A+B \mid + \mid A+C \mid + \mid B+C \mid - \mid A \mid - \mid B \mid - \mid C \mid$, proof by algebraic substitution is trivial. Note $\mid \mathbf{I}_i(k \mid k-1, \boldsymbol{u}_i) \mid = 0$, $i = 1, 2$ in this case.

[2] Although these are not equivalent to the formal definitions

information.

4.5.2 Modelling Observation Information

The observation model is $\mathbf{z}_i(k) = \mathbf{h}(\mathbf{x}(k), \boldsymbol{u}_i, \mathbf{v}_i(k))$ where $\mathbf{v}_i(k)$ is taken to be a zero-mean uncorrelated Gaussian sequence with variance $\mathrm{E}\{\mathbf{v}_i(k)\mathbf{v}_i^T(k)\} = \mathbf{R}_i = \boldsymbol{\sigma}_i^2$, The observation model is range only

$$\mathbf{h}(\mathbf{x}(k)) = \mathbf{r}(k) + \mathbf{v}_i(k) = \sqrt{(x(k) - x_i(k))^2 + (y(k) - y_i(k))^2} + \mathbf{v}_i(k).$$

The Jacobian with respect to feature state estimate is

$$\begin{aligned} \mathbf{H}_i(\hat{\mathbf{x}}(k \mid k-1)) &= \begin{bmatrix} \frac{-\hat{x}(k|k-1)+x_i(k)}{\sqrt{(\hat{x}(k|k-1)-x_i(k))^2+(\hat{y}(k|k-1)-y_i(k))^2}} & \frac{\hat{y}(k|k-1)-y_i(k)}{\sqrt{(\hat{x}(k|k-1)-x_i(k))^2+(\hat{y}(k|k-1)-y_i(k))^2}} \end{bmatrix} \\ &= \begin{bmatrix} -\cos(\boldsymbol{u}_i(k)) & \sin(\boldsymbol{u}_i(k)) \end{bmatrix} \end{aligned} \tag{4.21}$$

where $\boldsymbol{u}_i(k)$ is the bearing angle from observer i to the estimated feature location, which is taken to be the control variable. The expected observation information for this sensor model is given by

$$\begin{aligned} \mathbf{I}_i(k, \boldsymbol{u}_i) &= \mathbf{H}_i^T(\hat{\mathbf{x}}(k \mid k-1))\mathbf{R}_i^{-1}\mathbf{H}_i(\hat{\mathbf{x}}(k \mid k-1)) \\ &= \frac{1}{\boldsymbol{\sigma}_i^2}\begin{bmatrix} \cos^2(\boldsymbol{u}_i(k)) & \sin(\boldsymbol{u}_i(k))\cos(\boldsymbol{u}_i(k)) \\ \sin(\boldsymbol{u}_i(k))\cos(\boldsymbol{u}_i(k)) & \sin^2(\boldsymbol{u}_i(k)) \end{bmatrix}. \end{aligned} \tag{4.22}$$

4.5.3 Partial and Global Solutions

The models of sensor observation information are substituted into Equations 4.20 to provide the individual partial utility measures $\tilde{\mathbf{J}}_1(\boldsymbol{u}_1)$ and $\tilde{\mathbf{J}}_2(\boldsymbol{u}_2)$ and their coupling in team utility $\mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)$. The form of the utility representation and its derivatives is as follows. Let

$$\mathbf{Y}(k \mid k-1) = \begin{bmatrix} \mathbf{Y}_x & \mathbf{Y}_{xy} \\ \mathbf{Y}_{xy} & \mathbf{Y}_y \end{bmatrix}. \tag{4.23}$$

This prior must be positive semi-definite so

$$\mathbf{Y}_x \geq 0, \quad \mathbf{Y}_y \geq 0, \quad \mathbf{Y}_{xy}^2 \leq \mathbf{Y}_x \mathbf{Y}_y. \tag{4.24}$$

From Equation 4.20, the partial local utility measure for each sensor is

$$\tilde{\mathbf{J}}_i(\boldsymbol{u}_i) = \mathbf{Y}_x \mathbf{Y}_y - \mathbf{Y}_{xy}^2 + \frac{1}{\boldsymbol{\sigma}_i^2}(\mathbf{Y}_x \sin(\boldsymbol{u}_i)^2 + \mathbf{Y}_y \cos(\boldsymbol{u}_i)^2 - 2\mathbf{Y}_{xy} \sin(\boldsymbol{u}_i)\cos(\boldsymbol{u}_i)), \tag{4.25}$$

with coupling given by

$$\mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2) = \frac{1}{2\boldsymbol{\sigma}_1^2 \boldsymbol{\sigma}_2^2}(1 - \cos(2(\boldsymbol{u}_1 - \boldsymbol{u}_2))). \tag{4.26}$$

The elements of the Jacobian are

$$\frac{\partial \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i} = \frac{1}{\boldsymbol{\sigma}_i^2}(\mathbf{Y}_x \sin(2\boldsymbol{u}_i) + \mathbf{Y}_y \sin(2\boldsymbol{u}_i) - 2\mathbf{Y}_{xy} \cos(2\boldsymbol{u}_i)) \tag{4.27}$$

$$\frac{\partial \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_1} = \frac{1}{\boldsymbol{\sigma}_1^2 \boldsymbol{\sigma}_2^2} \sin(2(\boldsymbol{u}_1 - \boldsymbol{u}_2)) \tag{4.28}$$

$$\frac{\partial \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_2} = -\frac{1}{\boldsymbol{\sigma}_1^2 \boldsymbol{\sigma}_2^2} \sin(2(\boldsymbol{u}_1 - \boldsymbol{u}_2)). \tag{4.29}$$

The elements of the Hessian are:

$$\frac{\partial^2 \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial {\boldsymbol{u}_i}^2} = \frac{2}{\boldsymbol{\sigma}_i^2}(\mathbf{Y}_x \cos(2\boldsymbol{u}_i) - \mathbf{Y}_y \cos(2\boldsymbol{u}_i) + 2\mathbf{Y}_{xy} \sin(2\boldsymbol{u}_i)) \tag{4.30}$$

$$\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial {\boldsymbol{u}_i}^2} = \frac{2}{\boldsymbol{\sigma}_1^2 \boldsymbol{\sigma}_2^2} \cos(2(\boldsymbol{u}_1 - \boldsymbol{u}_2)) \tag{4.31}$$

$$\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_1 \partial \boldsymbol{u}_2} = \frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_2 \partial \boldsymbol{u}_1} = -\frac{2}{\boldsymbol{\sigma}_1^2 \boldsymbol{\sigma}_2^2} \cos(2(\boldsymbol{u}_1 - \boldsymbol{u}_2)). \tag{4.32}$$

These equations describe local and global utility. From which local and global optimal actions are found [3]. The partial local utility $\tilde{\mathbf{J}}_i(\boldsymbol{u}_i)$, has two stationary points, $\frac{\partial \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i} = 0$

$$\begin{aligned} \tilde{\boldsymbol{u}}^+ &= \arg\max_{\boldsymbol{u}_i} \tilde{\mathbf{J}}_i(\boldsymbol{u}_i) \\ &= \arctan\left(\frac{-1}{2\mathbf{Y}_{xy}}(\mathbf{Y}_x - \mathbf{Y}_y + \sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy})}\right) + k\pi, \quad k = 0, 1 \end{aligned} \tag{4.33}$$

$$\begin{aligned} \tilde{\boldsymbol{u}}^- &= \arg\min_{\boldsymbol{u}_i} \tilde{\mathbf{J}}_i(\boldsymbol{u}_i) \\ &= \arctan\left(\frac{-1}{2\mathbf{Y}_{xy}}(\mathbf{Y}_x - \mathbf{Y}_y - \sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy})}\right) + k\pi, \quad k = 0, 1 \end{aligned} \tag{4.34}$$

Note, $\tilde{\boldsymbol{u}}^+$ and $\tilde{\boldsymbol{u}}^-$ differ by $\pi/2$ and correspond to the directions of the eigenvectors of the prior distribution. On substitution it is found that, $\left.\frac{\partial^2 \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i{}^2}\right|_{\boldsymbol{u}_i = \tilde{\boldsymbol{u}}_i^+} < 0$, hence $\tilde{\boldsymbol{u}}^+$ is the maximising solution.

With the aid of the Matlab Symbolic Math Toolbox [16], it is possible to determine an algebraic form for the globally optimal actions. Solutions are found to the system of nonlinear equations $\frac{\partial \mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_1} = 0$ and $\frac{\partial \mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_2} = 0$. Applying second derivative tests to each leaves two optimal solutions for each action.

$$\begin{aligned} \boldsymbol{u}_1^* &= \arg\max_{\{\boldsymbol{u}_1, \boldsymbol{u}_2\}} \mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) \\ &= \frac{1}{2}\arctan\left(\frac{2\mathbf{Y}_{xy}\left[-\boldsymbol{\sigma}_1^4 + \boldsymbol{\sigma}_2^4 + \boldsymbol{\sigma}_1^4\boldsymbol{\sigma}_2^4\mathbf{X}_1 \mp (\mathbf{Y}_x - \mathbf{Y}_y)\mathbf{Y}_{xy}\sqrt{\mathbf{X}}_2\right]}{\boldsymbol{\sigma}_1^4\boldsymbol{\sigma}_2^4(\mathbf{Y}_x - \mathbf{Y}_y)\left[\mathbf{X}_1 + \frac{1}{\boldsymbol{\sigma}_1^4} - \frac{1}{\boldsymbol{\sigma}_2^4}\right] + 2\mathbf{Y}_{xy}\sqrt{\mathbf{X}}_2}\right) \\ &\quad + k\pi, \quad k = 0, 1 \end{aligned} \tag{4.35}$$

$$\begin{aligned} \boldsymbol{u}_2^* &= \arg\max_{\{\boldsymbol{u}_1, \boldsymbol{u}_2\}} \mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) \\ &= \frac{1}{2}\arctan\left(\frac{2\mathbf{Y}_{xy}\left[\boldsymbol{\sigma}_1^4 - \boldsymbol{\sigma}_2^4 + \boldsymbol{\sigma}_1^4\boldsymbol{\sigma}_2^4\mathbf{X}_1 \pm (\mathbf{Y}_x - \mathbf{Y}_y)\mathbf{Y}_{xy}\sqrt{\mathbf{X}}_2\right]}{\boldsymbol{\sigma}_1^4\boldsymbol{\sigma}_2^4(\mathbf{Y}_x - \mathbf{Y}_y)\left[\mathbf{X}_1 - \frac{1}{\boldsymbol{\sigma}_1^4} + \frac{1}{\boldsymbol{\sigma}_2^4}\right] - 2\mathbf{Y}_{xy}\sqrt{\mathbf{X}}_2}\right) \\ &\quad + k\pi, \quad k = 0, 1 \end{aligned} \tag{4.36}$$

[3] This example was selected to compare local and global optimal actions due to the existence of analytic solutions.

Where

$$\mathbf{X}_1 = (\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2$$

$$\mathbf{X}_2 = -\left(\boldsymbol{\sigma}_1^4\boldsymbol{\sigma}_2^4\mathbf{X}_1 - (\boldsymbol{\sigma}_1^2 - \boldsymbol{\sigma}_2^2)^2\right)\left(\boldsymbol{\sigma}_1^4\boldsymbol{\sigma}_2^4\mathbf{X}_1 - (\boldsymbol{\sigma}_1^2 + \boldsymbol{\sigma}_2^2)^2\right).$$

Surfaces of these solutions for a range of feature prior information are shown in Figure 4.17 for the situation corresponding to $\boldsymbol{\sigma}_1 = 1.5$, $\boldsymbol{\sigma}_2 = 2$ and Y_x=.75 fixed while varying Y_y and Y_{xy}.

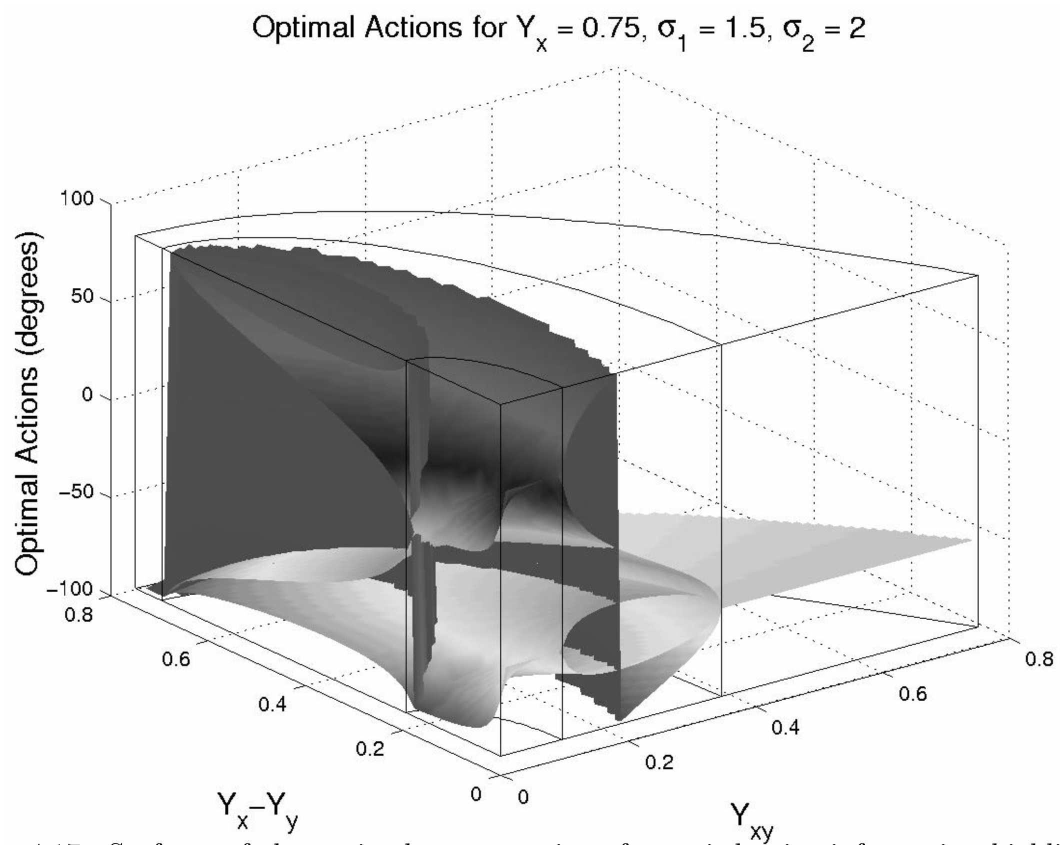


Figure 4.17: Surfaces of the optimal sensor actions for varied prior information highlighting the ocurrance bifurcation in the optimum group decisions. There are three distinct regions. In which the optimal sensing actions are equal, differ by 90 degrees or are one of two symmetrical solutions.

4.5.4 Determining Cooperation Boundaries

On inspecting the derivatives of the utility coupling term Equation 4.26 it is found that

$$\frac{\partial \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_1} = \frac{\partial \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_2} = 0, \text{ if } \boldsymbol{u}_1 - \boldsymbol{u}_2 = \pm k\pi/2,\ k = 0, 1, 2, \ldots$$

The local stationary solutions $\tilde{\boldsymbol{u}}^+$ and $\tilde{\boldsymbol{u}}^-$ given by Equations 4.33 and 4.34 differ by $\pi/2$. Hence, any combination of the local stationary solutions is a stationary solution of the global utility. Specifically,

$$\{\boldsymbol{u}_1^*, \boldsymbol{u}_2^*\} \in [\{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^+\}, \{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^-\}, \{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^+\}, \{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^-\}] \tag{4.37}$$

is a global stationary point, hence a potential global maximum. The conditions for this to occur will now be determined. Recall that $\{\boldsymbol{u}_1^*, \boldsymbol{u}_2^*\}$ is a maximum if

$$\left.\frac{\partial^2 \mathbf{J}(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_1}^2}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}} < 0 \quad \text{and}$$
$$\left.\frac{\partial^2 \mathbf{J}(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_1}^2}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}} \left.\frac{\partial^2 \mathbf{J}(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_2}^2}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}} - \left(\left.\frac{\partial^2 \mathbf{J}(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_1}\partial \boldsymbol{u_2}}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}}\right)^2 > 0.$$

For the coupled utility formulation Equation 4.19, this condition becomes

$$\begin{gathered}\left.\frac{\partial^2 \tilde{\mathbf{J}}_1(\boldsymbol{u_1})}{\partial \boldsymbol{u_1}^2}\right|_{\boldsymbol{u}_1^*} + \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_1}^2}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}} < 0 \quad \text{and} \\ \left(\left.\frac{\partial^2 \tilde{\mathbf{J}}_1(\boldsymbol{u_1})}{\partial \boldsymbol{u_1}^2}\right|_{\boldsymbol{u}_1^*} + \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_1}^2}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}}\right)\left(\left.\frac{\partial^2 \tilde{\mathbf{J}}_2(\boldsymbol{u_2})}{\partial \boldsymbol{u_2}^2}\right|_{\boldsymbol{u}_2^*} + \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_2}^2}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}}\right) - \cdots \\ \left(\left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_1}\partial \boldsymbol{u_2}}\right|_{\{\boldsymbol{u}_1^*,\boldsymbol{u}_2^*\}}\right)^2 > 0.\end{gathered} \tag{4.38}$$

Substituting the stationary partial utility combinations of Equation 4.37 into the Equation 4.38 leads to inequalities for the prior and sensor observation information that establish when these cases are global maximal solutions. The four combinations are considered as follows

Combination 1: $\{\boldsymbol{u}_1^*, \boldsymbol{u}_2^*\} = \{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^+\}$

$\boldsymbol{u}_1^* - \boldsymbol{u}_2^* = \pm k\pi,\ k = 0, 1, 2, \ldots$ so

$$\left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_i}^2}\right|_{\{\tilde{\boldsymbol{u}}^+,\tilde{\boldsymbol{u}}^+\}} = \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}$$
$$\left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_1}\partial \boldsymbol{u_2}}\right|_{\{\tilde{\boldsymbol{u}}^+,\tilde{\boldsymbol{u}}^+\}} = \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u_1},\boldsymbol{u_2})}{\partial \boldsymbol{u_2}\partial \boldsymbol{u_1}}\right|_{\{\tilde{\boldsymbol{u}}^+,\tilde{\boldsymbol{u}}^+\}} = -\frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}.$$

Substituting the partial maximal $\tilde{\boldsymbol{u}}^+$ and minimal $\tilde{\boldsymbol{u}}^-$ actions Equations 4.33 and 4.34 into the Hessian for the partial utilities Equation 4.30 gives

$$\begin{aligned} \left.\frac{\partial^2 \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i{}^2}\right|_{\tilde{\boldsymbol{u}}^+} &= -\frac{2}{\boldsymbol{\sigma}_i^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} \\ \left.\frac{\partial^2 \tilde{\mathbf{J}}_i(\boldsymbol{u}_i)}{\partial \boldsymbol{u}_i{}^2}\right|_{\tilde{\boldsymbol{u}}^-} &= \frac{2}{\boldsymbol{\sigma}_i^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} \end{aligned}$$

Condition 4.38 becomes

$$\begin{gathered} -\frac{2}{\boldsymbol{\sigma}_1^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} + \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} < 0 \\ (-\frac{2}{\boldsymbol{\sigma}_1^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} + \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2})(-\frac{2}{\boldsymbol{\sigma}_2^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} + \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}) - \frac{4}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} > 0 \end{gathered}$$

or

$$4\mathbf{Y}_{xy}^2 + (\mathbf{Y}_x - \mathbf{Y}_x)^2 > (\frac{1}{\boldsymbol{\sigma}_1^2} + \frac{1}{\boldsymbol{\sigma}_2^2}). \tag{4.39}$$

Combination 2: $\{\boldsymbol{u}_1^*, \boldsymbol{u}_2^*\} = \{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^-\}$

$\boldsymbol{u}_1^* - \boldsymbol{u}_2^* = \pm k\pi/2,\ k = 1, 3, 5\ldots$ so

$$\begin{gathered} \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_i{}^2}\right|_{\{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^-\}} = -\frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} \\ \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_1 \partial \boldsymbol{u}_2}\right|_{\{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^-\}} = \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_2 \partial \boldsymbol{u}_1}\right|_{\{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^-\}} = \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}. \end{gathered}$$

Condition 4.38 becomes

$$\begin{gathered} -\frac{2}{\boldsymbol{\sigma}_1^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} - \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} < 0 \\ (-\frac{2}{\boldsymbol{\sigma}_1^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} - \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2})(\frac{2}{\boldsymbol{\sigma}_2^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} - \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}) - \frac{4}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} > 0 \end{gathered}$$

or

$$4\mathbf{Y}_{xy}^2 + (\mathbf{Y}_x - \mathbf{Y}_y)^2 < (\frac{1}{\boldsymbol{\sigma}_1^2} - \frac{1}{\boldsymbol{\sigma}_2^2}). \tag{4.40}$$

Which can only occur if $\boldsymbol{\sigma}_1 < \boldsymbol{\sigma}_2$.

Combination 3: $\{\boldsymbol{u}_1^*, \boldsymbol{u}_2^*\} = \{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^+\}$

$\boldsymbol{u}_1^* - \boldsymbol{u}_2^* = \pm k\pi/2,\ k = 1, 3, 5\dots$ so

$$\left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial {\boldsymbol{u}_i}^2}\right|_{\{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^+\}} = -\frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}$$
$$\left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_1 \partial \boldsymbol{u}_2}\right|_{\{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^+\}} = \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_2 \partial \boldsymbol{u}_1}\right|_{\{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^+\}} = \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}.$$

Condition 4.38 becomes

$$\frac{2}{\boldsymbol{\sigma}_1^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} - \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} < 0$$
$$\left(\frac{2}{\boldsymbol{\sigma}_1^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} - \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}\right)\left(-\frac{2}{\boldsymbol{\sigma}_2^2}\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} - \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}\right) - \frac{4}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} > 0$$

or

$$4\mathbf{Y}_{xy}^2 + (\mathbf{Y}_x - \mathbf{Y}_y)^2 < \left(\frac{1}{\boldsymbol{\sigma}_2^2} - \frac{1}{\boldsymbol{\sigma}_1^2}\right). \tag{4.41}$$

Which can only occur if $\boldsymbol{\sigma}_1 > \boldsymbol{\sigma}_2$.

Combination 4: $\{\boldsymbol{u}_1^*, \boldsymbol{u}_2^*\} = \{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^-\}$

As in case 1, $\boldsymbol{u}_1^* - \boldsymbol{u}_2^* = \pm k\pi,\ k = 0, 1, 2, \dots$ so

$$\left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial {\boldsymbol{u}_i}^2}\right|_{\{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^-\}} = \frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}$$
$$\left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_1 \partial \boldsymbol{u}_2}\right|_{\{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^-\}} = \left.\frac{\partial^2 \mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)}{\partial \boldsymbol{u}_2 \partial \boldsymbol{u}_1}\right|_{\{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^-\}} = -\frac{2}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2}.$$

Condition 4.38 becomes

$$\sqrt{(\mathbf{Y}_x - \mathbf{Y}_y)^2 + 4\mathbf{Y}_{xy}^2} + \frac{1}{\boldsymbol{\sigma}_1^2\boldsymbol{\sigma}_2^2} < 0.$$

Which can never happen.

4.5.5 Summary of Problem Solution

In addition to the requirement for a proper prior, the developments in Section 4.5.4 lead to three conditions where the global optimal is a combination of the local extremum. These conditions are now summarised:

if $4\mathbf{Y}_{xy}^2 + (\mathbf{Y}_x - \mathbf{Y}_x)^2 \geq (\frac{1}{\boldsymbol{\sigma}_1^2} + \frac{1}{\boldsymbol{\sigma}_2^2})$

The globally optimal actions are the actions that maximise local partial utility. This situation is not fully cooperative. The decision makers do not influence each others' actions at the optimum solution. Utility of the group action is not changed by team members knowing other decsion makers exist. However, the individual partial utilities are unconservative due to the information common to the group observations.

$$\arg \max_{\{\boldsymbol{u}_1, \boldsymbol{u}_2\}} \mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) \equiv \arg\{\max_{\boldsymbol{u}_1} \tilde{\mathbf{J}}_1(\boldsymbol{u}_1), \max_{\boldsymbol{u}_2} \tilde{\mathbf{J}}_2(\boldsymbol{u}_2)\}$$

else if $\boldsymbol{\sigma}_1 < \boldsymbol{\sigma}_2$ and $4\mathbf{Y}_{xy}^2 + (\mathbf{Y}_x - \mathbf{Y}_x)^2 \leq (\frac{1}{\boldsymbol{\sigma}_1^2} - \frac{1}{\boldsymbol{\sigma}_2^2})$

The globally optimal actions are for the first sensor to maximise its local partial utility and the second sensor to minimise its local partial utility. The decision makers must cooperate to realise the optimum group utility.

$$\max_{\{\boldsymbol{u}_1, \boldsymbol{u}_2\}} \mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) \equiv \arg\{\max_{\boldsymbol{u}_1} \tilde{\mathbf{J}}_1(\boldsymbol{u}_1), \min_{\boldsymbol{u}_2} \tilde{\mathbf{J}}_2(\boldsymbol{u}_2)\}$$

else if $\boldsymbol{\sigma}_1 > \boldsymbol{\sigma}_2$ and $4\mathbf{Y}_{xy}^2 + (\mathbf{Y}_x - \mathbf{Y}_x)^2 \leq (\frac{1}{\boldsymbol{\sigma}_2^2} - \frac{1}{\boldsymbol{\sigma}_1^2})$

The globally optimal actions are for the first sensor to minimise its local partial utility and the second sensor to maximise its local partial utility. The decision makers must cooperate to realise the optimum group utility.

$$\max_{\{\boldsymbol{u}_1, \boldsymbol{u}_2\}} \mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) \equiv \arg\{\min_{\boldsymbol{u}_1} \tilde{\mathbf{J}}_1(\boldsymbol{u}_1), \max_{\boldsymbol{u}_2} \tilde{\mathbf{J}}_2(\boldsymbol{u}_2)\}$$

else

Coupling in utility has shifted the global maxima away from the partial utility stationay points. $\{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^+\}$, $\{\tilde{\boldsymbol{u}}^+, \tilde{\boldsymbol{u}}^-\}$ and $\{\tilde{\boldsymbol{u}}^-, \tilde{\boldsymbol{u}}^+\}$ are global saddle points. Alternative maxima must lie between pairs of them. The decision makers must cooperate to achive the optimum group utility

Example solutions are considered to illustrate how the stucture of the utility and optimal decisions varies with the prior feature location information. Figure 4.18 indicates the details of seven cases along with the cooperation boundaries for the situation corresponding to $\boldsymbol{\sigma}_1 = 1.5$, $\boldsymbol{\sigma}_2 = 2$ and Y_x=.75 fixed while varying Y_y and Y_{xy}. For each case, Figures 4.19 to 4.25 detail the utility topology over the range of sensing actions and the geometry of the optimal solution in terms of the prior and posterior feature location confidence. The utility topology is indicated by four sets of contours. The contours of global utility $\mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2)$, contours where the elements of the Jacobians $\nabla_{\boldsymbol{u}_1}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) = 0$ and $\nabla_{\boldsymbol{u}_2}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) = 0$ equal zero and a contour of zero Gaussian curvature $\mid \nabla^2_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2) \mid = 0$.

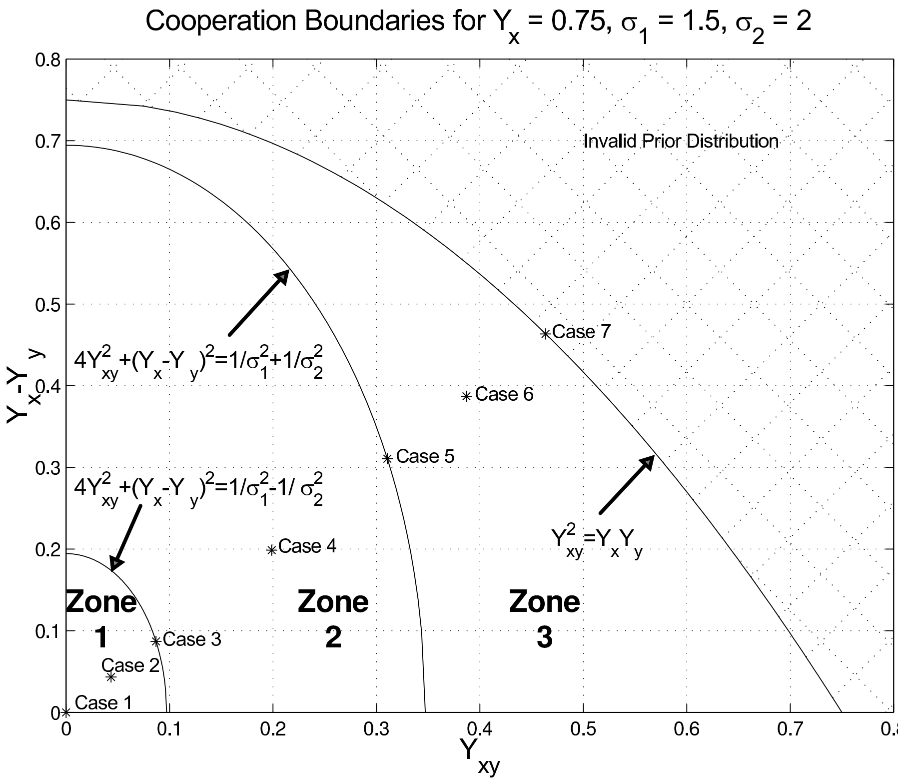


Figure 4.18: Boundaries for cooperative solution type in the two range-only feature localisation problem for varied prior information. The cases detailed in Figures 4.19 to 4.25 are indicated. Bifurcation in the structure of the optimal solution occurs at these boundaries. In 'zone 1' the optimum group solution is for the 1st sensor to maximise and the 2nd sensor to minimise their local partial utility measures. This changes in 'zone 2' and the maximising solutions move away from the partial utility stationary points. In 'zone 3' the group optimal actions are the maximum solutions to the partial utility measures.

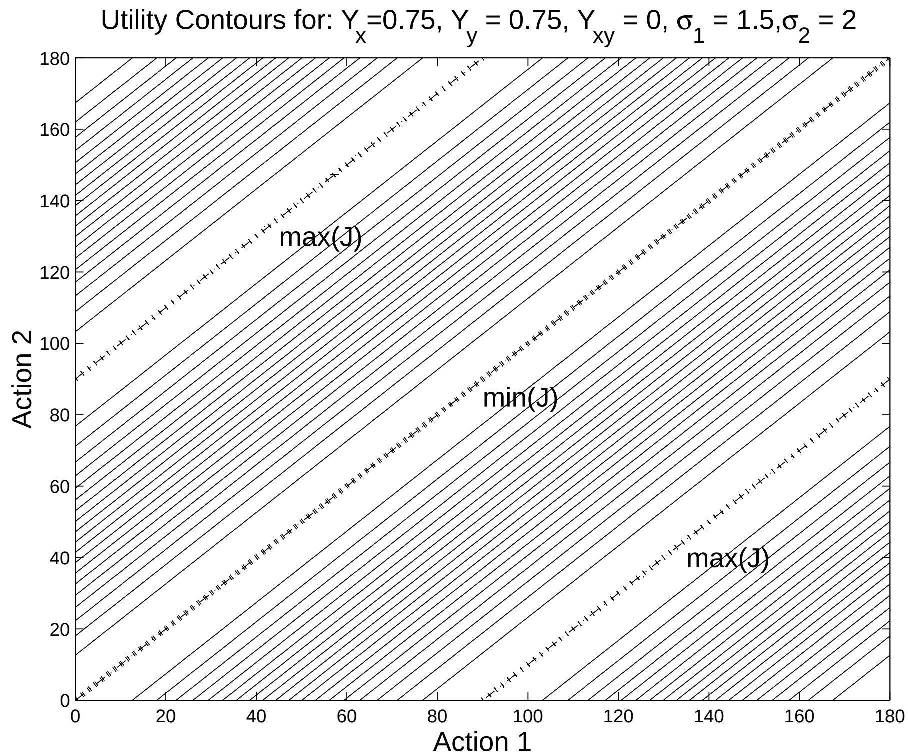


Figure 4.19: Utility contours for case 1. Note case 1 corresponds to the situation where the prior distribution confidence is circular. There are no unique local or global solutions. The global solution is the condition that the sensor actions differ by 90 degrees.

4.5.6 Cooperative Solution by Negotiation

Iterative solutions to team decision problems were discussed in section 2.5. *Better-Response* negotiation was highlighted as a mechanism for finding pure Nash equilibria. This method is now applied to the two sensor localisation problem under consideration.

Recall that the global utility 4.18 for this problem is the determinant of the sum of the prior information and the conditional observation information

$$\mathbf{J}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) = \mid \mathbf{Y}(k \mid k-1) + \sum_{i=1}^{n} \mathbf{I}_i(k \mid k-1, \boldsymbol{u}_i) \mid$$

Observe that the observation information of the other decision makers is additive and associative with the prior information. In effect, the other decision makers create a modified prior conditional on their actions. Hence, the optimal decision rule for a single sensor to a given prior is the best response function to the other team members given the prior

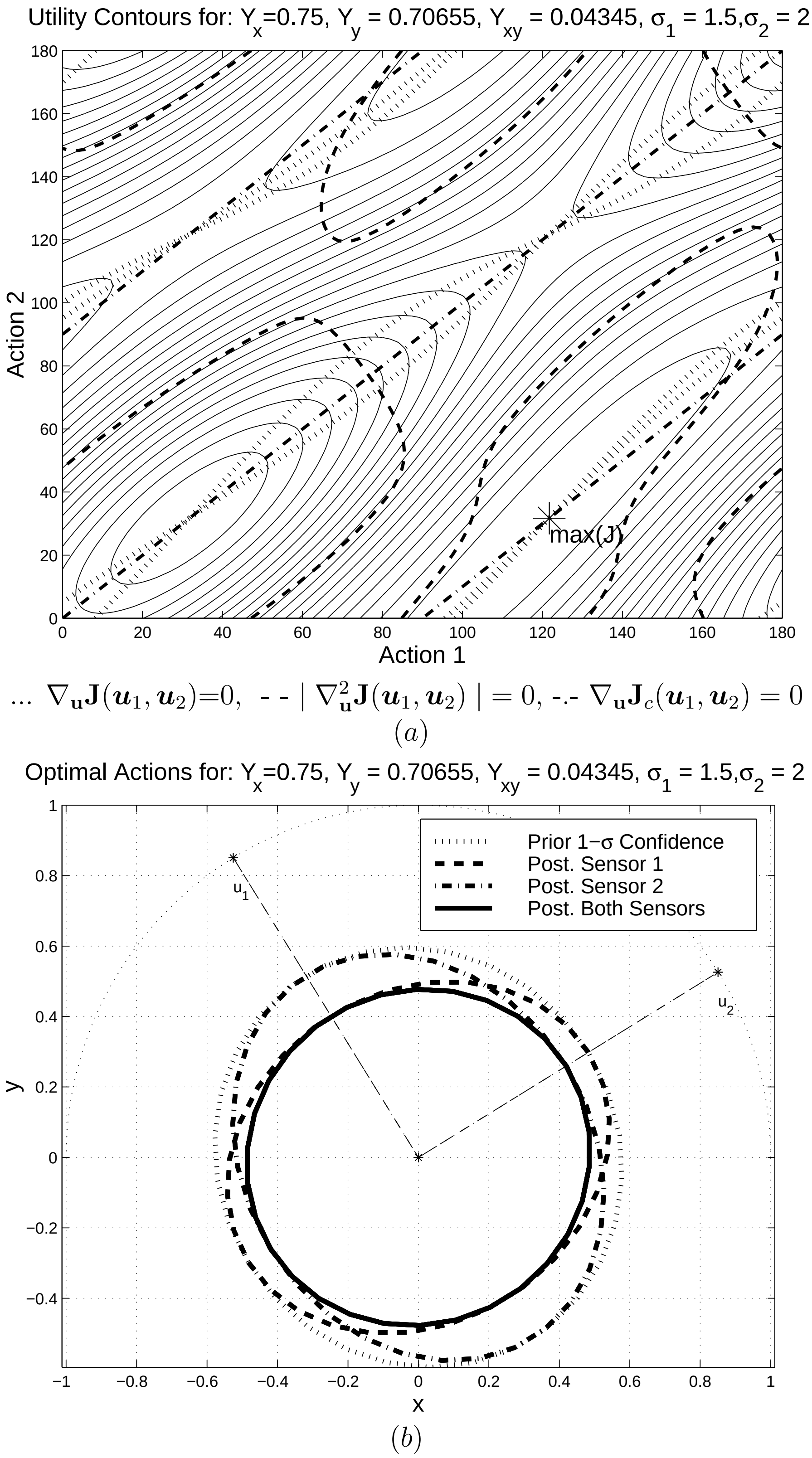
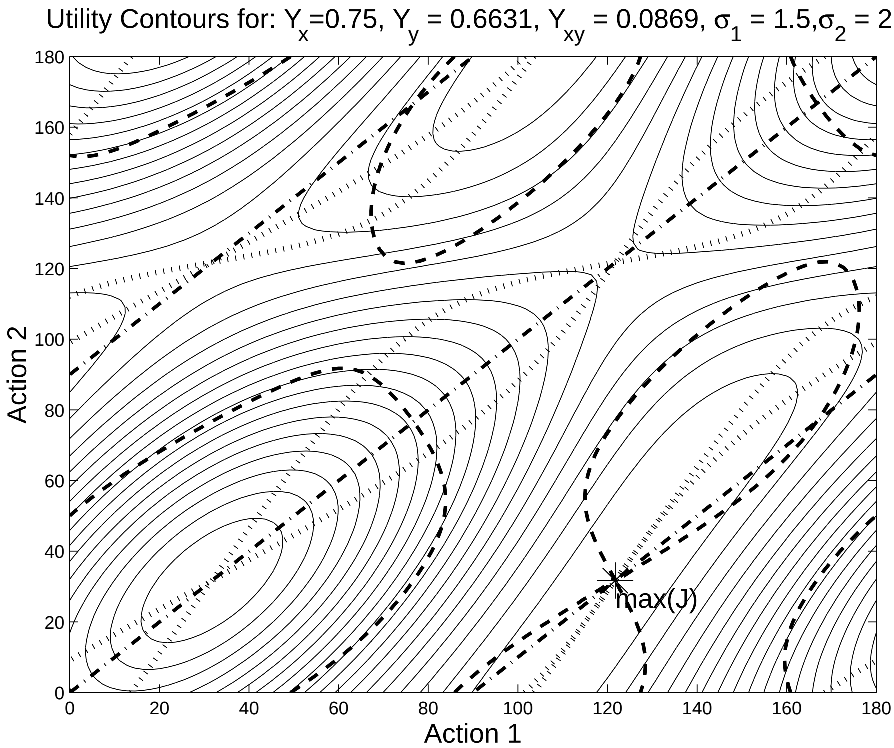
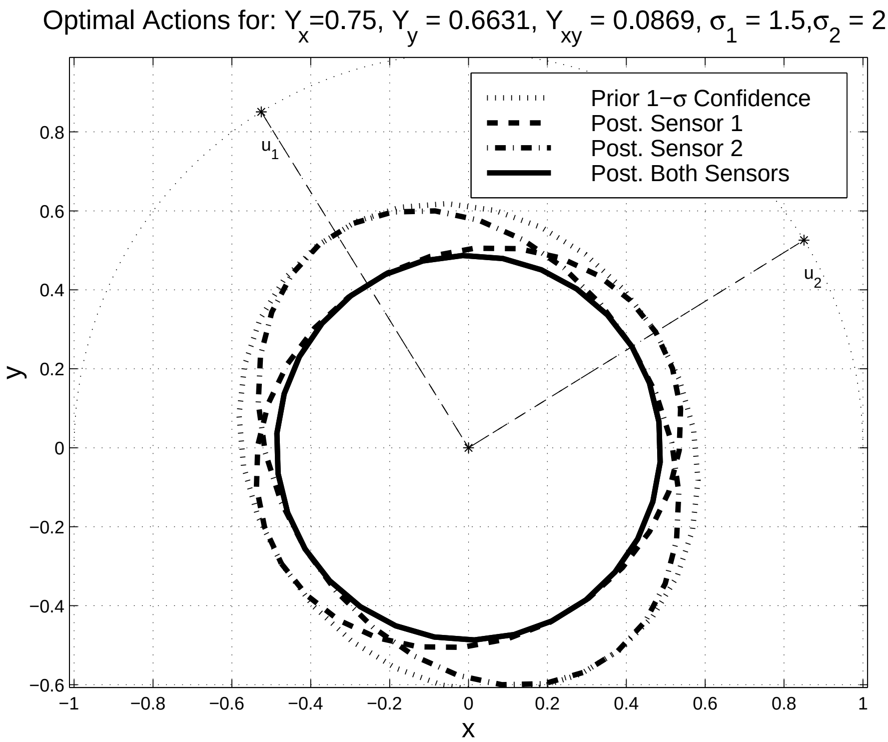


Figure 4.20: Utility contours and solution geometry (b) for case 2. (a) shows the contours in global utility (solid), along with zero contours of the Jacobian (dotted) and curvature (dashed). In this case the optimal global action corresponds to sensor 1 maximising local utility and sensor 2 minimising its local utility.



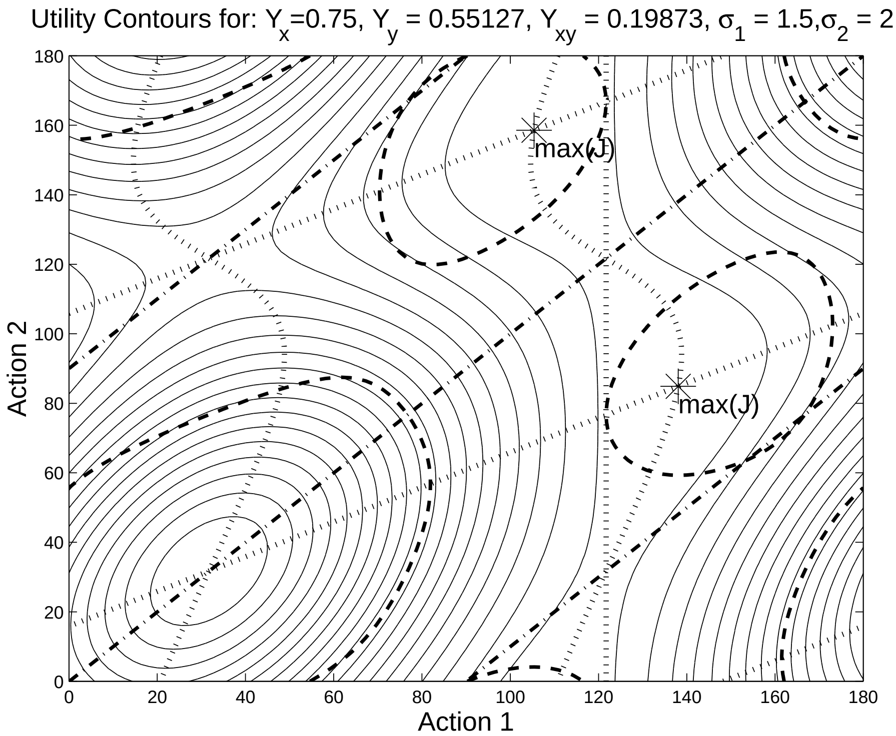
... $\nabla_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1,\boldsymbol{u}_2)$=0, - - $\mid \nabla^2_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1,\boldsymbol{u}_2) \mid = 0$, -.- $\nabla_{\mathbf{u}}\mathbf{J}_c(\boldsymbol{u}_1,\boldsymbol{u}_2) = 0$

(a)



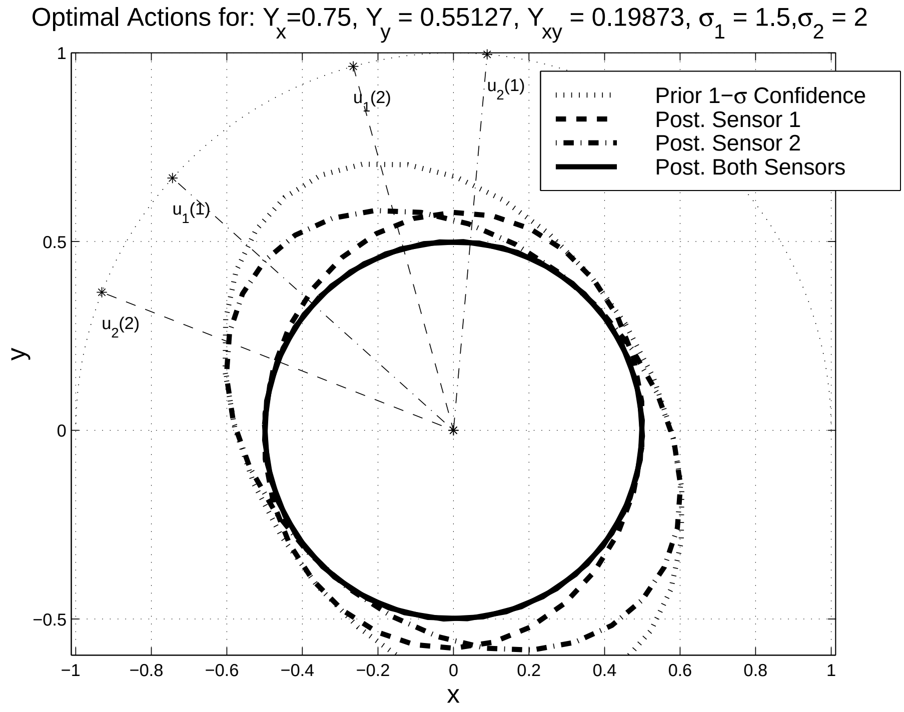
(b)

Figure 4.21: Utility contours and solution geometry (b) for case 3. (a) shows the contours in global utility (solid), along with zero contours of the Jacobian (dotted) and curvature (dashed). In this case the optimal global action corresponds to sensor 1 maximising local utility and sensor 2 minimising its local utility.



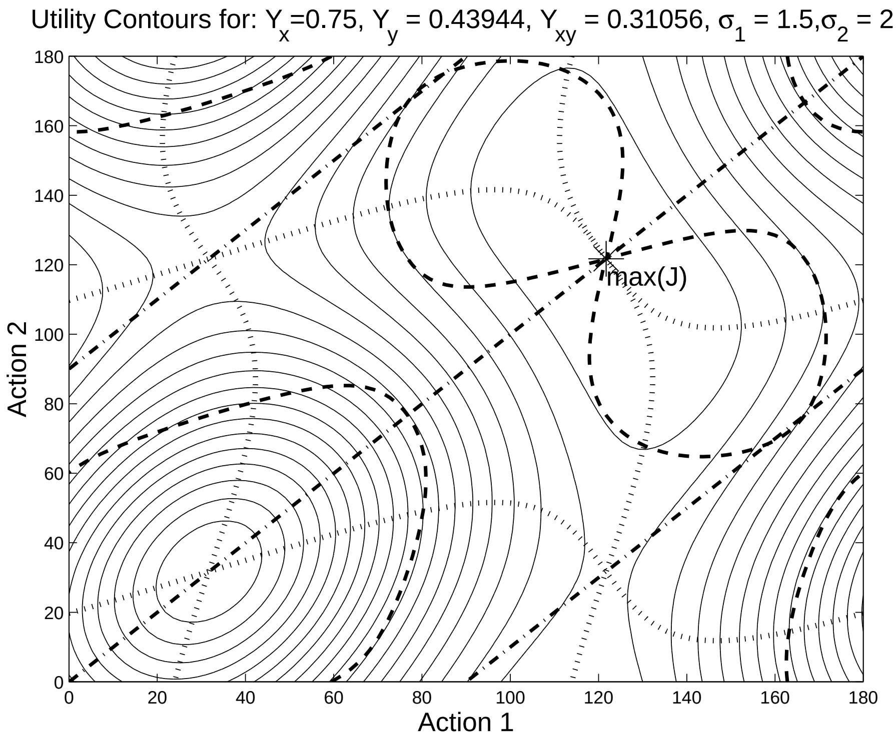
$$\ldots\ \nabla_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1,\boldsymbol{u}_2)=0,\ \ \text{- -}\ |\ \nabla^2_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1,\boldsymbol{u}_2)\ |=0,\ \text{-.-}\ \nabla_{\mathbf{u}}\mathbf{J}_c(\boldsymbol{u}_1,\boldsymbol{u}_2)=0$$

(a)



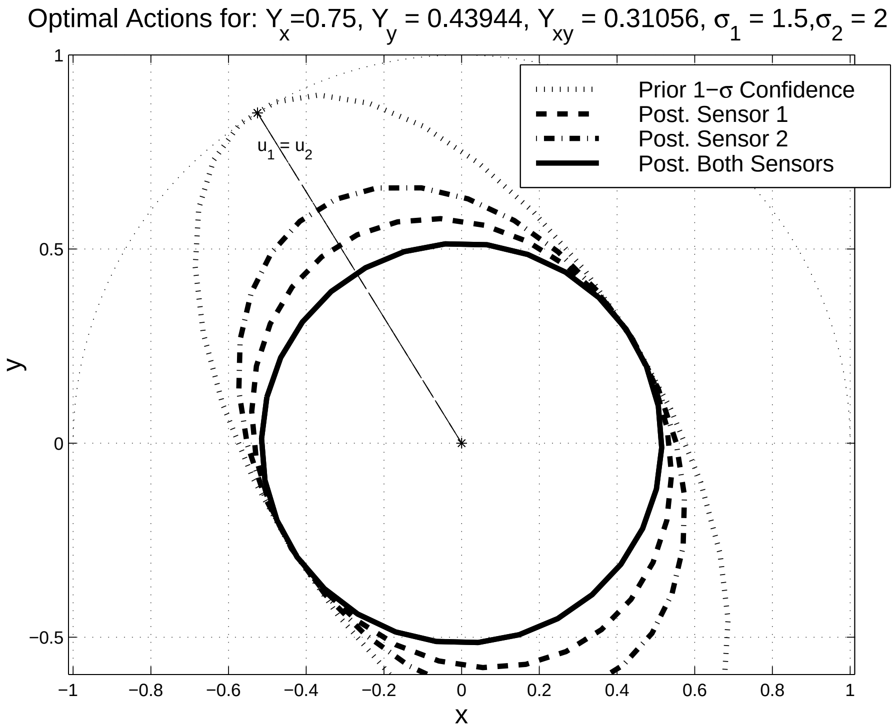
(b)

Figure 4.22: Utility contours and solution geometry (b) for case 4. (a) shows the contours in global utility (solid), along with zero contours of the Jacobian (dotted) and curvature (dashed). In this case the optimal global actions have moved away from the stationary solutions of the partial local problems.



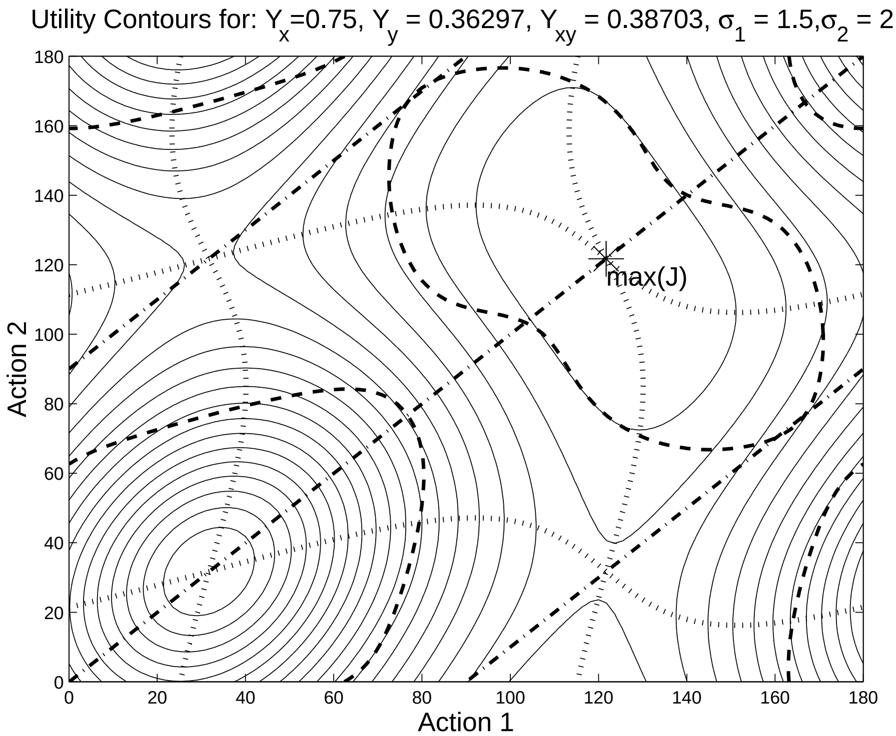
... $\nabla_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)$=0, - - | $\nabla^2_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)$ | $= 0$, -.- $\nabla_{\mathbf{u}}\mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2) = 0$

(a)



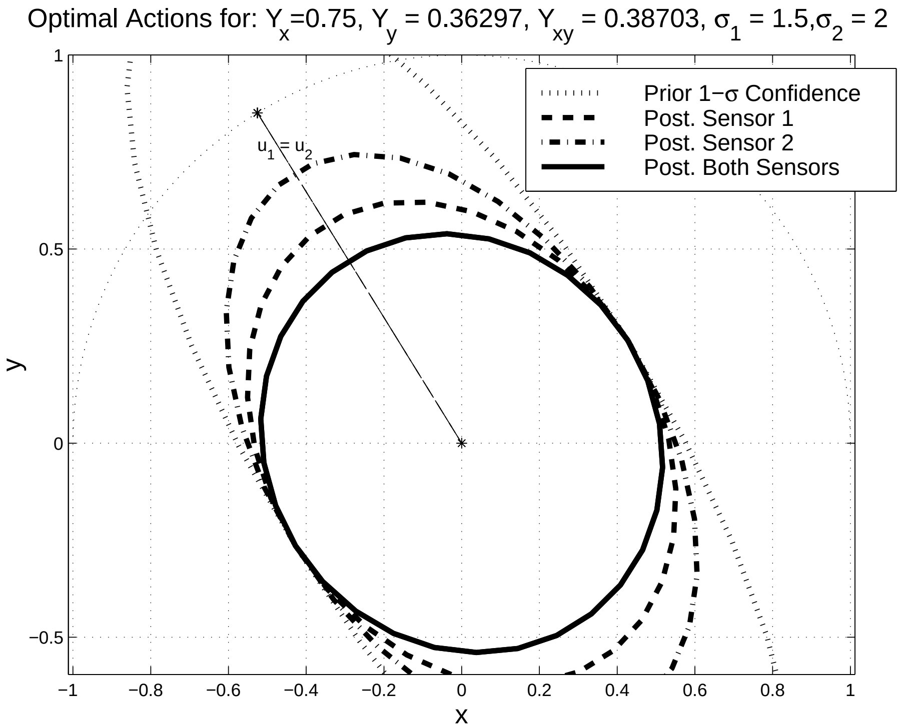
(b)

Figure 4.23: Utility contours and solution geometry (b) for case 5. (a) shows the contours in global utility (solid), along with zero contours of the Jacobian (dotted) and curvature (dashed). In this case the optimal global action corresponds to both sensors maximising partial local utility.



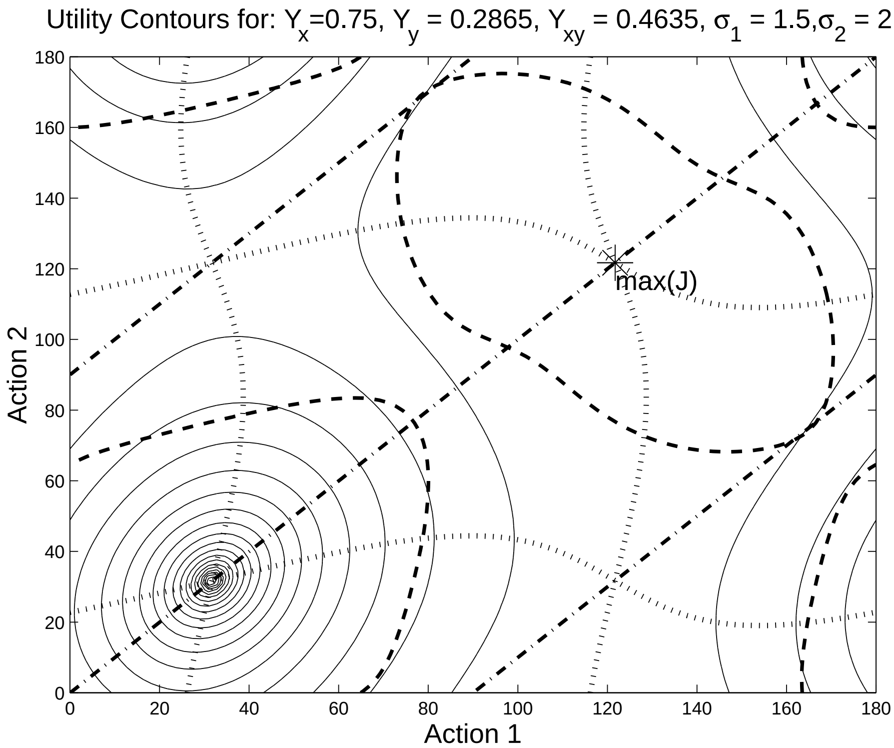
... $\nabla_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)=0$, - - $|\ \nabla^2_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)\ | = 0$, -.- $\nabla_{\mathbf{u}}\mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2) = 0$

(a)



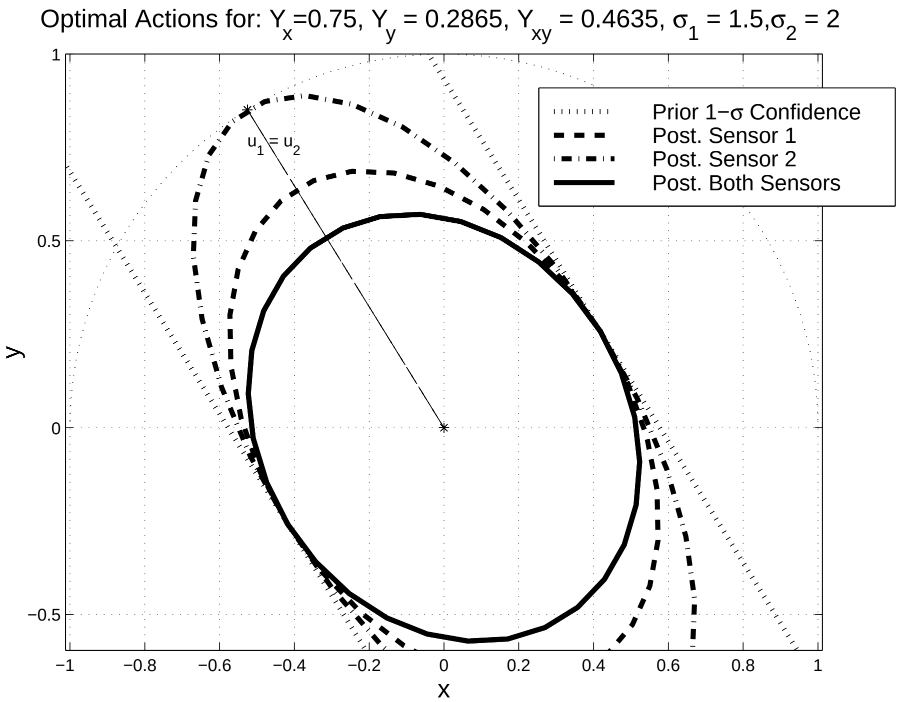
(b)

Figure 4.24: Utility contours and solution geometry (b) for case 6. (a) shows the contours in global utility (solid), along with zero contours of the Jacobian (dotted) and curvature (dashed). In this case the optimal global action corresponds to both sensors maximising partial local utility.



... $\nabla_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)$=0, - - $|\ \nabla^2_{\mathbf{u}}\mathbf{J}(\boldsymbol{u}_1, \boldsymbol{u}_2)\ | = 0$, -.- $\nabla_{\mathbf{u}}\mathbf{J}_c(\boldsymbol{u}_1, \boldsymbol{u}_2) = 0$

(a)



(b)

Figure 4.25: Utility contours and solution geometry (b) for case 7. (a) shows the contours in global utility (solid), along with zero contours of the Jacobian (dotted) and curvature (dashed). In this case the optimal global action corresponds to both sensors maximising partial local utility. Note, although utility is varied, the optimal actions are unchanged in Figures 4.23 to 4.25

accounting for their actions. The prior information available to the i^{th} decision maker is

$$\mathbf{Y}(k \mid \{k-1, \bar{\mathbf{u}}\}) = \begin{bmatrix} \bar{\mathbf{Y}}_x & \bar{\mathbf{Y}}_{xy} \\ \bar{\mathbf{Y}}_{xy} & \bar{\mathbf{Y}}_y \end{bmatrix} = \mathbf{Y}(k \mid k-1) + \sum_{j=1, j\neq i}^{n} \mathbf{I}_j(k \mid k-1, \bar{\mathbf{u}}_j).$$

The optimal action conditional on the other team members or *best response function* is given by equation 4.33

$$\begin{aligned} \boldsymbol{u}_i^* = \mathcal{B}_i(\bar{\mathbf{u}}) &= \arg\max_{\boldsymbol{u}_i} \mathbf{J}(\boldsymbol{u}_i, \bar{\mathbf{u}}) \\ &= \arctan\left(\frac{-1}{2\bar{\mathbf{Y}}_{xy}}(\bar{\mathbf{Y}}_x - \bar{\mathbf{Y}}_y + \sqrt{(\bar{\mathbf{Y}}_x - \bar{\mathbf{Y}}_y)^2 + 4\bar{\mathbf{Y}}_{xy})}\right). \end{aligned} \tag{4.42}$$

This can be verified through observation of the solution geometry in figures 4.20 to 4.25. These figures show the optimal sensor actions along with confidence ellipses associated with the probability densities. Ellipses are shown for the prior distribution, the distribution after each individual action and posterior after both sensor actions. In each case, optimal sensor action lies along the direction of most uncertainty after the prior is combined with the other sensors' observation information. This action corresponds to the solution given by equation 4.42.

This best response function has been used to implement a decentralised solution to the two sensor localisation problem. Two example solution sequences are shown in figures 4.26 and 4.27. Figure 4.26 examines case 3 where the optimal solution is for sensor 1 to maximise and for sensor 2 to minimise their local partial utility. In this case the solution is a singular stationary point with zero curvature. Gradient or better response iterative procedures suffer extremely poor convergence rates. This situation highlights the benefit of applying a line search technique to accelerate the convergence. Figure 4.27 examines case 4 where the optimal solution has moved away from the local stationary solutions. In this case the utility topology provides rapid convergence. However, the situation highlights a weakness with using the synchronous better response procedure.

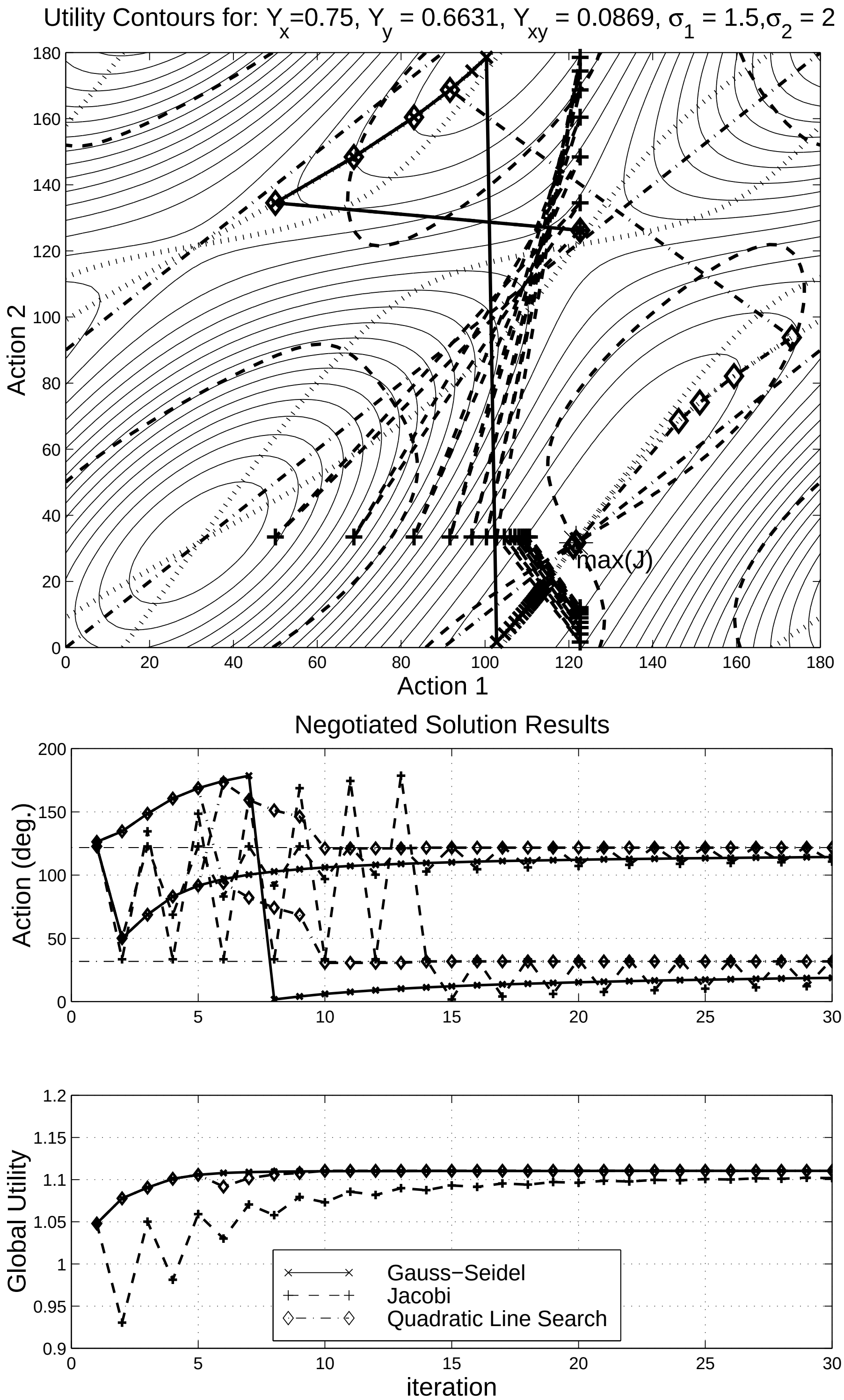


Figure 4.26: Comparison of three negotiated solution techniques for problem 4.5 for case 3. In this case the optimal solution is for sensor 1 to maximise and for sensor 2 to minimise their local partial utility. The initial actions are a small random distance from the local maximising solutions. Both the simultaneous Jacobi and sequential Gauss-Seidel best response methods converge to a global maximising solution. Significant improvement in solution time is achieved by combining the sequential best response with a quadratic line search.

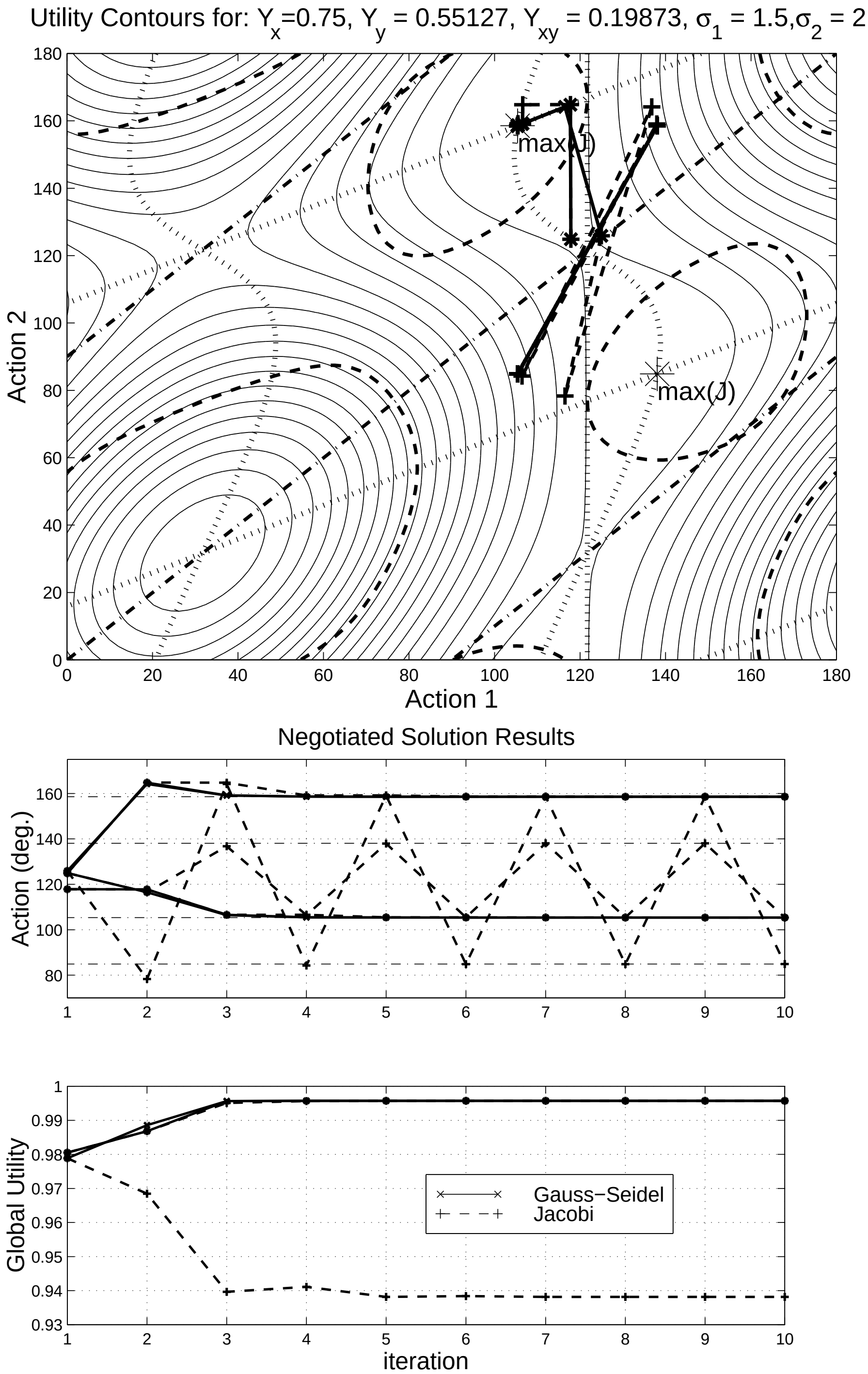


Figure 4.27: Negotiated solution to problem 4.5 for case 4. In this case the optimal solution has moved away from the stationary solutions to local partial utilities. The initial actions are a small random distance from the local maximising solutions. The sequential best response method always converges to a global maximising solution. While the simultaneous best response converges for one set of initial conditions, another highlights susceptibility to oscillations about symmetries in the stationary solutions. This problem for synchronous schemes is resolved by reducing $\boldsymbol{\gamma_k}$ and/or perturbing the updates $\boldsymbol{\beta_k} \neq 0$.

Solution sequences are shown for two different initial actions. The synchronous best response converges for one set of initial conditions. For the other, the procedure oscillates about symmetries in the global stationary solutions. This symmetry can be broken by reducing $\boldsymbol{\gamma_k}$ and/or perturbing the updates $\boldsymbol{\beta_k} \neq 0$. This is a significant advantage of the sequential or randomised sequential approaches over synchronous methods.

4.6 Summary

Information gathering was presented as an optimal control problem. Modelling of the environment, vehicles and sensors was combined with utility based on entropic information. The appropriateness of this method in active sensing was demonstrated by generating optimal *information seeking* trajectories for a single sensor platform. Attention was focused on the team utility structure and its role in cooperation among multiple sensor platforms. A proposed decomposition of the team utility was used to explore the influence of coupled utility on the optimal member decisions. This investigation and application to a two sensor range-localisation example established:

1. The conditions when the global solution is composed from stationary solutions to utilities that only consider local information and influence.

2. Situations exist where the worst possible action for a particular decision maker based on local partial information is the global optimal solution.

3. Locally optimal actions may be globally optimal regardless of coupled utility.

4. The *better-response* iterative procedure provides a decentralised mechanism for decision makers to reach the global solution without knowledge of their fellow team members strategy or utility.

5. Globlal analysis may reveal the team solution to be simply maximising independent local utility. But this violates the decentralised philosophy. The *better-response* method initialised with local solutions will identify this without global knowledge.

Chapter 5

Endogeneous Algorithms and Decentralised Architectures

5.1 Introduction

The goal of a decentralised control algorithm is to exert coordinated control over a scalable number of sensors and platforms, through the exchange of information and local decisions, without the need for a central arbiter. For example, in a multiple-platform surveillance task, each sensor and platform must make its own decision about where and what to sense, but by coordinating these decisions with other sensors and platforms must also arrive at a globally-optimal control for the system as a whole.

This chapter explores communication and coupled utility between decision makers as fundamental mechanisms underlying coordination and cooperation. Section 5.2 identifies the form of the information structure as critical to enabling coordination and cooperation. The implications of various information structures of practical interest are considered in Section 5.3. This leads to the coordinated and cooperative solution methods presented in Section 5.4. Section 5.5 applies information-theoretic modelling to an area exploration task and demonstrates a coordinated multi-vehicle solution. Coordinated and cooperative solutions to a multi-platform bearings-only localisation problem are investigated in Section 5.6. These solutions provide insight into a more general approach to decentralised cooperative control. In Section 5.7, all the elements considered through this thesis are brought together in the form of a general architecture for decentralised coordinated control of multi-sensor information gathering systems.

The required architecture must exchange information and decisions seamlessly across networks of inter-operating systems. The origin, state and physical nature and value of the information source is abstracted into the utility and information structures of the system architecture. Sub-systems may be added or removed dynamically. The information structure provides a means to propagate and fuse information from disparate sources. The utility structure values the actions of the individual systems with regard to mission objectives. Propagation of information through the network couples the value of the system actions. A decentralised decision making mechanism optimises the actions leading to coordinated interactions and potentially synergistic inter-operation of the component systems.

To fully realise the benefits of this approach, the architecture must adhere to the strict definition of decentralised systems. The means of interpreting, encoding, estimating, valuing and fusing information, states and actions must be internal to the individual systems. The use of the term *Endogeneous* in describing these systems is intended to emphasise that this functionality lies within each sub-system.

5.2 The Mechanisms Underlying Coordination and Cooperation

This section addresses the underlying mechanisms that give rise to coordinated and cooperative solutions to problems involving multiple decision makers. Coupled utility plays a fundamental role. The decision makers are informed of the variables on which their utility depends through observation and communication. Coupled utility is investigated with specific interest in the role played by *prior information, locally observed information* and *communicated information*. The dependence of coordination and cooperation on communication results in the leading role played by the information structure.

Coordination can occur through coupled system dynamics, constraints or coupled utility. The focus of this study is on coupled utility as the basis for coordination and cooperation. Coupled system dynamics may lead to a requirement for tightly coupled low level control. This is not addressed here. The work of Mutambara in decentralised vehicle control [58, 59], provides an example of coordinated control through distributed dynamic models and constraints. An alternative to treating constraints in the solution method directly is to incorporate them into the utility by means of penalty functions.

Problems involving decision makers with decoupled physical dynamics and coupled dynamic utility represent a wide range of practical situations. This motivates the following scrutiny of coupled utility as the basis for coordination and cooperation.

5.2.1 Coordination Through Coupled Utility

Fundamentally, coupled utility or value results in coordination between decision makers. The effect one decision maker has on another is captured through its influence on local utility. Given knowledge of this influence, it is possible to capture the effect of system-wide actions on the utility of individual decision makers. Thus utility and the coupling between utilities provides the underlying mechanism for coordination.

One approach to coordinated control is the behaviour-based method of Mataric [49] and the DAMN architecture of Rosenblatt [78]. These methods generate controllers based on utilities associated with individual objectives. A composite controller arises from the individual controller objectives by means of weighting or arbitration, to execute missions comprising multiple objectives. In many instances, the resulting controller exhibits interesting and useful behaviours. However, these behaviours are simply a consequence of the interaction between the dynamics of the component controllers. The arbitrator takes no account of the fundamental fact that the underlying utilities for the component controllers are coupled. Consequently, these 'group' controllers avoid and obscure the basic issue of interaction between utility measures in coordinated and cooperative decision making.

An alternative is to seek a globally optimal solution using parallel decentralised optimisation of a set of decomposed but coupled sub-problems. Such decomposition is the key to overcoming the "Curse of Dimensionality"[83]. This has been applied to multi-vehicle coordinated and cooperative control problems by McLain [52, 53]. This situation considers the rendezvous of multiple UAVs while minimising risk and fuel use. Coordination variables and coordination functions are introduced to enable a distributed solution. The functions and variables are communicated to inform the individual vehicle sub-problems of their interrelation. For the task considered, the coordination variable is time of rendezvous. The coordination functions relate variations in the rendezvous time to variation in individual vehicle fuel costs and risk. Communication of these allow each vehicle to determine the team optimal rendezvous time in a decentralised manner. The struc-

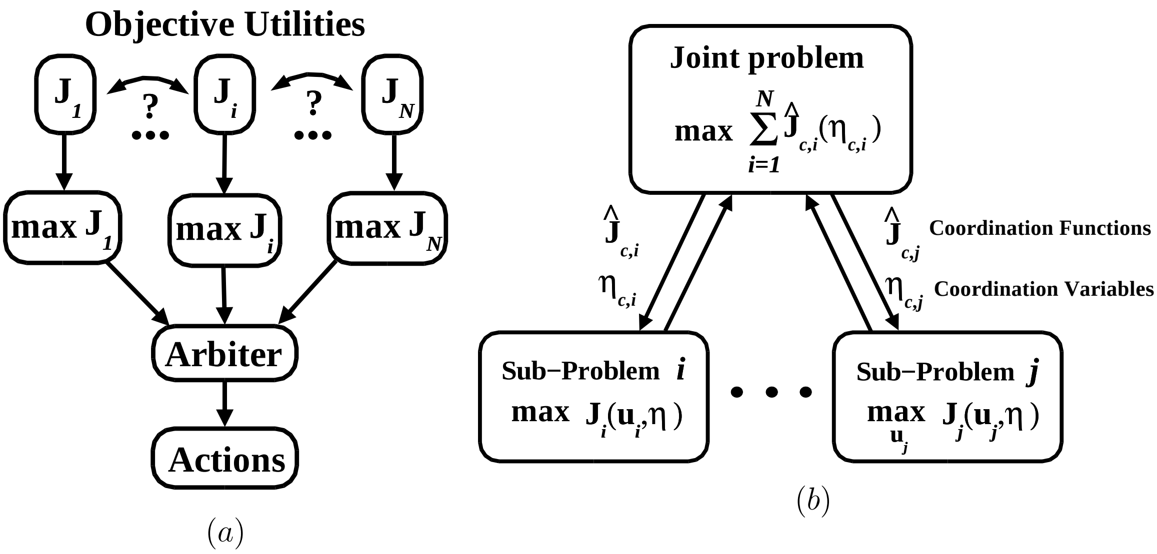


Figure 5.1: Two approaches to coordinated control. (a) Arbitration of controllers based on independent objectives. (b) Decomposition into coupled sub-problems.

ture of the arbitrator and decomposition approaches are summarised in Figure 5.1. The decomposition approach of Sobieski [84, 1, 82] and McLain [52, 53] is closely related to the team decision framework described in Section 2.3. A key issue in the formulation of such problems is to elucidate what information needs to be communicated between team members to capture the coupling between decision processes.

5.2.2 The Role of Prior, Local and Communicated Information

In sensing tasks; prior, local observation and communicated information can be combined to provide effective measures of information-based utility and coupling between sensing actions. The DDF information fusion algorithms suggest a particularly simple additive form for information-based utilities as

$$\mathbf{J}_i(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n) \propto \overset{\textbf{Prior}}{\mathbf{Y}(k \mid k-1)} + \overset{\textbf{Local}}{\mathbf{I}_i(k)} + \overset{\textbf{Communicated}}{\sum_{\substack{j=1 \\ j \neq i}}^{n} \mathbf{I}_j(k)}$$

In this utility function, prior and communicated information is additive and thus associative. Communicated information may include current observations, delayed information or future expected observation information. Current or delayed information will not affect the current local optimal action but will alter the prior information on which subsequent decisions are made.

Future expected information is fused in the prediction process for determining the current locally optimal action plan. As communicated information simply adds to the local prior at each stage, a control law or decision rule developed based only on prior and local information is the individual's best response to the communicating decision makers. A global equilibrium between decision makers can then be obtained by iteration.

An individual decision maker must understand how its local utility is influenced by information communicated from other decision makers. In information fusion problems, this requires that communicated information be associated with the local model of uncertainty. As this is a requirement for the underlying sensor fusion process, it must be implemented directly in the communications protocol for the DDF system.

5.2.3 Propagating Observation Information Leads to Coordination

It has been noted that current observation or delayed information does not alter the current local optimal action. The local control law or decision rule remains unchanged. Current or delayed communicated observation information will be fused with the local prior and observations. This alters the prior from which subsequent local decisions are generated and consequently the decision processes are coordinated over time. This process is illustrated in Figure 5.2. The DDF process propagates current and delayed information throughout the sensor and vehicle system network. Consequently, simply activating DDF with independent control rules on each sensor and vehicle leads to a coordinated solution.

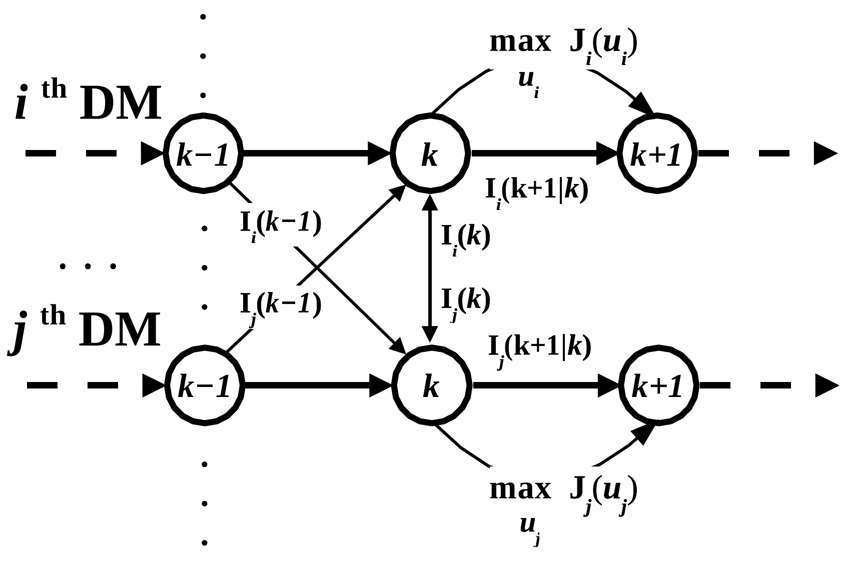


Figure 5.2: Coordination through propagated observation information

5.2.4 Exchanging Predicted Information Leads to Cooperation

Communicated predicted observation information does influence the current local optimal action. This has significant consequences. The local decision procedures and optimal actions of the communicating decision makers are coupled. This coupling occurs through exchange and evaluation of *a priori* information on individual utility functions. Communicating observation information predictions gives rise to coordinated actions.

A negotiation or bargaining procedure is required to reach the joint optimal solution. The exchanging and evaluation of *a priori* observation information, when combined with a negotiation procedure, is a coordination mechanism that leads to cooperation. This process is illustrated in Figure 5.3.

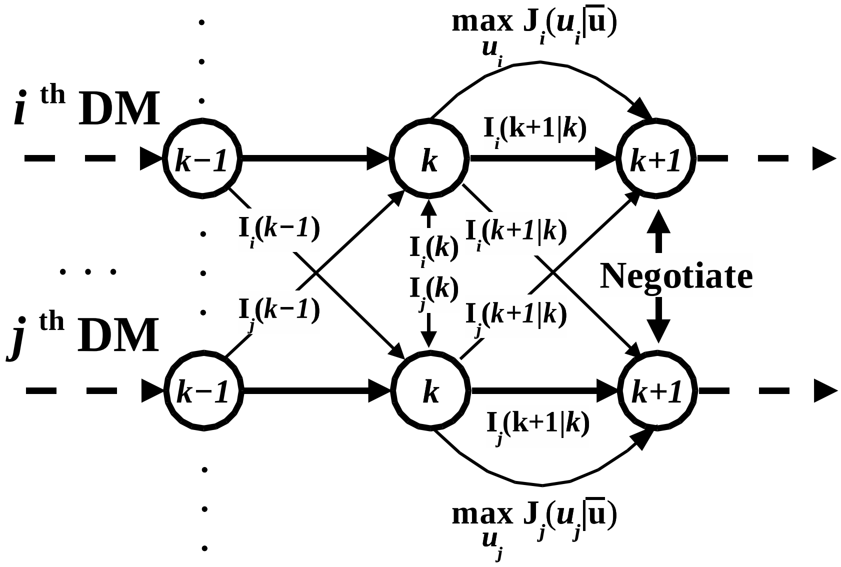


Figure 5.3: Negotiated cooperation based on exchanging *a priori* observation information.

5.2.5 Incentive to Cooperate

The incentive to cooperative arises when more information can be gained by coordinating a sensing action, then by simple exchange of information. Cooperation, while ultimately resulting in greater global reward, may involve a reduction in local utility. Simple examples of this particularly occur with sensing agents that have exactly the same capabilities. In these circumstances, each agent (being identical) will resolve on exactly the same course of action. However, having all sensors take the same action is unlikely to be globally optimal. Rather, observation of different aspects of the feature or target under consideration will give greater information to the group.

Accepting a lower individual reward in anticipation of receiving a higher global reward from the team is the essence of cooperation. Practically, in terms of information gain, the incentive to cooperate is captured by the increase in information hypothesised from a candidate's predicted observation.

5.3 Practical Information Structures

The information structure is one of the five key elements in the team decision problem identified in Section 2.3. It specifies the information available to a decision maker and the exchange of information among team members. Designing the system information structure is a critical task requiring a trade-off between system performance, scalability, computation and communication. The role of coupled utility and of communication are key elements in this design. Practical information structures must enable effective coordination and cooperation.

Recall that an information structure is referred to as *static* if communicated information does not influence the immediate decision processes and *dynamic* otherwise. Accepting the distinction between cooperation and coordination made in Section 2.4.1, static information structures preclude the possibility of cooperation. Yet, they are of considerable practical interest as they provide a scalable implementation of coordination with limited computational and communication costs. Thus, an application and situation dependent performance penalty may be justifiable. Dynamic information structures are coupled at a decision making level. This provides the potential to seek cooperative solutions through negotiation. The price is a potentially intensive and time consuming iterative procedure.

5.3.1 Static Information Structures

Imposing a static information structure on a team allows coordination through information exchange. A number of static information structures are possible, these include:

1. **Open loop multistage look ahead with communicated observations:**
   Decision makers plan local optimal actions n stages ahead, $n \geq 1$. The plan is developed based on local prior knowledge and local conditional sensor information. Information is communicated as observations are made during the execution of the sensing plan.

2. **Closed loop multistage look ahead with communicated observations:**
   As for the open loop multistage structure, but the local control plan is updated as information becomes available through the DDF process.

3. **Instant communication and action with zero look ahead:**
   This is a special case of the multistage structures. With zero look ahead, decision makers are not influenced by the action of others or their own dynamics. This significantly simplifies the coordination problem, offering a useful approximate solution.

4. **Any of these with delayed communication:**
   The DDF process allows for fusion of delayed information. Thus, any of these information structures can be implemented with communication of delayed information.

5.3.2 Dynamic Information Structures

Dynamic information structures permit coupled decision making. This allows improved solutions over static structures with respect to joint team optimality. The system designer has control over the level of optimality and solution process complexity. In the limit, the global Nash cooperative solution is sought through negotiation. Dynamic information structures of practical interest include:

1. **Sequential fixed shot multistage look ahead:**

   Decision makers plan n stages ahead, $n \geq 1$. The local decision process incorporates local prior knowledge, local conditional sensor information and conditional observation information communicated from other decision makers. On convergence, the conditional observation information associated with the sensing plan is communicated to the other decision makers. This process is repeated for a fixed number of iterations.

2. **Negotiated multistage look ahead:**

   As for the sequential fixed shot procedure, except the iterations are repeated until convergence criteria is met. This permits a multistage *better response* negotiation method to find an ϵ-optimal cooperative solution.

3. **Adaptive multistage structure:**

   The nature of the information structure becomes part of the decision problem. Utility measures incorporate rewards and costs associated with exchange of information. The information structure varies dynamically across the network of decision makers. Features of any previously mentioned information structures may be activated among sub-groups of the decision making team.

5.4 Solution Approaches

Consideration of the role of the information structure and coupled utility in decision making suggests two decentralised solution approaches: coordinated methods based on static information structures; and cooperative implementations based on dynamic information structures. Coordinated and cooperative approaches are presented. A special coordinated case corresponding to zero look-ahead is presented and its interpretation as a potential field method discussed.

5.4.1 Coordinated Solution Procedure

The coordinated solution procedure is illustrated in Figure 5.5. Local decision making is implemented in addition to the decentralised data fusion algorithm. This local control algorithm is the same as the single decision maker case. The information on which the actions are based is coupled through a static information structure. It should be emphasised that this solution approach is fully decentralised. The static information structure consists of a communications network, a communications protocol and an interface for each decision maker. The decision making and communications management mechanisms are internal to each team member. The only component external to the decision making nodes is the medium and protocol through which they communicate.

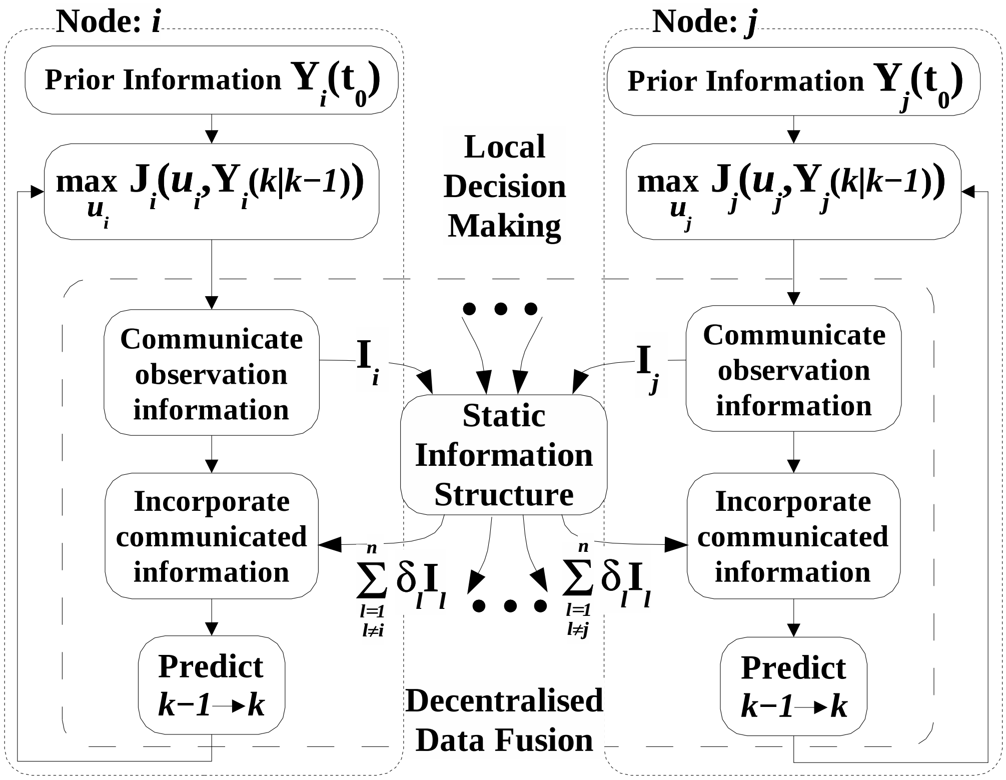


Figure 5.4: Multi-platform coordinated decision making with a static information structure. The information structure is formed through an interface on each decision maker. This allows the individual decision maker to incorporate the influence of other team members' observations over time, and inform the team of their own observations.

5.4.2 Cooperative Solution Procedure

A dynamic information structure allows coupling between the individual decision processes. This permits the propagation of each decision makers predicted observation information throughout the team. Each decision maker couples its individual solution procedure to the team observation information structure in an iterative loop. Negotiating towards the team solution. The solution procedure is illustrated in Figure 5.5. The coordinated solution structure of Figure 5.4 is a sub-set of this implementation.

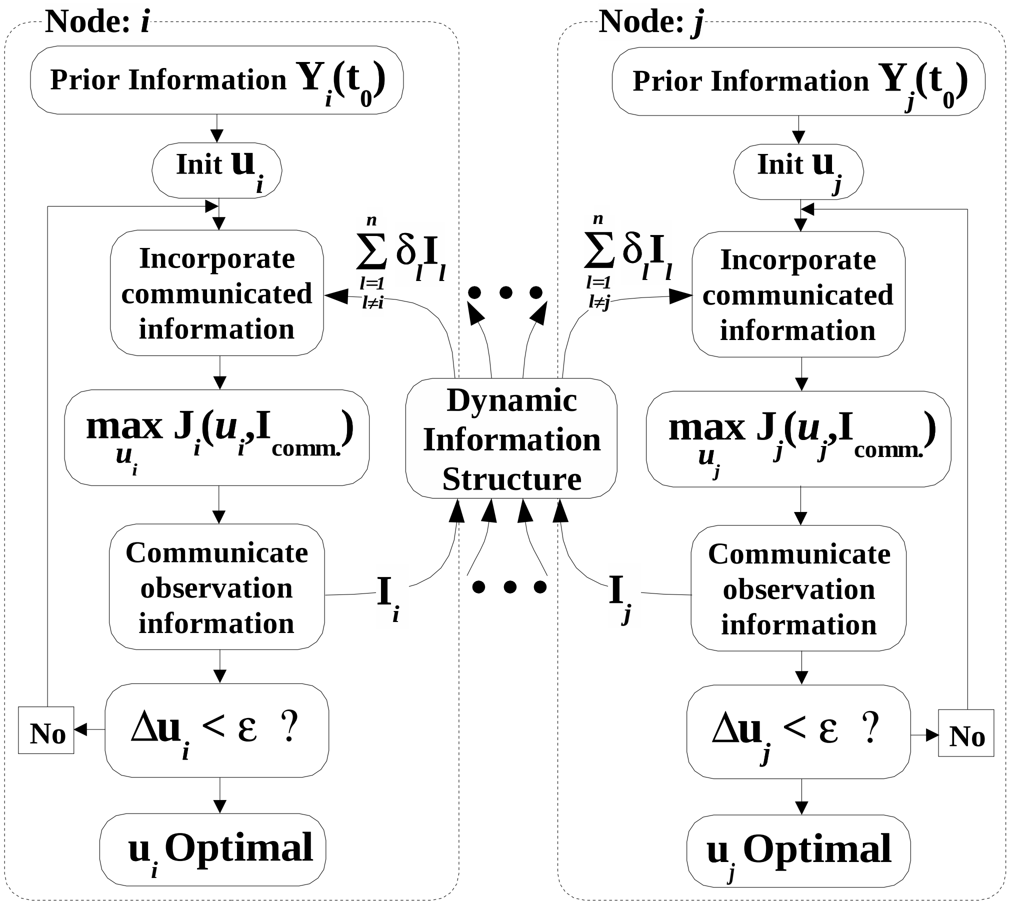


Figure 5.5: The role of the dynamic information structure in cooperative multi-platform decision making. The information structure is formed through an interface on each decision maker. This allows the individual decision maker to incorporate the influence of other team members predicted observations and in turn, inform the team of their own effect. This is combined with the *better-response* negotiation procedure to determine the cooperative solution.

5.4.3 Information Dynamics as a Potential Field

Planning with zero look ahead provides a special case in coordinated multi vehicle control. It will be shown how this can be used to form simple approximate solutions to coordinated sensing problems. The sensor platforms are directed by the dynamics of the mutual information rate gradient field.

The Fisher information evolution in continuous linearised filtering is given by the information form of the Kalman filter Ricatti equation [63].

$$\underbrace{\dot{\mathbf{Y}}}_{\substack{\textbf{Information}\\ \textbf{Rate}}} = \underbrace{-\mathbf{F}\mathbf{Y}-\mathbf{F}^T\mathbf{Y}}_{\substack{\textbf{Loss or Gain}\\ \textbf{System Dynamics}}} \quad \underbrace{-\mathbf{Y}\mathbf{G}\mathbf{Q}\mathbf{G}^T\mathbf{Y}}_{\substack{\textbf{Loss Through}\\ \textbf{Process Noise}}} \quad \underbrace{+\sum_{i=1}^{n}\mathbf{H}_i^T\mathbf{R}_i^{-1}\mathbf{H}_i}_{\substack{\textbf{Gain Through}\\ \textbf{Observations}}} \tag{5.1}$$

Where $\dot{\mathbf{Y}}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{H}_i$ are functions of time with time index suppressed for notational clarity. $\mathbf{F}$, $\mathbf{G}$, $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{H}_i$ are all also potentially functions of the system state $\mathbf{x}$, and the control inputs $\mathbf{u}$. Using matrix calculus identities from [42], the instantaneous rate of change of entropy, or mutual information rate is

$$\mathcal{I}(t) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\log|\,\mathbf{Y}(t)\,| = \frac{1}{2}\mathrm{trace}\left(\mathbf{Y}^{-1}(t)\dot{\mathbf{Y}}(t)\right). \tag{5.2}$$

Equation 5.2 represents a dynamic vector field $\mathcal{I}(\mathbf{x}, \mathbf{u}, t)$. It shows that the mutual information rate is determined by the current solution Equation 5.1. This relates the system state and control to the instantaneous rate of change of entropic information. Its gradient relates changes in the system state and control to changes in the rate of change of entropic information. Since $\mathbf{Y}(t)$ is not an explicit function of $\mathbf{x}$ or $\mathbf{u}$; the gradient field is given by

$$\begin{aligned}\nabla_{\mathbf{x}}\mathcal{I}(t) &= \frac{1}{2}\mathrm{trace}\left(\mathbf{Y}^{-1}(t)\nabla_{\mathbf{x}}\dot{\mathbf{Y}}(t)\right) \\ \nabla_{\mathbf{u}}\mathcal{I}(t) &= \frac{1}{2}\mathrm{trace}\left(\mathbf{Y}^{-1}(t)\nabla_{\mathbf{u}}\dot{\mathbf{Y}}(t)\right).\end{aligned} \tag{5.3}$$

This allows evaluation of the gradient field in terms of the current Fisher information and the partial derivatives of Equation 5.1. Control actions can be scheduled according to the direction and magnitude of the local gradient field. For example, considering the the constant velocity vehicle model Equation 5.4

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad \dot{\mathbf{x}}(t) = \begin{bmatrix} V\cos(\boldsymbol{u}(t)) \\ V\sin(\boldsymbol{u}(t)) \end{bmatrix}. \tag{5.4}$$

The best control with zero look ahead is the direction of the gradient vector of information rate with respect to the vehicle state $\{x, y\}$.

$$\boldsymbol{u}(t) = \arctan\left(\frac{\nabla_y \mathcal{I}(t)}{\nabla_x \mathcal{I}(t)}\right) \tag{5.5}$$

The concept of using information gain as a field for sensor platform control is related to other approaches in robotics. For example, Payton [69] uses artificial “pheromones” as a potential field for generating paths for platoons of robots. A possible weakness in potential field approaches is *ad hoc* methods for designing the fields. This is avoided in the information gain based approach. The field is formed directly from the models of the environment, vehicles and sensors.

5.5 Application to Area Exploration

To illustrate the applicability of the decentralised information-theoretic approach to general problems, an area exploration example is constructed. This problem represents the abstract task of estimating the value of some multi-variate characteristic distributed over a surface. The estimate of the characteristic and its associated uncertainty are now functions defined over an area. Entropy provides a time varying scalar measure of the information at a location on the surface. This can be used to construct utility measures by considering information and the information gain associated with sensing actions over an area. A multi-vehicle example is constructed to illustrate this.

5.5.1 Problem Formulation

A team of vehicles $i = 1, \ldots, n$ are exploring a terrain characteristic $\mathbf{T}(x, y)$ defined over area $\mathbf{S}$ on the (x, y) plane. For this example, the area is generalised to the unit square. The (two-dimensional) trajectory for the i^{th} vehicle is defined by $\mathbf{x}_i = [x_i(k), y_i(k)]^T,\ k = 1, \ldots, N$. Each vehicle makes observations $\mathbf{z}_i(k)$ of the terrain according to

$$\mathbf{z}_i(k) = \mathbf{T}(x, y) + \mathbf{v}_i(k), \tag{5.6}$$

where $\mathbf{v}_i(k)$ is taken to be a zero-mean uncorrelated Gaussian sequence with a variance that is a function of the range between the vehicle and terrain feature

$$\mathrm{E}\{\mathbf{v}_i(k)\mathbf{v}_i^T(k)\} \ = \mathbf{R}_{S,i}(k) = f_n\left(r\right),$$

where $r = \sqrt{(x_z(k) - x_i(k))^2 + (y_z(k) - y_i(k))^2}$ is the distance to the true terrain locations being observed $\{x_z, y_z\}$. The subscript 'S' is used to emphasise that these observations, estimates and information measures are quantities defined at every point x, y over $\mathbf{S}$.

It is required to generate estimates for the terrain characteristic $\mathbf{T}(x, y)$ over $\mathbf{S}$. The states of the vehicle trajectories are known exactly and the kinematics of the terrain are stationary. The observation model Equation 5.6, is an uncertain measurement of the true state. There is no transformation between the spaces of the measurement and state estimate. In this case, the state transition and observation Jacobian matrices are simply the identity matrix; $\mathbf{F}(k) = \mathbf{1}$ and $\mathbf{H}(k) = \mathbf{1}$. Thus, the observation information is simply the inverse of the observation variance, $\mathbf{I}_{S,i}(k) = \mathbf{R}_{S,i}^{-1}(k)$. So, the prediction and update stages of the information filter reduce to;

Prediction:

$$\mathbf{Y}_{S,i}(k \mid k-1) = \mathbf{Y}_{S,i}(k-1 \mid k-1)$$

Update:

$$\mathbf{Y}_{S}(k \mid k) = \mathbf{Y}_{S}(k \mid k-1) + \sum_{i=1}^{n} \mathbf{R}_{S,i}^{-1}(k)$$

Each vehicle maintains a local estimate of $\mathbf{T}(x, y)$ and propagates observation information through the team information structure.

5.5.2 Observation Model

Due to their underlying physical mechanisms, real world sensors typically exhibit exponential or quadratic variation in measurement uncertainty up to some finite range. A Gaussian function is used throughout this example.

$$\mathbf{R}_{S,i} = \sigma_{0,i}^{2} \exp\left(4.6(\frac{r}{r_{max,i}})^{2}\right) \tag{5.7}$$

where $\sigma_{0,i}$ and $r_{max,i}$ are the observation standard deviation at zero range and the maximum range of the sensor. The maximum range is taken to be the distance where the observation information is one percent of its maximum value. Examples of simple yet realistic models of sensor observation information are shown in Figure 5.6. It is important to note that this methodology places no restrictions on the usable sensor representations.

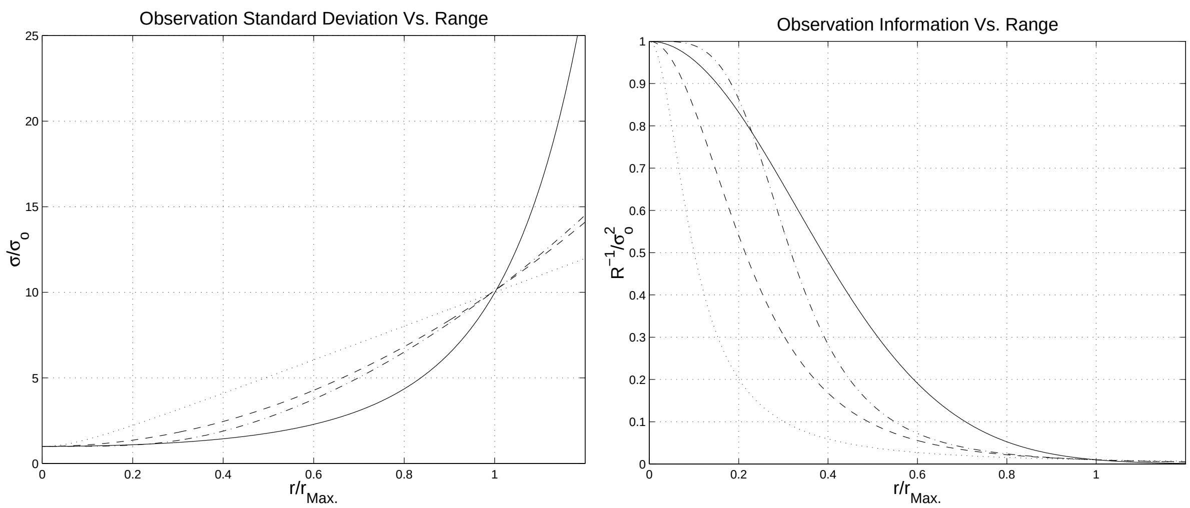


Figure 5.6: Exponential and quadratic modelling of range dependent measurement errors is a suitable approximation for a wide range of realistic sensors. Four representative models are shown to illustrate the associated spatial dependence of observation information:
$\mathbf{R} = \sigma_0^2 \exp\left(4.6(r/r_{max})^2\right)$ '*solid*', $\mathbf{R} = \sigma_0^2 \left(1 + 99(r/r_{max})^4\right)$ '$-\cdot-$',
$\mathbf{R} = \sigma_0^2 \left(1 + 99(r/r_{max})^2\right)$ '$\cdots$' and $\mathbf{R} = \sigma_0^2 \left(1 + 9(r/r_{max})^2\right)^2$ '$--$'.

5.5.3 Exploration Utility Metric

For the i^{th} sensor platform, its expected posterior Fisher information given the implemented information structure is

$$\mathbf{Y}_{S,i}(k \mid k) = \mathbf{Y}_{S,i}(k \mid k-1) + \mathbf{R}_{S,i}^{-1}(k) + \sum_{\substack{j=1 \\ j \neq i}}^{n} \delta_j \mathbf{R}_{S,j}^{-1}(k).$$

The posterior entropic information contained in the estimate of $\mathbf{T}(x, y)$ over $\mathbf{S}$ is given by

$$\boldsymbol{i}_{S,i}(k) = \frac{1}{2} \log \left[2\pi e \mathbf{Y}_{S,i}(k \mid k)\right].$$

The mutual information gain expected for an observation $\mathbf{z}_i(k)$ of a terrain element $\mathbf{T}(x, y)$ is

$$I_{S,i}(k) = \frac{1}{2} \log \left[\frac{\mathbf{Y}_{S,i}(k \mid k)}{\mathbf{Y}_{S,i}(k \mid k-1)}\right].$$

This information measure is a function over $\mathbf{S}$. Mutual information gives an expected utility measure for an observation made at $\mathbf{x}_i(k)$

$$\mathbf{U}(\mathbf{x}_i(k), k) = \int\int I_S(\mathbf{x}_i(k))\mathrm{d}x\mathrm{d}y.$$

This can now be employed to generate a performance metric to determine the trajectory of each vehicle to maximise the total information over the whole area for a number N, of observation stages. The trajectory utility is

$$\mathbf{J}(\mathbf{x}_i) = \sum_{k=1}^{N} \mathbf{U}(\mathbf{x}_i(k), k). \tag{5.8}$$

Note, maximising final entropic information is equivalent to maximising the utility given by 5.8. The mutual information formulation is preferred as it captures the value of each observation stage. This allows for combination with other cost criteria such as the required energy associated with the sensing action.

5.5.4 Visualisation of Information in Exploration

Two vehicles flying deliberately chosen non-optimal trajectories are used to illustrate the manner by which this information based utility formulation captures the exploration task. Figure 5.7 shows snapshots of the information measures over time. The vehicles start on opposite sides of the region. They travel at constant velocity over the indicated trajectories. The last leg of the first vehicle's path overlaps the first leg of the second vehicle's path. Plots (a, d, g, j) show the entropic information over the area. Plots (b, e, h, k) indicate the current observation information in the Fisher sense. Plots (c, f, i, l) indicate the mutual information gain for the current observation.

Mutual information gain is higher in locations with lower prior information. Vehicles are attracted to regions with low information. The value of future observations from the

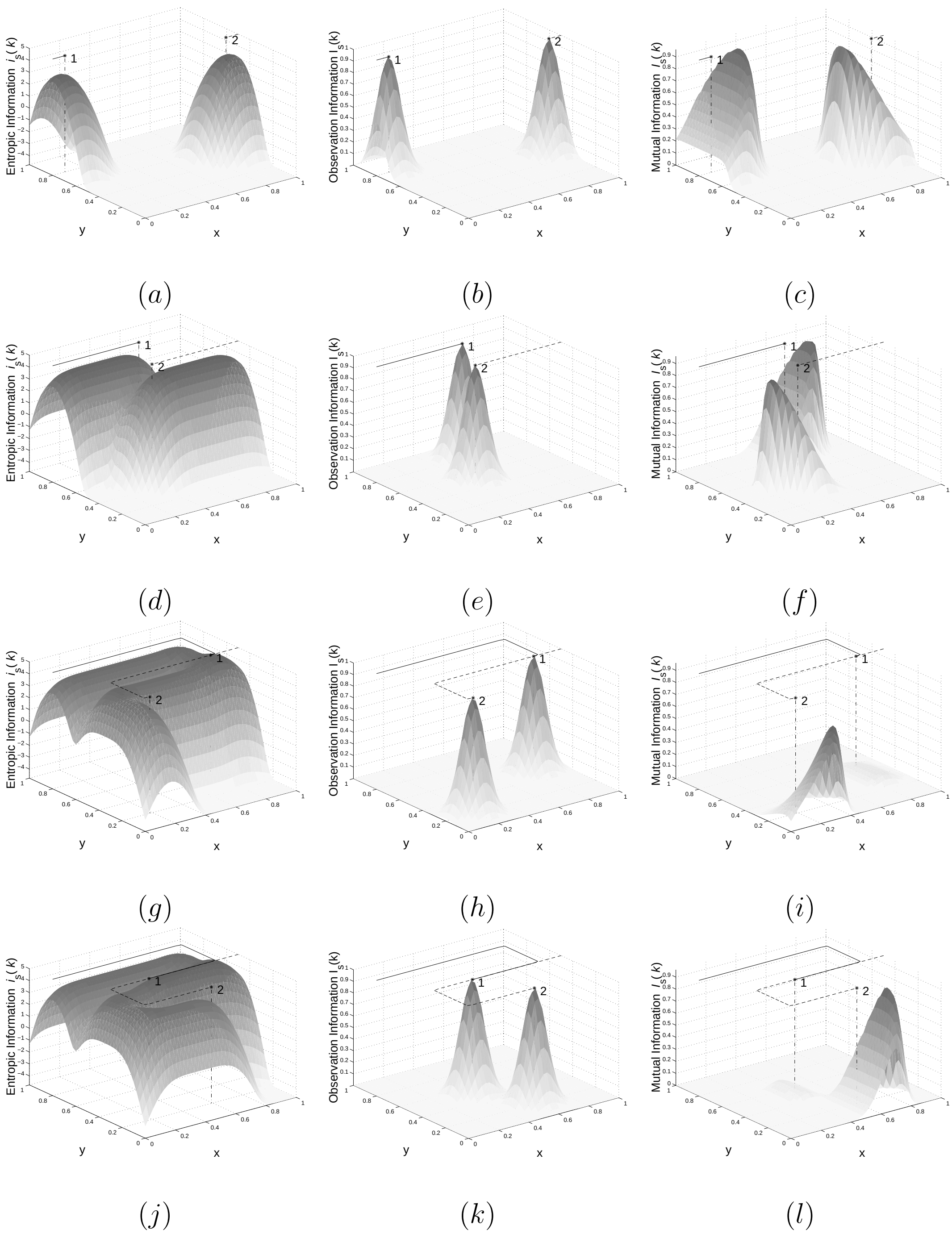


Figure 5.7: Snapshots of information measures for an area exploration example over time. Plots (a, d, g, j) show information $\boldsymbol{i}_S(k)$ (as negative entropy) over the area, (b, e, h, k) indicate the observation information for the current sensor actions $\mathbf{I}_{S,1}(k) + \mathbf{I}_{S,2}(k)$ and (c, f, i, l) display the mutual information gain $\mathcal{I}_{S,\{1,2\}}(k)$ associated with the current observations. Note that the mutual information gain is not centred at the maximum sensor observation information.

current location is reduced through the sensing action. Hence, vehicles are drawn away from regions they have explored so ensuring coverage. This highlights that the value of making observations at a location is time dependent and that the value of vehicle trajectories are coupled. Entropic information is revealed to provide a most suitable utility formulation for exploration. These metrics can be combined with other constraints, objectives and costs to form an optimal area exploration control problem and solution.

5.5.5 A Coordinated Team Solution

The area exploration utility measures can be included in the decentralised team architecture of Section 5.7 to develop a solution with the desired level of coordination and cooperation. A non-negotiated coordinated solution is pursued here.

The control and estimation process is conducted in discrete time with time step ΔT. Each vehicle moves at constant velocity $\mathbf{V}_i$ with its heading rate as the decision variable $\boldsymbol{u}_i(k)$. The decision is constrained by $\mid \boldsymbol{u}_i(k) \mid < \boldsymbol{u}_{i,Max}$. The vehicle state is governed by

$$\begin{bmatrix} \Delta x_i(k) \\ \Delta y_i(k) \\ \Delta \psi_i(k) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_i \cos \psi_i(k) \\ \mathbf{V}_i \sin \psi_i(k) \\ \boldsymbol{u}_i(k) \end{bmatrix} \Delta T. \tag{5.9}$$

Each vehicles sensor observation information is modelled by

$$\mathbf{R}_{S,i}^{-1} = \sigma_{0,i}^2 \exp\left(-4.6(\frac{r}{r_{max,i}})^2\right). \tag{5.10}$$

An information threshold $\boldsymbol{i}_{Thresh}$, is introduced to specify that the exploration requirements have been met at a location. The task is completed when entropic information is higher than this over the entire area $\mathbf{S}$. Each vehicle's action is decided based on maximising utility given by the instant mutual information gain with zero look-ahead. Under this approximation, the sensor platforms' decisions are decoupled. The global solution is to maximise individual mutual information gain.

$$\begin{aligned}\mathbf{J}_i(k) &= \iint\limits_{\mathbf{S}} \delta_{S,i}(k)\mathcal{I}_{S,i}(k)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{2}\iint\limits_{\mathbf{S}} \delta_{S,i}(k)\mathbf{Y}_{S,i}^{-1}(k)\mathbf{R}_{S,i}^{-1}(k)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{5.11}$$

Where $\delta_{S,i}(k)$ captures the explored region $\delta_{S,i}(k) = \begin{cases} 1 & \text{if } \boldsymbol{i}_{S,i}(k) < \boldsymbol{i}_{Thresh} \\ 0 & \text{if } \boldsymbol{i}_{S,i}(k) \geq \boldsymbol{i}_{Thresh} \end{cases}$

The vehicle heading and control that maximises the instant mutual information gain is

$$\psi_i^\star(k) = \arctan\frac{\nabla_{y_i}\mathbf{J}_i(k)}{\nabla_{x_i}\mathbf{J}_i(k)}, \quad \boldsymbol{u}_i^\star(k) = \frac{\psi_i^\star(k) - \psi_i(k-1)}{\Delta T} \tag{5.12}$$

Where,

$$\begin{aligned}\nabla_{x_i}\mathbf{J}_i(k) &= \frac{1}{2}\iint\limits_{\mathbf{S}} \delta_{S,i}(k)\mathbf{Y}_{S,i}^{-1}(k)\nabla_{x_i}\mathbf{R}_{S,i}^{-1}(k)\mathrm{d}x\mathrm{d}y \\ \nabla_{y_i}\mathbf{J}_i(k) &= \frac{1}{2}\iint\limits_{\mathbf{S}} \delta_{S,i}(k)\mathbf{Y}_{S,i}^{-1}(k)\nabla_{y_i}\mathbf{R}_{S,i}^{-1}(k)\mathrm{d}x\mathrm{d}y \\ \nabla_{x_i}\mathbf{R}_{S,i}^{-1}(k) &= \frac{9.2}{\sigma_{0,i}^2 r_{max,i}^2}(x_z - x_i(k))\exp\left[-4.6\left(\frac{r}{r_{max,i}}\right)^2\right] \\ \nabla_{y_i}\mathbf{R}_{S,i}^{-1}(k) &= \frac{9.2}{\sigma_{0,i}^2 r_{max,i}^2}(y_z - y_i(k))\exp\left[-4.6\left(\frac{r}{r_{max,i}}\right)^2\right]\end{aligned}$$

The constrained individual decision is

$$\boldsymbol{u}_i(k) = \begin{cases} \boldsymbol{u}_{i,Max} & \text{if } \boldsymbol{u}_i^\star(k) \geq \boldsymbol{u}_{i,Max} \\ \boldsymbol{u}_i^\star(k) & \text{if } -\boldsymbol{u}_{i,Max} < \boldsymbol{u}_i^\star(k) < \boldsymbol{u}_{i,Max} \\ -\boldsymbol{u}_{i,Max} & \text{if } \boldsymbol{u}_i^\star(k) \leq -\boldsymbol{u}_{i,Max} \end{cases} \tag{5.13}$$

5.5.6 Implementation Results

Figure 5.8 displays stages of an example solution to this exploration problem. Four vehicles seek information about the scalar characteristic. Equations 5.9 to 5.13 govern this process. The vehicles start from random initial conditions and have no initial information about the characteristic they are estimating. The task is terminated when the entropic information is above a specified threshold over the entire area. At each displayed solution stage, four plots indicate the various information measures, the history of the vehicle trajectories and the extent of the area explored. The unexplored portion of the area is indicated by the shaded region. The information measures shown are the current entropic information $\boldsymbol{i}_S(k)$, mutual information $\mathcal{I}_S(k)$ and observation information $\mathbf{I}_S(k)$.

This solution rationale provides the least complex coordinated outcome with regard to associated numerical effort. The vehicle trajectories indicate that the vehicle sensor platform decisions are indeed coordinated. The paths overlap as in this situation a single pass is not sufficient to fully explore the over flown region. A criticism of this solution would be that more information than required is gathered over some regions. This could be considered inefficient. Allowing the sensor platforms to plan their actions ahead in time and negotiate a cooperative solution would reduce this, at the cost of the associated computation and communication overhead. Without planning to the final time of task completion, it is not possible to ensure that increasing the planning horizon will ensure an improved solution.

In this static case it is possible to solve *a priori* for the entire cooperative team open-loop actions to final time. This becomes a daunting problem for exploring dynamic characteristics over arbitrary and dynamic shaped areas. The attraction of this approach is the relative ease with which these attributes can be handled.

The information-theoretic methodology achieves this solution to the exploration problem without imposing *ad hoc* rules. The attraction to unexplored regions is captured by the problem modelling and entropic utility, and discovered by the solution procedure.

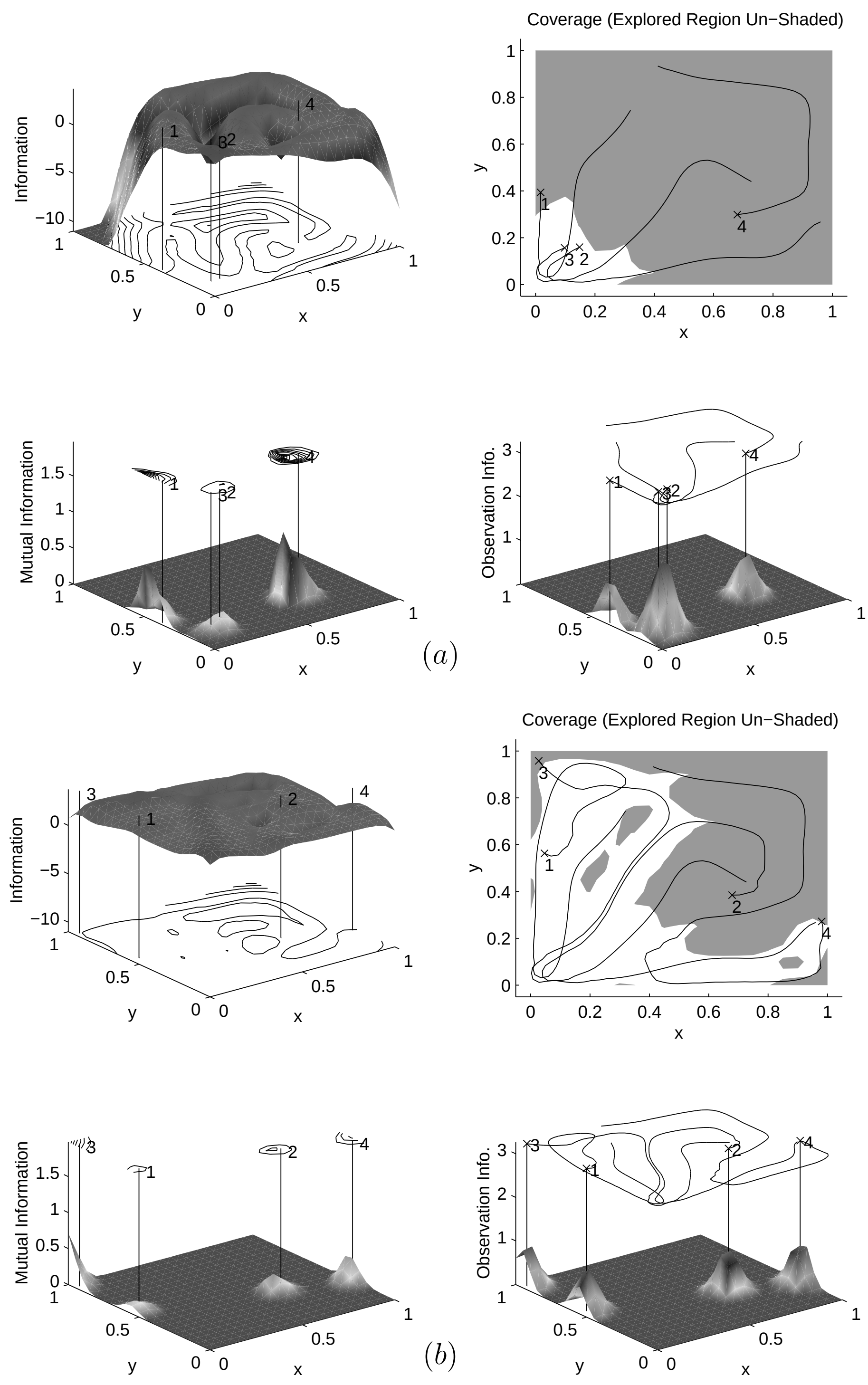


Figure 5.8: Four stages of an example coordinated team solution to an area exploration problem are shown. At each stage, four plots indicate the various information measures, the history of the vehicle trajectories and the extent of the area explored. A region is considered explored when the entropic information is above a threshold. The information measures shown are the current entropic information, mutual information and observation information

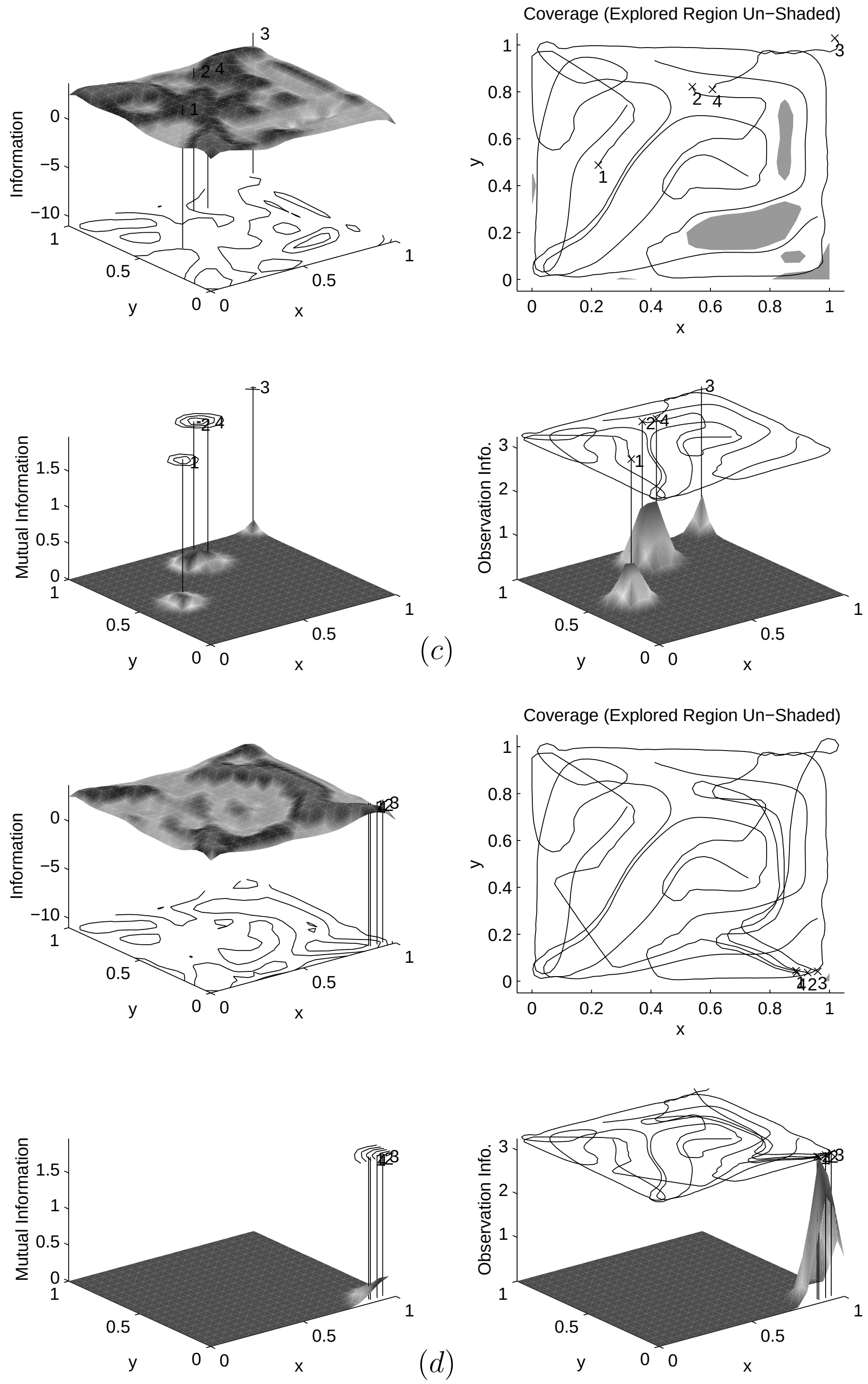


Figure 5.8: (continued)

5.6 Multi-Vehicle Multi-Feature Localisation: An Example

The feature localisation example of Section 4.3 is now extended to multiple features and multiple sensor platforms. This more complex example allows exploration and illustration of the issues raised in Section 5.2, regarding mechanisms underlying coordination and control, and in Section 5.3, regarding the role of different information structures.

5.6.1 Problem Formulation

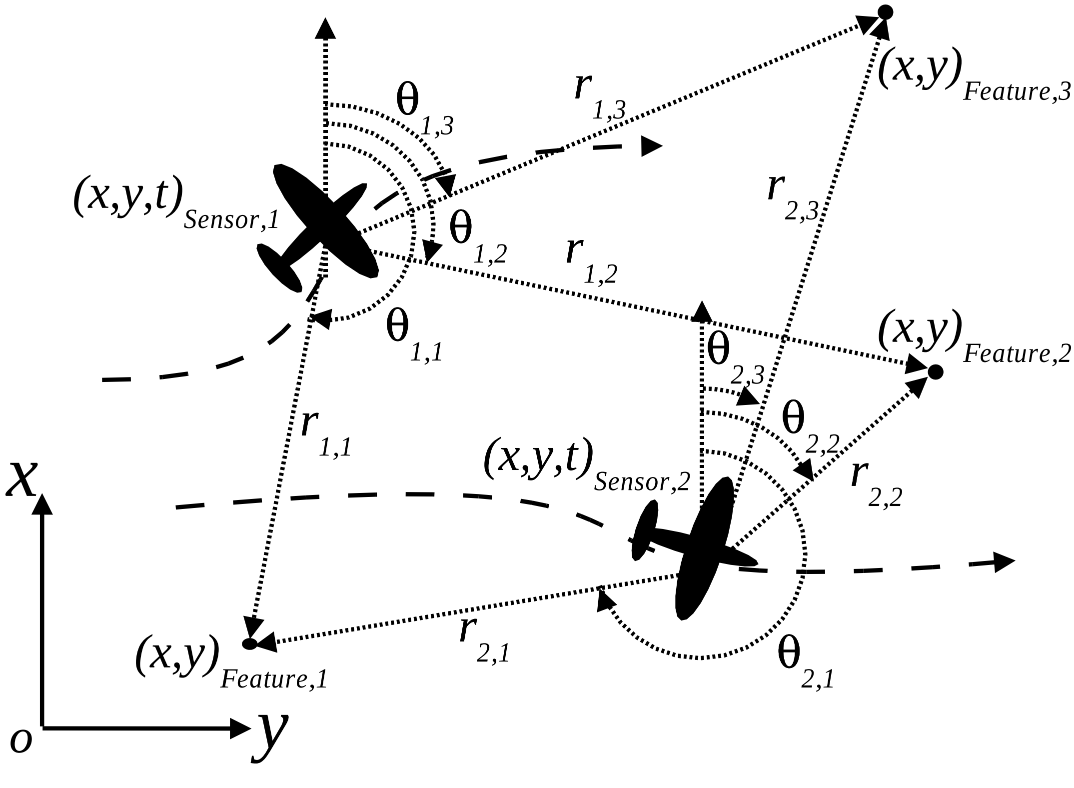


Figure 5.9: 2D Multi-Vehicle Multi-Feature Localisation Problem

The problem consists of n sensor platforms $i = 1, \ldots, n$, localising m point features $j = 1, \ldots, m$. The modelling of each individual vehicle, sensor and feature is the same as in the single vehicle example of Section 4.3. The global system equations are composed from these individual models. The global state consists of the current sensor platform locations and headings, feature location estimates and feature error covariance. Each vehicle maintains a local estimate of the feature states and a map of the feature information given by

$$\hat{\mathbf{x}}_f(t) = \begin{bmatrix} \hat{\mathbf{x}}_{f,1}(t) \\ \vdots \\ \hat{\mathbf{x}}_{f,m}(t) \end{bmatrix} = \mathbf{Y}_f^{-1}(t)\hat{\mathbf{y}}_f(t), \quad \mathbf{Y}_f(t) = \begin{bmatrix} \mathbf{Y}_{f,1}(t) & 0 & \dots & 0 \\ 0 & \mathbf{Y}_{f,2}(t) & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{Y}_{f,m}(t) \end{bmatrix} \tag{5.14}$$

The local information dynamics for utility prediction is

$$\dot{\mathbf{Y}}_{f,j}(t) = \mathbf{I}_{i,j}(t) + \sum_{\substack{k=1 \\ k \neq i}}^{n} \delta_{k,j}\mathbf{I}_{k,j}(t) \tag{5.15}$$

where $\delta_{i,j}$ allows for incorporation of different information structures. $\delta_{i,j} = 1$ *if* conditional observation information from sensor i regarding feature j is available, *else* $\delta_{i,j} = 0$. Each feature and observation Fisher information is a 2×2 block diagonal element

$$\mathbf{Y}_{f,j}(t) = \begin{bmatrix} \mathrm{Y}_{x,j}(t) & \mathrm{Y}_{xy,j}(t) \\ \mathrm{Y}_{xy,j}(t) & \mathrm{Y}_{y,j}(t) \end{bmatrix}, \quad \mathbf{I}_{i,j}(t) = \begin{bmatrix} \mathrm{I}_{x,i,j}(t) & \mathrm{I}_{xy,i,j}(t) \\ \mathrm{I}_{xy,i,j}(t) & \mathrm{I}_{y,i,j}(t) \end{bmatrix}$$

The state vector of each sensor platform is now the stacked vehicle state and feature location Fisher information.

$$\mathbf{x}_i(t) = \begin{bmatrix} \mathbf{x}_{s,i}(t) \\ \mathbf{x}_{f,1}(t) \\ \vdots \\ \mathbf{x}_{f,m}(t) \end{bmatrix}, \quad \text{Where, } \mathbf{x}_{s,i}(t) = \begin{bmatrix} x_i(t) \\ y_i(t) \\ \psi_i(t) \end{bmatrix}, \text{ and } \mathbf{x}_{f,j}(t) = \begin{bmatrix} \mathrm{Y}_{x,j}(t) \\ \mathrm{Y}_{xy,j}(t) \\ \mathrm{Y}_{y,j}(t) \end{bmatrix}$$

The combined state dynamics for the i^{th} sensor platform is

$$\dot{\mathbf{x}}_i(t) = \begin{bmatrix} V\cos(\psi_i(t)) \\ V\sin(\psi_i(t)) \\ \boldsymbol{u}_i(t) \\ \sum_{i=1}^{n} \mathrm{I}_{x,i,1}(t) \\ \sum_{i=1}^{n} \mathrm{I}_{xy,i,1}(t) \\ \sum_{i=1}^{n} \mathrm{I}_{y,i,1}(t) \\ \vdots \\ \sum_{i=1}^{n} \mathrm{I}_{x,i,m}(t) \\ \sum_{i=1}^{n} \mathrm{I}_{xy,i,m}(t) \\ \sum_{i=1}^{n} \mathrm{I}_{y,i,m}(t) \end{bmatrix} = \begin{bmatrix} V\cos(\psi_i(t)) \\ V\sin(\psi_i(t)) \\ \boldsymbol{u}_i(t) \\ \frac{1}{\sigma_\theta^2 r_{i,1}(t)^2}\sin^2(\theta_{i,1}(t)) + \sum_{k=1,k\neq i}^{n} \delta_{k,1}\mathrm{I}_{x,k,1}(t) \\ \frac{-1}{\sigma_\theta^2 r_{i,1}(t)^2}\sin(\theta_{i,1}(t))\cos(\theta_{i,1}(t)) + \sum_{k=1,k\neq i}^{n} \delta_{k,1}\mathrm{I}_{xy,k,1}(t) \\ \frac{1}{\sigma_\theta^2 r_{i,1}(t)^2}\cos^2(\theta_{i,1}(t)) + \sum_{k=1,k\neq i}^{n} \delta_{k,1}\mathrm{I}_{y,k,1}(t) \\ \vdots \\ \frac{1}{\sigma_\theta^2 r_{i,m}(t)^2}\sin^2(\theta_{i,m}(t)) + \sum_{k=1,k\neq i}^{n} \delta_{k,m}\mathrm{I}_{x,k,m}(t) \\ \frac{-1}{\sigma_\theta^2 r_{i,m}(t)^2}\sin(\theta_{i,m}(t))\cos(\theta_{i,m}(t)) + \sum_{k=1,k\neq i}^{n} \delta_{k,m}\mathrm{I}_{xy,k,m}(t) \\ \frac{1}{\sigma_\theta^2 r_{i,m}(t)^2}\cos^2(\theta_{i,m}(t)) + \sum_{k=1,k\neq i}^{n} \delta_{k,m}\mathrm{I}_{y,k,m}(t) \end{bmatrix}$$

As in the single vehicle case it is desired to maximise the feature estimate entropic information at the final time of the stage t_f.

$$\begin{aligned} \mathbf{J}_i(\boldsymbol{u}_i, t_f) &= \log|\,\mathbf{Y}_f(t_f)\,| \\ &= \sum_{j=1}^{m} \log|\,\mathbf{Y}_{f,j}(t_f)\,| \end{aligned} \tag{5.16}$$

5.6.2 Coordinated Solutions

Coordinated solutions to the multi-feature multi-sensor localisation problem are used to investigate properties of static information structures. Four coordinated solution implementations with different static information structures were explored:

1. Coordinated control can be achieved simply by employing the DDF algorithm among decision makers with local information seeking controllers. This is shown in Figure 5.10. Trajectories are shown for the same local controllers with and without the underlying DDF process activated.

2. Communication can be reduced and still result in an acceptable coordinated solution. Figure 5.11 shows an example of the variation in platform trajectories when the propagation of consolidated observation information is delayed in the DDF process. The impact on actions and performance reduces as uncertainty is decreased. This indicates that the value of communicating observations is determined by their mutual information gain.

3. The interpretation of information seeking as a dynamic potential field is illustrated in Figure 5.12. Six snapshots of the mutual information field are displayed of three bearings-only platforms localising four features. The field represents the information gain associated with an observation made at that location. As observations are made, the information propagates through the platforms via DDF. This process dynamically alters the field.

4. A claimed advantage of decentralised systems is tolerance to failure and addition of sub-systems. Figure 5.13 indicates how platform failure and addition is handled seamlessly through the definition of and interface to the team information structure. On failure, a platform ceases to contribute information. New members receive information through the DDF process and commence communicating their own expected and actual observation information. The remaining team members continue their individual decision making based on whatever information is available to them.

5.6.3 Comparative Cooperative and Coordinated Solutions

The cooperative and coordinated solution procedures from Section 5.4 are used to compare two solutions to a two platform, single feature bearings only localisation problem. Both solutions determine multistage open loop control sequences over a fixed time horizon. The two solutions differ in their information structure:

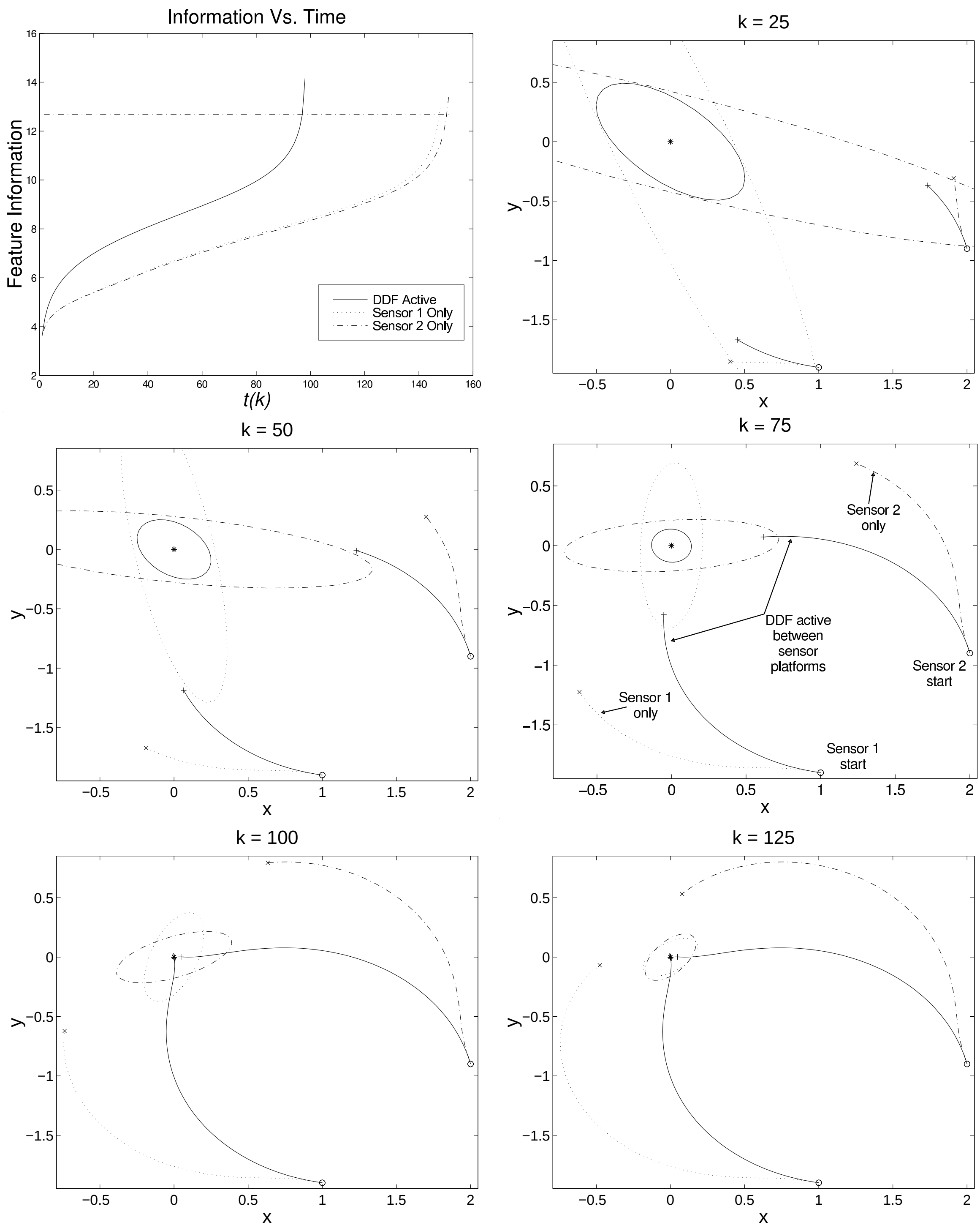


Figure 5.10: Illustration of coordinated control arising through decentralised data fusion (DDF). Feature information over time is shown along with five snapshots of the locally optimal trajectories, with and without DDF active. Both vehicles implement local control laws that maximise their individual information gain from bearings-only observations given local prior knowledge. Coordination results from the DDF process updating local prior knowledge from which the optimal action is generated. There is no change to the control laws between cases.

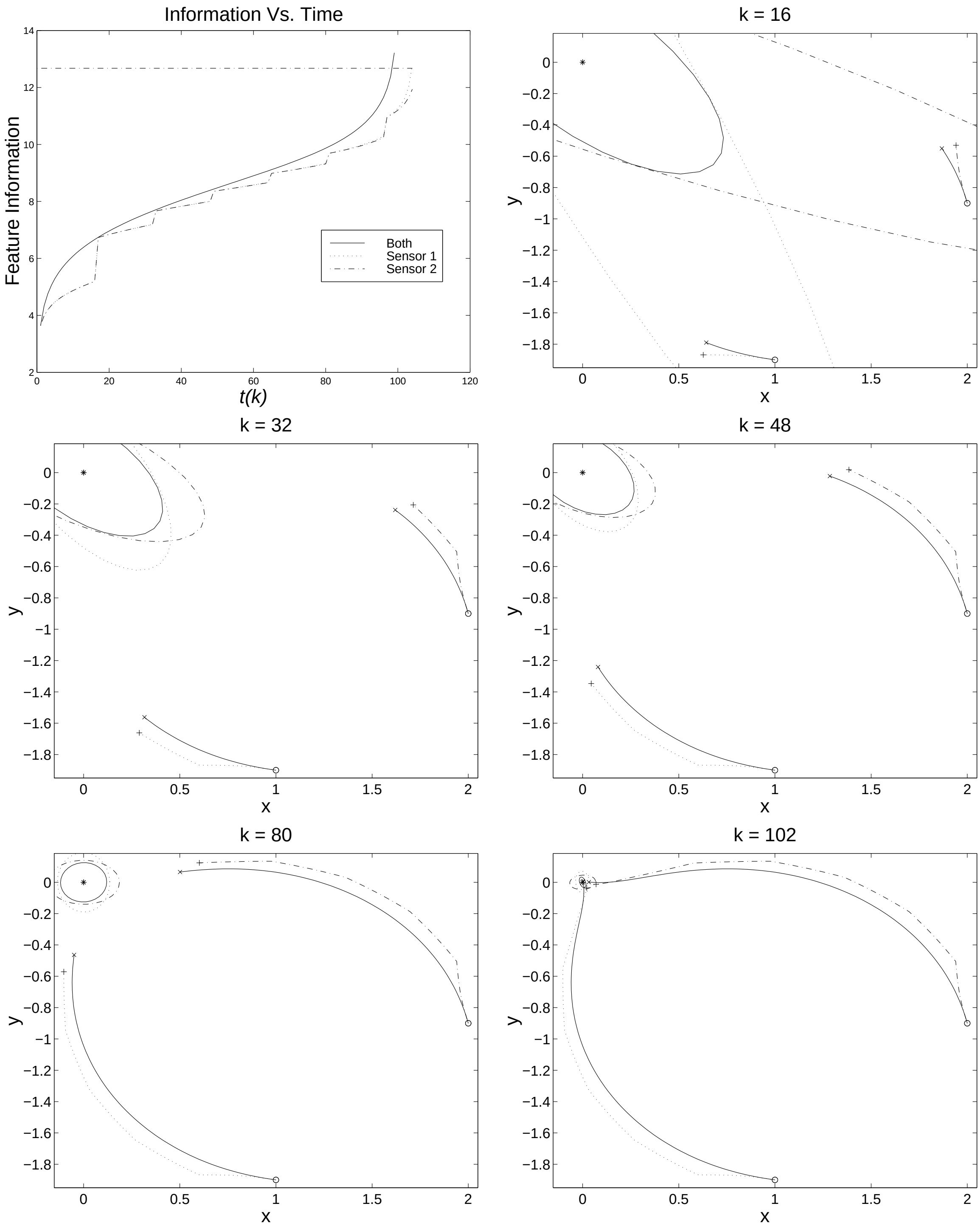


Figure 5.11: Illustration of the effect of delayed communication. The observation information of each sensor is consolidated and communicated every 15 solution steps. The trajectories are only significantly affected when the feature information is low. This offers significant communication savings with only a relatively small increase in the time required to perform the task. It also suggests that the communication rate can be varied according to the mutual information gain associated with the observations.

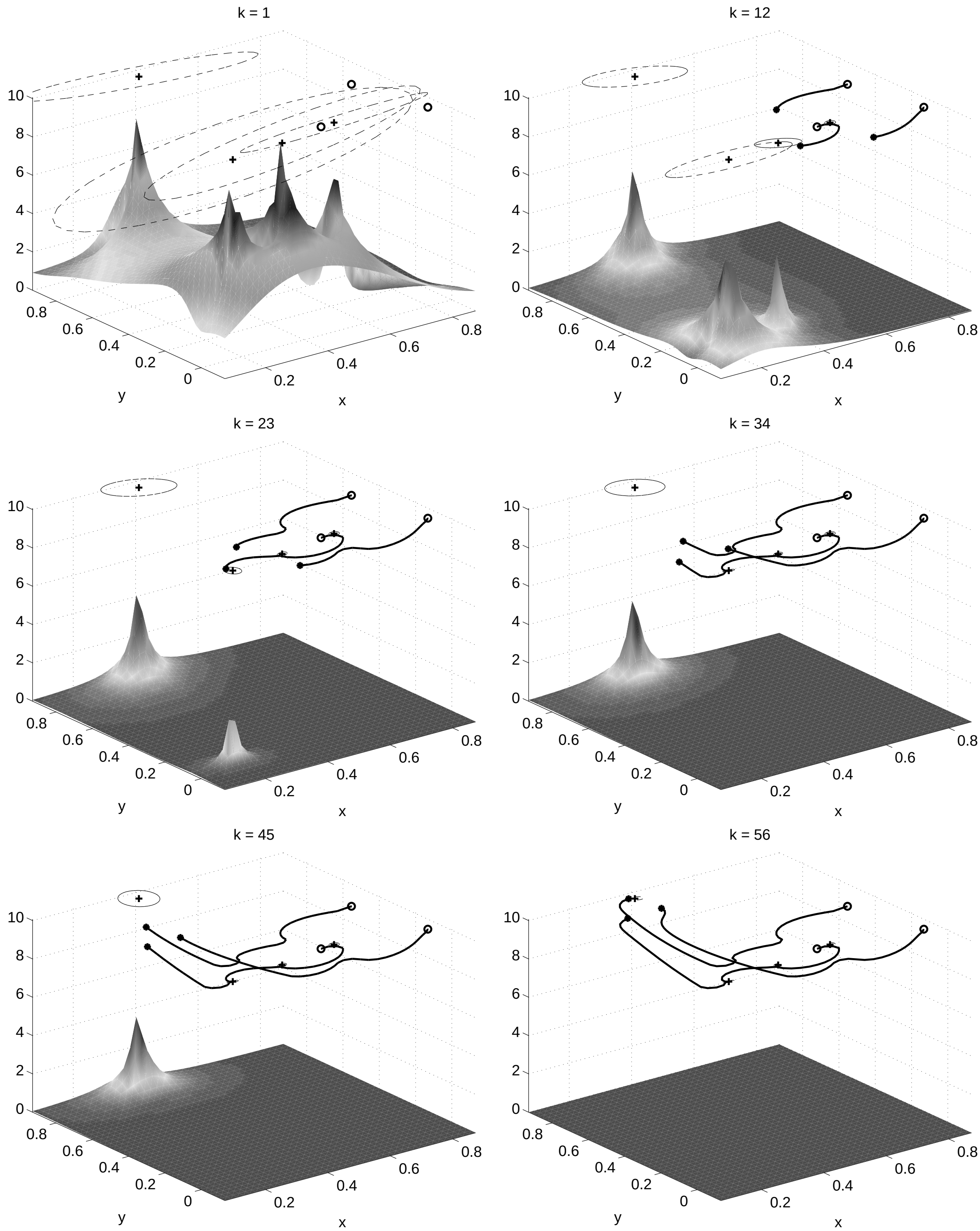


Figure 5.12: Interpretation (of mutual information gain) as a potential field. Six snapshots of the mutual information field as three bearings-only platforms localise four point features. The vehicles "surf" along the local gradient of this dynamic field. As observations are made, the information propagates through the platforms via DDF altering the field. This approach provides scalable coordinated control with very low computational requirements.

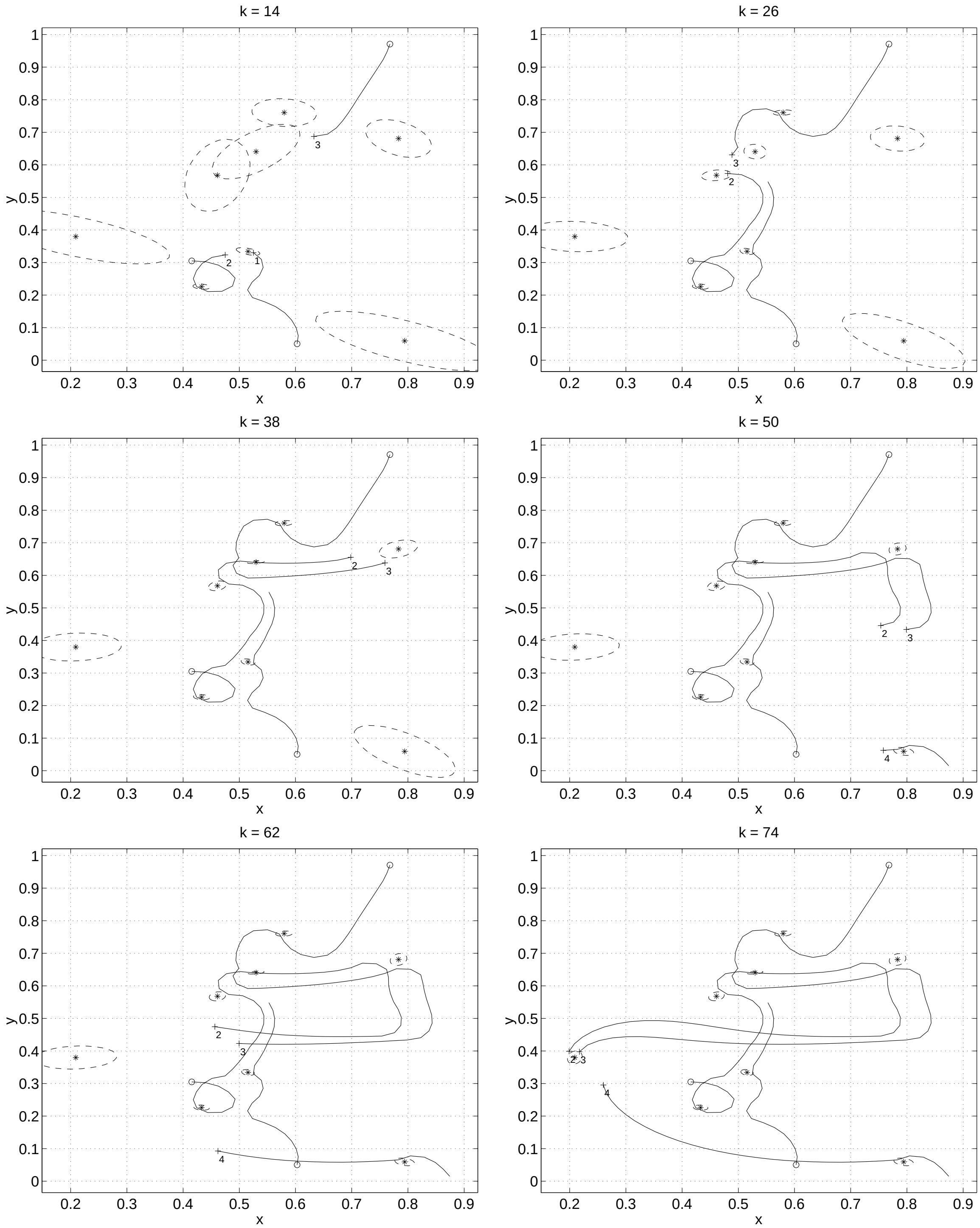


Figure 5.13: Six stages of a coordinated feature localisation solution. At stage $k = 25$, platform 1 fails. At stage $k = 45$, a fourth platform joins the sensing team. Each decision maker's interface to the team information structure facilitates seamless handling of platform failure and addition. Team members make decisions based on the information available to them through this structure. On failure, a platform ceases to contribute information. New members receive information through the DDF process and commence communicating their own expected and actual observation information.

1. **Coordinated solution:** A static information structure is implemented. Information is communicated as observations are made, but no predicted observation information is exchanged. The actions over each horizon are determined without accounting for the immediate influence of the other platform. The feature location is estimated by the DDF algorithm while the plan is executing. Subsequent horizons are based on information from this DDF process. This incorporates observation information from the other platform, resulting in a coordinated solution.

2. **Cooperative solution:** A dynamic information structure informs each sensor platform of the future expected observation information associated with the other platforms intended sensing plan. Each vehicle in turn, re-plans its open loop control sequence and communicates its new expected observation information. This iterative process is terminated when changes in each sensor platforms intended actions are below a convergence tolerance.

Results for the comparative solutions are presented in Figures 5.14 and 5.15. Figure 5.14 shows the platform trajectories along with the predicted and actual feature entropic information. Figure 5.15 indicates the state and observation innovations in the DDF process along with the control actions and vehicle headings. The platforms move with velocity $1m/s$ and make bearing observations at $10Hz$ with standard deviation $\sigma = 2.5^o$. The open loop plans are made over 4 second time horizons. The control is parameterised by $1Hz$ zero order stages.

The cooperative solution achieves higher information gain over time by jointly optimising the coupled team objective.

5.6.4 Influence of the Information Structure

This example highlights the importance of the form of the information structure. The coordinated and cooperative solutions indicated in Figure 5.14 are both optimal with

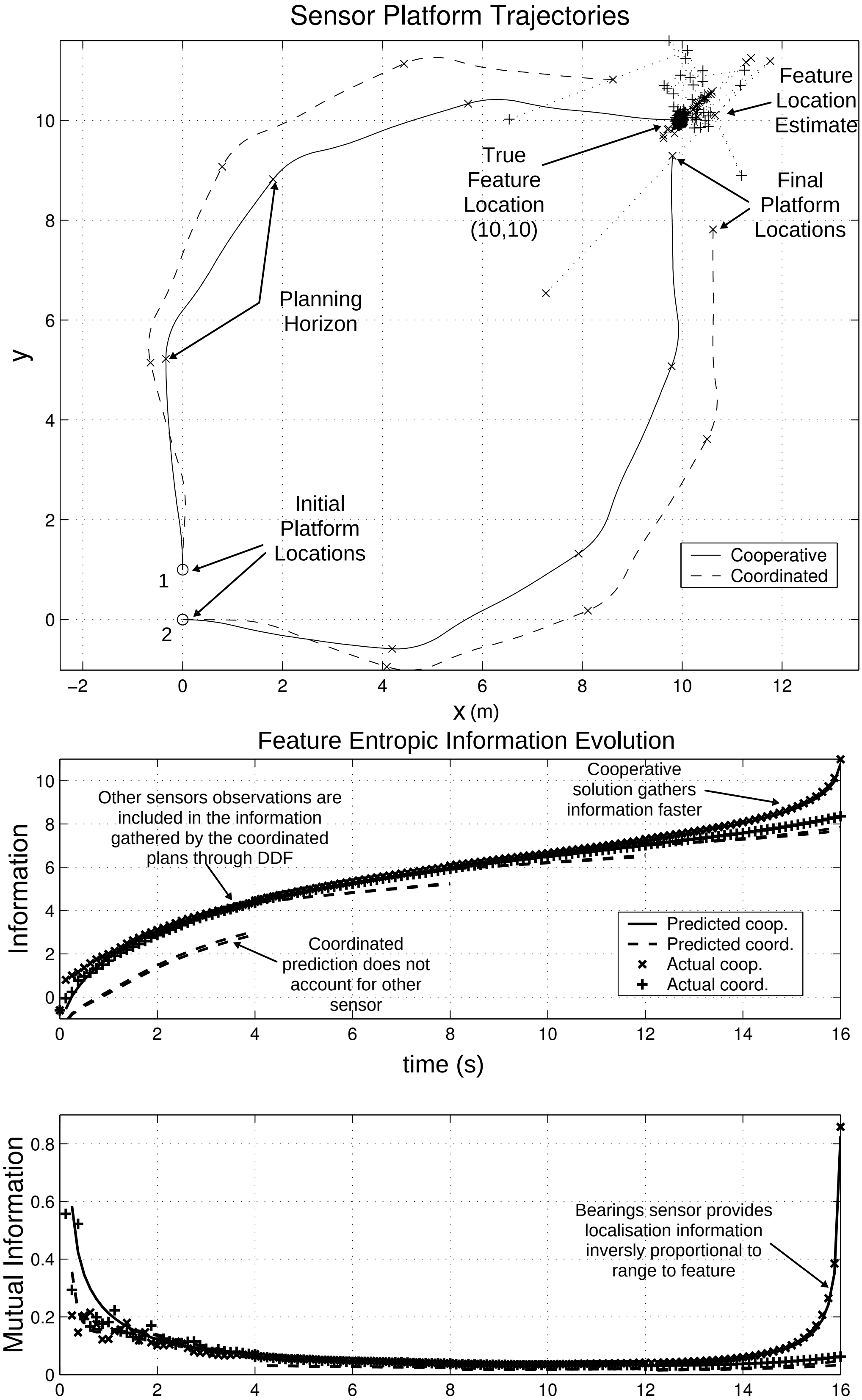


Figure 5.14: Trajectories of the sensor platform and location estimate along with the predicted and actual information evolution for the coordinated and cooperative bearings-only localisation example. Bearings observations are made at $10Hz$. The open-loop control is calculated at $1Hz$ over a 4 second time horizon. The difference between the solutions is most significant when the feature information is low.

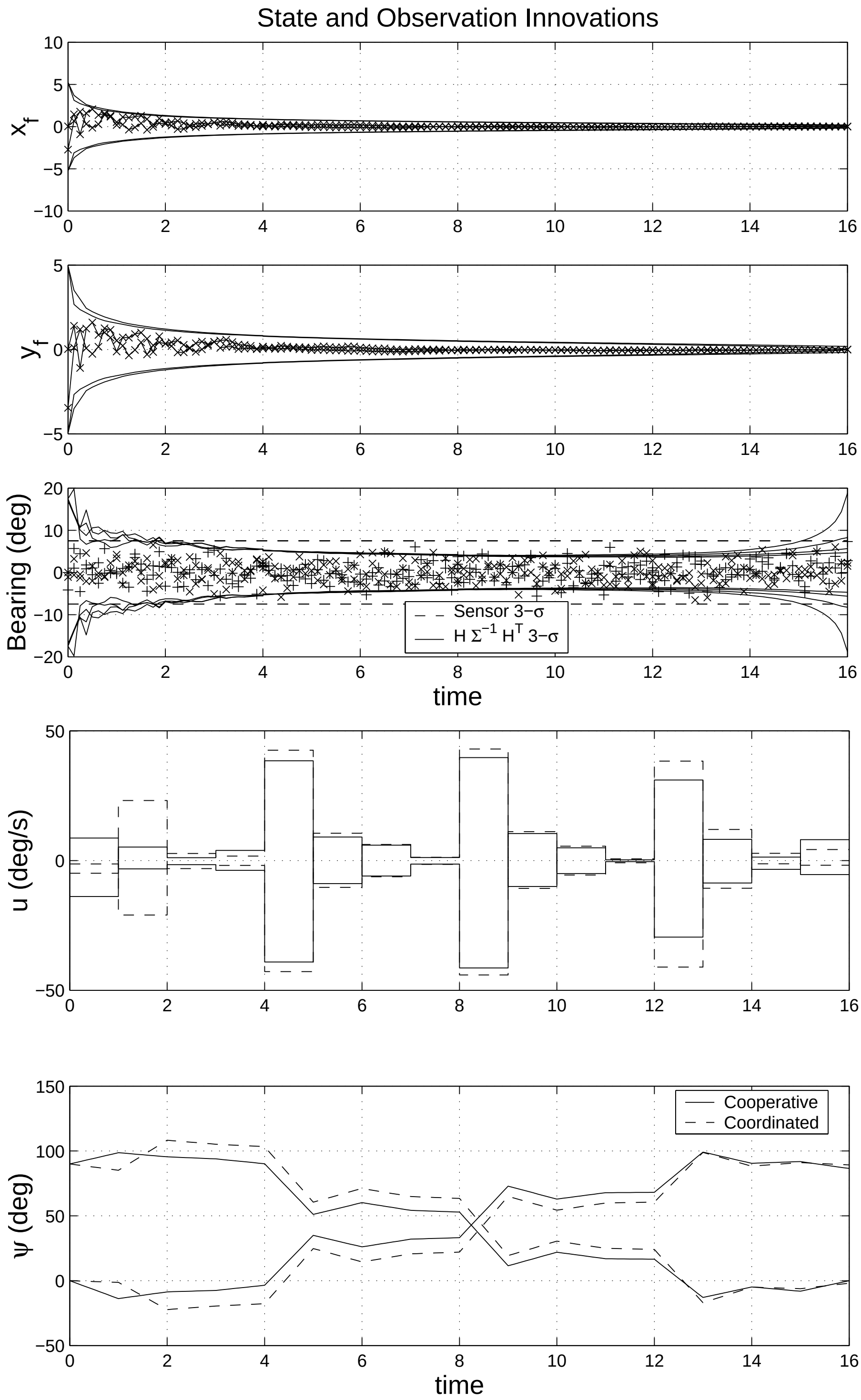


Figure 5.15: Feature state estimate and bearings observation innovations from the DDF process along with coordinated and cooperative control actions and vehicle heading.

respect to the available information. The information structure in the cooperative case allows the decision makers to reach a jointly optimal solution through exchange of predicted information. The degree to which choice of information structure influences sensor platform actions reduces with reduction in uncertainty.

The information structure clearly influences the performance and decisions made by the team. Information shared through the information structure enables coordination and cooperation between decision makers. The value of sharing information among decision makers is dynamic and depends both on observed and predicted information. Extending the planning horizon, using predicted information and seeking a negotiated solution increases the value of cooperation but incurs a cost in increased communication and computation. Design of the information structure is a trade-off between the attainable level of optimality and the required system resources.

5.7 Towards A General Active Sensor System Architecture

Decentralised team decision making and control is a logical extension of decentralised and information-theoretic modelling and fusion. Once information is made available locally, in a decentralised form, and information based utility functions have been defined, then it is possible to implement a decentralised team decision process.

Information-theoretic models offer a uniquely powerful method of mathematically modelling large-scale systems. Decentralised methods allow information gathering and decision making systems to be described in a mathematically rigorous and modular manner. The global system can be considered as a system of interacting systems or 'Systems of Systems'.

Information-theoretic methods provide three essential ingredients necessary to develop a usable 'theory' for such systems:

1. **Analytic:** Decentralised and information-theoretic methods provide an ability to analyse and reason about a system and its information gathering or decision making role. In particular, the process of local information formation, communication and assimilation, and decision making are well formulated.

2. **Composable and dynamically configurable:** Decentralised methods also provide an ability to compose mathematical descriptions of larger systems from descriptions of component sub-systems. Significantly this is a consequence of the inherent modularity and scalability of decentralised system algorithms.

3. **Predictive:** Information-theoretic methods provide a natural and powerful ability to predict expected "information" rewards associated with an action sequence.

The basis for this is provided by the Decentralised Data Fusion (DDF) architecture, Team Decision Theory, Information-Theoretic Utility and the Best-Response negotiation procedure. The amalgam of these ingredients offers a general methodology for the decentralised control of active sensor teams. A node based non-hierarchical system architecture is proposed and discussed.

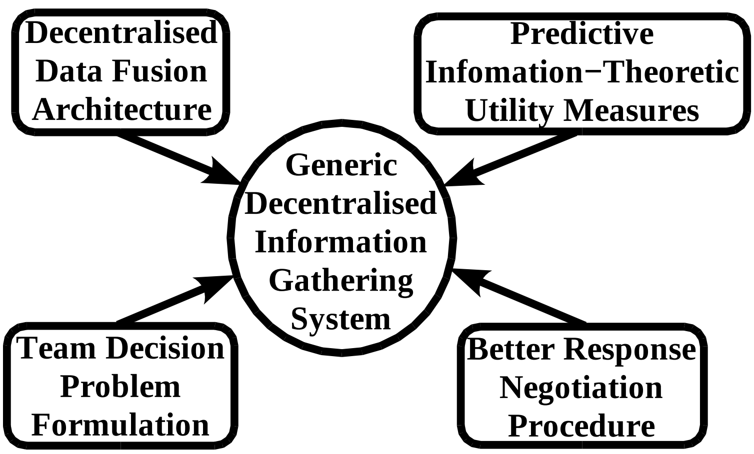


Figure 5.16: A generic sensor team architecture based on information theoretic utility, DDF, team decision theory and better response solution procedures

5.7.1 Extending the Decentralised Sensor Fusion Architecture

The key to scalable, decentralised data fusion is the additive and associative property of information. All system nodes can be made aware of global information through propagation of inter-node information differences through a communication network. This is studied in detail by Grime [28]. A channel filter at each fusion node manages the accumulation and communication of information. Regardless of the physical connectivity, the communications layer data can be routed in a virtual tree network. Dismissing arbitrary network topologies greatly simplifies the formulation of the channel filter algorithms. The inter-node communications requirement for this architecture is independent of the number of fusion nodes. Orgaisation and management of communication in sensor networks is addressed in detail by Ho [32] and Utete [91, 93, 94, 92, 95]

Communication of predicted observation information has been identified as the mechanism that enables cooperation between sensing team members. The channel filter concept in DDF methods can be extended to include propagation of predicted information. Information prediction propagation is simplified by the fact that the conditional observation information has not yet been fused with other knowledge. This alleviates the problem of identifying common information between nodes which pervades the DDF problem. Cooperative solutions involve an iterated negotiation procedure. At each iteration stage, the node negotiation manager simply propagates the difference between its current and last predicted information added to the predicted information it receives.

5.7.2 An Endogeneous Sensor Node

To realise decentralised decision making, an additional layer is added to the DDF architecture at each sensing node. The additional functionality comprises the individual decision process and a predictive channel filter. Better response negotiation and propagation of predicted observation information enable this node to participate in cooperative

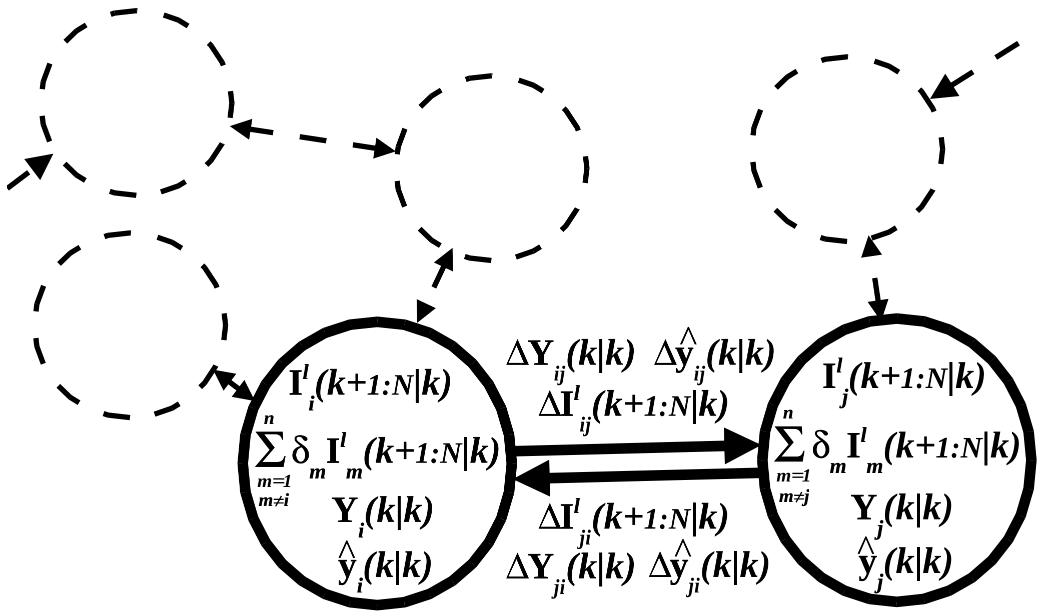


Figure 5.17: Channel filter algorithms implemented at each node manage accumulation and communication of information. Differences in information between nodes is transmitted and assimilated simply through summation. This ensures propagation of information through the tree networked team of sensing decision makers. The established DDF node structure is extended to include propagation of future expected observation information.

solutions to team decision problems. The internal components are illustrated in Figure 5.18. This structure is termed Endogeneous to emphasise that the enabling mechanisms reside internal to the nodes.

Figure 5.18 describes a generic architecture for a decision making node within a sensing team. Teams of inter-operating, coordinated or cooperative sensors and vehicles can be composed from elements engineered with this nodal structure. This deceptively simple architecture possesses a number of important properties. Each team member can internally measure the value of its observations and actions with respect to the team. This is achieved by comparing accumulated actual and predicted team information with local observation information. In combination with the information from other team members, communication of these observations can be included in the node decision process. Internally, nodes can decide what information about the environment is maintained lo-

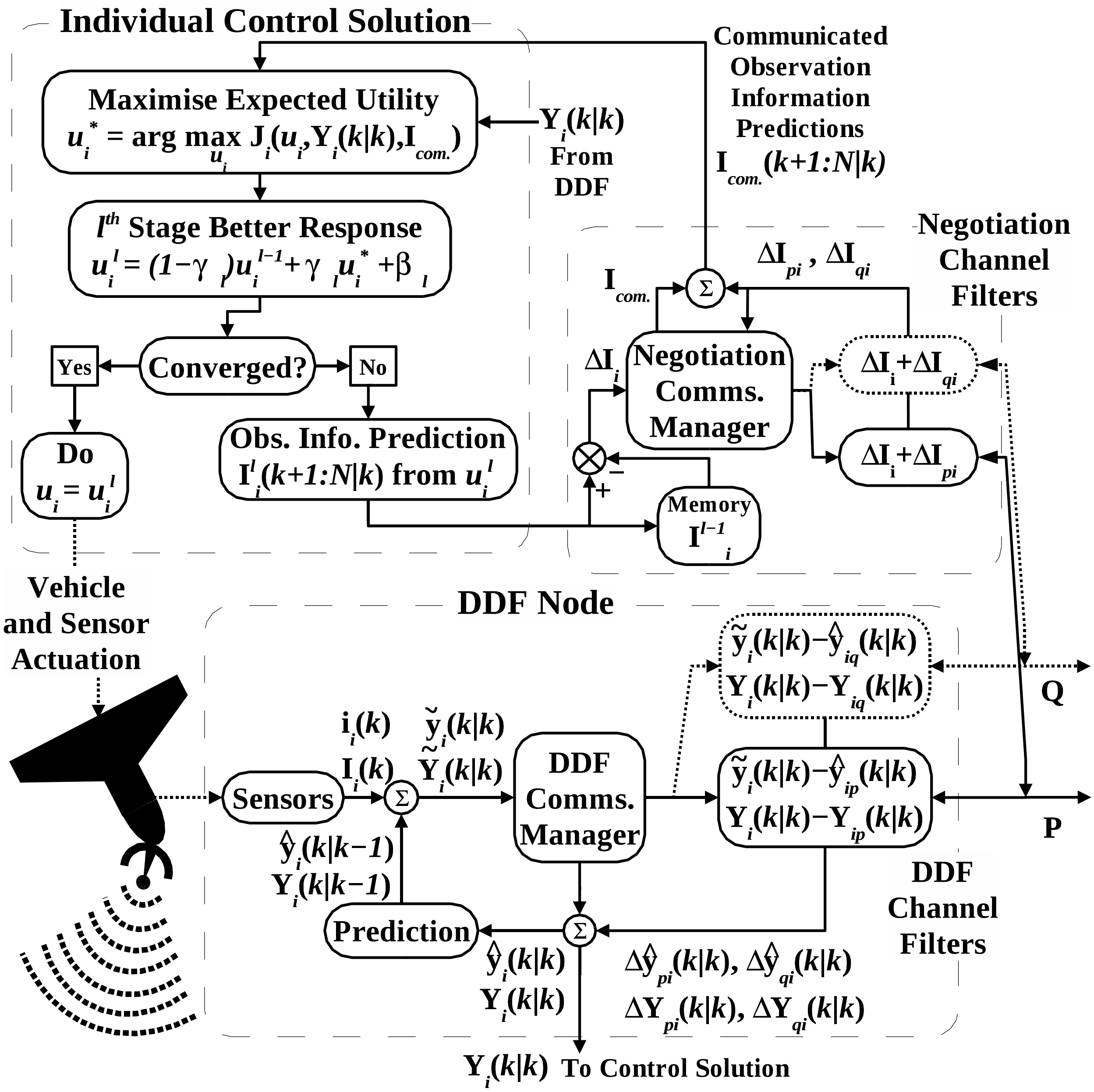


Figure 5.18: Structure of an Endogeneous Decentralised Decision Making Team Member. The DDF node is augmented with components to enable a decentralised solution to coordinated or cooperative control among team members. Two additional components are added. Firstly, a control solution that optimises individual actions accounting for available information. Secondly, a manager for communication of predicted observation information. This node structure allows for distributed decision makers to find the negotiated cooperative team solution.

cally. Information of no local interest is simply communicated. Through communication, nodes can be made aware of information they have never sensed or can not physically sense. Node plans can therefore involve intent to sense objects not currently seen and to incorporate the effects of capabilities the node itself does not possess.

5.7.3 Advantages of Cooperative Endogeneous Solutions

In this architecture, coordination and cooperation arise from propagation of anonymous information. Decision makers can account for and influence each other through transmitting and receiving information. Importantly, there is no knowledge of structure, state or size of the team other than of connections to adjacent nodes in the network. This has advantageous consequences when implementing coordinated or cooperative control policies and provides insight into why the decentralised architecture captures the essence of the global team control problem.

A non-hierarchical structure avoids the overheads, bottlenecks or catastrophic failure points associated with hierarchical or centralised arbitration. Hierarchical decomposition increases communication and computation requirements at points in the system. This also increases vulnerability of the system to failure. In the proposed approach, the solution process are made internal to each decision maker. The channel filter concept provides scalable communication and a means of recovery from component failure.

This architecture automatically takes advantage of natural sparsity in the utility and action coupling. If communicated information does not influence a decision makers action, they do not participate in the negotiation process. They simply act to propagate information through the team network.

Coalition formation is transparent. Multiple independent anonymous coalitions of negotiating cooperative decision makers can form, exist and dissipate. This approach completely avoids the intractable problem of trying to design coalitions through testing combinations of team members. The requirement and decision to cooperate is internal to the team member node. There is no specific physical knowledge of whom or with what they are cooperating.

There is no imposed solution structure. No imposed dependency exists between actions of individual team members. Decision making dependencies form and disperse dynamically from the interaction between system elements. If the right team action is to 'flock' or follow an individuals lead, this architecture allows that solution without imposing any situation specific structure or dependencies.

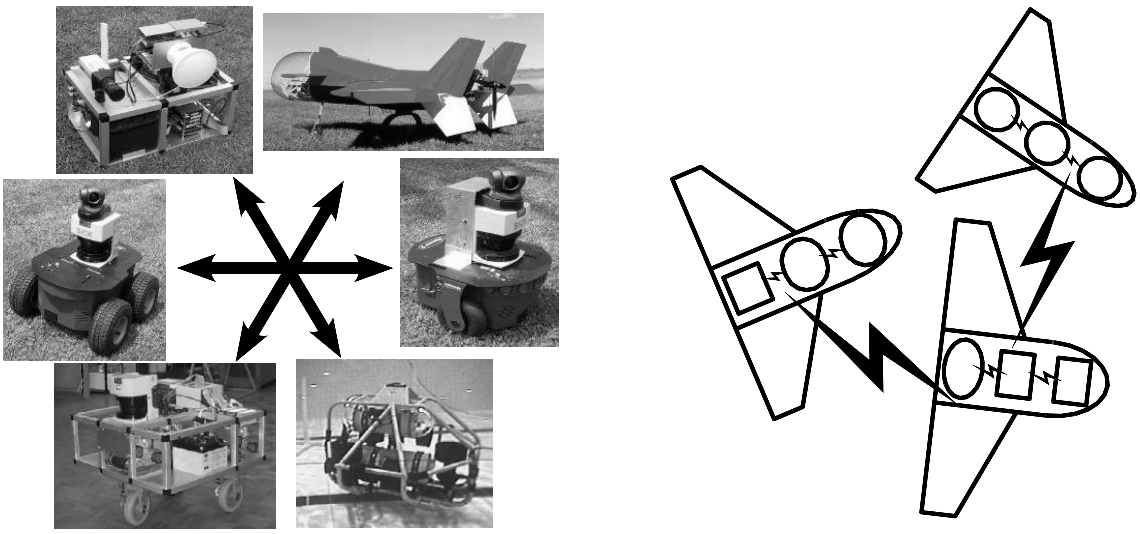


Figure 5.19: Decentralised team decision makers coordinate and cooperated through propagation of anonymous information. There is no knowledge of the physical nature of the information sources. Information-theoretic representations provide transparent inter-operation of decentralised decision making subsystems. This occurs between all system components within and across sensor platforms.

5.7.4 Advantages for Systems Engineering

Modelling of the vehicle, sensors, and environment encapsulates the relationship between component systems. A sensor model relates the sensor's observations to the estimated state of the environment, vehicles and sensors. Different sensors possess different characteristics and associated modelling. Information-theoretic representations allow abstraction of an information source's physical nature and state. In this form, communicated information can be interpreted and fused without any knowledge of its source. Decision makers can seamlessly understand, account for and influence each other. This provides transparent inter-operation between decentralised sensing and control sub-systems.

This has significant consequences for system functionality and systems engineering. Heterogeneity does not alter the architecture, but may yield significant performance benefits. Systems composed from heterogeneous sensors and vehicles are handled transparently by this approach. The same is true for redundancy. Combining redundant elements will likely improve accuracy and fault tolerance of the resulting system. It does not require changing the internal structure of the decision making systems.

A demonstration by Vaughan of an autonomous helicopter cooperating with ground vehicles to perform a localisation task illustrates the clear benefit of heterogeneity [97]. However, the proposed decentralised information-theoretic methodology suggests heterogeneity is not fundamental to the system architecture. This is a significantly different approach to that of Parker [67] and Sukhatme [87, 18].

From a systems engineering viewpoint, the decentralised paradigm greatly reduces the complexity of designing and combining subsystems. An interface to the communications protocol in information form is all that is required to allow incorporation of an additional system into this architecture. Different capabilities and characteristics can be configured through composition of existing complementary or redundant subsystems. Reconfiguration simply amounts to connecting or removing components having this decentralised interface.

5.8 Summary

This chapter presented a general architecture for decentralised coordinated control of multi-sensor information gathering systems. This endogeneous sensor architecture is an extension of the established decentralised data fusion methods. The term ‘endogeneous’ emphasises that the functionality enabling team decision making is internal. Each node is augmented with an individual distributed decision making procedure and a communication manager. This architecture is the key to achieving synergistic cooperation among decision makers.

Investigation of the mechanisms underlying coordination and cooperation demonstrated that the form of the information structure is crucial. Communication of observed information through a static information structure couples future decisions leading to coordinated actions. A dynamic information structure permitting communication of expected observation information couples the individual decision making processes. Negotiation through exchange of expected observation information allows decision makers to account for and influence each other leading to cooperation. Thus, coordinated or cooperative outcomes are determined through the design of the information structure. Significant outcomes from this analysis are:

- Demonstration of the use of static information structures to achieve scalable coordinated control of multiple sensor platforms.

- Showing that information seeking control based on zero look ahead provides useful and numerically simple solution with interpretation as a dynamic potential field.

- Demonstration of cooperative control by combining a dynamic team information structure with negotiation.

- Identification of components that extend the established data fusion architecture to decentralised cooperative multi-sensor control.

Chapter 6

Conclusion

6.1 Introduction

The objective of this thesis was to describe and explain the development of a consistent information-theoretic framework for engineering decentralised multi-sensor multi-vehicle systems. This chapter summarises the contributions of this thesis. Section 6.2 illuminates the major theoretical and practical solutions it has offered. Section 6.3 suggests areas of future work in this compelling field of research.

6.2 Summary of Contributions

The major contributions arise from the information-theoretic problem formulation, investigation of the relationship between individual and team utility, analysis of coordination and cooperation, and the development of a architecture for system implementation.

6.2.1 Information-Theoretic Approach to Control of Sensing

Information gathering is formulated and solved as an optimal control problem. The utility associated with a planned sequence of control actions is determined *a priori* from the modelling of the vehicles, sensors and environment. The sensing task is converted into a numerical representation suitable for systematic optimisation. Importantly, this approach considers control of information over time. Steady state analysis loses temporal properties critical to the information gathering problem. The information-theoretic methodology provides straight forward generation of utility measures for sensing problems in terms of entropic information.

6.2.2 Utility in Team Decision Making

The utility of a team decision maker is considered from both an individual and a team perspective. This establishes the relationship between the individual and team optimal actions, and the complexity of possible cooperation. Cooperation is only beneficial when coupling in utility results in team optimal actions that differ from the individual actions determined in isolation. It is observed that coupled utility does not necessarily alter the individual actions. A situation is demonstrated where the action associated with absolute minimum individual utility is team optimal.

6.2.3 Coordination Versus Cooperation

A distinction is made between coordination and cooperation. Coordination is considered to occur when a mechanism coupling the actions of the system gives rise to an increase in the utility of the system. The cooperative solution is taken to be the negotiated equilibrium between sensor action plans. This distinction permits a range of practically useful coordinated solutions without the effort associated with seeking cooperation.

Investigation of the mechanisms underlying coordination and cooperation revealed the form of the information structure and utility structure to be crucial. Communication of observed information through a static information structure couples future actions leading to coordinated actions. A dynamic information structure permitting communication of expected observation information couples the individual decision making processes. Negotiation through exchange of expected observation information allows decision makers to account for and influence each other leading to a cooperative solution. Hence, coordinated or cooperative outcomes are selected through the design of the information structure.

6.2.4 Scalable Coordinated Sensing

Scalable coordinated decision making is realised by addition of a local information seeking control layer to the established decentralised data fusion architecture. The decentralised data fusion network implements a static information structure that propagates observed information influencing subsequent locally optimised sensing plans. Benefits in lower computational and communication requirements are obtained by not seeking the cooperative solution. A special case is obtained when the decisions are made without looking ahead in time. This requires extremely low solution effort and can be interpreted as 'surfing' the mutual information vector field.

6.2.5 Endogeneous Cooperative Sensing

Decentralised cooperative sensing is achieved through a proposed endogeneous node based system architecture. This is an extension of the established decentralised data fusion node. The innovation is that the functionality enabling team decision making is internal to each node. An individual distributed decision making procedure with negotiation enabled through an interface to a dynamic information structure is added to each node. This architecture is the key to elegantly achieving transparent synergistic inter-operation among decision making elements of sensor teams.

6.3 Directions for Future Work

The research described in this thesis provides a quantitative and extensible basis for the discipline of cooperative, multi-robot systems. The work has immediate practical implications in addition to opening many rich avenues of future research.

6.3.1 Application to Practical Multi-Robot Missions

The methods developed in this thesis would find immediate application in multi-vehicle implementations of a number of practical robotics problems involving sensing and exploration. There is significant interest in the development of vehicles capable of autonomous navigation and exploration, In particular, Thrun, Fox and Burgard present a probabilistic approach to navigation based on Sequential Monte Carlo methods [19, 10], Leonard, Newman and Fenwick [23] and research at the ACFR [88] employ feature-based Kalman filtering approaches to localisation and mapping problems. The information-theoretic basis of the work described in this thesis is immediately extensible to multiple-platform versions of such probabilistic navigation and mapping methods.

The bearings-only localisation problem considered in Section 5.6is, in effect, the 'structure from motion' problem in computer vision [81]. A camera can be considered as a sensor providing bearings-only measurements. A multi-camera implementation of such a system could be composed in the proposed team decision structure. Each individual camera subsystem would then be capable of local control subject to some global objective in tracking image features.

6.3.2 Extensions to the Endogeneous Approach

The endogenous sensing and estimation approach can be extended to incorporate elements such as active feature classification, sensor data association and fault detection. Elfes [22] has applied information theoretic methods to single vehicle trajectory planning for target identification. Application of decentralised multi-vehicle systems to feature classification and data association is a logical and practical extension. Within this decentralised framework, all the information required to detect faults is available. If a fault is detectable in the equivalent global system, it is detectable in the decentralised architecture. Individuals can internally validate team information through checking the consistency between the estimate and observation information. Fault isolation is more difficult. Due to the summing of anonymous information contributions, an individual can not identify which teammate is at fault. A decision maker can however detect if they themselves are responsible for the fault. This offers an alternative approach to fault detection.

In addition, elements of the system design, such as parameterisation of the world and actions, can be incorporated into the decision problem. Overly complex representations are likely to incur significant solution effort for modest performance improvements. Decision making may be augmented to include consideration of the parametric representation of the environment and member actions.

6.3.3 Consideration of Tasks Other Than Sensing

The information-theoretic analysis extends to any quantifiable probabilistic characteristics. This include risk, energy and general commodities. In general, any problem involving allocation of limited resources with quantifiable rewards and costs can be treated within the team decision problem framework. Tasks of significant interest include:

- Pursuit and evasion between cooperative teams.
- Logistical problems such as transportation and mining.
- Risk management in resource allocation.

The robotics field provides structured problems and environments for experimental evaluation of theoretical developments. Analysis of cooperative robotic systems is likely to provide insight to other application domains.

6.4 Summary

Finally, this thesis provides a significant contribution to the realisation of cooperative control of multi-sensor teams through a consistent information theoretic framework. It is the implementation of the innovative endogeneous node based architecture which will achieve transparent synergistic inter-operation among decision making elements of sensor teams.

Appendix A

Numerical Solution Approach

A.1 Optimal Control

The problems considered in this thesis are not linear set point regulation or trajectory tracking problems. Linear quadratic solution forms do not exist. This leads to a requirement for nonlinear control approaches. The general problem formulation as specified by Bryson [9] follows. The solution is an open-loop, feed-forward action sequence $\boldsymbol{u}(t) \in \mathbb{R}^r$ that minimises the performance index $\mathbf{J}(\boldsymbol{x}(t), \boldsymbol{u}(t))$, where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $t \in [t_0, t_f]$.

Given initial conditions:

$$\boldsymbol{x}(0) = \boldsymbol{x}(t_0)$$

System Equations:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{A.1}$$

Subject to constraints:

$$\begin{gathered} \boldsymbol{\psi}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = 0 \\ \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \leq 0 \end{gathered}$$

With scalar final and path cost:

$$\mathbf{J}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \boldsymbol{\phi}(\boldsymbol{x}(t_f)) + \int_{t_0}^{t_f} \boldsymbol{L}(\boldsymbol{x}(t), \boldsymbol{u}(t))dt \tag{A.2}$$

A solution $\boldsymbol{u}(t)$ is sought to:

$$\arg\min_{\boldsymbol{u}(t)} \mathbf{J}(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{A.3}$$

This is a difficult problem in functional analysis. The problem requires solution to a multi-point boundary value problem. A wide variety of numerical approaches to optimal control problems exist. Reviews of methods are provided by Bryson [9] and Pesch [70].

A.1.1 Parameterised Solution by Mathematical Programming

Accepting that most optimal control problems require numerical solution, an approximate solution to the functional optimisation problem A.3 can be sought through suitable discreteisation or parameterisation of the control vector $\boldsymbol{u}(t)$

$$\boldsymbol{u}^i(t) = \sum_{j=1}^{m} \boldsymbol{p}^{ij} \chi_j(t), \quad i = 1, \ldots, r. \tag{A.4}$$

This approach is described by Sargent [79] and Goh and Teo [27]. The simplest practical scheme involves holding the control values constant over m equal time partitions Δt_u. In this case the basis function $\chi_j(t)$ is given by

$$\chi_j(t) = \begin{cases} 1 & \text{if}(j-1)\Delta t_u \leq t \leq j\Delta t_u \\ 0 & \text{otherwise} \end{cases}, \quad \Delta t_u = \frac{t_f - t_0}{m}. \tag{A.5}$$

An example of a more advanced parameterisation is provided by Bindera [89]. The

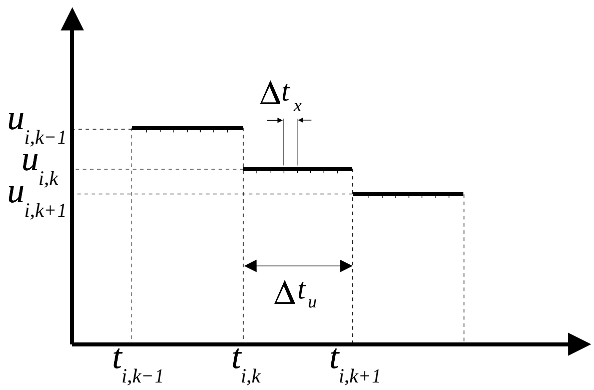


Figure A.1: Control parameterisation and time discreteisation.

system state and performance index may now be evaluated from equations A.1 and A.2

using standard solution techniques for ordinary differential equations. If a Heun integration scheme is used, the optimal control problem A.3 is converted into the nonlinear programming problem:

$$\min_{\boldsymbol{p}} \mathbf{J}(\boldsymbol{p}) = \boldsymbol{\phi}(\boldsymbol{x}_{t_f}) + \frac{1}{2}\Delta t_x \sum_{i=1}^{m.n_{steps}} (\boldsymbol{L}_i(\boldsymbol{x}_i, \boldsymbol{u}_k) + \boldsymbol{L}_{i-1}(\boldsymbol{x}_{i-1}, \boldsymbol{u}_k)). \tag{A.6}$$

Where the parameter vector is $\boldsymbol{p} = \mathrm{vec}\left([\boldsymbol{u}_1, \dots, \boldsymbol{u}_m]\right)_{r.m\mathrm{x}1}$. The state is evaluated on a possibly finer partition with time step $\Delta t_x = \frac{\Delta t_u}{n_{steps}}$, $\{n_{steps} \in \mathbb{Z} : n_{steps} \geq 1\}$ and the control index is $k = \mathrm{floor}(i\frac{\Delta t_u}{\Delta t_x})$. The minimisation is conducted subject to:

the system algebraic constraints:

$$\boldsymbol{x}_i = \boldsymbol{x}_{i-1} + \frac{1}{2}\Delta t_x[\boldsymbol{f}_i(\boldsymbol{x}_i, \boldsymbol{u}_k) + \boldsymbol{f}_{i-1}(\boldsymbol{x}_{i-1}, \boldsymbol{u}_k)], \quad i = 1, \dots, n_{steps}.m, \tag{A.7}$$

control parameter bounds:

$$[\boldsymbol{u}_{min}]_k \leq \boldsymbol{u}_k \leq [\boldsymbol{u}_{max}]_k\,, \quad k = 1, \dots, m, \tag{A.8}$$

and state constraints:

$$\begin{aligned} \boldsymbol{\psi}_i(\boldsymbol{x}_i) &= 0 \\ \boldsymbol{g}_i(\boldsymbol{x}_i) &\leq 0. \end{aligned} \tag{A.9}$$

This original optimal control problem A.3, is now in the form of a mathematical programing problem. The routine used throughout this thesis is *fmincon* from the *Matlab Optimization Toolbox* [8]. This implements the sequential quadratic programming method (SQP) see Gill [26, 2].

A.1.2 Gradient Determination

The efficiency and reliability of solving this nonlinear programming problem is aided by knowledge of the gradient of the cost with respect to the solution parameters $\nabla_{\boldsymbol{p}}\mathbf{J}$. Knowledge of the Hessian $\nabla^2_{\boldsymbol{p}}\mathbf{J}$ is required by routines such as Newton's exact method, hovever an alternative is to update an approximation from the gradient over sucessive iterations.

Three methods general techniqes exist for the calculation of the partial derivatives.

1. Numerical calulation by finite differences

2. Numerical approxiation of the gradient by integration of the adjoint (dual or costate) equation. See Sargent [79], Goh and Teo [27] or Roemisch [77]

3. Numerical integration of an enlarged system of ordinary differential equations, solving for the state $\boldsymbol{x}_i$ and derivatives $\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}}$ and $\frac{\partial^2 \boldsymbol{x}_i}{\partial \boldsymbol{p}^2}$ simultaneously. As described by Vassiliadis *et. al.* [96]

Method 1 is the simplest requiring no knowledge of the partial derivatives of the system equations with respect to state and control vectors. The drawback is the heavy computational load associated with integrating the purtubed system equations with sufficient accuracy. Method 2 is an elegant technique most suitable for problems with low state dimension and a large number of parameters. Method 3 is suited to the complementary situation. The implementation details of method 3, referred to a forward sensitivity analysis, follow. Differentiating A.6 with respect to $\boldsymbol{p}$ gives the gradient as

$$
\begin{aligned}
(\nabla_{\boldsymbol{p}}\mathbf{J})_{m.r\times 1} \quad = \quad & \frac{\partial \boldsymbol{\phi}(\boldsymbol{x}_{t_f})}{\partial \boldsymbol{x}_{t_f}}\frac{\partial \boldsymbol{x}_{t_f}}{\partial \boldsymbol{p}} + \frac{1}{2}\Delta t_x \sum_{i=1}^{m.n_{steps}} \left(\frac{\partial \boldsymbol{L}_i}{\partial \boldsymbol{x}_i}\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}} + \right. \\
& \left. \frac{\partial \boldsymbol{L}_i}{\partial \boldsymbol{u}_k}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}} + \frac{\partial \boldsymbol{L}_{i-1}}{\partial \boldsymbol{x}_{i-1}}\frac{\partial \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}} + \frac{\partial \boldsymbol{L}_{i-1}}{\partial \boldsymbol{u}_k}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}} \right). \qquad \text{(A.10)}
\end{aligned}
$$

Differentiating once more gives the Hessian

$$
\begin{aligned}
(\nabla^2_{\boldsymbol{p}}\mathbf{J})_{m.r\times m.r} = & \left[\mathbf{I}_{m.r} \otimes \frac{\partial \boldsymbol{\phi}(\boldsymbol{x}_{t_f})}{\partial \boldsymbol{x}_{t_f}}\right]\frac{\partial^2 \boldsymbol{x}_{t_f}}{\partial \boldsymbol{p}^2} + \left(\frac{\partial \boldsymbol{x}_{t_f}}{\partial \boldsymbol{p}}\right)^{\mathrm{T}} \frac{\partial^2 \boldsymbol{\phi}(\boldsymbol{x}_{t_f})}{\partial {\boldsymbol{x}_{t_f}}^2}\frac{\partial \boldsymbol{x}_{t_f}}{\partial \boldsymbol{p}} + \\
\frac{1}{2}\Delta t_x \sum_{i=1}^{m.n_{steps}} & \left\{ \left[\mathbf{I}_{m.r} \otimes \frac{\partial \boldsymbol{L}_i}{\partial \boldsymbol{x}_i}\right]\frac{\partial^2 \boldsymbol{x}_i}{\partial \boldsymbol{p}^2} + \left(\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}}\right)^{\mathrm{T}} \left[\frac{\partial^2 \boldsymbol{L}_i}{\partial {\boldsymbol{x}_i}^2}\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{L}_i}{\partial \boldsymbol{x}_i \partial \boldsymbol{u}_k}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right] + \right. \\
& \left(\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right)^{\mathrm{T}} \left[\frac{\partial^2 \boldsymbol{L}_i}{\partial \boldsymbol{u}_k \partial \boldsymbol{x}_i}\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{L}_i}{\partial {\boldsymbol{u}_k}^2}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right] + \left[\mathbf{I}_{m.r} \otimes \frac{\partial \boldsymbol{L}_{i-1}}{\partial \boldsymbol{x}_{i-1}}\right]\frac{\partial^2 \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}^2} + \\
& \left(\frac{\partial \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}}\right)^{\mathrm{T}} \left[\frac{\partial^2 \boldsymbol{L}_{i-1}}{\partial {\boldsymbol{x}_{i-1}}^2}\frac{\partial \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{L}_{i-1}}{\partial \boldsymbol{x}_{i-1} \partial \boldsymbol{u}_k}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right] + \\
& \left. \left(\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right)^{\mathrm{T}} \left[\frac{\partial^2 \boldsymbol{L}_{i-1}}{\partial \boldsymbol{u}_k \partial \boldsymbol{x}_{i-1}}\frac{\partial \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{L}_{i-1}}{\partial {\boldsymbol{u}_k}^2}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right] \right\}. \qquad \text{(A.11)}
\end{aligned}
$$

Applying the chain rule to the system equation A.1 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}} + \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{p}}, \quad \text{with } \left.\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}}\right|_{t=t_0} = \frac{\partial \boldsymbol{x}_0}{\partial \boldsymbol{p}}. \tag{A.12}$$

From which the first order parametric sensitivities are computed, e.g. by Heun scheme.

$$\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}} = \left[\mathbf{I}_n - \frac{1}{2}\Delta t_x \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}_i}\right]^{-1}\left[\left[\mathbf{I}_n + \frac{1}{2}\Delta t_x \frac{\partial \boldsymbol{f}_{i-1}}{\partial \boldsymbol{x}_{i-1}}\right] + \frac{1}{2}\Delta t_x \left(\frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{u}_k} + \frac{\partial \boldsymbol{f}_{i-1}}{\partial \boldsymbol{u}_k}\right)\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right]. \tag{A.13}$$

Diffentiating the forward sensitivity equation A.12 once again with respect to $\boldsymbol{p}$ leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial^2 \boldsymbol{x}}{\partial \boldsymbol{p}^2} \quad = \quad & \left[\mathbf{I}_{m.r} \otimes \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right]\frac{\partial^2 \boldsymbol{x}}{\partial \boldsymbol{p}^2} + \left[\mathbf{I}_n \otimes \left(\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}}\right)^{\mathrm{T}}\right]\left[\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{x}^2}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{x}\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{x}\partial \boldsymbol{p}}\right] + \\ & \left[\mathbf{I}_{m.r} \otimes \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right]\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{p}^2} + \left[\mathbf{I}_n \otimes \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{p}}\right)^{\mathrm{T}}\right]\left[\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{u}\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{u}^2}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{u}\partial \boldsymbol{p}}\right] + \\ & \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{p}\partial \boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{p}\partial \boldsymbol{u}}\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{p}^2}, \quad \text{with } \left.\frac{\partial^2 \boldsymbol{x}}{\partial \boldsymbol{p}^2}\right|_{t=t_0} = \frac{\partial^2 \boldsymbol{x}_0}{\partial \boldsymbol{p}^2}.\end{aligned} \tag{A.14}$$

For the cases considered, the system equations are not an explicit function of the parameters and the control is linear in the parameters. So $\frac{\partial^2 \boldsymbol{f}}{\partial \boldsymbol{p}\partial(\cdot)} = \mathbf{0}$ and $\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{p}^2} = \mathbf{0}$. From A.14, the second order parametric sensitivities can be computed recursively, e.g. by Heun scheme.

$$\begin{aligned}\frac{\partial^2 \boldsymbol{x}_i}{\partial \boldsymbol{p}^2} = & \left[\mathbf{I}_{n.m.r} - \frac{1}{2}\Delta t_x \left[\mathbf{I}_{m.r} \otimes \frac{\partial \boldsymbol{f}_i}{\partial \boldsymbol{x}_i}\right]\right]^{-1}\left\{\left[\mathbf{I}_{n.m.r} + \frac{1}{2}\Delta t_x \left[\mathbf{I}_{m.r} \otimes \frac{\partial \boldsymbol{f}_{i-1}}{\partial \boldsymbol{x}_{i-1}}\right]\right] + \right. \\ & \frac{1}{2}\Delta t_x \left\{\left[\mathbf{I}_n \otimes \left(\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}}\right)^{\mathrm{T}}\right]\left[\frac{\partial^2 \boldsymbol{f}_i}{\partial {\boldsymbol{x}_i}^2}\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_i}{\partial \boldsymbol{x}_i\partial \boldsymbol{u}_k}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_i}{\partial \boldsymbol{x}_i\partial \boldsymbol{p}}\right] + \right. \\ & \left[\mathbf{I}_n \otimes \left(\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right)^{\mathrm{T}}\right]\left[\frac{\partial^2 \boldsymbol{f}_i}{\partial \boldsymbol{u}_k\partial \boldsymbol{x}_i}\frac{\partial \boldsymbol{x}_i}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_i}{\partial {\boldsymbol{u}_k}^2}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_i}{\partial \boldsymbol{u}_k\partial \boldsymbol{p}}\right] + \\ & \left[\mathbf{I}_n \otimes \left(\frac{\partial \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}}\right)^{\mathrm{T}}\right]\left[\frac{\partial^2 \boldsymbol{f}_{i-1}}{\partial {\boldsymbol{x}_{i-1}}^2}\frac{\partial \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_{i-1}}{\partial \boldsymbol{x}_{i-1}\partial \boldsymbol{u}_k}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_{i-1}}{\partial \boldsymbol{x}_{i-1}\partial \boldsymbol{p}}\right] + \\ & \left.\left.\left[\mathbf{I}_n \otimes \left(\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}}\right)^{\mathrm{T}}\right]\left[\frac{\partial^2 \boldsymbol{f}_{i-1}}{\partial \boldsymbol{u}_k\partial \boldsymbol{x}_{i-1}}\frac{\partial \boldsymbol{x}_{i-1}}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_{i-1}}{\partial {\boldsymbol{u}_k}^2}\frac{\partial \boldsymbol{u}_k}{\partial \boldsymbol{p}} + \frac{\partial^2 \boldsymbol{f}_{i-1}}{\partial \boldsymbol{u}_k\partial \boldsymbol{p}}\right]\right\}\right\}.\end{aligned} \tag{A.15}$$

A.1.3 Example Solution

A simple example is presented to demonstrated the parameterised control solution. The example is from the textbook by Lewis [40]. Let a system obey Newton's law so that

$$\dot{\mathbf{y}} = \mathbf{v} \tag{A.16}$$

$$\dot{\mathbf{v}} = \mathbf{u} \tag{A.17}$$

with $\mathbf{y}$ the position, $\mathbf{v}$ the velocity and $\mathbf{u}$ the acceleration control input. The system state is $\mathbf{x} = [\mathbf{y}, \mathbf{v}]^T$. Select the performance index at final time T to be

$$\mathbf{J} = \frac{\mathbf{w}_y}{2}(\mathbf{y}(T) - \mathbf{r}_y)^2 + \frac{\mathbf{w}_v}{2}(\mathbf{v}(T) - \mathbf{r}_v)^2 + \frac{1}{2}\int_0^T \mathbf{u}^2(t)\mathrm{d}t \tag{A.18}$$

where $\mathbf{r}_y$ and $\mathbf{r}_v$ are desired reference values for $\mathbf{y}$ and $\mathbf{v}$ at final time T. $\mathbf{w}_y$ and $\mathbf{w}_v$ are weighting factors that adjust the trade-of between required control energy and the terminal state errors. Lewis finds a sample solution for $\mathbf{y}(0) = \mathbf{v}(0) = 1$, $\mathbf{r}_y(T) = \mathbf{r}_v(T) = 0$, $\mathbf{w}_y = \mathbf{w}_v = 100$ and $T = 10$ seconds to be

$$\mathbf{u}^{\star}(t) = -.4594 + .0718t$$

Tis is compared to a solution parameterised by eight control steps in Figure A.2.

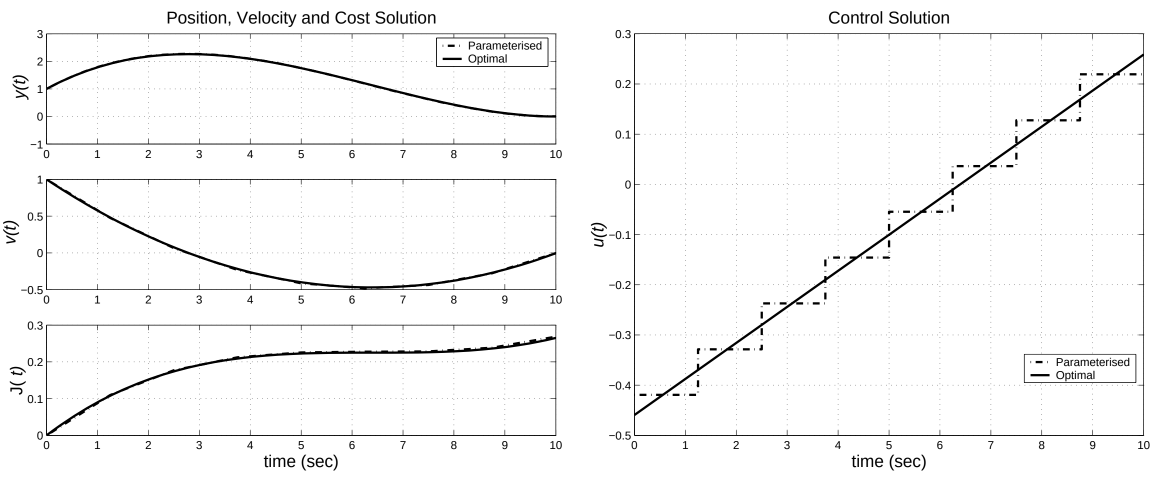


Figure A.2: Control Parameterisation Example

Appendix B

Decentralised Data Fusion

B.1 The Information Filter

A key tool in decentralised data fusion systems is the information filter. The information filter allows standard continuous estimation and control functions to be decentralised. The information filter is summarised in this section.

Consider a system described in standard linear form

$$\mathbf{x}(k) = \mathbf{F}(k)\mathbf{x}(k-1) + \mathbf{B}(k)\mathbf{u}(k) + \mathbf{G}(k)\mathbf{w}(k), \tag{B.1}$$

where $\mathbf{x}(j)$ is the state of interest at time j, $\mathbf{F}(k)$ is the state transition matrix from time $k-1$ to k, $\mathbf{B}(k)$ and $\mathbf{G}(k)$ the control input and noise input transition matrices, and where $\mathbf{u}(k)$ and $\mathbf{w}(k)$ are the associated control input and process noise input modeled as an uncorrelated white sequence with $\mathrm{E}\{\mathbf{w}(i)\mathbf{w}^T(j)\} = \delta_{ij}\mathbf{Q}(i)$. The system is observed by a sensor according to the linear equation

$$\mathbf{z}(k) = \mathbf{H}(k)\mathbf{x}(k) + \mathbf{v}(k) \tag{B.2}$$

where $\mathbf{z}(k)$ is the vector of observations made at time k, $\mathbf{H}(k)$ the observation matrix or model, and where $\mathbf{v}(k)$ is the associated observation noise modeled as an uncorrelated white sequence with $\mathrm{E}\{\mathbf{v}(i)\mathbf{v}^T(j)\} = \delta_{ij}\mathbf{R}(i)$.

The conventional Kalman filter algorithm generates estimates for the state $\hat{\mathbf{x}}(k \mid k)$ at a time k given all observations up to time k, together with a corresponding estimate covariance $\mathbf{P}(k \mid k)$ as:

$$\hat{\mathbf{x}}(k \mid k) = \hat{\mathbf{x}}(k \mid k-1) + \mathbf{W}(k)\left[\mathbf{z}(k) + \mathbf{H}(k)\hat{\mathbf{x}}(k \mid k-1)\right] \tag{B.3}$$

$$\mathbf{P}(k \mid k) = \mathbf{P}(k \mid k-1) - \mathbf{W}(k)\mathbf{S}(k)\mathbf{W}^T(k) \tag{B.4}$$

where $\mathbf{W}(k)$ is the gain matrix, $\mathbf{S}(k)$ the innovation covariance. The information form of the Kalman filter is obtained by re-writing the state estimate and covariance in terms of two new variables

$$\hat{\mathbf{y}}(i \mid j) \triangleq \mathbf{P}^{-1}(i \mid j)\hat{\mathbf{x}}(i \mid j), \qquad \mathbf{Y}(i \mid j) \triangleq \mathbf{P}^{-1}(i \mid j), \tag{B.5}$$

and also the information associated with an observation in the form

$$\mathbf{i}(k) \triangleq \mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{z}(k), \qquad \mathbf{I}(k) \triangleq \mathbf{H}^T(k)\mathbf{R}^{-1}(k)\mathbf{H}(k) \tag{B.6}$$

With these definitions, the information filter can be summarised

Prediction:

$$\begin{aligned}\hat{\mathbf{y}}(k \mid k-1) &= \left[\mathbf{1} - \mathbf{\Omega}(k)\mathbf{G}^T(k)\right]\mathbf{F}^{-T}(k)\hat{\mathbf{y}}(k-1 \mid k-1) + \\ &\quad + \mathbf{Y}(k \mid k-1)\mathbf{B}(k)\mathbf{u}(k)\end{aligned} \tag{B.7}$$

$$\mathbf{Y}(k \mid k-1) = \mathbf{M}(k) - \mathbf{\Omega}(k)\mathbf{\Sigma}(k)\mathbf{\Omega}^T(k) \tag{B.8}$$

where

$$\mathbf{M}(k) = \mathbf{F}^{-T}(k)\mathbf{P}^{-1}(k-1 \mid k-1)\mathbf{F}^{-1}(k), \tag{B.9}$$

$$\mathbf{\Omega}(k) = \mathbf{M}(k)\mathbf{G}(k)\mathbf{\Sigma}^{-1}(k), \tag{B.10}$$

and

$$\mathbf{\Sigma}(k) = \left[\mathbf{G}^T(k)\mathbf{M}(k)\mathbf{G}(k) + \mathbf{Q}^{-1}(k)\right]. \tag{B.11}$$

Estimate:

$$\hat{\mathbf{y}}(k \mid k) = \hat{\mathbf{y}}(k \mid k-1) + \mathbf{i}(k) \tag{B.12}$$

$$\mathbf{Y}(k \mid k) = \mathbf{Y}(k \mid k-1) + \mathbf{I}(k). \tag{B.13}$$

The information-filter form has the advantage that the update Equations B.12 and B.13 for the estimator are computationally simpler than the equations for the Kalman Filter, at the cost of increased complexity in prediction. The value of this in decentralized sensing is that estimation occurs locally at each node, requiring partition of the estimation equations which are simpler in their information form. Prediction, which is more complex in this form, relies on a propagation coefficient which is independent of the observations made and so is again simpler to decouple and decentralize amongst a network of sensor nodes.

The information form of the Kalman filter, while widely known, is not commonly used because the update terms are of dimension the state, whereas in the distributed Kalman filter updates are of dimension the observation. For single sensor estimation problems, this argues for the use of the Kalman filter over the information filter. However, in multiple sensor problems, the opposite is true. The reason is that with multiple sensor observations

$$\mathbf{z}_i(k) = \mathbf{H}_i(k)\mathbf{x}(k) + \mathbf{v}_i(k), \qquad i = 1, \cdots, N$$

the estimate can not be constructed from a simple linear combination of contributions from individual sensors

$$\hat{\mathbf{x}}(k \mid k) \neq \hat{\mathbf{x}}(k \mid k-1) + \sum_{i=1}^{N} \mathbf{W}_i(k) \left[\mathbf{z}_i(k) - \mathbf{H}_i(k)\hat{\mathbf{x}}(k \mid k-1) \right],$$

as the innovation $\mathbf{z}_i(k) - \mathbf{H}_i(k)\hat{\mathbf{x}}(k \mid k-1)$ generated from each sensor is correlated because they share common information through the prediction $\hat{\mathbf{x}}(k \mid k-1)$. However, in information form, estimates can be constructed from linear combinations of observation information

$$\hat{\mathbf{y}}(k \mid k) = \hat{\mathbf{y}}(k \mid k-1) + \sum_{i=1}^{N} \mathbf{i}_i(k),$$

as the information terms $\mathbf{i}_i(k)$ from each sensor are uncorrelated. Once the update equations have been written in this simple additive form, it is straight-forward to distribute the data fusion problem (unlike for a Kalman filter); each sensor node simply generates the information terms $\mathbf{i}_i(k)$, and these are summed at the fusion center to produce a global information estimate.

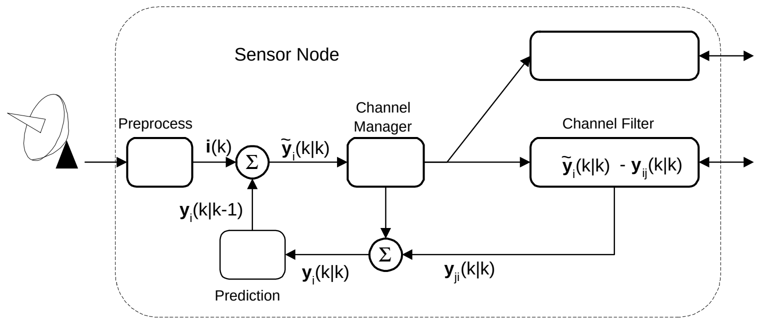


Figure B.1: Algorithmic structure of a decentralised sensing node.

To decentralise the information filter all that is necessary is to replicate the central fusion algorithm (summation) at each sensor node and simplify the result. This yields a surprisingly simple nodal fusion algorithm. The algorithm is described graphically in Figure B.1. Essentially, local estimates are first generated at each node by fusing (adding) locally available observation information $\mathbf{i}_i(k)$ with locally available prior information $\hat{\mathbf{y}}_i(k \mid k-1)$. This yields a local information estimate $\tilde{\mathbf{y}}_i(k \mid k)$. The difference between this local estimate and prediction (corresponding to new information gained) is then transmitted to other nodes in the network. In a fully connected or broadcast network, this results in every sensing node getting all new information. Communicated information is then assimilated simply by summing with the local information. An important point to note is that, after this, the locally available estimates are *exactly* the same as if the data fusion problem had been solved on a single central processor using a monolithic formulation of the conventional Kalman filter.

It is also worth noting the manner in which the control input enters the prediction stage of the information form; through the term $\mathbf{Y}(k \mid k-1)\mathbf{B}(k)\mathbf{u}(k)$. In general $\mathbf{H}_i(k)$ is a function of state which is dependent on control action. Thus, the control input also influences the observed information update.

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