Northumbria Research Link

www.northumbria.ac.uk/nrl

northumbria
UNIVERSITY NEWCASTLE

Automorphic Lie Algebras with dihedral symmetry

V Knibbeler[1], S Lombardo[1] and J A Sanders[2]

1 Department of Mathematics and Information Sciences, Northumbria University, UK
2 Department of Mathematics, Vrije Universiteit Amsterdam, The Netherlands

E-mail: `vknibbeler@gmail.com`

Abstract. The concept of *Automorphic Lie Algebras* arises in the context of reduction groups introduced in the early 1980s in the field of integrable systems. Automorphic Lie Algebras are obtained by imposing a discrete group symmetry on a current algebra of Krichever-Novikov type. Past work shows remarkable uniformity between algebras associated to different reduction groups. For example, if the base Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ and the poles of the Automorphic Lie Algebra are restricted to an exceptional orbit of the symmetry group, changing the reduction group does not affect the Lie algebra structure. In the present research we fix the reduction group to be the dihedral group and vary the orbit of poles as well as the group action on the base Lie algebra. We find a uniform description of Automorphic Lie Algebras with dihedral symmetry, valid for poles at exceptional and generic orbits.

1. Introduction

Automorphic Lie Algebras were introduced in the context of the classification of integrable partial differential equations [1, 2, 3]. In fact, the Zakharov-Shabat / Ablowitz-Kaup-Newell-Segur scheme, used to integrate these equations, requires a pair of matrix valued functions of the spectral parameter $\lambda \in \overline{\mathbb{C}}$ living on the Riemann sphere. Since a general pair of such λ-dependent matrices gives rise to an under determined system of differential equations, one requires additional constraints. By a well established scheme introduced by Mikhailov [4], and further developed in [3], this can be achieved by imposing a group symmetry on the matrices. Algebras consisting of all such symmetric matrices are called *Automorphic Lie Algebras*, by analogy with automorphic functions. Since their introduction they have been extensively studied (see, [5] and references therein, but also [6, 7]).

Definition 1.1 (Automorphic Lie Algebras, [1, 3]). *Let $\mathfrak{g}$ be a simple Lie algebra, $\mathcal{M}(\overline{\mathbb{C}})$ the field of meromorphic functions on the Riemann sphere, and G a finite subgroup of $Aut(\mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}}))$; the* Automorphic Lie Algebra *is the space of invariants*

$$\left(\mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}}) \right)_{\Gamma}^{G} = \left\{ f \in \mathfrak{g} \otimes \mathcal{M}(\overline{\mathbb{C}}) \mid f \text{ analytic on } \overline{\mathbb{C}} \setminus \Gamma, \ gf = f \ \forall g \in G \right\}$$

where $\Gamma \subset \overline{\mathbb{C}}$ is a single G-orbit where poles are allowed.

Algebras of meromorphic functions, taking values in a finite dimensional Lie algebra and having their poles restricted to a finite set, are *current algebras of Krichever-Novikov type* [8, 9]. Automorphic Lie Algebras are subalgebras thereof. They are infinite dimensional over the complex numbers but finitely generated over a ring. The infinite dimensional view dominates in the current literature. In this paper we rather consider the Lie algebras as modules over a polynomial ring (in the invariants, or the modular invariant); in fact, while this adds some computational complications, one is rewarded with classical looking Serre normal form results (see Section 4.1).

The case where the reduction group G is the symmetry group of a polygon, the dihedral group $\mathbb{D}_N$, has been studied before. Seminal but isolated results were obtained already in [4]; the first attempt towards a systematic classification can be found in [2], further development in [3], and [5]. However, all results so far have been obtained under the simplifying assumption that one can use the same dihedral representation to define an action on either the space of vectors or matrices, and on the polynomials. Here this restriction is lifted. In the context of integrable systems, dihedral symmetric Automorphic Lie Algebras have been studied the most (e.g. [2, 4, 10, 11, 12]) and together with icosahedral symmetry, dihedral symmetry seems the most relevant in physical systems.

In this paper the complete classification of Automorphic Lie Algebras with dihedral symmetry is tackled using classical invariant theory. The Riemann sphere is identified with the complex projective line $\mathbb{C}P^1$ consisting of quotients $\frac{X}{Y}$ of two complex variables by setting $\lambda = \frac{X}{Y}$. Möbius transformations on λ then correspond to linear transformations on the vector (X, Y) by the same matrix. Classical invariant theory is

used to find the G-invariant subspaces of $\mathbb{C}[X, Y]$-modules, where $\mathbb{C}[X, Y]$ is the ring of polynomials in X and Y. These ring-modules of invariants are then localised by a choice of multiplicative set of invariants. This choice corresponds to selecting the orbit Γ of poles. The set of elements in the localisation of degree zero, i.e. the set of elements which can be expressed as functions of λ, generate the Automorphic Lie Algebra. Once the algebra is identified, it is transformed into a normal form in the spirit of the standard Serre normal form [13]; we believe this is the most convenient form for analysis.

This investigation leads to our main result:

Let the dihedral group $\mathbb{D}_N$ act faithfully on the spectral parameter $\lambda \in \overline{\mathbb{C}}$ and let $\Gamma \subset \overline{\mathbb{C}}$ be an orbit thereof. Let V be an irreducible complex projective representation of $\mathbb{D}_N$ and $\mathfrak{g}(V)$ a simple Lie subalgebra and $\mathbb{D}_N$-submodule of $\mathrm{End}(V) \cong V \otimes V^*$. This leaves only $\mathfrak{sl}(V)$. Then the Automorphic Lie Algebra $(\mathfrak{g}(V) \otimes \mathcal{M}(\overline{\mathbb{C}}))_\Gamma^{\mathbb{D}_N}$ is generated by three matrices as a module over the ring of automorphic functions and there is a choice of generators e_0, e_+, e_- in which the bracket relations are

$$[e_0, e_\pm] = \pm 2 e_\pm \qquad [e_+, e_-] = f e_0$$

where f is the automorphic function with zeros of lowest possible order at all exceptional orbits not equal to Γ. To be more specific, $f = \frac{F_0^{\nu_0} F_1^{\nu_1} F_2^{\nu_2}}{F_\Gamma^{3\nu_\Gamma}}$ where Γ_i denotes an exceptional orbit and F_i respectively F_Γ is the form of lowest degree that vanishes on Γ_i, respectively Γ, as introduced in Section 2. If Γ is one of the exceptional orbits the Lie algebra structure can be summarised as in Table 1.

Table 1. Lie algebra structure of Automorphic Lie Algebras with dihedral symmetry: the structure constant f with poles at the exceptional orbits Γ_i.

Γ	Γ_0	Γ_1	Γ_2
f	$\dfrac{F_1^2 F_2^N}{F_0^4}$	$\dfrac{F_2^N F_0^2}{F_1^4}$	$\dfrac{F_0^2 F_1^2}{F_2^{2N}}$

In particular, if one evaluates the algebra elements at λ then $(\mathfrak{g}(V) \otimes \mathcal{M}(\overline{\mathbb{C}}))_\Gamma^{\mathbb{D}_N}(\lambda) \cong \mathfrak{sl}_2(\mathbb{C})$ if and only if the orbit $\mathbb{D}_N \lambda$ is generic and not equal to Γ. The result highlighted above (see Theorem 4.3 in Section 4.1) allows us to provide a complete classification of $\mathfrak{sl}_2(\mathbb{C})$-based Automorphic Lie Algebras with dihedral symmetry, which is an important step towards the complete classification of Automorphic Lie Algebras, as the simplifying assumption made in [5] to use the same matrices representing the reduction group to act on the spectral parameter as well as on the base Lie algebra is dropped. This simplifying assumption is no longer valid when considering higher dimensional Lie algebras. Moreover, there is no need to distinguish between *generic* and *exceptional* orbits, as we establish that they can be all treated in the same way, in contrast to what was previously generally accepted ([3, 6]), even if the corresponding algebras are non isomorphic.

This paper is organised as follows: Section 2 summarises definitions and basic notions used in the rest of the paper. Here notation is also fixed. Section 3 illustrates

our approach to the classification of Automorphic Lie Algebras, while Section 4 presents the results for the $\mathbb{D}_N$ case. In particular, in 4.1 the Serre normal forms are presented and in 4.3 the concept of *matrices of invariants* is introduced. We believe that the latter will play a fundamental role in the classification of higher dimensional Automorphic Lie Algebras. A section of Conclusions and an Appendix complete the paper.

2. Preliminaries

In this section, for the benefit of the reader, we recall definitions and basic notions from the theory of representations of finite groups, notably $\mathbb{D}_N$ representation theory, and from the classical theory of invariants which will be used in what follows. For representation theory of finite groups we refer the reader to [14], among many other books. The results from invariant theory are discussed in [15].

2.1. Representations of the dihedral group $\mathbb{D}_N$

The dihedral group of order $2N$ can be defined abstractly by

$$\mathbb{D}_N = \langle r, s \mid r^N = s^2 = (rs)^2 = 1 \rangle.$$

If one thinks of the representation as symmetries of a regular N-gon, then r stands for rotation and s for reflection. The group consists of N rotations r^ℓ and N reflections sr^ℓ, $0 \le \ell < N$. There are significant differences between the cases where N is odd or even. For instance, the square of a rotation of order N has again order N when N is odd, while it has order $N/2$ when N is even. If N is odd then $\mathbb{D}_N$ has two one-dimensional characters, χ_1 and χ_2. If N is even there are two additional one-dimensional characters, χ_3 and χ_4. The values of $\chi_1, \ldots, \chi_4$ are given in Table 2.

Table 2. $\mathbb{D}_N$ one-dimensional characters: χ_1, χ_2 if N is odd; $\chi_1, \ldots, \chi_4$ if N is even.

	χ_1	χ_2	χ_3	χ_4
r	1	1	-1	-1
s	1	-1	1	-1

The remaining irreducible characters are two-dimensional. We denote them by ψ_j, for $1 \le j < N/2$, and they take the values

$$\psi_j(r^\ell) = \omega^{j\ell} + \omega^{-j\ell}, \qquad \psi_j(sr^\ell) = 0,$$

where $\omega = e^{\frac{2\pi i}{N}}$. The characters of the faithful (injective) representations are precisely those ψ_j for which $\gcd(j, N) = 1$, where gcd stands for *greatest common divisor*.

When we consider explicit matrices for a representation $\rho : \mathbb{D}_N \to \mathrm{GL}(\mathbb{C}^2)$ with character ψ_j, there is an entire equivalence class to choose from. We will consider here the choice

$$\rho_r = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{N-j} \end{pmatrix}, \qquad \rho_s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1}$$

If one knows the space of invariants V^G, when G is represented by ρ, then one can easily find the space of invariants belonging to an equivalent representation ρ', because the invertible transformation T that relates the two representations, $T\rho_g = \rho'_g T$, also relates the spaces of invariants: $V^{\rho'(G)} = TV^{\rho(G)}$.

2.2. Elements of invariant theory

Definition 2.1. *Let V be a vector space and let $\mathbb{C}[X,Y]$ be the ring of polynomials in X and Y; then we define*
$$\widehat{V} = V \otimes \mathbb{C}[X,Y].$$

We denote, here and in what follows, $\widehat{V}^G$ the space of G-invariants in $\widehat{V}$. An important classical result in invariant theory is Molien's Theorem (see for example [15, 16, 17]).

Theorem 2.2 (Molien). *Let V and U be G-modules. Then the Poincaré series ‡ for the space of invariants in $V \otimes \mathbb{C}[U]$ is given by*
$$P((V \otimes \mathbb{C}[U])^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{\overline{\chi_V(g)}}{\det(1 - \sigma_U(g)t)}.$$

where χ_V is the character of V and σ_U the representation of U.

In this paper U will always be a two-dimensional module with dual basis $\{X,Y\}$.

Example 2.3 (see [16], p.93)**.** *Consider the cyclic group $\mathbb{Z}/N$ generated by the matrix* $\begin{pmatrix} \omega & 0 \\ 0 & \omega^{N-1} \end{pmatrix}$ *where ω is a primitive N-th root of unity. By direct investigation one finds that $\mathbb{C}[X,Y]^{\mathbb{Z}/N} = \bigoplus_{i=0}^{N-1} \mathbb{C}[X^N, Y^N](XY)^i$ and thus*
$$P(\mathbb{C}[X,Y]^{\mathbb{Z}/N}, t) = \frac{\sum_{i=0}^{N-1} t^{2i}}{(1-t^N)^2} = \frac{1-t^{2N}}{(1-t^2)(1-t^N)^2} = \frac{1+t^N}{(1-t^2)(1-t^N)}.$$

With this in mind one can determine the invariants for the group $\mathbb{D}_N$ generated by $\begin{pmatrix} \omega & 0 \\ 0 & \omega^{N-1} \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *using Molien's Theorem. Indeed, it follows that*

$$P(\mathbb{C}[X,Y]^{\mathbb{D}_N}, t) = \frac{1}{2N} \sum_{g \in \mathbb{D}_N} \frac{1}{\det(1 - \sigma_g t)}$$

$$= \frac{1}{2N} \sum_{\ell=0}^{N-1} \left(\begin{vmatrix} 1-\omega^\ell t & 0 \\ 0 & 1-\omega^{N-\ell}t \end{vmatrix}^{-1} + \begin{vmatrix} 1 & -\omega^\ell t \\ -\omega^{N-\ell}t & 1 \end{vmatrix}^{-1} \right)$$

$$= \frac{1}{2}\left(P(\mathbb{C}[X,Y]^{\mathbb{Z}/N}, t) + \frac{1}{(1-t^2)} \right) = \frac{1}{(1-t^2)(1-t^N)}.$$

‡ There is no unanimous consensus regarding the name of the generating function $P(M,t)$ of a graded module M over a graded algebra: in e.g. [15] the term *Hilbert series* is used, while in e.g. [16, 17] it is called *Poincaré series* and we prefer to follow this last convention. It is worth noticing that in the context of classical invariant theory, if $M = (V \otimes \mathbb{C}[U])^G$, it is also called *Molien series*.

If one can find two algebraically independent $\mathbb{D}_N$-invariant forms, one of degree 2 and one of degree N, e.g. XY and $X^N + Y^N$, then the above calculation proves that any $\mathbb{D}_N$-invariant polynomial is a polynomial in these two forms, i.e. $\mathbb{C}[X,Y]^{\mathbb{D}_N} = \mathbb{C}[XY, X^N + Y^N]$.

The $\mathbb{D}_N$-invariant spaces are known; they are listed in Table 3 and 4, where we use the definitions

$$F_0 = \frac{X^N - Y^N}{2} \qquad F_1 = \frac{X^N + Y^N}{2} \qquad F_2 = XY\,, \tag{2}$$

and where V_χ is the representation space of the character χ. These forms satisfy one algebraic relation:

$$F_0^2 - F_1^2 + F_2^N = 0\,. \tag{3}$$

The appendix contains an explicit calculation for the two-dimensional cases $\widehat{V}_{\psi_j}^{\mathbb{D}_N}$, for completeness.

Table 3. Module generators η_i in $\widehat{V}_\chi^{\mathbb{D}_N} = \bigoplus_i \mathbb{C}[F_1, F_2]\eta_i$, N odd

	χ_1	χ_2	ψ_j
η_i	1	F_0	$\begin{pmatrix} X^j \\ Y^j \end{pmatrix}, \begin{pmatrix} Y^{N-j} \\ X^{N-j} \end{pmatrix}$

Table 4. Module generators η_i in $\widehat{V}_\chi^{\mathbb{D}_{2N}} = \bigoplus_i \mathbb{C}[F_1^2, F_2]\eta_i$.

	χ_1	χ_2	χ_3	χ_4	ψ_j
η_i	1	$F_1 F_0$	F_1	F_0	$\begin{pmatrix} X^j \\ Y^j \end{pmatrix}, \begin{pmatrix} Y^{2N-j} \\ X^{2N-j} \end{pmatrix}$

Let χ_V denote the character of a G-module V. Then the character of $\mathfrak{sl}(V)$ is given by

$$\chi_{\mathfrak{sl}(V)}(g) = \chi_V(g)\overline{\chi_V}(g) - 1 \qquad \forall g \in G, \tag{4}$$

where the overline stands for complex conjugation. Indeed, $\mathfrak{gl}(V) = \mathrm{End}(V) \cong V \otimes V^*$ has character $\chi_V \overline{\chi_V}$, and all scalars $\mathbb{C}\mathrm{Id}$ are invariant.

Example 2.4 $\left(\widehat{\mathfrak{sl}(V_{\psi_j})}^{\mathbb{D}_N}\right)$. *Using (4) one finds that $\chi_{\mathfrak{sl}(V_{\psi_j})} = \chi_2 + \psi_{2j}$. In order to find the invariant matrices explicitly, it is sufficient to identify these two irreducible representations within the space $\mathfrak{sl}(V_{\psi_j})$ generated by $e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and then use Table 3 or 4, depending on whether N is odd or even.*

The χ_2 copy in $\chi_{\mathfrak{sl}(V_{\psi_j})}$ is given by $\mathbb{C}e_0$ and $\{e_+, e_-\}$ is a basis for the ψ_{2j} summand in which $\mathbb{D}_N$ acts according to our preferred choice of matrices (1). Thus, if e.g. N is odd, the $\mathbb{D}_N$ invariant algebra $\widehat{\mathfrak{sl}(V_{\psi_j})}^{\mathbb{D}_N}$ is a free $\mathbb{C}[F_1, F_2]$-module generated by the matrices

$$\frac{1}{2}\begin{pmatrix} X^N - Y^N & 0 \\ 0 & Y^N - X^N \end{pmatrix}, \begin{pmatrix} 0 & X^{2j} \\ Y^{2j} & 0 \end{pmatrix}, \begin{pmatrix} 0 & Y^{N-2j} \\ X^{N-2j} & 0 \end{pmatrix}. \tag{5}$$

2.3. Invariant theory on the sphere

Each noncyclyc finite Möbius group G, such as the dihedral group, is the symmetry group of a tessellation of the sphere with d_0 vertices, d_1 edges and d_2 faces. Let $\Gamma_i \subset \overline{\mathbb{C}}$ be the set of centroids of i-dimensional cells of the tessellation, i.e. an exceptional orbit.

Definition 2.5 (The form vanishing on an orbit). *For any G-orbit Γ, we define*

$$F_\Gamma \in \mathbb{C}_{|\Gamma|}[X, Y] \tag{6}$$

as the form vanishing on Γ. F_Γ is unique up to multiplicative constant. Moreover, there is a natural number ν_Γ such that

$$\nu_\Gamma |\Gamma| = |G|.$$

To shorten notation we use $F_i = F_{\Gamma_i}$ and $\nu_i = \nu_{\Gamma_i}$.

Under a suitable choice of constants $\sum_{i=0}^2 F_i^{\nu_i} = 0$ and there is no other relation between the forms F_i. Notice that for any form F, that is, any homogeneous polynomial of degree d, the set of zeros is well defined on the Riemann sphere $\mathbb{C}P^1$ since $F(cX, cY) = c^d F(X, Y)$ for any scalar c.

The forms F_i are relative invariants of a *Schur covering group* $G^\flat$ of G [18, 19], that is, $g\, F_i = \chi(g) F_i$, $\forall g \in G^\flat$ and where χ is a one-dimensional character of $G^\flat$. In fact, these forms generate the same ring as the relative invariants of any Schur cover $G^\flat$ do, i.e.

$$\mathbb{C}[F_0, F_1, F_2] \Big/ F_0^{\nu_0} + F_1^{\nu_1} + F_2^{\nu_2} = \mathbb{C}[X, Y]^{[G^\flat, G^\flat]},$$

where $[G, G]$ denotes the derived subgroup.

Example 2.6. *Let the equator of the Riemann sphere be an N-gon with a vertex at 1. Then the corresponding representation of $\mathbb{D}_N$ by Möbius transformations corresponds to the representation ψ_1 of $\mathbb{D}_N^\flat$ in the form (1). It follows that the relative invariants are*

$$\mathbb{C}[X, Y]^{[\mathbb{D}_N^\flat, \mathbb{D}_N^\flat]} = \mathbb{C}[F_1, F_2] \oplus \mathbb{C}[F_1, F_2]F_0$$

where F_0, F_1 and F_2 are given in (2).

Lemma 2.7. *Let F_i and ν_i be defined as above. Then $F_i^{\nu_i}$ is invariant:*

$$F_i^{\nu_i} \in \mathbb{C}_{|G|}[X, Y]^{G^\flat}.$$

The next lemma shows one can find all invariant vectors $(V \otimes \mathcal{M}(\overline{\mathbb{C}}))_\Gamma^G$ by considering quotients of invariant vectors and invariant forms whose degrees are multiples of $|G|$.

Lemma 2.8. *Let V and $\overline{\mathbb{C}}$ be G-modules and $\Gamma \subset \overline{\mathbb{C}}$ a single G-orbit. Suppose $\bar{v} \in (V \otimes \mathcal{M}(\overline{\mathbb{C}}))_\Gamma^G$. Then $\bar{v} = \frac{v}{F}$ for an invariant vector $v \in \widehat{V}^{G^\flat}$ and an invariant polynomial $F \in \mathbb{C}[X, Y]^{G^\flat}$ of the same degree, divisible by $|G|$.*

Proof. By the identification $\overline{\mathbb{C}} \cong \mathbb{C}P^1$, $\bar{v} = \frac{v'}{F'}$ for some (X, Y)-dependent vector v' and scalar F' of the same degree. Assume v' and F' have no common factors.

We first argue that F' is a relative invariant of $G^\flat$. Let $g \in G$. Then $g \cdot \bar{v} = g \cdot \frac{v'}{F'} = \frac{g' \cdot v'}{g' \cdot F'}$ for an element in the Schur covering group $g' \in G^\flat$. Hence, by invariance and because v' and F' have no common factor, $g' \cdot v' = Qv'$ and $g' \cdot F' = QF'$ for some $Q \in \mathbb{C}$. Moreover, $G^\flat$ is finite so that Q is a root of unity and hence $F' \in \mathbb{C}[X, Y]^{[G^\flat, G^\flat]} = \mathbb{C}[F_0, F_1, F_2] \big/ F_0^{\nu_0} + F_1^{\nu_1} + F_2^{\nu_2}$.

If F' vanishes on a generic orbit then its degree is a multiple of $|G|$ and the result follows. If F' vanishes on an exceptional orbit then it is a power of F_i for some i. One can multiply the numerator and denominator of $\bar{v}$ with a power of F_i to get a quotient of invariants and because the degree of F_i divides $|G|$ this can be done ending up with degrees divisible by $|G|$. $\qquad\square$

Example 2.9. *If we apply Lemma 2.8 to the trivial G-module $V = \mathbb{C}$, one obtains the automorphic functions of G with poles on a specified orbit. If $f \in \mathcal{M}(\overline{\mathbb{C}})_\Gamma^G$ the invariance and pole restriction imply that the denominator of f is a power of F_Γ. By Lemma 2.8 we can assume that it is in fact a power of $F_\Gamma^{\nu_\Gamma}$ and that the numerator is a form in $\mathbb{C}[F_0^{\nu_0}, F_1^{\nu_1}, F_2^{\nu_2}]$. The numerator factors into elements of the two-dimensional vector space $\mathbb{C}F_0^{\nu_0} + \mathbb{C}F_1^{\nu_1} + \mathbb{C}F_2^{\nu_2} \big/ F_0^{\nu_0} + F_1^{\nu_1} + F_2^{\nu_2}$. One line in this space is $\mathbb{C}F_\Gamma^{\nu_\Gamma}$. Any vector outside this line will generate the ring of automorphic functions, e.g.*

$$\mathcal{M}(\overline{\mathbb{C}})_\Gamma^G = \mathbb{C}\left[\frac{F_i^{\nu_i}}{F_\Gamma^{\nu_\Gamma}}\right] \qquad \Gamma_i \neq \Gamma.$$

3. Towards Automorphic Lie Algebras

Our approach to investigate Automorphic Lie Algebras was briefly sketched in the Introduction. In this section we provide the necessary justification.

The field of meromorphic functions on the Riemann sphere $\mathcal{M}(\overline{\mathbb{C}})$ equals the set of rational functions on $\overline{\mathbb{C}}$, that is, quotients of polynomials. Using the identification

$$\overline{\mathbb{C}} \ni \lambda = [(X, Y)] \in \mathbb{C}P^1$$

this is the set of quotients of two homogeneous polynomials in X and Y of the same degree. Moreover, Möbius transformations on λ correspond to linear transformation on (X, Y) by the same matrix. However, two matrices yield the same Möbius transformation if and only if they are scalar multiples of one another. Therefore we allow the action on (X, Y) to be projective in order to cover all possibilities. That is, we require projective representations $\rho : G \to \mathrm{PGL}(\mathbb{C}^2)$. The same holds for the action on the Lie algebra $\mathfrak{g}(V) < \mathfrak{gl}(V)$.

Projective representations of G are in one to one correspondence with linear representations of a *Schur covering group* $G^\flat$. The Schur covering groups of the dihedral groups are well known [20].

Proposition 3.1. *If N is odd then the dihedral group and its Schur covering group coincide: $\mathbb{D}_N^\flat = \mathbb{D}_N$. If N is even then $\mathbb{D}_{2N}$ is one of multiple Schur covering groups for $\mathbb{D}_N$.*

We consider a faithful linear representations of $G^\flat$

$$\sigma : G^\flat \to \mathrm{GL}(\mathbb{C}^2)$$

which restricts G to the groups

$$\mathbb{Z}/N, \quad \mathbb{D}_N, \quad \mathbb{T}, \quad \mathbb{O}, \quad \mathbb{Y}$$

of Klein's classification [21, 22], where $\mathbb{Z}/N$ is the cyclic group, $\mathbb{D}_N$ the dihedral group, $\mathbb{T}$ the tetrahedral group, $\mathbb{O}$ the octahedral group and $\mathbb{Y}$ the icosahedral group.

Let

$$\tau : G^\flat \to \mathrm{GL}(V)$$

be any irreducible linear representation and consider $\mathfrak{g}(V)$ to be a Lie subalgebra and $G^\flat$-submodule of $\mathfrak{gl}(V)$. The representations σ and τ induce an action on the space $\mathfrak{g}(V) \otimes \mathbb{C}[X,Y]$, which concretely reads

$$g \cdot (M \otimes F(X,Y)) = \tau_g M \tau_{g^{-1}} \otimes F(\sigma_{g^{-1}}(X,Y))$$

where $M \in \mathfrak{g}(V)$ and $F(X,Y) \in \mathbb{C}[X,Y]$. Notice that the space of invariants $(\mathfrak{g}(V) \otimes \mathbb{C}[X,Y])^{G^\flat}$ is a Lie subalgebra because the $G^\flat$-action respects the Lie bracket.

Definition 3.2 (Localisation). *Let $F \in \mathbb{C}_d[X,Y]^G$ and define the multiplicative set $U_F = \bigcup_{\ell \geq 0} \mathbb{C}^* F^\ell$. The space of invariants $\widehat{V}^G$ is a $\mathbb{C}[X,Y]^G$-module and one can consider its localisation by U_F, which we denote*

$$\widehat{V}_F^G = U_F^{-1} \widehat{V}^G = \left\{ \frac{v}{F^\ell} \mid v \in \widehat{V}^G, \ell \geq 0 \right\}.$$

The localisation of a ring by a multiplicative subset U is the smallest extension of the ring in which all elements in U are units, that is, have a multiplicative inverse.

Definition 3.3. *Let $G^\flat$ be the Schur cover of G; consider the localisation $\widehat{V}_{F_\Gamma}^{G^\flat}$. We define $\overline{V}_\Gamma^G < \widehat{V}_{F_\Gamma}^{G^\flat}$ as the subset of elements that factor through $\mathbb{C}P^1$*

$$\overline{V}_\Gamma^G = \left\{ v \in \widehat{V}_{F_\Gamma}^{G^\flat} \mid v(zX, zY) = v(X,Y), \forall z \in \mathbb{C}^*, \forall (X,Y) \in \mathbb{C}^2 \setminus (0,0) \right\}.$$

Notice that $v \in \overline{V}_\Gamma^G$ has no poles outside Γ.

Finally, we define two operators, *prehomogenisation p* and *homogenisation h*, which will take us from $\widehat{V}^{G^\flat}$ to $\overline{V}_\Gamma^G$ in two steps. By taking just the first step, one can study $\overline{V}_\Gamma^G$ while holding on to the degree information of the homogeneous polynomials in (X,Y).

Definition 3.4 (Prehomogenisation). *Define* $p_d : \widehat{V}^G \to \widehat{V}^G$ *on basis vectors* v *of homogeneous degree by*

$$p_d v = \begin{cases} v & \text{if } d \mid \deg v \\ 0 & \text{otherwise}. \end{cases}$$

Then extend linearly.

Definition 3.5 (Homogenisation). *Let* $\Gamma \subset \overline{\mathbb{C}}$ *be a* G-*orbit,* F_Γ *defined as in (6), and suppose* $|\Gamma| \mid d$. *Define* $h_\Gamma : p_d \widehat{V}^{G^\flat} \to \overline{V}_\Gamma^G$ *on basis vectors* v *of homogeneous degree* $\ell|\Gamma|$ *by*

$$h_\Gamma v = \frac{v}{F_\Gamma^\ell} .$$

Then extend linearly.

We define equivalence classes *mod F* on $\widehat{V}^G$ by

$$u \bmod F = v \bmod F \Leftrightarrow \exists \ell \in \mathbb{Z} : u = F^\ell v .$$

Lemma 3.6. *Let* $\Gamma \subset \overline{\mathbb{C}}$ *be a* G-*orbit,* F_Γ *defined as in (6), and suppose* $|\Gamma| \mid d$. *There is a ring-module isomorphism*

$$p_d \widehat{V}^{G^\flat} \bmod F_\Gamma \cong \overline{V}_\Gamma^G .$$

Proof. The linear map $h_\Gamma : p_d \widehat{V}^{G^\flat} \to \overline{V}_\Gamma^G$ respects products and is therefore a homomorphism of modules. Its kernel is the equivalence class of the identity and by Lemma 2.8 the map is surjective. ☐

In particular, since the Lie algebra structure of the Automorphic Lie Algebras depends only on the ring structure, that is, addition and multiplication, the previous Lemma also gives Lie algebra isomorphisms. Notice also that it will always suffice to use $d = |G|$ since the order of any orbit divides the order of the group. Thus we have

Corollary 3.7. *Adopting the notation from above, there is a Lie algebra isomorphism*

$$p_{|G|} \widehat{\mathfrak{g}(V)}^{G^\flat} \bmod F_\Gamma \cong \overline{\mathfrak{g}(V)}_\Gamma^G$$

Because $p_{|G|} \widehat{\mathfrak{g}(V)}^{G^\flat}$ is independent of the orbit Γ, one can study all Automorphic Lie Algebras $\left\{ \overline{\mathfrak{g}(V)}_\Gamma^G \mid \Gamma \text{ an orbit} \right\}$ by studying the one Lie algebra $p_{|G|} \widehat{\mathfrak{g}(V)}^{G^\flat}$.

The prehomogenisation projection p_d only concerns degrees. Therefore it makes sense to define it on Poincaré series. Since there is little opportunity for confusion, we use the same name:

$$p_d : \mathbb{Z}_{\geq 0}[t] \to \mathbb{Z}_{\geq 0}[t], \quad \sum_{\ell \geq 0} k_\ell t^\ell \mapsto \sum_{d \mid \ell} k_\ell t^\ell .$$

Example 3.8 $(\overline{V_{\chi_1 \Gamma}}^{\mathbb{D}_N})$. *The case of the trivial representation* V_{χ_1} *corresponds to automorphic functions. First we assume that* N *is odd so that* $\mathbb{D}_N^\flat = \mathbb{D}_N$ *and* $\widehat{V}_{\chi_1}^{\mathbb{D}_N^\flat} = \widehat{V}_{\chi_1}^{\mathbb{D}_N} = \mathbb{C}[F_1, F_2]$. *Recall that* $\deg F_1 = N$ *and* $\deg F_2 = 2$ *and the two forms are algebraically independent. The prehomogenisation projection* p_{2N} *maps*

$$P(\widehat{V}_{\chi_1}^{\mathbb{D}_N^\flat}, t) = \frac{1}{(1 - t^2)(1 - t^N)} = \frac{(1 + t^2 + t^4 + \ldots + t^{2N-2})(1 + t^N)}{(1 - t^{2N})^2} \mapsto \frac{1}{(1 - t^{2N})^2}$$

and one finds

$$p_{2N} \mathbb{C}[F_1, F_2] = \mathbb{C}[F_1^{\nu_1}, F_2^{\nu_2}].$$

If N *is even we use the Schur cover* $\mathbb{D}_{2N}$ *with invariant forms* $\widehat{V}_{\chi_1}^{\mathbb{D}_{2N}} = \mathbb{C}[F_1^2, F_2]$, *which are mapped onto the same ring under* p_{2N}.

If the equivalence relation mod F_Γ *is introduced one is left with* $\mathbb{C}[F_k^{\nu_k}]$ *for* $F_k \neq F_\Gamma$. *Notice that this ring is equivalent to* $\mathcal{M}(\overline{\mathbb{C}})_\Gamma^G = \mathbb{C}\left[\frac{F_k^{\nu_k}}{F_\Gamma^{\nu_\Gamma}}\right]$, $\Gamma_k \neq \Gamma$ *from Example 2.9.*

4. Automorphic Lie Algebras with dihedral symmetry

Recall from Example 2.4 that

$$\widehat{\mathfrak{sl}(V_{\psi_j})}^{\mathbb{D}_N} = \mathbb{C}[F_1, F_2]\eta_1 \oplus \mathbb{C}[F_1, F_2]\eta_2 \oplus \mathbb{C}[F_1, F_2]\eta_3$$

where, in our preferred basis (1), F_0, F_1 and F_2 are given in (2) and where

$$\eta_1 = \begin{pmatrix} 0 & X^{2j} \\ Y^{2j} & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & Y^{N-2j} \\ X^{N-2j} & 0 \end{pmatrix}, \quad \eta_3 = {}^1\!/\!{}_2(X^N - Y^N)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrices η_1, η_2 and η_3 generate a Lie algebra over $\mathbb{C}[F_1, F_2]$, with commutator brackets

$$[\eta_2, \eta_3] = 2(-F_2^{N-2j}\eta_1 + F_1\eta_2)$$
$$[\eta_3, \eta_1] = 2(F_1\eta_1 - F_2^{2j}\eta_2)$$
$$[\eta_1, \eta_2] = 2\eta_3.$$

Notice that this is not an Automorphic Lie Algebra yet, but it can be made into one using the homogenisation operator h (see Definition 3.5).

Example 4.1. *It is useful to compare ours with previous results, starting from this algebra. In particular, let us consider [3] and [6], which contain explicit descriptions of Automorphic Lie Algebras with dihedral symmetry with poles restricted to the orbit of two points, which we call* $\Gamma_2 = \{0, \infty\}$.

The generators (27) in [3], page 190, where $N = 2$ *and* $j = 1$ *are nothing but* $h_{\Gamma_2}(\eta_1)$, $h_{\Gamma_2}(\eta_2)$ *and* $2h_{\Gamma_2}(\eta_3)$, *respectively.*

Similarly, generators (4.10) in [6], page 81, for general N *and* j, *are* $h_{\Gamma_2}(\eta_1)$, $h_{\Gamma_2}(2\eta_2 - \eta_1)$ *and* $4h_{\Gamma_2}(\eta_3)$ *respectively, if* N *is odd, and* $h_{\Gamma_2}(\eta_1)$, $h_{\Gamma_2}(\eta_2)$ *and* $2h_{\Gamma_2}(\eta_3)$ *respectively, if* N *is even.*

Observe that he image of $\widehat{\mathfrak{sl}(V_{\psi_j})}^{\mathbb{D}_N^\flat}$ under $p_{|\mathbb{D}_N|} = p_{2N}$ is

$$\mathbb{C}[F_1^2, F_2^N]\left(F_2^{N-j}\eta_1 \oplus F_1 F_2^j \eta_2 \oplus F_1 \eta_3\right)$$

regardless of whether N is odd, in which case $\mathbb{D}_N^\flat = \mathbb{D}_N$, or even, in which case we use $\mathbb{D}_N^\flat = \mathbb{D}_{2N}$. This reflects the fact that $p_{|G|}\widehat{\mathfrak{sl}(V_{\psi_j})}^{G^\flat} \bmod F_\Gamma \cong \overline{\mathfrak{sl}(V_{\psi_j})}^G_\Gamma$ is a space of G-invariants rather than $G^\flat$-invariants (see also Example 3.8).

The module generators of $p_{|\mathbb{D}_N|}\widehat{\mathfrak{sl}(V_{\psi_j})}^{\mathbb{D}_N^\flat}$ will be denoted with a tilde:

$$\tilde{\eta}_1 = F_2^{N-j}\eta_1 = F_2^N \begin{pmatrix} 0 & \lambda^j \\ \lambda^{-j} & 0 \end{pmatrix};$$

$$\tilde{\eta}_2 = F_1 F_2^j \eta_2 = F_2^N \begin{pmatrix} 0 & {}^1\!/\!{}_2(\lambda^j + \lambda^{j-N}) \\ {}^1\!/\!{}_2(\lambda^{-j} + \lambda^{N-j}) & 0 \end{pmatrix};$$

$$\tilde{\eta}_3 = F_1 \eta_3 = F_0 F_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One readily computes the structure constants

$$[\tilde{\eta}_2, \tilde{\eta}_3] = 2(-F_1{}^2 \tilde{\eta}_1 + F_1{}^2 \tilde{\eta}_2)$$
$$[\tilde{\eta}_3, \tilde{\eta}_1] = 2(F_1{}^2 \tilde{\eta}_1 - F_2{}^N \tilde{\eta}_2)$$
$$[\tilde{\eta}_1, \tilde{\eta}_2] = 2F_2{}^N \tilde{\eta}_3.$$

At this stage, one could apply the homogenisation operator h to obtain the wanted Automorphic Lie Algebra. However, before doing so, we prefer to define a *normal form* notion.

4.1. A normal form for Automorphic Lie Algebras

In order to compare Automorphic Lie Algebras with each other it is useful to define a normal form. Once two algebras are transformed to a particular normal form one can immediately see whether they are isomorphic. We define a normal form for Automorphic Lie Algebras as similar to the Serre normal form for simple Lie algebras as possible, and we will use here the same name.

Definition 4.2 (Serre normal form). *Let $\mathfrak{g}(V)$ be a simple Lie algebra with Cartan subalgebra (CSA) $\mathfrak{h}$. We say that the module generators of an Automorphic Lie Algebra $\overline{\mathfrak{g}(V)}^G_\Gamma$ are in Serre normal form if the adjoint action of $\dim_{\mathbb{C}} \mathfrak{h}$ of the $\dim_{\mathbb{C}} \mathfrak{g}(V)$ generators is diagonal on all generators, with the same roots as $\mathfrak{h}$ in Serre normal form.*

This definition relies on the fact that the Automorphic Lie Algebra $\overline{\mathfrak{g}(V)}^G_\Gamma$ can be generated by $\dim \mathfrak{g}(V)$ elements. In the previous section we saw that this is the case if $G = \mathbb{D}_N$: we found the $\tilde{\eta}_i$. In [23] it is shown that this holds for all the polyhedral groups.

Despite this fact, it is not obvious to us for which Automorphic Lie Algebras a Serre normal form exists. Moreover, we find that the normal form depends on the

Automorphic Lie Algebra CSA; this is, to the best of our knowledge, a new phenomenon in Lie algebra theory, and it deserves further investigation. The fact that we work over a ring complicates matters compared to the classical situation. The main result of this paper describes the Lie algebra structure of Automorphic Lie Algebras with dihedral symmetry by a construction of this normal form. We reformulate this theorem, which was already sketched in the Introduction, as follows:

Theorem 4.3. *Let $\mathbb{D}_N$ act faithfully on the Riemann sphere and let Γ be a single orbit therein. Let $(c_1, c_2) \in \mathbb{C}P^1$ be such that*

$$F = c_1 F_1^2 + c_2 F_2^N$$

vanishes on Γ, if F_i is given by (2).

Then, adopting the previous notation, for $\Gamma_k \neq \Gamma$,

$$\overline{\mathfrak{sl}(V_{\psi_j})}_\Gamma^{\mathbb{D}_N} = (e_0 \oplus e_+ \oplus e_-) \otimes \mathbb{C} \left[\frac{F_k^{\nu_k}}{F} \right]$$

and

$$[e_0, e_\pm] = \pm 2 e_\pm$$

$$[e_+, e_-] = e_0 \prod_{i=0}^{2} \frac{F_i^{\nu_i}}{F} .$$

A set of module-generators for this normal form is given by

$$e_0 = \frac{1}{F} \begin{pmatrix} c_1 F_0 F_1 & F_2^N(c_2 \lambda^j + {}^{c_1}/_2 (\lambda^j + \lambda^{j-N})) \\ F_2^N(c_2 \lambda^{-j} + {}^{c_1}/_2 (\lambda^{-j} + \lambda^{N-j})) & -c_1 F_0 F_1 \end{pmatrix}$$

$$e_+ = \frac{F_0 F_1 F_2^N}{2F^2} \begin{pmatrix} 1 & -\lambda^j \\ \lambda^{-j} & -1 \end{pmatrix}$$

$$e_- = \frac{F_0 F_1 F_2^N}{2F^2} \begin{pmatrix} c_1 + c_2 + c_1 c_2 \frac{F_0^2}{F} & \left(c_2(c_1+c_2)\frac{F_2^N}{F} + c_1 \lambda^{-N} \right) \lambda^j \\ -\left(c_2(c_1+c_2)\frac{F_2^N}{F} + c_1 \lambda^N \right) \lambda^{-j} & -c_1 - c_2 - c_1 c_2 \frac{F_0^2}{F} \end{pmatrix} .$$

Proof. We prove the theorem by an explicit construction. Consider the prehomogenised Lie algebra $p_{|2N|}\widehat{\mathfrak{sl}(V_{\psi_j})}^{\mathbb{D}_N^\flat} = (\tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3) \mathbb{C}[F_1^2, F_2^N]$. Define $(\tilde{e}_0, \tilde{e}_+, \tilde{e}_-)$ by

$$(\tilde{e}_0, \tilde{e}_+, \tilde{e}_-) = (\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3) T$$

where

$$T = \frac{1}{2} \begin{pmatrix} 2c_2 & -F_1^2 & c_2^2 F_0^2 F_2^N + F^2 \\ 2c_1 & F_2^N & -c_1^2 F_0^2 F_1^2 - F^2 \\ 2c_1 & F_2^N & F_2^N(c_1 c_2 F_0^2 + (c_1 + c_2)F) \end{pmatrix} .$$

It is a straightforward though very tedious exercise to check that this transformation has determinant $^1/_2 F^3$, thus is invertible in the localised ring, and that

$$[\tilde{e}_0, \tilde{e}_\pm] = \pm 2F \tilde{e}_\pm$$

$$[\tilde{e}_+, \tilde{e}_-] = F F_0^2 F_1^2 F_2^N \tilde{e}_0 .$$

Hence the homogenised matrices $e. = h_\Gamma(\tilde{e}.)$ satisfy the normal form described in the theorem. $\qquad\square$

We observe that different choices of transformation groups have been made in previous works. Here we follow [5] and allow invertible transformations T on the generators over the ring of invariant forms, contrary to [6], where only linear transformations over $\mathbb{C}$ are considered.

To find the desired isomorphism T one first looks for a matrix $\tilde{e}_0 \in (\tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3)\,\mathbb{C}[F_1^2, F_2^N]$ which has eigenvalues $\pm\mu$ that are units in the localised ring, that is, powers of F. This yields the first column of T. The other two columns are found by diagonalising $\mathrm{ad}(\tilde{e}_0)$, i.e. solving $[\tilde{e}_0, \tilde{e}_\pm] = \pm 2\mu\tilde{e}_\pm$ for $\tilde{e}_\pm \in (\tilde{\eta}_1 \oplus \tilde{\eta}_2 \oplus \tilde{\eta}_3)\,\mathbb{C}[F_1^2, F_2^N]$.

In Table 5 we present the invariant matrices when Γ is one of the exceptional orbits. In other words, we evaluate the matrices in Theorem 4.3 in $(c_1, c_2) = (1, -1)$ for poles at Γ_0, in $(c_1, c_2) = (1, 0)$ for poles at Γ_1 and in $(c_1, c_2) = (0, 1)$ for poles at Γ_2.

Table 5. Generators in normal form of Automorphic Lie Algebras with dihedral symmetry and exceptional poles

Γ	e_0	e_+	e_-
Γ_0	$\frac{1}{\lambda^N-1}\begin{pmatrix} \lambda^N+1 & -2\lambda^j \\ 2\lambda^{N-j} & -\lambda^N-1 \end{pmatrix}$	$\frac{2(\lambda^N+1)\lambda^N}{(\lambda^N-1)^3}\begin{pmatrix} 1 & -\lambda^j \\ \lambda^{-j} & -1 \end{pmatrix}$	$\frac{2(\lambda^N+1)\lambda^N}{(\lambda^N-1)^3}\begin{pmatrix} -1 & \lambda^{j-N} \\ -\lambda^{N-j} & 1 \end{pmatrix}$
Γ_1	$\frac{1}{\lambda^N+1}\begin{pmatrix} \lambda^N-1 & -2\lambda^j \\ 2\lambda^{N-j} & -\lambda^N+1 \end{pmatrix}$	$\frac{2(\lambda^N-1)\lambda^N}{(\lambda^N+1)^3}\begin{pmatrix} 1 & -\lambda^j \\ \lambda^{-j} & -1 \end{pmatrix}$	$\frac{2(\lambda^N-1)\lambda^N}{(\lambda^N+1)^3}\begin{pmatrix} 1 & \lambda^{j-N} \\ -\lambda^{N-j} & -1 \end{pmatrix}$
Γ_2	$\begin{pmatrix} 0 & \lambda^j \\ \lambda^{-j} & 0 \end{pmatrix}$	$\frac{\lambda^{2N}-1}{8\lambda^N}\begin{pmatrix} 1 & -\lambda^j \\ \lambda^{-j} & -1 \end{pmatrix}$	$\frac{\lambda^{2N}-1}{8\lambda^N}\begin{pmatrix} 1 & \lambda^j \\ -\lambda^{-j} & -1 \end{pmatrix}$

The three generators given in Theorem 4.3 can also be presented explicitly in λ as

$$e_0 = \frac{1}{\frac{c_1}{4}(\lambda^N+1)^2 + c_2\lambda^N}\begin{pmatrix} \frac{c_1}{4}(\lambda^{2N}-1) & \lambda^N(c_2\lambda^j + {c_1}/{2}(\lambda^j + \lambda^{j-N})) \\ \lambda^N(c_2\lambda^{-j} + {c_1}/{2}(\lambda^{-j} + \lambda^{N-j})) & \frac{c_1}{4}(1 - \lambda^{2N}) \end{pmatrix}$$

$$e_+ = \frac{\frac{1}{4}\lambda^N(\lambda^{2N}-1)}{2(\frac{c_1}{4}(\lambda^N+1)^2 + c_2\lambda^N)^2}\begin{pmatrix} 1 & -\lambda^j \\ \lambda^{-j} & -1 \end{pmatrix}$$

$$e_- = \frac{\frac{1}{4}\lambda^N(\lambda^{2N}-1)}{2(\frac{c_1}{4}(\lambda^N+1)^2 + c_2\lambda^N)^2}$$
$$\times \begin{pmatrix} c_1 + c_2 + \frac{\frac{c_1c_2}{4}(\lambda^N-1)^2}{\frac{c_1}{4}(\lambda^N+1)^2+c_2\lambda^N} & c_2(c_1+c_2)\frac{\lambda^{N+j}}{\frac{c_1}{4}(\lambda^N+1)^2+c_2\lambda^N} + c_1\lambda^{-N+j} \\ -c_2(c_1+c_2)\frac{\lambda^{N-j}}{\frac{c_1}{4}(\lambda^N+1)^2+c_2\lambda^N} - c_1\lambda^{N-j} & -c_1 - c_2 - \frac{\frac{c_1c_2}{4}(\lambda^N-1)^2}{\frac{c_1}{4}(\lambda^N+1)^2+c_2\lambda^N} \end{pmatrix}.$$

Theorem 4.3 describes Automorphic Lie Algebras with dihedral symmetry, with poles at any orbit. Its proof exhibits the consequence of Lemma 2.8 that one can compute all these algebra in one go. In particular, one does not need to distinguish between exceptional orbits and generic orbits.

The resulting Lie algebras differ only in the bracket $[e_+, e_-] = \frac{F_0^{\nu_0}}{F} \frac{F_1^{\nu_1}}{F} \frac{F_2^{\nu_2}}{F} e_0$. Here one can discriminate between the case of generic and exceptional orbits since precisely one factor $\frac{F_i^{\nu_i}}{F}$ equals 1 if and only if Γ is an exceptional orbit. In other words, $\frac{F_0^{\nu_0}}{F} \frac{F_1^{\nu_1}}{F} \frac{F_2^{\nu_2}}{F}$ is a polynomial in $\frac{F_k^{\nu_k}}{F}$, $\Gamma_k \neq \Gamma$, of degree 2 or 3 if Γ is exceptional or generic respectively. Notice that this degree equals the complex dimension of the quotient

$$\overline{\mathfrak{sl}(V_{\psi_j})}_\Gamma^{\mathbb{D}_N} \bigg/ \left[\overline{\mathfrak{sl}(V_{\psi_j})}_\Gamma^{\mathbb{D}_N}, \overline{\mathfrak{sl}(V_{\psi_j})}_\Gamma^{\mathbb{D}_N} \right] \cong \mathbb{C}\left[\frac{F_k^{\nu_k}}{F} \right] \bigg/ \mathbb{C}\left[\frac{F_k^{\nu_k}}{F} \right] \frac{F_0^{\nu_0}}{F} \frac{F_1^{\nu_1}}{F} \frac{F_2^{\nu_2}}{F}$$

in agreement with results in [6].

Another way to analyze the Automorphic Lie Algebras is by considering their values at a particular point λ. This can be done in the Lie algebra, i.e. the strucure constants, or in the representation of the Lie algebra, i.e. the invariant matrices. In the first case we have a three-dimensional Lie algebra, equivalent to $\mathfrak{sl}_2(\mathbb{C})$ if $\lambda \in \overline{\mathbb{C}} \setminus (\Gamma \cup \Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$. When on the other hand $\lambda \in \Gamma_i \neq \Gamma$ we obtain the Lie algebra

$$[e_0, e_\pm] = \pm 2 e_\pm, \quad [e_+, e_-] = 0.$$

Evaluating the invariant matrices in $\lambda \in \overline{\mathbb{C}} \setminus (\Gamma \cup \Gamma_0 \cup \Gamma_1 \cup \Gamma_2)$ also results in $\mathfrak{sl}_2(\mathbb{C})$. If, on the other hand, $\lambda \in \Gamma_i \neq \Gamma$ then two generators $e_\pm$ vanish and one is left with a one-dimensional, and in particular commutative, Lie algebra. For completeness, we mention that if $\lambda \in \Gamma$ then all generators of $\overline{\mathfrak{sl}(2, \mathbb{C})}_\Gamma^{\mathbb{D}_N}(\lambda)$ contain clearly singularities, by construction.

4.2. Other base Lie algebras

From the offset of this paper we have restricted our attention to simple Lie algebras $\mathfrak{g}(V)$, leaving in the dihedral case only $\mathfrak{sl}_2(\mathbb{C})$. It is however straightforward if not particularly interesting to consider other possibilities, hence we do this here for completeness.

Let V be an irreducible $\mathbb{D}_N$-module. We are interested in subspaces $\mathfrak{g}(V) < \mathfrak{gl}(V)$ which are both a Lie algebra and a $\mathbb{D}_N$-module. The dimension of V is 1 or 2. In the first case the action on $\mathfrak{gl}(V)$ is trivial and the Automorphic Lie Algebras $\overline{\mathfrak{g}(V)}_\Gamma^G = \mathcal{M}(\overline{\mathbb{C}})_\Gamma^G$ are the rings of automorphic functions analytic outside Γ.

Let V now be a two-dimensional $\mathbb{D}_N$-module affording the character ψ_j. Then the $\mathbb{D}_N$-module $\mathfrak{gl}(V)$ has character $\psi_j^2 = \chi_1 + \chi_2 + \psi_{2j}$ and all $\mathbb{D}_N$-submodules $\mathfrak{g}(V) < \mathfrak{gl}(V)$ are given by a subset of these characters. Now we add the condition that $\mathfrak{g}(V)$ be a Lie algebra. To this end we look at the matrices affording the available characters $\chi_1 + \chi_2 + \psi_{2j}$ as in Example 2.4, where we noticed that the bracket of the ψ_{2j} summand equals the χ_2 summand. Any Lie algebra containing the first therefore contains the latter. This turns out to be the only restriction on the submodule $\mathfrak{g}(V)$ coming from the requirement that it be a Lie algebra. Lie algebras of all dimensions ≤ 4 are available. The only noncommutative ones are $\mathfrak{sl}(V)$, affording $\chi_2 + \psi_{2j}$, and $\mathfrak{gl}(V)$. Since $\overline{\mathfrak{gl}(V)}_\Gamma^G = \mathcal{M}(\overline{\mathbb{C}})_\Gamma^G \mathrm{Id} \oplus \overline{\mathfrak{sl}(V)}_\Gamma^G$ considering $\mathfrak{sl}(V)$ is rather natural.

4.3. Matrices of invariants

We conclude this section with yet another representation of the algebra. Invariant matrices act on invariant vectors by multiplication. The description of the invariant matrices in terms of this action yields greatly simplified matrices, which we will call *matrices of invariants*, while preserving the structure constants of the Lie algebra. The entries of these matrices are indeed invariant, but the matrices are not invariant under the group action.

There are different ways to compute matrices of invariants. Here, for example, we choose two vectors, v_1 and v_2, satisfying three conditions: they are invariant, independent eigenvectors of the matrices in the Cartan subalgebra of the Automorphic Lie Algebras (in this case only e_0) and the difference of their degrees is a multiple of $|G| = 2N$. This will ensure that the transformed matrices have entries in the ring $p_{|G|}\mathbb{C}[X,Y]^{G^\flat} = \mathbb{C}[F_1^2, F_2^N]$ and the matrices in the Cartan subalgebra will be diagonal. Alternatively, one could look at the Molien series for the equivariant vectors and consider the lowest degree at which there are two different equivariant vectors, as a guiding principle.

Let us choose

$$v_1 = \begin{pmatrix} X^j \\ Y^j \end{pmatrix}, \qquad v_2 = (c_1 + c_2)F_1^2 F_2^N \begin{pmatrix} X^j \\ Y^j \end{pmatrix} - FF_1 F_2^j \begin{pmatrix} Y^{N-j} \\ X^{N-j} \end{pmatrix}.$$

resulting in

$$e_0 \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_+ \sim \begin{pmatrix} 0 & \frac{F_0^{\nu_0} F_1^{\nu_1} F_2^{\nu_2}}{F^3} \\ 0 & 0 \end{pmatrix}, \qquad e_- \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This approach is especially useful when investigating Automorphic Lie Algebras with $\mathbb{T}$, $\mathbb{O}$ or $\mathbb{Y}$ symmetry, because then the invariant matrices do not fit a page and in the worst case it takes a top of the range workstation months to calculate the structure constants. However, if one can construct the matrices of invariants first, it is easy to find the structure constants.

5. Conclusions

Among Automorphic Lie Algebras those with dihedral symmetry have been studied the most (e.g. [2, 4, 10, 11, 12]) and, together with icosahedral symmetry, dihedral symmetry seems the most relevant in physical systems. From a mathematical point of view, $\mathbb{D}_N$ is the only non-abelian group in Klein's classification [21, 22], whose order depends on N. On the classification side, many results are available, starting with the seminal results in [4]; the first attempt towards a systematic classification can be found in [2], further development in [3, 5, 6].

In this paper we provide a complete classification of Automorphic Lie Algebras with dihedral symmetry, with poles at any orbit (see Theorem 4.3). We find that there exists

a *normal form* for the algebras given by

$$[e_0, e_\pm] = \pm 2e_\pm$$
$$[e_+, e_-] = \frac{F_0^{\nu_0}}{F} \frac{F_1^{\nu_1}}{F} \frac{F_2^{\nu_2}}{F} e_0$$

where $\{e_0\,,\, e_\pm\}$ is a set of generators, $F = c_1 F_1^2 + c_2 F_2^N$ is a nonzero invariant form vanishing on Γ, F_i are the relative automorphic functions given by (2), after identification of the Riemann sphere with the complex projective line $\mathbb{C}P^1$ consisting of quotients $\frac{X}{Y}$ of two complex variables by setting $\lambda = \frac{X}{Y}$, and where ν_i are given in Definition 2.5. It is worth pointing out that the symmetric Lie algebras differ only in the bracket $[e_+, e_-] = \frac{F_0^{\nu_0}}{F} \frac{F_1^{\nu_1}}{F} \frac{F_2^{\nu_2}}{F} e_0$. Thus one can discriminate between the case of generic and exceptional orbits since precisely one factor $\frac{F_i^{\nu_i}}{F}$ equals 1 if and only if Γ is an exceptional orbit.

Theorem 4.3 is an important step towards the complete classification of Automorphic Lie Algebras, as the simplifying assumption made in [5] to use the same matrices representing the reduction group to act on the spectral parameter as well as on the base Lie algebra is dropped. This simplifying assumption is no longer valid when considering higher dimensional Lie algebras, so the result is also a step forward in the classification beyond $\mathfrak{sl}_2(\mathbb{C})$. Moreover, there is no need to distinguish between *generic* and *exceptional* orbits, as they can be all treated in the same way. The classification is thus *uniform*, both in the choice of representations and in the choice of the orbits.

The introduction of a normal form is also a fundamental step, even more so in higher dimensional cases, as it allows to consider whether algebras are isomorphic or not. Preliminary results show that having a normal form makes the problem feasible. Similarly, the classification of dihedral Automorphic Lie Algebras associated to *reducible* $\mathbb{D}_N$-representations and any simple (or semisimple) Lie algebra $\mathfrak{g}$ seems within reach.

We also introduce the concept of *matrices of invariants* (see Section 4.3); they describe the (multiplicative) action of invariant matrices on invariant vectors. The description of the invariant matrices in terms of this action yields greatly simplified matrices, while preserving the structure constants of the Lie algebra. We believe that the matrices of invariants will play a fundamental role in the classification of higher dimensional Automorphic Lie Algebras.

Appendix A. Invariant vectors

Theorem Appendix A.1 (Invariant vectors)**.** *Let* $\mathbb{D}_N$ *act on* (X, Y) *with character* ψ_1*. In the basis corresponding to (1), one finds the relative invariant forms* F_i *as in (2) and the space of invariant vectors*

$$(V_{\psi_j} \otimes \mathbb{C}[X, Y])^{\mathbb{D}_N} = \left(\begin{pmatrix} X^j \\ Y^j \end{pmatrix} \oplus \begin{pmatrix} Y^{N-j} \\ X^{N-j} \end{pmatrix} \right) \otimes \mathbb{C}[F_1, F_2]$$

where the sum is direct over the ring $\mathbb{C}[F_1, F_2]$*.*

Proof. An object is invariant under a group action if it is invariant under the action of all generators of a group. To find all $\mathbb{D}_N = \langle r, s \rangle$- invariant vectors we first look for the $\langle r \rangle = \mathbb{Z}/N$- invariant vectors and then average over the action of s to obtain all dihedral invariant vectors.

The space of invariant vectors is a module over the ring of invariant forms. When searching for $\mathbb{Z}/N$-invariant vectors, one can therefore look for invariants modulo powers of the $\mathbb{Z}/N$-invariant forms XY, X^N and Y^N. Moreover, we represent $\mathbb{Z}/N$ by diagonal matrices (which is possible because $\mathbb{Z}/N$ is abelian). Hence, if $\mathbb{Z}/N$ acts on $V = \langle e_1, e_2 \rangle$ then $\mathbb{Z}/N e_i \subset \langle e_i \rangle$. Therefore, one only needs to investigate the vectors $X^d e_i$ and $Y^d e_i$ for $d \in \{0, \ldots, N-1\}$ and $i \in \{1, 2\}$.

Let σ be the action on (X, Y) and τ the action on the vectors. We use the basis in which $\sigma_r = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{N-1} \end{pmatrix}$ and $\tau_r = \begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{N-j} \end{pmatrix}$, so that

$$r X^d e_1 = \omega^j \omega^{-d} X^d e_1.$$

We want to solve $\omega^j \omega^{-d} = 1$, i.e. $d - j \in N\mathbb{Z}$. Hence $d \in (j + N\mathbb{Z}) \cap \{0, \ldots, N-1\} = j$. That is, $X^j e_1$ is invariant under the action of $\langle r \rangle \cong \mathbb{Z}/N$.

Let us now consider the next one, $r Y^d e_1 = \omega^j \omega^d Y^d e_1$. We solve $\omega^j \omega^d = 1$ i.e. $d + j \in N\mathbb{Z}$. This implies $d \in (N - j + N\mathbb{Z}) \cap \{0, \ldots, N-1\} = N - j$, thus $r Y^{N-j} e_1 = Y^{N-j} e_1$ is invariant.

Similarly one finds the invariant vectors $Y^j e_2$ and $X^{N-j} e_2$, resulting in the space

$$\left(\mathbb{C}^2 \otimes \mathbb{C}[X, Y] \right)^{\mathbb{Z}/N} = \left(\begin{pmatrix} X^j \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ Y^j \end{pmatrix} + \begin{pmatrix} Y^{N-j} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ X^{N-j} \end{pmatrix} \right) \otimes \mathbb{C}[X, Y]^{\mathbb{Z}/N}.$$

If we use the fact $\mathbb{C}[X, Y]^{\mathbb{Z}/N} = (1 \oplus F_0) \otimes \mathbb{C}[F_1, F_2]$ this space is generated as a $\mathbb{C}[F_1, F_2]$ module by the vectors

$$\begin{pmatrix} X^j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Y^j \end{pmatrix}, \begin{pmatrix} Y^{N-j} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ X^{N-j} \end{pmatrix}, F_0 \begin{pmatrix} X^j \\ 0 \end{pmatrix}, F_0 \begin{pmatrix} 0 \\ Y^j \end{pmatrix}, F_0 \begin{pmatrix} Y^{N-j} \\ 0 \end{pmatrix}, F_0 \begin{pmatrix} 0 \\ X^{N-j} \end{pmatrix}.$$

It turns out the above vectors are dependent over the ring $\mathbb{C}[F_1, F_2]$. One finds that

$$F_0 \begin{pmatrix} X^j \\ 0 \end{pmatrix} = f(F_1, F_2) \begin{pmatrix} X^j \\ 0 \end{pmatrix} + g(F_1, F_2) \begin{pmatrix} Y^{N-j} \\ 0 \end{pmatrix}$$

if $f(F_1, F_2) = F_1$ and $g(F_1, F_2) = -F_2^j$, hence this vector is redundant. Similarly, the other vectors with a factor F_0 can be expressed in terms of the vectors without this factor.

The remaining vectors between the brackets are independent over the ring $\mathbb{C}[F_1, F_2]$. Indeed, let $f, g \in \mathbb{C}[F_1, F_2]$ and consider the equation

$$f X^j + g Y^{N-j} = 0.$$

If the equation is multiplied by Y^j we find

$$f F_2^j + g Y^N = f F_2^j + g(F_1 - F_0) = 0.$$

Now one can use the fact that all terms are invariant under the action of s except for gF_0 to see that $g = 0$, and hence $f = 0$. Similarly $fY^j + gX^{N-j} = 0$ implies $f = g = 0$. Thus we have a direct sum

$$\left(\mathbb{C}^2 \otimes \mathbb{C}[X,Y]\right)^{\mathbb{Z}/N} = \left(\begin{pmatrix} X^j \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ Y^j \end{pmatrix} \oplus \begin{pmatrix} Y^{N-j} \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ X^{N-j} \end{pmatrix}\right) \otimes \mathbb{C}[F_1, F_2].$$

To obtain $\mathbb{D}_N$-invariants we apply the projection $\frac{1}{2}(1+s)$. Observe that the $\mathbb{D}_N$-invariant polynomials move through this operator so that one only needs to compute

$$\frac{1}{2}(1+s)\begin{pmatrix} X^j \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} X^j \\ Y^j \end{pmatrix} = \frac{1}{2}(1+s)\begin{pmatrix} 0 \\ Y^j \end{pmatrix}$$

and

$$\frac{1}{2}(1+s)\begin{pmatrix} Y^{N-j} \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} Y^{N-j} \\ X^{N-j} \end{pmatrix} = \frac{1}{2}(1+s)\begin{pmatrix} 0 \\ X^{N-j} \end{pmatrix}$$

These two vectors are independent over the ring by the previous reasoning.

$\square$

Remark. If one allows σ to be non-faithful, several more cases appear. However, they are not more interesting than what we have seen so far, which is why we decided not to include this in the theorem. In words, it is as follows. If the character of σ is $\psi_{j'}$ and $\mathbb{D}_N / \ker \tau$ is a subgroup of $\mathbb{D}_N / \ker \sigma$ then everything is the same as above except that N will be replaced by $N' = \frac{N}{\gcd(N,j')}$. If on the other hand $\mathbb{D}_N / \ker \tau$ is not a subgroup of $\mathbb{D}_N / \ker \sigma$, then the only invariant vector is the zero vector.

References

[1] Sara Lombardo. *Reductions of Integrable Equations and Automorphic Lie Algebras.* PhD in Applied Mathematics, The University of Leeds, School of Mathematics, Department of Applied Mathematics, 2004.

[2] S. Lombardo and A. V. Mikhailov. Reductions of integrable equations: dihedral group. *J. Phys. A*, 37(31):7727–7742, 2004.

[3] S. Lombardo and A. V. Mikhailov. Reduction groups and Automorphic Lie Algebras. *Comm. Math. Phys.*, 258(1):179–202, 2005.

[4] Alexander V. Mikhailov. The reduction problem and the inverse scattering method. *Physica D*, 3(1&2):73–117, 1981.

[5] S. Lombardo and J. A. Sanders. On the Classification of Automorphic Lie Algebras. *Comm. Math. Phys.*, 29(3):793, 2010.

[6] Rhys Bury. *Automorphic Lie Algebras, Corresponding Integrable Systems and their Soliton Solutions.* PhD in Applied Mathematics, The University of Leeds, School of Mathematics, Department of Applied Mathematics, 2010.

[7] Mikaël Chopp. *Lie-Admissible Structures on Witt type algebras and Automorphic Algebras.* PhD in Mathematics, The Faculty of Sciences, Technology and Communication, Université du Luxembourg and Paul Verlaine Université-Metz, 2011.

[8] Martin Schlichenmaier. Higher genus affine algebras of Krichever-Novikov type. *Moscow Math. J*, **3** (2003), No.4, 1395–1427.

[9] Martin Schlichenmaier. From the Virasoro Algebra to Krichever–Novikov type Algebras and beyond. *arXiv preprint arXiv:1301.7725*, 2013, and (in) Vasil'ev, A.(ed), Harmonic and Complex Analysis and its Applications, pp.325-358, Springer 2014.

[10] Jing Ping Wang. Lenard scheme for two-dimensional periodic Volterra chain. *J. Math. Phys.*, 50(2):023506, 25, 2009.

[11] S. Konstantinou-Rizos and A. V. Mikhailov. Darboux transformations, finite reduction groups and related Yang–Baxter maps. *Journal of Physics A: Mathematical and Theoretical*, 46(42):425201, 2013.

[12] A. V. Mikhailov, G. Papamikos, and J. P. Wang. Darboux transformation with Dihedral reduction group. *arXiv preprint arXiv:1402.5660*, 2014.

[13] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer-Verlag, New York, 1972. Graduate Texts in Mathematics, Vol. 9.

[14] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[15] Richard P. Stanley. Invariants of finite groups and their applications to combinatorics. *Bulletin of the American Mathematical Society*, 1(3):475–511, 1979.

[16] Larry Smith. *Polynomial invariants of finite groups*, volume 6 of *Research Notes in Mathematics*. A K Peters Ltd., Wellesley, MA, 1995.

[17] Mara D. Neusel. *Invariant theory*, volume 36, *Student Mathematical Library*, AMC 10:12, 2007.

[18] J Schur. Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen. *J. Reine Angew. Math.*, 139:155–250, 1911.

[19] Charles W. Curtis. *Pioneers of representation theory: Frobenius, Burnside, Schur, and Brauer*, volume 15 of *History of Mathematics*. American Mathematical Society, Providence, RI, 1999.

[20] E. W. Read. On the Schur multipliers of the finite imprimitive unitary reflection groups $G(m, p, n)$. *J. London Math. Soc. (2)*, 13(1):150–154, 1976.

[21] Felix Klein. *Lectures on the icosahedron and the solution of equations of the fifth degree*. Dover Publications Inc., New York, N.Y., revised edition, 1956. Translated into English by George Gavin Morrice.

[22] Felix Klein. *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Birkhäuser Verlag, Basel, 1993. Reprint of the 1884 original, Edited, with an introduction and commentary by Peter Slodowy.

[23] V. Knibbeler, S. Lombardo, and J. A. Sanders. Higher dimensional Automorphic Lie Algebras. In preparation (2014).