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Parametric and nonparametric A-Laplace problems: Existence of solutions and asymptotic analysis

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Abstract. We give sufficient conditions for the existence of weak solutions to quasilinear elliptic Dirichlet problem driven by the A-Laplace operator in a bounded domain Ω. The techniques, based on a variant of the symmetric mountain pass theorem, exploit variational methods. We also provide information about the asymptotic behavior of the solutions as a suitable parameter goes to 0^+. In this case, we point out the existence of a blow-up phenomenon. The analysis developed in this paper extends and complements various qualitative and asymptotic properties for some cases described by homogeneous differential operators.

Keywords: Dirichlet boundary value problem, A-Laplace operator, asymptotic analysis, Orlicz–Sobolev space

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. In this paper, we study the following quasilinear elliptic Dirichlet problem

$$-\operatorname{div}\big(a\big(|\nabla u|\big)\nabla u\big) = f(z,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

This problem is driven by a differential operator $a(t)t \in C(\mathbb{R})$, the so-called A-Laplace operator. We ask about the existence of the solutions in $L^\infty(\Omega)$. To this goal the reaction $f(z,t) \in C(\overline{\Omega} \times \mathbb{R})$ obeys assumption (f_1). On the other hand, we impose that $a : [0,+\infty) \to [0,+\infty)$ satisfy suitable hypotheses to include relevant classes of functions. Motivations arise from the literature review as follows. We recall the nice work of Cencelj–Rădulescu–Repovš [5] on double phase problems in variable exponent Lebesgue–Sobolev spaces, where the authors point out as the study of nonlinear problems is strongly related to the description of significant phenomena in applied sciences (see also Papageorgiou–Rădulescu–Repovš [12,13,15], and the book of Breit [4, Chapter 2]). For anisotropic double-phase problems we refer to Bahrouni–Rădulescu–Repovš [3], Ragusa–Tachikawa [18], and Zhang–Rădulescu [23].

Here, we mention that existence and multiplicity results for quasilinear elliptic problems were established by Tan–Fang [20], in the Orlicz–Sobolev spaces. Papageorgiou–Vetro [16] proved multiplicity results in variable exponent Lebesgue–Sobolev spaces, Vetro [21] studied semilinear Robin problems of Laplace operator using Lyapunov–Schmidt reduction method, and Vetro [22] considered mixed

Dirichlet–Neumann problems with the (p, q)-Laplace operator. Also, the existence of multiple positive solutions for quasilinear elliptic problems with nonhomogeneous principal part a was established by Fukagai–Narukawa [8] in the Orlicz–Sobolev spaces. Very recently, Alves–De Holanda–Santos [2] proved the existence of positive weak solutions for a semipositone problem driven by a A-Laplace operator, with subcritical growth of the reaction.

We recall that the Orlicz spaces are a genuine extension of L^p spaces $(1 \leqslant p < +\infty)$, whenever a N-function (that is, a convex, even function $A : \mathbb{R} \to [0, +\infty)$ satisfying $A(t) = 0$ if and only if $t = 0$, $\lim_{t\to 0} \frac{A(t)}{t} = 0$, and $\lim_{t\to +\infty} \frac{A(t)}{t} = +\infty$) replaces the function $t \to |t|^p$ in the definition of the L^p space. Under suitable conditions, Orlicz–Sobolev spaces (extension of the $W^{1,p}$ spaces) are an interesting source of solutions of constrained optimization problems for the energy functional related to (1). Indeed, as stated in Fukagay–Ito–Narukawa [7] the usual Sobolev space is not useful to deal with general forms of the operator a in (1). For example let $a(t)t \in C(\mathbb{R})$ be a function whose primitive is the function $A(t) = [1 + t^2]^\eta - 1$ with $\eta \in \mathbb{R} \setminus \{1\}$, which means nonlinear elasticity in a physical setting if $\eta > 1/2$. We know that $A(t)$ acts as $2\eta t^2$ as t goes to zero, and acts as $t^{2\eta}$ as t goes to $\pm\infty$. Thus, the Eulero energy functional associated to problem (1) (namely (6) of Proposition 1 below) cannot be well-defined in both the Sobolev spaces $W_0^{1,2}(\Omega)$ and $W_0^{1,2\eta}(\Omega)$ (since no one of these spaces includes the other). This fact motivates the use of the Orlicz–Sobolev space defined in Section 2 to deal with problem (1) (see again [7]).

In this paper we establish some existence results using variational tools together with growth conditions on the reaction. In details the paper is organized as follows. In Section 2 we collect the basic facts on the working spaces and N-functions. In Section 3, by using the Palais–Smale condition and a mountain pass theorem for the energy functional associated to problem (1), we establish the existence of at least one nontrivial weak solution of (1) in $C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. The working conditions on the reaction f concern its behavior near zero and at infinity, plus some technical hypotheses. In Section 4, introducing a parameter in the reaction, we prove two results concerning the asymptotic behavior of the solutions as the parameter goes to zero. For closely related results work, we refer to Papageorgiou–Vetro–Vetro [17]. Some of the abstract methods used in this paper can be found in the recent monograph Papageorgiou–Rădulescu–Repovš [14].

2. Mathematical background

We introduce the function space framework for problem (1). So, we recall some facts on Orlicz and Orlicz–Sobolev spaces (see also Adams–Fournier [1] and Rao–Ren [19]).

For a N-function $A : \mathbb{R} \to [0, +\infty)$, we have the representation

$$A(t) = \int_0^{|t|} \zeta(\xi)\, d\xi, \quad t \in \mathbb{R}, \tag{2}$$

with $\zeta : [0, +\infty) \to [0, +\infty)$ being a right derivative of A. Also, it is non-decreasing and right continuous such that

$$\zeta(\xi) > 0 \quad \text{for all } \xi > 0, \qquad \lim_{\xi\to +\infty} \zeta(\xi) = +\infty, \qquad \zeta(0) = 0.$$

Of course, whenever ζ meets the above conditions, then A, given by (2), is a N-function. For our further use, we put $\zeta(\xi) = a(\xi)\xi$ for all $\xi \in [0, +\infty)$ so that (2) reduces to

$$A(t) = \int_0^{|t|} a(\xi)\xi \, d\xi, \quad t \in \mathbb{R}. \tag{3}$$

The hypotheses on $a : [0, +\infty) \to [0, +\infty)$ are as follows:

(a_1) $a \in C^1(0, +\infty)$, $a(t) > 0$, $(a(t)t)' > 0$ for any $t > 0$;
(a_2) there exist $q, p \in (1, N)$, with $q \leqslant p < q^*$, such that $q \leqslant \frac{A'(t)t}{A(t)} \leqslant p$ for any $t > 0$, where A is defined by (3) and $q^* = Nq/(N - q)$;
(a_3) there exist $a_0, a_1 > 0$ such that $a_0 \leqslant \frac{A''(t)t}{A'(t)} \leqslant a_1$ for any $t > 0$.

The real function $a(t) = qct^{q-1} + pCt^{p-1}$ for all $t \in [0, +\infty)$, with $q < p$ and $c, C \geqslant 0$ where $c + C > 0$, satisfies the above hypotheses.

We mention that (a_1) and (a_2) imply that A in (3) is a N-function which satisfies the inequality

$$A(2t) \leqslant kA(t), \quad \text{for all } t > 0, \text{ some } k > 0 \text{ (say } \Delta_2\text{-condition).}$$

Now, A admits a conjugate $\widetilde{A}$ given as

$$\widetilde{A}(t) = \sup\{\tau t - A(\tau) : \tau \geqslant 0\}.$$

Remark 1. Hypothesis (a_1) implies that $\frac{A(s)}{s}$ is increasing for $s > 0$ and so

$$\widetilde{A}\left(\frac{A(s)}{s}\right) \leqslant \frac{A(s)}{s}s = A(s) \quad \text{for } s > 0.$$

A finite-valued N-function Ψ is said to increase essentially more slowly than another N-function A near infinity if

$$\lim_{t\to+\infty} \frac{\Psi(\lambda t)}{A(t)} = 0 \quad \text{for every } \lambda \in \mathbb{R} \text{ with } \lambda > 0.$$

The Orlicz space $L^A(\Omega)$, associated with a N-function A satisfying the Δ_2-condition, is the Banach function space of those measurable functions $u : \Omega \to \mathbb{R}$ such that the Luxemburg norm

$$\|u\|_A = \inf\left\{\lambda > 0 : \int_\Omega A\left(\frac{|u|}{\lambda}\right) dz \leqslant 1\right\}$$

is finite. We note that $L^A(\Omega) = L^p(\Omega)$ if $A(t) = |t|^p$ for some $p \in [1, +\infty)$, and $L^A(\Omega) = L^\infty(\Omega)$ if $A(t) = 0$ for $t \in [0, 1]$ and $A(t) = +\infty$ for $t > 1$. Later on, we denote with $\|\cdot\|_p$ the norm in $L^p(\Omega)$.

The Δ_2-condition leads us to say that the dual space $L^A(\Omega)^*$ is identified with $L^{\widetilde{A}}(\Omega)$.

We also recall the Hölder type inequality

$$\int_\Omega |uv| \, dz \leqslant 2\|u\|_A \|v\|_{\widetilde{A}} \quad \text{for all } u \in L^A(\Omega), \text{ all } v \in L^{\widetilde{A}}(\Omega).$$

Let $V^{1,A}(\Omega)$ be the Sobolev type space

$$V^{1,A}(\Omega) = \big\{u : u \text{ is weakly differentiable on } \Omega \text{ and } |\nabla u| \in L^A(\Omega)\big\}.$$

We consider the Orlicz–Sobolev space $W^{1,A}(\Omega)$ defined by

$$W^{1,A}(\Omega) = V^{1,A}(\Omega) \cap L^A(\Omega),$$

equipped with the norm $\|u\|_{1,A} = \||\nabla u|\|_A + \|u\|_A$.

As usual, $W_0^{1,A}(\Omega)$ stands for the closure in $W^{1,A}(\Omega)$ of the set of smooth compactly supported functions on Ω. Hypothesis (a_2) implies that $\widetilde{A}$ satisfies the Δ_2-condition. So, $L^A(\Omega)$, $W^{1,A}(\Omega)$ and $W_0^{1,A}(\Omega)$ are separable and reflexive Banach spaces and the functional

$$I(u) = \int_\Omega A\big(|\nabla u|\big)\,dz \quad \text{for all } u \in W_0^{1,A}(\Omega)$$

is Fréchet differentiable. Hypotheses (a_1)–(a_3) ensure the validity of some elementary inequalities listed in the following lemmas (see [7,8]).

Lemma 1. *If (a_1), (a_2) hold, then whenever $m_1(t) = \min\{t^q, t^p\}$ and $m_2(t) = \max\{t^q, t^p\}$, $t > 0$, we have:*

(i) $m_1(k)A(t) \leqslant A(kt) \leqslant m_2(k)A(t)$ *for all* $k, t \geqslant 0$;
(ii) $m_1(\|u\|_A) \leqslant \int_\Omega A(|u|)\,dz \leqslant m_2(\|u\|_A)$ *for all* $u \in L^A(\Omega)$.

Lemma 2. *If (a_1)–(a_3) hold, one can find $k_0 > 0$ satisfying*

$$\big(a\big(|w|\big)w - a\big(|v|\big)v\big)(w-v) \geqslant k_0 \frac{A(|w-v|)^{(q+1)/q}}{(A(|w|)+A(|v|))^{1/q}}$$

for all $v, w \in \mathbb{R}^N$ with $w \neq 0$.

Now, the Poincaré inequality for A can be stated as follows (see the details in Gossez [9], Lemma 2): There exists $\Theta > 0$ such that

$$\int_\Omega A\big(|u|\big)\,dz \leqslant \Theta \int_\Omega A\big(|\nabla u|\big)\,dz \quad \text{for all } u \in W_0^{1,A}(\Omega). \tag{4}$$

We will use $\|\cdot\| = \||\nabla u|\|_A$ as the norm of $W_0^{1,A}(\Omega)$ (recall that this norm is equivalent to $\|u\|_{1,A}$). By A_* we mean the Sobolev's conjugate N-function of A given as

$$A_*^{-1}(t) = \int_0^t \frac{A^{-1}(s)}{s^{(N+1)/N}}\,ds \quad \text{for } t > 0.$$

Hypotheses (a_1) and (a_2) imply that A_* and $\widetilde{A}_*$ are N-functions satisfying the Δ_2-condition (see [7], Lemma 2.7). Note that

$$q^* \leqslant \frac{A_*'(t)t}{A_*(t)} \leqslant p^* \quad \text{for all } t > 0.$$

We recall that Donaldson–Trudinger [6] showed that there exists a constant $S_N > 0$ such that

$$\|u\|_{A_*} \leqslant S_N \|u\| \quad \text{for all } u \in W_0^{1,A}(\Omega),$$

which means that the embedding $W_0^{1,A}(\Omega) \hookrightarrow L^{A_*}(\Omega)$ is continuous. If Ω is a bounded domain and B is a N-function satisfying

$$\limsup_{t\to+\infty} \frac{B(t)}{A_*(t)} = 0, \tag{5}$$

then $W_0^{1,A}(\Omega) \hookrightarrow L^B(\Omega)$ is a compact embedding. In particular, $W_0^{1,A}(\Omega) \hookrightarrow L^A(\Omega)$ is compact too.

Remark 2. We note that Lemma 1(ii) implies:

(j) $\int_\Omega A(|u|)\,dz < +\infty$ for all $u \in L^A(\Omega)$;

(jj) a sequence $\{u_n\}_{n\geqslant 1} \subset L^A(\Omega)$ converges to some $u \in L^A(\Omega)$ if and only if

$$\lim_{n\to+\infty} \int_\Omega A\big(|u_n - u|\big)\,dz = 0;$$

(jjj) a sequence $\{u_n\}_{n\geqslant 1} \subset L^A(\Omega)$ is bounded in $L^A(\Omega)$ if and only if

$$\left\{\int_\Omega A\big(|u_n|\big)\,dz\right\}_{n\geqslant 1} \quad \text{is bounded.}$$

Moreover, we point out that A increases essentially more slowly than A_* near infinity. In fact, for $t \geqslant 1$ we have

$$0 \leqslant \frac{A(\lambda t)}{A_*(t)} \leqslant \frac{A(\lambda)t^p}{A_*(1)t^{q^*}} \to 0 \quad \text{as } t \to +\infty, \text{ since } p < q^*.$$

The next lemma states a convergence result in a Orlicz–Sobolev space.

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and suppose that (a_1)–(a_2) hold true. If $u \in W_0^{1,A}(\Omega)$ and $\{u_n\}_{n\geqslant 1}$ is such that $u_n \xrightarrow{w} u$ in $W_0^{1,A}(\Omega)$ and*

$$\lim_{n\to+\infty} \int_\Omega a\big(|\nabla u_n|\big)\nabla u_n(\nabla u_n - \nabla u)\,dz = 0,$$

then u_n converges to u in $W_0^{1,A}(\Omega)$.

Proof. Firstly, we note that $u_n \xrightarrow{w} u$ in $W_0^{1,A}(\Omega)$ yields

$$\lim_{n\to+\infty} \int_\Omega a\big(|\nabla u|\big)\nabla u \nabla(u_n - u)\,dz = 0.$$

So, we get

$$\lim_{n\to+\infty}\int_\Omega \big(a(|\nabla u_n|)\nabla u_n - a(|\nabla u|)\nabla u\big)(\nabla u_n - \nabla u)\,dz = 0.$$

Since the sequence $\{u_n\}_{n\geqslant 1}$ is bounded, the Hölder inequality and Lemma 2 lead to

$$\begin{aligned}
&\int_\Omega A\big(|\nabla u_n - \nabla u|\big)\,dz \\
&\quad\leqslant \left(\int_\Omega \frac{A(|\nabla u_n - \nabla u|)^{(q+1)/q}}{(A(|\nabla u_n|) + A(|\nabla u|))^{1/q}}\,dz\right)^{q/(q+1)} \left(\int_\Omega \big(A\big(|\nabla u_n|\big) + A\big(|\nabla u|\big)\big)\,dz\right)^{1/(q+1)} \\
&\quad\leqslant M\left(\frac{1}{k_0}\int_\Omega \big(a\big(|\nabla u_n|\big)\nabla u_n - a\big(|\nabla u|\big)\nabla u\big)(\nabla u_n - \nabla u)\,dz\right)^{q/(q+1)} \quad \text{for some } M > 0 \\
&\quad\to 0 \quad \text{as } n \to +\infty.
\end{aligned}$$

So, by Remark 2(jj), we conclude that $u_n \to u$ in $W_0^{1,A}(\Omega)$, as $n \to +\infty$. $\square$

3. One nontrivial weak solution

We recall that $u \in W_0^{1,A}(\Omega)$ is a weak solution of (1) whenever

$$\int_\Omega a\big(|\nabla u|\big)\nabla u \nabla v\,dz = \int_\Omega f(z,u)v\,dz \quad \text{for any } v \in W_0^{1,A}(\Omega).$$

For the sake of clarity, we recall the Palais–Smale condition too.

Definition 1. Let $W_0^{1,A}(\Omega)^*$ be the topological dual of $W_0^{1,A}(\Omega)$. Then, $I : W_0^{1,A}(\Omega) \to \mathbb{R}$ satisfies the Palais–Smale condition if any sequence $\{u_n\}_{n\geqslant 1}$ such that

(i) $\{I(u_n)\}_{n\geqslant 1}$ is bounded;
(ii) $\lim_{n\to+\infty} \|I'(u_n)\|_{W_0^{1,A}(\Omega)^*} = 0$,

has a convergent subsequence.

A sequence $\{u_n\}_{n\geqslant 1}$ satisfying Definition 1(i)–(ii), is called a Palais–Smale sequence for the functional I. Here, we use the following inequality:

$$\limsup_{|t|\to+\infty}\left(\sup_{z\in\Omega} \frac{\beta F(z,t) - tf(z,t)}{A(|t|)}\right) < \frac{\beta - p}{\Theta}, \tag{f_0}$$

for some $\beta > p$ where Θ is as in (4) and $F(z,t) = \int_0^t f(z,\xi)d\xi$.

Now, we consider the following condition (see assumption (f_*) of [20]):

(f_1) $f(z,0) = 0$ for all $z \in \overline{\Omega}$ and there are a N-function B such that $B' : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism, and constants $\alpha_0, \alpha_1 \geqslant 0$ with $|f(z,t)| \leqslant \alpha_0 + \alpha_1 B'(|t|)$ for all $z \in \overline{\Omega}$, $t \in \mathbb{R}$, and

$$\lim_{t\to+\infty} \frac{B(t)}{A_*(t)} = 0,$$

and

$$p < b^- := \inf_{t>0} \frac{tB'(t)}{B(t)} \leqslant \sup_{t>0} \frac{tB'(t)}{B(t)} := b^+ < q^*.$$

We establish the following result.

Proposition 1. *If* (f_0), (f_1) *hold, then the functional* $I : W_0^{1,A}(\Omega) \to \mathbb{R}$ *defined by*

$$I(u) = \int_\Omega A\big(|\nabla u|\big)\,dz - \int_\Omega F(z,u)\,dz \quad \textit{for all } u \in W_0^{1,A}(\Omega) \tag{6}$$

satisfies the Palais–Smale condition.

Proof. Using (f_0), we choose $\rho \in (0, \frac{\beta-p}{\Theta})$ such that

$$\rho > \limsup_{|t|\to+\infty} \left(\sup_{z\in\Omega} \frac{\beta F(z,t) - tf(z,t)}{A(|t|)} \right).$$

Then, we can find $t^* > 0$ such that

$$\beta F(z,t) - tf(z,t) \leqslant \rho A\big(|t|\big) \quad \text{for all } |t| \geqslant t^*, z \in \Omega.$$

So, there exists $\delta > 0$ satisfying

$$\beta F(z,t) - tf(z,t) \leqslant \rho A\big(|t|\big) + \delta \quad \text{for all } t \in \mathbb{R}, z \in \Omega. \tag{7}$$

Let $\{u_n\}_{n\geqslant1}$ be a Palais–Smale sequence in $W_0^{1,A}(\Omega)$ for the functional I. Set $\varepsilon_n := \|I'(u_n)\|$. Since $\{I(u_n)\}_{n\geqslant1}$ is bounded, (6) and

$$\big\langle I'(u_n), v\big\rangle = \int_\Omega a\big(|\nabla u_n|\big)\nabla u_n \nabla v\,dz - \int_\Omega f(z,u_n)v\,dz \quad \text{for all } u_n, v \in W_0^{1,A}(\Omega), n \in \mathbb{N},$$

imply that we can find a constant L satisfying

$$\begin{aligned} L + \varepsilon_n\|u_n\| &= L + \varepsilon_n \big\| |\nabla u_n| \big\|_A \\ &\geqslant \beta I(u_n) - \big\langle I'(u_n), u_n\big\rangle \\ &= \beta \int_\Omega A\big(|\nabla u_n|\big)\,dz - \beta \int_\Omega F(z,u_n)\,dz - \int_\Omega a\big(|\nabla u_n|\big)|\nabla u_n|^2\,dz + \int_\Omega f(z,u_n)u_n\,dz \end{aligned}$$

$$\begin{aligned}
&\geqslant (\beta - p)\int_\Omega A\big(|\nabla u_n|\big)\,dz - \int_\Omega \big[\beta F(z,u_n) - f(z,u_n)u_n\big]\,dz \quad \big(\text{by } (a_2)\big)\\
&\geqslant (\beta - p)\int_\Omega A\big(|\nabla u_n|\big)\,dz - \rho\int_\Omega A\big(|u_n|\big)\,dz - \delta|\Omega| \quad \big(\text{by } (7)\big)\\
&\geqslant (\beta - p)\int_\Omega A\big(|\nabla u_n|\big)\,dz - \rho\Theta\int_\Omega A\big(|\nabla u_n|\big)\,dz - \delta|\Omega| \quad \big(\text{by } (4)\big)\\
&\geqslant (\beta - p - \rho\Theta)\int_\Omega A\big(|\nabla u_n|\big)\,dz - \delta|\Omega|\\
&\geqslant (\beta - p - \rho\Theta)m_1\big(\||\nabla u_n|\|_A\big) - \delta|\Omega|\\
&\geqslant (\beta - p - \rho\Theta)\big\||\nabla u_n|\big\|_A^q - \delta|\Omega| \quad (\text{by Lemma 1, if } \||\nabla u_n|\|_A \geqslant 1),
\end{aligned}$$

where $|\Omega|$ is the Lebesgue measure of Ω. If the sequence $\{\|u_n\|\}_{n\geqslant 1}$ is not bounded, from

$$L + \varepsilon_n\|u_n\| = L + \varepsilon_n\big\||\nabla u_n|\big\|_A \geqslant (\alpha - p - \rho\Theta)\big\||\nabla u_n|\big\|_A^q - \delta|\Omega|$$

for infinite values of n large enough,

we obtain a contradiction (recall that $\|u_n\| = \||\nabla u_n|\|_A$). So, the sequence $\{u_n\}_{n\geqslant 1}$ is bounded in $W_0^{1,A}(\Omega)$. Consequently, $\{u_n\}_{n\geqslant 1}$ admits a subsequence (namely $\{u_n\}_{n\geqslant 1}$ too) such that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,A}(\Omega) \text{ and } u_n \to u \text{ in } L^B(\Omega) \quad \big(\text{recall (5), since } B(t)/A_*(t) \to 0 \text{ as } t \to +\infty\big).$$

We note that the condition (f_1) ensures:

- $f(\cdot, u_n(\cdot)) \in L^{\widetilde{B}}(\Omega)$ for all $n \in \mathbb{N}$, where $\widetilde{B}$ is the conjugate of B;
- $\{f(\cdot, u_n(\cdot))\}_{n\geqslant 1}$ is bounded in $L^{\widetilde{B}}(\Omega)$.

Using Hölder inequality, we infer that

$$\lim_{n\to+\infty}\int_\Omega \big|f(z,u_n)\big||u_n - u|\,dz = 0.$$

From

$$\int_\Omega a\big(|\nabla u_n|\big)\nabla u_n\nabla(u_n - u)\,dz = \big\langle I'(u_n), u_n - u\big\rangle + \int_\Omega f(z,u_n)(u_n - u)\,dz,$$

we get

$$\lim_{n\to+\infty}\int_\Omega a\big(|\nabla u_n|\big)\nabla u_n\nabla(u_n - u)\,dz = 0$$

and by Lemma 3, we conclude that $u_n \to u$ in $W_0^{1,A}(\Omega)$ as $n \to +\infty$. □

Remark 3. Note that the condition (f_0) is motivated by Assumption 2.1(iv) of [10]. Also, (f_0) is weaker than the Ambrosetti–Rabinowitz condition (see Remark 2.3 of [10]).

In the sequel we will use the following conditions:

(f_2) there exist $\varepsilon \in (0, \Theta^{-1})$ and $\delta_\varepsilon > 0$ such that $F(z,t) \leqslant \varepsilon A(|t|)$ for a.a. $z \in \Omega$, all $|t| \leqslant \delta_\varepsilon$;
(f_3) $\lim_{|t|\to+\infty} \frac{F(z,t)}{|t|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
(f_3^+) $\lim_{t\to+\infty} \frac{F(z,t)}{t^p} = +\infty$ uniformly for a.a. $z \in \Omega$;
(f_3^-) $\lim_{t\to-\infty} \frac{F(z,t)}{|t|^p} = +\infty$ uniformly for a.a. $z \in \Omega$.

We establish our next result in the form of a lemma.

Lemma 4. *If (f_1), (f_2) hold and f satisfies also (f_3^+) or (f_3^-), then*

(i) *there exist $\rho > 0$ and $\sigma > 0$ such that $I(u) \geqslant \sigma$ for each $u \in W_0^{1,A}(\Omega)$ with $\|u\| = \rho$;*
(ii) *there exists $v \in W_0^{1,A}(\Omega)$ such that $0 > I(v)$ and $\rho < \|v\|$.*

Proof. (i). Since $W_0^{1,A}(\Omega) \hookrightarrow L^B(\Omega)$ continuously, there is a constant $C_B > 0$ satisfying

$$\|u\|_B \leqslant C_B\|u\| \quad \text{for all } u \in W_0^{1,A}(\Omega). \tag{8}$$

Using (f_1) and (f_2), we can find a constant $C_\varepsilon > 0$ satisfying

$$F(z,t) \leqslant \varepsilon A\big(|t|\big) + C_\varepsilon B\big(|t|\big) \quad \text{for a.a. } z \in \Omega, \text{ all } t \in \mathbb{R}. \tag{9}$$

If $u \in W_0^{1,A}(\Omega)$ is such that $\max\{\|u\|, C_B\|u\|\} < 1$, by (8) and (9) we have

$$\begin{aligned}
I(u) &= \int_\Omega A\big(|\nabla u|\big)\,dz - \int_\Omega F(z,u)\,dz \\
&\geqslant \int_\Omega A\big(|\nabla u|\big)\,dz - \varepsilon\int_\Omega A\big(|u|\big)\,dz - C_\varepsilon\int_\Omega B\big(|u|\big)\,dz \\
&\geqslant (1-\varepsilon\Theta)\int_\Omega A\big(|\nabla u|\big)\,dz - C_\varepsilon\|u\|_B^{b^-} \quad \big(\text{see (4) and Lemma 1(ii)}\big) \\
&\geqslant (1-\varepsilon\Theta)m_1\big(\||\nabla u|\|_A\big) - C_\varepsilon C_B^{b^-}\|u\|^{b^-} \\
&= (1-\varepsilon\Theta)\|u\|^p - C_\varepsilon C_B^{b^-}\|u\|^{b^-} \\
&= \big[(1-\varepsilon\Theta) - C_\varepsilon C_B^{b^-}\|u\|^{b^- - p}\big]\|u\|^p.
\end{aligned}$$

Choosing $0 < \rho < \min\{1, C_B^{-1}\}$ with

$$\vartheta = (1-\varepsilon\Theta) - C_\varepsilon C_B^{b^-}\rho^{b^- - p} > 0,$$

then we have $I(u) \geqslant \vartheta\rho^p = \sigma > 0$ for all $u \in W_0^{1,A}(\Omega)$ such that $\|u\| = \rho$.

(ii). Assume that f satisfies (f_3^+). By (f_1) and (f_3^+), for all $L > 0$ we can find a constant $C_L > 0$ satisfying

$$F(z,t) \geqslant Lt^p - C_L \quad \text{for a.a. } z \in \Omega, \text{ all } t > 0. \tag{10}$$

Set $w \in W_0^{1,A}(\Omega)$ with $w(z) > 0$ for all $z \in \Omega$. From (10), for all $t > 1$ we have

$$\begin{aligned} I(tw) &= \int_\Omega A\big(|\nabla tw|\big)\,dz - \int_\Omega F(z, tw)\,dz \\ &\leqslant t^p \int_\Omega A\big(|\nabla w|\big)\,dz - Lt^p \|w\|_p^p + C_L|\Omega| \\ &= t^p\bigg[\int_\Omega A\big(|\nabla w|\big)\,dz - L\|w\|_p^p\bigg] + C_L|\Omega|. \end{aligned}$$

Choosing $L > 0$ with

$$\int_\Omega A\big(|\nabla w|\big)\,dz - L\|w\|_p^p < 0,$$

then $I(tw) \to -\infty$ as $t \to +\infty$. Consequently there is $v = t_0 w \in W_0^{1,A}(\Omega)$ with $0 > I(v)$ and $\rho < \|v\|$.

The same conclusion holds if we assume that f satisfies (f_3^-). □

For reader convenience, we recall the following version of Mountain Pass Theorem (see Theorem 5.40 of [11]).

Theorem 1. *If $I \in C^1(W_0^{1,A}(\Omega))$ satisfies the (C_c)-condition, there exist $u_0, u_1 \in W_0^{1,A}(\Omega)$ and $\rho > 0$ such that*

$$\|u_0 - u_1\| > \rho, \qquad \max\big\{I(u_0), I(u_1)\big\} < \inf\big\{I(u) : \|u - u_0\| = \rho\big\} = m_\rho, \quad \text{and}$$
$$c = \inf_{\gamma\in\Gamma}\max_{0\leqslant t\leqslant 1} I\big(\gamma(t)\big) \quad \text{with } \Gamma = \big\{\gamma \in C\big([0,1], W_0^{1,A}(\Omega)\big) : \gamma(0) = u_0, \gamma(1) = u_1\big\},$$

then $c \geqslant m_\rho$ and c is a critical value of I (that is, there exists $\widehat{u} \in W_0^{1,A}(\Omega)$ such that $I'(\widehat{u}) = 0$ and $I(\widehat{u}) = c$).

Remark 4. We recall that $I \in C^1(W_0^{1,A}(\Omega))$ satisfies the (C_c)-condition, if every sequence $\{u_n\}_{n\geqslant 1} \subset W_0^{1,A}(\Omega)$ such that $I(u_n) \to c \in \mathbb{R}$ and $(1 + \|u_n\|_A)I'(u_n) \to 0$ in $W_0^{1,A}(\Omega)^*$ as $n \to +\infty$, admits a convergent subsequence. Note that if I satisfies the Palais–Smale condition then it satisfies the (C_c)-condition.

By Proposition 1, Lemma 4 and Remark 4, the functional I defined in (6) satisfies the assumptions of Theorem 1. So, it admits a critical value $c \geqslant m_\rho > 0$.

Resuming we establish the existence of one nontrivial weak solution of (1) in the following result. By Corollary 3.1 of [20] this solution is in $C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

Theorem 2. *If (f_0)–(f_2), (f_3^+) (or (f_3^-)) hold, then problem (1) admits at least one nontrivial weak solution $\widehat{u} \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.*

4. The parametric case: Existence and blow-up of solutions

In this section, we study the following parametric version of problem (1):

$$-\operatorname{div}\big(a\big(|\nabla u|\big)\nabla u\big) = \lambda f(z,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{11}$$

where $\lambda > 0$. In particular, we are interested in the existence of high energy solutions, that is, solutions with higher and higher energies as the positive parameter becomes smaller and smaller.

As a consequence of Theorem 2 we deduce the following existence result.

Theorem 3. *If (f_0)–(f_2), (f_3^+) (or (f_3^-)) hold, then problem (11) admits for all $\lambda \in (0,1]$ at least one nontrivial weak solution $\widehat{u}_\lambda \in C_0^{1,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$.*

Now, we show that for small values of the parameter $\lambda > 0$ problem (1) has a solution $u_\lambda \in W_0^{1,A}(\Omega)$ such that $\lim_{\lambda\to 0^+} \|u_\lambda\| = +\infty$.

Lemma 5. *If (f_1) holds, then there exist positive constants m_λ and ρ_λ such that $\lim_{\lambda\to 0^+} m_\lambda = +\infty$ and $I_\lambda(u) \geqslant m_\lambda > 0$ for all $u \in W_0^{1,A}(\Omega)$ such that $\|u\| = \rho_\lambda$.*

Proof. Let $u \in W_0^{1,A}(\Omega)$ with $\|u\| > 1$. From (f_1), we deduce that there is $C > 0$ with

$$\big|F(z,t)\big| \leqslant C\big(1 + B\big(|t|\big)\big), \tag{12}$$

for all $(x,t) \in \Omega \times \mathbb{R}$, $p < b^- \leqslant b^+ < q^*$. Consequently, we have

$$\begin{aligned}
I_\lambda(u) &= \int_\Omega A\big(|\nabla u|\big)\,dz - \lambda \int_\Omega F(z,u)\,dz \\
&\geqslant \int_\Omega A\big(|\nabla u|\big)\,dz - \lambda C \int_\Omega B\big(|u|\big)\,dz - \lambda C|\Omega| \\
&\geqslant m_1\big(\||\nabla u|\|\big) - \lambda C m_2\big(\|u\|_B\big) - \lambda C|\Omega| \quad \big(\text{see Lemma 1(ii)}\big) \\
&\geqslant \|u\|^q - \lambda C m_2\big(C_B\|u\|\big) - \lambda C|\Omega| \\
&\geqslant \|u\|^q - \lambda C \max\big\{C_B^{b^+}, C_B^{b^-}\big\}\|u\|^{b^+} - \lambda C|\Omega|.
\end{aligned}$$

Let $\rho_\lambda = \lambda^{-\sigma}$ with $0 < \sigma < \frac{1}{b^+ - q}$, so that $\rho_\lambda > 1$ for $\lambda > 0$ small enough. Putting $\|u\| = \rho_\lambda = \lambda^{-\sigma}$ in the above inequality, we get

$$I_\lambda(u) \geqslant \lambda^{-\sigma q} - C \max\big\{C_B^{b^+}, C_B^{b^-}\big\}\lambda^{1-\sigma b^+} - \lambda C|\Omega|.$$

Now, set $m_\lambda = \lambda^{-\sigma q} - C \max\{C_B^{b^+}, C_B^{b^-}\}\lambda^{1-\sigma b^+} - \lambda C|\Omega|$. As $0 < \sigma < \frac{1}{b^+ - q}$, then we can find λ_0 small enough such that $m_\lambda > 0$ for all $0 < \lambda < \lambda_0$ and $m_\lambda \to +\infty$ as $\lambda \to 0^+$. $\square$

Theorem 4. *If (f_0), (f_1), (f_3) hold, then there exists $\lambda_0 \in (0,1]$ such that, for all $0 < \lambda < \lambda_0$, Problem (11) has at least one weak solution $u_\lambda \in W_0^{1,A}(\Omega)$ and $\|u_\lambda\| \to +\infty$ as $\lambda \to 0^+$.*

Proof. By Proposition 1, the functional I_λ satisfies the (C_c)-condition for all $\lambda \in (0,1]$. Thanks to Proposition 1, Lemma 5 and Lemma 4(ii) all the hypotheses of the mountain pass theorem are satisfied and so, there exists a nontrivial critical point u_λ for I_λ such that

$$I_\lambda(u_\lambda) = c_\lambda \geqslant m_\lambda.$$

On the other hand, from (12), we have

$$\begin{aligned} I_\lambda(u_\lambda) &\leqslant \int_\Omega A(|\nabla u_\lambda|)\,dz + \lambda \int_\Omega |F(z,u_\lambda)|\,dz \\ &\leqslant m_2(\|\nabla u_\lambda\|_A) + \lambda C m_2(\|u_\lambda\|_B) + \lambda C|\Omega|. \end{aligned}$$

Taking the limit as $\lambda \to 0^+$ in the previous inequality, and using Lemma 5 one has $\lim_{\lambda\to 0^+}\|u_\lambda\| = +\infty$. □

The new condition on the function $f(z,t)$ in the reaction is the following:
(f_4): There exists $\tau \in (1,p)$ and $\delta, \widehat{c}$ such that

$$\widehat{c}|t|^\tau \leqslant F(z,t) \quad \text{for a.a. } z \in \Omega, \text{ all } |t| \leqslant \delta.$$

Theorem 5. *If hypotheses (f_1), (f_4) hold, then we can find $\widehat{\lambda} \in (0,1)$ such that for all $\lambda \in (0,\widehat{\lambda})$ problem (11) has a nontrivial solution $\widehat{u}_\lambda \in W_0^{1,A}(\Omega)$ and $\|\widehat{u}_\lambda\| \to 0^+$ as $\lambda \to 0^+$.*

Proof. We consider again the functional $I_\lambda : W_0^{1,A}(\Omega) \to \mathbb{R}$ related to problem (11) and given as

$$I_\lambda(u) = \int_\Omega A(|\nabla u|)\,dz - \lambda \int_\Omega F(z,u)\,dz \quad \text{for all } u \in W_0^{1,A}(\Omega).$$

We know that there is $C_\tau > 0$ such that $\|u\|_\tau \leqslant C_\tau\|u\|$. Hypotheses (f_1), (f_4) imply that

$$|F(z,t)| \leqslant C\big[|t|^\tau + B(|t|)\big] \quad \text{for a.a. } z \in \Omega, \text{ all } t \in \mathbb{R}, \text{ some } C > 0. \tag{13}$$

Let $0 < \sigma < \frac{1}{p}$. Then for $u \in W_0^{1,A}(\Omega)$ with $\|u\| = \lambda^\sigma < 1$, we have

$$\begin{aligned} I_\lambda(u) &\geqslant \int_\Omega A(|\nabla u|)\,dz - \lambda C\|u\|_\tau^\tau - \lambda C \int_\Omega B(|u|)\,dz \quad (\text{see } (13)) \\ &\geqslant m_1(\||\nabla u|\|) - \lambda C\|u\|_\tau^\tau - \lambda C m_2(C_B\|u\|) \\ &= \|u\|^p - \lambda C C_\tau^\tau \|u\|^\tau - \lambda C \max\{C_B^{b^+}, C_B^{b^-}\}\|u\|^{b^-}. \end{aligned}$$

As $\sigma p - 1 < 0$, then one can find $\widehat{\lambda} > 0$ such that for all $\lambda \in (0,\widehat{\lambda})$ we get

$$I_\lambda(u) > 0 \quad \text{for all } u \in W_0^{1,A}(\Omega) \text{ with } \|u\| = \lambda^\sigma. \tag{14}$$

Let $B_\lambda = \{u \in W_0^{1,A}(\Omega) : \|u\| < \lambda^\sigma\}$. The reflexivity of $W_0^{1,A}(\Omega)$ and the Eberlein–Smulian theorem imply that $\overline{B}_\lambda$ is sequentially weakly compact. Now, I_λ is sequentially weakly lower semicontinuous (note that $W_0^{1,A}(\Omega)) \hookrightarrow L^p(\Omega)$ compactly). By the Weierstrass–Tonelli theorem, we have $\widehat{u}_\lambda \in W_0^{1,A}(\Omega)$ such that

$$I_\lambda(\widehat{u}_\lambda) = \min\bigl[I_\lambda(u) : u \in \overline{B}_\lambda\bigr]. \tag{15}$$

Let $u \in C_0^1(\overline{\Omega}) \subset W_0^{1,A}(\Omega)$ with $u(z) > 0$ for all $z \in \Omega$. Then we can find $t \in (0,1)$ small such that $0 < tu(z) \leqslant \delta$ for all $z \in \Omega$, where $\delta > 0$ is as postulated by hypothesis H_2(ii). We have

$$\begin{aligned} I_\lambda(tu) &\leqslant \int_\Omega A\bigl(t|u|\bigr)\,dz - \lambda C\|tu\|_\tau^\tau \quad \bigl(\text{see hypothesis } H_2(\text{ii})\bigr) \\ &\leqslant m_2(t)\int_\Omega A\bigl(|u|\bigr)\,dz - \lambda C t^\tau \|u\|_\tau^\tau \\ &= t^\tau\biggl[t^{p-\tau}\int_\Omega A\bigl(|u|\bigr)\,dz - \lambda C\|u\|_\tau^\tau\biggr]. \end{aligned}$$

Since $1 < \tau < p$, choosing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &I_\lambda(tu) < 0, \\ &\quad\Rightarrow \quad I_\lambda(\widehat{u}_\lambda) < 0 = I_\lambda(0) \quad (\text{see } (15)), \\ &\quad\Rightarrow \quad \widehat{u}_\lambda \neq 0. \end{aligned} \tag{16}$$

Also from (14) and (16) it follows that

$$\|\widehat{u}_\lambda\| < \lambda^\sigma. \tag{17}$$

Therefore $\widehat{u}_\lambda \in B_\lambda \setminus \{0\}$. On account of (15) we have

$$\begin{aligned} &\widehat{u}_\lambda \in K_{I_\lambda}, \\ &\quad\Rightarrow \quad \widehat{u}_\lambda \text{ is a nontrivial solution of (11)}, \lambda \in (0, \widehat{\lambda}). \end{aligned}$$

From (17) we see that $\|u_{\widehat{\lambda}}\| \to 0^+$ as $\lambda \to 0^+$. $\square$

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