


Third Way to Interacting p -Form TheoriesMatteo Broccoli¹, Nihat Sadik Deger², and Stefan Theisen¹¹Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut Am Mühlenberg 1, D-14476 Potsdam, Germany²Department of Mathematics, Bogazici University, Bebek, 34342 Istanbul, Turkey (Received 30 March 2021; revised 31 May 2021; accepted 27 July 2021; published 27 August 2021)

We construct a class of interacting $(d - 2)$ -form theories in d dimensions that are “third-way” consistent. This refers to the fact that the interaction terms in the p -form field equations of motion neither come from the variation of an action nor are they off-shell conserved on their own. Nevertheless, the full equation is still on-shell consistent. Various generalizations, e.g., coupling them to $(d - 3)$ -forms, where three algebras play a prominent role, are also discussed. The method to construct these models also easily recovers the modified three-dimensional Yang-Mills theory obtained earlier and straightforwardly allows for higher derivative extensions.

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Introduction.—Study of three-dimensional gravity theories brought out a new class of theories, labeled as third-way consistent, whose field equations contain tensors whose divergences vanish only on shell [1]. These models do not have actions that contain only the metric field and the Bianchi identity alone does not guarantee the consistency of the equations of motion. The first such example was the minimal massive gravity (MMG) [2], which avoids the clash between bulk and boundary unitarity, which is a feature of generic three-dimensional higher derivative gravity theories; in its parameter space, there is a region where both left and right central charges of the dual conformal field theory and the energy of the massive bulk graviton mode are positive [3]. These desirable features motivated further investigation of this type of theories and more gravity examples were constructed [4,5]. However, this structure is not unique to gravity; in [6] a Yang-Mills theory with this property was found. (See also [7,8] where special cases of the systems discussed in [6] had been constructed previously.) Since all these are in three dimensions, it is natural to ask whether such models exist in higher dimensions. In this Letter, we answer this question affirmatively for p -form theories.

Our construction is based on shifting a flat Yang-Mills gauge connection with a Lie algebra valued 1-form. The resulting equation from the vanishing field strength condition is third-way consistent thanks to the Jacobi identity and cannot be derived from an action. This shifting idea is inspired by the fact that, in the first order formulation of the

third-way consistent gravity examples [2,4,5], their spin connection is similarly shifted. In the next section, we first show that in $d = 3$ choosing the shift appropriately it is possible to recover the model constructed in [6]. With this approach, finding extensions of [6] becomes straightforward and systematic, for which we give one example. Then, we study this issue in general d -dimensions. If the shifting 1-form is a dual field strength of a $(d - 2)$ -form, then one obtains a theory of $(d - 2)$ -forms coupled Yang-Mills gauge fields. However, requiring the compatibility of the gauge symmetries enforces to set the Yang-Mills field to zero and one ends up with an interacting $(d - 2)$ -form theory, which in $d = 3$ corresponds to the ungauged version of [6]. p -form theories have been studied extensively in the past and they are important ingredients of supergravity and string theory. References to some of the early work include [9–17]. The novelty of our model is that interaction is possible without modifying the standard p -form gauge transformation, unlike, for example, the well-known four-dimensional Freedman-Townsend (FT) model with 2-forms [9–11]. Although the field equation of this model has the same form as ours, namely, a flatness condition, theirs comes from a nonpolynomial action. We also discuss extensions of our model with additional fields and, in particular, with $(d - 3)$ -forms in which three-algebras enter when one imposes the consistency condition. We conclude with some future directions.

Shifting the connection.—In this section, we review the third-way consistent three-dimensional gauge theories that were found in [6]. We present them in a way that can be easily generalized. Some generalizations will be presented in this section; others, in particular those to higher dimensions with higher form gauge field, will be the subject of the next section.

The starting point is a flat connection of an arbitrary gauge group G , i.e., a connection that solves the equation

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$$F(A) = dA + A^2 = 0. \quad (2.1)$$

As it simplifies the calculations, we will mostly use form notation. Here $A = A_\mu^i T_i dx^\mu$ is the gauge field 1-form and $i = 1, \dots, \dim(G)$ is a gauge index. The T_i are (anti-Hermitian) representation matrices that satisfy $[T_i, T_j] = f^k_{ij} T_k$. The structure constants satisfy the Jacobi identity $f^i_{jk} f^j_{mn} = 0$, where we antisymmetrize over the underlined indices. We will be mostly considering compact semisimple gauge groups where we can lower the first index with the Cartan-Killing metric such that f_{ijk} is totally antisymmetric.

Consider now an arbitrary Lie algebra valued 1-form $C = C_\mu^i T_i dx^\mu$ and the equation

$$F(A + C) = F(A) + D_A C + C^2 = 0, \quad (2.2)$$

where $D_A C = dC + AC + CA$ is the covariant derivative of C .

For consistency, Eq. (2.2) should satisfy $D_A F(A + C) = 0$. While $D_A F(A) \equiv 0$ (Bianchi identity), we now find the condition

$$D_A^2 C + D_A C^2 = [F(A), C] + [D_A C, C] = 0, \quad (2.3)$$

where we have used $D_A^2 C = [F(A), C]$ and $D_A C^2 = [D_A C, C]$. Using the equation of motion (2.2) and $[C^2, C] = 0$, we find that (2.3) is indeed satisfied on shell.

In three dimensions, a simple generalization of (2.2) exists when C is conserved. In this case

$$F(A + C) + \tau * C = 0 \quad \text{with} \quad D_A * C = 0, \quad (2.4)$$

is also on-shell consistent, using $[*C, C] = 0$. Here τ is an arbitrary constant parameter. Equations (2.2) and (2.4) are gauge covariant if C transforms under gauge transformations as F , i.e., $C \rightarrow g^{-1} C g$, $g \in G$.

Note that our starting point, namely (2.1), is the equation of motion following from the Chern-Simons action. For the choices of C that we will discuss below, and which depend on the field strength $F(A)$, this is not the case for (2.2) and (2.4). In other words, they cannot be derived from a local action unless one introduces auxiliary fields similar to the model discussed in [6]. If they could be derived from an action, consistency would be automatic.

We now consider some special choices for C . This recovers models constructed in [6], but also new third-way consistent theories. (i) The choice $C = \kappa * F(A) \equiv \kappa \tilde{F}$, which satisfies $D_A * C = 0$; κ is an arbitrary parameter of dimension 1/mass. This from (2.4) leads to the third-way consistent equation

$$(1 - \kappa\tau)F + \kappa D\tilde{F} + \kappa^2 \tilde{F}^2 = 0, \quad (2.5)$$

where here and below $D = D_A$. In components this becomes

$$\varepsilon^{\mu\nu\rho} \left(D_\nu \tilde{F}_\rho + \frac{1}{2m} \tilde{F}_\nu \times \tilde{F}_\rho \right) + \mu \tilde{F}^\mu = 0, \quad (2.6)$$

To make contact with the notation used in [6], we have defined $\tilde{F}_\mu \times \tilde{F}_\nu = [\tilde{F}_\mu, \tilde{F}_\nu]$, $\kappa = 1/m$, and $\tau = m - \mu$, where m and μ are mass parameters. For this choice of C , the shift $A \rightarrow A + \kappa * F(A)$ was used as a parity transformation of the gauge field A in [6]. For the special case $m = 2\mu$ this system had already appeared in [8]. (ii) If we use instead $C = \kappa \tilde{F} + \alpha j$, where j is a conserved matter current, i.e., $D * j = 0$ and, therefore, $D * C = 0$, we find from (2.4) the equation

$$F(A + \kappa \tilde{F} + \alpha j) + \tau * (\kappa \tilde{F} + \alpha j) = 0. \quad (2.7)$$

To make contact with Ref. [6] we define $\kappa = 1/m$, $\tau = m - \mu$ and normalize the matter current by choosing $\alpha = \kappa/\tau$. We then find the following component expression:

$$\varepsilon^{\mu\nu\rho} \left(D_\nu \tilde{F}_\rho + \frac{1}{2m} \tilde{F}_\nu \times \tilde{F}_\rho \right) + \mu \tilde{F}^\mu = J^\mu, \quad (2.8)$$

where

$$J^\mu = j^\mu - \frac{\varepsilon^{\mu\nu\rho}}{m - \mu} \left(D_\nu j_\rho + \frac{\tilde{F}_\nu \times j_\rho}{m} + \frac{j_\nu \times j_\rho}{2m(m - \mu)} \right). \quad (2.9)$$

This agrees (up to the sign of the last term) with Ref. [6], but it has now been derived without any effort. Note that any choice of C in (2.2) that is not conserved would result in a third-way consistent model that is not of the form (2.8).

(iii) So far we have shown how to obtain the models constructed in Ref. [6] by the method of ‘‘shifting the connection.’’ Since this method is very general, we can easily construct new third-way consistent gauge models in three dimensions. For instance, the choice $C = \kappa * D\tilde{F}$ in (2.4), which satisfies $D * C = \kappa D^2 \tilde{F} = \kappa [F, \tilde{F}] = 0$, leads to

$$F(A + \kappa * D\tilde{F}) - \tau \kappa D\tilde{F} = 0 \quad (2.10)$$

(in odd dimensions with Minkowski signature, $** = -1$ for any p -form). In components we find

$$\varepsilon^{\mu\nu\rho} \left(-\tau D_\nu \tilde{F}_\rho + \frac{1}{2} \tilde{F}_\nu \times \tilde{F}_\rho + \frac{1}{2} \kappa \varepsilon_{\nu\alpha\beta} \varepsilon_{\rho\sigma\delta} D^\alpha \tilde{F}^\beta \times D^\sigma \tilde{F}^\delta \right) + \frac{1}{\kappa} \tilde{F}^\mu = -D^2 \tilde{F}^\mu, \quad (2.11)$$

which is a higher derivative extension of (2.6). We remark that the coefficient of \tilde{F}^μ cannot be set to zero unlike in (2.8).

The third-way with p -forms.—The discussion so far has been in three spacetime dimensions. An immediate question is whether the construction given there, which uses a shifted connection, also works in dimensions other than three. This is indeed the case. In $d \geq 3$ dimensions we choose C as the dual of a Lie algebra valued $(d-1)$ -form H , i.e., $C \equiv \kappa \tilde{H} = \kappa * H$ with $H = H^i T_i$; κ is a coupling constant. From $F(A + \kappa \tilde{H}) = 0$ we obtain the equation of motion

$$F(A)^i + \kappa D_A \tilde{H}^i = -\frac{1}{2} \kappa^2 f^i{}_{jk} \tilde{H}^j \wedge \tilde{H}^k, \quad (3.1)$$

which is the d -dimensional version of (2.2) that we considered above [18]. As it stands, the equation is Yang-Mills gauge covariant if \tilde{H}^i transforms as $\delta_\lambda \tilde{H}^i = f^i{}_{jk} \tilde{H}^j \lambda^k$ under $\delta_\lambda A^i = d\lambda^i + f^i{}_{jk} A^j \lambda^k$.

We now want to elevate H^i to a dynamical field. To this end we define it to be the Yang-Mills covariant field strength of a $(d-2)$ -form B^i ; i.e., $H^i = (DB)^i = dB^i + f^i{}_{jk} A^j B^k$. B^i transforms covariantly under Yang-Mills transformations, i.e., like H^i . Being the curl of a $p = (d-2)$ -form, we would like H^i to have the p -form gauge symmetry. To make it compatible with the Yang-Mills symmetry, we need to define the p -form transformation rules for B^i and A^i as

$$\delta_\xi B^i = D\xi^i = d\xi^i + f^i{}_{jk} A^j \xi^k, \quad \delta_\xi A^i = 0, \quad (3.2)$$

where ξ^i are $(d-3)$ -forms. But this implies

$$\delta_\xi H^i = f^i{}_{jk} F^j \xi^k. \quad (3.3)$$

Covariance of H^i and of the equations of motion require $F(A) = 0$ [19]. The simplest way to achieve this is by setting $A = 0$. This leaves an equation that involves only $H^i = dB^i$ in (3.1), namely,

$$d\tilde{H}^i = -\frac{1}{2} \kappa \tilde{H}^j \wedge \tilde{H}^k f^i{}_{jk}. \quad (3.4)$$

This is a system of second order equations for a collection of interacting $(d-2)$ -forms B^i . If the coupling constant $\kappa = 0$, we get a collection of free $(d-2)$ -forms.

Third-way consistency of the deformed system follows from the construction and can be easily checked explicitly: acting with the exterior derivative, the left-hand side vanishes identically due to $d^2 = 0$, but the right-hand side vanishes only on shell, after using the Jacobi identity. Obviously H^i is invariant under $\delta B^i = d\xi^i$. The system of equations has a global symmetry under G , under which B^i transforms as before, but with constant parameters λ^i . However, gauging this symmetry fails for the reasons given above.

It is not difficult to show that the equations of motion cannot be obtained as the variation of an action that does not contain additional (auxiliary) fields. The equations are of the form $\mathcal{L}(B)_\mu^i = 0$, and one needs to check the integrability condition, i.e., whether $\delta/\delta B_\nu^j(y) \mathcal{L}(B)_\mu^i(x) - (x \leftrightarrow y, i \leftrightarrow j, \mu \leftrightarrow \nu)$ vanishes. If it does not, then there is no action from which the equations of motion follow. In this way one verifies that neither (3.4) nor the other equations in this section can be derived from an action that does not contain additional, auxiliary fields.

Note that (3.4) has the form of a flatness condition[20]

$$F(\tilde{H}) = 0, \quad (3.5)$$

pretending that \tilde{H} is a gauge field. From this it follows that \tilde{H} is pure gauge, i.e.,

$$\tilde{H} = U^{-1} dU, \quad (3.6)$$

where U is a group element and where we have set $\kappa = 2$ for simplicity. The Bianchi identity is $dH = 0$. This immediately reminds us of the principal chiral sigma model (for a review, see, e.g., [21]) which is defined by the Lagrangian

$$L = \text{Tr}(\partial_\mu \hat{U}^{-1} \partial^\mu \hat{U}), \quad (3.7)$$

where \hat{U} is an arbitrary Lie group element. One can define a connection 1-form that takes values in the corresponding Lie algebra as $\hat{A} = \hat{U}^{-1} d\hat{U}$ and then the equations of motion derived from (3.7) state that \hat{A} is coclosed, i.e., $d * \hat{A} = 0$. Moreover, flatness $F(\hat{A}) = d\hat{A} + \hat{A} \wedge \hat{A} = 0$ is satisfied identically, as \hat{A} is pure gauge. We therefore see that conditions that \hat{A} and \tilde{H} satisfy are the same, but with the role of the equations of motion and the Bianchi identity reversed. In this sense the two models are dual to each other.

This brings us to the comparison of our third-way consistent model of $p = (d-2)$ -forms B^i with the Lagrangian model of Freedman and Townsend [11] in $d = 4$. They also considered interacting Lie-algebra valued antisymmetric tensor fields B^i , which are coupled to a 2-form current $j = *(* dB^i \wedge * dB^i)$. Minimal coupling, which modifies the equations of motion, destroys current conservation and requires further nonminimal couplings, eventually leading to a nonpolynomial action and equations of motion. Current conservation guarantees the invariance of the action under a deformed, field-dependent p -form symmetry and the commutator of two variations on B vanishes on shell [22]. The equations of motion in the FT model reduce to the ones we have been considering, namely (3.5), if one formally identifies their composite ‘‘gauge field’’ A^i with our \tilde{H}^i . This amounts to dropping all higher order terms in A^i , which has an infinite power series

expansion in the p -form field B^i . The model is dual to the principal chiral model in the sense that both can be derived from the same first order action. However, the equation of motion of the principal chiral model, $\partial^\mu A_\mu^i = 0$, is not satisfied identically in the dual FT model, but it holds on shell.

We now discuss various generalizations of our third-way consistent system of equations (3.4), which consist of including additional fields.

Coupling to gravity: To couple the third-way consistent system of $(d-2)$ -forms to an external gravitational field is trivial. In that case, (3.4) becomes

$$\nabla_\mu \tilde{H}_\nu^i - \nabla_\nu \tilde{H}_\mu^i = -\kappa f^i_{jk} \tilde{H}_\mu^j \tilde{H}_\nu^k. \quad (3.8)$$

Because of the symmetry of the Christoffel symbols, they drop out and the only appearance of the metric is through the Hodge star implicit in \tilde{H}^i . On-shell consistency can easily be shown for (3.8). We act with ∇_ρ and antisymmetrize in all three indices. The lhs is identically zero: we replace the product of the two covariant derivatives by half of their commutator and use the Bianchi identity $R_{\mu\nu\rho}^\sigma = 0$ to show that it vanishes. The on-shell vanishing of the rhs is as in the flat case.

If gravity is dynamical, we need to construct an energy-momentum tensor that is on-shell conserved. A possible candidate is $T_{\mu\nu} = \tilde{H}_\mu^i \tilde{H}_\nu^i - \frac{1}{2} g_{\mu\nu} \tilde{H}^{i\rho} \tilde{H}_\rho^i$ [6]. On-shell conservation is straightforward to demonstrate.

Adding lower form fields: A simple generalization, which is inspired by [15,16], is to add p -forms b^a (with $p < d-2$) whose field strengths and dual field strengths we denote by h^a and \tilde{h}^a , respectively. They come with their own gauge symmetry. The following system of equations is easily shown to be third-way consistent:

$$d\tilde{H}^i = -\frac{1}{2} \kappa f^i_{jk} \tilde{H}^j \tilde{H}^k, \quad d\tilde{h}^a = -\kappa t_{ib}^a \tilde{H}^i \tilde{h}^b, \quad (3.9)$$

provided $[t_j, t_k] = f^i_{jk} t_i$; i.e., b^a is a vector in some representation of G .

Other generalizations containing p -forms of degree lower than $d-2$ are also possible. Consider, for instance, in addition to the B^i , a single $(d-3)$ -form b with field strength $h = db$ and dual field strength \tilde{h} . Postulate its equation of motion as

$$d\tilde{h} = \tilde{H}^i \wedge \tilde{H}^j \wedge \tilde{H}^k f_{ijk}, \quad (3.10)$$

together with (3.4). Consistency now requires $d(\tilde{H}^i \wedge \tilde{H}^j \wedge \tilde{H}^k f_{ijk}) = 0$. Indeed, on shell, we have

$$\begin{aligned} & d(\tilde{H}^i \wedge \tilde{H}^j \wedge \tilde{H}^k f_{ijk}) \\ &= -\frac{3\kappa}{2} \tilde{H}^m \wedge \tilde{H}^n \wedge \tilde{H}^i \wedge \tilde{H}^k f^i_{mn} f_{ijk} = 0, \end{aligned} \quad (3.11)$$

with the help of the Jacobi identity, where we used (3.4).

A slightly less trivial generalization is to include a collection of $(d-3)$ -forms b^i and to impose the equations of motion

$$d\tilde{h}^i = f^i_{jkl} \tilde{H}^j \wedge \tilde{H}^k \wedge \tilde{H}^l, \quad (3.12)$$

where f^i_{jkl} is totally antisymmetric in its lower indices. Consistency now requires

$$\begin{aligned} 0 &= d(f^i_{jkl} \tilde{H}^j \wedge \tilde{H}^k \wedge \tilde{H}^l) \\ &= -\frac{3\kappa}{2} f^i_{jkl} f^j_{mn} \tilde{H}^m \wedge \tilde{H}^n \wedge \tilde{H}^k \wedge \tilde{H}^l, \end{aligned} \quad (3.13)$$

which is satisfied if

$$f^i_{jkl} f^j_{mn} = 0. \quad (3.14)$$

A simple special solution of this condition is

$$f^i_{jkl} = v^i f_{jkl}, \quad (3.15)$$

where, as before, f_{jkl} are the structure constants of a simple Lie algebra. Then (3.14) is satisfied by virtue of the Jacobi identity. v^i is an arbitrary vector that transforms in the adjoint representation of the Lie algebra. If we choose it to have only one nonzero component, say $v^i = \delta^{i1}$, the system reduces to (3.10) plus a collection of free $(d-3)$ -forms.

A more interesting solution to (3.14) can be obtained as follows. The appearance of f^i_{jkl} suggests a relation to three algebras, whose essential features we briefly recall; for review and references see [23,24]. Denoting its generators by T_i , one has

$$[i, j, k] \equiv [T_i, T_j, T_k] = f^l_{ijk} T_l, \quad (3.16)$$

and the f^i_{jkl} satisfy the fundamental identity

$$\begin{aligned} [i, j, [k, l, m]] &= [[i, j, k], l, m] + [k, [i, j, l], m] \\ &+ [k, l, [i, j, m]]. \end{aligned} \quad (3.17)$$

Equivalently,

$$f^n_{klm} f^p_{ijn} = f^n_{ijk} f^p_{nlm} + f^n_{ijl} f^p_{knm} + f^n_{ijm} f^p_{kln}. \quad (3.18)$$

Following [25], one introduces a bi-invariant metric h_{ij} on the three algebra to lower the first index and arrives at f_{ijkl} , which is antisymmetric in all its indices. Still following [25] we choose it to be $h_{ij} = \text{diag}(-1, 1, \dots, 1)$. One then splits the indices into $i = (0, a, \phi)$ and chooses

$$f_{0abc} = f_{\phi abc} = f_{abc}, \quad f_{0\phi ab} = f_{abcd} = 0, \quad (3.19)$$

and f_{abc} are the structure constants of a compact Lie algebra. This solves the fundamental identity (3.18). Note that the above choices imply that the Cartan-Killing metric on the Lie algebra is δ_{ab} .

We choose, furthermore, that all components of $f^i{}_{jk}$ vanish except for $f^a{}_{bc}$, which are totally antisymmetric structure constants $f^a{}_{bc} = f_{abc}$. With these choices, one verifies that (3.14) is also satisfied.

The vanishing of some of the components of the structure constants $f^i{}_{jk}$ and $f^i{}_{jkl}$ implies that $b^0 + b^\phi$ is a free and completely decoupled field, while B^0 and B^ϕ satisfy source-free equations, but they appear as sources in the equation for b^a .

Instead of adding $(d-3)$ -form fields, we can add a set of $(d-q)$ -forms c^i with dual field strengths \tilde{k}^i and postulate the equations of motion

$$d\tilde{k}^i = \tilde{H}^{j_1} \wedge \cdots \wedge \tilde{H}^{j_q} f^i{}_{j_1, \dots, j_q}. \quad (3.20)$$

Consistency now requires the following relation for the “structure constants”:

$$f^i{}_{j_1 j_2 \dots j_q} f^j{}_{m_1 m_2 \dots m_q} = 0. \quad (3.21)$$

We have not explored this possibility, i.e., the existence of such algebraic structures.

Outlook.—In this Letter, we constructed third-way consistent interacting $(d-2)$ -form theories in d dimensions and studied some of their properties. Third-way consistency allows us to have interactions without modifying the standard p -form gauge symmetry, but the price is the lack of an action. However, it might be possible to find an action by introducing auxiliary fields as it happened in [6]. It would be useful to determine such an action that would allow studying its dynamical couplings with other fields such as gravity [26,27], which may lead to some interesting applications of these models. Another reason why an action is desirable is that it might be a viable starting point for the (path-integral) quantization of these models, a question that is otherwise not clear how to address. Constructing supersymmetric versions of our model is another open problem.

Higher order form fields are sources for branes. Adding $(d-3)$ -brane sources to the equations of motion for $(d-2)$ -forms and studying the resulting systems of branes would also be of interest.

Constructing third-way interacting p -form theories where all transformations are compatible with nontrivial charges is another direction worth exploring. To achieve this, the so-called embedding tensor formalism [28,29], which is used in the classification of gauged supergravities and where several p -forms are required, might be useful. In $d=4$ only 1- and 2-forms are needed [30] and hence this

could be a suitable setup to couple our model covariantly to Yang-Mills fields.

The three-dimensional Yang-Mills model of [6] is closely related to non-Abelian Chern-Simons theories considered in [7] where a novel Higgs mechanism was found. It would be interesting to see if something similar happens for our model and try to explain the interaction terms we have as spontaneous breaking of a local symmetry as it was shown for the MMG model [2] in [31].

Finding third-way consistent gravity theories in $d > 3$ is another challenge. In $d=3$, using the first order formulation of gravity was useful [2,4,5], but in higher dimensions the metric description might be more practical. Note that the Christoffel symbol $\Gamma^\nu{}_{\mu\rho}$ can also be thought of as the (ν, ρ) component of a Yang-Mills field: $\Gamma^\nu{}_{\mu\rho} = (A_\mu)^\nu{}_\rho$ [32]; hence, to obtain such models we expect a shift in the Christoffel connection, starting from the vacuum Einstein equation or a modification of it that comes from an action. Indeed, we checked that the MMG model [2] can be obtained via the shift: $\Gamma^\nu{}_{\mu\rho} \rightarrow \Gamma^\nu{}_{\mu\rho} + \kappa \epsilon_\mu^{\nu\alpha} S_{\alpha\rho}$, where $S_{\alpha\rho} = R_{\alpha\rho} - (1/4)Rg_{\alpha\rho}$ is the Schouten tensor. However, there is an important difference with the gauge theory case that we studied in this Letter, namely, not every shift leads to a third-way consistent theory. For example, doing this shift with the Ricci instead of the Schouten tensor does not work. We hope to come back to these issues in the near future.

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- [19] Including the term $f^i{}_{jk} \tilde{H}^j \xi^k$ in the transformation of B^i as in the FT model [11] does not change this conclusion. Also note that the $d = 3$ case is exceptional where B^i and A^i can be identified.
- [20] We may also construct higher derivative extensions of this model by adding further terms in the shift C , e.g., $\kappa * d * (\tilde{H} \wedge \tilde{H})$. The equation of motion (3.5) will be modified to $F[\tilde{H} + \kappa * d * (\tilde{H} \wedge \tilde{H})] = 0$. As long as the shift is built from the field strength H , p -form symmetry is guaranteed to work off shell.
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