Subelliptic harmonic maps, morphisms, and vector fields

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Abstract. We review the salient properties of subelliptic harmonic maps and morphisms, both from a domain in $\mathbb{R}^N$ endowed with a Hörmander system and from a strictly pseudoconvex CR manifold. We also report on several generalizations of harmonic morphisms such as heat equation morphisms from a CR manifold and $\Box_b$-harmonic morphisms. We discuss subelliptic harmonic vector fields within pseudohermitian geometry.

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1 Subelliptic harmonic maps

It is our purpose in this section to recall the notion of a *subelliptic harmonic map* as introduced by J. Jost & C-J. Xu, [22], as well as their main existence and regularity results for weak solutions to the subelliptic harmonic maps system. Let $U \subseteq \mathbb{R}^N$ be an open subset and $X = \{X_1, \cdots, X_m\} \in \mathfrak{X}^\infty(U)$ a system of smooth vector fields. We call X a *Hörmander system* if the vector fields X_j together with their commutators of length $\leq r$ span $T_x(\mathbb{R}^N)$ at each $x \in U$. Here a commutator of the form $[X_i, X_j]$ has length 2 and by convention each X_j has length 1. Let $X = \{X_1, \cdots, X_m\}$ be a Hörmander system on U and let us set $X_j = b_j^A(x) \, \partial/\partial x^A$ for some $b_j^A \in C^\infty(U)$ where $(x^1, \cdots x^N)$ are the natural coordinates on $\mathbb{R}^N$. Let X_j^* be the formal adjoint of X_j i.e.

$$X_j^* f \equiv -\frac{\partial}{\partial x^A} \left(b_j^A(x) f \right) = -X_j f + \varphi_j f, \quad f \in C_0^1(U).$$

Next let us consider the *Hörmander operator* (often referred loosely as a *sum of squares* of vector fields)

$$Hu \equiv -\sum_{j=1}^m X_j^* X_j u = \sum_{j=1}^m X_j^2 u + X_0 u, \tag{1}$$

where $X_0 = -\sum_{j=1}^{m} \varphi_j X_j$. It may be written as

$$Hu = \sum_{A,B=1}^{N} \frac{\partial}{\partial x^A} \left(a^{AB}(x) \frac{\partial u}{\partial x^B} \right),$$

where $a^{AB}(x) = \sum_{j=1}^{m} b_j^A(x) b_j^B(x)$ (a positive semi-definite matrix) hence H is a *degenerate elliptic* (in the sense of J.M. Bony, [7]) second order differential operator. Nevertheless, by a result of L. Hörmander, [19], H is hypoelliptic i.e. if $Hu = f$ in distributional sense and f is C^∞ smooth then u is C^∞ smooth, as well. Hypoellipticity is the main property enjoyed both by H and the ordinary Laplacian on $\mathbb{R}^N$, and pleading for the similarity among Hörmander sums of squares of vector fields and elliptic operators. In an attempt to generalize known results on nonlinear elliptic systems (such as the harmonic maps system) of variational origin to the hypoelliptic setting J. Jost & C-J. Xu, [22], introduced the following notion. Let $U \subseteq \mathbb{R}^N$ be an open set and Ω a domain such that $\overline{\Omega} \subset U$. Let $X = \{X_1, \cdots, X_m\}$ be a Hörmander system on U and $\phi : \Omega \to N$ a C^∞ map from Ω into a Riemannian manifold N. We say ϕ is a *subelliptic harmonic map* if

$$(H_N \phi)^\alpha \equiv H\phi^\alpha + \sum_{j=1}^{m} \left(\Gamma_{\beta\gamma}^\alpha \circ \phi \right) X_j(\phi^\beta) X_j(\phi^\gamma) = 0, \tag{2}$$

on $U \equiv \phi^{-1}(V)$ for any local coordinate system $(V, y^1, \cdots, y^\nu)$ on N. Here $\phi^\alpha = y^\alpha \circ \phi$. Also (2) is referred to as the *subelliptic harmonic map system*. A word on the adopted terminology. First it should be observed that when $m = N$ and $X_j = \partial/\partial x^j$ then H is the ordinary Laplacian $\Delta = \sum_{j=1}^{N} \partial^2/\partial x_j^2$ and a subelliptic harmonic map is nothing but a harmonic map (cf. e.g. J. Jost, [21], p. 4). As to the label "subelliptic", it is motivated by the fact that for any $x \in U$ there is an open subset $O \subseteq U$ such that $x \in O$ and

$$\|u\|_{1/2}^2 \le C \left(|(Hu, u)| + \|u\|^2 \right), \quad u \in C_0^\infty(O),$$

i.e. H is *subelliptic* of order $\epsilon = 1/2$ (and subelliptic operators are known to be hypoelliptic, [19]). Here $\| \cdot \|_\epsilon$ denotes the Sobolev norm of order ϵ

$$\|u\|_\epsilon = \left(\int (1 + |\xi|^2)^\epsilon |\hat{u}(\xi)|^2 \right)^{1/2},$$

cf. e.g. [27], p. 217. See G.B. Folland, [16], for the general notion of a subelliptic operator, the *a priori* estimates satisfied by such operators, and a basic example of subelliptic operator on a Heisenberg group (the *sublaplacian*, an object on

which we shall come back later on in the context of CR geometry). It may be shown that $H_N\phi = 0$ are the Euler-Lagrange equations of the variational principle $\delta E(\phi) = 0$ where

$$E(\phi) = \frac{1}{2}\int_\Omega \sum_{j=1}^m X_j(\phi^\alpha)X_j(\phi^\beta)\,(h_{\alpha\beta}\circ\phi)\ dx.$$

In the spirit of variational calculus one needs appropriate function spaces where one may look for weak solutions to (2). Let us consider the Sobolev-type spaces

$$W_X^{1,p}(\Omega) \equiv \{u \in L^p(\Omega) : X_j u \in L^p(\Omega),\ \ 1 \le j \le m\},$$

where $X_j u$ is meant in distributional sense. By a result of C-J. Xu, [30], if $1 \le p < \infty$ then $W_X^{1,p}(\Omega)$ is a separable Banach space with the norm

$$\|u\|_{W_X^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)}^p + \sum_{j=1}^m \|X_j u\|_{L^p(\Omega)}^p\right)^{1/p}. \tag{3}$$

Also if $1 < p < \infty$ then $W_X^{1,p}(\Omega)$ is reflexive and $W_X^{1,2}(\Omega)$ is a separable Hilbert space. For further use we let $W_0^{1,2}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ with respect to the norm (3). Although the system (2) is nonlinear an appropriate notion of weak solution does exist. Indeed let us assume that N is covered by a single coordinate neighborhood $\chi = (y^1,\cdots,y^\nu) : N \to \mathbb{R}^\nu$ and define

$$W_X^{1,2}(\Omega, N) \equiv \{\phi : \Omega \to N : \phi^\alpha = y^\alpha \circ \phi \in W_X^{1,2}(\Omega),\ \ 1 \le \alpha \le \nu\}.$$

Then $\phi : \Omega \to N$ is said to be a *weak solution* to $H_N\phi = 0$ if $\phi \in W_X^{1,2}(\Omega, N)$ and

$$\sum_{j=1}^m \int_\Omega \{X_j(\phi^\alpha)X_j(\varphi) - \left(\Gamma_{\beta\gamma}^\alpha \circ \phi\right)X_j(\phi^\beta)X_j(\phi^\gamma)\varphi\}dx = 0$$

for any $\varphi \in C_0^\infty(\Omega)$. Given $f \in C^0(\partial\Omega, N)$ J. Jost & C-J. Xu, [22], considered the following *Dirichlet problem* for the subelliptic harmonic maps system

$$H_N\phi = 0\ \ \text{in}\ \Omega,\ \ \phi = f\ \ \text{on}\ \partial\Omega.$$

The main result in [22] may be stated as follows. Let us assume that 1) N is a complete ν-dimensional ($\nu \ge 2$) Riemannian manifold without boundary ($\partial N = \emptyset$) covered by a single coordinate neighborhood and $\mathrm{Sect}(N) \le \kappa^2$ for some $\kappa > 0$, where $\mathrm{Sect}(N)$ is the sectional curvature of N. Also 2) let $p \in N$ be a point and $0 < \mu < \min\{\pi(2\kappa),\ i(p)\}$ where $i(p)$ denotes the injectivity radius at

p. Finally 3) we consider $f \in C^0(\overline{\Omega}, N) \cap W_X^{1,2}(\Omega, N)$ such that $f(\overline{\Omega}) \subset B(p, \mu)$. Then there is a unique $\phi_f \in W_X^{1,2}(\Omega, N) \cap L^\infty(\Omega, N)$ such that

$$\phi_f^\alpha - f^\alpha \in W_0^{1,2}(\Omega), \quad 1 \le \alpha \le \nu, \quad \phi_f(\overline{\Omega}) \subset B(p, \mu),$$

and ϕ_f minimizes $E(\phi)$ among all such maps. Also ϕ_f is a weak solution to $H_N \phi = 0$ and $\phi_f \in C^0(\Omega, N)$. Cf. S. Hildebrand et al., [17], [18], for the results on harmonic maps brought here to the realm of Hörmander systems of vector fields.

Higher regularity of continuous solutions to a class of quasilinear subelliptic systems including (2) has been studied by C-J. Xu & C. Zuily, [31]. To recall their result let $X = \{X_1, \cdots, X_m\}$ be a Hörmander system on $U \subseteq \mathbb{R}^N$, $N \ge 2$, $U \supset \overline{\Omega}$, and let $a^{ij}(x, y)$ be symmetric and positive definite. If

$$|f(x, y, p)| \le a|p|^2 + b, \quad (x, y, p) \in \Omega \times \mathbb{R}^\nu \times \mathbb{R}^{m\nu},$$

then any C^0 solution $\phi = (\phi^1, \cdots, \phi^\nu)$ to

$$\sum_{i,j=1}^m X_j^* \left(a^{ij}(x, \phi(x)) X_i \phi^\alpha(x) \right) = f^\alpha(x, \phi(x), X\phi(x)) \quad \text{in} \quad \Omega$$

is actually C^∞ smooth. As a corollary to J. Jost & C-J. Xu and C-J. Xu & C. Zuily's results the solution $\phi_f : \Omega \to N$ to the Dirichlet problem for the subelliptic harmonic map system is smooth, thus settling the question of the existence of subelliptic harmonic maps.

It is the proper place to give a both fundamental and classical example of a Hörmander system of vector fields, appearing on the Heisenberg group. Let us set $\mathbb{H}_n \equiv \mathbb{C}^n \times \mathbb{R}$ and let us endow $\mathbb{H}_n$ with the group law $(z, t) \circ (w, s) \equiv (z + w, t + s + 2\text{Im}(z \cdot \overline{w}))$ so that $(\mathbb{H}_n, \circ)$ becomes a Lie group. Next we consider the *Lewy operators*

$$Z_j = \frac{\partial}{\partial z^j} + i\overline{z}^j \frac{\partial}{\partial t}, \quad 1 \le j \le n,$$

and set

$$X_j = \frac{1}{\sqrt{2}}(Z_j + \overline{Z}_j), \quad X_{n+j} = \frac{i}{\sqrt{2}}(Z_j - \overline{Z}_j).$$

Then $\{X_a : 1 \le a \le 2n\}$ is a Hörmander system on $\mathbb{R}^{2n+1}$ and each X_a is left invariant. To give a geometric interpretation we may consider the *Siegel domain*

$$\mathcal{D}_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \rho(z) \equiv \text{Im}(z_n) - |z'|^2 > 0\}$$

and observe that the mapping $F : \mathbb{H}_{n-1} \to \partial \mathcal{D}_n$ given by $F(z, t) = (z, t - i|z'|^2)$, is a C^∞ diffeomorphism. As we shall emphasize later on the Lewy operators

span a left invariant CR structure on $\mathbb{H}_{n-1}$. On the other hand the boundary $\partial \mathcal{D}_n$ of the Siegel domain is a smooth orientable real hypersurface in $\mathbb{C}^n$ so that it carries a CR structure naturally induced by the complex structure of $\mathbb{C}^n$. Then F becomes a CR isomorphism.

Subelliptic harmonic maps may be looked at as boundary values of harmonic maps. Let us endow the Siegel domain $\mathcal{D}_n \subset \mathbb{C}^n$ ($n \geq 2$) with the Bergman metric i.e. the Kähler metric g whose Kähler 2-form is $\frac{i}{2} \partial \overline{\partial}(1/\rho)$). By a result M. Soret et al., [15], if $\phi : \mathcal{D}_n \to N$ is a Bergman-harmonic map into a Riemannian manifold N such that $\phi \in C^\infty(\overline{\mathcal{D}}_n, N)$ and the boundary values $f : \partial \mathcal{D}_n \to N$, $f \equiv \phi|_{\partial \mathcal{D}}$, have a vanishing normal derivative $N_\rho f^\alpha = 0$ (with $N_\rho = JT$ and T the characteristic direction of $\frac{i}{2}(\overline{\partial} - \partial)\rho$) then $f \circ F : \mathbb{H}_{n-1} \to N$ is a subelliptic harmonic map. A similar result holds for Bergman-harmonic maps $\phi : \Omega \to N$ from an arbitrary smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ into a Riemannian manifold (cf. Y. Kamishima et al., [13]).

2 Subelliptic harmonic morphisms

A map $\phi : \Omega \to N$ is said to be a *subelliptic harmonic morphism* if $\phi \in C^0(\Omega, N)$ and for any local harmonic function $v : V \to \mathbb{R}$ ($V \subseteq N$ open and $\Delta_N v = 0$ in V) one has $H(v \circ \phi) = 0$ in $\phi^{-1}(V)$ in distributional sense. The notion is due to E. Barletta, [2]. As H is hypoelliptic and for any $p \in N$ there is a local coordinate system whose coordinate functions are harmonic any subelliptic harmonic morphism is C^∞ smooth.

We need to recall (cf. e.g. A. Bonfiglioli et al., [6]) that a second order partial differential operator $\mathcal{L} \equiv \sum_{j=1}^p X_j^2$ is said to be a real *sublaplacian* on $\mathbb{R}^n$ if 1) there is a group structure $\circ$ on $\mathbb{R}^n$ making $\mathbb{G} = (\mathbb{R}^n, \circ)$ into a Lie group such that each X_j is a first order differential operator with smooth real valued coefficients and X_j is left invariant, 2) the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$ is stratified and nilpotent i.e. there is an integer $r \geq 1$ and there are linear subspaces $V_j \subset \mathfrak{g}$, $1 \leq i \leq r$, such that $\mathfrak{g}$ admits the decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$ and i) $[V_1, V_j] = V_{j+1}$, $1 \leq j \leq r-1$, ii) $[V_j, V_r] = 0$, $1 \leq j \leq r$, and $\{X_1, \cdots, X_p\}$ is a basis of V_1 (as a real linear space). $\mathbb{G}$ is a *Carnot group* and the smallest integer $r \geq 1$ as above is its *step*. If $m_j = \dim_{\mathbb{R}} V_j$ then $Q = \sum_{j=1}^r j m_j$ is the *homogeneous dimension* of $\mathbb{G}$.

Let N be a Riemannian manifold. We say that N has the *Liouville property* if any harmonic function $f : N \to [0, +\infty)$ is constant. For instance any closed (i.e. compact, without boundary) Riemannian manifold has the Liouville property (as an elementary consequence of the Hopf maximum principle). Also if N is a complete Riemannian manifold of nonnegative Ricci curvature then any bounded harmonic function on N is a constant (cf. S-T. Yau, [32]). An

extension of the Liouville property to the case of the Hörmander operator on the Heisenberg group was proved by A. Korányi & N.K. Stanton, [24].

Next we need to recall the following Harnack-type inequality (due to A. Bonfiglioli & E. Lanconelli, [5]). For any $p \in (Q/2, +\infty]$ there exist constants $C > 0$ and $\theta > 0$ (depending only on $\mathcal{L}$ and p) such that

$$\sup_{|x| \leq r} u(x) \leq C \left\{ \inf_{|x| \leq r} u(x) + r^{2-Q/p} \|\mathcal{L}u\|_{L^p(D(0,\theta r))} \right\} \qquad (4)$$

for any C^2 function $u : \mathbb{R}^n \to [0, +\infty)$ and any $r > 0$. Here $D(x, r) = \{y \in \mathbb{R}^n : |x^{-1} \circ y| \leq r\}$. The inequality (4) is the main ingredient in the proof of the following result (due to E. Lanconelli et al., [14]). Let N be a Riemannian manifold. If there is a surjective $\mathcal{L}$-harmonic morphism from a Carnot group into N then N has the Liouville property.

3   Subelliptic harmonic maps in CR geometry

Let $(M, T_{1,0}(M))$ be a real $(2n + 1)$-dimensional CR manifold of CR dimension n i.e. $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ is a complex subbundle of complex rank n such that i) $T_{1,0}(M) \cap T_{0,1}(M) = (0)$ where $T_{0,1}(M) \equiv \overline{T_{1,0}(M)}$ and ii) if $Z, W \in \Gamma^\infty(T_{1,0}(M))$ then $[Z, W] \in \Gamma^\infty(T_{1,0}(M))$. For instance each real hypersurface $M \subset \mathbb{C}^n$ is a CR manifold of CR dimension $n - 1$ with the CR structure $T_{1,0}(M) = T^{1,0}(\mathbb{C}^n) \cap [T(M) \otimes \mathbb{C}]$ where $T^{1,0}(\mathbb{C}^n)$ is the holomorphic tangent bundle over $\mathbb{C}^n$ i.e. the span of $\{\partial/\partial z^j : 1 \leq j \leq n\}$. Cf. e.g. G. Tomassini et al., [12]. The Heisenberg group $\mathbb{H}_n$ is a CR manifold with the CR structure $T_{1,0}(\mathbb{H}_n)_x = \sum_{j=1}^n \mathbb{C}Z_{j,x}$, $x \in \mathbb{H}_n$. The natural C^∞ diffeomorphism $F : \mathbb{H}_{n-1} \to \partial\mathcal{D}_n$ is a CR isomorphism i.e.

$$(d_x F)T_{1,0}(\mathbb{H}_{n-1})_x = T_{1,0}(\partial\mathcal{D}_n)_{F(x)}, \quad x \in \mathbb{H}_{n-1}.$$

Let us briefly recall a few notions of pseudohermitian and contact geometry. The real rank $2n$ subbundle $H(M) \equiv \operatorname{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ of the tangent bundle is the *Levi distribution*. The Levi distribution carries the complex structure $J : H(M) \to H(M)$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$. Let $H(M)^\perp \equiv \{\omega \in T^*(M) : \operatorname{Ker}(\omega) \supseteq H(M)\}$ be the conormal bundle associated to $H(M)$. A *pseudohermitian structure* is a globally defined nowhere zero section $\theta \in \Gamma^\infty(H(M)^\perp)$. A pseudohermitian structure is a *contact form* if $\theta \wedge (d\theta)^n$ is a volume form. The *Levi form* is

$$L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W}), \quad Z, W \in T_{1,0}(M).$$

A CR manifold $(M, T_{1,0}(M))$ is *nondegenerate* if L_θ is nondegenerate for some θ. If $(M, T_{1,0}(M))$ is nondegenerate then any pseudohermitian structure on M is a contact form.

Let $(M, T_{1,0}(M))$ be an oriented nondegenerate CR manifold and θ a contact form on M. Let $T \in \mathfrak{X}(M)$ such that $\theta(T) = 1$ and $T \rfloor d\theta = 0$ (the *characteristic direction* of $d\theta$). Let g_θ be the semi-Riemannian metric determined by

$$g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

$$g_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M),$$

(the *Webster metric* of (M, θ)). There is a unique linear connection ∇ (the *Tanaka-Webster connection* of (M, θ)) such that i) $H(M)$ is parallel with respect to ∇, ii) $\nabla g_\theta = 0$, $\nabla J = 0$, iii) $T_\nabla(Z, W) = 0$, $T_\nabla(Z, \overline{W}) = 2iL_\theta(Z, \overline{W})T$, for any $Z, W \in T_{1,0}(M)$.

We may use the Tanaka-Webster connection to relate the nondegeneracy of the given CR manifold to Hörmander's bracket generating property in section 1. Let M be a CR manifold. Let $\{X_a : 1 \leq a \leq 2n\}$ be a local frame of the Levi distribution $H(M)$, defined on the domain $U \subseteq M$ of a local chart (U, χ) with $\chi = (x^1, \cdots, x^{2n+1}) : U \to \mathbb{R}^{2n+1}$, with $X_{j+n} = JX_j$ for $1 \leq j \leq n$. If M is nondegenerate then $\{(d\chi)X_a : 1 \leq a \leq 2n\}$ is a Hörmander system on $\chi(U)$. Indeed let us set $T_j \equiv X_j - JX_j \in T_{1,0}(M)$ for any $1 \leq j \leq n$. Let θ be a contact form on M and ∇ the corresponding Tanaka-Webster connection. By the torsion property (iii) of ∇

$$\nabla_{T_j}\overline{T}_k - \nabla_{\overline{T}_k}T_j - 2ig_{j\overline{k}}T = [T_j, \overline{T}_k]$$

where $g_{j\overline{k}} = L_\theta(T_j, \overline{T}_k)$. Also we set $\nabla_{T_j}\overline{T}_k = \Gamma^{\overline{\ell}}_{j\overline{k}}T_{\overline{\ell}}$ so that

$$\Gamma^{\overline{\ell}}_{j\overline{k}}T_{\overline{\ell}} - \Gamma^{\ell}_{\overline{k}j}T_\ell - 2ig_{j\overline{k}}T = [T_j, \overline{T}_k]. \tag{5}$$

As $\{T_j, \overline{T}_k, T\}$ span $T(M) \otimes \mathbb{C}$ over U it follows (as a consequence of (5)) that $\{T_j, \overline{T}_k, [T_j, \overline{T}_k]\}$ span $T(M) \otimes \mathbb{C}$ over U as well. Therefore $\{X_a, [X_a, X_b]\}$ span $T(M)$ over U.

Let us recall that a CR manifold $(M, T_{1,0}(M))$ is *strictly pseudoconvex* if L_θ is positive definite for some θ. Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold and θ a contact form with L_θ positive definite. The *sublaplacian* is the operator Δ_b given by $\Delta_b u \equiv \mathrm{div}\left(\nabla^H u\right)$ for any $u \in C^2(M)$. The divergence operator is given by $\mathcal{L}_X\Psi = \mathrm{div}(X)\Psi$ where Ψ is the volume form $\Psi \equiv \theta \wedge (d\theta)^n$ and $\mathcal{L}$ denotes the Lie derivative. Also $\nabla^H u \equiv \pi_H \nabla u$ is the *horizontal gradient* (and $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$) while $\pi_H : T(M) = H(M) \oplus$

$\mathbb{R}T \to H(M)$ is the natural projection associated to the decomposition $T(M) = H(M) \oplus \mathbb{R}T$.

The sublaplacian Δ_b is a formally self-adjoint second order partial differential operator. Also Δ_b is subelliptic of order $\epsilon = 1/2$. If $\{X_a : 1 \le a \le 2n\}$ is a local orthonormal $(g_\theta(X_a, X_b) = \delta_{ab})$ frame of $H(M)$ with $X_{n+j} = JX_j$, $1 \le j \le n$, then

$$\Delta_b = H \equiv -\sum_{a=1}^{2n} X_a^* X_a. \tag{6}$$

At this point we should recall the following result of E. Barletta et al., [1]. Let $(M, T_{1,0}(M))$ be a compact strictly pseudoconvex CR manifold and let θ be a contact form on M such that the corresponding Levi form L_θ is positive definite. Let F_θ be the *Fefferman metric* associated to θ (cf. Definition 2.15 in [12], p. 128). This is a Lorentzian metric on $C(M)$, the total space of the canonical circle bundle $S^1 \to C(M) \xrightarrow{\pi} M$ (cf. Definition 2.8 in [12], p. 119), such that the *restricted* conformal class $[F_\theta] = \{e^{u \circ \pi} F_\theta : u \in C^\infty(M)\}$ is a CR invariant. Let $\phi : M \to N$ be a smooth map into a Riemannian manifold (N, h). By a result in [1] the following assertions are equivalent

i) $\Phi \equiv \phi \circ \pi : (C(M), F_\theta) \to N$ is a harmonic map.

ii) $\phi : M \to N$ is a critical point of

$$E(\phi) = \frac{1}{2} \int_M \text{trace}_{g_\theta} (\pi_H \phi^* h) \, \Psi.$$

iii) $\phi : M \to N$ satisfies

$$\Delta_b \phi^\alpha + \sum_{a=1}^{2n} \left(\Gamma_{\beta\gamma}^\alpha \circ \phi \right) X_a(\phi^\beta) X_a(\phi^\gamma) = 0$$

for any local coordinate systems (U, x^A) on M and (V, y^α) on N such that $U = \phi^{-1}(V)$. A smooth map $\phi : M \to N$ satisfying one of the equivalent statements (i)-(iii) is called a *pseudoharmonic map*. As $\Delta_b = H$ this is of course formally similar to the notion of a subelliptic harmonic map. It should be noted however that the two notions are quantitatively distinct (the formal adjoint of X_a in (1) is meant with respect to the Lebesgue measure on $\mathbb{R}^N$ while formal adjoints in (6) are with respect to the volume form $\Psi = \theta \wedge (d\theta)^n$).

A study of the geometry of CR orbifolds has been started by J. Masamune at al, [9]. Cf. also E. Barletta et al., [4]. The problem of studying subelliptic harmonic maps from a CR orbifold into a Riemnannian manifold (by analogy to the work of Y-J. Chiang, [8], in Riemannian geometry) is open.

The notion of a subelliptic harmonic morphism may be recast within CR geometry as follows. A smooth map $\phi : M \to N$ from a strictly pseudoconvex

CR manifold M into a Riemannian manifold N is a *pseudoharmonic morphism* if for any $v : V \subseteq N \to \mathbb{R}$ such that $\Delta_N v = 0$ in V and $\phi^{-1}(V) \neq \emptyset$ then $\Delta_b(v \circ \phi) = 0$ in $\phi^{-1}(V)$. The contents of the Fuglede-Ishihara theorem may be recovered (cf. E. Barletta, [3]). Indeed let M be a connected strictly pseudo-convex CR manifold and θ a contact form with L_θ positive definite. Let N be a ν-dimensional Riemannian manifold. Then i) any pseudoharmonic morphism is a pseudoharmonic map and there is a C^0 function $\lambda : M \to [0, +\infty)$ (the θ-*dilation* of ϕ) such that λ^2 is C^∞ and

$$g_\theta(\nabla^H \phi^i, \nabla^H \phi^j)_x = \lambda(x)^2 \delta^{ij}, \quad 1 \leq i, j \leq \nu, \tag{7}$$

for any $x \in M$ and any local system of normal coordinates (V, y^i) on N at x (here $\phi^i = y^i \circ \phi$). Viceversa ii) any subelliptic harmonic map $\phi : M \to N$ satisfying (7) is a pseudoharmonic morphism. Moreover iii) if $\nu > 2n$ then there are no nonconstant pseudoharmonic morphisms from M into N while if $\nu \leq 2n$ then for any $x \in M$ such that $\lambda(x) \neq 0$ there is an open neighborhood $U \subseteq M$ such that $\phi : U \to N$ is a submersion. Finally iv) for any pseudoharmonic morphism $\phi : M \to N$ and any $f \in C^2(N)$

$$\Delta_b(f \circ \phi) = \lambda^2(\Delta_N f) \circ \phi \tag{8}$$

where Δ_N is the Laplace-Beltrami operator on N.

4   Further generalizations of harmonic morphisms

4.1   Heat equation morphisms

The *heat equation* on M is

$$\left(\frac{\partial}{\partial t} - \Delta_b \right) u(x, t) = 0, \quad x \in M, \ t > 0, \tag{9}$$

where Δ_b is the sublaplacian associated to θ. A smooth map $\Psi : M \times (0, +\infty) \to N \times (0, +\infty)$ is a *heat equation morphism* if for any open set $V \subseteq N$ and any solution $f : V \times (0, +\infty) \to \mathbb{R}$ to $\partial f / \partial t - \Delta_N f = 0$ it follows that $u = f \circ \Psi$ is a solution to (9). Extending work by E. Loubeau, [25], in Riemannian geometry one has the following result (cf. E. Lanconelli et al., [14]). Let M be a strictly pseudoconvex CR manifold and θ a contact form on M with L_θ positive definite. Let N be a Riemannian manifold. Let $\Psi : M \times (0, +\infty) \to N \times (0, +\infty)$ be a smooth map of the form $\Psi(x, t) = (\phi(x), h(t))$, $x \in M$, $t > 0$. Then Ψ is a heat equation morphism if and only if $\phi : M \to N$ is a pseudoharmonic morphism of constant θ-dilation λ and $h(t) = \lambda^2 t + C$ for some $C \in \mathbb{R}$.

Let $p_t(x)$ be the fundamental solution to the heat equation on the Heisenberg group i.e. $u(x,t) = (p_t * f)(x)$ solves

$$\left(\frac{\partial}{\partial t} - H\right) u(x,t) = 0, \quad u(x,0) = f(x), \quad x \in \mathbb{H}_n,$$

for $f \in L^\infty(\mathbb{H}_n)$. The fundamental solution $p_t(x)$ was explicitly computed (cf. A. Hulanicki, [20]) as

$$\hat{p}_t(\alpha + i\beta, s) = (\cosh 2ts)^{-n/2} \times$$

$$\times \exp\left\{ (2s\cosh 2ts)^{-1} \left[-\frac{1}{2}(|\alpha|^2 + |\beta|^2)\cosh 2ts + i(\alpha \cdot \beta)(\sinh ts)^2 \right] \right\}$$

for any $(\alpha + i\beta, s) \in \mathbb{H}_n$, where a hat denotes the Fourier transform.

Let $p^N : N \times N \times (0, +\infty) \to \mathbb{R}$ be a heat kernel of a connected Riemannian manifold (N, h) i.e. $p^N \in C^0$, $u(x, y, t)$ is C^2 in y, C^1 in t, and

$$\left(\frac{\partial}{\partial t} - \Delta_{N,y}\right) p^N = 0,$$

$$\lim_{t \to 0^+} \int_N p^N(x, y, t)\varphi(y) \, d\operatorname{vol}_h(y) = \varphi(x), \quad x \in N,$$

for any bounded C^0 function φ on N. A heat kernel always exists and if N is compact it is also unique.

A *heat kernel morphism* is a C^∞ map $\Phi : \mathbb{H}_n \times \mathbb{H}_n \times (0, +\infty) \to N \times N \times (0, +\infty)$ such that $p = p^N \circ \Phi$ where $p(x, y, t) = p_t(xy^{-1})$. Again building on the analogy to Riemannian geometry (cf. E. Loubeau, [25]) one has (cf. E. Lanconelli et al., [14]) the following result. Let $\Phi : \mathbb{H}_n \times \mathbb{H}_n \times (0, +\infty) \to N \times N \times (0, +\infty)$ be a heat kernel morphism of the form $\Phi(x, y, t) = (\phi(x), \phi(y), h(t))$ for some surjective C^∞ map $\phi : \mathbb{H}_n \to N$ and some C^∞ function $h(t) > 0$ for $t > 0$. Then $\Psi(x, t) = (\phi(x), h(t))$ is a heat equation morphism.

4.2 $\square_b$-harmonic morphisms

Let M be a compact nondegenerate CR manifold, θ a contact form on M, and T the characteristic direction of $d\theta$. A $(0, q)$-*form* on M is a $\mathbb{C}$-valued differential q-form ω such that $\mathbb{C}T \oplus T_{1,0}(M) \rfloor \omega = 0$. Let $\Lambda^{0,q}(M) \to M$ be the bundle of all $(0, q)$-forms and let us set $\Omega^{0,q}(M) \equiv \Gamma^\infty(\Lambda^{0,q}(M))$. Let

$$\overline{\partial}_b : \Omega^{0,q}(M) \to \Omega^{0,q+1}(M), \quad q \geq 0,$$

be the *tangential Cauchy-Riemann operator* i.e. if $\omega \in \Omega^{0,q}(M)$ then $\overline{\partial}_b \omega$ is the $(0, q+1)$-form on M coinciding with $d\omega$ on $T_{0,1}(M) \otimes \cdots \otimes T_{0,1}(M)$ ($q+1$ terms). Then

$$C^\infty(M) \otimes \mathbb{C} \xrightarrow{\overline{\partial}_b} \Omega^{0,1}(M) \xrightarrow{\overline{\partial}_b} \cdots \xrightarrow{\overline{\partial}_b} \Omega^{0,n}(M)$$

is a cochain complex (the *tangential Cauchy-Riemann complex* of M). Let

$$H^{0,q}(M) = H^q(\Omega^{0,\bullet}(M), \overline{\partial}_b) = \frac{\mathrm{Ker}\{\overline{\partial}_b : \Omega^{0,q}(M) \to \cdot\}}{\overline{\partial}_b \Omega^{0,q-1}(M)}, \quad q \geq 1,$$

be the cohomology of the tangential Cauchy-Riemann complex (the *Kohn-Rossi cohomology* of M). Let $\overline{\partial}_b^*$ be the formal adjoint of $\overline{\partial}_b$ with respect to the L^2 inner product

$$(\alpha, \beta) = \int_M G_\theta^*(\alpha, \overline{\beta})\, \theta \wedge (d\theta)^n, \quad \alpha, \beta \in \Omega^{0,q}(M),$$

that is

$$(\overline{\partial}_b^* \psi, \varphi) = (\psi, \overline{\partial}_b \varphi), \quad \varphi \in \Omega^{0,q}(M), \quad \psi \in \Omega^{0,q+1}(M).$$

Next let us consider the *Kohn-Rossi laplacian*

$$\Box_b \varphi = \left(\overline{\partial}_b^* \overline{\partial}_b + \overline{\partial}_b \overline{\partial}_b^* \right) \varphi, \quad \varphi \in \Omega^{0,q}(M),$$

and set $\mathcal{H}^{0,q}(M) = \mathrm{Ker}\{\Box_b : \Omega^{0,q}(M) \to \cdot\}$, the space of all $\Box_b$-harmonic $(0, q)$-forms on M.

Let M and N be two CR manifolds with contact forms θ and θ_N. A *pseudohermitian* map is i) a smooth CR map $\phi : M \to N$ i.e.

$$(d_x \phi) T_{1,0}(M)_x \subseteq T_{1,0}(N)_{\phi(x)}, \quad x \in M,$$

such that ii) $\phi^* \theta_N = c\theta$ for some $c \in \mathbb{R} \setminus \{0\}$. Given a pseudohermitian map $\phi : M \to N$ of nondegenerate CR manifolds it may be easily shown that $\phi^* \overline{\partial}_b^N \varphi = \overline{\partial}_b(\phi^* \varphi)$ for any $\varphi \in \Omega^{0,q}(N)$ hence ϕ induces a linear map $\phi^* : H^{0,q}(N) \to H^{0,q}(M)$, $q \geq 1$.

A smooth map $\phi : M \to N$ is said to be a $\Box_b$-*harmonic morphism* if the pullback by ϕ of any local $\Box_b^N$-harmonic function on N is a local $\Box_b$-harmonic function on M. We may state (cf. E. Lanconelli et al., [14]) the following result. Let $\phi : M \to N$ be a pseudohermitian map of a compact strictly pseudoconvex CR manifold M into a compact strictly pseudoconvex real hypersurface $N \subset \mathbb{C}^M$. Let $\phi^* : H^{0,1}(N) \to H^{0,1}(M)$ the induced map on Kohn-Rossi cohomology. If ϕ is a submersive $\Box_b$-harmonic morphism then ϕ^* is injective. The proof relies on the Poincaré lemma for the $\overline{\partial}_b^N$-complex (cf. M. Nacinovich, [26]) and J.J. Kohn's Hodge-de Rham theory for the $\overline{\partial}_b$-complex (cf. [23]).

4.3   CR-pluriharmonic morphisms

A C^1 function $u : M \to \mathbb{R}$ is *CR-pluriharmonic* if for any $x \in M$ there is an open neighborhood U of x in M and a C^1 function $v : U \to \mathbb{R}$ such that $\overline{\partial}_b(u + iv) = 0$ in U i.e. $u + iv$ is a CR function. CR functions may be thought of as boundary values of holomorphic functions. Therefore CR-pluriharmonic functions may be thought of as boundary values of pluriharmonic functions. On the other hand pluriharmonic functions are several complex variables analogs of harmonic functions. One is led to the following natural generalization of the notion of a harmonic morphism.

A function $f \in L^1_{\mathrm{loc}}(\mathbb{H}_n)$ is a weak solution to the tangential Cauchy-Riemann equations $\overline{\partial}_b f = 0$ (a *weak CR function*) if

$$\int_{\mathbb{H}_n} f(x)(Z_j\varphi)(x)\, dx = 0, \quad 1 \le j \le n,$$

for any $\varphi \in C_0^\infty(\mathbb{H}_n)$. A *weak* CR-pluriharmonic function is locally the real part of a weak CR function.

Let N be a CR manifold. Let $\mathcal{PM}(N)$ be the class of all C^0 maps $\phi : U \subseteq \mathbb{H}_n \to N$ (with U open) such that $u \circ \phi$ is a weak CR-pluriharmonic function, for any CR-pluriharmonic function $u : V \to \mathbb{R}$ with $V \subseteq N$ open and $\phi^{-1}(V) \ne \emptyset$. The properties of the class $\mathcal{PM}(N)$ are unknown as yet.

5   Subelliptic harmonic vector fields

Let M be a compact strictly pseudoconvex CR manifold and X a C^2 vector field on M. Let $\Delta_b X$ be the vector field locally given by

$$(\Delta_b X)^i = \Delta_b X^i + 2a^{jk}\Gamma^i_{j\ell}\frac{\partial X^\ell}{\partial x^k} + a^{jk}\Big(\frac{\partial\Gamma^i_{j\ell}}{\partial x^k} + \Gamma^s_{j\ell}\Gamma^i_{ks} - \Gamma^s_{jk}\Gamma^i_{s\ell}\Big)X^\ell,$$

with respect to a local coordinate system (U, x^i) on M. Here $X = X^i\,\partial/\partial x^i$ for some $X^i \in C^\infty(U)$, Γ^i_{jk} are the local coefficients of the Tanaka-Webster connection ∇ of (M, θ) and $a^{ij} = g^{ij} - T^i T^j$ with $[g^{ij}] = [g_{ij}]^{-1}$ and $g_{ij} = g_\theta(\partial_i\,, \partial_j)$, $\partial_i = \partial/\partial x^i$.

Next let $\mathcal{U}(M, \theta) = \{X \in \mathcal{X}^\infty(M) : g_\theta(X, X) = 1\}$ be the set of all C^∞ unit vector fields on (M, g_θ). By analogy to the Riemannian case (cf. R. Wiegmink, [28]) the *pseudohermitian biegung*, or *total bending*, is the functional $\mathcal{B} : \mathcal{U}(M, \theta) \to [0, +\infty)$ given by

$$\mathcal{B}(X) = \frac{1}{2}\int_M \|\nabla^H X\|^2\,\Psi, \quad X \in \mathcal{U}(M, \theta). \tag{10}$$

Here $\nabla^H X \in \Gamma^\infty(H(M)^* \otimes T(M))$ is the restriction of ∇X to $H(M)$. The central notion in this section may be introduced as follows. A *subelliptic harmonic vector field* is a C^∞ unit vector field $X \in \mathcal{U}(M, \theta)$ which is a critical point of $\mathcal{B}$ with respect to 1-parameter variations of X through unit vector fields.

Let S_θ be the *Sasaki metric* on $T(M)$ i.e.

$$S_\theta(\mathcal{X}, \mathcal{Y}) = g_\theta(L\mathcal{X}, L\mathcal{Y}) + g_\theta(K_\theta \mathcal{X}, K_\theta \mathcal{Y}), \tag{11}$$

for any $\mathcal{X}, \mathcal{Y} \in T(T(M))$. To recall the definition of the vector bundle morphisms L and K_θ entering (11) let us consider the natural projection $\Pi : T(M) \to M$ and let $\Pi^{-1}TM \to T(M)$ be the pullback of $T(M) \to M$ by Π. We set

$$L : T(T(M)) \to \Pi^{-1}TM, \quad L\mathcal{X} = (v, (d_v\Pi)\mathcal{X}),$$

for any $\mathcal{X} \in T_v(T(M))$ and any $v \in T(M)$. Moreover let $K_\theta : T(T(M)) \to \Pi^{-1}TM$ be the *Dombrowski map* associated to the Tanaka-Webster connection of (M, θ) i.e. $K_\theta = \gamma^{-1} \circ \Pi_{\mathcal{V}}$. Here $\gamma : \Pi^{-1}TM \to \mathcal{V}$ is the *vertical lift* i.e.

$$\gamma_v(v, X) = \frac{dC}{dt}(0) \in T_v(T(M)), \quad (v, X) \in (\Pi^{-1}TM)_v,$$

$$C : (-\epsilon, \epsilon) \to T(M), \quad C(t) = v + tX, \quad |t| < \epsilon,$$

and $\Pi_{\mathcal{V}} : T(T(M)) \to \mathcal{V}$ is the projection associated to the direct sum decomposition $T(T(M)) = \mathcal{H} \oplus \mathcal{V}$. Also $\mathcal{V} = \text{Ker}(d\Pi)$ (the *vertical distribution*) and

$$\mathcal{H}_v = \{V \in T_v(T(M)) : \left(\hat{\nabla}_{\tilde{V}} \mathcal{L}\right)_v = 0\}, \quad v \in T(M),$$

(the *horizontal distribution*). Here $\tilde{V}$ is a smooth extension of V to $T(M)$ i.e. $\tilde{V} \in \mathfrak{X}(T(M))$ and $\tilde{V}_v = V$ and $\mathcal{L} \in \Gamma^\infty(\Pi^{-1}TM)$ is the *Liouville vector* i.e. $\mathcal{L}(v) = (v, v)$ for any $v \in T(M)$. Finally $\hat{\nabla} = \Pi^{-1}\nabla$ is the connection induced in $\Pi^{-1}TM \to T(M)$ by the Tanaka-Webster connection ∇ of (M, θ).

Let $E : \mathcal{U}(M, \theta) \to [0, +\infty)$ be the energy functional defined by

$$E(X) = \frac{1}{2} \int_M \text{trace}_{G_\theta} (\pi_H X^* S_\theta) \; \Psi, \tag{12}$$

where $\pi_H X^* S_\theta$ denotes the restriction of $X^* S_\theta$ to $H(M) \otimes H(M)$. The following results are due to Y. Kamishima et al., [13]. Let M be a compact strictly pseudoconvex CR manifold and θ a contact form with L_θ positive definite. Let us set $\text{Vol}(M, \theta) = \int_M \Psi$. Then for any $X \in \mathcal{U}(M, \theta)$

$$E(X) = n \, \text{Vol}(M, \theta) + \mathcal{B}(X). \tag{13}$$

Moreover let $\mathfrak{X} : M \times (-\delta, \delta) \to T(M)$ be a smooth 1-parameter variation of X through unit vector fields so that $\mathfrak{X}(x, 0) = X(x)$ for any $x \in M$. Let V be the infinitesimal variation induced by $\mathfrak{X}$ i.e. $V = (\partial X_t / \partial t)_{t=0}$ where $X_t(x) = \mathfrak{X}(x, t)$ for any $x \in M$, $|t| < \delta$. Then $g_\theta(V, X) = 0$ and

$$\frac{d}{dt} \{E(X_t)\}_{t=0} = - \int_M g_\theta(V, \Delta_b X) \, \Psi. \tag{14}$$

Consequently a C^∞ unit vector field X on M is subelliptic harmonic if and only if X is a C^∞ solution to

$$\Delta_b X + \|\nabla^H X\|^2 X = 0. \tag{15}$$

Cf. C.M. Wood, [29], for the Riemannian counterpart of (12) and (15). As a corollary (cf. again [13]) the characteristic direction T of $d\theta$ is a subelliptic harmonic vector field and an absolute minimum of the energy functional $E : \mathcal{U}(M, \theta) \to [0, +\infty)$. Moreover, for any nonempty open subset $\Omega \subseteq M$ and any unit vector field X on M such that $X \in H(M)$ there is a sequence $\{Y_\nu\}_{\nu \geq 1}$ of unit vector fields such that each Y_ν coincides with X outside Ω and $E(Y_\nu) \to \infty$ for $\nu \to \infty$. In particular the energy functional E is unbounded from above.

The problem of existence and regularity of weak solutions to (15) is open.

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