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A Conjecture of Brian Hartley and developments arising

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Abstract. Around 1980 Brian Hartley conjectured that if the unit group of a torsion group algebra FG satisfies a group identity, then FG satisfies a polynomial identity. In this short survey we shall review some results dealing with the solution of this conjecture and the extensive activity that ensued. Finally, we shall discuss special polynomial identities satisfied by FG (or by some of its subsets) and the corresponding group identities satisfied by its unit group (or by some of its subsets).

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1 A Conjecture of Brian Hartley

Let $\langle x_1, x_2, \ldots\rangle$ be the free group on a countable set of generators. If S is any subset of a group G, we say that S satisfies a group identity if there exists a non-trivial reduced word $w(x_1, \ldots, x_n) \in \langle x_1, x_2, \ldots\rangle$ such that $w(g_1, \ldots, g_n) = 1$ for all $g_i \in S$. For elements $y_1, \ldots, y_n$ of a group G, set $(y_1, y_2) = y_1^{-1}y_2^{-1}y_1y_2$, the *group commutator* of y_1 and y_2, and inductively $(y_1, \ldots, y_n) = ((y_1, \ldots, y_{n-1}), y_n)$. Obviously, abelian groups and nilpotent groups are examples of groups satisfying a group identity $((x_1, x_2)$ and $(x_1, \ldots, x_c)$ for some c, respectively).

In an attempt to give a connection between the additive and the multiplicative structure of a group algebra FG of a group G over a field F, Brian Hartley made the following famous conjecture.

Conjecture 1. *Let G be a torsion group and F an infinite field. If the unit group $\mathcal{U}(FG)$ of FG satisfies a group identity, then FG satisfies a polynomial identity.*

We recall that a subset R of FG satisfies a polynomial identity (PI) if there exists a non-trivial element $f(x_1, \ldots, x_n)$ in the free algebra $F\{x_1, x_2, \ldots\}$ on non-commuting indeterminates $x_1, x_2, \ldots$ such that $f(a_1, \ldots, a_n) = 0$ for all $a_i \in R$. The conditions under which FG satisfies a polynomial identity were determined in classical results due to Passman and Isaacs-Passman (see Corollaries 5.3.8 and 5.3.10 of [41]) summarized in the following

Theorem 1. *Let F be a field of characteristic $p \geq 0$ and G a group. Then FG satisfies a polynomial identity if and only if G has a p-abelian subgroup of finite index.*

For the sake of completness recall that, for any prime p, a group G is said to be p-abelian if its commutator subgroup G' is a finite p-group, and that 0-abelian means abelian.

The Hartley's Conjecture was first studied by Warhurst in his PhD thesis [48] where special words satisfied by $\mathcal{U}(FG)$ were investigated. Pere Menal [39] suggested a possible solution for some p-groups. When the field is infinite, Goncalves and Mandel [21] verified it in the special case that the group identity is actually a semigroup identity (that is, an identity of the form $x_{i_1}x_{i_2}\cdots x_{i_k} = x_{j_1}x_{j_2}\cdots x_{j_l}$). Giambruno, Jespers and Valenti [11] handled the characteristic 0 case as well as the characteristic $p > 0$ case when G has no elements of p-power order. In fact, under these assumptions FG is semiprime and the fact that $\mathcal{U}(FG)$ satisfies a group identity forces G to be abelian. By using the Menal's construction, Giambruno, Sehgal and Valenti [18] solved the conjecture, by proving the following

Theorem 2. *Let G be a torsion group and F an infinite field. If $\mathcal{U}(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

A positive answer to Hartley's Conjecture having been established, it was natural to look for necessary and sufficient conditions for $\mathcal{U}(FG)$ to satisfy a group identity. Clearly, satisfying a polynomial identity cannot be sufficient. We see from Theorem 1 that if G is finite, then FG always satisfies a polynomial identity, but if $\operatorname{char} F = 0$, then $\mathcal{U}(FG)$ does not satisfy a group identity unless G is abelian. The question was solved by Passman [42], by using the results of [18], in the following

Theorem 3. *Let F be an infinite field of characteristic $p > 0$ and G a torsion group. Then the following are equivalent:*

(i) $\mathcal{U}(FG)$ satisfies a group identity;

(ii) $\mathcal{U}(FG)$ satisfies the group identity $(x, y)^{p^r} = 1$, for some $r \geq 0$;

(iii) G has a normal p-abelian subgroup of finite index and G' is a p-group of bounded exponent.

The fact that F is assumed to be infinite allowed the authors to apply a *Vandermonde determinant argument* (see, for instance, Proposition 1 of [11] and the roles played by its implications in [18] and [42]). On the other hand, by Theorem 3, for any non-abelian finite group G, if $\mathcal{U}(FG)$ satisfies a group identity then G is p-abelian. This is obviously no longer true if F has finitely many elements: in this case, for any finite group G, $\mathcal{U}(FG)$ is finite, hence it satisfies a group identity. Subsequently a lot of work has been done to generalize the above results to

- arbitrary fields
- arbitrary groups
- special subsets of $\mathcal{U}(FG)$

1.1 Arbitrary Fields F

By modifying the original proof of [18], Liu [36] confirmed the Hartley's Conjecture for fields of all sizes. His arguments were decisive to generalize the results of [42] to group algebras over non-necessarily infinite fields. This was done by Liu and Passman in [37]. It turns out that the solution is different if G' is not a p-group. Their main results are the following.

Theorem 4. *Let F be a field of characteristic $p > 0$ and G a torsion group. If G' is a p-group, then the following are equivalent:*

(i) $\mathcal{U}(FG)$ satisfies a group identity;

(ii) $\mathcal{U}(FG)$ satisfies the group identity $(x, y)^{p^r} = 1$, for some $r \geq 0$;

(iii) G has a p-abelian subgroup of finite index and G' has bounded exponent.

Theorem 5. *Let F be a field of characteristic $p > 0$ and G a torsion group. If G' is not a p-group, then the following are equivalent:*

(i) $\mathcal{U}(FG)$ satisfies a group identity;

(ii) $\mathcal{U}(FG)$ has bounded exponent;

(iii) G has a p-abelian subgroup of finite index, G has bounded exponent and F is a finite field.

1.2 Non-torsion Groups

In general, the Hartley's Conjecture is not expected to hold for arbitrary groups. For instance, if G is a torsion-free nilpotent group, then the only units in FG are *trivial*, namely αg, with $0 \neq \alpha \in F$ and $g \in G$, and $\mathcal{U}(FG)$ is nilpotent. But FG need not satisfy a polynomial identity. The main obstruction in trying to characterize group algebras of non-torsion groups whose units satisfy a group identity is the difficulty in handling the torsion free part of the group. It is worth noting that for any such result, a restriction will be required for the sufficiency, pending a positive answer to the following very famous (and difficult) conjecture due to Kaplansky.

Conjecture 2. *If G is a torsion-free group and F a field, then the only units in FG are trivial.*

Anyway, for groups with elements of infinite order the question was studied by Giambruno, Sehgal and Valenti in [20]. They proved that, if $\mathcal{U}(FG)$ satisfies a group identity, then the torsion elements of G form a subgroup, T. For the converse, a suitable restriction upon G/T is required, namely that it is a u.p. (unique product) group, i.e., for every pair of non-empty finite subsets S_1 and S_2 of G/T, there exists an element $g \in G/T$ that can be uniquely written as $g = s_1 s_2$, with each $s_i \in S_i$. We have to separate two cases according as FG is semiprime (by virtue of Theorems 4.2.12 and 4.2.13 of [41] this means that either $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 0$ and G has no normal subgroups with order divisible by p) or not.

Theorem 6. *Let FG be semiprime and suppose that F is infinite or G has an element of infinite order. If $\mathcal{U}(FG)$ satisfies a group identity then*

(1) all the idempotents of FG are central;

(2) T is an abelian p'-subgroup of G.

Conversely, if G is a group satisfying (1) and (2) and G/T is nilpotent of class c, then $\mathcal{U}(FG)$ satisfies the group identity $((x_1, \dots, x_c), (x_{c+1}, \dots, x_{2c})) = 1$.

The characteristic zero case having been dealt with, in the next result assume that F is a field of characteristic $p \geq 2$.

Theorem 7. *Suppose that F is infinite or G has an element of infinite order. We have the following*

(1) If $\mathcal{U}(FG)$ satisfies a group identity then P, the set of the p-elements of G, is a subgroup.

(2) If P is of unbounded exponent and $\mathcal{U}(FG)$ satisfies a group identity then

(a) G contains a p-abelian subgroup of finite index;

(b) G' is of bounded p-power exponent.

Conversely, if P is a subgroup and G satisfies (a) and (b), then $\mathcal{U}(FG)$ satisfies a group identity.

(3) If P is of bounded exponent and $\mathcal{U}(FG)$ satisfies a group identity then

(a′) P is finite or G contains a p-abelian subgroup of finite index;

(b′) $T(G/P)$ is an abelian p'-subgroup and so T is a group;

(c′) every idempotent of $F(G/P)$ is central.

Conversely, if P is a subgroup, G satisfies (a′), (b′) and (c′) and G/T is a u.p. group, then $\mathcal{U}(FG)$ satisfies a group identity.

1.3 Special Subsets of $\mathcal{U}(FG)$

A natural question of interest is to ask if group identities satisfied by some special subset of the unit group of a group algebra FG can be lifted to $\mathcal{U}(FG)$ or force FG to satisfy a polynomial identity. A motivation for this study is the classical theorem of Amitsur regarding an identity on symmetric elements of a ring with involution forcing an identity of the whole ring. In this framework, the *symmetric units* have been the subject of a good deal of attention.

Let FG be the group ring of a group G over a field F of characteristic different from 2. If G is endowed with an involution $\star$, then it can extended F-linearly to an involution of FG, also denoted by $\star$. An element $\alpha \in FG$ is said to be *symmetric with respect to* $\star$ if $\alpha^\star = \alpha$. We write FG^+ for the set of symmetric elements, which are easily seen to be the linear combinations of the terms $g + g^\star$, for all $g \in G$. Let $\mathcal{U}^+(FG)$ denote the set of symmetric units. Prior to the last couple of years, attention had largely been devoted to the *classical involution* induced from the map $g \mapsto g^{-1}$ on G. Giambruno, Sehgal and Valenti [19] confirmed a stronger version of Hartley's Conjecture by proving

Theorem 8. *Let FG be the group algebra of a torsion group G over an infinite field F of characteristic different from* 2 *endowed with the classical involution. If $\mathcal{U}^+(FG)$ satisfies a group identity, then FG satisfies a polynomial identity.*

Under the same restrictions as in the above theorem, they also obtained necessary and sufficient conditions for $\mathcal{U}^+(FG)$ to satisfy a group identity. They get different answers depending on whether G contains a copy of the quaternion group Q_8. More precisely, it is effected by the presence in G of a copy of a *Hamiltonian* 2*-group.* We recall that a non-abelian group G is a Hamiltonian group if every subgroup of G is normal. It is well-known that in this case $G = O \times E \times Q_8$, where O is an abelian group with every element of odd order and E is an elementary abelian 2-group. In fact, a crucial remark for the classification of torsion group algebras FG whose symmetric units satisfy a group identity is that, for any commutative ring R and Hamiltonian 2-group H, RH^+ is commutative. The main result of [19] is the following

Theorem 9. *Let FG be the group algebra of a torsion group G over an infinite field F of characteristic different from 2 endowed with the classical involution.*

(a) If $\operatorname{char} F = 0$, $\mathcal{U}^+(FG)$ satisfies a group identity if and only if G is either abelian or a Hamiltonian 2-group.

(b) If $\operatorname{char} F = p > 2$, then $\mathcal{U}^+(FG)$ satisfies a group identity if and only if FG satisfies a polynomial identity and either $Q_8 \not\subseteq G$ and G' is of bounded exponent p^k for some $k \geq 0$ or $Q_8 \subseteq G$ and

(1) the p-elements of G form a (normal) subgroup P of G and G/P is a Hamiltonian 2-group;

(2) G is of bounded exponent $4p^s$ for some $s \geq 0$.

Obviously, group identities on $\mathcal{U}^+(FG)$ do not force group identities on $\mathcal{U}(FG)$. To see this it is sufficient to observe that, for any infinite field F of characteristic $p > 2$, FQ_8^+ is commutative, hence $\mathcal{U}^+(FQ_8)$ satisfies a group identity but, according to Theorem 3, $\mathcal{U}(FG)$ does not satisfy a group identity.

The above results were extended to non-torsion groups in [44] under the usual restriction for the only if part related to Kaplansky's Conjecture. We do not review here the statements of that paper, but we confine ourselves to report the following result, which goes in the direction of the Hartley's Conjecture and Theorem 8.

Theorem 10. *Let FG be the group algebra of a group G with an element of infinite order over an infinite field F of characteristic different from 2 endowed with the classical involution. If $\mathcal{U}^+(FG)$ satisfies a group identity, then the set P of p-elements of G forms a normal subgroup and, if P is infinite, then FG satisfies a polynomial identity.*

Recently, there has been a considerable amount of work on involutions of FG obtained as F-linear extension of arbitrary group involutions on G other than the classical one. In particular, Broche Cristo, Jespers, Polcino Milies and Ruiz Marin have studied the interesting question as to when FG^+ and $FG^- = \{\alpha \mid \alpha \in FG \quad \alpha^\star = -\alpha\}$ the Lie subalgebra of the *skew-symmetric* elements of FG are commutative ([25] and [5]). Goncalves and Passman [22] considered the existence of bicyclic units u in the integral group rings such that the group $\langle u, u^\star \rangle$ is free. Marciniak and Sehgal in [38] had proved that, with respect to the classical involution, $\langle u, u^\star \rangle$ is always free if $u \neq 1$.

In the classification results on group algebras whose symmetric units with respect to the classical involution satisfy a group identity in some sense the exceptional cases turned out to involve Hamiltonian 2-groups, because they are non-abelian groups such that the symmetric elements in the group rings commute. When one works with linear extensions of arbitrary involutions of the base group of the group algebra, one finds a larger class of groups such that the symmetric elements of the related group algebra have the same property. In order to state the next results, a definition is required. We recall that a group G is said to be an LC-group (that is, it has the “lack of commutativity” property) if it is not abelian, but if $g, h \in G$, and $gh = hg$, then at least one of g, h and gh must be central. These groups were introduced by Goodaire. By Proposition III.3.6 of [23], a group G is an LC-group with a unique non-identity commutator (which must, obviously, have order 2) if and only if $G/\zeta(G) \cong C_2 \times C_2$. Here, $\zeta(G)$ denotes the centre of G.

Definition 1. A group G endowed with an involution $*$ is said to be a special LC-group, or SLC-group, if it is an LC-group, it has a unique non-identity commutator z, and for all $g \in G$, we have $g^* = g$ if $g \in \zeta(G)$, and otherwise, $g^* = zg$.

The SLC-groups arise naturally in the following result proved by Jespers and Ruiz Marin [25] for an arbitrary involution on G.

Theorem 11. *Let R be a commutative ring of characteristic different from 2, G a non-abelian group with an involution $*$ which is extended linearly to RG. Then the following are equivalent:*

(i) RG^+ is commutative;

(ii) RG^+ is the centre of RG;

(iii) G is an SLC-group.

This is crucial for the classification of torsion group algebras endowed with an involution induced from an arbitrary involution on G with symmetric units satisfying a group identity. The question was originally studied by Dooms and Ruiz [8] and completely solved by Giambruno, Polcino Milies and Sehgal [14].

Theorem 12. *Let F be an infinite field of characteristic $p \neq 2$, G a torsion group with an involution $*$ which is extended linearly to FG. Then the symmetric units of FG satisfy a group identity if and only if one of the following holds:*

(a) FG is semiprime and G is abelian or an SLC-group;

(b) FG is not semiprime, the p-elements of G form a (normal) subgroup P, G has a p-abelian normal subgroup of finite index, and either

(1) G' is a p-group of bounded exponent, or

(2) *G/P is an SLC-group and G contains a normal $*$-invariant p-subgroup B of bounded exponent, such that P/B is central in G/B and the induced involution acts as the identity on P/B.*

2 Lie Properties in FG

Any associative algebra A over a field F may be regarded as a Lie algebra by defining the Lie multiplication

$$[a,b] = ab - ba \qquad \forall a, b \in A.$$

For any two subspaces S and T of A, we define $[S,T]$ to be the additive subgroup of A generated by all the Lie products $[s,t]$ with $s \in S$ and $t \in T$. Obviously $[S,T]$ is a F-subspace of A. We can define inductively the *Lie central series* and the *Lie derived series* of A by

$$A^{[1]} = A, \qquad A^{[n+1]} = [A^{[n]}, A]$$

and

$$\delta^{[0]}(A) = A, \qquad \delta^{[n+1]}(A) = [\delta^{[n]}(A), \delta^{[n]}(A)],$$

respectively. One may also enlarge the terms of this series by making them associative at every stage. More precisely, we define by induction the series

$$A^{(1)} = A, \qquad A^{(n+1)} = \langle [A^{(n)}, A] \rangle$$

and

$$\delta^{(0)}(A) = A, \qquad \delta^{(n+1)}(A) = \langle [\delta^{(n)}(A), \delta^{(n)}(A)] \rangle,$$

where, for any two associative ideals S, T of A, $\langle [S,T] \rangle$ denotes the associative ideal of A generated by $[S,T]$.

We say that A is *Lie nilpotent* if $A^{[n]} = 0$ for some integer n and, similarly, A is *Lie solvable* if $\delta^{[m]}(A) = 0$ for some integer m. In a similar fashion, A is said to be *strongly Lie nilpotent* (*strongly Lie solvable*, respectively) if $A^{(n)} = 0$ (if $\delta^{(n)}(A) = 0$, respectively) for some integer n. If A is strongly Lie nilpotent, the smallest integer m such that $A^{[m+1]} = 0$ ($A^{(m+1)} = 0$, respectively) is called the *Lie nilpotency class* (the *strong Lie nilpotency class*, respectively) of A and is denoted by $cl_L(A)$ ($cl^L(A)$, respectively). We make at once the following simple observations: an algebra A which is strongly Lie nilpotent (solvable, respectively) is Lie nilpotent (solvable, respectively) and the (strong) Lie nilpotency property implies the (strong) Lie solvable property. It is apparent that algebras which are Lie solvable satisfy a certain multilinear polynomial identity.

At the beginning of 70s, thanks to the classification by Passman and Isaacs of PI group algebras, Passi, Passman and Sehgal [40] solved the question of when a group algebra FG of a group G over a field F is Lie solvable and Lie nilpotent by proving the following

Theorem 13. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Then FG is Lie nilpotent if and only if G is nilpotent and p-abelian.*

Theorem 14. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Then FG is Lie solvable if and only if either G is p-abelian or $p = 2$ and G contains a 2-abelian subgroup of index 2.*

For the sake of completness we recall that the original results of [40] were established for arbitrary group rings over commutative rings with identity. For an overview we refer to Chapter V of [43], where the conditions so that a group algebra satisfies the strong Lie identities were also stated, namely

Theorem 15. *Let FG be the group algebra of a group G over a field F. Then FG is strongly Lie nilpotent if and only if FG is Lie nilpotent.*

Theorem 16. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Then FG is strongly Lie solvable if and only if G is p-abelian.*

Another question of interest was to find necessary and sufficient conditions so that a group algebra FG is *bounded Lie Engel.* We recall that, for a positive integer n, a ring R (or a subset of it) is said to be Lie n-Engel if

$$[a, \underbrace{b, \dots, b}_{n \text{ times}}] = 0$$

for all $a, b \in R$. A ring R is *bounded Lie Engel* if it is Lie n-Engel for some positive integer n. This was done by Sehgal (Theorem V.6.1 of [43]).

Theorem 17. *Let FG be the group algebra of a group G over a field F. If $\operatorname{char} F = 0$, then FG is bounded Lie Engel if and only if G is abelian. If $\operatorname{char} F = p > 0$, then FG is bounded Lie Engel if and only if G is nilpotent and G has a p-abelian normal subgroup of finite p-power index.*

We have already seen in Section 1 the connection between group identities on units and polynomial identities on the group algebra. Furthermore it is possible frequently to reduce problems concerning specific group identities to problems concerning specific Lie identities. This is evident in particular when the group algebra is Lie nilpotent. To this purpose, let us consider the unit group $\mathcal{U}(FG)$ of a group algebra FG and let $u, v \in \mathcal{U}(FG)$. Then

$$(u, v) - 1 = u^{-1}v^{-1}[u, v].$$

A consequence of this fact is that, for any positive integer n,

$$\gamma_n(\mathcal{U}(FG)) - 1 \subseteq FG^{(n)}, \tag{1}$$

where $\gamma_n(\mathcal{U}(FG))$ denotes the n-th term of the lower central series of the group $\mathcal{U}(FG)$. It immediately follows that if FG is strongly Lie nilpotent then $\mathcal{U}(FG)$ is nilpotent and, if $cl(\mathcal{U}(FG))$ denotes the nilpotency class of $\mathcal{U}(FG)$, $cl(\mathcal{U}(FG)) \leq cl^L(FG)$. Gupta and Levin [24] improved the result of (1) by proving that

$$\gamma_n(\mathcal{U}(FG)) - 1 \subseteq \langle FG^{[n]} \rangle$$

and, consequently, if FG is Lie nilpotent then $cl(\mathcal{U}(FG)) \leq cl_L(FG)$. The implication between the Lie nilpotency property of FG and the nilpotency of $\mathcal{U}(FG)$ is true also in the other direction, at least in the modular case (if $\operatorname{char} F = p > 0$, a group algebra FG is said to be *modular* if G contains at least one element of order p) as established by Khripta [26].

Theorem 18. *Let FG be the modular group algebra of a group G over a field F. Then $\mathcal{U}(FG)$ is nilpotent if and only if FG is Lie nilpotent.*

The semiprime case was settled by Fisher, Parmenter and Sehgal [10] and involves more conditions.

Theorem 19. *Let FG be the group algebra of a group G over a field F of characteristic $p \geq 0$. Suppose that G has no elements of order p (if $p > 0$). Then $\mathcal{U}(FG)$ is nilpotent if and only if G is nilpotent and one of the following holds:*

(a) T, the set of the elements of finite order of G, is a central subgroup of G;

(b) $|F| = 2^\beta - 1$ is a Mersenne prime, T is an abelian subgroup of G of exponent $p^2 - 1$ and, for all $x \in G$ and $t \in T$, $x^{-1}tx = t$ or t^p.

At the end of 1980s, Shalev (see [45] for a general discussion) proposed a systematic study of the nilpotency class of the unit group of a group algebra of a finite p-group G over the field with p elements $\mathbb{F}_p$. Even in the case in which the group G is rather simple, $\mathcal{U}(\mathbb{F}_pG)$ is a finite p-group whose structure is rather complicated and its nilpotency class in some way measures its complexity. For a long time, a line of research has been that of considering the existence of a given groups L *involved* in $\mathcal{U}(\mathbb{F}_pG)$. Using this approach, Coleman and Passman [7] proved that, if G is non-abelian, then the wreath product $C_p \wr C_p$, where C_p is the group of order p, is involved in $\mathcal{U}(\mathbb{F}_pG)$, from which it follows that $cl(\mathcal{U}(\mathbb{F}_pG)) \geq p$. Subsequently Baginski [2] has proved the equality in the case in which the commutator subgroup of G has order p. Based on the original idea of Coleman and Passman, Shalev conjectured that $\mathcal{U}(\mathbb{F}_pG)$ always possesses a section isomorphic to the wreath product $C_p \wr G'$ and proved the result in [46] when G' is cyclic and p is odd.

A fundamental contribution in this framework was given by the solution of Jennings' Conjecture on radical rings by Du [9], which allowed to conclude that, for any field F of characteristic p and finite p-group G,

$$cl_L(FG) = cl(\mathcal{U}(FG)). \tag{2}$$

In this way, group commutator computations were replaced by ones involving Lie commutators, which are considerably easier. But this is not the only advantage. Indeed, in [3] Bahandari and Passi proved for an arbitrary Lie nilpotent group algebra FG that

$$cl_L(FG) = cl^L(FG)$$

provided $\operatorname{char} F = p > 3$ (it is an *open question* to decide if the equality holds for arbitrary p). Thus, according to (2), when $\operatorname{char} F = p > 3$ and G is a finite p-group the computation of the nilpotency class of $\mathcal{U}(FG)$ is reduced to that of $cl^L(FG)$. For this an extension of Jennings's theory provides a rather satisfactory formula based on the size of the Lie dimension subgroups of the underlying group G. In confirmation of all this, the most prominent results in this direction, presented in [47], were just deduced on the basis of the breakthrough of Du and Bhandari and Passi.

The equality (2) is easily seen to be satisfied when G is a (not necessarily finite) p-group. Recently, Catino, Siciliano and Spinelli [6] settled the case in which G is an arbitrary torsion group by proving the following

Theorem 20. *Let F be a field of positive characteristic p and G a torsion group containing an element of order p such that $\mathcal{U}(FG)$ is nilpotent. Then $cl_L(FG) = cl(\mathcal{U}(FG))$.*

One cannot expect that Theorem 20 is valid for arbitrary modular group algebras. In fact, Theorems 4.3, 4.4 and 5.2 of [4] provide examples in which the equality does not hold.

2.1 Amitsur Theorem and Lie identities for FG^+ and FG^-

Let A be an F-algebra with involution $*$. A question of general interest is which properties of A^+ or A^- can be lifted to A. One of the most celebrated results in this direction is the following theorem due to Amitsur [1] dealing with algebras satisfying a $*$-polynomial identity. We recall that an F-algebra A with involution $*$ satisfies a $*$-polynomial identity if there exists a non-zero polynomial $f(x_1, x_1^*, \ldots, x_t, x_t^*)$ in $F\{x_1, x_1^*, x_2, x_2^*, \ldots\}$, the *free associative algebra with involution* on the countable set of variables $\{x_1, x_2, \ldots\}$, such that $f(r_1, r_1^*, \ldots, r_t, r_t^*) = 0$ for all $r_i \in A$.

Theorem 21. *Let F be a field and A an F-algebra with involution (with or without an identity). If A satisfies a $*$-polynomial identity, then A satisfies a polynomial identity.*

Obviously if A^+ or A^- satisfy a polynomial identity, then A satisfies a $*$-polynomial identity and, by the above theorem, it is PI.

Since the second half of the 90s there have a been a number of papers devoted to investigate the extent to which the Lie identities satisfied by the symmetric and the skew-symmetric elements of a group algebra FG with respect to the classical involution determine the Lie identities satisfied by the whole group ring. Work on Lie nilpotence was begun by Giambruno and Sehgal in [16] with the following

Theorem 22. *Let FG be the group algebra of a group G with no 2-elements over a field F of characteristic different from 2 endowed with the classical involution. Then FG^+ or FG^- are Lie nilpotent if and only if FG is Lie nilpotent.*

It is easy enough to see that the above result does not hold if G has 2-elements. Indeed, as observed in Section 1.3, if G is a Hamiltonian 2-group, then the symmetric elements of FG commute. But Theorem 13 tells us that FG is not Lie nilpotent. Moreover, if D_8 denotes the dihedral group of order 8, for any field F of odd characteristic FD_8^- is commutative, but again FD_8 is not Lie nilpotent. In [27] Lee showed that Theorem 22 can be extended to groups not containing the quaternions, and then classified the groups G containing Q_8 such that FG^+ is Lie nilpotent.

Theorem 23. *Let FG be the group algebra of a group G not containing Q_8 over a field F of characteristic different from 2 endowed with the classical involution. Then FG^+ is Lie nilpotent if and only if FG is Lie nilpotent.*

Theorem 24. *Let FG be the group algebra of a group G containing Q_8 over a field F of characteristic $p \neq 2$ endowed with the classical involution. Then FG^+ is Lie nilpotent if and only*

(a) $p = 0$ and $G \cong Q_8 \times E$, where E is an elementary abelian 2-group, or

(b) $p > 2$ and $G \cong Q_8 \times E \times P$, where E is an elementary abelian 2-group and P is a finite p-group.

Work on group algebras of groups containing 2-elements whose Lie subalgebra of skew-symmetric elements is nilpotent is much more complicated and took a rather long time. It was begun by Giambruno and Polcino Milies in [12] and recently completed by Giambruno and Sehgal [17] with the proof of the following

Theorem 25. *Let FG be the group algebra of a group G over a field F of characteristic $p \neq 2$ endowed with the classical involution. Then FG^- is Lie nilpotent if and only*

(a) G has a nilpotent p-abelian subgroup H with $(G \setminus H)^2 = 1$, or

(b) G has an elementary abelian 2-subgroup of index 2, or

(c) the p-elements of G form a finite normal subgroup P and G/P is an elementary abelian 2-group.

The same questions concerning the bounded Lie Engel property were investigated a bit later by Lee [28]. Also in this case for the symmetric elements the answer depends on the fact that G contains Q_8 or not.

Theorem 26. *Let FG be the group algebra of a group G not containing Q_8 over a field F of characteristic different from 2 endowed with the classical involution. Then FG^+ is bounded Lie Engel if and only if FG is bounded Lie Engel.*

Theorem 27. *Let FG be the group algebra of a group G containing Q_8 over a field F of characteristic $p \neq 2$ endowed with the classical involution. Then FG^+ is bounded Lie Engel if and only*

(a) $p = 0$ and $G \cong Q_8 \times E$, where E is an elementary abelian 2-group, or

(b) $p > 2$ and $G \cong Q_8 \times E \times P$, where E is an elementary abelian 2-group and P is a p-group of bounded exponent having a p-abelian subgroup of finite index.

Up to now the best known result as when the skew-symmetric elements of a group algebra are Lie n-Engel is again in the same paper by Lee [28]. It deals with groups without elements of even order and is in the same direction as Theorem 22.

Theorem 28. *Let FG be the group algebra of a group G with no 2-elements over a field F of characteristic different from 2 endowed with the classical involution. Then FG^- is bounded Lie Engel if and only if FG is bounded Lie Engel.*

For any F-algebra with involution A it is easy to see that $[A^+, A^+] \subseteq A^-$. Thus, as A^- is a Lie subalgebra of A, if it is Lie solvable then so is A^+. This simple observation is very useful for the classification of group algebras whose skew and symmetric elements are Lie solvable. The question has been recently investigated by Lee, Sehgal and Spinelli in [31]. It was solved under a restriction upon the orders of the group elements. Their first theorem deals with the characteristic zero case and two different prime characteristic cases.

Theorem 29. *Let FG be the group algebra of a group G with no 2-elements over a field F of characteristic $p \neq 2$ endowed with the classical involution. Suppose either that $p = 0$ or else $p > 2$ and either*

(a) G has only finitely many p-elements, or

(b) G contains an element of infinite order.

Then the following are equivalent:

(i) FG^+ is Lie solvable;

(ii) FG^- is Lie solvable;

(iii) FG is Lie solvable.

We can assume now that the group G is torsion. No result is known that completely covers the remaining case, but the following theorem, also from [31], gives a partial answer.

Theorem 30. *Let F be a field of characteristic $p > 2$. Let G be a torsion group containing an infinite p-subgroup of bounded exponent, but no non-trivial elements of order dividing $p^2 - 1$. Let FG have the classical involution. Then the following are equivalent:*

(i) FG^+ is Lie solvable;

(ii) FG^- is Lie solvable;

(iii) FG is Lie solvable.

No result is currently known for groups with 2-elements except for what concerns the skew-symmetric elements. In fact, if $\operatorname{char} F = 0$ or $\operatorname{char} F = p > 2$ and G has only finitely many p-elements, Lee, Sehgal and Spinelli (Theorem 1.2 of [31]) classified the groups G containing 2-elements such that FG^- is Lie solvable. They also observed that, in order to remove the condition that G contains an infinite p-subgroup of bounded exponent in Theorem 30, it is sufficient to consider the case in which G has a normal subgroup A which is a direct product of finitely many copies of the quasicyclic p-group, C_{p^∞}, and $G/A = \langle Ag \rangle$, where the order of g is a prime power. This case, however, remains open. Indeed, the restriction can be dropped whenever G does not have C_{p^∞} as a subhomomorphic image.

Of course, for any field F of characteristic different from 2, FQ_8^+ is Lie solvable (being commutative) but, according to Theorem 14, FQ_8 is not. Unfortunately, the usual criterion that G does not contain Q_8 will not be sufficient.

Indeed, as observed after Theorem 22, if G is the dihedral group of order 8, then FG^- is commutative; hence, FG^+ is Lie solvable. However, it seems reasonable to conjecture that if G has no 2-elements, then the conclusions of Theorem 29 hold without any other restriction.

Work on Lie identities for symmetric elements is very useful also in discussing the corresponding group identities for the symmetric units. We do not review the details of this in the present survey, but we confine ourselves to report the principal results showing how, in some sense, polynomial identities satisfied by FG^+ reflect group identities satisfied by $\mathcal{U}^+(FG)$ and the latter ones can be lifted to the whole unit group of FG. For the following result see [34].

Theorem 31. *Let F be an infinite field of characteristic $p > 2$. Let G be a group containing an infinite p-subgroup of bounded exponent, but no non-trivial elements of order dividing $p^2 - 1$. Let FG have the classical involution. Then the following are equivalent:*

(i) $\mathcal{U}^+(FG)$ is solvable;

(ii) $\mathcal{U}(FG)$ is solvable;

(iii) FG^+ is Lie solvable;

(iv) FG is Lie solvable.

If one replaces the hypothesis that G contains an infinite p-subgroup of bounded exponent with the weaker assumption that G contains infinitely many p-elements, Lee and Spinelli (Theorem 4 of [34]) proved that (i), (ii) and (iv) are equivalent. The case in which G contains finitely many p-elements was completely solved again in [34], but for the details we refer the reader to the original paper.

Along this line, Lee and Spinelli [35] determined the conditions under which the subgroup generated by $\mathcal{U}^+(FG)$, $\langle \mathcal{U}^+(FG) \rangle$, is bounded Engel, when G is torsion and F infinite.

Theorem 32. *Let FG be the group algebra of a torsion group G not containing Q_8 over an infinite field F of characteristic different from 2 endowed with the classical involution. Then the following are equivalent:*

(i) $\langle \mathcal{U}^+(FG) \rangle$ is bounded Engel;

(ii) $\mathcal{U}(FG)$ is bounded Engel;

(iii) FG^+ is bounded Lie Engel;

(iv) FG is bounded Lie Engel.

Theorem 33. *Let FG be the group algebra of a torsion group G containing Q_8 over an infinite field F of characteristic different from 2 endowed with the classical involution. Then $\langle \mathcal{U}^+(FG) \rangle$ is bounded Engel if and only if FG^+ is bounded Lie Engel.*

The result by Jespers and Ruiz Marin (Theorem 11) on group algebras FG endowed with F-linear extensions of arbitrary group involutions whose symmetric elements commute is fundamental for the investigation of more general properties of FG^+. The first results of this type were obtained by Giambruno, Polcino Milies and Sehgal [13] for groups without 2-elements. They confirm that Theorem 22 and Theorem 26 can be extended to this general setting.

Theorem 34. *Let F be a field of characteristic different from 2, G a group without 2-elements with an involution $*$ and let FG have the induced involution. Then FG^+ is Lie nilpotent (bounded Lie Engel, respectively) if and only if FG is Lie nilpotent (bounded Lie Engel, respectively).*

Obviously we cannot expect that the result is true for an arbitrary group G. According to the discussion after Theorem 10, the answer will depend on the presence of SLC-groups in G. A complete answer has been given by Lee, Sehgal and Spinelli [32] with the proof of the following

Theorem 35. *Let F be a field of characteristic $p > 2$, G a group with an involution $*$ and let FG have the induced involution. Suppose that FG is not Lie nilpotent. Then FG^+ is Lie nilpotent if and only if G is nilpotent, and G has a finite normal $*$-invariant p-subgroup N such that G/N is an SLC-group.*

Theorem 36. *Let F be a field of characteristic $p > 2$, G a group with an involution $*$ and let FG have the induced involution. Suppose that FG is not bounded Lie Engel. Then FG^+ is bounded Lie Engel if and only if G is nilpotent, G has a p-abelian $*$-invariant normal subgroup A of finite index, and G has a normal $*$-invariant p-subgroup N of bounded exponent, such that G/N is an SLC-group.*

As when FG is endowed with the classical involution, the link between Lie identities satisfied by FG^+ and group identities satisfied by $\mathcal{U}^+(FG)$ appears strong. In confirmation of this, by using Theorem 12, Lee, Sehgal and Spinelli [33] found necessary and sufficient conditions so that $\mathcal{U}^+(FG)$ is nilpotent by proving the following

Theorem 37. *Let F be an infinite field of characteristic different from 2, G a torsion group with an involution $*$ and let FG have the induced involution. Then $\mathcal{U}^+(FG)$ is nilpotent if and only if FG^+ is Lie nilpotent.*

We recall that Theorem 37 was originally proved for group algebras over arbitrary fields (non-necessarily infinite) endowed with the classical involution by Lee [29]. The same question was investigated for non-necessarily torsion groups by Lee, Polcino Milies and Sehgal in [30].

Finally, work on group algebras whose skew-symmetric elements satisfy a certain Lie identity is rather complicated. Recently Giambruno, Polcino Milies and Sehgal [15] have classified the torsion groups G with no 2-elements for which FG^- is Lie nilpotent. It turns out that the conclusion is much more involved than for the classical involution (Theorem 22). Their main result is the following.

Theorem 38. *Let F be a field of characteristic $p \neq 2$ and G a torsion group with no elements of order 2. Let $*$ be an involution on FG induced by an involution of G. Then the Lie algebra FG^- is nilpotent if and only if FG is Lie nilpotent or $p > 2$ and the following conditions hold:*

(1) the set P of p-elements in G is a subgroup;

(2) $*$ is trivial on G/P;

(3) there exist normal $*$-invariant subgroups A and B, $1 \leq B \leq A$ such that B is a finite central p-subgroup of G, A/B is central in G/B with both G/A and $\{a \mid a \in A \quad aa^* \in B\}$ finite.

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