

*Note di Matematica* **23**, n. 2, 2004/2005, 173–181.

# Quantum Bi-Hamiltonian systems, alternative Hermitian structures and Bi-Unitary transformations

**G. Marmo**

*Dipartimento di Scienze Fisiche, Università di Napoli Federico II,  
Complesso Universitario di Monte S Angelo, Napoli, Italy;  
I.N.F.N., Sezione di Napoli, Italy  
giuseppe.marmo@na.infn.it*

**G. Scolarici**

*Dipartimento di Fisica, Università di Lecce, 73100, Lecce;  
I.N.F.N., Sezione di Lecce, 73100 Lecce, Italy  
giuseppe.scolarici@le.infn.it*

**A. Simoni**

*Dipartimento di Scienze Fisiche, Università di Napoli Federico II,  
Complesso Universitario di Monte S Angelo, Napoli, Italy;  
I.N.F.N., Sezione di Napoli, Italy  
alberto.simoni@na.infn.it*

**F. Ventriglia**

*Dipartimento di Scienze Fisiche, Università di Napoli Federico II,  
Complesso Universitario di Monte S Angelo, Napoli, Italy;  
I.N.F.M and I.N.F.N., Sezione di Napoli, Italy  
franco.ventriglia@na.infn.it*

Received: 30/5/2004; accepted: 30/5/2004.

**Abstract.** We discuss the dynamical quantum systems which turn out to be bi-unitary with respect to the same alternative Hermitian structures in a infinite-dimensional complex Hilbert space. We give a necessary and sufficient condition so that the Hermitian structures are in generic position. Finally the transformations of the bi-unitary group are explicitly obtained.

**Keywords:** functional analysis, quantum mechanics, symplectic geometry, spectral theory.

**MSC 2000 classification:** 37K05 (primary), 58D19.

## 1 Introduction

In the study of bi-Hamiltonian systems (see ref. [1] for quantum systems) one starts with a given dynamics and looks for alternative Hamiltonian descriptions. In this paper, we deal with the “inverse problem” recently considered by some of us in different combinations [2]. We start with two Hermitian structures on complex Hilbert spaces arising from two admissible triples in the corresponding

real Hilbert space and look for all dynamical quantum systems which turn out to be bi-unitary with respect to them. However, we limit ourselves to consider here the case of admissible triples sharing the same complex structure. This study is a preliminary step to generalize some results in Ref. [3], to the infinite-dimensional case.

This paper is organized as follows. In sec. 2 we show how to construct an admissible Hermitian structure (admissible triple) on a real Hilbert space starting from a metric tensor and a complex structure [3]. Moreover we show how to recover a Hermitian scalar product after the complexification of the Hilbert space. In sec. 3, we briefly discuss the bi-unitary group in the case of finite dimensional spaces and relate the genericity condition of the Hermitian structures with the cyclicity of the corresponding connecting Hermitian operator. In sec. 4, we discuss bi-unitary groups in the infinite-dimensional case, showing how the direct integral decomposition of the Hilbert space with respect to a normed ring is a suitable theoretical tool to deal with such a problem [4]. In particular we show how this decomposition with respect to the bicommutant of the operator connecting two Hermitian structures can be constructed and used for a full comparison of the two Hermitian structures. In sec. 5, we prove that the component spaces in the decomposition are unidimensional if and only if the Hermitian structures are in generic position, what allows to conclude that all the generic bi-unitary quantum systems commute. This construction resembles that of the algebra one obtains for completely integrable systems by means of the recursion operator [5]. Moreover, the bi-unitary group is explicitly exhibited both in the generic and non generic case. Finally, in the last section, some concluding remarks are posed.

## 2 Admissible Hermitian structures on the Hilbert space

Let us show that given a metric tensor  $g$  and a complex structure  $J$  in a infinite-dimensional real Hilbert space  $\mathcal{H}^{\mathcal{R}}$ , one can construct an admissible Hermitian structure  $h$  on  $\mathcal{H}^{\mathcal{R}}$  which becomes an Hermitian scalar product in a corresponding complex Hilbert space  $\mathcal{H}$ .

Given a couple  $g$  and  $J$  on  $\mathcal{H}^{\mathcal{R}}$ , we can always construct out of them an admissible couple  $g_s, J$  where  $g_s$  is defined in the following way

$$g_s =: \frac{1}{2}\{g(J., J.) + g(., .)\}, \quad (1)$$

which will be positive and nondegenerate if  $g$  is positive and nondegenerate. The metric  $g_s$  and  $J$  are admissible in the sense that  $J$  is an anti-Hermitian operator

with respect to  $g_s$ . Hereafter we drop  $s$  in  $g_s$ . Such an admissible couple may be completed defining a symplectic structure  $\omega$  on  $\mathcal{H}^{\mathcal{R}}$  as

$$\omega = g \circ J.$$

Then the triple  $(g, J, \omega)$  is *admissible*.

It is possible to obtain an admissible triple even starting from  $g$  and  $\omega$ . In fact by Riesz's theorem a nonsingular linear operator  $B$  exists such that

$$\omega(x, y) = g(Bx, y) \quad (2)$$

and the antisymmetry of  $\omega$  together with symmetry of  $g$  imply that

$$g(Bx, y) = -g(x, By) \quad (3)$$

so  $B$  is skew-Hermitian and  $-B^2 > 0$ . Then a (symmetric) non-negative square root of  $B$ ,  $R$ , will be injective and densely defined. So  $R^{-1}$  is well defined [6] and putting  $J =: BR^{-1}$  and

$$g_{\omega}(\cdot, \cdot) =: g(R\cdot, \cdot) \quad (4)$$

we recover

$$\omega(x, y) = g(Bx, y) = g_{\omega}(Jx, y) \quad (5)$$

and in conclusion the triple  $(g_{\omega}, J, \omega)$  is admissible. In other words  $J$  is orthogonal and symplectic and also infinitesimally orthogonal and infinitesimally symplectic, i.e. it is unitary and skew-Hermitian:  $J^{\dagger} = -J$ ,  $J^2 = -\mathbf{1}$ .

Now, from a metric tensor  $g$  and an admissible symplectic form  $\omega$ , an Hermitian structure on  $\mathcal{H}^{\mathcal{R}}$  can be obtained. This is a map  $h : \mathcal{H}^{\mathcal{R}} \times \mathcal{H}^{\mathcal{R}} \rightarrow \mathbf{R}^2$  defined as follows

$$h : (x, y) \longmapsto (g(x, y), \omega(x, y)). \quad (6)$$

Equivalently (and having in mind a quantum system) one can complexify  $\mathcal{H}^{\mathcal{R}}$  by defining, for any complex number  $z = \alpha + i\beta$  and any vector  $x$ :

$$z.x = (\alpha + i\beta)x =: \alpha x + J\beta x. \quad (7)$$

Then  $h$  becomes an Hermitian scalar product, linear in the second factor, on this complex Hilbert space  $\mathcal{H}$ , and we get

$$h(x, y) = g(x, y) + ig(Jx, y). \quad (8)$$

### 3 Bi-unitary descriptions in $\mathcal{H}$

In quantum mechanics the Hilbert space  $\mathcal{H}$  is given as a *complex* vector space. Now we consider on  $\mathcal{H}$  two different Hermitian structures, coming from two admissible triples  $(g_1, J_1, \omega_1)$  and  $(g_2, J_2, \omega_2)$  on  $\mathcal{H}^{\mathcal{R}}$  with the same complex structure:  $J_1 = J_2 = J$ . The assumption  $J_1 = J_2$  implies that the corresponding Hermitian structures on  $\mathcal{H}$  are compatible, in the sense that the group of bi-unitary transformations is non void. [3]

Denoting with  $h_1(., .)$  and  $h_2(., .)$  the Hermitian structures given on  $\mathcal{H}$  (both linear, for instance, in the second factor), we search for the group that leaves both  $h_1$  and  $h_2$  invariant, that is the bi-unitary group.

Again by Riesz's theorem a bounded, positive operator  $G$  may be defined, which is self-adjoint both with respect to  $h_1$  and  $h_2$ , as

$$h_2(x, y) = h_1(Gx, y) \quad \forall x, y \in \mathcal{H}. \quad (9)$$

Moreover, any bi-unitary operator  $U$  must commute with  $G$ . Indeed

$$\begin{aligned} h_2(x, U^\dagger G U y) &= h_2(Ux, G U y) = h_1(Ux, U y) = \\ &h_1(x, y) = h_2(Gx, y) = h_2(x, G y) \end{aligned}$$

and from this

$$U^\dagger G U = G, \quad [G, U] = 0. \quad (10)$$

Therefore the group of bi-unitary operators belongs to the commutant  $G'$  of the operator  $G$ .

Let us discuss the bi-unitary group when  $\mathcal{H}$  is finite-dimensional. In this case  $G$  is diagonalizable and the two Hermitian structures result proportional in each eigenspace of  $G$  *via* the eigenvalue. Then the group of bi-unitary transformations is

$$U(n_1) \times U(n_2) \times \cdots \times U(n_k) \quad n_1 + n_2 + \cdots + n_k = n = \dim \mathcal{H},$$

where  $n_l$  denotes the degeneracy of the  $l$ -th eigenvalue of  $G$ .

In finite-dimensional complex Hilbert spaces the following definition can be stated [3]:

**1 Definition.** Two Hermitian forms are in generic position iff the eigenvalues of  $G$  are nondegenerate.

Then, if  $h_1$  and  $h_2$  are in generic position the group of bi-unitary transformations becomes

$$\underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{n \text{ factors}}.$$

In other words this means that  $G$  generates a complete set of observables.

Now we show that:

*Two Hermitian forms are in generic position if and only if their connecting operator  $G$  is cyclic.*

In fact the non singular operator  $G$  has a discrete spectrum and is diagonalizable, so when  $h_1$  and  $h_2$  are in generic position  $G$  admits  $n$  distinct eigenvalues  $\lambda_k$ . Let now  $\{e_k\}$  be the eigenvector basis of  $G$  and  $\{\mu^k\}$  an  $n$ -ple of nonzero complex numbers. Then the vector

$$x_0 = \mu^k e_k \quad (11)$$

is a cyclic vector for  $G$ , that is the vectors  $x_0, Gx_0, \dots, G^{n-1}x_0$  are  $n$  linearly independent vectors. In fact one obtains

$$G^m x_0 = \mu^k \lambda_k^m e_k, \quad (12)$$

and the coefficient determinant is given by

$$\left(\prod_k \mu^k\right) V(\lambda_1, \dots, \lambda_n), \quad (13)$$

where  $V$  denotes the Vandermonde determinant which is different from zero when all the eigenvalues  $\lambda_k$  are distinct. The converse is also true.

## 4 Infinite-dimensional case

Now we deal with the infinite-dimensional case, when the connecting operator  $G$  may have a point part and a continuum part in its spectrum. As to the point part, the bi-unitary group is  $U(n_1) \times U(n_2) \times \cdots$  where now  $n_l$  can also be  $\infty$ . When  $G$  admits a continuum spectrum, the characterization of the bi-unitary group is more involved and suitable mathematical tools are needed, such as spectral theory of operators and theory of rings of operators in Hilbert spaces. In particular in the infinite-dimensional case definition 1 of generic position of two Hermitian forms is no more valid and has to be generalized in the following way [3]:

**2 Definition.** Two Hermitian forms are in generic position iff  $G'' = G'$ , that is when the bicommutant  $G''$  coincides with the commutant  $G'$  of  $G$ .

First of all, observe that the commutant  $G'$  and the bicommutant  $G''$  of the operator  $G$  are weakly closed rings of bounded operators in the ring of all bounded operators  $\mathcal{B}(\mathcal{H})$  on  $\mathcal{H}$  and  $G'' \subset G'$ . Let  $E_0$  be the principal identity of a set  $S \subset \mathcal{B}(\mathcal{H})$ : by definition  $E_0$  is the projection operator on the orthogonal complement of the set  $\text{Ker}S \cap \text{Ker}S^\dagger$  and satisfies for all  $A \in S$  the following relation:

$$E_0A = AE_0 = A. \quad (14)$$

It can be proved [4] that the minimal weakly closed ring  $R(S)$  containing  $S$  is the strongest closure of the ring  $R_{a^*}(S)$ , which is the minimal ring containing  $S \cup S^\dagger$ .  $R(S)$  contains only those elements  $A \in S''$  which satisfy the above condition (14).

Now, by the spectral theorem we express the selfadjoint operator  $G$  in terms of its spectral family  $\{P(\lambda)\}$ :

$$G = \int_{\Delta} \lambda dP(\lambda) \quad (15)$$

where  $\Delta = [a, b]$  is a closed interval containing the entire spectrum of  $G$ . Moreover, the positiveness of  $G$  ensures that  $\text{Ker}G = 0$ . This implies that  $\mathbf{1} \in R(G)$  and hence  $R(G) \equiv G''$ . Therefore  $G''$  is commutative (in fact  $R(G)$  is the w-closure of the symmetric commutative subring  $G$  in the ring of  $\mathcal{B}(\mathcal{H})$ ).

We recall that each commutative weakly closed ring of operators  $C$  in the Hilbert space corresponds to a direct integral of Hilbert spaces.

The following theorems hold [4]:

**3 Theorem.** *To each direct integral of Hilbert spaces*

$$\mathcal{H} = \int_{\Delta} H_{\lambda} d\sigma(\lambda),$$

*with respect to a measure  $\sigma$  there corresponds a commutative weakly closed ring  $C = L_{\sigma}^{\infty}(\Delta)$  where to each  $\varphi \in L_{\sigma}^{\infty}(\Delta)$  there corresponds the operator  $L_{\varphi} : (L_{\varphi}\xi) = \varphi(\lambda)\xi_{\lambda}$  where  $\xi \in \mathcal{H}$ ,  $\xi_{\lambda} \in H_{\lambda}$  and  $\|L_{\varphi}\| = \|\varphi\|$ .*

*Vice versa:*

**4 Theorem.** *To each commutative weakly closed ring  $C$  of operators in a Hilbert space  $\mathcal{H}$  there corresponds a decomposition of  $\mathcal{H}$  into a direct integral, for which  $C$  is the set of operators of the form  $L_{\varphi}$ ,  $\varphi \in L^{\infty}$ .*

From the previous theorems we get that *the weakly closed commutative ring  $R(G)$  corresponds to a decomposition of the Hilbert space  $\mathcal{H}$  into the direct integral*

$$\mathcal{H} = \int_{\Delta} H_{\lambda} d\sigma(\lambda), \quad (16)$$

where  $\Delta = [a, b]$  is the entire spectrum of the positive selfadjoint operator  $G$ .

The measure  $\sigma(\lambda)$  is obtained by the spectral family  $\{P(\lambda)\}$  and cyclic vectors in the usual way. Now every operator  $A$  belonging to the commutant  $G'$  is representable in the form of a direct integral of operators

$$A = \int_{\Delta} A(\lambda) \cdot d\sigma(\lambda), \quad (17)$$

where  $A(\lambda)$  is, for almost all  $\lambda$ , a bounded operator in  $H_{\lambda}$ .

Every operator  $B$  of the bicommutant  $G'' = R(G)$  is then a multiplication by a number  $b(\lambda)$  on  $H_{\lambda}$  for almost all  $\lambda$ .

Thus the bi-unitary transformations are in general a direct integral of unitary operators  $U(\lambda)$  acting on  $H_{\lambda}$ .

## 5 Bi-unitary group transformations

More insight can be gained by a more specific analysis of the direct integral decomposition.

Let  $G'(\lambda)$  be the totality of all the operators  $A(\lambda)$  corresponding to  $G'$ , for fixed  $\lambda$ . Then as  $R(G)$  is a maximal commutative ring in itself, we may conclude [7] that the family  $G'(\lambda)$  is irreducible for almost all  $\lambda$ , and the direct integral of  $\mathcal{H}$  with respect to  $R(G)$  can be written as follows

$$\mathcal{H} = \int_{\Delta} H_{\lambda} d\sigma(\lambda) = \sum_k \oplus \mathcal{H}_k = \sum_k \oplus \int_{\Delta_k} H_{\lambda} d\sigma(\lambda), \quad (18)$$

where now the spectrum  $\Delta$  decomposes into a sum of finite or countable number of measurable sets  $\Delta_k$ , such that for  $\lambda \in \Delta_k$  the spaces  $H_{\lambda}$  have constant dimension  $k$  ( $k$  can be  $\infty$ ). Then any operator  $A$  belonging to the commutant  $G'$  is representable as follows:

$$A = \sum_k \oplus \int_{\Delta_k} A(\lambda) \cdot d\sigma(\lambda). \quad (19)$$

As a consequence of Eq. (18), remembering that every operator  $B$  of  $G'' = R(G)$  is a multiplication by a number  $b(\lambda)$  on  $H_{\lambda}$ , we get the following result:

**5 Proposition.** *Let two Hermitian structures  $h_1$  and  $h_2$  be given on the Hilbert space  $\mathcal{H}$ . Then there exists a decomposition of  $\mathcal{H}$  into a direct integral of Hilbert spaces  $H_\lambda$ , of dimension  $k$ , such that in each space  $H_\lambda$ ,  $h_1$  and  $h_2$  are proportional:  $h_2 = \lambda h_1$ .*

Proposition 5 partially generalizes the Lemma in Ref. [3] to the case of infinite-dimensional complex Hilbert spaces  $\mathcal{H}$ . This result belongs to the constellation of propositions connected with the so called ‘‘Quadratic Hamiltonian theorems’’. [8]

As a consequence of Proposition 5, the elements  $U$  of the bi-unitary group can be written as follows (see Eq. (19)):

$$U = \sum_k \oplus \int_{\Delta_k} U_k(\lambda) \cdot d\sigma(\lambda), \quad (20)$$

where  $U_k : \Delta_k \rightarrow U(H_\lambda)$ , that is  $U_k(\lambda)$  is an element of the unitary group  $U(H_\lambda)$  on the spaces  $H_\lambda$  of dimension  $k$ .

Moreover, when the Hermitian forms are in generic position the following statement holds:

**6 Proposition.** *The component spaces  $H_\lambda$  of the decomposition of  $\mathcal{H}$  into a direct integral with respect to  $R(G)$  are unidimensional if and only if the two Hermitian forms  $h_1$  and  $h_2$  are in generic position.*

PROOF. Let us suppose that two Hermitian forms are given in generic position, then by definition  $R(G) = G'' = G'$ , therefore  $G'$  must be commutative. Hence the totality of all irreducible operators  $A(\lambda)$  corresponding to  $G'$  are unidimensional, then  $\dim H_\lambda = 1$  for all  $\lambda \in \Delta$ .

In order to prove the converse, observe that if  $R(G) = G'' \neq G'$ , there exists a non zero subset  $\Delta_0$  of  $\Delta$  such that for  $\lambda \in \Delta_0$  the totality of irreducible operators  $A(\lambda)$  corresponding to  $G'$  is not commutative. Hence  $\dim H_\lambda \neq 1$ . □QED

Proposition 6 fully generalizes the Proposition in Ref. [3] to the case of infinite-dimensional complex Hilbert space, so that we can say that all the generic different admissible quantum dynamical systems are pairwise commuting.

Finally, Proposition 6 implies that the unitary operators  $U_k(\lambda)$  in Eq. (20), reduce to a multiplication by a phase factor  $e^{i\varphi(\lambda)}$  on  $H_\lambda$  for almost all  $\lambda$ , so that the elements of the bi-unitary group read

$$U = \int_{\Delta} e^{i\varphi(\lambda)} \cdot d\sigma(\lambda). \quad (21)$$



## 6 Concluding remarks

In this paper we have shown how to extend to the more realistic case of infinite dimensions our results of a previous paper dealing mainly with finite level quantum systems.

Our approach shows how to deal with “pencils of compatible Poisson Brackets” [9] in the framework of quantum systems.

Our formulation is already in a form suitable to deal with quantum electrodynamics.

We hope to be able to extend these results to the evolutionary equations for classical and quantum field theories.

## References

- [1] F. MAGRI: *A simple model of integrable Hamiltonian equation*, J. Math. Phys. **19** (1978), 1156–1162; F. MAGRI: *A geometrical approach to the nonlinear solvable equations*, Lect. Notes in Phys. **120** (1980), 233–263; B. FUCHSSTEINER: *The Lie algebra structure and degenerate Hamiltonian and bi-Hamiltonian systems*, Progr. Theor. Phys. **68** (1982), 1082–1104.
- [2] F. VENTRIGLIA: *Alternative Hamiltonian descriptions for quantum systems and non-Hermitian operators with real spectrum*, Mod. Phys. Lett. **A 17** (2002), 1589–1599; G. MARMO, A. SIMONI, F. VENTRIGLIA: *Quantum systems: real spectra and nonhermitian (Hamiltonian) operators*, Rep. Math. Phys., **51** (2003), 275–285; G. MARMO, A. SIMONI, F. VENTRIGLIA: *Quantum systems and alternative unitary descriptions*, Int. J. Mod. Phys. A, **19** (2004), 2561–2578.
- [3] G. MARMO, G. MORANDI, A. SIMONI, F. VENTRIGLIA: *Alternative structures and bi-Hamiltonian systems*, J. Phys. A: Math. Gen., **35** (2002), 8393–8406.
- [4] M. A. NAIMARK: *Normed Rings*, Wolters-Noordhoff Publishing Groningen, 1970.
- [5] S. DE FILIPPO, G. MARMO, M. SALERNO, G. VILASI: *A new characterization of completely integrable systems*, Nuovo Cimento **B 83** (1984), 97–112.
- [6] R. ABRAHAM, J. E. MARSDEN: *Foundation of Mechanics*, Benjamin-Cummings, New York, 1978.
- [7] See Ref. [4] p. 503.
- [8] G. MARMO, E. J. SALETAN, R. SCHMID, A. SIMONI: *Bi-Hamiltonian Dynamical Systems and the Quadratic-Hamiltonian Theorem*, Nuovo Cimento **B 100** (1987), 297–317.
- [9] I. M. GEL’FAND, I ZAKHAROVICH: *On Local Geometry of Bihamiltonian Structure*, Gel’fand Mathematical Seminar vol. 1, ed. L. Corwin, I. M. Gel’fand and J. Lepowski, Birkhauser, Basel 1993; A. IBORT, F. MAGRI, G. MARMO, *Bihamiltonian structures and Stackel separability*, J. Geom. Phys. **33** (2000), 210–228.