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Representation growth and representation zeta functions of groups

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Abstract. We give a short introduction to the subject of representation growth and representation zeta functions of groups, omitting all proofs. Our focus is on results which are relevant to the study of arithmetic groups in semisimple algebraic groups, such as the group $\mathrm{SL}_n(\mathbb{Z})$ consisting of $n \times n$ integer matrices of determinant 1. In the last two sections we state several results which were recently obtained in joint work with N. Avni, U. Onn and C. Voll.

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1 Introduction

Let G be a group. For $n \in \mathbb{N}$, let $r_n(G)$ denote the number of isomorphism classes of n-dimensional irreducible complex representations of G. We suppose that G is *representation rigid*, i.e., that $r_n(G) < \infty$ for all positive integers n.

If the group G is finite then G is automatically representation rigid and the sequence $r_n(G)$ has only finitely many non-zero terms, capturing the distribution of irreducible character degrees of G. The study of finite groups by means of their irreducible character degrees and conjugacy classes is a well established research area; e.g., see [10] and references therein. Interesting asymptotic phenomena are known to occur when one considers the irreducible character degrees of suitable infinite families of finite groups, for instance, families of finite groups H of Lie type as $|H|$ tends to infinity; see [14].

In the present survey we are primarily interested in the situation where G is infinite, albeit G may sometimes arise as an inverse limit of finite groups. Two fundamental questions in this case are: what are the arithmetic properties of the sequence $r_n(G)$, $n \in \mathbb{N}$, and what is the asymptotic behaviour of $R_N(G) = \sum_{n=1}^{N} r_n(G)$ as N tends to infinity? To a certain degree this line of investigation is inspired by the subject of subgroup growth and subgroup zeta functions which,

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in a similar way, is concerned with the distribution of finite index subgroups; e.g., see [16, 8].

In order to streamline the investigation it is convenient to encode the arithmetic sequence $r_n(G)$, $n \in \mathbb{N}$, in a suitable generating function. The *representation zeta function* of G is the Dirichlet generating function

$$\zeta_G(s) = \sum_{n=1}^{\infty} r_n(G)\, n^{-s} \qquad (s \in \mathbb{C}).$$

If the group G is such that there is a one-to-one correspondence between isomorphism classes of irreducible representations and irreducible characters then, writing $\mathrm{Irr}(G)$ for the space of irreducible characters of G, we can express the zeta function also in the suggestive and slightly more algebraic form

$$\zeta_G(s) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^{-s} \qquad (s \in \mathbb{C}).$$

The function $\zeta_G(s)$ is a suitable vehicle for studying the distribution of character degrees of the group G whenever the representation growth of G is 'not too fast', a condition which is made precise in Section 3. Groups which meet this requirement include, for instance, arithmetic groups in semisimple algebraic groups with the Congruence Subgroup Property and open compact subgroups of semisimple p-adic Lie groups. In recent years, several substantial results have been obtained concerning the representation growth and representation zeta functions of these types of groups; see [12, 13, 2, 1, 3, 4, 5, 6]. In the present survey we discuss some of these results and we indicate what kinds of methods are involved in proving them.

2 Finite groups of Lie type

Our primary focus is on infinite groups, but it is instructive to briefly touch upon representation zeta functions of finite groups of Lie type. For example, the representation theory of the general linear group $\mathrm{GL}_2(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ is well understood and one deduces readily that

$$\zeta_{\mathrm{GL}_2(\mathbb{F}_q)}(s) = (q-1)\left(1 + q^{-s} + \tfrac{q-2}{2}(q+1)^{-s} + \tfrac{q}{2}(q-1)^{-s}\right). \qquad (2.1)$$

It is remarkable that the formula (2.1) is uniform in q in the sense that both the irreducible character degrees and their multiplicities can be expressed in terms of polynomials in q over the rational field $\mathbb{Q}$. In general, Deligne-Lusztig theory provides powerful and sophisticated tools to study the irreducible characters of finite groups of Lie type. In [14], Liebeck and Shalev obtained, for instance, the following general asymptotic result.

Theorem 1 (Liebeck and Shalev)**.** *Let L be a fixed Lie type and let h be the Coxeter number of the corresponding simple algebraic group $\mathbf{G}$, i.e., $h+1 = \dim \mathbf{G}/\mathrm{rk}\mathbf{G}$. Then for the finite quasi-simple groups $L(q)$ of type L over $\mathbb{F}_q$,*

$$\zeta_{L(q)}(s) \to \begin{cases} 1 & \text{for } s \in \mathbb{R}_{>2/h} \\ \infty & \text{for } s \in \mathbb{R}_{<2/h} \end{cases} \qquad \text{as } q \to \infty.$$

The Coxeter number h is computed easily. For example, for $\mathbf{G} = \mathrm{SL}_n$ and $L(q) = \mathrm{SL}_n(\mathbb{F}_q)$ one has $h = n$. In the smallest interesting case $n = 2$ and for odd q, the zeta function of $\mathrm{SL}_2(\mathbb{F}_q)$ is

$$\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s) = 1+q^{-s}+\tfrac{q-3}{2}(q+1)^{-s}+\tfrac{q-1}{2}(q-1)^{-s}+2(\tfrac{q+1}{2})^{-s}+2(\tfrac{q-1}{2})^{-s}, \quad (2.2)$$

which is approximately the expression in (2.1) divided by $(q-1)$. From the explicit formula one can verify directly the assertion of Theorem 1 in this special case.

3 Abscissa of convergence and polynomial representation growth

In Section 1 we introduced the zeta function $\zeta_G(s)$ of a representation rigid group G as a formal Dirichlet series. Clearly, if G is finite – or more generally if G has only finitely many irreducible complex representations – then the Dirichlet polynomial $\zeta_G(s)$ defines an analytic function on the entire complex plane.

Now suppose that G is infinite and that $r_n(G)$ is non-zero for infinitely many $n \in \mathbb{N}$. Naturally, we are interested in the convergence properties of $\zeta_G(s)$ for $s \in \mathbb{C}$. The general theory of Dirichlet generating functions shows that the region of convergence is always a right half plane of $\mathbb{C}$, possibly empty, and that the resulting function is analytic. If the region of convergence is non-empty, one is also interested in meromorphic continuation of the function to a larger part of the complex plane.

The *abscissa of convergence* $\alpha(G)$ of $\zeta_G(s)$ is the infimum of all $\alpha \in \mathbb{R}$ such that the series $\zeta_G(s)$ converges (to an analytic function) on the right half plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > \alpha\}$. The abscissa $\alpha(G)$ is finite if and only if G has *polynomial representation growth*, i.e., if $R_N(G) = \sum_{n=1}^{N} r_n(G)$ grows at most polynomially in N. In fact, if the growth sequence $R_N(G)$, $N \in \mathbb{N}$, is unbounded then

$$\alpha(G) = \limsup_{N\to\infty} \frac{\log R_N(G)}{\log N}$$

gives the *polynomial degree of growth*: $R_N(G) = O\left(N^{\alpha(G)+\varepsilon}\right)$ for every $\varepsilon > 0$.

Two fundamental problems in the subject are: to characterise groups of polynomial representation growth – motivated by Gromow's celebrated theorem on groups of polynomial word growth – and to link the actual value of the abscissa of convergence $\alpha(G)$ of a group G to structural properties of G. In general these questions are still very much open. However, in the context of semisimple algebraic groups and their arithmetic subgroups a range of results have been obtained. A selection of these are discussed in the following sections.

4 Witten zeta functions

In [20], Witten initiated in the context of quantum gauge theories the study of certain representation zeta functions. Let $\mathbf{G}$ be a connected, simply connected, complex almost simple algebraic group and let $G = \mathbf{G}(\mathbb{C})$. It is natural to focus on rational representations of the algebraic group G and one can show that G is representation rigid in this restricted sense. The *Witten zeta function* $\zeta_G(s)$ counts irreducible rational representations of the complex algebraic group G. These zeta functions also appear naturally as archimedean factors of representation zeta functions of arithmetic groups, as explained in Section 8.

For example, the group $\mathrm{SL}_2(\mathbb{C})$ has a unique irreducible representation of each possible degree. Hence

$$\zeta_{\mathrm{SL}_2(\mathbb{C})}(s) = \sum_{n=1}^{\infty} n^{-s},$$

the famous Riemann zeta function. In particular, the abscissa of convergence is 1 and there is a meromorphic continuation to the entire complex plane.

In general, the irreducible representations V_λ of G are parametrised by their highest weights $\lambda = \sum_{i=1}^{r} a_i \overline{\omega}_i$, where $\overline{\omega}_1, \ldots, \overline{\omega}_r$ denote the fundamental weights and the coefficients $a_1, \ldots, a_r$ range over all non-negative integers. Moreover, $\dim V_\lambda$ is given by the Weyl dimension formula. By a careful analysis, Larsen and Lubotzky prove in [13] the following result.

Theorem 2 (Larsen and Lubotzky)**.** *Let* $\mathbf{G}$ *be a connected, simply connected, complex almost simple algebraic group and let* $G = \mathbf{G}(\mathbb{C})$*. Then* $\alpha(G) = 2/h$*, where* h *is the Coxeter number of* $\mathbf{G}$*.*

It is known that Witten zeta functions can be continued meromorphically to the entire complex plane. Further analytic properties of these functions, such as the location of singularities and functional relations, have been investigated in some detail using multiple zeta functions; e.g., see [18, 11]. It is remarkable that the same invariant $2/h$ features in Theorems 1 and 2. Currently there appears to be no conceptual explanation for this.

5 The group $\mathrm{SL}_2(R)$ for discrete valuation rings R

If G is a topological group it is natural to focus attention on continuous representations. A finitely generated profinite group G is representation rigid in this restricted sense if and only if it is *FAb*, i.e., if every open subgroup H of G has finite abelianisation $H/[H,H]$. This is a consequence of Jordan's theorem on abelian normal subgroups of bounded index in finite linear groups. We tacitly agree that the representation zeta function $\zeta_G(s)$ of a finitely generated FAb profinite group G counts irreducible continuous complex representations of G.

Let R be a complete discrete valuation ring, with residue field $\mathbb{F}_q$ of odd characteristic. This means that R is either a finite integral extension of the ring of p-adic integers $\mathbb{Z}_p$ for some prime p or a formal power series ring $\mathbb{F}_q[\![t]\!]$ over a finite field of cardinality q.

In [12], Jaikin-Zapirain showed by a hands-on computation of character degrees that the representation zeta function $\zeta_{\mathrm{SL}_2(R)}(s)$ equals

$$\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s) + \left(4q\left(\tfrac{q^2-1}{2}\right)^{-s} + \tfrac{q^2-1}{2}(q^2-q)^{-s} + \tfrac{(q-1)^2}{2}(q^2+q)^{-s}\right)/(1-q^{1-s}),$$

where the Dirichlet polynomial $\zeta_{\mathrm{SL}_2(\mathbb{F}_q)}(s)$ is described in (2.2). It is remarkable that the above formula is uniform in q, irrespective of the characteristic, absolute ramification index or isomorphism type of the ring R. In the case where R has characteristic 0, Lie-theoretic techniques combined with Clifford theory can be used to gain an insight into the features of this specific example which hold more generally; see Sections 6 and 9.

Clearly, the explicit formula for the function $\zeta_{\mathrm{SL}_2(R)}(s)$ provides a meromorphic extension to the entire complex plane. The abscissa of convergence is 1 and, in view of Theorems 1 and 2, this value could be interpreted as $2/h$, the Coxeter number of SL_2 being $h=2$. But such an interpretation is misleading, as can be seen from the following general result obtained in [13].

Theorem 3 (Larsen and Lubotzky)**.** *Let* $\mathbf{G}$ *be a simple algebraic group over a non-archimedean local field* F*. Suppose that* $\mathbf{G}$ *is* F*-isotropic, i.e.,* $\mathrm{rk}_F\mathbf{G} \geq 1$*. Let* H *be a compact open subgroup of* $\mathbf{G}(F)$*. Then* $\alpha(H) \geq 1/15$*.*

Taking $\mathbf{G} = \mathrm{SL}_n$ and $F = \mathbb{Q}_p$, we may consider the compact p-adic Lie groups $\mathrm{SL}_n(\mathbb{Z}_p)$. For these groups $2/h = 2/n \to 0$ as $n \to \infty$, whereas $\alpha(\mathrm{SL}_n(\mathbb{Z}_p))$ is uniformly bounded away from 0. Currently, the only explicit values known for $\alpha(\mathrm{SL}_n(\mathbb{Z}_p))$ are: 1 for $n=2$ (as seen above), and 2/3 for $n=3$ (see [4]). Unfortunately, these do not yet indicate the general behaviour.

6 FAb compact p-adic Lie groups

Let G be a compact p-adic Lie group. Then one associates to G a $\mathbb{Q}_p$-Lie algebra as follows. The group G contains a uniformly powerful open pro-p subgroup U. By the theory of powerful pro-p groups, U gives rise to a $\mathbb{Z}_p$-Lie lattice $L = \log(U)$ and the induced $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ does not depend on the specific choice of U. It is a fact that G is FAb if and only if $\mathcal{L}(G)$ is perfect, i.e., if $[\mathcal{L}(G), \mathcal{L}(G)] = \mathcal{L}(G)$. Conversely, for any $\mathbb{Q}_p$-Lie algebra $\mathcal{L}$ one can easily produce compact p-adic Lie groups G such that $\mathcal{L}(G) = \mathcal{L}$, using the exponential map. This supplies a large class of compact p-adic Lie groups which are FAb and hence have polynomial representation growth.

Using the Kirillov orbit method and techniques from model theory, Jaikin-Zapirain established in [12] that the representation zeta function of a FAb compact p-adic analytic pro-p group can always be expressed as a rational function in p^{-s} over $\mathbb{Q}$. More generally, he proved the following result, which is illustrated by the explicit example $G = \mathrm{SL}_2(R)$ given in Section 5.

Theorem 4 (Jaikin-Zapirain). *Let G be an FAb compact p-adic Lie group, and suppose that $p > 2$. Then there are finitely many positive integers $n_1, \ldots, n_k$ and rational functions $f_1, \ldots, f_k \in \mathbb{Q}(X)$ such that*

$$\zeta_G(s) = \sum_{i=1}^{k} f_i(p^{-s})\, n_i^{-s}.$$

In particular, the theorem shows that the zeta function of a FAb compact p-adic Lie group G extends meromorphically to the entire complex plane. The invariant $\alpha(G)$ is the largest real part of a pole of $\zeta_G(s)$. It is natural to investigate the whole spectrum of poles and zeros of $\zeta_G(s)$.

Currently, very little is known about the location of the zeros of representation zeta functions. In 2010 Kurokawa and Kurokawa observed from the explicit formula given in Section 5 that $\zeta_{\mathrm{SL}_2(\mathbb{Z}_p)}(s) = 0$ for $s \in \{-1, -2\}$. We note that if G is a finite group then $\zeta_G(-2) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2 = |G|$. Based on this fact and the results in [12] one can prove the following general result.

Theorem 5 (Jaikin-Zapirain and Klopsch). *Let G be an infinite FAb compact p-adic Lie group and suppose that $p > 2$. Then $\zeta_G(-2) = 0$.*

7 Rational representations of the infinite cyclic group

Before considering arithmetic subgroups of semisimple algebraic groups, let us look at representations of the simplest infinite group, i.e., the infinite cyclic group C_∞. The group C_∞ has already infinitely many 1-dimensional representations. Hence in order to say anything meaningful we need to slightly adapt our basic definitions.

We make two modifications: firstly let us only consider representations with finite image and secondly let us consider irreducible representations over $\mathbb{Q}$ rather than $\mathbb{C}$. More precisely, for any finitely generated nilpotent group Γ let $\hat{r}_n^{\mathbb{Q}}(\Gamma)$ denote the number of n-dimensional irreducible representations of Γ over $\mathbb{Q}$ with finite image. Then it turns out that $\hat{r}_n^{\mathbb{Q}}(\Gamma)$ is finite for every $n \in \mathbb{N}$ and we can define the *$\mathbb{Q}$-rational representation zeta function*

$$\zeta_\Gamma^{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \hat{r}_n^{\mathbb{Q}}(\Gamma)\, n^{-s}.$$

Using that every finite nilpotent group is the direct product of its Sylow p-subgroups and basic facts from character theory, one can show that $\zeta_\Gamma^{\mathbb{Q}}(s)$ admits an *Euler product decomposition*

$$\zeta_\Gamma^{\mathbb{Q}}(s) = \prod_{p \text{ prime}} \zeta_{\Gamma,p}^{\mathbb{Q}}(s), \tag{7.1}$$

where for each prime p the local factor $\zeta_{\Gamma,p}^{\mathbb{Q}}(s) = \sum_{k=0}^{\infty} \hat{r}_{p^k}^{\mathbb{Q}}(\Gamma)\, p^{-ks}$, enumerating irreducible representations of p-power dimension, can be re-interpreted as the $\mathbb{Q}$-rational representation zeta function of the pro-p completion $\widehat{\Gamma}_p$ of Γ. For more details and deeper results in this direction we refer to the forthcoming article [9].

Let us now return to the simplest case: $\Gamma = C_\infty$, the infinite cyclic group. Since the group C_∞ is abelian, its irreducible representations over $\mathbb{Q}$ with finite image can be effectively described by means of Galois orbits of irreducible complex characters. In the general setting, one would also need to keep track of Schur indices featuring in the computation of $\zeta_{\Gamma,2}^{\mathbb{Q}}(s)$. A short analysis yields

$$\zeta_{C_\infty}^{\mathbb{Q}}(s) = \sum_{m=1}^{\infty} \varphi(m)^{-s},$$

where φ denotes Euler's function familiar from elementary number theory.

The Dirichlet series $\psi(s) = \sum_{m=1}^{\infty} \varphi(m)^{-s}$ is of independent interest in analytic number theory and has been studied by many authors; e.g., see [7]. The Euler product decomposition (7.1) can be established directly

$$\psi(s) = \prod_{p \text{ prime}} \left(1 + (p-1)^{-s}/(1-p^{-s})\right).$$

The abscissa of convergence of $\psi(s)$, which can be interpreted as the degree $\alpha^{\mathbb{Q}}(C_\infty)$ of $\mathbb{Q}$-rational representation growth, is equal to 1. In fact, writing

$$\psi(s) = \underbrace{\prod_{p \text{ prime}} \left(1 + (p-1)^{-s} - p^{-s}\right)}_{\text{converges for } \mathrm{Re}(s) > 0} \cdot \underbrace{\prod_{p \text{ prime}} (1 - p^{-s})^{-1}}_{\text{Riemann zeta function } \zeta(s)} ,$$

one sees that $\psi(s)$ admits a meromorphic continuation to $\mathrm{Re}(s) > 0$ (but not to the entire complex plane) and has a simple pole at $s = 1$ with residue $c = \zeta(2)\zeta(3)/\zeta(6) = 1.9435964\ldots$ This yields very precise asymptotics for the $\mathbb{Q}$-rational representation growth of C_∞; in particular,

$$\sum_{n=1}^{N} \hat{r}_n^{\mathbb{Q}}(C_\infty) = \#\{m \mid \varphi(m) \leq N\} \sim cN \qquad \text{as } N \to \infty.$$

One may regard this simple case and its beautiful connections to classical analytic number theory as a further motivation for studying representation zeta functions of arithmetic groups.

8 Arithmetic lattices in semisimple groups

In this section we turn our attention to lattices in semisimple locally compact groups. These lattices are discrete subgroups of finite co-volume and often, but not always, have arithmetic origin. For instance, $\mathrm{SL}_n(\mathbb{Z})$ is an arithmetic lattice in the real Lie group $\mathrm{SL}_n(\mathbb{R})$. More generally, let Γ be an arithmetic irreducible lattice in a semisimple locally compact group G of characteristic 0. Then Γ is commensurable to $\mathbf{G}(\mathcal{O}_S)$, where $\mathbf{G}$ is a connected, simply connected absolutely almost simple algebraic group defined over a number field k and $\mathcal{O}_S$ is the ring of S-integers for a finite set S of places of k. By a theorem going back to Borel and Harish-Chandra, any such $\mathbf{G}(\mathcal{O}_S)$ forms an irreducible lattice in the semisimple locally compact group $G = \prod_{\wp \in S} \mathbf{G}(k_\wp)$ under the diagonal embedding, as long as S is non-empty and contains all archimedean places $\wp$ such that $\mathbf{G}(k_\wp)$ is non-compact. Examples of this construction are $\mathrm{SL}_n(\mathbb{Z}[\sqrt{2}]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$. Margulis has shown that in the higher rank situation all irreducible lattices are arithmetic and arise in this way. For precise notions and a more complete description see [17].

Throughout the following we assume, for simplicity of notation, that $\Gamma = \mathbf{G}(\mathcal{O}_S)$ as above. In [15], Lubotzky and Martin gave a characterisation of arithmetic groups of polynomial representation growth, linking them to the classical Congruence Subgroup Problem.

Theorem 6 (Lubotzky and Martin)**.** *Let Γ be an arithmetic group as above. Then $\alpha(\Gamma)$ is finite if and only if Γ has the Congruence Subgroup Property.*

The group Γ has the Congruence Subgroup Property (CSP) if, essentially, all its finite index subgroups arise from the arithmetic structure of the group. Technically, this means that the congruence kernel $\ker(\widehat{\mathbf{G}(\mathcal{O}_S)} \to \overline{\mathbf{G}(\mathcal{O}_S)})$ is finite; here $\widehat{\mathbf{G}(\mathcal{O}_S)}$ is the profinite completion and $\overline{\mathbf{G}(\mathcal{O}_S)} \cong \prod_{\mathfrak{p} \notin S} \mathbf{G}(\mathcal{O}_\mathfrak{p})$, with $\mathfrak{p}$ running over non-archimedean places, denotes the congruence completion of $\mathbf{G}(\mathcal{O}_S)$. For instance, it was shown by Bass-Lazard-Serre and Mennicke that the group $\mathrm{SL}_n(\mathbb{Z})$ has the CSP if and only if $n \geq 3$. That $\mathrm{SL}_2(\mathbb{Z})$ does not have the CSP was discovered by Fricke and Klein. Retrospectively this is not surprising, because $\mathrm{SL}_2(\mathbb{Z})$ contains a free subgroup of finite index. We refer to [19] for a comprehensive survey of the Congruence Subgroup Problem, i.e., the problem to decide precisely which arithmetic groups have the CSP.

Suppose that Γ has the CSP. Using Margulis' super-rigidity theorem, Larsen and Lubotzky derived in [13] an Euler product decomposition for $\zeta_\Gamma(s)$, which takes a particularly simple form whenever the congruence kernel is trivial.

Theorem 7 (Larsen and Lubotzky)**.** *Let Γ be an arithmetic group as above and suppose that Γ has the CSP. Then $\zeta_\Gamma(s)$ admits an Euler product decomposition. In particular, if the congruence kernel for $\Gamma = \mathbf{G}(\mathcal{O}_S)$ is trivial then*

$$\zeta_\Gamma(s) = \zeta_{\mathbf{G}(\mathbb{C})}(s)^{[k:\mathbb{Q}]} \prod_{\mathfrak{p} \notin S} \zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s). \tag{8.1}$$

For instance, for the groups $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, the Euler product takes the form

$$\zeta_{\mathrm{SL}_n(\mathbb{Z})}(s) = \zeta_{\mathrm{SL}_n(\mathbb{C})}(s) \prod_{p \text{ prime}} \zeta_{\mathrm{SL}_n(\mathbb{Z}_p)}(s).$$

In Sections 4 and 5 we already encountered individually the factors of these Euler products: $\zeta_{\mathbf{G}(\mathbb{C})}(s)$ is the Witten zeta function capturing rational representations of the algebraic group $\mathbf{G}(\mathbb{C})$ and, for each $\mathfrak{p}$, the function $\zeta_{\mathbf{G}(\mathcal{O}_\mathfrak{p})}(s)$ enumerates continuous representations of the compact p-adic Lie group $\mathbf{G}(\mathcal{O}_\mathfrak{p})$. Larsen and Lubotzky's results for the abscissae of convergence of these local zeta functions include Theorems 2 and 3 stated above.

Regarding the abscissa of convergence of the global representation zeta function, Avni employed in [1] model-theoretic techniques to prove that the abscissa of convergence of $\zeta_\Gamma(s)$ is always a rational number. In [13], Larsen and Lubotzky made the following conjecture, which can be regarded as a refinement of Serre's conjecture on the Congruence Subgroup Problem.

Conjecture 1 (Larsen and Lubotzky)**.** *Let G be a higher-rank semisimple locally compact group. Then, for any two irreducible lattices Γ_1 and Γ_2 in G, $\alpha(\Gamma_1) = \alpha(\Gamma_2)$.*

Roughly speaking, the conjecture states that the ambient semisimple locally compact group does not only control whether lattices contained in it have the CSP (as in Serre's conjecture), but also what their polynomial degree of representation growth is. A concrete example of a lattice in $\mathrm{SL}_n(\mathbb{R})$ which is rather different from the most familiar one $\mathrm{SL}_n(\mathbb{Z})$ is the special unitary group $\mathrm{SU}_n(\mathbb{Z}[\sqrt{2}], \mathbb{Z})$, consisting of all matrices $A = (a_{ij})$ over the ring $\mathbb{Z}[\sqrt{2}]$ with $\det A = 1$ and $A^{-1} = (a_{ji}^{\sigma})$, where σ is the Galois automorphism of $\mathbb{Q}(\sqrt{2})$ swapping $\sqrt{2}$ and $-\sqrt{2}$.

9 New results for arithmetic groups and compact p-adic Lie groups

The short announcement [2] summarises a number of results obtained recently by the author in joint work with Avni, Onn and Voll. Details are appearing in [3, 4, 6]. The toolbox which we use to prove our results comprises a variety of techniques which can only be hinted at: they include, for instance, the Kirillov orbit method for p-adic analytic pro-p groups, methods from $\mathfrak{p}$-adic integration and the study of generalised Igusa zeta functions, the theory of sheets of simple Lie algebras, resolution of singularities in characteristic 0, aspects of the Weil conjectures regarding zeta functions of smooth projective varieties over finite fields, approximative and exact Clifford theory.

In summary our main results are

- a global *Denef formula* for the zeta functions of principal congruence subgroups of compact p-adic Lie groups, such as $\mathrm{SL}_n^m(\mathbb{Z}_p) \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$;
- local *functional equations* for the zeta functions of principal congruence subgroups of compact p-adic Lie groups, such as $\mathrm{SL}_n^m(\mathbb{Z}_p) \subseteq \mathrm{SL}_n(\mathbb{Z}_p)$;
- *candidate pole sets* for the non-archimedean factors occurring in the Euler product (8.1), e.g., the zeta functions $\zeta_{\mathrm{SL}_n(\mathbb{Z}_p)}(s)$;
- *explicit formulae* for the zeta functions of compact p-adic Lie groups of type A_2, such as $\mathrm{SL}_3(\mathbb{Z}_p)$ and $\mathrm{SU}_3(\mathfrak{O}, \mathbb{Z}_p)$ for unramified $\mathfrak{O}$;
- *meromorphic continuation* of zeta functions and a precise *asymptotic description* of the representation growth for arithmetic groups of type A_2, such as $\mathrm{SL}_3(\mathbb{Z})$.

These results are clearly relevant in the context of the Euler product (8.1). Moreover, a large part of our work applies in a more general context than discussed so far. We recall from Section 6 that a compact p-adic Lie group G is representation rigid if and only if its $\mathbb{Q}_p$-Lie algebra $\mathcal{L}(G)$ is perfect. Let k be a number field, and let $\mathcal{O}$ be its ring of integers. Let Λ be an $\mathcal{O}$-Lie lattice such that $k \otimes_{\mathcal{O}} \Lambda$ is perfect of dimension d. Let $\mathfrak{o}$ be the completion $\mathcal{O}_{\mathfrak{p}}$ of $\mathcal{O}$ at a non-archimedean place $\mathfrak{p}$. Let $\mathfrak{O}$ be a finite integral extension of $\mathfrak{o}$, corresponding to a place $\mathfrak{P}$ lying above $\mathfrak{p}$. For $m \in \mathbb{N}$, let $\mathfrak{g}^m(\mathfrak{O})$ denote the mth principal congruence Lie sublattice of the $\mathfrak{O}$-Lie lattice $\mathfrak{O} \otimes_{\mathcal{O}} \Lambda$. For sufficiently large m, let $\mathsf{G}^m(\mathfrak{O})$ be the p-adic analytic pro-p group $\exp(\mathfrak{g}^m(\mathfrak{O}))$.

Using the Kirillov orbit method for permissible $\mathsf{G}^m(\mathfrak{O})$, e.g., $\mathrm{SL}_n^1(\mathbb{Z}_p)$, we can 'linearise' the problem of enumerating irreducible characters of the group $\mathsf{G}^m(\mathfrak{O})$ by their degrees. We then set up a generalised Igusa zeta function, i.e., a p-adic integral of the form

$$\mathcal{Z}_{\mathfrak{O}}(r,t) = \int_{(x,\mathbf{y}) \in V(\mathfrak{O})} |x|_{\mathfrak{P}}^t \prod_{j=1}^{\lfloor d/2 \rfloor} \frac{\|F_j(\mathbf{y}) \cup F_{j-1}(\mathbf{y}) x^2\|_{\mathfrak{P}}^r}{\|F_{j-1}(\mathbf{y})\|_{\mathfrak{P}}^r} \, d\mu(x,\mathbf{y}),$$

where $V(\mathfrak{O}) \subset \mathfrak{O}^{d+1}$ is a union of cosets modulo $\mathfrak{P}$, $F_j(\mathbf{Y}) \subset \mathcal{O}[\mathbf{Y}]$ are polynomial sets defined in terms of the structure constants of the underlying $\mathcal{O}$-Lie lattice Λ, $\|\cdot\|_{\mathfrak{P}}$ is the $\mathfrak{P}$-adic maximum norm and μ is the additive Haar measure on $\mathfrak{O}^{d+1}$ with $\mu(\mathfrak{O}^{d+1}) = 1$. The integral $\mathcal{Z}_{\mathfrak{O}}(r,t)$ allows us to treat 'uniformly' the representation zeta functions of the different groups $\exp(\mathsf{G}^m(\mathfrak{O}))$ arising from the global $\mathcal{O}$-Lie lattice Λ under variation of the place $\mathfrak{p}$ of $\mathcal{O}$, the local ring extension $\mathfrak{O}$ of $\mathcal{O}_{\mathfrak{p}}$ and the congruence level m. In particular, we derive from our analysis a Denef formula and local functional equations.

Theorem 8 (Avni, Klopsch, Onn and Voll [4]). *In the setup described, there exist $r \in \mathbb{N}$ and a rational function $R(X_1, \ldots, X_r, Y) \in \mathbb{Q}(X_1, \ldots, X_r, Y)$ such that for almost every non-archimedean place $\mathfrak{p}$ of k the following holds.*

There are algebraic integers $\lambda_1, \ldots, \lambda_r$ such that for all finite extensions $\mathfrak{O}$ of $\mathfrak{o} = \mathcal{O}_{\mathfrak{p}}$ and all permissible m one has

$$\zeta_{\mathsf{G}^m(\mathfrak{O})}(s) = q_{\mathfrak{p}}^{fdm} R(\lambda_1^f, \ldots, \lambda_r^f, q_{\mathfrak{p}}^{-fs}),$$

where $q_{\mathfrak{p}}$ is the residue field cardinality of $\mathfrak{o}$, f denotes the inertia degree of $\mathfrak{O}$ over $\mathfrak{o}$ and $d = \dim_k(k \otimes_{\mathcal{O}} \Lambda)$. Moreover, there is the functional equation

$$\zeta_{\mathsf{G}^m(\mathfrak{O})}(s)\big|_{\substack{q_{\mathfrak{p}} \to q_{\mathfrak{p}}^{-1} \\ \lambda_i \to \lambda_i^{-1}}} = q_{\mathfrak{p}}^{fd(1-2m)} \zeta_{\mathsf{G}^m(\mathfrak{O})}(s).$$

Furthermore, we obtain candidate pole sets and we show that, locally, abscissae of convergence are monotone under ring extensions.

Theorem 9 (Avni, Klopsch, Onn and Voll [4]). *In the setup described, there exists a finite set $P \subset \mathbb{Q}_{>0}$ such that the following is true.*

For all non-archimedean places $\mathfrak{p}$ of k, all finite extensions $\mathfrak{O}$ of $\mathfrak{o} = \mathcal{O}_\mathfrak{p}$ and all permissible m one has

$$\big\{\mathrm{Re}(z) \mid z \in \mathbb{C} \text{ a pole of } \zeta_{\mathsf{G}^m(\mathfrak{O})}(s)\big\} \subseteq P.$$

In particular, one has $\alpha(\mathsf{G}^m(\mathfrak{O})) \leq \max P$, and equality holds for a set of positive Dirichlet density.

Furthermore, if $\mathfrak{p}$ is any non-archimedean place of k and if $\mathcal{O}_\mathfrak{p} = \mathfrak{o} \subseteq \mathfrak{O}_1 \subseteq \mathfrak{O}_2$ is a tower of finite ring extensions, then for every permissible m one has

$$\alpha(\mathsf{G}^m(\mathfrak{O}_1)) \leq \alpha(\mathsf{G}^m(\mathfrak{O}_2)).$$

By a more detailed study of groups of type A_2, we obtain the following theorems addressing, in particular, the conjecture of Larsen and Lubotzky stated in Section 8. Analysing the unique subregular sheet of the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ and using approximative Clifford theory, we prove the next result.

Theorem 10 (Avni, Klopsch, Onn and Voll [4]). *Let Γ be an arithmetic subgroup of a connected, simply connected simple algebraic group of type A_2 defined over a number field. If Γ has the CSP, then $\alpha(\Gamma) = 1$.*

Employing exact Clifford theory, we obtain the following more detailed result for the special linear group $\mathrm{SL}_3(\mathcal{O})$ over the ring of integers of a number field.

Theorem 11 (Avni, Klopsch, Onn and Voll [6]). *Let $\mathcal{O}$ be the ring of integers of a number field k. Then there exists $\varepsilon > 0$ such that the representation zeta function of $\mathrm{SL}_3(\mathcal{O})$ admits a meromorphic continuation to the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) > 1 - \varepsilon\}$. The continued function is analytic on the line $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1\}$, except for a double pole at $s = 1$.*

Consequently, there is a constant $c \in \mathbb{R}_{>0}$ such that

$$R_N(\mathrm{SL}_3(\mathcal{O})) = \sum_{n=1}^{N} r_n(\mathrm{SL}_3(\mathcal{O})) \sim c \cdot N(\log N) \qquad \text{as } N \to \infty.$$

A key step in proving this result consists in deriving explicit formulae for the representation zeta function of groups $\mathrm{SL}_3(\mathfrak{o})$, where $\mathfrak{o}$ is a compact discrete valuation ring of characteristic 0 and residue field characteristic different from 3. In fact, we also derive similar results for special unitary groups $\mathrm{SU}_3(\mathcal{O}, \mathfrak{o})$.

10 New results regarding the conjecture of Larsen and Lubotzky

Very recently, in joint work with Avni, Onn and Voll we prove the following theorem in connection with the conjecture of Larsen and Lubotzky which is stated in Section 8.

Theorem 12 (Avni, Klopsch, Onn and Voll [5]). *Let Φ be an irreducible root system. Then there exists a constant α_Φ such that for every number field k with ring of integers $\mathcal{O}$, every finite set S of places of k and every connected, simply connected absolutely almost simple algebraic group $\mathbf{G}$ over k with absolute root system Φ the following holds.*

If the arithmetic group $\mathbf{G}(\mathcal{O}_S)$ has polynomial representation growth, then $\alpha(\mathbf{G}(\mathcal{O}_S)) = \alpha_\Phi$.

On the one hand, Theorem 12 is weaker than the conjecture of Larsen and Lubotzky, because it does not resolve Serre's conjecture on the Congruence Subgroup Problem. However, Serre's conjecture is known to be true in many cases and we have the following corollary.

Corollary 1. *Serre's conjecture on the Congruence Subgroup Problem implies Larsen and Lubotzky's conjecture on the degrees of representation growth of lattices in higher rank semisimple locally compact groups.*

On the other hand, Theorem 12 is stronger than the conjecture of Larsen and Lubotzky, because it shows that many arithmetic groups with the CSP have the same degree of representation growth, even when they do not embed as lattices into the same semisimple locally compact group. For instance, fixing Φ of type A_{n-1} for some $n \geq 3$, all of the following groups (for which we also display their embeddings as lattices into semisimple locally compact groups) have the same degree of representation growth:

(1) $\mathrm{SL}_n(\mathbb{Z}) \subseteq \mathrm{SL}_n(\mathbb{R})$,

(2) $\mathrm{SL}_n(\mathbb{Z}[\sqrt{2}]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$,

(3) $\mathrm{SL}_n(\mathbb{Z}[i]) \subseteq \mathrm{SL}_n(\mathbb{C})$,

(4) $\mathrm{SL}_n(\mathbb{Z}[1/p]) \subseteq \mathrm{SL}_n(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{Q}_p)$,

(5) $\mathrm{SU}_n(\mathbb{Z}[\sqrt{2}], \mathbb{Z}) \subseteq \mathrm{SL}_n(\mathbb{R})$.

Presently, the only known explicit values of α_Φ are: 2 for Φ of type A_1 (see [13]), and 1 for Φ of type A_2 (see Theorem 10). It remains a challenging problem to find a conceptual interpretation of α_Φ for general Φ.

For the proof of Theorem 12 and further details we refer to the preprint [5].

References

[1] N. Avni: *Arithmetic groups have rational representation growth*, Ann. of Math., **174** (2011), 1009–1056.

[2] N. Avni, B. Klopsch, U. Onn, C. Voll: *On representation zeta functions of groups and a conjecture of Larsen-Lubotzky*, C. R. Math. Acad. Sci. Paris, Ser. I, **348** (2010), 363–367.

[3] N. Avni, B. Klopsch, U. Onn, C. Voll: *Representation zeta functions of some compact p-adic analytic groups*, in: Zeta Functions in Algebra and Geometry, Contemporary Mathematics 566, Amer. Math. Soc., Providence, RI, 2012, 295–330.

[4] N. Avni, B. Klopsch, U. Onn, C. Voll: *Representation zeta functions of compact p-adic analytic groups and arithmetic groups*, to appear in Duke. Math. J. **162** (2013), 111–197.

[5] N. Avni, B. Klopsch, U. Onn, C. Voll: *Arithmetic groups, base change and representation growth*, preprint `arXiv:1110.6092v2`, 2012.

[6] N. Avni, B. Klopsch, U. Onn, C. Voll: *Representation zeta functions for* SL_3 *and* SU_3, in preparation.

[7] P. T. Bateman: *The distribution of values of the Euler function*, Acta Arith., **21** (1972), 329–345.

[8] M. du Sautoy, F. Grunewald: *Zeta functions of groups and rings*, in: International Congress of Mathematicians, Vol. II, Eur. Math. Soc., Zürich, 2006, 131–149.

[9] J. González-Sánchez, A. Jaikin-Zapirain, B. Klopsch: *Zeta functions of rational representations*, in preparation.

[10] B. Huppert: Character theory of finite groups, Walter de Gruyter & Co., Berlin, 1998.

[11] Y. Komori, K. Matsumoto, H. Tsumura: *On Witten multiple zeta-functions associated with semisimple Lie algebras II*, J. Math. Soc. Japan, **62** (2010), 355–394.

[12] A. Jaikin-Zapirain: *Zeta function of representations of compact p-adic analytic groups*, J. Amer. Math. Soc., **19** (2006), 91–118.

[13] M. Larsen, A. Lubotzky: *Representation growth of linear groups*, J. Eur. Math. Soc. (JEMS), **10** (2008), 351–390.

[14] M. Liebeck, A. Shalev: *Character degrees and random walks in finite groups of Lie type*, Proc. London Math. Soc., **90** (2005), 61–86.

[15] A. Lubotzky, B. Martin: *Polynomial representation growth and the congruence subgroup problem*, Israel J. Math., **144** (2004), 293–316.

[16] A. Lubotzky, D. Segal: Subgroup growth, Birkhäuser-Verlag, Basel, 2003.

[17] G. A. Margulis: *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, Berlin, 1991.

[18] K. Matsumoto: *On Mordell-Tornheim and other multiple zeta-functions*, in: Proceedings of the Session in Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften, **360**, Univ. Bonn, Bonn, 2003.

[19] G. Prasad, A. Rapinchuk: *Developments on the Congruence Subgroup Problem after the work of Bass, Milnor and Serre*, in: Collected papers of John Milnor. V: Algebra (ed. H. Bass and T. Y. Lam), Amer. Math. Soc., Providence, RI, 2010.

[20] E. Witten: *On quantum gauge theories in two dimensions*, Comm. Math. Phys., **141** (1991), 153–209.