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Contact symmetries of the elliptic Euler-Darboux equation

Diego Catalano Ferraioli
Dipartimento di Matematica e Informatica, University of Salerno
dcatalan@unisa.it

Gianni Manno
Dipartimento di Matematica "Ennio De Giorgi", University of Lecce
gianni.manno@kcl.ac.uk

Fabrizio Pugliese
Dipartimento di Matematica e Informatica, University of Salerno
fpugliese@unisa.it

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Abstract. We study contact symmetries of an elliptic equation parametrizing some Ricci flat metrics with bidimensional Killing orbits.The variational nature of such symmetries is investigated as well.

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1 Introduction

In this paper we completely describe the algebra of contact symmetries of the following linear elliptic equation $\mathcal{E}$

$$(x+y)(u_{xx}+u_{yy})+u_x+u_y=0. \tag{1}$$

This equation appears in two recent papers [13, 14] devoted to the study of Ricci flat metrics under particular symmetry assumptions. Namely, it is assumed that the spacetime (M, g), with g being a Lorentzian metric on M, admits a non-abelian bidimensional Killing algebra $Kil\,(g)$.

Furthermore, it is assumed that the distribution $\mathcal{D}^{\perp}$ orthogonal to Killing distribution $\mathcal{D}$ is integrable and transversal to $\mathcal{D}$. Under these assumptions, it can be proved [13] that there exist local coordinates (ξ, η, p, q) on M such that:

a) Killing leaves are described by equations $\xi = c_1$, $\eta = c_2$, with $c_1, c_2 \in \mathbf{R}$.

b) Vector fields $X = \frac{\partial}{\partial p}$, $Y = e^p \frac{\partial}{\partial q}$ are the generators of $Kil\,(g)$, linked by relation $[X, Y] = Y$

c) $\mathcal{D}^{\perp}$ is spanned by vectors $\frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta}$

Such coordinates are called adapted. If Killing field Y is light-like, then the most general Ricci flat metric of the above type takes, in adapted coordinates, the form:

$$g = 2f\left(dx^2 + dy^2\right) + \mu\left[\left(u\left(x,y\right) - 2q\right)dp^2 + 2dpdq\right], \tag{2}$$

where: $\mu = D\,\Phi + B$, with D, B constants and Φ harmonic function; $f = \pm\left(\nabla\Phi\right)^2/\sqrt{|\mu|}$; $u\left(\xi,\eta\right)$ is a solution of equation

$$\Delta u + \left(\partial_\xi \ln|\mu|\right)\partial_\xi u + \left(\partial_\eta \ln|\mu|\right)\partial_\eta u = 0 \tag{3}$$

Consider the coordinate change $x = \mu+\widetilde{\mu}$, $y = \mu-\widetilde{\mu}$, with $\widetilde{\mu}$ harmonic conjugate of μ. Then, it is easily checked that, in the new coordinates, equation (3) is transformed into (1). Therefore, in order to find concrete Einstein metrics of the form (2), it is necessary to find exact solutions of (1). The most efficient way to do this consists in finding symmetries of (1) and then using them to generate solutions.

The paper is structured as follows. In section 2 the necessary preliminary definitions and theorems on jet spaces, contact symmetries and conservation laws are briefly recalled. Then in section 3 these techniques are applied to compute contact symmetries of equation (1) which are shown to be point ones. The corresponding flows are used to explicitly generate new solutions of (1). Finally, in the last section, Lagrangian formalism for equation (1) is discussed.

2 Preliminary Definitions

Jet spaces are fundamental objects in differential geometry. Namely they are the basis for a geometrization of partial differential equations. Here, we give the definition of a jet bundle (for further details see [3], [10]). Roughly speaking, a jet bundle of order r can be seen as a smooth manifold whose coordinate functions of a chart can be interpreted as "independent" and "dependent" variables, and by the derivatives of the latter with respect to the former up to order r. More precisely, let $\pi\colon E \to M$ be a vector bundle, with $\dim M = n$ and $\dim E = n+m$. Let $\mathcal{U} \subset M$ be a neighborhood of M such that $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{R}^m$. Let (x^λ, u^i), $\lambda = 1\ldots n$, $i = 1\ldots m$, be coordinates on $\pi^{-1}(\mathcal{U})$, with (x^λ) coordinates on $\mathcal{U}$. A local section of π is locally given by $u^i = f^i(x^1, x^2, \ldots, x^n)$.

Two local sections s and $\tilde{s}$ of π are said to be *r-contact equivalent* at the point $x \in M$ if their Taylor expansions around this point coincide up to order r. This relation is of course an equivalence relation. We shall denote by $[s]^r_x$ an equivalence class. The definition of r-contact is intrinsic.

Let us examine the above definition in coordinates. Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_r)$, with $1 \le \sigma_i \le n$ and $r \in \mathbb{N}$, be a multi-index, and $|\boldsymbol{\sigma}| \overset{\text{def}}{=} r$. Then s and $\tilde{s}$ are r-equivalent at the point x if

$$\frac{\partial^{|\boldsymbol{\sigma}|} f^i}{\partial x^{\sigma_1} \cdots \partial x^{\sigma_r}}(x) = \frac{\partial^{|\boldsymbol{\sigma}|} \tilde{f}^i}{\partial x^{\sigma_1} \cdots \partial x^{\sigma_r}}(x), \qquad 0 \le |\boldsymbol{\sigma}| \le r. \tag{4}$$

where s and $\tilde{s}$ are locally expressed by

$$u^i = f^i(x^1, x^2, \dots, x^n) \quad \text{and} \quad u^i = \tilde{f}^i(x^1, x^2, \dots, x^n).$$

The set $J^r(\pi)$ of the all the equivalence classes $[s]^r_x$ is called the *jet bundle* of order r and it has a natural manifold structure. A chart (V, x^λ, u^i) on E induces the chart $(V^r_n, x^\lambda, u^i_{\boldsymbol{\sigma}})$ on $J^r(\pi)$ where $V^r_n = \{[s]^r_x \colon s(x) \in V\}$. The set of such charts is an atlas on $J^r(\pi)$. The coordinates $u^i_{\boldsymbol{\sigma}}$ on V^r_n are determined by

$$u^i_{\boldsymbol{\sigma}}([s]^r_x) = \left.\frac{\partial^{|\boldsymbol{\sigma}|} f^i}{\partial x^{\boldsymbol{\sigma}}}\right|_x$$

where $u^i = f^i(x^\lambda)$ describes locally the section s. See [12,17] for technical details.

We have the following natural maps:

(1) the embedding $j_r s \colon M \longrightarrow J^r(\pi),\ x \longmapsto [s]^r_x$,

(2) the projection $\pi_{k,h} \colon J^k(\pi) \longrightarrow J^h(\pi),\ [s]^k_x \longmapsto [s]^h_x \quad k \ge h$.

The *Cartan plane* $\mathcal{C}_\theta$ at the point $\theta \in J^r(\pi)$ is the span of the planes $T_\theta(j_r s(M))$. We have the *Cartan distribution* $\theta \to \mathcal{C}_\theta$ on $J^r(\pi)$. A diffeomorphism of $J^r(\pi)$ is called a *contact transformation* if it is a symmetry of the Cartan distribution. A vector field on $J^r(\pi)$ which preserves the Cartan distribution is called a *contact field*.

1 Remark. A point $\theta = [s]^{r+1}_x$ of $J^{r+1}(\pi)$ is completely characterized by $T_{\pi_{r+1,r}(\theta)}(j_r s(M))$.

We can lift a contact transformation G of $J^r(\pi)$ to a contact transformation $G^{(1)}$ of $J^{r+1}(\pi)$. In fact in view of Remark 1 we can interpret a point $\theta \in J^{r+1}(\pi)$ as the pair $\big(\pi_{r+1,r}(\theta), T_{\pi_{r+1,r}(\theta)}(j_r s(M))\big)$. Then we can define $G^{(1)}$ by:

$$\begin{aligned} G^{(1)}(\theta) &= G^{(1)}\left(\pi_{r+1,r}(\theta), T_{\pi_{r+1,r}(\theta)}(j_r s(M))\right) \\ &= \left(G(\pi_{r+1,r}(\theta)), G_*(T_{\pi_{r+1,r}(\theta)}(j_r s(M)))\right) \end{aligned}$$

Of course we can lift contact fields by lifting their local flows.

2 Theorem (Lie-Bäcklund). *Any contact field on $J^r(\pi)$ with $r \ge 1$ is:*

(1) an $(r-1)$-lift of some contact field on $J^1(\pi)$ if $m=1$ and $r>1$;

(2) an r-lift of some vector field on E if $m>1$ and $r>0$.

A *differential equation* $\mathcal{E}$ of order r is a submanifold of $J^r(\pi)$. A *linear* equation is a linear subbundle of $J^r(\pi)\to M$. A (local) *solution* of $\mathcal{E}$ is a section s of π such that $j_r s(M)\subset\mathcal{E}$. The *1-prolongation* $\mathcal{E}^1$ of the equation $\mathcal{E}$ is the set

$$\mathcal{E}^1=\{\theta\in J^{r+1}(\pi)\,|\,\pi_{r+1,r}(\theta)\in\mathcal{E},\ T_{\pi_{r+1,r}(\theta)}(j_r s(M))\subset T_{\pi_{r+1,r}(\theta)}\mathcal{E}\}.$$

By iteration we can define the l-prolongation $\mathcal{E}^l$.

A *classical symmetry* of $\mathcal{E}$ is a contact field on $J^r(\pi)$ tangent to $\mathcal{E}$. A symmetry which is a lift of a vector field of E is called a *point symmetry.* A contact field on $J^\infty(\pi)$ tangent to $\mathcal{E}^\infty$ is called a *higher symmetry.*

A vector field on $J^\infty(\pi)$ lying in the Cartan distribution $\mathcal{C}$ is called a *trivial field* as it is tangent to all integral manifolds of $\mathcal{C}$. We notice that the Cartan distribution on $J^\infty(\pi)$ is integrable and it is spanned by the *total derivatives*

$$D_\lambda=\frac{\partial}{\partial x^\lambda}+\sum_{j,\boldsymbol{\sigma}}u^j_{\boldsymbol{\sigma},\lambda}\frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}}$$

In the case of $J^r(\pi)$ with $r<\infty$ the unique trivial field is the null vector field.

3 Theorem. *Any contact field X on $J^\infty(\pi)$ is of the form*

$$X=\mathcal{Э}_\varphi+H$$

where $\mathcal{Э}_\varphi=\sum_{j,\boldsymbol{\sigma}}D_{\boldsymbol{\sigma}}(\varphi^j)\frac{\partial}{\partial u^j_{\boldsymbol{\sigma}}}$, *$H$ is a trivial vector field and $D_{\boldsymbol{\sigma}}$ stands for* $D_{\sigma_1}\circ D_{\sigma_2}\circ\cdots\circ D_{\sigma_r}$.

Now we are interested in non-trivial symmetries, that is symmetries of the form $\mathcal{Э}_\varphi$. Locally, if the equation $\mathcal{E}$ is described by

$$\begin{cases} F^1=0\\ F^2=0\\ \quad\vdots\\ F^k=0\end{cases}$$

then the vector field $\mathcal{Э}_\varphi$ is a symmetry of $\mathcal{E}$ if

$$\mathcal{Э}_\varphi(F^i)|_{\mathcal{E}^\infty}=0.$$

In this case we call $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_m)$ the *generating section* of the symmetry $\ni_\varphi$.

If we define the matrix operator

$$\ell_F \varphi = \ell_{F^i} \varphi \stackrel{\text{def}}{=} \ni_\varphi F^i \, ,$$

we have that φ is a higher symmetry of $\mathcal{E}$ if and only if

$$(\ell_F \varphi)|_{\mathcal{E}^\infty} = 0 \tag{5}$$

The operator $\ell_{\mathcal{E}} = \ell_F|_{\mathcal{E}^\infty}$ is called the operator of *universal linearization* of $\mathcal{E}$. Locally we have that φ is a symmetry if

$$\ell_{\mathcal{E}}(\bar{\varphi}) = \sum_{j,\boldsymbol{\sigma}} \frac{\partial F^i}{\partial u^j_{\boldsymbol{\sigma}}} \bar{D}_{\boldsymbol{\sigma}}(\bar{\varphi}^j) = 0$$

where the bar denotes the restriction to $\mathcal{E}^\infty$.

In view of Lie-Bäcklund theorem, it is easy to realize that, in the case $m = 1$, generating functions of contact symmetries are all functions on $J^1(\pi)$ satisfying equation (5). If, in particular, such functions are linear in the 1-jet variables, it can be shown [3] that the corresponding symmetries are liftings of point ones.

We also define the formal adjoint ℓ^*_F of ℓ_F by

$$\ell^*_F = \left(\sum_{j,\boldsymbol{\sigma}} (-1)^{\boldsymbol{\sigma}} D_{\boldsymbol{\sigma}} \circ \frac{\partial F^i}{\partial u^j_{\boldsymbol{\sigma}}} \right)^{\mathrm{T}}$$

Equation $\mathcal{E}$ is said *self adjoint* iff its linearization operator is self adjoint.

A vector function $(P_1, P_2 \ldots P_n)$, where $P_\lambda \in C^\infty(J^\infty(\pi))$ is called a *conserved quantity* if $\sum_{\lambda=1}^n \bar{D}_\lambda(P_\lambda) = 0$ and a *trivial* conserved quantity if $\sum_{\lambda=1}^n D_\lambda(P_\lambda) = 0$.

3 Contact symmetries

In this section we shall calculate contact symmetries of Equation $\mathcal{E}$ defined by (1). According to definitions given in section 2, it can be interpreted as a submanifold of the second order jets space $J^2(\pi)$ of the trivial bundle $\pi : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2$. Let $(x, y, u_{lx,my})_{l,m \geq 0}$, where $u_{lx,my} = u_{\underbrace{xx\ldots x}_{l-times}\underbrace{yy\ldots y}_{m-times}}$, be a chart on $J^\infty(\pi)$.

With respect to these coordinates the total differentiation operators D_x and D_y read

$$D_x = \frac{\partial}{\partial x} + \sum_{l,m\geq 0} u_{(l+1)x,my}\frac{\partial}{\partial u_{lx,my}},$$
$$D_x = \frac{\partial}{\partial x} + \sum_{l,m\geq 0} u_{lx,(m+1)y}\frac{\partial}{\partial u_{lx,my}}$$

and the infinite prolongation of $\mathcal{E}$ is

$$\mathcal{E}^\infty = \left\{ D_x^l D_y^m(F) = 0, \quad l,m \geq 0 \right\}$$

where $F = 0$ defines $\mathcal{E}$.

Since the functions $u_{(l+2)x,my}$ on $\mathcal{E}^\infty$ can be expressed in terms of $(x, y, u_{x,sy}, u_{sy})_{s\geq 0}$ then we take the functions

$$x, y, u, u_x, u_y, u_{xy}, u_{yy}, u_{xyy}, u_{yyy}, \ldots, u_{x,hy}, u_{(h+1)y}, \ldots \qquad h = 3, 4, \ldots$$

as internal coordinates on $\mathcal{E}^\infty$.

Symmetries of $\mathcal{E}$ are functions φ satisfying (5). In our case

$$\ell_{\mathcal{E}} = (x+y)\left(\bar{D}_x \circ \bar{D}_x + \bar{D}_y \circ \bar{D}_y\right) + \bar{D}_x + \bar{D}_y.$$

In particular any solution of $\mathcal{E}$ is itself a symmetry of it.

In view of the Lie-Bäcklund theorem, the generating functions of contact symmetries of the equation (1) have the form

$$\varphi = \varphi(x, y, u, u_x, u_y) \tag{6}$$

We notice that in this case $\varphi = \bar{\varphi}$. A straightforward computation shows that $\ell_{\mathcal{E}}(\varphi)$ is a second degree polynomial in u_{xy} and u_{yy} of the following form

$$A_1\left(u_{xy}^2 + u_{yy}^2\right) + A_2 u_{xy} + A_3 u_{yy} + A_4 \tag{7}$$

Since the A_i's are functions of x, y, u, u_x, u_y and derivatives of φ, (5) holds iff

$$A_1 = A_2 = A_3 = A_4 = 0.$$

Equations $A_1 = A_2 = A_3 = 0$ respectively read

$$\begin{aligned}
\varphi_{u_x u_x} &= -\varphi_{u_y u_y} \\
\varphi_{x u_y} &= -\left(u_x \varphi_{u u_y} + u_y \varphi_{u u_x} + \varphi_{y u_x} - \frac{u_x + u_y}{x+y}\varphi_{u_x u_y} \right) \\
\varphi_{x u_x} &= u_y \varphi_{u u_y} - u_x \varphi_{u u_x} + \varphi_{y u_y} - \frac{u_x + u_y}{x+y}\varphi_{u_y u_y}
\end{aligned} \tag{8}$$

where the φ_{ab}'s, with $a, b = x, y, u, u_x, u_y$, denote partial derivations of φ with respect to the jet variables.

By considering the compatibility conditions

$$\begin{aligned}&\left(\varphi_{xu_y}\right)_{u_y} + \left(\varphi_{xu_x}\right)_{u_x} = 0\\&\left(\varphi_{xu_y}\right)_{u_x} = \left(\varphi_{xu_x}\right)_{u_y}\end{aligned}$$

one immediately gets

$$\begin{aligned}\varphi_{uu_x} &= \frac{-\varphi_{u_y u_y} + \varphi_{u_x u_y}}{2\,(x+y)},\\ \varphi_{uu_y} &= \frac{\varphi_{u_y u_y} + \varphi_{u_x u_y}}{2\,(x+y)}.\end{aligned} \tag{9}$$

In view of (9) the second and third equations in (8) become

$$\begin{aligned}\varphi_{xu_y} &= \frac{u_x + u_y}{x+y}\varphi_{u_x u_y} - \varphi_{yu_x},\\ \varphi_{xu_x} &= -\frac{u_x + u_y}{x+y}\varphi_{u_y u_y} + \varphi_{yu_y}\end{aligned} \tag{10}$$

respectively.

Then a straightforward computation shows that in virtue of the first equation of (8) the compatibility conditions for equations (9) and (10) imply that

$$\varphi_{u_y u_y} = 0, \quad \varphi_{u_x u_y} = 0.$$

There follows, in view of equations (8) and (9), that φ has the form

$$\varphi = a(x, y, u) + b(x, y)u_x + c(x, y)u_y$$

with b and c satisfying conditions

$$\begin{aligned}c_x &= -b_y,\\ b_x &= c_y.\end{aligned} \tag{11}$$

Now equation $A_4 = 0$ is a second degree polynomial in u_x and u_y of the following form

$$a_{uu}(u_x^2 + u_y^2) + B_1(x, y, u)u_x + B_2(x, y, u)u_y + B_3(x, y, u) = 0. \tag{12}$$

Hence $a_{uu} = B_1 = B_2 = B_3 = 0$ and by setting $a = a_1(x, y)u + \alpha(x, y)$ one gets:

$$B_1 = 2a_{1x} - \frac{c_x + c_y}{x + y} + \frac{b + c}{(x + y)^2},$$

$$B_2 = 2a_{1y} + \frac{c_x - c_y}{x + y} + \frac{b + c}{(x + y)^2},$$

$$B_3 = \left(a_{1xx} + a_{1yy} + \frac{a_{1x} + a_{1y}}{x + y}\right) u + \alpha_{xx} + \alpha_{yy} + \frac{\alpha_x + \alpha_y}{x + y}$$

where we have used the compatibility condition

$$c_{xx} + c_{yy} = 0 \tag{13}$$

for equations (11).

Then by deriving equations $B_1 = 0$ and $B_2 = 0$ with respect to x and y, respectively, and adding the results one gets

$$a_{1xx} + a_{1yy} + 2\left(\frac{a_{1x} + a_{1y}}{x + y}\right) = 0. \tag{14}$$

Note that $B_3 = 0$ iff

$$\begin{aligned} a_{1xx} + a_{1yy} + \frac{a_{1x} + a_{1y}}{x + y} &= 0 \\ \alpha_{xx} + \alpha_{yy} + \frac{\alpha_x + \alpha_y}{x + y} &= 0 \end{aligned} \tag{15}$$

therefore the first equation of (15) together with (14) implies

$$\begin{aligned} a_{1x} + a_{1y} &= 0 \\ a_{1xx} + a_{1yy} &= 0 \end{aligned}$$

that is

$$\begin{aligned} a_{1x} + a_{1y} &= 0 \\ a_{1xy} = a_{1xx} = a_{1yy} &= 0 \end{aligned}$$

In the end, by summing up the previous results and integrating the remaining equations $B_1 = 0$ and $B_2 = 0$, one can rearrange arbitrary constants to get the most general solution of (5) in the following form

$$\begin{aligned} \varphi = [k_1(x - y) + k_2]\, u &+ \left[k_1\left(x^2 - y^2 - 2xy\right) + k_3 x + k_4\right] u_x \\ &+ \left[k_1\left(x^2 - y^2 + 2xy\right) + k_3 y - k_4\right] u_y + \alpha \end{aligned} \tag{16}$$

where the k_i's are constants and $\alpha = \alpha(x, y)$ is an arbitrary solution of $\mathcal{E}$.

4 Proposition. *Contact symmetries* $\varphi = \varphi(x, y, u, u_x, u_y)$ *of the equation* $\mathcal{E}$ *are point symmetries of form (16).*

Therefore, the algebra of contact symmetries splits into the semidirect sum of the infinite dimensional Abelian ideal of solutions of equation (1) and of the subalgebra generated by:

$$u\,,\ u_x - u_y\,,\ xu_x + yu_y\,,\ (y^2 - x^2 + 2xy)u_x + (y^2 - x^2 - 2xy)u_y + (y - x)u. \tag{17}$$

Vector fields on E corresponding to (17) are:

$$X_1 = u\frac{\partial}{\partial u}\,,\ X_2 = -\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\,,\ X_3 = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$

$$X_4 = -(y^2 - x^2 + 2xy)\frac{\partial}{\partial x} - (y^2 - x^2 - 2xy)\frac{\partial}{\partial y} + (y - x)u\frac{\partial}{\partial u}$$

Each one-parameter subgroup generated by X_i sends every solution $u = f(x, y)$ of (1) into the following solutions:

$$\begin{aligned} u^1 &= e^k f(x, y) \\ u^2 &= f(x - k, y + k) \\ u^3 &= f(e^k x, e^k y) \\ u^4 &= \frac{f\left(\frac{x+k(x^2+y^2)}{z}, \frac{-y+k(x^2+y^2)}{z}\right)}{z} \end{aligned}$$

where $z = 2k^2(x^2 + y^2) + 2k(x - y + 1)$.

4 Lagrangian formalism

In this section we shall discuss the variational aspects of the equation $\mathcal{E}$. First, we prove that $\mathcal{E}$ is variational, that is $\mathcal{E} = \{\mathbf{E}(L) = 0\}$ for some Lagrangian $L = L(x, y, u_{\boldsymbol{\sigma}})$, where $\mathbf{E}$ is the Euler operator. Let us recall that, in our situation, the Euler operator is

$$\mathbf{E} = \sum_{\boldsymbol{\sigma}} (-1)^{|\boldsymbol{\sigma}|} D_{\boldsymbol{\sigma}} \frac{\partial}{\partial u_{\boldsymbol{\sigma}}}$$

5 Proposition. *The equation* $\mathcal{E}$ *is self-adjoint.*

Proof. This amount to prove that $\ell_F = \ell_F^*$, that is that the equation $\mathcal{E}$ satisfies the Helmholtz conditions. For any $\varphi \in C^\infty(J^\infty(\pi))$, we have that

$$\begin{aligned} \ell_F^*(\varphi) &= D_{xx}\left((x + y)\varphi\right) + D_{yy}\left((x + y)\varphi\right) - D_x(\varphi) - D_y(\varphi) \\ &= D_x\left(D_x((x + y)\varphi)\right) + D_y\left(D_y((x + y)\varphi)\right) - D_x(\varphi) - D_y(\varphi) \\ &= (x + y)(D_{xx}(\varphi) + D_{yy}(\varphi)) + D_x(\varphi) + D_y(\varphi) = \ell_F(\varphi). \end{aligned}$$

QED

A self-adjoint equation is (locally) variational [3,7,10], the Lagrangian being $L = \int_0^1 u\, F(\lambda u)\, d\lambda = uF$, where $F = 0$ defines the equation. This Lagrangian, in our case, is of course a second order one. However, in our case a first order Lagrangian $L = L(x, y, u, u_x, u_y)$ can be easily found. In fact, for such an L, $\mathbf{E}(L)$ must be linear with respect to second derivatives. Hence, by condition $\mathbf{E}(L) = F$ one gets:

$$L = \frac{1}{2}(-x - y)(u_x^2 + u_y^2) + F_1 u_y + F_2 + F_3 u_x$$

with $F_i = F_i(x, y, u)$ satisfying

$$\frac{\partial F_2}{\partial u} - \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial y} = 0.$$

Since two Lagrangians define the same Euler-Lagrange equation if they differ for a total divergence one can factor out such total divergence, and obtain the following

6 Proposition. *The equation $\mathcal{E}$ is the Euler-Lagrange equation of the Lagrangian $L = \frac{1}{2}(-x - y)(u_x^2 + u_y^2)$.*

Now we see if symmetries (16) are *divergence symmetries.* This means that they preserve the Lagrangian density $Ldx \wedge dy$ up to total divergences. The condition that a symmetry $\varphi = b - a_1 u_x - a_2 u_y$ is a divergence symmetry is

$$X_\varphi^{(1)} L + L\, \mathrm{Div}(a_1, a_2) = \mathrm{Div}(q_1, q_2) \tag{18}$$

for some vector functions $(q_1(x, y, u), q_2(x, y, u))$. We call *variational symmetries* the symmetries for which the right hand side term of (18) is equal to zero. Below we discuss the variationality of symmetries (17).

(1) The symmetries $xu_x + yu_y$ and u are not divergence symmetries. In fact their left hand side term of (18) are equal respectively to $-L$ and $2L$. Then $(u + 2xu_x + 2yu_y)$ is a variational symmetry. By virtue of Noether theorem we can associate a conserved quantity. It has the following form:

$$\begin{aligned} p &= g + fu_y + (x^2 + xy)(u_x^2 - u_y^2) + ((u + 2yu_y)(x + y))u_x \\ q &= h - fu_x + (y^2 + xy)(u_y^2 - u_x^2) + ((u + 2xu_x)(x + y))u_y \end{aligned} \tag{19}$$

with $f, g, h \in C^\infty(E)$ such that

$$h_u + f_x = 0\,,\ h_y + g_x = 0\,,\ f_y - g_u = 0 \tag{20}$$

We notice that the conditions (20) make $(g + fu_y, h - fu_x)$ a trivial conserved quantity, then we can factor it out from (19).

(2) The symmetry $u_x - u_y$ is variational. Taking into account the considerations of previous case, a straightforward computation shows that

$$p = \frac{1}{2}(x+y)u_x^2 - (x+y)u_x u_y - \frac{1}{2}(x+y)u_y^2$$
$$q = \frac{1}{2}(x+y)u_x^2 + (x+y)u_y u_x - \frac{1}{2}(x+y)u_y^2$$

is the a non trivial conserved quantity associated to such a symmetry.

(3) The symmetry $(y^2 - x^2 + 2xy)u_x + (y^2 - x^2 - 2xy)u_y + (y-x)u$ is a divergence symmetry but not variational. In fact the left hand side term of (18) is equal to

$$u(x+y)(u_x - u_y) = \mathrm{Div}\left(\frac{1}{2}(x+y)u^2, -\frac{1}{2}(x+y)u^2\right).$$

In this case also the Noether theorem holds true.

$$\begin{aligned} p =& g + f u_y + \frac{3}{2}u_x^2 y^2 x - \frac{1}{2}u_x^2 x^3 + \frac{1}{2}u_x^2 x^2 y + \frac{1}{2}u_x^2 y^3 \\ & - u_x u_y x^3 - 3u_x x^2 u_y y + -u_x x^2 u - u_x u_y y^2 x + u_x u y^2 \\ & + u_x u_y y^3 + \frac{1}{2}u_y^2 x^3 - \frac{1}{2}u_y^2 x^2 y - \frac{3}{2}u_y^2 y^2 x - \frac{1}{2}u_y^2 y^3 \end{aligned}$$

$$\begin{aligned} q =& - f u_x + h - \frac{1}{2}u_y^2 x^3 - \frac{3}{2}u_y^2 x^2 y - \frac{1}{2}u_y^2 y^2 x + \frac{1}{2}u_y^2 y^3 \\ & - u_x u_y x^3 - u_y x^2 u + u_x x^2 u_y y + 3u_x u_y y^2 x + u_x u_y y^3 + u_y u y^2 \\ & + \frac{1}{2}u_x^2 x^3 + \frac{3}{2}u_x^2 x^2 y + \frac{1}{2}u_x^2 y^2 x - \frac{1}{2}u_x^2 y^3 \end{aligned}$$

with $f, g, h \in C^\infty(E)$ such that

$$h_u + f_x = -u(x+y)\,,\ g_u - f_y = u(x+y)\,,\ g_x + h_y = 0$$

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