ABELIAN HYPERCOMPLEX STRUCTURES ON CENTRAL EXTENSIONS OF H-TYPE LIE ALGEBRAS

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Abstract. It is the aim of this work to give a characterization of the 2-step nilpotent Lie algebras carrying abelian hypercomplex structures. In the special case of trivial extensions of irreducible H-type Lie algebras this characterization is given in terms of the dimension of the commutator subalgebra. As a consequence, we obtain the corresponding theorem for arbitrary H-type Lie algebras, extending a result in [1].

1. Preliminaries

We start by giving the basic definitions. An *abelian* complex structure on a real Lie algebra $\mathfrak{g}$ is an endomorphism of $\mathfrak{g}$ satisfying

$$J^2 = -I, \qquad [Jx, Jy] = [x, y], \quad \forall x, y \in \mathfrak{g}. \tag{1}$$

The above conditions automatically imply the vanishing of the Nijenhuis tensor, that is, J is integrable. By an abelian hypercomplex structure we mean a pair of anticommuting abelian complex structures. Our main motivation for studying abelian hypercomplex structures comes from the fact that such structures provide examples of homogeneous HKT-geometries (where HKT stands for hyper-Kähler with torsion, cf. [4]).

It was proved in [1] that if $\dim [\mathfrak{g}, \mathfrak{g}] \leq 2$ then every hypercomplex structure on $\mathfrak{g}$ must be abelian. To complete the classification of the Lie algebras $\mathfrak{g}$ with $\dim [\mathfrak{g}, \mathfrak{g}] \leq 2$ carrying hypercomplex structures (cf. [1]) it remained to give a characterization in the case when $\mathfrak{g}$ is 2-step nilpotent and $\dim [\mathfrak{g}, \mathfrak{g}] = 2$: this is obtained by taking $m = 2$ in Corollary 1.2 below.

It is a result of [3] that the only 8-dimensional non-abelian nilpotent Lie algebras carrying abelian hypercomplex structures are trivial central extensions of H-type Lie algebras. We show that this does not hold for higher dimensions: there exist 2-step nilpotent Lie algebras which are not of type H carrying such structures.

Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a two-step nilpotent Lie algebra endowed with an inner product $\langle\ ,\ \rangle$ and consider the orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$, where $\mathfrak{z}$ is the center of $\mathfrak{n}$ and $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{z}$. Define a linear map $j : \mathfrak{z} \to \text{End}\,(\mathfrak{v})$, $z \mapsto j_z$, where j_z is determined as follows:

$$\langle j_z v, w \rangle = \langle z, [v, w]\, \rangle, \quad \forall v, w \in \mathfrak{v}. \tag{2}$$

Observe that $j_z,\ z \in \mathfrak{z}$, are skew-symmetric so that $z \to j_z$ defines a linear map $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{v})$. Note that $\text{Ker}(j)$ is the orthogonal complement of $[\mathfrak{n}, \mathfrak{n}]$ in $\mathfrak{z}$. In particular, $[\mathfrak{n}, \mathfrak{n}] = \mathfrak{z}$ if and only if j is injective. Conversely, any linear map j :

1991 *Mathematics Subject Classification.* Primary 17B30, 53C15; Secondary 32C10, 53C56.
Key words and phrases. Two-step nilpotent, hypercomplex, hyperhermitian.
The author was partially supported by CONICET (Argentina).

$\mathbb{R}^m \to \mathfrak{so}(k)$ gives rise to a 2-step nilpotent Lie algebra $\mathfrak{n}$ by means of (2). It follows that the center of $\mathfrak{n}$ is $\mathbb{R}^m \oplus (\cap_{z\in\mathbb{R}^m} \operatorname{Ker} j_z)$ and $[\mathfrak{n},\mathfrak{n}] \subseteq \mathbb{R}^m$ where equality holds precisely when j is injective. We say that $(\mathfrak{n}, \langle\ ,\ \rangle)$ is irreducible when $\mathfrak{v}$ has no proper subspaces invariant by all j_z, $z \in \mathfrak{z}$.

We show in the following lemma that a two-step nilpotent Lie algebra carrying an abelian complex structure amounts to a linear map $j : \mathfrak{z} \to \mathfrak{u}(k)$ (where $\dim \mathfrak{v} = 2k$ and $\mathfrak{u}(k)$ denotes the Lie algebra of the unitary group $U(k)$).

Lemma 1.1. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra carrying an abelian complex structure J. Then, for any Hermitian inner product $\langle\ ,\ \rangle$ on $\mathfrak{n}$, the endomorphisms j_z, $z \in \mathfrak{z}$, defined as in (2), commute with J.*

Proof. Given a Hermitian inner product $\langle\ ,\ \rangle$ on $\mathfrak{n}$ decompose $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ where $\mathfrak{v}$ is the orthogonal complement of $\mathfrak{z}$. It follows from (1) that J leaves $\mathfrak{z}$ stable and therefore, since J is orthogonal, also $\mathfrak{v}$ is J-stable. We show next that the restriction of J to $\mathfrak{v}$ commutes with the endomorphisms j_z, $z \in \mathfrak{z}$:

$$\begin{aligned} \langle J j_z v, w\rangle &= -\langle j_z v, Jw\rangle = -\langle z, [v, Jw]\rangle \\ &= \langle z, [Jv, w]\rangle = \langle j_z J v, w\rangle \end{aligned}$$

for all $v, w \in \mathfrak{v}$, $z \in \mathfrak{z}$, and this implies $J j_z = j_z J$ for every $z \in \mathfrak{z}$. □

As a consequence of the above lemma we obtain the following corollary, where we denote by $\mathfrak{sp}(k)$ the Lie algebra of the symplectic group $Sp(k)$:

Corollary 1.2. *Every injective linear map $j : \mathbb{R}^m \to \mathfrak{sp}(k)$ ($m \leq k(2k+1)$) gives rise to a two-step nilpotent Lie algebra $\mathfrak{n}$ with $\dim[\mathfrak{n},\mathfrak{n}] = m$ carrying an abelian hypercomplex structure. Conversely, any two step nilpotent Lie algebra carrying an abelian hypercomplex structure arises in this manner.*

Proof. Given $j : \mathbb{R}^m \to \mathfrak{sp}(k)$, fix $0 \leq s \leq 3$ with $s + m \equiv 0 \pmod 4$, set $\mathfrak{n} = \mathbb{R}^s \oplus \mathbb{R}^m \oplus \mathbb{R}^{4k}$ (orthogonal direct sum) with the canonical inner product on each summand and define the Lie bracket on $\mathfrak{n}$ by

$$\langle z, [v, w]\,\rangle = \langle j_z v, w\rangle, \quad \forall v, w \in \mathbb{R}^{4k},\ z \in \mathbb{R}^m$$

and $\langle z, [\mathfrak{n},\mathfrak{n}]\,\rangle = 0$ for $z \in \mathbb{R}^s$ ($\mathbb{R}^s \oplus \mathbb{R}^m$ is central). Let J_α, $\alpha = 1, 2$ be the anticommuting complex endomorphisms of $\mathbb{R}^{4k}$ defining $\mathfrak{sp}(k)$, that is, $\mathfrak{sp}(k) = \{T \in \mathfrak{so}(4k) : T J_\alpha = J_\alpha T,\ \alpha = 1, 2\}$. Extend J_α to all of $\mathfrak{n}$ with arbitrary (orthogonal) anticommuting endomorphisms on $\mathbb{R}^s \oplus \mathbb{R}^m$ satisfying $J_\alpha^2 = -I$. This defines an abelian hypercomplex structure on $\mathfrak{n}$. Observe that $[\mathfrak{n},\mathfrak{n}] = \mathbb{R}^m$ since j is injective.

Conversely, if $\mathfrak{n}$ is a two-step nilpotent Lie algebra carrying an abelian hypercomplex structure J_1, J_2, then there exists an inner product $\langle\ ,\ \rangle$ on $\mathfrak{n}$ which is hyperhermitian, that is, Hermitian with respect to both, J_1 and J_2. Let $\mathfrak{v}$ denote the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ and consider the linear map j defined by (2). It follows that j is injective on $[\mathfrak{n},\mathfrak{n}]$ and, from the above lemma, its image is contained in $\mathfrak{sp}(k) = \{T \in \mathfrak{so}(4k) : T J_\alpha = J_\alpha T,\ \alpha = 1, 2\}$, where $\dim \mathfrak{v} = 4k$. In particular, $\dim [\mathfrak{n},\mathfrak{n}] \leq \dim \mathfrak{sp}(k) = k(2k+1)$. □

Remark 1.3. Using the same idea as in the above corollary it is possible to construct hypercomplex structures on certain solvable Lie algebras. In fact, given a two step nilpotent Lie algebra $(\mathfrak{n}, \langle\ ,\ \rangle)$ set $\mathfrak{s} = \mathbb{R}a \oplus \mathfrak{n}$ with $[a, z] = z$, $\forall\, z \in \mathfrak{z}$, $[a, v] = \frac{1}{2} v$, $\forall\, v \in \mathfrak{v}$, where the inner product on $\mathfrak{v}$ is extended to $\mathfrak{s}$ by decreeing $a \perp \mathfrak{v}$ and

$\langle a, a\rangle = 1$. This solvable extension of $\mathfrak{n}$ has been studied by various authors ([2]). In the special case when $\dim \mathfrak{z} \equiv 3 \pmod 4$, $\dim \mathfrak{v} = 4k$ and the the endomorphisms j_z, $z \in \mathfrak{z}$, defined as in (2), belong to $\mathfrak{sp}(k)$, it can be shown that $\mathfrak{s}$ carries a hypercomplex (hyperhermitian) structure. The procedure is analogous to that in the preceding corollary (one can easily check the integrability conditions using results of [1]). It should be noted that these structures cannot be abelian and the corresponding metrics are not hyper-Kähler (since they are not flat).

2. A special class of two-step nilpotent Lie algebras

Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a two-step nilpotent Lie algebra with the orthogonal decomposition $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ as before and the endomorphisms j_z, $z \in \mathfrak{z}$, as in (2). Then $(\mathfrak{n}, \langle\ ,\ \rangle)$ is said to be an H-type algebra (or a Lie algebra of Heisenberg type, cf. [6]) if for any nonzero z in $\mathfrak{z}$ we have that $j_z^2 = -\langle z, z\rangle I$. In this case, the linear map $j : \mathfrak{z} \to \mathrm{End}(\mathfrak{v})$ extends to an algebra homomorphism $j : \mathsf{Cl}_m \to \mathrm{End}(\mathfrak{v})$, where $m = \dim \mathfrak{z}$ and Cl_m denotes the real Clifford algebra associated to $(\mathfrak{z}, \langle\ ,\ \rangle)$. The extended homomorphism j is a unitary representation of Cl_m on $\mathfrak{v}$, in other words, $\mathfrak{v}$ becomes a Cl_m-module with the elements of unit length in $\mathfrak{z}$ acting as orthogonal transformations of $\mathfrak{v}$. Conversely, any unitary representation of the Clifford algebra Cl_m on a real vector space $\mathfrak{v}$ gives rise to an H-type Lie algebra in the obvious way ([7]).

We say that two H-type algebras are isomorphic if there is an orthogonal Lie isomorphism between them.

Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a trivial central extension of an H-type algebra, that is, $(\mathfrak{n}, \langle\ ,\ \rangle)$ is a two step nilpotent Lie algebra such that $[\mathfrak{n}, \mathfrak{n}] \oplus \mathfrak{v}$ is of type H. The following theorem shows that, when $(\mathfrak{n}, \langle\ ,\ \rangle)$ is irreducible, the existence of an abelian complex or hypercomplex structure on $(\mathfrak{n}, \langle\ ,\ \rangle)$ depends on $m = \dim [\mathfrak{n}, \mathfrak{n}]$.

Theorem 2.1. *Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a trivial central extension of an H-type algebra and let $m = \dim [\mathfrak{n}, \mathfrak{n}]$. Assume that $(\mathfrak{n}, \langle\ ,\ \rangle)$ is irreducible.*

(i) $(\mathfrak{n}, \langle\ ,\ \rangle)$ carries an abelian complex structure such that $\langle\ ,\ \rangle$ is Hermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 2$ and $m \equiv 1, 2, 3, 4$ or $5 \pmod 8$.

(ii) $(\mathfrak{n}, \langle\ ,\ \rangle)$ carries an abelian hypercomplex structure such that $\langle\ ,\ \rangle$ is hyperhermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 4$ and $m \equiv 2, 3$ or $4 \pmod 8$.

The above complex structures arise naturally by extending certain endomorphisms which belong to the algebra $\mathcal{K}_m = \{T \in \mathrm{End}(\mathfrak{v}) : TS = ST\ \forall S \in \mathsf{Cl}_m\}$. We devote the next section to the study of this algebra (see, for example, [9] Chapter I, §5).

Remark 2.2. The Lie algebras obtained from an injective linear map $j : \mathbb{R}^m \to \mathfrak{sp}(1)$ with $m \leq 3$, as in Corollary 1.2, are trivial extensions of H-type algebras. This follows from the identification of $\mathfrak{sp}(1)$ with the space of pure imaginary quaternions (see also [3]).

Remark 2.3. Let d_m denote the dimension of any irreducible Cl_m-module and take k such that $4k$ is not a multiple of d_m and $k(2k+1) \geq m$. Then the Lie algebras arising from any injective linear map $j : \mathbb{R}^m \to \mathfrak{sp}(k)$ such that $\cap_{z\in\mathbb{R}^m} \mathrm{Ker}\, j_z = \{0\}$ are not trivial extensions of H-type algebras. For example, take $m = 10$ and $k = 2$ (in this case $d_{10} = 64$; see Table 1).

2.1. **The algebra $\mathcal{K}_{\mathbf{m}}$.** We start by recalling the definition of the Clifford algebra $\mathsf{Cl}_m = \mathsf{Cl}(E, \langle\ ,\ \rangle)$, where E is an m dimensional real vector space endowed with an inner product $\langle\ ,\ \rangle$. Fix an orthonormal basis $e_1, \dots, e_m$ of E and define Cl_m as the algebra generated by $1, e_1, \dots, e_m$ with relations

$$e_i e_j + e_j e_i = 0, \qquad i \neq j, \qquad\qquad e_i^2 = -1, \qquad i = 1, \dots, m.$$

Let $\mathcal{T}(E)$ be the tensor algebra of E and $\Im$ the two-sided ideal of $\mathcal{T}(E)$ generated by the elements $x \otimes x + \langle x, x\rangle 1$ for $e \in E$. Then the Clifford algebra Cl_m is defined to be the quotient $\mathcal{T}(E)/\Im$ and $\dim \mathsf{Cl}_m = 2^m$ (see [5] or [9] for details).

Let $\mathfrak{v}$ be an irreducible Cl_m-module. We want to study the algebra $\mathcal{K}_m = \{T \in \mathrm{End}(\mathfrak{v}) : TS = ST\ \forall\, S \in \mathsf{Cl}_m\}$. We observe that, being a real division associative algebra, $\mathcal{K}_m$ must be isomorphic to either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

Recall from [9] that when $m \not\equiv 3$ or $7 \pmod 8$ there is exactly one irreducible Cl_m-module $\mathfrak{v}$ up to isomorphism. If $m \equiv 3$ or $7 \pmod 8$ there are two equivalence classes of irreducible Cl_m-modules $\mathfrak{v}_1$, $\mathfrak{v}_2$ which give rise to isomorphic (irreducible) H-type algebras, so for our purposes we may assume $\mathfrak{v} = \mathfrak{v}_1$.

2.1.1. $\mathbf{m} \not\equiv \mathbf{3}$ *or* $\mathbf{7} \pmod{\mathbf{8}}$. Assume first that $m \not\equiv 3$ or $7 \pmod 8$. It follows from [9] that Cl_m is the real algebra $M_n(\mathbb{F})$ of $n \times n$ $\mathbb{F}$-matrices for some n, where $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and $\mathfrak{v} = \mathbb{F}^n$ (see Table 1). The action of Cl_m on $\mathfrak{v}$ is given by matrix multiplication, viewing $\mathbb{F}^n$ as $n \times 1$ matrices. It then follows that when $\mathbb{F} = \mathbb{R}$ then $\mathcal{K}_m$ is the centralizer of $M_n(\mathbb{R})$ in $M_n(\mathbb{R})$, hence $\mathcal{K}_m = \mathbb{R}$. It follows from [9] that this happens precisely when $m \equiv 6$ or $8 \pmod 8$.

We next study the cases $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. Consider the inclusions $M_n(\mathbb{C}) \hookrightarrow M_{2n}(\mathbb{R})$ and $M_n(\mathbb{H}) \hookrightarrow M_{4n}(\mathbb{R})$ given as follows:

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}, \quad A + iB + jC + kD \mapsto \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix} \tag{3}$$

where A, B, C, D are real $n \times n$ matrices. It then follows that when $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{F} = \mathbb{H}$) $\mathcal{K}_m$ is the centralizer of $M_n(\mathbb{C})$ (resp. $M_n(\mathbb{H})$) in $M_{2n}(\mathbb{R})$ (resp. $M_{4n}(\mathbb{R})$). It can be shown by direct calculation that this centralizer equals $\mathbb{C}$ (resp. $\mathbb{H}$) when $\mathbb{F} = \mathbb{C}$ (resp. $\mathbb{F} = \mathbb{H}$). We observe that $\mathbb{F} = \mathbb{C}$ precisely when $m \equiv 1$ or $5 \pmod 8$ and $\mathbb{F} = \mathbb{H}$ when $m \equiv 2$ or $4 \pmod 8$ (cf. [9]).

2.1.2. $\mathbf{m} \equiv \mathbf{3}$ *or* $\mathbf{7} \pmod{\mathbf{8}}$. If $m \equiv 3 \pmod 8$ then $\mathsf{Cl}_m = M_n(\mathbb{H}) \oplus M_n(\mathbb{H})$ for some n (Table 1) and we can take $\mathfrak{v} = \mathbb{H}^n \oplus 0$ with the obvious action. It follows that $\mathcal{K}_m = \mathbb{H}$.

On the other hand, when $m \equiv 7 \pmod 8$ then $\mathsf{Cl}_m = M_n(\mathbb{R}) \oplus M_n(\mathbb{R})$ for some n (Table 1) and we can take $\mathfrak{v} = \mathbb{R}^n \oplus 0$. It follows that $\mathcal{K}_m = \mathbb{R}$.

The above paragraphs can be summarized as follows (see also Table 1):

Proposition 2.4. *Let $\mathfrak{v}$ be an irreducible Cl_m-module. Then*

$$\mathcal{K}_m = \begin{cases} \mathbb{R} & \text{if } m \equiv 6, 7 \text{ or } 8 \pmod 8, \\ \mathbb{C} & \text{if } m \equiv 1 \text{ or } 5 \pmod 8, \\ \mathbb{H} & \text{if } m \equiv 2, 3 \text{ or } 4 \pmod 8. \end{cases}$$

2.1.3. **The inner product on $\mathfrak{v}$.** It follows from the irreducibility of the Cl_m-module $\mathfrak{v}$ that it admits a unique inner product (up to a positive multiple) such that the elements of unit length in E act by orthogonal transformations. This inner product is given as follows (cf. [10]):

$$\langle v, w\rangle = \begin{cases} v^t w & \text{if } \mathbb{F}=\mathbb{R}, \\ \Re\mathrm{e}(v^t \bar{w}) & \text{if } \mathbb{F}=\mathbb{C} \text{ or } \mathbb{H} \end{cases}$$

where $\mathfrak{v}=\mathbb{F}^n$ is thought of as $n\times 1$ matrices, v^t denotes the transpose of $v\in\mathbb{F}^n$ and $\bar{w}$ is the conjugate of $w\in\mathbb{F}^n$. Let $O(\mathfrak{v})$ denote the orthogonal group of endomorphisms of $\mathfrak{v}$ relative to $\langle\ ,\ \rangle$ and set $\mathcal{C}=\{J\in \mathrm{End}\,(\mathfrak{v}) \,:\, J^2=-1\}$. The aim is to obtain a parametrization of the intersection $\mathcal{K}_m\cap O(\mathfrak{v})\cap\mathcal{C}$, which could be empty (in fact, this is the case when $\mathbb{F}=\mathbb{R}$). We study first $\mathcal{K}_m\cap O(\mathfrak{v})$ (see also [9], Chapter I, §5.16).

Lemma 2.5.

$$\mathcal{K}_m\cap O(\mathfrak{v}) = \begin{cases} \mathbb{Z}_2 & \text{if } \mathbb{F}=\mathbb{R}, \\ U(1) & \text{if } \mathbb{F}=\mathbb{C}, \\ Sp(1) & \text{if } \mathbb{F}=\mathbb{H}. \end{cases}$$

Proof. Let I denote the identity $n\times n$ matrix and $C_{\mathcal{B}}(\mathcal{A})$ the centralizer of $\mathcal{A}$ in $\mathcal{B}$. Using the inclusions (3) and the inner product on $\mathfrak{v}$, it is not hard to verify the following equalities:

$$\begin{aligned}
C_{M_n(\mathbb{R})}(M_n(\mathbb{R}))\cap O(\mathfrak{v}) &= \{\pm I\}\cong\mathbb{Z}_2, \\
C_{M_{2n}(\mathbb{R})}(M_n(\mathbb{C}))\cap O(\mathfrak{v}) &= \left\{\begin{pmatrix} aI & -bI \\ bI & aI\end{pmatrix} : a,b\in\mathbb{R},\ a^2+b^2=1\right\}\cong U(1), \\
C_{M_{4n}(\mathbb{R})}(M_n(\mathbb{H}))\cap O(\mathfrak{v}) &= \left\{\begin{pmatrix} aI & bI & cI & dI \\ -bI & aI & -dI & cI \\ -cI & dI & aI & -bI \\ -dI & -cI & bI & aI\end{pmatrix} : \begin{array}{c} a,b,c,d\in\mathbb{R}, \\ a^2+b^2+c^2+d^2=1\end{array}\right\}\cong Sp(1).
\end{aligned}$$

□

Corollary 2.6.

$$\mathcal{K}_m\cap O(\mathfrak{v})\cap\mathcal{C} = \begin{cases} \emptyset & \text{if } \mathbb{F}=\mathbb{R}, \\ \mathbb{Z}_2 & \text{if } \mathbb{F}=\mathbb{C}, \\ \text{the 2-sphere } S^2 & \text{if } \mathbb{F}=\mathbb{H}. \end{cases}$$

Proof. We must intersect the sets obtained in the previous lemma with $\mathcal{C}$. We thus obtain

$$\begin{aligned}
C_{M_n(\mathbb{R})}(M_n(\mathbb{R}))\cap O(\mathfrak{v})\cap\mathcal{C} &= \emptyset, \\
C_{M_{2n}(\mathbb{R})}(M_n(\mathbb{C}))\cap O(\mathfrak{v})\cap\mathcal{C} &= \left\{\pm\begin{pmatrix} 0 & -I \\ I & 0\end{pmatrix}\right\}\cong\mathbb{Z}_2, \\
C_{M_{4n}(\mathbb{C})}(M_n(\mathbb{H}))\cap O(\mathfrak{v})\cap\mathcal{C} &= \left\{\begin{pmatrix} 0 & bI & cI & dI \\ -bI & 0 & -dI & cI \\ -cI & dI & 0 & -bI \\ -dI & -cI & bI & 0\end{pmatrix} : \begin{array}{c} b,c,d\in\mathbb{R}, \\ b^2+c^2+d^2=1\end{array}\right\}\cong S^2.
\end{aligned}$$

□

TABLE 1

$m \pmod 8$	n	Cl_m	$\mathfrak{v}$	$\mathcal{K}_m$	$\mathcal{K}_m \cap O(\mathfrak{v})$	$\mathcal{K}_m \cap O(\mathfrak{v}) \cap \mathcal{C}$
1	16^q	$M_n(\mathbb{C})$	$\mathbb{C}^n$	$\mathbb{C}$	$U(1)$	$\mathbb{Z}_2$
2	16^q	$M_n(\mathbb{H})$	$\mathbb{H}^n$	$\mathbb{H}$	$Sp(1)$	S^2
3	16^q	$M_n(\mathbb{H}) \oplus M_n(\mathbb{H})$	$\mathbb{H}^n$	$\mathbb{H}$	$Sp(1)$	S^2
4	$2(16)^q$	$M_n(\mathbb{H})$	$\mathbb{H}^n$	$\mathbb{H}$	$Sp(1)$	S^2
5	$4(16)^q$	$M_n(\mathbb{C})$	$\mathbb{C}^n$	$\mathbb{C}$	$U(1)$	$\mathbb{Z}_2$
6	$8(16)^q$	$M_n(\mathbb{R})$	$\mathbb{R}^n$	$\mathbb{R}$	$\mathbb{Z}_2$	$\emptyset$
7	$8(16)^q$	$M_n(\mathbb{R}) \oplus M_n(\mathbb{R})$	$\mathbb{R}^n$	$\mathbb{R}$	$\mathbb{Z}_2$	$\emptyset$
8	16^{q+1}	$M_n(\mathbb{R})$	$\mathbb{R}^n$	$\mathbb{R}$	$\mathbb{Z}_2$	$\emptyset$

Table 1 summarizes the above results. The integer q is obtained from $m = 8q+r$ where $1 \leq r \leq 8$.

Proof of Theorem 2.1. (i) Assume J is an abelian complex structure on $(\mathfrak{n}, \langle\ ,\ \rangle)$ such that $\langle\ ,\ \rangle$ is Hermitian. Since J leaves $\mathfrak{z}$ stable, it follows that $\dim \mathfrak{z} \equiv 0 \pmod 2$. Lemma 1.1 says that $J \in \mathcal{K}_m \cap O(\mathfrak{v}) \cap \mathcal{C}$, which is nonempty if and only if $m \equiv 1, 2, 3, 4$ or $5 \pmod 8$ by Corollary 2.6. This proves the if part of (i). For the only if part, fix an endomorphism $J \in \mathcal{K}_m \cap O(\mathfrak{v}) \cap \mathcal{C}$, which always exists when $m \equiv 1, 2, 3, 4$ or $5 \pmod 8$, and extend J to all of $\mathfrak{n}$ with any orthogonal endomorphism of $\mathfrak{z}$ satisfying $J^2 = -I$.

(ii) It follows from Corollary 2.6 that there exists a pair of anticommuting endomorphisms in $\mathcal{K}_m \cap O(\mathfrak{v}) \cap \mathcal{C}$ if and only if $m \equiv 2, 3$ or $4 \pmod 8$. This proves the if part of (ii). For the only if part, extend a pair of anticommuting endomorphisms $J_1, J_2 \in \mathcal{K}_m \cap O(\mathfrak{v}) \cap \mathcal{C}$ with anticommuting orthogonal endomorphisms of $\mathfrak{z}$ satisfying $J_\alpha^2 = -I,\ \alpha = 1, 2$.

□

3. General H-type algebras

Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be an H-type algebra with center $\mathfrak{z}$, $\dim \mathfrak{z} = m$, and let $\mathfrak{v}_0$ denote an irreducible Cl_m-module. It is well known that $\mathfrak{v}$, the orthogonal complement of $\mathfrak{z}$, is formed by taking several copies of $\mathfrak{v}_0$ (cf. [8]). The precise situation is as follows.

If $m \not\equiv 3$ or $7 \pmod 8$ then the general H-type algebra is, modulo isomorphisms, $\mathfrak{n} = (\mathfrak{v}_0)^r \oplus \mathfrak{z}$ with bracket

$$[v, w] = [v_1, w_1] + \cdots + [v_r, w_r]$$

for $v = (v_1, \ldots, v_r)$, $w = (w_1, \ldots, w_r) \in (\mathfrak{v}_0)^r$, where the bracket on each copy of $\mathfrak{v}_0$ is given by the Cl_m-module structure on $\mathfrak{v}_0$.

If $m \equiv 3$ or $7 \pmod 8$ then the general H-type algebra is, modulo isomorphisms, $\mathfrak{n} = (\mathfrak{v}_0)^p \oplus (\mathfrak{v}_0)^q \oplus \mathfrak{z}$ with bracket

$$[(v, x), (w, y)] = [v_1, w_1] + \cdots + [v_p, w_p] - [x_1, y_1] - \cdots - [x_q, y_q]$$

for $v = (v_1, \ldots, v_p)$, $w = (w_1, \ldots, w_p) \in (\mathfrak{v}_0)^p$, $x = (x_1, \ldots, x_q)$, $y = (y_1, \ldots, y_q) \in (\mathfrak{v}_0)^q$, where two pairs of exponents p, q and r, s give isomorphic algebras if and only if $\{p, q\} = \{r, s\}$.

Table 2 is obtained from the results in the previous section.

We now state the general theorem about existence of abelian complex and hypercomplex structures on trivial central extensions of H-type Lie algebras. The proof,

TABLE 2

$m \pmod 8$	$\mathfrak{v}$	$\mathcal{K}_m$	$\mathcal{K}_m \cap O(\mathfrak{v})$
1 or 5	$(\mathfrak{v}_0)^r$	$\mathfrak{gl}(r,\mathbb{C})$	$U(r)$
2 or 4	$(\mathfrak{v}_0)^r$	$\mathfrak{gl}(r,\mathbb{H})$	$Sp(r)$
6 or 8	$(\mathfrak{v}_0)^r$	$\mathfrak{gl}(r,\mathbb{R})$	$O(r)$
3	$(\mathfrak{v}_0)^p \oplus (\mathfrak{v}_0)^q$	$\mathfrak{gl}(p,\mathbb{H}) \oplus \mathfrak{gl}(q,\mathbb{H})$	$Sp(p) \times Sp(q)$
7	$(\mathfrak{v}_0)^p \oplus (\mathfrak{v}_0)^q$	$\mathfrak{gl}(p,\mathbb{R}) \oplus \mathfrak{gl}(q,\mathbb{R})$	$O(p) \times O(q)$

which is analogous to that of Theorem 2.1, follows from Table 2. Observe that when $m \equiv 2, 3$ or $4 \pmod 8$ (case (ii) below) then there is no obstruction (except for the obvious ones) for the existence of such structures. The integers p, q and r below are those introduced at the beginning of this section (see also Table 2).

Theorem 3.1. *Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a trivial central extension of an H-type algebra and let $m = \dim\,[\mathfrak{n},\mathfrak{n}]$.*

(i) If $m \equiv 1$ or $5 \pmod 8$ then $(\mathfrak{n}, \langle\ ,\ \rangle)$ carries an abelian complex structure such that $\langle\ ,\ \rangle$ is Hermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 2$. It carries an abelian hypercomplex structure such that $\langle\ ,\ \rangle$ is hyperhermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 4$ and $r \equiv 0 \pmod 2$.

(ii) If $m \equiv 2, 3$ or $4 \pmod 8$ then $(\mathfrak{n}, \langle\ ,\ \rangle)$ carries an abelian complex structure such that $\langle\ ,\ \rangle$ is Hermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 2$. It carries an abelian hypercomplex structure such that $\langle\ ,\ \rangle$ is hyperhermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 4$.

(iii) If $m \equiv 6$ or $8 \pmod 8$ then $(\mathfrak{n}, \langle\ ,\ \rangle)$ carries an abelian complex structure such that $\langle\ ,\ \rangle$ is Hermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 2$ and $r \equiv 0 \pmod 2$. It carries an abelian hypercomplex structure such that $\langle\ ,\ \rangle$ is hyperhermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 4$ and $r \equiv 0 \pmod 4$.

(iv) If $m \equiv 7 \pmod 8$ then $(\mathfrak{n}, \langle\ ,\ \rangle)$ carries an abelian complex structure such that $\langle\ ,\ \rangle$ is Hermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 2$, $p \equiv 0 \pmod 2$ and $q \equiv 0 \pmod 2$. It carries an abelian hypercomplex structure such that $\langle\ ,\ \rangle$ is hyperhermitian if and only if $\dim \mathfrak{z} \equiv 0 \pmod 4$, $p \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 4$.

Remark 3.2. The hyperhermitian metrics just considered, which are obtained as the riemannian product of the euclidean metric by a metric of Heisenberg type, are not flat. Therefore, these metrics are not hyper-Kähler.

Remark 3.3. Let $(\mathfrak{n}, \langle\ ,\ \rangle)$ be a trivial central extension of an H-type algebra and N the corresponding connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ endowed with the left invariant metric induced by $\langle\ ,\ \rangle$. It is well known that N admits a discrete cocompact subgroup Γ (cf. [8], [10]). Any invariant complex or hypercomplex structure on N induces one on the compact nilmanifold $\Gamma\backslash N$. Theorem 3.1 thus extends Corollary 1.4 in [1], yielding a rich family of compact hyperhermitian nilmanifolds (which are not hyper-Kähler, by the preceding remark).

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