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A symmetric Bloch–Okounkov theorem

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Abstract

The algebra of so-called shifted symmetric functions on partitions has the property that for all elements a certain generating series, called the q-bracket, is a quasimodular form. More generally, if a graded algebra A of functions on partitions has the property that the q-bracket of every element is a quasimodular form of the same weight, we call A a quasimodular algebra. We introduce a new quasimodular algebra $\mathcal{T}$ consisting of symmetric polynomials in the part sizes and multiplicities.

Keywords: Modular forms, Eisenstein series, Partitions, Symmetric functions, Möbius inversion, Border strip tableaux

1 Introduction

Partitions of integers are related in interesting ways to modular forms, starting with the observation that the generating series of partitions is closely related to the Dedekind η-function, i.e.,

$$\sum_{\lambda\in\mathscr{P}} q^{|\lambda|} \;=\; \prod_{n>0}(1-q^n)^{-1} \;=\; q^{1/24}\eta(\tau)^{-1} \qquad (q=e^{2\pi i\tau}),$$

where $\mathscr{P}$ denotes set of all partitions and $|\lambda|$ denotes the integer λ is a partition of. Another example is the occurrence of modular forms in the proof of the partition congruences which go back to Ramanujan [1].

More recently, partitions were connected to (quasi)modular forms via the q-bracket. Given a function $f:\mathscr{P}\to\mathbb{Q}$, the *$q$-bracket* of f is defined as the following power series

$$\langle f\rangle_q \;=\; \frac{\sum_{\lambda\in\mathscr{P}} f(\lambda)q^{|\lambda|}}{\sum_{\lambda\in\mathscr{P}} q^{|\lambda|}} \in \mathbb{Q}[[q]]. \tag{1}$$

Before continuing, note that it is not surprising at all that for a well-chosen function f the q-bracket $\langle f\rangle_q$ is a quasimodular form, since it is easily seen that the map (1) from $\mathbb{Q}^{\mathscr{P}}$ to $\mathbb{Q}[[q]]$ is surjective. What *is* surprising is that one can find graded subalgebras A of $\mathbb{Q}^{\mathscr{P}}$ which (i) are "interesting" in the sense that they have an interpretation in combinatorics, enumerative geometry or another field of mathematics and (ii) have the property that the q-bracket of a homogeneous function $f\in A$ is quasimodular of the same weight as f. In this case we call A a *quasimodular algebra*. Note that the q-bracket is linear but not multiplicative, so in order to show that an algebra is quasimodular, it is not sufficient to show that the q-brackets of the generators of such an algebra are quasimodular. The aim of this paper is to introduce new quasimodular algebras.

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The Bloch–Okounkov theorem [3, Theorem 0.5] provided the first quasimodular algebra Λ^*. Write a partition λ as a non-increasing sequence $(\lambda_1, \lambda_2, \ldots)$ of non-negative integers with $|\lambda| = \sum_{i=1}^{\infty} \lambda_i$ finite. The $\mathbb{Q}$-algebra Λ^* is freely generated by the so-called *shifted symmetric power sums*

$$Q_k(\lambda) \;=\; c_k \;+\; \sum_{i=1}^{\infty}\big((\lambda_i - i + \tfrac{1}{2})^{k-1} - (-i + \tfrac{1}{2})^{k-1}\big) \qquad (k \geq 2), \tag{2}$$

where the c_k are constants given by $\frac{1}{x} + \sum_k c_k \frac{x^{k-1}}{(k-1)!} \;=\; \frac{1}{2\sinh(x/2)}$. The function Q_3 naturally occurs in the simplest case of the Gromov–Witten theory of an elliptic curve, as discovered by Dijkgraaf [7] and for which quasimodularity was proven rigorously in [14]. Quasimodularity of Λ^* is used in many recent works in enumerative geometry [4–6,12,13]. There are many other functions in invariants of partitions which turn out to be elements of Λ^*, for example symmetric polynomials in de modified Frobenius coordinates [23, Eq. 19]; the hook-length moments [5, Theorem 13.5] (see Sect. 7.1); central characters of the symmetric group [15, Proposition 3] and symmetric polynomials in the content vector of a partition [15, Proof of Theorem 4].

Previously, the Bloch–Okounkov algebra Λ^* and some generalizations to higher levels (see, e.g., [8,9]), were the only known quasimodular algebras. However, there are many examples of functions on partitions admitting a quasimodular q-bracket (and in general not belonging to Λ^*) [23, Sect. 9], for example the Möller transformation of functions with quasimodular q-bracket (defined by [23, Eq. 45] and recalled in Sect. 7), invariants $\mathcal{A}_P$ for every even polynomial defined in terms of the arm- and leg-lengths of a partition and the moment functions

$$S_k(\lambda) \;=\; -\frac{B_k}{2k} \;+\; \sum_{i=1}^{\infty} \lambda_i^{k-1} \qquad (k \text{ even}, B_k = k\text{th Bernoulli number}) \tag{3}$$

that also occur in the study of so-called spin Hurwitz numbers in the algebra of *supersymmetric polynomials* [10] (in that reference, these functions are only evaluated at strict partitions—partitions without repeated parts—and quasimodularity is shown for a correspondingly adapted q-bracket).

In this paper, we prove the stronger result that the algebra $\mathcal{S}$ generated by these moment functions S_k is quasimodular. Moreover, besides the pointwise product of functions on partitions, we define a second associative product $\odot$, called the *induced product* as it is inherited from the product of power series. The vector space $\mathrm{Sym}^{\odot}(\mathcal{S})$ generated by the elements in $\mathcal{S}$ under the induced product is strictly bigger than $\mathcal{S}$, is a quasimodular algebra for either of the two products, and has a particularly nice description in terms of functions $T_{k,l}$ depending not only on the parts of a partition, but also on their multiplicities. Here, the *multiplicity* $r_m(\lambda)$ of parts of size m in a partition λ is defined as the number of parts of λ of size m. More precisely, let $\mathcal{F}_l$ be the *Faulhaber polynomial* of positive integer degree l, defined by $\mathcal{F}_l(n) = \sum_{i=1}^{n} i^{l-1}$ for all $n \in \mathbb{Z}_{>0}$. Then, $T_{k,l}$ is given by

$$T_{k,l}(\lambda) \;=\; C_{k,l} \;+\; \sum_{m=1}^{\infty} m^k \mathcal{F}_l(r_m(\lambda)) \qquad (k \geq 0, l \geq 1, k+l \text{ even}) \tag{4}$$

with $C_{k,l}$ a constant equal to $-\frac{B_{k+l}}{2(k+l)}$ if $k=0$ or $l=1$ and 0 else. Let $\mathcal{T}$ be the algebra generated by all these $T_{k,l}$ under the pointwise product.

We show that $\mathrm{Sym}^{\odot}(\mathcal{S})$ and $\mathcal{T}$ are algebras for the pointwise product as well as for the induced product. In fact, the expression of elements of $\mathrm{Sym}^{\odot}(\mathcal{S})$ in terms of the $T_{k,l}$ implies that $\mathrm{Sym}^{\odot}(\mathcal{S})$ is a strict subalgebra of $\mathcal{T}$ (with respect to both products). Our main result is the following:

Theorem 1.1 *The algebras* $\mathrm{Sym}^{\odot}(\mathcal{S})$ *and* $\mathcal{T}$ *are quasimodular algebras with respect to the induced product.*

With respect to the pointwise product, these algebras are not quasimodular because of the following subtlety: The q-bracket of a homogeneous function f in $\mathcal{T}$ (with respect to the pointwise product) often is of mixed weight (i.e., a linear combination of quasimodular forms of weights bounded by the weight of f). By making use of the induced product, one can explain these lower weight quasimodular forms, as we do in Sect. 6. For example,

$$\langle T_{0,2}^2\rangle_q \;=\; G_2^2 \,+\, \frac{5}{6}G_4 \,+\, \frac{1}{6}G_2 \,+\, \frac{1}{288},$$

where G_2 and G_4 are the Eisenstein series defined by (6). The right-hand side is a quasimodular form of mixed weight, which is explained by the fact that

$$T_{0,2}^2 \;=\; T_{0,2}\odot T_{0,2} \,+\, \frac{5}{6}T_{0,4} \,+\, \frac{1}{6}T_{0,2} \,+\, \frac{1}{288},$$

is a linear combination of elements of $\mathcal{T}$ of different weights with respect to the induced product.

A main theme throughout this paper is the principle to establish all identities in $\mathbb{Q}^{\mathscr{P}}$ or $\mathcal{T}$ *before* taking the q-bracket, instead of doing these computations in $\mathbb{Q}[[q]]$ or the space of quasimodular forms $\widetilde{M}$. By doing so, we discover the algebraic structure of $\mathcal{T}$. Without having the induced product at one's disposal, for example when studying the shifted symmetric algebra Λ^*, this seems impossible. See the following table for an overview of situations where the principle is applied:

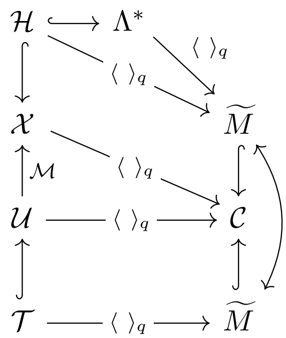
Previous definitions and results	Definitions and results in this work	Sections
Multiplication in $\mathbb{Q}[[q]]$	*Induced product* $\odot$ on $\mathbb{Q}^{\mathscr{P}}$	3.2
q-bracket: $\mathbb{Q}^{\mathscr{P}} \to \mathbb{Q}[[q]]$	*$\underline{u}$-bracket*: $\mathbb{Q}^{\mathscr{P}} \to \mathbb{Q}[[u_1,u_2,\ldots]]$	3.2
Connected q-bracket: $\mathrm{Sym}^{\otimes}(\mathbb{Q}^{\mathscr{P}}) \to \mathbb{Q}[[q]]$	*Connected product*: $\mathrm{Sym}^{\otimes}(\mathbb{Q}^{\mathscr{P}}) \to \mathbb{Q}^{\mathscr{P}}$	3.2
Derivative $q\frac{\mathrm{d}}{\mathrm{d}q}$ on $\mathbb{Q}[[q]]$	*Derivative* on $\mathbb{Q}^{\mathscr{P}}$	5.1
$\mathfrak{sl}_2$*-action* on $\widetilde{M}$	$\mathfrak{sl}_2$*-action* on $\mathcal{T}$	5.2
Rankin–Cohen brackets on $\widetilde{M}$	*Rankin–Cohen brackets* on $\mathcal{T}$	5.3
Formula for $\langle H_p f\rangle_q$ in [5, Eq. 152][a]	Formula for $T_{k,l}f$	6.2

[a]In that work the hook-length moment H_p (see also Sect. 7.1) was denoted by T_{p-1}

A further main result of the paper is the following:

Theorem 1.2 *The q-bracket is an equivariant mapping* $\mathcal{T} \to \widetilde{M}$ *with respect to* $\mathfrak{sl}_2$*-actions by derivations on both spaces.*

Motivated by the fact that many functions in invariants of partitions are elements of Λ^*, in Sect. 7 we describe many functions on partitions which are elements of $\mathcal{T}$ or are closely related. Among those are the border strip moments, generalizing the hook-length moments, which are defined in terms of the representation theory of the symmetric group. The corresponding space $\mathcal{X}$ of border strip moments is the image of a space $\mathcal{U}$ under the aforementioned Möller transform $\mathcal{M}$, where $\mathcal{U}$ is generated by the double moment functions $T_{k,l} \in \mathcal{T}$ as well as the odd double moments functions (those for which $k+l$ is odd). The q-brackets of these functions are contained in the space $\mathcal{C}$ of so-called *combinatorial Eisenstein series*, having the space of quasimodular forms as a subspace. Moreover, the space of hook-length moments $\mathcal{H}$ is contained in both Λ^* and $\mathcal{X}$—this contrasts the situation for $\mathcal{T}$, which by Remark 4.1.6 has a trivial intersection with Λ^*. See the commutative diagram below for an overview of the spaces related to $\mathcal{T}$ with their corresponding mappings.



We hope that this work—besides advocating the notion of a 'quasimodular algebra' by giving a new example of such an algebra and studying its algebraic structure—may serve as a tool for enumerative geometers trying to show that generating series are quasimodular forms.

The contents of the paper are as follows. In Sect. 2 we recall notions (known to the experts) related to quasimodular forms, partitions and special families of polynomials. Next, in Sect. 3 we motivate all new notions in this work and prove quasimodularity of the algebra $\mathcal{S}$. A study of the symmetric algebra $\mathcal{T}$, including a proof of our main theorem, can be found in Sect. 4. The $\mathfrak{sl}_2$-action by differential operators, the proof of Theorem 1.2 and Rankin–Cohen brackets are the content of Sect. 5. In Sect. 6 further results that arise from comparing the two different products on $\mathcal{T}$ are given, and finally, in Sect. 7 we provide many examples of functions in or closely related to $\mathcal{T}$.

2 Preliminaries

2.1 Quasimodular forms

Let $\mathrm{Hol}_0(\mathfrak{H})$ be the ring of holomorphic functions φ of moderate growth on the complex upper half plane $\mathfrak{H}$, i.e., for all $C>0$ one has $\varphi(x+iy)=O(e^{Cy})$ as $y\to\infty$ and $\varphi(x+iy)=O(e^{C/y})$ as $y\to 0$. A *quasimodular form* of *weight* k and *depth* at most p for $\mathrm{SL}_2(\mathbb{Z})$ is a function $\varphi\in\mathrm{Hol}_0(\mathfrak{H})$ such that there exist $\varphi_0,\ldots,\varphi_p\in\mathrm{Hol}_0(\mathfrak{H})$ so that for all $\tau\in\mathfrak{H}$ and all $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, one has

$$(c\tau+d)^{-k}\varphi\Big(\frac{a\tau+b}{c\tau+d}\Big) = \varphi_0(\tau)+\varphi_1(\tau)\frac{c}{c\tau+d}+\ldots+\varphi_p(\tau)\Big(\frac{c}{c\tau+d}\Big)^p. \qquad (5)$$

Equation (5) is called the *quasimodular transformation property*. Note that if φ is a quasimodular form, the functions $\varphi_0, \ldots, \varphi_p$ are quasimodular forms uniquely determined by φ (the function φ_r has weight $k-2r$ and depth $\leq p-r$). For example, taking the identity $I \in \Gamma$ yields $\varphi_0 = \varphi$. Quasimodular forms of depth 0 are called *modular forms*. Besides the constant functions, the simplest examples are the *Eisenstein series*

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{r=1}^{\infty}\sum_{m=1}^{\infty} m^{k-1}q^{mr} \qquad (B_k = k\text{th Bernoulli number and } q = e^{2\pi i\tau}) \tag{6}$$

for positive even integers k. For $k > 2$ the Eisenstein series are modular forms of weight k. The Eisenstein series G_2 is a quasimodular form of weight 2 and depth 1.

Denote by $\widetilde{M}_k^{(\leq p)}$ the vector space of quasimodular forms of weight k and depth at most p. Often we omit the depth and/or weight and simply write $\widetilde{M}_k$ for the vector space of all quasimodular forms of weight k or $\widetilde{M}$ for the graded algebra of all quasimodular forms. Let M denote the graded algebra of modular forms. The quasimodular form G_2 generates the algebra of quasimodular forms as an algebra over the subalgebra of modular forms, that is, $\widetilde{M} = M[G_2]$.

Often, when encountering an indexed collection of numbers or functions, we study its generating series. The generating series corresponding to the Eisenstein series is called the *propagator* or the *Kronecker–Eisenstein series of weight* 2 and given by

$$P(z,\tau) \;=\; P(z) \;:=\; \frac{1}{z^2} + 2\sum_{k=2}^{\infty} G_k \frac{z^{k-2}}{(k-2)!}. \tag{7}$$

The propagator is closely related to the Weierstrass $\wp$-function and Jacobi theta series

$$\wp(z,\tau) = \wp(z) := \frac{1}{z^2} + \sum_{\substack{\omega\in\mathbb{Z}\tau+\mathbb{Z}\\ \omega\neq 0}} \Big(\frac{1}{(z+\omega)^2} - \frac{1}{\omega^2}\Big), \tag{8}$$

$$\theta(z) \;:=\; \sum_{\nu\in\mathbb{Z}+\frac{1}{2}} (-1)^{\lfloor \nu\rfloor} e^{\nu z} q^{\nu^2/2}$$

by

$$P(z) \;=\; \frac{1}{2\pi i}\wp\big(\tfrac{z}{2\pi i},\tau\big) + 2G_2, \qquad P(z) \;=\; -\frac{\partial}{\partial z}\frac{\theta'(z)}{\theta(z)}.$$

2.2 The action of $\mathfrak{sl}_2$ on quasimodular forms by derivations

A way to produce examples of quasimodular forms is by taking derivatives of (quasi)modular forms under the differential operator $D : \widetilde{M}_k^{(\leq p)} \to \widetilde{M}_{k+2}^{(\leq p+1)}$, given by

$$D \;=\; \frac{1}{2\pi i}\frac{\mathrm{d}}{\mathrm{d}\tau} \;=\; q\frac{\mathrm{d}}{\mathrm{d}q}.$$

In fact, every quasimodular form can uniquely be written as a linear combination of derivatives of modular forms and derivatives of G_2. For more details, see [22, p. 58–60]. It may happen that a polynomial in the derivatives of two modular forms $f \in M_k$ and $g \in M_l$ is actually modular. This is the case for the *Rankin–Cohen brackets* of f and g, defined by

$$[f,g]_n \;=\; \sum_{\substack{r,s\geq 0\\ r+s=n}} (-1)^r \binom{k+n-1}{s}\binom{l+n-1}{r} D^r f\, D^s g \qquad (n \geq 0).$$

That is, for all $f \in M_k, g \in M_l$ and $n \geq 0$, one has that $[f,g]_n$ is a modular form of weight $k+l+2n$.

Besides the differential operator D, an important differential operator on quasimodular forms is the operator $\mathfrak{d} : \widetilde{M}_k^{(\leq p)} \to \widetilde{M}_{k-2}^{(\leq p-1)}$ defined by $\varphi \mapsto 2\pi i \varphi_1$ (with φ_1 defined in the quasimodular transformation property (5)). For example $\mathfrak{d}G_2 = -\frac{1}{2}$ and in fact this property together with the fact that $\mathfrak{d}$ annihilates modular forms defines $\mathfrak{d}$ completely since $\mathfrak{d}$ is a derivation and $\widetilde{M} = M[G_2]$.

Let W be the weight operator, which multiplies a quasimodular form by its weight. The triple $(D, \mathfrak{d}, W)$ forms an $\mathfrak{sl}_2$-*triple* with respect to the commutator bracket $[A,B] = AB - BA$:

Definition 2.2.1 A triple (X, Y, H) of operators is called an $\mathfrak{sl}_2$-*triple* if

$$[H, X] \;=\; 2X, \quad [H, Y] \;=\; -2Y, \quad [Y, X] \;=\; H.$$

Remark 2.2.2 By these commutation relations, for all $n \geq 1$ one has

$$[\mathfrak{d}, D^n] \;=\; n(W - n + 1)D^{n-1}, \tag{9}$$

which turns out to be useful later.

Following a suggestion of Zagier, we make the following definition:

Definition 2.2.3 Given a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-algebra is an algebra A together with a Lie homomorphism $\mathfrak{g} \to \mathrm{Der}(A)$.

As D, $\mathfrak{d}$ and W satisfy the Leibniz rule, the algebra $\widetilde{M}$ becomes an $\mathfrak{sl}_2$-algebra.

2.3 Partitions as a partially ordered set

Given $n \in \mathbb{Z}_{\geq 0}$, let $\mathscr{P}(n)$ denote the set of all integer partitions of n and $\Pi(n)$ the set of all partitions of the set $[n] := \{1, 2, \ldots, n\}$. Let $\mathscr{P} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathscr{P}(n)$ and $\Pi = \bigcup_{n \in \mathbb{Z}_{\geq 0}} \Pi(n)$ be the sets of all such partitions. Given $\lambda \in \mathscr{P}(n)$ we write $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geq \lambda_2 \geq \ldots$ and $|\lambda| := \sum_{i=1}^{\infty} \lambda_i = n$. The largest index k such that $\lambda_k > 0$ is called the length of λ, denoted by $\ell(\lambda)$. Similarly, for $\alpha \in \Pi(n)$ we write $\ell(\alpha)$ for the cardinality of α. Moreover, for $\lambda \in \mathscr{P}$ we let $r_m(\lambda)$ denote the number of parts of λ equal to m, i.e., $r_m(\lambda) = \#\{i \mid \lambda_i = m\}$, and denote by λ' the conjugate partition of λ. We call a partition λ *strict* if there are no repeated parts, i.e., $r_m(\lambda) \in \{0, 1\}$ for all m. For two partitions κ, λ we write $\kappa \cup \lambda$ for the union of κ and λ as multisets, i.e., $r_m(\kappa \cup \lambda) = r_m(\kappa) + r_m(\lambda)$ for all $m \in \mathbb{N}$.

Both $\mathscr{P}$ and $\Pi(n)$ form a locally finite partially ordered set, i.e., a partially ordered set P for which for all $x, z \in P$ there exists finitely many $y \in P$ such that $x \leq y \leq z$. Namely, on $\mathscr{P}$ we define a partial order by $\kappa \leq \lambda$ if $r_m(\kappa) \leq r_m(\lambda)$ for all $m \geq 1$. The ordering on $\Pi(n)$ is given by $\alpha \leq \beta$ if for all $A \in \alpha$ there exists a $B \in \beta$ such that $A \subseteq B$. For instance, we have $\alpha \leq \mathbf{1}_n$ for all $\alpha \in \Pi(n)$, where $\mathbf{1}_n = \{[n]\}$.

Recall that on a locally finite partially ordered set P the Möbius function $\mu : P^2 \to \mathbb{Z}$ is defined recursively by (see for example [16]): $\mu(x,z) \;=\; -\sum_{x \leq y \leq z} \mu(x,y)$ if $x < z$ with initial conditions $\mu(x,x) = 1$ and $\mu(x,z) = 0$ else. For the above partial order on $\mathscr{P}$ the value of $\mu(\kappa, \lambda)$ depends on whether the difference of κ and λ considered as multisets, denoted by $\lambda - \kappa$, is a strict partition. That is,

$$\mu(\kappa,\lambda) \;=\; \begin{cases} (-1)^{\ell(\lambda)-\ell(\kappa)} & \lambda-\kappa \text{ is a strict partition} \\ 0 & \text{else.} \end{cases} \tag{10}$$

The Möbius function $\mu(\alpha,\beta)$ of two elements $\alpha,\beta\in\Pi(n)$ is given by

$$\mu(\alpha,\beta) \;=\; \prod_{B\in\beta}(-1)^{\ell(\alpha_B)-1}(\ell(\alpha_B)-1)!,$$

where α_B for $B\subset[n]$ is the partition on B induced by α. A Möbius function satisfies the following two properties:

Theorem 2.3.1 *Let f, g be functions on a partially ordered set P. Then*

(i) $\displaystyle\sum_{\alpha\le\gamma\le\beta}\mu(\alpha,\gamma) \;=\; \delta_{\alpha,\beta} \;=\; \sum_{\alpha\le\gamma\le\beta}\mu(\gamma,\beta)$ *for all* $\alpha,\beta\in P$;

(ii) $\displaystyle f(\alpha)=\sum_{\gamma\le\alpha}g(\gamma)\quad\forall\alpha\in P \quad\iff\quad g(\beta)=\sum_{\gamma\le\beta}\mu(\gamma,\beta)f(\gamma)\quad\forall\beta\in P.$

2.4 The connected q-bracket

The q-bracket defined in the introduction (Eq. 1) is a map $\mathbb{Q}^{\mathscr{P}}\to\mathbb{Q}[[q]]$. In this section we define the connected q-bracket following [5, p. 55–57], which naturally arises in enumerative geometric when counting *connected* coverings. In our setting, the connected q-bracket turns out to be easier to compute than the usual q-bracket.

For $A\subset[n]$ we denote $f_A=\prod_{a\in A}f_a$.

Definition 2.4.1 Given an integer $n\ge 1$, the *connected q-bracket* is defined as the multilinear map

$$\langle\;\rangle_q : \underbrace{\mathbb{Q}^{\mathscr{P}}\otimes\cdots\otimes\mathbb{Q}^{\mathscr{P}}}_{n}\to\mathbb{Q}$$

extending the q-bracket such that for all $f,f_1,\ldots,f_n\in\mathbb{Q}^{\mathscr{P}}$ any of the following two equivalent conditions hold:

(i) $\displaystyle\langle f_1\otimes\cdots\otimes f_n\rangle_q \;=\; \sum_{\alpha\in\Pi(n)}\mu(\alpha,\mathbf{1}_n)\prod_{A\in\alpha}\langle f_A\rangle_q\,;$

(ii) $\langle f_1\otimes\cdots\otimes f_n\rangle_q$ is the coefficient of $x_1\cdots x_n$ in $\log\langle\exp(\sum_{i=1}^n x_if_i)\rangle_q\,.$

By invoking the Möbius inversion formula (Theorem 2.3.1(ii)) condition (i) in Definition 2.4.1 implies that

$$\prod_{B\in\beta}\langle\otimes_{b\in B}f_b\rangle_q \;=\; \sum_{\alpha\le\beta}\mu(\alpha,\beta)\prod_{A\in\alpha}\langle f_A\rangle_q\,,\qquad \prod_{A\in\alpha}\langle f_A\rangle_q \;=\; \sum_{\beta\le\alpha}\prod_{B\in\beta}\langle\otimes_{b\in B}f_b\rangle_q\,.$$

For example,

$$\begin{aligned}\langle f\otimes g\rangle_q &\;=\; \langle fg\rangle_q-\langle f\rangle_q\langle g\rangle_q\,,\\ \langle f\otimes g\otimes h\rangle_q &\;=\; \langle fgh\rangle_q-\langle f\rangle_q\langle gh\rangle_q-\langle g\rangle_q\langle fh\rangle_q-\langle h\rangle_q\langle fg\rangle_q+2\langle f\rangle_q\langle g\rangle_q\langle h\rangle_q\,,\end{aligned}$$

and

$$\langle fg\rangle_q \;=\; \langle f\otimes g\rangle_q+\langle f\rangle_q\langle g\rangle_q\,,$$

$$\langle fgh\rangle_q \;=\; \langle f\otimes g\otimes h\rangle_q + \langle f\rangle_q\langle g\otimes h\rangle_q + \langle g\rangle_q\langle f\otimes h\rangle_q + \langle h\rangle_q\langle f\otimes g\rangle_q + \langle f\rangle_q\langle g\rangle_q\langle h\rangle_q\,.$$

We often make use of the fact that the connected q-bracket of functions $f_1,\ldots,f_n$ vanishes if one of the f_i is constant.

Lemma 2.4.2 *For all* $f_1,\ldots,f_n\in\mathbb{Q}^{\mathscr{P}}$ *one has*

$$\langle 1\otimes f_1\otimes\cdots\otimes f_n\rangle_q = 0.$$

Proof Write $f_{n+1}=1$. Observe that $\prod_{A\in\alpha}\langle f_A\rangle_q$ takes the same value for all $\alpha\in\Pi(n+1)$ which agree on $[n]$ (but differ in the subset A of α containing $n+1$). Then, summing $\mu(\alpha,\mathbf{1}_n)$ over all such α yields

$$a\cdot(-1)^{a-1}(a-1)! + (-1)^a a! = 0$$

as there are a choices for α for which $\{n+1\}$ is not a subset of α, where a is the length of such an α, and there is only one choice for α for which $\{n+1\}$ is a subset. By Definition 2.4.1(i) the result follows. □

We will use the second condition in Definition 2.4.1 in our proof that $\mathcal{S}$ is a quasimodular algebra.

2.5 The discrete convolution product and Faulhaber polynomials

Let $\mathbb{N}$ denote the set of strictly positive integers. Given $f,g:\mathbb{N}\to\mathbb{Q}$ we denote by $f\cdot g$ or fg the pointwise product of f and g. We define the *discrete convolution product* of f and g by

$$(f*g)(n) \;=\; \sum_{i=1}^{n-1} f(i)\,g(n-i)$$

and denote the convolution product of functions $f_1,\ldots,f_n$ by

$$\mathop{\Huge *}_{i=1}^{n} f_i \;=\; f_1*\cdots*f_n. \tag{11}$$

Let the *discrete derivative* ∂ of $f:\mathbb{N}\to\mathbb{Q}$ be defined by $\partial f(n)=f(n)-f(n-1)$ for $n\geq 2$ and $\partial f(1)=f(1)$ and denote by id the identity function $\mathbb{N}\to\mathbb{N}\subset\mathbb{Q}$. Observe that

$$\partial(f*g) = (\partial f)*g \;=\; f*(\partial g), \tag{12}$$

$$\partial(fg) = \partial(f)\,g + f\,\partial(g) - \partial(f)\,\partial(g), \tag{13}$$

$$\mathrm{id}\cdot(f*g) = (\mathrm{id}\cdot f)*g + f*(\mathrm{id}\cdot g), \tag{14}$$

$$\partial^2(f*\mathrm{id}) = f - \partial f. \tag{15}$$

The *Faulhaber polynomials* $\mathcal{F}_l$ for $l\geq 1$ are defined as the unique polynomials with vanishing constant term satisfying $\partial\mathcal{F}_l(n)=n^{l-1}$ for all $n\in\mathbb{N}$, or equivalently by $\mathcal{F}_l(n)=\sum_{i=1}^n i^{l-1}$. The first four are given by

$$\mathcal{F}_1(x)=x,\quad \mathcal{F}_2(x)=\frac{x(x+1)}{2},\quad \mathcal{F}_3(x)=\frac{x(x+1)(2x+1)}{6},\quad \mathcal{F}_4(x)=\frac{x^2(x+1)^2}{4}.$$

Note that these polynomials are related to the Bernoulli polynomials $B_n(x)$, the unique family of polynomials satisfying $\int_x^{x+1} B_n(u)\,\mathrm{d}u = x^n$, by the formula $l\mathcal{F}_l(x) = B_l(x+1) - B_l$. Hence, the Faulhaber polynomials admit the symmetry

$$\mathcal{F}_l(x) \;=\; (-1)^l \mathcal{F}_l(-x-1) \qquad (l \geq 2), \tag{16}$$

which can also be deduced directly from the definition. The generating series $\mathcal{F}(n)$ of the Faulhaber polynomials equals

$$\mathcal{F}(n) \;:=\; \sum_{l=1}^{\infty} \mathcal{F}_l(n) \frac{z^{l-1}}{(l-1)!} \;=\; e^z \frac{1-e^{nz}}{1-e^z}. \tag{17}$$

3 The moment functions, their *q*-bracket and a second product

3.1 Three proofs of the quasimodularity of the moment functions

The q-bracket of the moment function S_k defined in (3) equals the Eisenstein series G_k. To motivate the results in the rest of this work, we provide three different proofs—and three generalizations—of this statement using three different approaches. In the first approach, we motivate the definition of the $T_{k,l}$ (see (4)), the second approach gives an interpretation for these functions, and the last approach gives an example of our main principle of establishing all identities before taking the q-bracket.

First approach The key observation in this first proof is that S_k can be rewritten as

$$S_k(\lambda) \;=\; -\frac{B_k}{2k} + \sum_{m=1}^{\infty} m^{k-1} r_m(\lambda).$$

More generally, for $k > 0$ and $f : \mathbb{N} \to \mathbb{Q}$ we set $f(0) = 0$ and we let

$$S_{k,f}(\lambda) \;=\; -\frac{B_{k+1}}{2(k+1)} \delta_{f,\mathrm{id}} \;+\; \sum_{m=1}^{\infty} m^k f(r_m(\lambda)). \tag{18}$$

In case when f is the identity, $S_{k,f} = S_{k+1}$. Our first method of proof gives the following more general statement:

Proposition 3.1.1 *Let f be a polynomial of degree l without constant term and k a positive integer satisfying $k \equiv l \bmod 2$. Then,*

(i) *if f equals a Faulhaber polynomial $\mathcal{F}_l$, then $\langle S_{k,f} \rangle_q$ equals*

$$-\frac{B_{k+1}}{2(k+1)} \delta_{l,1} + \sum_{m,r \geq 1} m^k r^{l-1} q^{mr} \;=\; \begin{cases} D^{l-1} G_{k-l+2} & k-l \geq 0, \\ D^k G_{l-k} & k-l \leq 2; \end{cases}$$

(ii) *if $\langle S_{k,f} \rangle_q$ is a quasimodular form, then f is a multiple of the Faulhaber polynomial $\mathcal{F}_l$.*

Proof Let $|x| \leq 1$ and $m \geq 1$. We compute

$$\langle x^{r_m} \rangle_q \;=\; \frac{\sum_{\lambda \in \mathscr{P}} x^{r_m(\lambda)} q^{|\lambda|}}{\sum_{\lambda \in \mathscr{P}} q^{|\lambda|}}. \tag{19}$$

Observe that the multiplicities $r_1(\lambda), r_2(\lambda), \ldots$ uniquely determine the partition λ. Hence, for $|q| < 1$ we have that

$$\begin{aligned}\sum_{\lambda\in\mathscr{P}} x^{r_m(\lambda)}q^{|\lambda|} &= \sum_{r_1,r_2,\ldots\geq 0} x^{r_m}q^{r_1+2r_2+\ldots+mr_m+\ldots}\\ &= \left(\sum_{r_m=0}^{\infty} x^{r_m}q^{mr_m}\right)\prod_{i\neq m}\left(\sum_{r_i=0}^{\infty} q^{ir_i}\right)\\ &= \frac{1}{1-xq^m}\prod_{i\neq m}\frac{1}{1-q^i}.\end{aligned}$$

Substituting this result in the numerator of (19), we obtain

$$\langle x^{r_m}\rangle_q = \frac{1-q^m}{1-xq^m}.$$

Hence,

$$\left\langle \tfrac{x}{1-x}(1-x^{r_m})\right\rangle_q = \frac{xq^m}{1-xq^m}. \tag{20}$$

Observe that applying $x\frac{\partial}{\partial x}$ to the right-hand side of (20) has the same effect as applying $\frac{1}{m}D$, where D is defined in §2.1. After setting $x = e^z$, we find that $\frac{x}{1-x}(1-x^{r_m})$ equals $\mathcal{F}(r_m)$ (see 17). Hence, by taking $l-1$ derivatives $x\frac{\partial}{\partial x} = \frac{\partial}{\partial z}$ and setting $z = 0$, it follows that

$$\begin{aligned}\langle S_{k,\mathcal{F}_l}\rangle_q + \frac{B_{k+1}}{2(k+1)}\delta_{l,1} &= \sum_{m\geq 0} m^k\langle \mathcal{F}_l(r_m)\rangle_q\\ &= \sum_{m\geq 0} m^k\left(x\frac{\partial}{\partial x}\right)^{l-1}\frac{xq^m}{1-xq^m}\Big|_{x=1}\\ &= \sum_{m\geq 0} m^k\left(\frac{1}{m}D\right)^{l-1}\frac{q^m}{1-q^m}\\ &= \sum_{m,r\geq 1} m^k r^{l-1}q^{mr}.\end{aligned}$$

Part (ii) of the statement follows by writing f as a linear combination of Faulhaber polynomials. □

Second approach The *double moment functions* $T_{k,l}$ (see (4)) are by definition equal to $S_{k,\mathcal{F}_l}$ if $k>0$. Given a partition λ, let $c_i(\lambda) = \#\{j\leq i \mid \lambda_i = \lambda_j\}$. Then, one has

$$T_{k,l}(\lambda) = -\frac{B_{k+l}}{2(k+l)}(\delta_{l,1}+\delta_{k,0}) + \sum_{i=1}^{\infty}\lambda_i^k c_i(\lambda)^{l-1}.$$

In this section we give a direct proof for the quasimodularity of the q-brackets of $T_{k,l}$:

Proposition 3.1.2 *For all $k\geq 0$, $l\geq 1$ and $k+l$ even, one has*

$$\langle T_{k,l}\rangle_q = \begin{cases} D^{l-1}G_{k-l+2} & \text{if } k-l\geq 0,\\ D^k G_{l-k} & \text{if } k-l\leq 2.\end{cases}$$

Proof Denote by $T^0_{k,l}(\lambda) = \sum_{i=1}^{\infty} \lambda_i^k c_i(\lambda)^{l-1}$. The generating series of $T^0_{k,l}$ is given by

$$W(X,Y)(\lambda) \;=\; \sum_{i=1}^{\infty} X^{\lambda_i} Y^{c_i(\lambda)},$$

that is, $T^0_{k,l}(\lambda)$ is the coefficient of $\frac{x^k y^{l-1}}{k!(l-1)!}$ in $W(e^x, e^y)(\lambda)$. Consider

$$\sum_{\lambda\in\mathscr{P}} W(X,Y)(\lambda)\, q^{|\lambda|} \;=\; \sum_{\lambda\in\mathscr{P}} \sum_{i=1}^{\infty} X^{\lambda_i} Y^{c_i(\lambda)} q^{|\lambda|}. \tag{21}$$

Given $a, b, n \in \mathbb{Z}_{\geq 0}$, denote by $C_{a,b}(n)$, the coefficient in front of $X^a Y^b q^n$ in (21), that is

$$\sum_{\lambda\in\mathscr{P}} \sum_{i=1}^{\infty} X^{\lambda_i} Y^{c_i(\lambda)} q^{|\lambda|} \;=:\; \sum_{a,b,n\geq 0} C_{a,b}(n) X^a Y^b q^n.$$

Let $p(n)$ denote the number of partitions of n. The coefficient $C_{a,b}(n)$ equals the number of partitions of n with at least b parts of size a, i.e., $C_{a,b}(n) = p(n-ab)$. Hence, writing $m = n - ab$ we obtain

$$\sum_{\lambda\in\mathscr{P}} \sum_{i=1}^{\infty} X^{\lambda_i} Y^{c_i(\lambda)} q^{|\lambda|} \;=\; \left(\sum_{m=0}^{\infty} p(m) q^m\right)\left(\sum_{a,b\geq 0} X^a Y^b q^{ab}\right).$$

In other words,

$$\langle W(X,Y)\rangle_q \;=\; \sum_{a,b\geq 0} X^a Y^b q^{ab}$$

so that expanding this equation for $X = e^x$ and $Y = e^y$ yields

$$\langle T^0_{k,l}\rangle_q \;=\; \sum_{a,b\geq 0} a^k b^{l-1} q^{ab}.$$

As $T_{k,l}(\lambda) = -\frac{B_{k+l}}{(k+l)}(\delta_{l,1} + \delta_{k,0}) + T^0_{k,l}(\lambda)$ we obtain the desired result. □

Third approach In this last proof we start with the observation that one can rewrite the q-bracket as

$$\langle f\rangle_q \;=\; \frac{\sum_{\lambda\in\mathscr{P}} f(\lambda)\, u_{\lambda_1} u_{\lambda_2}\cdots}{\sum_{\lambda\in\mathscr{P}} u_{\lambda_1} u_{\lambda_2}\cdots}\Big|_{u_i = q^i}. \tag{22}$$

In contrast to the previous two proofs, it is only in the last step of this proof that we take the q-bracket: First we rewrite (22) considering $u_1, u_2, \ldots$ to be formal variables, and in the last step we let $u_i = q^i$. We start with the denominator, where we encounter the Möbius function on partitions also defined in [17].

Proposition 3.1.3 *There exists a function $\mu : \mathscr{P} \to \{-1, 0, 1\}$ defined by any one of the following three equivalent definitions:*

(i) *$\mu(\lambda)$ is given by the Möbius function $\mu(\emptyset, \lambda)$ on the partial order on the set of partitions in (10);*

(ii) $$\mu(\lambda) \;=\; \begin{cases} (-1)^{\ell(\lambda)} & \lambda \text{ is a strict partition,} \\ 0 & \text{else;} \end{cases}$$

(iii) $$\frac{1}{\sum_{\lambda\in\mathscr{P}} u_{\lambda_1}u_{\lambda_2}\cdots} \;=\; \sum_{\lambda\in\mathscr{P}} \mu(\lambda)\, u_{\lambda_1}u_{\lambda_2}\cdots.$$

Proof The first two definitions clearly coincide using (10). For the latter, it suffices to show that

$$\sum_{\alpha\cup\beta=\lambda} \mu(\alpha) \;=\; \delta_{\lambda,\emptyset}.$$

Let $f(\lambda) = 1$ and $g(\lambda) = \delta_{\lambda,\emptyset}$ for $\lambda \in \mathscr{P}$. Then, $f(\alpha) = \sum_{\gamma\leq\alpha} g(\gamma)$ for all $\alpha \in \mathscr{P}$, so that by Möbius inversion and by using $\mu(\gamma,\beta) = \mu(\emptyset, \beta-\gamma)$ the last definition is equivalent. □

The fact that $\langle S_k\rangle_q = G_k$ follows directly from the following proposition:

Proposition 3.1.4 *For all $m \geq 1$ and $f : \mathbb{N} \to \mathbb{Q}$ extended by $f(0) = 0$, one has*

$$\frac{\sum_{\lambda\in\mathscr{P}} f(r_m(\lambda))\, u_{\lambda_1}u_{\lambda_2}\cdots}{\sum_{\lambda\in\mathscr{P}} u_{\lambda_1}u_{\lambda_2}\cdots} \;=\; \sum_{r=1}^{\infty} \partial f(r)\, u_m^r.$$

Proof Fix $m \geq 1$. By the previous proposition, we have

$$\frac{\sum_{\lambda\in\mathscr{P}} f(r_m(\lambda))\, u_{\lambda_1}u_{\lambda_2}\cdots}{\sum_{\lambda\in\mathscr{P}} u_{\lambda_1}u_{\lambda_2}\cdots} \;=\; \Bigg(\sum_{\lambda\in\mathscr{P}} f(r_m(\lambda))\, u_{\lambda_1}u_{\lambda_2}\cdots\Bigg)\Bigg(\sum_{\lambda\in\mathscr{P}} \mu(\lambda)\, u_{\lambda_1}u_{\lambda_2}\cdots\Bigg).$$

Denote by $C(\lambda)$ the coefficient of $u_{\lambda_1}u_{\lambda_2}\cdots$ after expanding the right-hand side of above equation. Observe that

$$C(\lambda) \;=\; \sum_{\alpha\cup\beta=\lambda} (-1)^{\ell(\beta)} f(r_m(\alpha)),$$

where $\alpha\cup\beta$ denotes the union of α and β considered as multisets and it is understood that β is a strict partition. Suppose λ admits a part equal to $m' \neq m$. Then, define an involution ω on all pairs (α, β) satisfying that $\alpha\cup\beta = \lambda$ and β is strict by

$$\omega(\alpha,\beta) \;=\; \begin{cases} (\alpha\backslash\{m'\}, \beta\cup\{m'\}) & \text{if } r_{m'}(\beta) = 0, \\ (\alpha\cup\{m'\}, \beta\backslash\{m'\}) & \text{if } r_{m'}(\beta) = 1. \end{cases}$$

As ω changes the sign of $(-1)^{\ell(\beta)} f(r_m(\alpha))$, it follows that $C(\lambda) = 0$.

Observe that $C(\emptyset) = 0$ and that in case $\lambda = (m, m, \ldots)$ consists of a strictly positive number of parts all equal to m one has

$$C(\lambda) \;=\; f(r_m(\lambda)) - f(r_m(\lambda)-1) \;=\; \partial f(r_m(\lambda)).$$

Therefore, the desired result follows. □

3.2 The induced and connected product

Motivated by the last of the three approaches in the previous section, we define the $\underline{u}$-*bracket* of a function $f \in \mathbb{Q}^{\mathscr{P}}$ by

$$\langle f\rangle_{\underline{u}} \;=\; \frac{\sum_{\lambda\in\mathscr{P}} f(\lambda)\, u_\lambda}{\sum_{\lambda\in\mathscr{P}} u_\lambda} \qquad (u_\lambda = u_{\lambda_1}u_{\lambda_2}\cdots).$$

Then, for all $f \in \mathbb{Q}^{\mathscr{P}}$ one has $\langle f\rangle_q = \langle f\rangle_{(q,q^2,q^3,\ldots)}$. Observe that the $\underline{u}$-bracket defines an isomorphism of vector spaces

$$\mathbb{Q}^{\mathscr{P}} \xrightarrow{\sim} \mathbb{Q}[[u_1, u_2, u_3, \ldots]], \qquad f \mapsto \langle f\rangle_{\underline{u}}.$$

We now use the algebra structure of $\mathbb{Q}[[u_1, u_2, u_3, \ldots]]$ to define a product on $\mathbb{Q}^{\mathscr{P}}$.

Definition 3.2.1 Given $f, g \in \mathbb{Q}^{\mathscr{P}}$ we define their *induced product* $f \odot g$ by

$$\langle f \odot g\rangle_{\underline{u}} \;=\; \langle f\rangle_{\underline{u}}\langle g\rangle_{\underline{u}},$$

where the product of $\langle f\rangle_{\underline{u}}$ and $\langle g\rangle_{\underline{u}}$ is the usual product of power series.

Remark 3.2.2 Observe that $\mathbb{Q}^{\mathscr{P}}$ is a commutative algebra with the constant function 1 as the identity for both the pointwise and the induced product. This observation should be compared with the q-bracket arithmetic in [17].

The following proposition gives an alternative definition for the induced product.

Proposition 3.2.3 *For all* $\lambda \in \mathscr{P}$, *one has*

$$(f \odot g)(\lambda) \;=\; \sum_{\alpha\cup\beta\cup\gamma=\lambda} f(\alpha)\, g(\beta)\, \mu(\gamma).$$

Proof By definition

$$\sum_{\lambda\in\mathscr{P}} (f \odot g)(\lambda)\, u_\lambda \;=\; \frac{\left(\sum_{\lambda\in\mathscr{P}} f(\lambda)\, u_\lambda\right)\left(\sum_{\lambda\in\mathscr{P}} g(\lambda)\, u_\lambda\right)}{\sum_{\lambda\in\mathscr{P}} u_\lambda}.$$

By Proposition 3.1.3 this equals

$$\left(\sum_{\lambda\in\mathscr{P}} f(\lambda)\, u_\lambda\right)\left(\sum_{\lambda\in\mathscr{P}} g(\lambda)\, u_\lambda\right)\left(\sum_{\lambda\in\mathscr{P}} \mu(\lambda)\, u_\lambda\right).$$

The result follows by expanding the products. □

Analogous to the connected q-bracket, we define the connected product. For a set S and functions $f_s \in \mathbb{Q}^{\mathscr{P}}$ for all $s \in S$, we denote $f_S = \prod_{s\in S} f_s$.

Definition 3.2.4 For $f_1, \ldots, f_n \in \mathbb{Q}^{\mathscr{P}}$, define the *connected product* $f_1 | \ldots | f_n$ to be the following function $\mathscr{P} \to \mathbb{Q}$:

$$f_1 | \ldots | f_n \;:=\; \sum_{\alpha\in\Pi(n)} \mu(\alpha, \mathbf{1}) \bigodot_{A\in\alpha} f_A\,. \tag{23}$$

For example, for $f, g, h \in \mathbb{Q}^{\mathscr{P}}$ one has

$$\begin{aligned} f\,|\,g \;&=\; fg - f \odot g,\\ f\,|\,g\,|\,h \;&=\; fgh - f \odot gh - g \odot fh - h \odot fg + 2f \odot g \odot h. \end{aligned}$$

The induced and connected product allow us to establish many identities before taking the q-bracket, as follows from the following result.

Proposition 3.2.5 *For all* $f_1, \ldots, f_n \in \mathbb{Q}^{\mathscr{P}}$ *one has*

- $\langle f_1 \odot f_2 \odot \cdots \odot f_n\rangle_q \;=\; \langle f_1\rangle_q\langle f_2\rangle_q \cdots \langle f_n\rangle_q\,;$
- $\langle f_1 | \ldots | f_n\rangle_q \;=\; \langle f_1 \otimes \cdots \otimes f_n\rangle_q\,.$

Proof Both statements follow directly from the definitions. For the first, note that for all $f, g \in \mathbb{Q}^{\mathscr{P}}$ one has

$$\langle f \odot g\rangle_q \;=\; \langle f\rangle_{\underline{u}}\langle g\rangle_{\underline{u}}|_{u_i=q^i} \;=\; \langle f\rangle_{\underline{u}}|_{u_i=q^i}\langle g\rangle_{\underline{u}}|_{u_i=q^i} \;=\; \langle f\rangle_q\langle g\rangle_q,$$

so that the statement follows inductively. The second follows from the first, as

$$\langle f_1 \,|\, \ldots \,|f_n\rangle_q \;=\; \sum_{\alpha\in\Pi(n)} \mu(\alpha,\mathbf{1}) \prod_{A\in\alpha} \langle f_A\rangle_q \;=\; \langle f_1 \otimes \cdots \otimes f_n\rangle_q\,. \qquad \square$$

Remark 3.2.6 Let $\mathcal{R}$ be the space of functions having a quasimodular form as q-bracket, i.e., $\mathcal{R} = \langle\,\cdot\,\rangle_q^{-1}(\widetilde{M})$. Then, $\mathcal{R}$ is a graded algebra with multiplication given by the induced product. Namely, if $f\in\mathcal{R}$ and $\langle f\rangle_q \in \widetilde{M}_k$, we define the weight of f to be equal to k. Note that if $f,g\in\mathcal{R}$ and $\langle f\rangle_q$ and $\langle g\rangle_q$ are quasimodular forms of weight k and l, respectively, then $\langle f\odot g\rangle_q = \langle f\rangle_q\langle g\rangle_q$ is a quasimodular form of weight $k+l$.

When establishing identities on the level of functions on partitions (before taking the q-bracket), it turns out to be very useful to express the connected product of pointwise products of elements of $\mathbb{Q}^{\mathscr{P}}$ in terms of connected and induced products. This can be done recursively using the following result.

Proposition 3.2.7 *For all $f_1,\ldots f_n\in\mathbb{Q}^{\mathscr{P}}$ one has*

$$\begin{aligned} f_1f_2\,|f_3\,|f_4\,|\ldots|f_n \;&=\; f_1\,|f_2\,|\ldots|f_n \\ &\quad+\sum_{A\sqcup B=\{3,\ldots,n\}} (f_1\,|f_{A_1}\,|f_{A_2}\,|\ldots)\odot(f_2\,|f_{B_1}\,|f_{B_2}\,|\ldots), \end{aligned} \tag{24}$$

where $A_1,A_2,\ldots$ enumerate the elements of A (and similarly for B).

Proof Observe that both sides of the equation in the statement are a linear combination of terms of the form $\bigodot_{C\in\gamma} f_C$ over $\gamma\in\Pi(n)$. We determine the coefficient of such a term on both sides of the equation.

First of all, assume γ is such that $\{1,2\}\subset C$ for some $C\in\gamma$. Then, on the right-hand side such a term only occurs in $f_1\,|\,\ldots\,|f_n$ with coefficient $\mu(\gamma,\mathbf{1})$. Moreover, let $\tilde\gamma\in\Pi(n-1)$ be given by $\gamma\cap\{2,\ldots,n\}$ subject to replacing i by $i-1$ for all $i=2,\ldots,n$. Note that the coefficient on the left-hand side equals $\mu(\tilde\gamma,\mathbf{1})$. As $\ell(\tilde\gamma)=\ell(\gamma)$, the coefficients on both sides agree.

Next, assume $C_1,C_2\in\gamma$ with $1\in C_1$ and $2\in C_2$. Then, the coefficient of $\bigodot_{C\in\gamma} f_C$ on right-hand side of (24) equals

$$\mu(\gamma,\mathbf{1})+\sum \mu(\gamma|_A,\mathbf{1})\mu(\gamma|_B,\mathbf{1}), \tag{25}$$

where the sum is over all $I\subset\{2,3,\ldots,\ell(\gamma)\}$ and A and B are given by $A=C_1\cup\bigcup_{i\in I}C_i$ and $B=C_2\cup\bigcup_{i\in I^c}C_i$. Letting i be the number of elements of I, we find that (25) equals

$$\begin{aligned} &\mu(\gamma,\mathbf{1})+\sum_{i=0}^{\ell(\gamma)-2}\binom{\ell(\gamma)-2}{i}\cdot(-1)^i i!\cdot(-1)^{\ell(\gamma)-i-2}(\ell(\gamma)-i-2)! \\ &\quad=\mu(\gamma,\mathbf{1})+\sum_{i=0}^{\ell(\gamma)-2}(\ell(\gamma)-2)!(-1)^{\ell(\gamma)-2} \\ &\quad=\mu(\gamma,\mathbf{1})-\mu(\gamma,\mathbf{1})=0. \end{aligned}$$

Correspondingly, the coefficient of $\bigodot_{C\in\gamma}\underline{f_C}$ on the left-hand side of (24) vanishes if there are $C_1,C_2\in\gamma$ with $1\in C_1$ and $2\in C_2$. $\square$

3.3 Quasimodularity of pointwise products of moment functions

Not only do the moment functions S_k admit quasimodular q-brackets, but also the homogeneous polynomials in the moment functions admit quasimodular q-brackets; here, each moment function S_k has weight k in accordance with the fact that $\langle S_k\rangle_q$ has weight k. Given a tuple $\underline{k} = (k_1, ..., k_n)$ of even integers, we write $S_{\underline{k}} = S_{k_1} \cdots S_{k_n}$. Note that, as a vector space, $\mathcal{S}$ is spanned by these functions $S_{\underline{k}}$. We provide two approaches to proving the quasimodularity of the q-brackets of the $S_{\underline{k}}$. First, we give a direct proof of the statement in Theorem 3.3.1, after which, in accordance with our main principle of establishing all identities before taking the q-bracket, we prove a more general result which will be used frequently in the next section.

Theorem 3.3.1 *The algebra $\mathcal{S}$ is a quasimodular algebra. More precisely, for $\underline{k} \in (2\mathbb{N})^n$ one has*

$$\langle S_{\underline{k}}\rangle_q \;=\; \sum_{\alpha\in\Pi(n)} \prod_{A\in\alpha} D^{\ell(A)-1} G_{|\underline{k}_A|-2\ell(A)+2}\,. \tag{26}$$

Proof Observe that it suffices to show that

$$\left\langle \bigotimes\nolimits_{k\in\underline{k}} S_k \right\rangle_q \;=\; D^{n-1} G_{|\underline{k}|-2n+2} \tag{27}$$

as (26) follows from (27) by Möbius inversion. Recall that $\langle f_1 \otimes \cdots \otimes f_n\rangle_q$ is the coefficient of $x_1 \cdots x_n$ in $\log\langle \exp(\sum_{i=1}^n x_i f_i)\rangle_q$ (see Definition 2.4.1(ii)). Consider $S_k^0(\lambda) = \sum_{i=1}^\infty \lambda_i^{k-1}$ for all positive even k. Euler's formula for the generating series of partitions

$$\sum_{\lambda\in\mathscr{P}} q^{|\lambda|} \;=\; \prod_{m=1}^{\infty} (1-q^m)^{-1}$$

follows from writing $|\lambda| = \sum_{m\geq 1} m r_m(\lambda)$ and summing over all possible values of $r_1(\lambda), r_2(\lambda)$, etc. By the same idea, we find

$$\sum_{\lambda\in\mathscr{P}} \exp\left(\sum\nolimits_k S_k^0(\lambda)\, x_k\right) q^{|\lambda|} \;=\; \prod_{m=1}^{\infty} \left(1-\exp\left(\sum\nolimits_k m^{k-1} x_k\right) q^m\right)^{-1}. \tag{28}$$

The logarithm of this expression equals

$$\sum_{m,r=1}^{\infty} \exp\left(r \sum\nolimits_k m^{k-1} x_k\right) \frac{q^{mr}}{r}. \tag{29}$$

Now, assume all parts of $\underline{k}$ are distinct. In the expansion of (29) the coefficient of $x_{k_1}\cdots x_{k_n}$ equals

$$\sum_{m,r=1}^{\infty} m^{|\underline{k}|-n} r^{n-1} q^{mr} \;=\; D^{n-1} G_{|\underline{k}|-2n+2}\,.$$

Hence,

$$\left\langle \bigotimes\nolimits_{k\in\underline{k}} S_k^0 \right\rangle_q \;=\; D^{n-1} G_{|\underline{k}|-2n+2}.$$

By introducing distinct variables in Eq. (28) for each repeated part of $\underline{k}$, we obtain the same result if not all parts of $\underline{k}$ are distinct.

Note that if $n \geq 2$, by Lemma 2.4.2 both sides of the equation do not change if one replaces S^0_k by S_k. In case $n = 1$ we have established (27) in Proposition 3.1.1 or in Proposition 3.1.2. Hence, (27) holds and (26) is then implied by Möbius inversion. □

Denoting

$$p_k(z) = \begin{cases} \frac{z^{k-2}}{(k-2)!} & k \geq 0, \\ \frac{z^{-2}}{2} & k = 0 \end{cases}$$

and setting $S_0(\lambda) \equiv 1$, one has the following expression for the generating series of the q-bracket of the generators of $\mathcal{S}$:

Corollary 3.3.2

$$\sum_{k_1,\ldots,k_n \geq 0} \langle S_{k_1} \cdots S_{k_n} \rangle_q \, p_{k_1}(z_1) \cdots p_{k_n}(z_n) = \sum_{\alpha \in \Pi(n)} \prod_{A \in \alpha} D^{|A|-1} \frac{P^{even}(\tau; z_A)}{2},$$

where $z_A = \sum_{a \in A} z_a$ *and*

$$P^{even}(\tau; z_1, \ldots, z_n) = \frac{1}{2^n} \sum_{s \in \{-1,1\}^n} P(\tau, s_1 z_1 + \ldots + s_n z_n)$$

is the totally even part of the propagator in (7).

3.4 Intermezzo: surjectivity of the *q*-bracket

We deduce from Theorem 3.3.1 the surjectivity of the q-bracket: Every quasimodular form is the q-bracket of some $f \in \mathcal{S}$.

Theorem 3.4.1 *The* q*-bracket* $\langle \cdot \rangle_q : \mathcal{S} \to \widetilde{M}$ *is surjective.*

Note that this is not obvious since the q-bracket is not an algebra homomorphism. Denote by $\vartheta_k : M_k \to M_{k+2}$ the Serre derivative, given by $\vartheta_k = D + 2kG_2$. Extend this notation by letting $\vartheta_x : \widetilde{M} \to \widetilde{M}$ for $x \in \mathbb{Q}$ be given by $\vartheta_x = D + 2xG_2$.

Proposition 3.4.2 *Let* $x \in \mathbb{Q} \backslash 2\mathbb{Z}_{\geq 0}$. *Then*

$$\widetilde{M}_k^{(\leq p)} = \bigoplus_{r=0}^{p} \vartheta_x^r M_{k-2r} \, .$$

Proof Let $f \in M_k$ with $f \neq 0$. Observe that $\vartheta_x f$ is modular precisely if $k = x$. By our assumption on x, this is not the case. Hence, ϑ_x increases the depth strictly by one. The result follows by induction on p by the same argument as in [22, Proposition 20]. Namely, if $\varphi \in \widetilde{M}_k^{\leq p}$, then the last coefficient φ_p in the quasimodular transformation (5) is a modular form of weight $k - 2p$. Hence, φ is a linear combination of $\vartheta_x^p \varphi_p$ and a quasimodular form of depth strictly smaller than p. □

Proof of Theorem 3.4.1 First observe that $(D + G_2)\langle f \rangle_q = \langle S_2 f \rangle_q$. As $D + G_2$ is not a Serre derivative, by Proposition 3.4.2 it follows that it suffices to show that the q-bracket is surjective on modular forms. Every modular form can be written as a polynomial of degree at most 2 in Eisenstein series, see [19, Section 5]. Hence, we show that the q-bracket is surjective on polynomials of degree at most 2 in all Eisenstein series, possibly involving the quasimodular Eisenstein series G_2.

Eisenstein series are in the image of the q-bracket by Theorem 3.3.1. Note that DG_k can be written a polynomial of degree 2 in Eisenstein series, explicitly:

$$DG_k \;=\; \frac{k+3}{2(k+1)}G_{k+2} - \sum_{\substack{0<j<k\\ j\equiv 1\,(2)}} \binom{k}{j} G_{j+1}G_{k+1-j}\,.$$

Also, we have an explicit formula for the q-bracket of S_kS_l:

$$\langle S_kS_l\rangle_q \;=\; G_kG_l + DG_{k+l-2} \tag{30}$$

so that this q-bracket is expressible as a polynomial of degree at most 2 in the Eisenstein series.

Now fix an integer $m\geq 4$. We consider the Eqs. (30) for all $k+l=m$. It suffices to show that we can invert these equations, i.e., write G_kG_l as a linear combination of q-brackets of products of at most two S_i. A direct computation shows that the determinant of the matrix corresponding to the equations above equals

$$1-\sum_{\substack{0<j<m\\ j\equiv 1\,(2)}}\binom{m}{j} \;=\; 1-2^{m-3} \;<\; 0.$$

Hence, the q-bracket is surjective. □

Remark 3.4.3 Only the last step of above proof uses the explicit formula (30) for the derivative of Eisenstein series. The author expects one could conclude the proof by an abstract argument, but he is not aware of such an argument.

3.5 The connected product of moment functions

In the second approach we compute the connected product $S_{k_1}\mid \ldots \mid S_{k_n}$, which by Proposition 3.2.5 yields the left-hand side of (26) after taking the q-bracket. The result is formulated in Theorem 3.5.4 and depends on two technical lemma's which we state first.

In order to do so, we start by introducing the following notation. For a partition λ and a subset A of $\mathbb{N}$, we write $\lambda|_A$ for the partition where a part of size m occurs $r_m(\lambda)$ times if $m\in A$ and does not occur if $m\notin A$. For example, $(5,4,3,3,1,1,1)|_{\{4,1\}}=(4,1,1,1)$.

Definition 3.5.1 We say $f:\mathscr{P}\to\mathbb{Q}$ is *supported on* A if $f(\lambda)=f(\lambda|_A)$ for all partitions λ.

The first lemma expresses the induced product of two functions F and G supported on disjoint sets as the *pointwise product* of these functions, and of two functions F and G supported on the same singleton set as a *convolution product* of functions.

Lemma 3.5.2 *Suppose X and Y are subsets of $\mathbb{N}$ and $F, F', G, G':\mathscr{P}\to\mathbb{Q}$ are supported on X, X, Y and Y, respectively. Then*

(i) *$F\odot F'$ is supported on X;*
(ii) *If X and Y are disjoint, then*

$$FG\odot F'G' \;=\; (F\odot F')(G\odot G'), \qquad \textit{in particular} \qquad F\odot G \;=\; FG;$$

(iii) *If $X = Y = \{m\}$, then*

$$(F \odot G)(\lambda) \;=\; \partial(f * g)(r_m(\lambda)),$$

where f and g are such that $F(\lambda) = f(r_m(\lambda))$, $G(\lambda) = g(r_m(\lambda))$.

Proof By Proposition 3.2.3, we have

$$(F \odot F')(\lambda) \;=\; \sum_{\alpha\cup\beta\cup\gamma=\lambda} (-1)^{\ell(\gamma)}\, F(\alpha)\, F'(\beta),$$

where it is understood that γ is a strict partition. We have that

$$\begin{aligned}(F \odot F')(\lambda) \;&=\; \Big(\sum_{\alpha\cup\beta\cup\gamma=\lambda|_X} (-1)^{\ell(\gamma)}\, F(\alpha)\, F'(\beta)\Big)\Big(\sum_{\alpha\cup\beta\cup\gamma=\lambda|_{X^c}} (-1)^{\ell(\gamma)}\Big)\\ &=\; (F \odot F')(\lambda|_X)\cdot(1\odot 1)(\lambda|_{X^c}).\end{aligned}$$

Recall $f \odot 1 = f$ for all functions f, hence $(F \odot F')(\lambda) = (F \odot F')(\lambda|_X)$, which is the first statement.

Next, we have that

$$(FG \odot F'G')(\lambda) \;=\; \sum_{\alpha\cup\beta\cup\gamma=\lambda} (-1)^{\ell(\gamma)}\, (FG)(\alpha)\, (F'G')(\beta),$$

where again it is understood that γ is a strict partition. Using the fact that F, F', G and G' are supported on X, X, Y and Y, respectively, we obtain

$$(FG \odot F'G')(\lambda) \;=\; \sum_{\alpha\cup\beta\cup\gamma=\lambda} (-1)^{\ell(\gamma|_X)+\ell(\gamma|_Y)+\ell(\gamma|_Z)}\, F(\alpha|_X)\, G(\alpha|_Y)\, F'(\beta|_X)\, G'(\beta|_Y), \tag{31}$$

where Z denotes the complement of $X\cup Y$ in $\mathbb{N}$. We factor the right-hand side of (31) as

$$\Big(\sum_{\alpha\cup\beta\cup\gamma=\lambda|_X} (-1)^{\ell(\gamma)}\, F(\alpha)\, F'(\beta)\Big)\Big(\sum_{\alpha\cup\beta\cup\gamma=\lambda|_Y} (-1)^{\ell(\gamma)}\, G(\alpha)\, G'(\beta)\Big)\Big(\sum_{\alpha\cup\beta\cup\gamma=\lambda|_Z} (-1)^{\ell(\gamma)}\Big).$$

By definition of the product $\odot$, we conclude

$$(FG \odot F'G')(\lambda) = (F \odot F')(\lambda|_X)\, (G \odot G')(\lambda|_Y)\, (1\odot 1)(\lambda|_Z) = (F \odot F')(\lambda)\, (G \odot G')(\lambda).$$

By taking F' and G to be the constant function 1 (which is supported on every X and Y), we see that $F \odot G' = FG'$ is implied by $FG \odot F'G' = (F \odot F')(G \odot G')$.

Next, for iii we have

$$\begin{aligned}(F \odot G)(\lambda) &= \sum_{\alpha\cup\beta\cup\gamma=\lambda} (-1)^{\ell(\gamma)} f(r_m(\alpha))\, g(r_m(\beta))\\ &= \sum_{\alpha\cup\beta\cup\gamma=\lambda|_{\{m\}}} (-1)^{\ell(\gamma)} f(r_m(\alpha))\, g(r_m(\beta))\end{aligned}$$

Letting $i = r_m(\alpha)$ and $j = r_m(\beta)$, we have

$$(F \odot G)(\lambda) = \sum_{i+j=r_m(\lambda)} f(i)\, g(j) - \sum_{i+j+1=r_m(\lambda)} f(i)\, g(j)$$

$$= (f*g)(r_m(\lambda)) - (f*g)(r_m(\lambda)-1)$$
$$= \partial(f*g)(r_m(\lambda)). \qquad \square$$

The second lemma is concerned with the vanishing of certain sums of the Möbius functions of *set* partitions. Given $\alpha \in \Pi(n)$ and a subset Z of $[n]$, we let

$$\alpha|_Z = \{A \cap Z \mid A \in \alpha \text{ s.t. } A \cap Z \neq \emptyset\} \in \Pi(Z),$$

where $\Pi(Z)$ denotes the set of all partitions of the set Z. Observe that

$$\ell(\alpha) \;=\; \ell(\alpha|_Z) + |\{A \in \alpha \mid A \cap Z = \emptyset\}|,$$

in particular $\ell(\alpha|_Z) \leq \ell(\alpha)$. Given $Z \subset [n]$, define an equivalence relation on $\Pi(n)$ by writing $\alpha \sim \beta$ if

$$\alpha|_Z = \beta|_Z \quad \text{and} \quad \alpha|_{Z^c} = \beta|_{Z^c}. \tag{32}$$

Lemma 3.5.3 *Let $Z \subseteq [n]$. If $Z \neq \emptyset$ and $Z \neq [n]$, then for all $\beta \in \Pi(n)$ we have*

$$\sum_{\alpha\sim\beta} \mu(\alpha, \mathbf{1}) = 0.$$

Proof Observe that $\alpha \sim \beta$ precisely if for all $A \in \alpha$ we have ($A \cap Z = \emptyset$ or $A \cap Z \in \beta|_Z$) and similarly we have ($A \cap Z^c = \emptyset$ or $A \cap Z^c \in \beta|_{Z^c}$). Hence, every $A \in \alpha$ is the union of some $A_1 \in \alpha|_Z \cup \{\emptyset\}$ and $A_2 \in \alpha|_{Z^c} \cup \{\emptyset\}$ with not both $A_1 = \emptyset$ and $A_2 = \emptyset$. Write $a = \ell(\beta|_Z)$, $b = \ell(\beta|_{Z^c})$, and assume without loss of generality that $a \leq b$. Write k for the number of $A \in \alpha$ for which both $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Now, $\ell(\alpha) = a + b - k$. Moreover, given k, Z and β, there are

$$\binom{a}{k}\binom{b}{k}k!$$

ways to choose $\alpha \sim \beta$ with $\ell(\alpha) = a + b - k$. Hence, we find

$$\begin{aligned}\sum_{\alpha\sim\beta} \mu(\alpha, \mathbf{1}) \;&=\; \sum_{k=0}^{a} (-1)^{a+b-k-1}(a+b-k-1)!\binom{a}{k}\binom{b}{k}k! \\ &=\; (-1)^{a+b-1}(a+b-1)! \sum_{k=0}^{a} \frac{(-a)_k(-b)_k}{(-a-b+1)_k(1)_k},\end{aligned}$$

where $(d)_k = \prod_{i=0}^{k-1}(d+i)$ is the rising Pochhammer symbol. This expression equals up to the constant $(-1)^{a+b-1}(a+b-1)!$ the special value ${}_2F_1(-a, -b, -a-b+1; 1)$ of the hypergeometric function ${}_2F_1(-a, -b, -a-b+1; z)$, which vanishes by Gauss's theorem subject to $a, b > 0$. As $Z \neq \emptyset$, we have $a > 0$. Also, $b > 0$ as $Z \neq [n]$. $\square$

The following result not only computes the connected product of the moment functions S_k, but also is one of the main technical results needed to prove Theorem 1.1.

Theorem 3.5.4 *Let k_i, f_i for $i = 1, \ldots, n$ be such that (18) defines S_{k_i,f_i}. Then,*

(i) *There exists a function $g : \mathbb{N} \to \mathbb{Q}$ such that*

$$S_{k_1,f_1} \mid \ldots \mid S_{k_n,f_n} \;=\; S_{|\underline{k}|,g}.$$

In fact,

$$g \;=\; \sum_{\alpha\in\Pi(n)} \mu(\alpha,\mathbf{1})\, \partial^{\ell(\alpha)-1} \mathop{\ast}_{A\in\alpha} f_A,$$

where $f_A=\prod_{a\in A} f_a$ *and* $\ast$ *denotes the convolution product (11).*

(ii) *If* $f_1(x)=x$*, then* $\partial g = f_1\,\partial\tilde{g}$ *with* $\tilde{g}$ *given by* $S_{k_2,f_2}\mid\ldots\mid S_{k_n,f_n}=S_{|\underline{k}|,\tilde{g}}$.

Remark 3.5.5 We extend g by $g(0)=0$. Here and later in this work, we usually omit the dependence of g on $f_1,\ldots,f_n$ in the notation.

Proof For the first part, we let $\underline{m}^{\underline{k_A}}\underline{f_A}\circ r_{\underline{m}}$ denote $\prod_i m_i^{k_{A_i}}\cdot f_{A_i}\circ r_{m_i}$, where r_{m_i} is considered as a function $\mathscr{P}\to\mathbb{Q}$. In case $n=1$ the result (i) is trivially true, so we assume $n\geq 2$. By definition of the connected product and $S_{k,f}$ (see (23) and (18) respectively), we have

$$\begin{aligned} S_{k_1,f_1}\mid\ldots\mid S_{k_n,f_n} \;&=\; \sum_{\alpha\in\Pi(n)} \mu(\alpha,\mathbf{1}) \bigodot_{A\in\alpha}\Big(\sum_{\underline{m}\in\mathbb{N}^{\ell(A)}} \underline{m}^{\underline{k_A}}\underline{f_A}\circ r_{\underline{m}}\Big)\\ &=\; \sum_{\underline{m}\in\mathbb{N}^n}\sum_{\alpha\in\Pi(n)} \mu(\alpha,\mathbf{1})\bigodot_{A\in\alpha} \underline{m_A}^{\underline{k_A}}\underline{f_A}\circ r_{\underline{m}}. \end{aligned} \tag{33}$$

For all $m\geq 0$, the function $r_m:\mathscr{P}\to\mathbb{Q}$ is supported on $\{m\}$. Having Lemma 3.5.2 in mind, we aim to factor the functions in (33) as a product of functions supported on a singleton set. Given $\underline{m}\in\mathbb{N}^n$, we start by all functions supported on $\{m_1\}$, that is, we let $Z(\underline{m})=\{i\mid m_i=m_1\}\subset[n]$. Note that $Z(\underline{m})$ determines all i for which the support of r_{m_i} contains m_1. Denote by $E(\underline{m})$ the set of equivalence classes of $\Pi(n)$ for this choice of $Z=Z(\underline{m})$. We split the sum over $\alpha\in\Pi(n)$ in (33) as a sum over the elements of $E(\underline{m})$, i.e.,

$$S_{k_1,f_1}\mid\ldots\mid S_{k_n,f_n} \;=\; \sum_{\underline{m}\in\mathbb{N}^n}\sum_{[\beta]\in E(\underline{m})}\sum_{\alpha\in[\beta]} \mu(\alpha,\mathbf{1})\bigodot_{A\in\alpha} \underline{m_A}^{\underline{k_A}}\underline{f_A}\circ r_{\underline{m_A}}. \tag{34}$$

Then, given $\underline{m}\in\mathbb{N}^n$, $Z=Z(\underline{m})$ and $A\in\alpha|_Z$, the function $\lambda\mapsto \underline{m_A}^{\underline{k_A}}\underline{f_A}(r_{\underline{m_A}}(\lambda))$ is supported on $\{m_1\}$, whereas for $A\in\alpha|_{Z^c}$ the function $\lambda\mapsto \underline{m_A}^{\underline{k_A}}\underline{f_A}(r_{\underline{m_A}(\lambda)})$ is supported on $\mathbb{N}\backslash\{m_1\}$. Hence, by Lemma 3.5.2(ii) we find that (34) equals

$$\sum_{\underline{m}\in\mathbb{N}^n}\sum_{[\beta]\in E(\underline{m})}\sum_{\alpha\in[\beta]} \mu(\alpha,\mathbf{1})\Big(\bigodot_{A\in\alpha|_Z} \underline{m_A}^{\underline{k_A}}\underline{f_A}\circ r_{\underline{m_A}}\Big)\Big(\bigodot_{A\in\alpha|_{Z^c}} \underline{m_A}^{\underline{k_A}}\underline{f_A}\circ r_{\underline{m_A}}\Big). \tag{35}$$

Instead of writing the second factor as a product of functions which are all supported on a singleton set, we make the following observation.

As $\alpha|_Z=\beta|_Z$ and $\alpha|_{Z^c}=\beta|_{Z^c}$, the only dependence on α in the above equation is in $\mu(\alpha,\mathbf{1})$. By construction $Z(\underline{m})$ is non-empty. Hence, by Lemma 3.5.3 we have that if $Z\neq[n]$ then for all $\beta\in E(\underline{m})$ we have $\sum_{\alpha\in[\beta]}\mu(\alpha,\mathbf{1})=0$. This implies that we can restrict the first sum in (35) to $\underline{m}\in\mathbb{N}^n$ for which $m_i=m_j$ for all i,j, that is,

$$S_{k_1,f_1}\mid\ldots\mid S_{k_n,f_n} \;=\; \sum_{m\in\mathbb{N}}\sum_{\alpha\in\Pi(n)} \mu(\alpha,\mathbf{1})\bigodot_{A\in\alpha}\prod_{a\in A} m^{k_a}\cdot f_a\circ r_m.$$

Applying Lemma 3.5.2iii $\ell(\alpha)-1$ times and using (12), we obtain the desired result.

For the second part, let $Z = \{1\}$ and consider an equivalence class $[\beta]$ for the equivalence relation (32) determined by Z. We split the sum

$$\partial g \;=\; \sum_{\alpha\in\Pi(n)} \mu(\alpha,\mathbf{1})\,\partial^{\ell(\alpha)} \mathop{\ast}_{A\in\alpha} f_A$$

over all conjugacy classes. Write A_1 for the element of α for which $1\in A_1$. Denote $\hat{A}_1 = A_1\backslash\{1\}$ and $\gamma = \beta|_{\{2,\ldots,n\}}$. In case $A_1 = \{1\}$ one has by (15) that

$$\mu(\alpha,\mathbf{1})\,\partial^{\ell(\alpha)} \mathop{\ast}_{A\in\alpha} f_A \;=\; -\ell(\gamma)\,\mu(\gamma,\mathbf{1})\,\partial^{\ell(\gamma)-1}(1-\partial) \mathop{\ast}_{A\in\gamma} f_A. \tag{36}$$

In case $A_1 \neq \{1\}$ (i.e., $|A_1|\geq 2$), one finds by (13) that

$$\mu(\alpha,\mathbf{1})\,\partial^{\ell(\alpha)} \mathop{\ast}_{A\in\alpha} f_A \;=\; \mu(\gamma,\mathbf{1})\,\partial^{\ell(\gamma)-1}(f_1\,\partial f_{\hat{A}_1} + (1-\partial)f_{\hat{A}_1}) \ast \mathop{\ast}_{A\in\gamma\backslash\hat{A}_1} f_A. \tag{37}$$

As $[\beta]$ contains one element for which (36) holds and $\ell(\gamma)$ elements for which (37) holds, one finds

$$\sum_{\alpha\in[\beta]} \mu(\alpha,\mathbf{1})\partial^{\ell(\alpha)} \mathop{\ast}_{A\in\alpha} f_A \;=\; \mu(\gamma,\mathbf{1})\,\partial^{\ell(\gamma)-1} \sum_{C\in\gamma}\Big(f_1\,\partial f_C \ast \mathop{\ast}_{A\in\gamma\backslash C} f_A\Big).$$

By (12) and (14), this equals

$$\mu(\gamma,\mathbf{1})\sum_{C\in\gamma}\Big(f_1\,\partial f_C \ast \mathop{\ast}_{A\in\gamma\backslash C} \partial f_A\Big) \;=\; \mu(\gamma,\mathbf{1})f_1\,\partial^{\ell(\gamma)} \mathop{\ast}_{A\in\gamma} f_A.$$

Hence, summing over all conjugacy classes, we obtain

$$\partial g \;=\; f_1 \sum_{\gamma\in\Pi(n-1)} \mu(\gamma,\mathbf{1})\,\partial^{\ell(\gamma)} \mathop{\ast}_{A\in\gamma} f_A \;=\; f_1\,\partial\tilde{g}.$$

The case when $f_1(x) = \ldots = f_n(x) = x$ is the easiest example (for arbitrary $n\in\mathbb{N}$) of the above result. In this case one generalizes Theorem 3.3.1 by a result which, in accordance with our main principle of establishing identities before the q-bracket, yields this theorem after taking the q-bracket.

Corollary 3.5.6 *For all positive even $k_1,\ldots,k_n$, one has*

$$S_{k_1}\,|\,\ldots\,|\,S_{k_n} \;=\; S_{|\underline{k}|-n,\mathcal{F}_n}\,.$$

Proof Recall $S_k = S_{k-1,\mathrm{id}}$ and apply Theorem 3.5.4(ii) $n-1$ times. □

Later we will use Theorem 3.3.1 when the f_i are Faulhaber polynomials. This is the situation in which we prove the main result of this paper, in which case the following lemma is useful.

Lemma 3.5.7 *If $f_1,\ldots,f_n$ are Faulhaber polynomials of degrees $d_1,\ldots,d_n$, respectively, and $g:\mathbb{N}\to\mathbb{Q}$ is as in Theorem 3.5.4, then there exists a polynomial p such that $\partial g(m) = p(m)$ for all $m\in\mathbb{N}$. Moreover, p is strictly of degree $|\underline{d}|-1$, is even or odd and $p(0)=0$.*

Proof By Theorem 3.5.4(ii) we can assume w.l.o.g. that none of the degrees d_i equals 1. Now, consider a monomial $\partial^{\ell(\alpha)} \bigast_{A\in\alpha} f_A$ in ∂g. Note that both $*$ and ∂ are operators on the space of polynomials, more precisely:

$$*: \mathbb{Q}[x]_{\leq k} \times \mathbb{Q}[x]_{\leq l} \to \mathbb{Q}[x]_{\leq k+l+1} \qquad \text{and} \qquad \partial: \mathbb{Q}[x]_{\leq k} \to \mathbb{Q}[x]_{\leq k-1}$$

as

$$x^k * x^l \;=\; \frac{k!l!}{(k+l+1)!}x^{k+l+1} + O(x^{k+l}) \qquad \text{and} \qquad \partial(x^k) \;=\; kx^{k-1} + O(x^{k-2}).$$

Hence, the degree of such a monomial is $|\underline{d}| - 1$. Now observe that by the symmetry (16) one has

$$\partial f_A(x) \;=\; f_A(x) - f_A(x-1) \;=\; f_A(x) - (-1)^{|A|} f_A(-x).$$

Therefore, we see that ∂f_A is even or odd and as the convolution product preserves this property, every monomial is even or odd. By the same arguments $\partial f_A(0) = 0$ and hence the constant term of every monomial vanishes. Therefore, every monomial $\partial^{\ell(\alpha)-1} \bigast_{A\in\alpha} f_A$ in g satisfies the desired properties, so that it remains to show that the leading coefficient does not vanish.

As $\mathcal{F}_l = \frac{1}{l}x^l + O(x^{l-1})$, the leading coefficient of a monomial as above equals

$$\frac{|\underline{d}|}{\prod_i d_i} \frac{\prod_{i=1}^n d_{A_i}!}{|\underline{d}|!},$$

where for a set B we have set $d_B = \sum_{b\in B} d_b$. Hence, the leading coefficient of ∂g equals

$$\frac{|\underline{d}|}{\prod_i d_i} \cdot \sum_{\alpha\in\Pi(n)} \mu(\alpha, \mathbf{1}) \binom{|\underline{d}|}{d_{A_1}, \ldots, d_{A_r}}^{-1}, \tag{38}$$

where $\alpha = \{A_1, \ldots, A_r\}$. Note that this number has the following combinatorial interpretation. Let n balls be given which are colored such that d_1 balls are colored in the first color, d_2 in the second color, etc. Suppose we use the same multiset of colors to additionally mark each ball with a dot (possibly of the same color), that is, d_1 balls are marked with a dot of the first color, d_2 with a dot of the second color, etc. Given a subset C of the set of all colors, it may happen that if we consider all balls colored by the colors of C, all the dots on these balls are colored by the same set of colors C. We then say that the balls are *well-colored with respect to* C. For example, both the empty set of colors and the set of all possible colors give rise to a well-coloring of balls. If we independently at random color and mark the balls as above, the probability that the balls colored by a subset C are well-colored is $\binom{|\underline{d}|}{d_C}^{-1}$. Hence, by applying Möbius inversion the number

$$\sum_{\alpha\in\Pi(n)} \mu(\alpha, \mathbf{1}) \binom{|\underline{d}|}{d_{A_1}, \ldots, d_{A_r}}^{-1}$$

equals the probability that if we independently at random color and mark the balls as above, there does not exist a proper non-empty subset C of the colors such that the balls colored by C are well-colored. If we mark at least one ball of every color i with color $i+1$ (modulo n), such a set C cannot exist. Hence, the number (38) is positive, so the polynomial p is strictly of degree $|\underline{d}| - 1$. □

4 Three quasimodular algebras

4.1 Introduction

Given integers k, l with $k \geq 0$ and $l \geq 1$ recall the definition of the *double moment functions* in (4) by

$$T_{k,l}(\lambda) = -\frac{B_{k+l}}{2(k+l)}(\delta_{l,1} + \delta_{k,0}) + \sum_{m=1}^{\infty} m^k \mathcal{F}_l(r_m(\lambda)).$$

Unless stated explicitly, we always assume that

$$k \in \mathbb{Z}_{\geq 0}, l \in \mathbb{Z}_{\geq 1}, k + l \in 2\mathbb{Z}. \tag{39}$$

Moreover, it turns out to be useful to define $T_{0,0} \equiv T_{-1,1} \equiv -1$ and $T_{k,l} \equiv 0$ for other pairs (k, l) with $k < 0$ or $l < 1$.

Remark 4.1.1 The double moment functions specialize to the moment functions studied in the previous section whenever $l = 1$, i.e., $T_{k,1} = S_{k+1}$. Also, as $\mathcal{F}_l(1) = 1$, for a strict partition λ one has $T_{k,l}(\lambda) = S_k(\lambda)$. Hence, our functions $T_{k,l}$ can be seen as an extension of the *algebra of supersymmetric polynomials*, mentioned in the introduction, to functions on all partitions (and not only on strict partitions).

Remark 4.1.2 In case $k + l$ is odd, the q-bracket of $T_{k,l}$ does not vanish—in contrast to the shifted symmetric functions for which the q-bracket vanishes for all odd weights. However, the q-bracket of a polynomial involving the double moment functions in both even and odd weights also is a polynomial in the so-called combinatorial Eisenstein series, defined in Definition 7.2.4.

These double moment functions give rise to three different graded algebras, which turn out to be quasimodular (see page 1).

Definition 4.1.3 Define the $\mathbb{Q}$-algebras $\mathcal{S}$, $\mathrm{Sym}^{\odot}(\mathcal{S})$ and $\mathcal{T}$ by the condition that

- $\mathcal{S}$ is generated by the moment functions S_k under the pointwise product;
- $\mathrm{Sym}^{\odot}(\mathcal{S})$ is generated by the elements of $\mathcal{S}$ under the induced product;
- $\mathcal{T}$ is generated by the double moment functions under the pointwise product.

Our main result Theorem 1.1 is slightly refined by the following statement.

Theorem 4.1.4 *Let X be any of the algebras $\mathcal{S}$, $\mathrm{Sym}^{\odot}(\mathcal{S})$ and $\mathcal{T}$. Then, X is*

- *quasimodular;*
- *closed under the pointwise product;*
- *closed under the induced product if $X \neq \mathcal{S}$.*

Moreover, the three algebras are related by $\mathcal{S} \subsetneq \mathrm{Sym}^{\odot}(\mathcal{S}) \subsetneq \mathcal{T}$.

Remark 4.1.5 Observe that being closed under the pointwise product is not implied by being quasimodular. For example, the algebra $\mathcal{R} = \langle \cdot \rangle_q^{-1}(\widetilde{M})$ in Remark 3.2.6 is quasimodular, closed under the induced product and $\mathcal{T} \subset \mathcal{R}$, but $\mathcal{R}$ is not closed under the pointwise product [23, Section 9].

In the next section we provide different bases for these algebras: in this way we obtain many examples of functions with a quasimodular q-bracket, and moreover, the study of these bases leads to a proof of Theorem 4.1.4.

Remark 4.1.6 The algebras $\mathcal{T}$ and Λ^* are different algebras, as follows from the observation that $f(\lambda) = (-1)^k f(\lambda')$ for all $f \in \Lambda^*_k$, which follows by writing a shifted symmetric polynomial as a symmetric polynomial in the Frobenius coordinates. This does not hold for all $f \in \mathcal{T}$, as can easily be checked numerically. On the other hand, it is not true that $f(\lambda) \neq \pm f(\lambda')$ for all $f \in \mathcal{T}$, as $Q_2 = T_{1,1}$ with Q_k defined by Eq. (2). More precisely, one has

$$\mathcal{T} \cap \Lambda^* = \mathbb{Q}[Q_2].$$

Namely, if $f \in \mathcal{T} \cap \Lambda^*$, consider a *strict* partition λ (i.e., a partition for which $r_m(\lambda) \leq 1$ for all m). Then, we have that $f(\lambda)$ is symmetric polynomial in the parts $\lambda_1, \lambda_2, \ldots$. On the other hand, as $f \in \Lambda^*$, it follows that $f(\lambda)$ is a shifted symmetric polynomial in the parts $\lambda_1, \lambda_2, \ldots$. The only polynomials of degree d in the variables x_i that are both symmetric and shifted symmetric are up to a constant given by $\left(\sum_i x_i\right)^d$, hence $f \in \mathbb{Q}[Q_2]$.

4.2 The basis given by double moment functions

In this section we show that $\mathcal{T}$ is closed under the induced product. Moreover, we show that $\mathcal{S}$ and $\mathrm{Sym}^{\odot}(\mathcal{S})$ are subalgebras of $\mathcal{T}$. In the next section, we use these results to define a weight grading on $\mathcal{T}$. Observe that as a vector space $\mathcal{T}$ is spanned by the functions $T_{\underline{k},\underline{l}}$, defined by $T_{\underline{k},\underline{l}} = \prod_i T_{k_i,l_i}$, for all $\underline{k}, \underline{l} \in \mathbb{Z}^n$ satisfying the conditions (39) for all pairs $(k, l) = (k_i, l_i)$.

Theorem 4.2.1 *The algebra $\mathcal{T}$ is closed under the induced product.*

Proof Observe that

$$T_{\underline{k},\underline{l}} \odot T_{\underline{k}',\underline{l}'} \;=\; T_{\underline{k},\underline{l}} T_{\underline{k}',\underline{l}'} - T_{\underline{k},\underline{l}} \,|\, T_{\underline{k}',\underline{l}'}\,.$$

Hence, it suffices to show that $T_{\underline{k},\underline{l}} \,|\, T_{\underline{k}',\underline{l}'}$ can be expressed in terms of elements of $\mathcal{T}$.

By Theorem 3.5.4 and Lemma 3.5.7, we have that an expression of the form:

$$T_{k_1,l_1} \,|\, \cdots \,|\, T_{k_n,l_n}$$

is an element of $\mathcal{T}$. Proposition 3.2.7 implies that $f_1 f_2 \,|\, f_3 \,|\, f_4 \,|\, \ldots \,|\, f_n$ equals

$$(f_1 \,|\, f_2 \,|\, \ldots \,|\, f_n) + \sum_{A \sqcup B = \{3,\ldots,n\}} \Big((f_1 \,|\, f_{A_1} \,|\, f_{A_2} \,|\, \ldots) \cdot (f_2 \,|\, f_{B_1} \,|\, f_{B_2} \,|\, \ldots) \\ - (f_1 \,|\, f_{A_1} \,|\, f_{A_2} \,|\, \ldots) \,\big|\, (f_2 \,|\, f_{B_1} \,|\, f_{B_2} \,|\, \ldots) \Big).$$

Hence, by using this proposition recursively, we can replace the pointwise products in $T_{\underline{k},\underline{l}}$ and $T_{\underline{k}',\underline{l}'}$ by a linear combination of connected products of double moment functions $T_{k,l}$, showing that $T_{\underline{k},\underline{l}} \,|\, T_{\underline{k}',\underline{l}'}$ is an element of $\mathcal{T}$. □

Now, we determine a basis for the three algebras. Let $\mathcal{T}^{\mathrm{mon}}$ be the set of all monomials for the pointwise product in $\mathcal{T}$. Two elements of $\mathcal{T}^{\mathrm{mon}}$ are considered to be the same if one can reorder the products so that they agree, for example $T_{1,1}T_{3,5}$ and $T_{3,5}T_{1,1}$ are the same function. In other words, every elements of $\mathcal{T}^{\mathrm{mon}}$ can be written as $T_{\underline{k},\underline{l}}$ in a unique way up to commutativity of the (pointwise) product.

Theorem 4.2.2 *We have*

$$\mathcal{S} \subsetneq \mathrm{Sym}^{\odot}(\mathcal{S}) \subsetneq \mathcal{T}. \tag{40}$$

Moreover, a basis for

- $\mathcal{T}$ *is given by* $\mathcal{T}^{\mathrm{mon}}$*;*
- $\mathrm{Sym}^{\odot}(\mathcal{S})$ *is given by all* $T_{\underline{k},\underline{l}} \in \mathcal{T}^{\mathrm{mon}}$ *satisfying* $k_i \geq l_i$ *for all* i*;*
- $\mathcal{S}$ *is given by all* $T_{\underline{k},\underline{l}} \in \mathcal{T}^{\mathrm{mon}}$ *satisfying* $l_i = 1$ *for all* i*.*

Proof It suffices to prove the second part, as from the stated bases statement (40) follows immediately.

By definition the elements of $\mathcal{T}^{\mathrm{mon}}$ generate $\mathcal{T}$ as a vector space. Hence, it suffices to show that they are linearly independent, i.e., that if

$$\sum_{\alpha \in I} c_\alpha T_\alpha(\lambda) \;=\; 0 \tag{41}$$

for all $\lambda \in \mathscr{P}$, where I is the set of all pairs $(\underline{k}, \underline{l})$ up to simultaneous reordering and $c_\alpha \in \mathbb{Q}$, we have that $c_\alpha = 0$ for all α.

First of all, let $\lambda = (N_1, N_2)$ and consider (41) as $N_1 \to \infty$. Note that $T_{\underline{k},\underline{l}}(\lambda)$ grows as

$$N_1^{|\underline{k}|} + N_2^{k_{\min}} N_1^{|\underline{k} \backslash k_{\min}|}$$

plus lower-order terms, where $k_{\min}$ is the smallest of the k_i in $\underline{k}$. Hence, $|\underline{k}|$ should be constant among all T_α in (41). Moreover, we conclude that $k_{\min}$ should be constant among all T_α in (41). Continuing by considering the lower-order terms, we conclude that $\underline{k}$ is constant among all T_α. Similarly, by instead considering partitions consisting of N_1 times the part 1 and N_2 times the part 2, we conclude that $\underline{l}$ is constant among all T_α. Hence, there is at most one α with nonzero coefficient c_α. We conclude that $c_\alpha = 0$ for all $\alpha \in I$.

For $\mathrm{Sym}^{\odot}(\mathcal{S})$ we show, first of all, that indeed $T_{\underline{k},\underline{l}} \in \mathrm{Sym}^{\odot}(\mathcal{S})$ if $k_i \geq l_i$ for all i. Let $k \geq l$ of the same parity be given. By Corollary 3.5.6 we find that

$$\underbrace{T_{1,1} \,|\, T_{1,1} \,|\, \ldots \,|\, T_{1,1}}_{l-1} \,|\, T_{k-l+1,1} \;=\; \underbrace{S_2 \,|\, S_2 \,|\, \ldots \,|\, S_2}_{l-1} \,|\, S_{k-l+2} \;=\; T_{k,l}\,.$$

Therefore, $T_{k,l} \in \mathrm{Sym}^{\odot}(\mathcal{S})$ for all $k \geq l$. Moreover, by applying Möbius inversion on Eq. (23), which defines the connected product, we find

$$T_{\underline{k},\underline{l}} \;=\; \sum_{\alpha \in \Pi(n)} \bigodot_{A \in \alpha} (T_{k_{A_1},l_{A_1}} \,|\, T_{k_{A_2},l_{A_1}} \,|\, \ldots). \tag{42}$$

As we already showed that $T_{k,l} \in \mathrm{Sym}^{\odot}(\mathcal{S})$ if $k \geq l$, we find $T_{\underline{k},\underline{l}} \in \mathrm{Sym}^{\odot}(\mathcal{S})$ if $k_i \geq l_i$ for all i.

Next, we show that all elements in $\mathrm{Sym}^{\odot}(\mathcal{S})$ are a linear combination of the $T_{\underline{k},\underline{l}}$ satisfying $k_i \geq l_i$. As $\mathcal{S}$ clearly is contained in the space generated by the $T_{\underline{k},\underline{l}}$ for which $k_i \geq l_i$, it suffices to show that the latter space is closed under $\odot$. For this we follow the proof of Theorem 4.2.1 observing that in each step $k_i \geq l_i$, so that indeed the $T_{\underline{k},\underline{l}}$ for which $k_i \geq l_i$ form a generating set for $\mathrm{Sym}^{\odot}(\mathcal{S})$.

As we already showed that the $T_{\underline{k},\underline{l}}$ are linearly independent, we conclude that the $T_{\underline{k},\underline{l}} \in \mathcal{T}^{\mathrm{mon}}$ satisfying $k_i \geq l_i$ for all i form a basis for $\mathrm{Sym}^{\odot}(\mathcal{S})$.

The last part of the statement follows directly, as by definition all $T_{\underline{k},\underline{l}} \in \mathcal{T}^{\mathrm{mon}}$ satisfying $l_i = 1$ for all i generate $\mathcal{S}$, and by the above they are linearly independent. □

4.3 The basis defining the weight grading

By definition, the double moment functions generate $\mathcal{T}$ under the pointwise product. In this section we show that we can replace the pointwise product in the latter statement by the induced product. Again we will consider every reordering of the factors in $T_{k_1,l_1} \odot \cdots \odot T_{k_n,l_n}$ due to commutativity of the products to be the same element. Then, we have:

Theorem 4.3.1 *The elements $T_{k_1,l_1} \odot \cdots \odot T_{k_n,l_n}$ form a basis for $\mathcal{T}$. A basis for the subspace* $\mathrm{Sym}^{\odot}(\mathcal{S})$ *is given by the subset of elements for which $k_i \geq l_i$ for all i.*

Proof Assign to $T_{k,l}$ weight $k + l$. This defines a weight filtering on $\mathcal{T}$ with respect to the pointwise product. Consider the subspace of elements of weight at most w in $\mathcal{T}$. The number of basis elements in the basis given by the pointwise product in the previous section equals the number of induced products of the $T_{k,l}$. Hence, it suffice that the induced products of the $T_{k,l}$ generate $\mathcal{T}$. For this we proceed by induction first on the weight and then on the depth. Here, by depth we mean the unique filtering under the pointwise product for which every $T_{k,l}$ has depth 1, usually called the *total depth*.

Trivially, every element of weight 0 or depth 0 is generated by (empty) induced products of the $T_{k,l}$. Next, consider $T_{\underline{k},\underline{l}} \in \mathcal{T}$ and assume all elements of lower weight and of the same weight and lower depth are generated by induced product of the $T_{k,l}$. Let $T_{\underline{k},\underline{l}} \in \mathcal{T}$ of weight w be given and write $\underline{k}', \underline{l}'$ for $\underline{k}, \underline{l}$ after omitting the last (nth) entry. Then

$$T_{\underline{k},\underline{l}} \;=\; T_{\underline{k}',\underline{l}'} \odot T_{k_n,l_n} - T_{\underline{k}',\underline{l}'} \,|\, T_{k_n,l_n}\,.$$

Note that $T_{\underline{k}',\underline{l}'}$ is of weight strictly less than w, hence is generated by induced products of the $T_{k,l}$. Moreover, by Proposition 3.2.7 and Theorem 3.5.4 it follows that the depth of $T_{\underline{k}',\underline{l}'} \,|\, T_{k_n,l_n}$ is at most $n - 1$. Hence, by our induction hypothesis, it is generated by induced products of the $T_{k,l}$. We conclude that $T_{\underline{k},\underline{l}}$ is generated by induced products of the $T_{k,l}$, which proves the first part of the theorem.

The second part follows by the same proof, everywhere restricting to those $T_{k,l}$ for which $k \geq l$. □

By the above theorem, we can define a weight grading on $\mathcal{T}$.

Definition 4.3.2 Define a *weight grading* on $\mathcal{T}$ by assigning to $T_{k,l}$ weight $k + l$ and extending under the induced product.

Note that both the grading on $\mathcal{T}$ and the grading on $\mathcal{S}$ correspond to the grading on quasimodular forms after taking the q-bracket. Hence, the grading on $\mathcal{S}$ is the restriction of the grading on $\mathcal{T}$.

The weight grading defines a weight operator. In Sect. 5 we extend this weight operator to an $\mathfrak{sl}_2$-triple acting on $\mathcal{T}$, so that $\mathcal{T}$ becomes an $\mathfrak{sl}_2$-algebra.

4.4 The *n*-point functions

As induced products of the $T_{k,l}$ form a basis for $\mathcal{T}$, knowing $\langle f\rangle_q$ for all $f\in\mathcal{T}$ is equivalent to knowing the following generating function, called the *n-point function*

$$F_n(u_1,\ldots u_n,v_1,\ldots v_n) \;=\; \sum_{\underline{k},\underline{l}}\langle T_{k_1,l_1}\odot\cdots\odot T_{k_n,l_n}\rangle_q\,\frac{u_1^{k_1}\cdots u_n^{k_n}v_1^{l_1-1}\cdots v_n^{l_n-1}}{k_1!\cdots k_n!(l_1-1)!\cdots(l_n-1)!}$$

for all $n\geq 0$. Here the sum is over all k_i,l_i such that k_i+l_i is even and $m!$ is consider to be 1 for $m<0$. As the q-bracket is a homomorphism with respect to the induced product, we directly conclude that

$$F_n(\underline{u},\underline{v}) \;=\; \prod_{i=1}^n F_1(u_i,v_i). \tag{43}$$

We also define the *partition function* by

$$\Phi(\underline{t}) \;=\; \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\underline{k},\underline{l}}\langle T_{k_1,l_1}\odot\cdots\odot T_{k_n,l_n}\rangle_q t_{k_1,l_1}\cdots t_{k_n,l_n}\,.$$

The following result (together with (43)) expresses these functions in terms of the Jacobi theta series (see (8)).

Theorem 4.4.1 *For all $n\geq 0$ one has*

$$F_1(u,v) \;=\; -\frac{1}{2}\frac{\theta'(0)\theta(u+v)}{\theta(u)\theta(v)},\qquad \Phi(\underline{t}) \;=\; \exp\Bigl([x^0y^0]F_1\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\sum_{k,l}t_{k,l}x^ky^l\Bigr),$$

where $[x^0y^0]$ denotes taking the constant coefficient.

Proof We have that

$$F_1(u,v) \;=\; \prod_{i=1}^n\Bigl(-\frac{1}{2u}-\frac{1}{2v}+\Bigl(\sum_{k,l}D^{l-1}G_{k-l+2}+\sum_{k,l}D^kG_{l-k}\Bigr)\frac{u^kv^{l-1}}{k!(l-1)!}\Bigr),$$

where in the sum it is understood that $k+l$ is even, $k\geq 0$, $l\geq 1$. The expression for $F_1(u,v)$ in the statement now follows from [20, ♧♧3]. The expression for Φ follows immediately from this result. □

5 Differential operators

5.1 The derivative of a function on partitions

Note that for all $f\in\mathbb{Q}^{\mathscr{P}}$ one has

$$D\langle f\rangle_q \;=\; \langle S_2f\rangle_q-\langle S_2\rangle_q\langle f\rangle_q\,. \tag{44}$$

Hence, by letting $Df:=S_2\,|\,f=S_2f-S_2\odot f$ for $f\in\mathbb{Q}^{\mathscr{P}}$, we have that $D\langle f\rangle_q=\langle Df\rangle_q$. Moreover, D acts as a derivation:

Proposition 5.1.1 *The map $D:\mathbb{Q}^{\mathscr{P}}\to\mathbb{Q}^{\mathscr{P}}$ is an equivariant derivation, i.e., D is linear, satisfies the Leibniz rule and*

$$D\langle f\rangle_q \;=\; \langle Df\rangle_q.$$

In fact, for all $k \geq 1$, the mapping $f \mapsto S_k \,|\, f$ is a derivation. Recall the definition of the Möbius function μ defined in Proposition 3.1.3 and denote $S_k^0 = S_k - S_k(\emptyset)$.

Lemma 5.1.2 *For all even $m \geq 2$ one has*

- (i) $S_m^0 \odot \mu = -S_m^0\,\mu$;
- (ii) *The mapping $(\mathbb{Q}^{\mathscr{P}}, \odot) \to (\mathbb{Q}^{\mathscr{P}}, \odot), f \mapsto S_m \,|\, f$ is a derivation, uniquely determined on $\mathcal{T}$ by*

$$S_m \,|\, T_{k,l} \;=\; T_{k+m-1,l+1}\,.$$

Remark 5.1.3 In case $m \geq 4$, the derivation $f \mapsto S_m \,|\, f$ does not correspond to a derivation on $\widetilde{M}$, i.e., a derivation $\mathfrak{d}_m$ such that $\mathfrak{d}_m \langle f\rangle_q = \langle S_m \,|\, f\rangle_q$ for all $f \in \mathcal{T}$. For instance, although the q-brackets of $T_{m,m}$ and $T_{m-1,m+1}$ are the same, the q-brackets of $S_m \,|\, T_{m,m} = T_{2m-1,m+1}$ and $S_m \,|\, T_{m-1,m+1} = T_{2m-2,m+2}$ are different.

Proof First of all, by Proposition 3.1.4, one has

$$\Big(\sum_{\lambda\in\mathscr{P}} u_\lambda\Big)\langle S_k^0 \odot \mu\rangle_{\underline{u}} \;=\; \Big(\sum_{m,r\geq 1} m^{k-1} u_m^r\Big)\Big(\sum_{\lambda\in\mathscr{P}} \mu(\lambda)\, u_\lambda\Big). \tag{45}$$

Let $\mathscr{S}_m$ be the set of strict partitions not containing m as a part. Then, we can rewrite (45) as

$$\sum_{m}\sum_{\lambda\in\mathscr{S}_m} m^{k-1}\mu(\lambda)\, u_m u_\lambda \;=\; -\sum_{\lambda\in\mathscr{P}} S_k^0(\lambda)\,\mu(\lambda)\, u_\lambda\,,$$

since $\mu(\lambda\cup(m)) = -\mu(\lambda)$ for $\lambda \in \mathscr{S}_m$, so that for $r \geq 2$ the coefficient of $u_m^r u_\lambda$ cancels in pairs. We conclude that $S_k^0 \odot \mu = -S_k^0\,\mu$.

For the second part, note that (i) implies that

$$S_k \odot \mu \;=\; -\Big(S_k + \frac{B_k}{k}\Big)\,\mu.$$

Let $f, g \in \mathbb{Q}^{\mathscr{P}}$ be given. Then

$$S_k \,|\, (f \odot g) \;=\; S_k(f \odot g) - S_k \odot f \odot g.$$

If $\alpha \cup \beta \cup \gamma = \lambda$ then $S_k(\lambda) = S_k(\alpha) + S_k(\beta) + S_k(\gamma) + \frac{B_k}{k}$, hence

$$\begin{aligned}
S_k(\lambda)\,(f \odot g)(\lambda) \;&=\; \sum_{\alpha\cup\beta\cup\gamma=\lambda} \Big(S_k(\alpha) + S_k(\beta) + S_k(\gamma) + \frac{B_k}{k}\Big) f(\alpha)\,g(\beta)\,\mu(\gamma)\\
&=\; (S_k f) \odot g \,+ f \odot (S_k g) \,+ \sum_{\alpha\cup\beta\cup\gamma=\lambda} \Big(S_k(\gamma) + \frac{B_k}{k}\Big) f(\alpha)\,g(\beta)\,\mu(\gamma)\\
&=\; (S_k f) \odot g \,+ f \odot (S_k g) \,- \sum_{\alpha\cup\beta\cup\gamma=\lambda} (S_k \odot \mu)(\gamma) f(\alpha)\,g(\beta)\\
&=\; (S_k f) \odot g + f \odot (S_k g) \;-\; S_k \odot f \odot g.
\end{aligned}$$

Therefore,

$$S_k \,|\, (f \odot g) \;=\; (S_k f) \odot g + f \odot (S_k g) - 2\, S_k \odot f \odot g \;=\; (S_k \,|\, f) \odot g + f \odot (S_k \,|\, g),$$

i.e., the mapping $f \mapsto S_k \,|\, f$ is a derivation. The formula $S_m \,|\, T_{k,l} = T_{k+m-1,l+1}$ follows directly from Theorem 3.5.4. □

Proof of Proposition 5.1.1 As $S_2 \,|\, f = S_2 f - S_2 \odot f$ is derivation by the above lemma, the results follows directly from (44). □

5.2 The equivariant q-bracket

In this section we extend the action by the $\mathfrak{sl}_2$-triple $(D, \mathfrak{d}, W)$ on quasimodular forms to $\mathcal{T}$. As the derivation $\mathfrak{d}$ does not act on all power series in q, but only on quasimodular forms, we cannot hope to define $\mathfrak{d}$ on all functions on partitions as we did with D. On the algebra $\mathcal{T}$, however, this is possible. We define an $\mathfrak{sl}_2$-action on this space and we show that the q-bracket restricted to $\mathcal{T}$ is an equivariant map of $\mathfrak{sl}_2$-algebras.

Note that the following definition agrees with the definition of D in the previous section:

Definition 5.2.1 Define the derivations $D, W, \mathfrak{d}$ on $\mathcal{T}$ by

$$\begin{aligned} D\,T_{k,l} &= T_{k+1,l+1}, \\ W\,T_{k,l} &= (k+l)T_{k,l}, \\ \mathfrak{d}\,T_{k,l} &= k(l-1)T_{k-1,l-1} - \tfrac{1}{2}\delta_{k+l-2}. \end{aligned}$$

One immediately checks that D, W and $\mathfrak{d}$ satisfy the commutation relation of an $\mathfrak{sl}_2$-triple on $\mathcal{T}$. The corresponding acting of $\mathfrak{sl}_2$ on $\mathcal{T}$ makes the q-bracket equivariant, so that a refined version of Theorem 1.2 is:

Theorem 5.2.2 (The $\mathfrak{sl}_2$-equivariant symmetric Bloch–Okounkov theorem) *The algebra $\mathcal{T}$ is an $\mathfrak{sl}_2$-algebra with respect to the above action of $\mathfrak{sl}_2$ on $\mathcal{T}$. Moreover, the q-bracket becomes an equivariant map of $\mathfrak{sl}_2$-algebras, i.e., for $f \in \mathcal{T}$ one has*

$$D\langle f\rangle_q = \langle Df\rangle_q, \quad W\langle f\rangle_q = \langle Wf\rangle_q, \quad \mathfrak{d}\langle f\rangle_q = \langle \mathfrak{d}f\rangle_q\,.$$

Proof We already observed that the first of the three equality holds and the second is the homogeneity statement. Hence, it suffices to prove the last statement. Using (9) we find that for $a \geq 0, b \geq 2$ one has

$$\mathfrak{d}(D^a G_b) = a(a+b-1)D^{a-1}G_b - \tfrac{1}{2}\delta_{a+b-2}.$$

Hence,

$$\mathfrak{d}\langle T_{k,l}\rangle_q = k(l-1)\langle T_{k-1,l-1}\rangle_q - \delta_{k+l-2} = \langle \mathfrak{d}\,T_{k,l}\rangle_q$$

and the last statement follows from the Leibniz rule. □

5.3 Rankin–Cohen brackets

The $\mathfrak{sl}_2$-action allows us to define Rankin–Cohen brackets on $\mathcal{T}$.

Definition 5.3.1 For two elements $f, g \in \mathcal{T}$ and $n \geq 0$ the nth Rankin–Cohen bracket is given by

$$[f,g]_n = \sum_{\substack{r,s\geq 0\\ r+s=n}} (-1)^r \binom{k+n-1}{s}\binom{l+n-1}{r} D^r f \odot D^s g. \tag{46}$$

Note that the formula (46) would have defined the Rankin–Cohen brackets on $\widetilde{M}$ if D acts by $q\frac{\partial}{\partial q}$ and the induced product is replaced by the usual product, whereas in this line D acts on $\mathcal{T}$ as explained in the previous sections.

If $f, g \in \ker \mathfrak{d}$, then $\langle f\rangle_q$ and $\langle g\rangle_q$ are modular forms. The Rankin–Cohen bracket of two modular forms is a modular form; analogously, we have:

Proposition 5.3.2 *If $f, g \in \ker \mathfrak{d}$, then $[f, g]_n \in \ker \mathfrak{d}$.*

Proof Using (9), we find that

$$\begin{aligned}\mathfrak{d}[f,g]_n \;=\; \sum_{\substack{r,s\geq 0\\ r+s=n}} &(-1)^r \frac{(k+n-1)!}{s!(k+r-2)!}\frac{(l+n-1)!}{(r-1)!(k+s-1)!} D^{r-1}f \odot D^s g \\ &+ (-1)^r \frac{(k+n-1)!}{(s-1)!(k+r-1)!}\frac{(l+n-1)!}{r!(l+s-2)!} D^r f \odot D^{s-1} g,\end{aligned}$$

where $\frac{1}{(-1)!}$ should taken to be 0. This is a telescoping sum, vanishing identically. □

Remark 5.3.3 The above bracket makes the algebra $\mathcal{T}$ into a Rankin–Cohen algebra, meaning the following. Let $A_* = \oplus_{k\geq 0} A_k$ be a graded K-vector space with $A_0 = K$ and $\dim A_k < \infty$ (for us $A = \mathcal{T}$). We say A is a *Rankin–Cohen algebra* if there are bilinear operations $[\,,\,]_n : A_k \otimes A_l \to A_{k+l+2n}$ $(k, l, n > 0)$ which satisfy all the algebraic identities satisfied by the Rankin–Cohen brackets on $\widetilde{M}$ [21].

5.4 A restricted $\mathfrak{sl}_2$-action

Theorem 5.2.2 does not make $\mathcal{S}$ into an $\mathfrak{sl}_2$-algebra. Namely, D does not preserve $\mathcal{S}$. However, if we allow ourselves to deform the $\mathfrak{sl}_2$-triple $(D, \mathfrak{d}, W)$ as in [18], we can define an $\mathfrak{sl}_2$-action on $\mathcal{S}$. This action, however, does not make $\mathcal{S}$ into an $\mathfrak{sl}_2$-algebra, as the deformed operators are not derivations.

The operator taking the role of $\mathfrak{d}$ is the operator $\mathfrak{s} : \mathcal{S}_k \to \mathcal{S}_{k-2}$ defined by

$$\mathfrak{s} \;=\; \frac{1}{2}\sum_{k,l\geq 0} (k+l) S_{k+l} \frac{\partial^2}{\partial S_{k+1}\,\partial S_{l+1}} - \frac{1}{2}\frac{\partial}{\partial S_2}.$$

The operator D is replaced by multiplication with S_2.

Lemma 5.4.1 *The triple $(S_2, \mathfrak{s}, W - \frac{1}{2})$ forms an $\mathfrak{sl}_2$-triple of operators acting on $\mathcal{S}$.*

Proof Observe that

$$[\mathfrak{s}, S_2]f \;=\; \sum_k (k+1) S_{k+1} \frac{\partial}{\partial S_{k+1}} f - \tfrac{1}{2} f \;=\; (W - \tfrac{1}{2})f.$$

As $\mathfrak{s}$ and S_2 decrease, respectively, increasing the weight by 2, the claim follows. □

Theorem 5.4.2 *The q-bracket $\langle\cdot\rangle_q : \mathcal{S} \to \widetilde{M}$ is an equivariant mapping with respect to the $\mathfrak{sl}_2$-triple $(S_2, \mathfrak{s}, W - \frac{1}{2})$ on $\mathcal{S}$ and the $\mathfrak{sl}_2$-triple $(D + G_2, \mathfrak{d}, W - \frac{1}{2})$ on $\widetilde{M}$, i.e., for all $f \in \mathcal{S}$ one has*

$$(D + G_2)\langle f\rangle_q \;=\; \langle S_2 f\rangle_q, \quad (W - \tfrac{1}{2})\langle f\rangle_q \;=\; \langle (E - \tfrac{1}{2}) f\rangle_q, \quad \mathfrak{d}\langle f\rangle_q \;=\; \langle \mathfrak{s} f\rangle_q. \tag{47}$$

Proof The first of the three equalities in (47) follows from the definition of the q-bracket; the second is the homogeneity statement of Theorem 5.2.2. Hence, it remains to prove the last equation $\mathfrak{d}\langle f\rangle_q = \langle \mathfrak{s}f\rangle_q$.

Given $\underline{k} \in \mathbb{N}^n$, let $\underline{k}^i \in \mathbb{N}^{n-1}$ be given by $\underline{k}^i := (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n)$ omitting k_i. Similarly, define $\underline{k}^{i,j} \in \mathbb{N}^{n-2}$ by omitting k_i and k_j. Then

$$\mathfrak{s}S_{\underline{k}} = \sum_{i\neq j}(k_i + k_j - 2)S_{k_i+k_j-2}S_{\underline{k}^{i,j}} - \frac{1}{2}\sum_{i:\,k_i=2} S_{\underline{k}^i}.$$

By Theorem 3.3.1, one finds

$$\left\langle S_{k_i+k_j-2}S_{\underline{k}^{i,j}}\right\rangle_q = \sum_{\substack{\beta\in\Pi(n)\\ \exists I\in\beta:\,\{i,j\}\subset I}} D^{\ell(I)-2}G_{|k_I|-2\ell(I)+2}\prod_{B\neq I} D^{\ell(A)-1}G_{|k_B|-2\ell(A)+2}.$$

For $I \in \beta$ and $\underline{l} \in \mathbb{N}^I$, let

$$C(I,\underline{l}) := \sum_{i,j\in I,i\neq j}(l_i + l_j - 2) = (\ell(I) - 1)(|\underline{l}| - \ell(I)).$$

It follows that $\sum_{i\neq j}(k_i + k_j - 2)\left\langle S_{k_i+k_j-2}S_{\underline{k}^{i,j}}\right\rangle_q$ equals

$$\sum_{\beta\in\Pi(n)}\sum_{I\in\beta} 2C(I,\underline{k})D^{\ell(I)-2}G_{|k_I|-2\ell(I)+2}\prod_{B\neq I} D^{\ell(B)-1}G_{|k_B|-2\ell(B)+2}.$$

On the other hand, observe that if f is of weight $|\underline{l}| - 2\ell(I) + 2$, Eq. (9) yields

$$[\mathfrak{d}, D^{\ell(I)-1}]f = C(I,\underline{l})D^{\ell(I)-2}f.$$

Hence, using $\mathfrak{d}G_k = -\frac{1}{2}\delta_{k,2}$, we obtain

$$[\mathfrak{d}, D^{\ell(I)-1}]G_{|k_I|-2\ell(I)+2} = C(B,\underline{k}_I)D^{\ell(I)-2}G_{|k_I|-2\ell(I)+2} - \frac{1}{2}\delta_{\underline{k}_I,(2)}.$$

Therefore,

$$\begin{aligned}\mathfrak{d}\langle S_{\underline{k}}\rangle_q = &\sum_{\beta\in\Pi(n)}\sum_{I\in\beta} C(I,\underline{k}_I)D^{\ell(I)-2}G_{|k_I|-2\ell(I)+2}\prod_{B\neq I} D^{\ell(B)-1}G_{|k_B|-2\ell(B)+2}\\ &-\frac{1}{2}\sum_{i:\,k_i=2}\ \sum_{\beta\in\Pi([n]\backslash\{i\})}\ \prod_{B\in\beta} D^{\ell(B)-1}G_{|k_B|-2\ell(B)+2},\end{aligned}$$

which by the above reasoning is exactly equal to $\langle\mathfrak{s}S_{\underline{k}}\rangle_q$. □

6 Relating the two products

6.1 The structure constants

In Theorem 3.5.4, we deduced that

$$T_{k_1,f_1}\,|\,\ldots\,|\,T_{k_n,f_n} = T_{|\underline{k}|,g} \quad\text{with}\quad g(f_1,\ldots,f_n) = \sum_{\alpha\in\Pi(n)}\mu(\alpha,\mathbf{1})\partial^{\ell(\alpha)-1}\mathop{\Large\ast}_{A\in\alpha} f_A.$$

In the particular case that $f_1 = \ldots = f_n$ is the identity function, we saw in Corollary 3.5.6 that $g = \mathcal{F}_n$. If $f_1,\ldots,f_n$ are Faulhaber polynomials, the function g is not necessarily equal Faulhaber polynomial on all $m \in \mathbb{N}$, but, by Lemma 3.5.7, ∂g equals some polynomial. Also, using g is uniquely determined by ∂g, the function g equals some polynomial. We expand g as a linear combination of Faulhaber polynomials.

Definition 6.1.1 Given integers $l_1,\ldots,l_n$, we define *the structure constants* $C_i^{\underline{l}}$ by

$$g(\mathcal{F}_{l_1},\ldots,\mathcal{F}_{l_n}) \;=\; \sum_{i=0}^{|\underline{l}|-1} C_i^{\underline{l}}\,\mathcal{F}_{|\underline{l}|-i}\,.$$

Observe that $C_i^{\underline{l}}=0$ for odd i, as ∂g is even or odd. Corollary 3.5.6 is the statement

$$C_i^{(1,\ldots,1)} \;=\; \begin{cases}1 & i=0\\ 0 & \text{else.}\end{cases}$$

More generally, by Theorem 3.5.4(ii) one has $C_i^{1,\underline{l}}=C_i^{\underline{l}}$, so that w.l.o.g. we can assume $l_i>1$. In this section, we give an explicit, but involved, formula for these coefficients in terms of Bernoulli numbers and binomial coefficients. In order to do so, for $l_1,l_2\geq 1$ and $i\in\mathbb{Z}_{\geq 0}$, we introduce the following numbers:

$$\mathcal{B}_i^{l_1,l_2} \;:=\; \begin{cases}\frac{(l_1-1)!(l_2-1)!}{(l_1+l_2-1)!} & i=0,\\ \zeta(1-i)\Big((-1)^{l_2}\binom{l_1-1}{i-l_2}+(-1)^{l_1}\binom{l_2-1}{i-l_1}\Big) & i>0,\end{cases}$$

which by [2, Proposition A.10] satisfy

$$\sum_{i=0}^{l_1+l_2-2}\mathcal{B}_i^{l_1,l_2}\,\frac{B_{l_1+l_2-i}}{l_1+l_2-i}=(-1)^{l_1l_2}\,\frac{B_{l_1+l_2}-B_{l_1}B_{l_2}}{l_1l_2}.$$

Note that $\zeta(1-i)=(-1)^{i+1}\frac{B_i}{i}$ for $i\geq 1$. The following polynomials can be expressed in terms of these coefficients:

Lemma 6.1.2 *For all $l_1,l_2,\ldots,l_r\geq 2$ one has the following identities:*

(i) $\mathcal{F}_{l_1}(x) \;=\; \sum_{i=0}^{\infty}\mathcal{B}_i^{l_1,1}x^{l_1-i};$

(ii) $(\partial\mathcal{F}_{l_1}*\partial\mathcal{F}_{l_2})(x) \;=\; \sum_{i=0}^{\infty}\mathcal{B}_i^{l_1,l_2}x^{l_1+l_2-i-1};$

(iii) $\partial(\mathcal{F}_{l_1}\cdots\mathcal{F}_{l_r})(x) \;=\; 2\sum_{|\underline{i}|\equiv 1\,(2)}\mathcal{B}_{i_1}^{l_1,1}\cdots\mathcal{B}_{i_r}^{l_r,1}x^{|\underline{l}|-|\underline{i}|}.$

Proof The first two equations, of which the former is the well-known expansion of the Faulhaber polynomials, follow by considering the corresponding generating series. In order to prove (ii), we let $n\in\mathbb{N}$ and consider

$$\begin{aligned}\mathcal{G}(n) \;&:=\; \sum_{l_1,l_2=1}^{\infty}(\partial\mathcal{F}_{l_1}*\partial\mathcal{F}_{l_2})(n)\frac{z_1^{l_1-1}}{(l_1-1)!}\frac{z_2^{l_2-1}}{(l_2-1)!}\\ &=\sum_{m_1+m_2=n}e^{m_1z_1+m_2z_2}\\ &=\frac{e^{nz_1}}{e^{z_1-z_2}-1}+\frac{e^{nz_2}}{e^{z_2-z_1}-1}.\end{aligned}$$

As the generating series of the Bernoulli numbers $\sum_{j=0}^{\infty}B_j\frac{z^j}{j!}=z(e^z-1)^{-1}$ implies that

$$\frac{1}{e^{z_1-z_2}-1} \;=\; \frac{1}{z_1-z_2}+\sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\frac{B_j}{j}(-1)^i\frac{z_1^{j-i-1}z_2^i}{(j-1-i)!i!},$$

we find

$$\mathcal{G}(n) = \sum_{k=1}^{\infty}\Bigg(\sum_{i=0}^{k-1}\frac{z_1^i z_2^{k-i-1}}{i!(k-i-1)!} + \sum_{j=1}^{\infty}\sum_{i=0}^{j-1}\frac{B_j}{j}(-1)^i\Big(\frac{z_1^{k+j-i-1}z_2^i}{(j-i-1)!i!} + \frac{z_1^i z_2^{k+j-i-1}}{i!(j-i-1)!}\Big)\Bigg)\frac{n^k}{k!}$$
$$= \sum_{l_1,l_2=1}^{\infty}\sum_{i=0}^{\infty}(-1)^i\mathcal{B}_i^{l_1,l_2}\frac{z_1^{l_1-1}}{(l_1-1)!}\frac{z_2^{l_2-1}}{(l_2-1)!}n^{l_1+l_2-i-1}.$$

Since $\mathcal{B}_i^{l_1,l_2}$ vanishes for odd i if $l_1, l_2 > 1$, this proves the second equation. The third equation follows from the first by noting that

$$\partial(\mathcal{F}_{l_1}\cdots\mathcal{F}_{l_r})(x) = (\mathcal{F}_{l_1}\cdots\mathcal{F}_{l_r})(x) - (-1)^{|\underline{l}|}(\mathcal{F}_{l_1}\cdots\mathcal{F}_{l_r})(-x). \qquad \square$$

Using these identities, one obtains.

$$C_i^l = \mathcal{B}_i^{1,1} = \delta_{i,0}, \quad C_i^{l_1,l_2} = \mathcal{B}_i^{l_1,1} + \mathcal{B}_i^{l_2,1} - \mathcal{B}_i^{l_1,l_2}$$

These easy expressions for small n are misleading, as $6C_i^{l_1,l_2,l_3}$ equals

$$\frac{1}{4}\delta_{i,2} + 3\sum_{\substack{i_1,i_2\equiv 0\,(2)\\ i_1+i_2=i}}\mathcal{B}_{i_1}^{l_1,1}\mathcal{B}_{i_2}^{l_2,1} - \sum_{\substack{i_1\equiv 1\,(2),j_1\\ i_1+j_1=i}}\mathcal{B}_{i_1}^{l_1,1}\mathcal{B}_{j_1}^{l_1,l_2+l_3-i_1} + 2\sum_{j_1+j_2=i}\mathcal{B}_{j_1}^{l_1,l_2}\mathcal{B}_{j_2}^{l_1+l_2-j_1,l_3}$$

up to full symmetrization, i.e., summing over all $\sigma \in S_3$ with l_i replaced by $l_{\sigma(i)}$. In general, given $\alpha \in \Pi(n)$, write $\alpha = \{A_1,\ldots,A_r\}$ and denote $A^j = \cup_{i=1}^j A_j$. Also, for a vector $\underline{k}$ and a set B we let $k_B = \sum_{b\in B} k_b$. Then, the above observations allows us to write down the following formula, which is very amenable to computer calculation:

Proposition 6.1.3 *Let $l_1,\ldots,l_n > 1$. Then,*

$$C_i^{\underline{l}} = \sum_{\alpha\in\Pi(n)} 2^r\mu(\alpha,\mathbf{1})\sum_{\substack{i_1,\ldots,i_n\\ |\underline{i}_A|\equiv 1\,(2)}}\left(\prod_{k=1}^n \mathcal{B}_{i_k}^{l_k,1}\right)\left(\sum_{\substack{j_1,\ldots,j_{\ell(\alpha)-1}\\ |\underline{i}|+|\underline{j}|=i+r}}\prod_{s=1}^{r-1}\mathcal{B}_{j_s}^{l_{A^s}-j_{s-1},\,l_{A_{s+1}}-i_{A_{s+1}}+1}\right)$$

Here, $j_0 := l_{A_0} - i_{A_0}$.

Note that the latter formula is written in an asymmetric way, but (by associativity of the convolution product) is symmetric in the l_i.

6.2 From the pointwise product to the induced product

Suppose an element of $\mathcal{T}$ is given, written in the basis with respect to the pointwise product. How do we determine its (possibly mixed) weight and its representation in terms of the basis with respect to the $\odot$ product? A first answer is given by applying Möbius inversion to Eq. (23), as given by Eq. (42), i.e.,

$$T_{\underline{k},\underline{l}} = \sum_{\alpha\in\Pi(n)}\bigodot_{A\in\alpha}(T_{k_{A_1},l_{A_1}} \mid T_{k_{A_2},l_{A_2}} \mid \ldots). \tag{48}$$

However, as every factor $T_{k_{A_1},l_{A_1}} \mid T_{k_{A_2},l_{A_2}} \mid \ldots$ in the above equation is a linear combination of generators of different weights, it is useful to have a recursive version of this result. For this, we write $\frac{\partial}{\partial T_{k,l}}$ for the derivative of $f \in \mathcal{T}$ in the former basis (with respect to the pointwise product) and $\frac{\partial}{\partial T_{\underline{k},\underline{l}}}$ for $\prod_i \frac{\partial}{\partial T_{k_i,l_i}}$.

Proposition 6.2.1 *Let $k, l \geq 1$. There exist differential operators $\mathfrak{s}_{i,j}$ for all $i, j \in \mathbb{Z}$ such that $\mathfrak{s}_{i,j} = 0$ if $j < 0$ and for all $f \in \mathcal{T}$ one has*

$$T_{k,l}f \;=\; \sum_{i\geq 0}\sum_{j\geq -l+1} T_{k+i,l+j} \odot \mathfrak{s}_{i,j}(f).$$

Explicitly,

$$\mathfrak{s}_{i,j} \;=\; \sum_{|\underline{a}|=i} \mathfrak{t}_{\underline{a},j}, \qquad \mathfrak{t}_{\underline{a},j} \;=\; \sum_{\underline{b}} C^{l,\underline{b}}_{|\underline{b}|-j}\frac{\partial}{\partial T_{\underline{a},\underline{b}}},$$

where $\underline{a}$ and $\underline{b}$ are vectors of integers of the same length and with $|\underline{a}| = i$, the structure constants $C^{l,\underline{b}}_{|\underline{b}|-j}$ are as in Proposition 6.1.3 and $l, \underline{b}$ denotes the vector $(l, b_1, b_2, \ldots)$.

Proof By linearity, it suffices to prove the statement for monomials $T_{\underline{k},\underline{l}}$. Hence, assume $f = T_{\underline{k},\underline{l}}$. Applying (48), extracting the factor containing $T_{k,l}$ and applying (48) again, yields

$$\begin{aligned} T_{k,l}f \;&=\; \sum_{A\subset[n]} (T_{k,l} \mid T_{k_{A_1},l_{A_1}} \mid T_{k_{A_2},l_{A_2}} \mid \ldots) \odot T_{\underline{k}_{[n]\backslash A},\underline{l}_{[n]\backslash A}} \\ &=\; \sum_{\underline{a},\underline{b}} (T_{k,l} \mid T_{a_1,b_1} \mid T_{a_2,b_2} \mid \ldots) \odot \frac{\partial}{\partial T_{\underline{a},\underline{b}}} f. \end{aligned}$$

By Definition 6.1.1 this equals

$$T_{k,l}f \;=\; \sum_{\underline{a},\underline{b}}\sum_{j\in\mathbb{Z}} C^{l,\underline{b}}_{j}\, T_{|\underline{a}|+k,|\underline{b}|+l-j} \odot \frac{\partial}{\partial T_{\underline{a},\underline{b}}} f$$

Replacing j by $-j + |\underline{b}|$ and writing $i = |\underline{a}|$, one obtains

$$T_{k,l}f \;=\; \sum_{i\geq 0}\sum_{j\in\mathbb{Z}} T_{k+i,l+j} \odot \sum_{|\underline{a}|=i}\sum_{\underline{b}} C^{l,\underline{b}}_{|\underline{b}|-j}\frac{\partial}{\partial T_{\underline{a},\underline{b}}} f,$$

as desired. □

Corollary 6.2.2 *For all $k, l \geq 1$ and $f \in \mathcal{T}$ one has*

$$\langle T_{k,l}f\rangle_q \;=\; \sum_{a\geq 0}\sum_{b\geq 2} D^a G_b \langle \mathfrak{T}^{a,b}_{k,l} f\rangle_q\,,$$

where $\mathfrak{T}^{a,b}_{k,l} = \mathfrak{s}_{a-l+1,a+b-k-1} + \mathfrak{s}_{a+b-l,a-k}$.

Proof Distinguishing two cases in the previous result yields

$$\langle T_{k,l}f\rangle_q = \sum_{j<k+i-l} D^{l+j-1}G_{k+i-l-j+2}\odot \mathfrak{s}_{i,j}(f) + \sum_{i\geq 0}\sum_{j\geq k+i-l} D^{k+i}G_{l+j-k-i}\odot \mathfrak{s}_{i,j}(f)$$
$$= \sum_{a\geq 0}\sum_{b\geq 2} D^a G_b \langle(\mathfrak{s}_{a+b-k-1,a-l+1}+\mathfrak{s}_{a-k,a+b-l})(f)\rangle_q\,. \qquad \square$$

7 Related functions on partitions

We apply our results to interesting functions on partitions.

7.1 Hook-length moments

First of all, we focus on the hook-length moments H_k [5, part III]. These functions form a bridge between the symmetric algebra studied in this note and the shifted symmetric functions: The H_k themselves are shifted symmetric as

$$H_k(\lambda) = \frac{1}{2}\sum_{i=0}^{k}\binom{k-2}{i-1}(-1)^i Q_i(\lambda)Q_{k-i}(\lambda) \tag{49}$$

and they are also equal to the Möller transform of the symmetric S_k, i.e., $H_k = \mathcal{M}(S_k)$, meaning the following. Denote $z_\nu = \frac{n!}{|C_\nu|}$ with $|C_\nu|$ the size of the conjugacy class corresponding to ν. Recall that

$$z_\nu = \prod_{m=1}^{\infty} m^{r_m(\nu)} r_m(\nu)!.$$

Given $f\in\mathbb{Q}^{\mathscr{P}}$, *the Möller transform* of f at a partition $\lambda\in\mathscr{P}(n)$ is given by [23, Eqn (45)]

$$\mathcal{M}(f)(\lambda) = \sum_{\nu\vdash n} z_\nu^{-1}\chi^\lambda(\nu)^2 f(\nu),$$

where the sum $\nu\vdash n$ is over all partitions of size n and $\chi^\lambda(\rho)$ denotes the character of the representation corresponding to the partition λ evaluated at the conjugacy class corresponding to ρ. Then $\langle\mathcal{M}(f)\rangle_q$ is a quasimodular form if and only if $\langle f\rangle_q$ is a quasimodular form (which follows directly by the column orthogonality relations for the symmetric group). In the next section, we study the Möller transform of elements of $\mathcal{T}$, but first, we explain the Murnaghan–Nakayama rule, used in [5, part III] to show equality between $\mathcal{M}(S_k)$ and (49) and give two other expressions for the hook-length moments.

To start with the latter, *the hook-length moments*, as their name suggests, are defined as moments of the hook-lengths, i.e.,

$$H_k(\lambda) = -\frac{B_k}{2k} + \sum_{\xi\in Y_\lambda} h(\xi)^{k-2},$$

where Y_λ denotes the Young diagram of a partition λ and $h(\xi)$ denotes the hook-length of a cell $\xi\in Y_\lambda$.

Next, the following constructions related to the Young diagram, give rise to the Murnaghan–Nakayama rule for the characters of the symmetric group. Given partitions λ,ν with $\nu_i\leq\lambda_i$ for all i, we define the *skew Young diagram* λ/ν by removing the cells of Y_ν from Y_λ. Denote by $|\lambda/\nu| = |\lambda|-|\nu|$ the number of cells of this diagram. We call λ/ν a *border strip* of λ if it is connected (through edges, not only through vertices) and contains no 2×2-block. If $\gamma=\lambda/\nu$ we write $\lambda\setminus\gamma$ for ν. The *height* of a border strip γ is defined to be one less than the number of columns and denoted by $\mathrm{ht}(\gamma)$. Given $\underline{m}\in\mathbb{N}^s$, we let a *border strip tableau* γ of type $\underline{m}$ be a sequence $\gamma_1,\ldots,\gamma_s$ such that γ_i is a border

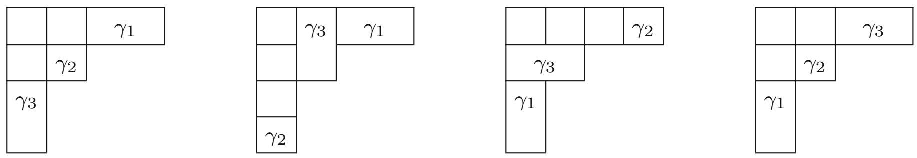


Fig. 1 The Young diagrams corresponding to the border strip tableaux of type (2, 1, 2) within $\lambda = (4, 2, 1, 1)$

strip of $\lambda \setminus \gamma_1 \setminus \cdots \setminus \gamma_{i-1}$ and $|\gamma_i| = m_i$. Write Y_γ for the skew Young diagram consisting of all boxes of all the γ_i and write $\mathrm{ht}(\gamma) = \mathrm{ht}(\gamma_1) + \ldots + \mathrm{ht}(\gamma_s)$. Denote by $\mathrm{BST}(\lambda, \underline{m})$ and $\mathrm{BST}(\lambda/\nu, \underline{m})$ the set of all border strip tableau of type $\underline{m}$ within λ and λ/ν, respectively (Fig. 1).

The *Murnaghan–Nakayama rule* (recursively) expresses the characters of the symmetric groups in terms the heights of border strip tableau. Namely, if $\rho' \subseteq \rho$ (both ρ' and ρ considered as multisets)

$$\chi^\lambda(\rho) \;=\; \sum_{\gamma \in \mathrm{BST}(\lambda, \rho')} (-1)^{\mathrm{ht}(\gamma)} \chi^{\lambda \setminus \gamma}(\rho - \rho'),$$

where $\rho - \rho'$ denotes the difference of (multi)sets. Of particular interest are the cases $\rho' = \rho$ and $\rho' = (\rho_1)$, yielding a direct or recursive combinatorial formula for $\chi^\lambda(\rho)$, respectively:

$$\chi^\lambda(\rho) \;=\; \sum_{\gamma \in \mathrm{BST}(\lambda, \rho)} (-1)^{\mathrm{ht}(\gamma)} \qquad \text{and} \qquad \chi^\lambda(\rho) = \sum_{|\gamma| = \rho_1} (-1)^{\mathrm{ht}(\gamma)}\, \chi^{\lambda \setminus \gamma}(\rho_2, \rho_3, \ldots),$$

where the latter sum is over all borders strips γ of λ of length ρ_1. The *skew character* $\chi^{\lambda/\nu}(\rho')$ is defined by ($|\lambda/\nu| = |\rho'|$)

$$\chi^{\lambda/\nu}(\rho') \;=\; \sum_{\gamma \in \mathrm{BST}(\lambda/\nu, \rho')} (-1)^{\mathrm{ht}(\gamma)}$$

so that

$$\chi^\lambda(\rho) \;=\; \sum_{|\nu| = |\rho'|} \chi^{\lambda/\nu}(\rho')\, \chi^\nu(\rho - \rho').$$

To conclude, we have the following definitions of the hook-length moments:

Definition 7.1.1 The hook-length moments H_k ($k \geq 2$ even) are defined by either of the following equivalent definitions [5, Section 13]:

(i) $H_k(\lambda) \;=\; -\dfrac{B_k}{2k} + \displaystyle\sum_{\xi \in Y_\lambda} h(\xi)^{k-2}\,;$

(ii) $H_k(\lambda) \;=\; -\dfrac{B_k}{2k} + \displaystyle\sum_{m=1}^{\infty} |\mathrm{BST}(\lambda, m)|\, m^{k-2}\,;$

(iii) $H_k \;=\; \dfrac{1}{2} \displaystyle\sum_{i=0}^{k} \binom{k-2}{i-1} (-1)^i Q_i\, Q_{k-i}\,;$

(iv) $H_k \;=\; \mathcal{M}(S_k).$

7.2 Border strip moments

The hook-length moments are Möller transformations of the S_k. In this section we study the Möller transformation of the algebra $\mathcal{T}$, which contains the vector space spanned by all the S_k. In order to do so, we express elements of $\mathcal{T}$ in terms of functions $U_{\underline{k},\underline{l}}$ for which the induced product and Möller transformation are easy to compute. However, these function do not admit the property that the q-bracket is quasimodular if $k_i + l_i$ is even for all i: each $U_{\underline{k},\underline{l}}$ lies in the space generated by all the $T_{\underline{k},\underline{l}}$ (possibly with $k_i + l_i$ odd).

Let

$$\mathbb{N}(\underline{l}) = \{(\underbrace{m_1,\ldots,m_1}_{l_1},\underbrace{m_2,\ldots,m_2}_{l_2},\ldots) \mid m_i \geq 1\}$$

the set of tuples of $n := |\underline{l}|$ positive integers, where the first l_1, the second l_2, etc., integers agree. For $\underline{k} \in \mathbb{Z}^n_{\geq 0}$, define

$$U_{\underline{k},\underline{l}} = \sum_{\underline{m}\in\mathbb{N}(\underline{l})} \underline{m}^{\underline{k}} \prod_{a=1}^{\infty} \binom{r_a(\lambda)}{r_a(\underline{m})}.$$

Observe that this product converges since $r_a(\underline{m}) = 0$ for all but finitely many values of a. Let $\mathcal{U}$ be the algebra generated by the $U_{\underline{k},\underline{l}}$.

Generalize the hook-length moments in Definition 7.1.1(ii) by the following notion:

Definition 7.2.1 The *border strip moments* are given by

$$X_{\underline{k},\underline{l}}(\lambda) = \sum_{\underline{m}\in\mathbb{N}(\underline{l})} \sum_{\gamma\in\mathrm{BST}(\lambda,\underline{m})} \frac{\chi^{\gamma}(\underline{m})^2}{z_{\underline{m}}} \underline{m}^{\underline{k}}.$$

Let $\mathcal{X}$ be the vector space spanned by all the $X_{\underline{k},\underline{l}}$. Define a filtration on $\mathcal{X}$ by assigning to $X_{\underline{k},\underline{l}}$ degree $|\underline{k}| + |\underline{l}|$.

Remark 7.2.2 Observe that for $n = 1$ and $l = 1$, the sum restricts to a sum over all border strips γ of λ and for such a border strip γ the factor $\chi^{\gamma}(\underline{m})^2$ equals 1 and z_m equals m. As the set of hook-lengths is in bijection with the set of all border strip lengths, one has that $-\frac{B_k}{2k} + X_{k,1} = H_{k+1}$.

Denote by $\left\{ {n \atop j} \right\}$ the Stirling numbers of the second kind (i.e., the number of elements in $\Pi(n)$ of length j).

Proposition 7.2.3 *For all $k \geq 0, l \geq 1, \underline{k}, \underline{k}' \in \mathbb{Z}^n_{\geq 0}$, and integer vectors $\underline{l}, \underline{l}'$ with $|\underline{l}| = |\underline{l}'| = n$, one has*

(i) $T_{k,l} = -\dfrac{B_{k+l}}{2(k+l)}(\delta_{l,1} + \delta_{k,0}) + \displaystyle\sum_{j=1}^{l} \left\{ {l \atop j} \right\} (j-1)!\, U_{k,j}\,;$

(ii) $U_{\underline{k},\underline{l}} \odot U_{\underline{k}',\underline{l}'} = U_{\underline{k}\cup\underline{k}',\underline{l}\cup\underline{l}'}\,;$

(iii) $\mathcal{M}(U_{\underline{k},\underline{l}}) = X_{\underline{k},\underline{l}}\,.$

Proof For the first property, we use the known identity

$$x^{l-1} = \sum_{j=1}^{l} \left\{ {l \atop j} \right\} (j-1)! \binom{x-1}{j-1}.$$

As $\mathcal{F}_l(x)$ and $\binom{x}{j}$ are the unique polynomials with constant term equal to zero and such that $\partial\mathcal{F}_l(x) = x^{l-1}$ and $\partial\binom{x}{j} = \binom{x-1}{j-1}$, respectively, we find

$$\mathcal{F}_l(x) = \sum_{j=1}^{l} \left\{ {l \atop j} \right\} (j-1)! \binom{x}{j},$$

which yields property (i).

Next, we show that for all $i, j \geq 0$ one has

$$\partial\binom{x}{i} * \binom{x}{j} = \binom{x}{i+j}.$$

Both

$$\binom{x}{i} * \binom{x}{j} = \sum_{m=0}^{x} \binom{m}{i}\binom{x-m}{j} \qquad \text{and} \qquad \binom{x+1}{i+j+1}$$

are polynomials of degree at most $i+j+1$ taking the value 0 for $x = 0, 1, \ldots, i+j-1$ and the value 1 for $x = i+j$; hence, they are equal. Therefore,

$$\partial\binom{x}{i} * \binom{x}{j} = \partial\binom{x+1}{i+j+1} = \binom{x}{i+j}.$$

By Lemma 3.5.2 property (ii) follows.

Finally, we have that

$$\mathcal{M}(U_{\underline{k},\underline{l}})(\lambda) = \sum_{\underline{m}\in\mathbb{N}(\underline{l})} \underline{m}^{\underline{k}} \sum_{\nu\vdash n} z_\nu^{-1} \chi^\lambda(\nu)^2 \prod_{a=1}^{\infty} \binom{r_a(\nu)}{r_a(\underline{m})}.$$

Observe that given $\underline{m}$ and ν the term

$$z_\nu^{-1} \chi^\lambda(\nu)^2 \prod_{a=1}^{\infty} \binom{r_a(\nu)}{r_a(\underline{m})} \tag{50}$$

vanishes unless $r_a(\nu) \geq r_a(\underline{m})$ for all positive a. Let ν' be the partition obtained from ν by removing $r_a(\underline{m})$ parts of size a from ν for all positive a. Denote by $n' = n - |\underline{m}|$ the size of ν'. By the Murnaghan–Nakayama rule one has

$$\chi^\lambda(\nu) = \sum_{\xi\in\mathrm{BS}(\lambda,\underline{m})} \chi^\xi(\underline{m})\, \chi^{\lambda\backslash\xi}(\nu').$$

One has

$$z_\nu^{-1} \prod_{a=1}^{\infty} \binom{r_a(\nu)}{r_a(\underline{m})} = \prod_{a=1}^{\infty} \frac{1}{a^{r_a(\nu)} r_a(\underline{m})!\,(r_a(\nu) - r_a(\underline{m}))!} = \frac{1}{z_{\nu'} z_{\underline{m}}}.$$

Hence, (50) equals

$$\sum_{\xi\in\mathrm{BS}(\lambda,\underline{m})} \sum_{\rho\in\mathrm{BS}(\lambda,\underline{m})} \frac{\chi^\xi(\underline{m})\,\chi^\rho(\underline{m})}{z_{\underline{m}}} \sum_{\nu'\vdash n'} z_{\nu'} \chi^{\lambda\backslash\xi}(\nu')\,\chi^{\lambda\backslash\rho}(\nu').$$

The orthogonality relation for the symmetric group is the statement

$$\sum_{\nu'\vdash n'} z_{\nu'}\, \chi^{\lambda\backslash\xi}(\nu')\, \chi^{\lambda\backslash\rho}(\nu') = \delta_{\lambda\backslash\xi,\lambda\backslash\rho}.$$

Hence, we obtain the desired result. □

The q-bracket of an element in $\mathcal{X}$ is not necessarily a quasimodular form. However, it always lies in the following space of q-analogues of zeta values, see [11].

Definition 7.2.4 Let $\mathcal{C}_{\leq \ell}$ be the $\mathbb{Q}$-vector space consisting of all polynomials in the *combinatorial Eisenstein series*

$$G_k(\tau) \;=\; -\frac{B_k}{2k} + \sum_{r=1}^{\infty}\sum_{m=1}^{\infty} m^{k-1}q^{mr} \qquad (k \geq 1,\ \text{not necessarily even})$$

and their derivatives of weight $\leq \ell$, where to $D^r G_k$ we assign the weight $k+2r$.

Now, Proposition 7.2.3 implies the following result:

Theorem 7.2.5 *For all $f \in \mathcal{X}_{\leq k}$, one has $\langle f\rangle_q \in \mathcal{C}_{\leq k}$.*

Proof By Proposition 7.2.3, f equals the Möller transform of some polynomial in the $T_{k,l}$ with respect to the product $\odot$. Here, however, it may happen that $k+l$ is odd. Mutatis mutandis in either of three approaches in Sect. 3.1, we find that the q-bracket of $T_{k,l}$ lies in $\mathcal{C}_{k+l}$, which proves the result. □

Theorem 7.2.6 *For all weights k one has $\mathcal{M}(\mathcal{T}_k) \subset \mathcal{X}_{\leq k}$. More precisely,*

$$\frac{\mathcal{M}(T_{k_1,l_1} \odot \cdots \odot T_{k_n,l_n})}{(l_1-1)!\cdots(l_n-1)!} \;=\; X_{\underline{k},\underline{l}} + \textit{elements in } \mathcal{X} \textit{ of lower degree.} \tag{51}$$

Proof Observe that Proposition 7.2.3 implies that $\mathcal{M}(\mathcal{T}_k) \subset \mathcal{X}_{\leq k}$. Equation (51) follows from this proposition after noting that the Möller transformation of $T_{k,l} - (l-1)!\,\mathcal{U}_{k,l}$ has degree strictly smaller than $k+l$. □

Example 7.2.7 The following two equations provide examples of linear combinations of elements of $\mathcal{X}$ with a quasimodular q-bracket whenever $k+l$ and k_i are even integers.

$$\mathcal{M}(T_{k,l}) \;=\; -\frac{B_{k+l}}{2(k+l)}(\delta_{k,1}+\delta_{l,0}) + \sum_{j=1}^{l} \left\{ {l \atop j} \right\} (j-1)!X_{k,j}\,,$$

$$\mathcal{M}(S_{k_1} \odot S_{k_2} \odot \cdots \odot S_{k_n}) \;=\; \sum_{A\subset[n]} \Big(\prod_{i\notin A} \frac{B_{k_i}}{2k_i}\Big) X_{k_A,(1,1,\ldots,1)}.$$

See Appendix A for a table of elements in $\mathcal{X}$ with quasimodular q-bracket and of small degree.

Remark 7.2.8 In many examples the $X_{\underline{k},\underline{l}}$ are not shifted symmetric functions or generated by shifted symmetric functions under the induced product. For example, $\mathcal{M}(T_{0,2}) \neq \mathcal{M}(S_2)$ and besides $Q_2 = \mathcal{M}(S_2) = S_2$ there are no other non-trivial functions generated by Λ^* under the pointwise product. It remains an open question whether the elements of $\mathcal{X}$ are in some sense related to shifted symmetric functions.

7.3 Moments of other partition invariants

So far we provided many examples of functions on partitions in Λ^* and $\mathcal{T}$ related to the representation theory of the symmetric group. Now, we see that many purely combinatorial notions lead to different bases for $\mathcal{S}$. We compare these bases to corresponding bases of Λ^*. Most of these bases take the following form. Suppose an index set I and a sequence $\{s_i\}_{i\in I}^{\infty}$ of elements of $\mathbb{Q}^{\mathscr{P}}$ are given. Then, we define the kth moment of $\underline{s}$ by

(whenever this sum converges)

$$M_k(\underline{s})(\lambda) \;=\; \sum_{i\in I}\Big(s_i(\lambda)^k - s_i(\emptyset)^k\Big).$$

For example, let the functions $\underline{p}, \underline{q}$ for the index set $\mathbb{N}$ be given by

$$p_i(\lambda) = \lambda_i\,, \qquad q_i(\lambda) = \lambda_i - i.$$

Then, by definition,

$$S_k \;=\; S_k(\emptyset) + M_{k-1}(\underline{p}), \qquad Q_k \;=\; Q_k(\emptyset) + M_{k-1}(\underline{q}).$$

Note that by definition $M_k(\underline{s})(\emptyset) = 0$. As the functions below will not respect the weight grading anyway, we will not include a constant term.

The sequences $\underline{a}, \underline{c}, \underline{h}, \underline{x}$ of functions on partitions are of further interest. Define these sequence, indexed by $\xi = (i,j) \in \mathbb{Z}^2_{\geq 0}$, by 0 if $\xi \notin Y_\lambda$ and

$$\begin{aligned} &a_\xi(\lambda):\ \text{arm length of } \xi && h_\xi(\lambda):\ \text{hook-length of } \xi\\ &x_\xi(\lambda) \;=\; i && c_\xi(\lambda):\ \text{content of } \xi,\ \text{i.e., } i-j \end{aligned}$$

if $\xi \in Y_\lambda$. For $\underline{h}$ and $\underline{c}$ it is known that the corresponding moment functions are shifted symmetric, for the latter see [15, Theorem 4]. The moment functions corresponding to $\underline{a}$ and $\underline{x}$ turn out to be equal and to be elements of $\mathcal{S}$.

Theorem 7.3.1

$$\mathcal{S} \;=\; \mathbb{Q}[M_k(\underline{a}) \mid k \geq 0 \text{ } even] \;=\; \mathbb{Q}[M_k(\underline{x}) \mid k \geq 0 \text{ } even].$$

Proof As the Faulhaber polynomials $\mathcal{F}_k$ with k odd form a basis for the space of all odd polynomials, the functions

$$\sum_{i=1}^{\infty}\mathcal{F}_k(\lambda_i) \;=\; \sum_{i=1}^{\infty}\sum_{a=1}^{\lambda_i} a^{k-1}$$

generate $\mathcal{S}$, which corresponds to the first equality in the statement. By interchanging the sums one obtains

$$\sum_{i=1}^{\infty}\mathcal{F}_k(\lambda_i) \;=\; \sum_{a=1}^{\infty} a^{k-1}\sum_{m=a}^{\infty} r_m(\lambda) \;=\; \sum_{(i,j)\in Y_\lambda} i^{k-1}. \tag{52}$$

Hence, the result is also true for $\underline{s} = \underline{x}$. □

Remark 7.3.2 Note that for a given i the number of $(i,j) \in Y_\lambda$ equals λ'_i, where λ' is the conjugate partition of λ. Hence, (52) can be written as

$$\sum_{i=1}^{\infty} i^{k-1}\lambda'_i$$

and consequently these functions for k odd generate $\mathcal{S}$. Note that these functions are different from the $S_k(\lambda')$. In fact, the algebra generated by the $S_k(\lambda')$ is distinct from the algebra $\mathcal{S}$, in contrast to the algebra of shifted symmetric functions, for which $Q_k(\lambda') = (-1)^k Q_k(\lambda)$.

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A Table of double moment functions up to weight 4

For all basis elements $f \in \mathcal{T}_{\leq 4}$ in the basis provided by Theorem 4.3.1, we compute its representation in the basis consisting of double moment function and the quantities $\langle f\rangle_{\underline{u}}$, $\langle f\rangle_q$, $D(f)$, $\mathfrak{d}(f)$ and $\mathcal{M}(f)$.

Weight at most 2

f	1	$T_{1,1}$	$T_{0,2}$
$\langle f\rangle_{\underline{u}}$	1	$-\frac{1}{24}+\sum_{m,r\geq 1} mu_m^r$	$-\frac{1}{24}+\sum_{m,r\geq 1} ru_m^r$
$\langle f\rangle_q$	1	G_2	G_2
$D(f)$	0	$T_{2,2}$	$T_{1,3}$
$\mathfrak{d}(f)$	0	$-\frac{1}{2}$	$-\frac{1}{2}$
$\mathcal{M}(f)$	1	$X_{1,1}-\frac{1}{24}$	$X_{0,2}+X_{0,1}-\frac{1}{24}$

Weight 4

f	$T_{3,1}$	$T_{2,2}$	$T_{1,3}$	$T_{0,4}$
$\langle f\rangle_{\underline{u}}$	$\frac{1}{240}+\sum m^3u_m^r$	$\sum m^2ru_m^r$	$\sum mr^2u_m^r$	$\frac{1}{240}+\sum r^3u_m^r$
$\langle f\rangle_q$	G_4	$\frac{5}{6}G_4-2G_2^2$	$\frac{5}{6}G_4-2G_2^2$	G_4
$D(f)$	$T_{4,2}$	$T_{3,3}$	$T_{2,4}$	$T_{1,5}$
$\mathfrak{d}(f)$	0	$2T_{1,1}$	$2T_{0,2}$	0
$\mathcal{M}(f)$	$X_{3,1}+\frac{1}{240}$	$X_{2,2}+X_{2,1}$	$2X_{1,3}+3X_{1,2}+X_{1,1}$	$6X_{0,4}+12X_{0,3}+7X_{0,2}+X_{0,1}+\frac{1}{240}$

f	$T_{1,1}\odot T_{1,1} = T_{1,1}^2-T_{2,2}$	$T_{1,1}\odot T_{0,2} = T_{1,1}T_{0,2}-T_{1,3}$
$\langle f\rangle_{\underline{u}}$	$\sum m_1m_2u_{m_1}^{r_1}u_{m_2}^{r_2}-\frac{1}{12}\sum mu_m^r+\frac{1}{576}$	$\sum m_1r_2u_{m_1}^{r_1}u_{m_2}^{r_2}-\frac{1}{24}\sum(m+r)u_m^r+\frac{1}{576}$
$\langle f\rangle_q$	G_2^2	G_2^2
$D(f)$	$2\cdot T_{2,2}\odot T_{1,1}$	$T_{2,2}\odot T_{0,2}+T_{1,1}\odot T_{1,3}$
$\mathfrak{d}(f)$	$-T_{1,1}$	$-\frac{1}{2}T_{1,1}-\frac{1}{2}T_{0,2}$
$\mathcal{M}(f)$	$X_{(1,1),(1,1)}-\frac{1}{12}X_{1,1}+\frac{1}{576}$	$X_{(1,0),(1,2)}+X_{(1,0),(1,1)}+ -\frac{1}{24}\left(X_{1,1}+X_{0,2}\right)+\frac{1}{576}$

f	$T_{0,2}\odot T_{0,2} = T_{0,2}^2-\frac{5}{6}T_{0,4}-\frac{1}{6}T_{0,2}-\frac{1}{288}$
$\langle f\rangle_{\underline{u}}$	$\sum r_1r_2u_{m_1}^{r_1}u_{m_2}^{r_2}-\frac{1}{12}\sum ru_m^r+\frac{1}{576}$
$\langle f\rangle_q$	G_2^2
$D(f)$	$2\cdot T_{1,3}\odot T_{0,2}$
$\mathfrak{d}(f)$	$-T_{0,2}$
$\mathcal{M}(f)$	$X_{(0,0),(2,2)}+X_{(0,0),(2,1)}+X_{(0,0),(1,2)}+X_{(0,0),(1,1)}-\frac{1}{12}X_{0,2}-\frac{1}{12}X_{0,1}+\frac{1}{576}$

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