Large deviations of the KPZ equation, Markov duality and SPDE limits of the vertex models

Yier Lin

Submitted in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy
under the Executive Committee
of the Graduate School of Arts and Sciences

COLUMBIA UNIVERSITY

2021

Abstract

Large deviations of the KPZ equation, Markov duality and SPDE limits of the vertex models

Yier Lin

The Kardar-Parisi-Zhang (KPZ) equation is a stochastic PDE describing various objects in statistical mechanics such as random interface growth, directed polymers, interacting particle systems. We study large deviations of the KPZ equation, both in the short time and long time regime. We prove the first short time large deviations for the KPZ equation and detects a Gaussian - $\frac{5}{2}$ power law crossover in the lower tail rate function. In the long-time regime, we study the upper tail large deviations of the KPZ equation starting from a wide range of initial data and explore how the rate function depends on the initial data.

The KPZ equation plays a role as the weak scaling limit of various models in the KPZ universality class. We show the stochastic higher spin six vertex model, a class of models which sit on top of the KPZ integrable systems, converges weakly to the KPZ equation under certain scaling. This extends the weak universality of the KPZ equation. On the other hand, we show that under a different scaling, the stochastic higher spin six vertex model converges to a hyperbolic stochastic PDE called stochastic telegraph equation. One key tool behind the proof of these two stochastic PDE limits is a property called Markov duality.

Table of Contents

Acknowledgements

First of all, I want to express my gratitude to my advisor, Ivan Corwin. This thesis will be impossible without his continuing support, encouragement and advice. Ivan is very generous with his time and he is always available when I am in need of advice. I benefit greatly from many conversations with Ivan, on the topics of research and more generally about academic career. I am also very grateful to him for his many valuable suggestions and comments for my first few papers. His guidance is crucial for my transition to an independent researcher.

Special thanks to Li-Cheng Tsai. I was inspired greatly from many discussion with him and also learned a lot from our collaboration. I am indebted to Li-Cheng for generously sharing his knowledge on research as well as his general advice on academic career.

I want to thank Sayan Das, Promit Ghosal, Shalin Parekh, Evgeni Dimitrov and Konstantin Mateski for many enjoyable discussion on mathematics. I want to thank all members of the Columbia probability group for creating a great environment for learning and discussing.

I want to thank Amol Aggarwal, Ivan Corwin for helpful comments of the thesis. I am thankful to Mark Rychnovsky for a careful read of the introduction part of the thesis and many helpful comments.

I want to thank Julien Dubédat, Ioannis Karatzas, Daniel Lacker, Sumit Mukherjee and Hao Shen for teaching me things in probability and statistics. I want to thank Amol Aggarwal, Julien Dubédat, Ioannis Karatzas and Li-Cheng Tsai for taking the time to be on my thesis committee.

Finally, I want to thank my family for their continuing and unconditional support.

Chapter 1: Introduction

1.1 Overview

The Kardar–Parisi–Zhang (KPZ) equation is a non-linear stochastic PDE (SPDE) which reads

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi, \tag{1.1.1}$$

where $\mathcal{H} = \mathcal{H}(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}$. Here, $\xi = \xi(t,x)$ denotes the space-time white noise, which is a Gaussian field with Dirac delta correlation function $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$. The equation was introduced by [KPZ86] as a model which describes the evolution of a randomly growing interface, where $\mathcal{H}(t,x)$ represents the height function of the interface at time t and at location x. Given the expression of (1.1.1), the growing surface is subject to: 1. smoothing; 2. slope-dependent growth and 3. space-time independent noise. The KPZ equation is connected to many physical systems including directed polymers in a random environment, last passage percolation, randomly stirred fluids, and interacting particle systems.

Mathematically, the solution to (1.1.1) is ill-posed in the classical sense due to the appearance of the space-time white noise and the non-linearity term. In fact, by classical Schauder estimate, one would expect the solution to the KPZ equation has the same regularity of a Brownian motion. As a result, $\partial_x \mathcal{H}$ should be interpreted as a distribution and the problem arises because one can not make sense the term $(\partial_x \mathcal{H})^2$, which is the square of a distribution. One way to properly define the KPZ equation is through the *Hopf-Cole transform*

$$\mathcal{H}(t,x) := \log \mathcal{Z}(t,x), \tag{1.1.2}$$

where $\mathcal{Z}(t,x)$ is the *mild solution* of the stochastic heat equation (SHE)

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \xi(t,x)\mathcal{Z}(t,x),$$

which exists and is unique. We say the KPZ equation starts from the narrow wedge initial data if $\mathcal{Z}(0,x) = \delta(x)$ where δ is the Dirac-Delta function. In this case, we denote the solution to be $\mathcal{H}^{\mathbf{nw}}$.

Other equivalent ways to define the KPZ equation include the theory of regularity structure [Hai14], paracontrolled equation [GIP15] or the energy solution [GJ14].

As an important model for the random interface growth, it is valuable to understand the behavior of the KPZ equation. One interesting a question is about the asymptotic behavior. [ACQ11] proves that for as $t \to \infty$,

$$\frac{\mathcal{H}^{\mathbf{nw}}(2t,0) + \frac{t}{12}}{t^{\frac{1}{3}}} \Rightarrow \text{Tracy-Widom GUE.}$$

Having the above as a fluctuation limit theorem for the KPZ equation, a natural following question is to understand the large deviation principle (LDP). In other words, as $t \to \infty$, we want to study the probability of a rare event where $\mathcal{H}(2t,0) + \frac{t}{12}$ deviates from 0 at scale t. [DT19] proves the LDP for the upper tail: for every $s > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{\mathbf{nw}}(2t,0) + \frac{t}{12} > st\Big) = -\phi_+(s)$$

where $\phi_+(s) = \frac{4}{3}s^{\frac{3}{2}}$. Based on this result, we ask the following question: What is the upper tail LDP for the KPZ equation starting from other initial data? In addition, how will the LDP rate function depend on the initial data? We will briefly discuss our answer to this question in Section 1.2.

Note that [Tsa18, CC19] prove a LDP for the lower tail

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big(\mathcal{H}^{\mathbf{nw}}(2t,0)+\frac{t}{12}<-st\Big)=-\phi_-(-s)$$

where the rate function is

$$\phi_-(z)=\frac{4}{15\pi}(1-\pi^2 z)^{\frac{5}{2}}-\frac{4}{15\pi^6}+\frac{2}{3\pi^4}z-\frac{1}{2\pi}z^2. \tag{1.1.3}$$

It is interesting to ask what is the lower tail LDP for the KPZ equation starting from general initial data other than narrow wedge. In particular, how does the rate function depend on the initial data. We leave the study of this question to the future.

For the KPZ equation starting from the narrow wedge initial data, [ACQ11] shows that the short time fluctuation limit theorem: as $t\longrightarrow 0$,

$$\frac{\mathcal{H}^{\mathbf{nw}}(2t,0)+\log\sqrt{4\pi t}}{t^{\frac{1}{4}}}\Rightarrow\mathcal{N}(0,\sqrt{\frac{\pi}{2}}),$$

where the right hand side is a Gaussian random variable with mean zero and variance $\frac{\pi}{2}$. It is a natural question to ask what is the large deviation behavior of $\mathcal{H}^{\mathbf{nw}}$ as $t\longrightarrow 0$. In Section 1.3, we will establish a LDP for the KPZ equation in the short time regime. It turns out that our LDP also reflects a crossover phenomenon in the lower tail of the KPZ equation. Such result and phenomenon was proved earlier in the $t\longrightarrow\infty$ regime by [CG20b, Tsa18, CC19].

The KPZ equation is not a scaling invariant object. More precisely, if we define $\mathcal{H}_\epsilon(t,x)=\epsilon^z\mathcal{H}(\epsilon^{-b}t,\epsilon^{-1}x)$, using the scaling of space-time white noise $\xi(\epsilon^{-b}t,\epsilon^{-1}x)\overset{d}{=}\epsilon^{\frac{b+1}{2}}\xi(t,x)$, we get

$$\partial_t\mathcal{H}_\epsilon(t,x)=\frac{1}{2}\epsilon^{2-b}\partial_x^2\mathcal{H}_\epsilon(t,x)+\frac{1}{2}\epsilon^{-z+2-b}(\partial_x\mathcal{H}_\epsilon(t,x))^2+\epsilon^{z+\frac{1}{2}-\frac{b}{2}}\xi(t,x). \tag{1.1.4}$$

It is clear that there is no b,z such that the coefficients in the above equation match with those in (1.1.1). However, if we simultaneously scale some of the parameters δ, κ, D, it is possible that the

KPZ equation remains unchanged: such scaling is called *weak scaling*. It is thus natural to believe that the KPZ equation is the weak scaling limit of microscopic models with similar properties such as relaxation and lateral growth. Roughly speaking, this is the *weak universality of the KPZ equation*.

The weak universality of the KPZ equation has been verified for a number of interacting particle systems. The first result was given in the work of [BG97], for Asymmetric Simple Exclusion Process (ASEP).

The SHS6V model introduced in [CP16] (also see [Bor17]) belongs to the family of vertex models which themselves are examples of quantum integrable systems. In general, the R-matrices (which can be thought of as the weights associated to the vertex) are not stochastic. The authors of [CP16] worked with the L-matrices, which is a stochastic version of the R-matrices and they defined the SHS6V model. The stochasticity allows us to define the vertex model on the entire line as an interacting particle system which follows sequential Markov update rule. Moreover, the L-matrices in [CP16] satisfy the Yang-Baxter equation which implies the integrability of the model. In particular, the transfer matrices are diagonalizable by a complete set of Bethe ansatz eigenfunctions [BCPS15, CP16]. The model also enjoys Markov duality. The L-matrices of the SHS6V model have four parameters. By specifying these parameters, the SHS6V model degenerates to known integrable systems such as stochastic six vertex (S6V) model, ASEP, q-Hahn TASEP, q-TASEP. Indeed, it is on top of a hierarchy of KPZ class integrable probabilistic systems.

We prove that under weakly asymmetric scaling, the SHS6V model converges to the KPZ equation. This significantly extends the class of models which belongs to the weak universality class of the KPZ equation.

On the other hand, we show that under a different scaling, the SHS6V model converges to a different SPDE called the stochastic telegraph equation. Our results extend the universality of the STE beyond the S6V model [BG19]. Unlike the KPZ equation which is parabolic, the STE is hyperbolic. It is very interesting to see whether there is an interpolation between these two SPDE limits in the SHS6V model.

The rest of the introduction is organized as the following. In Section 1.2, we talk about the large deviation of the KPZ equation with general initial data. In Section 1.3, we study the short time large deviations of the KPZ equation. Section 1.4 will concern the KPZ limit of the SHS6V model. Section 1.5 explains how the stochastic telegraph equation arises as a SPDE limit of the SHS6V model.

1.2 Large deviation of the KPZ equation with general initial data

This section serves as a summary of Chapter 3. We look at the KPZ equation (1.1.1) with general function valued initial data $\mathcal{H}(0,x) = f(x)$. We obtain the upper tail LDP of the KPZ equation under certain condition over f. For the purpose of simplification, here we will formulate our condition on the initial data in an imprecise way. We expect that this formulation will deliver our idea in a more intuitive way. The precise statement is recorded in Definition 3.1.1. Roughly speaking, the core condition that we impose on f is the existence of a function $g : \mathbb{R}_+ \to \mathbb{R}$ such that for every $p > 0$,

$$g(p) := \lim_{t\to\infty} \frac{1}{t} \log \int_{\mathbb{R}} e^{\frac{-px^2}{2t}} \mathbb{E}\Big[e^{pf(x)}\Big] dx. \tag{1.2.1}$$

We also require on certain regularity condition over f. The class of f satisfying these conditions includes any bounded deterministic initial data and the two sided Brownian motion (stationary initial data).

Theorem 1.2.1. *Suppose that $g(p) \in C^1(\mathbb{R}_{>0})$ and $\zeta := \lim_{p\to 0} g'(p)$ is finite. Then, for $s > \zeta$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^f(t,0) + \frac{t}{24} > st\Big) = -\max_{p\geq 0} \Big\{sp - \frac{p^3}{24} - g(p)\Big\}$$

Remark 1.2.2. Without proof, ζ should be the limit of $t^{-1}(\mathcal{H}^f(t,0) + \frac{t}{24})$ as $t \to \infty$. Note that we have $\max_{p\geq 0}\{\zeta p - \frac{p^3}{24} - g(p)\} = 0$ (for detail, see Chapter 3). In particular, we also know that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^f(t,0) + \frac{t}{24} > st\Big) = 0$$

when $s \le \zeta$.

Remark 1.2.3. The theorem above implies that if the initial data f is bounded, then $g(p) = 0$ and

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^f(t,0) + \frac{t}{24} > st\Big) = -\frac{4\sqrt{2}}{3} s^{\frac{3}{2}}$$

If f is a two-sided Brownian motion, then $g(p) = \frac{p^3}{8}$ and

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^f(t,0) + \frac{t}{24} > st\Big) = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}}$$

Let us talk about the proof idea. We denote $\mathcal{Z}^f$ to be the solution to the SHE

$$\partial_t \mathcal{Z}^f = \frac{1}{2}\partial_{xx}\mathcal{Z}^f + \mathcal{Z}^f \xi \quad , \mathcal{Z}^f(0,x) = \exp\big(f(x)\big)$$

Note that $\mathcal{H}^f := \log \mathcal{Z}^f$. The large deviation in Theorem 1.2.1 is closely related to the *Lyapunov exponent* of the SHE. For $p \in \mathbb{R}_{>0}$, we define the p-th moment Lyapunov exponent of the $\mathcal{Z}^f$ as

$$\gamma_p^f := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\mathcal{Z}^f(t,0)^p\Big] \tag{1.2.2}$$

It is well-known that for a sum of i.i.d. random variables, the large deviation rate function equals to the Legendre transform of the log moment generating function. Given $\mathcal{Z}^f(t,0) = \exp\big(\mathcal{H}^f(t,0)\big)$, γ_p^f defined above plays a role of the asymptotic log moment generating function of $\mathcal{H}^f(t,0)$. So we are able to conclude Theorem 1.2.1 once we prove the following.

Theorem 1.2.4. *We have* $\gamma_p^f = \frac{p^3-p}{24} + g(p)$ *for every* $p > 0$.

Even without the connection to the LDP, the Lyapunov exponent is an important quantity which describes the behavior of random systems. Many multiplicative type random fields or SPDEs exhibit a universal phenomenon called intermittency. Roughly speaking, the intermittency corresponds to the random field that exhibits unusual high peak in a small region. The nature of intermittency is characterized by the Lyapunov exponent. In the context of the SHE, the study goes back to [BC95], in which paper the authors compute all the integer Lyapunov exponent of the SHE. However, there is a mistake in their computation beyond the second moment due to an incorrect use of the Skorohod's lemma. This was fixed later by [Che15], who computes the Lyapunov exponent of the SHE for all bounded initial data. The integer and the fractional Lyapunov exponent of the SHE starting from narrow wedge initial data was obtained by [CG20a] and [DT19]. Our theorem was the first result which computes the fractional Lyapunov exponent of the SHE for a wide class of initial data.

Let us explain the heuristics of the proof. We start with the convolution formula in [CH16] to represent Z^f in terms of a convolution of coupling of the fundamental solution of the SHE and the initial data:

$$Z^f(t,0) \overset{d}{=} \int_{-\infty}^{\infty} Z^{\mathbf{nw}}(t,x) e^{f(x)} dx$$

This being the case, to compute the Lyapunov exponent of $\mathcal{Z}^f(t,0)$, it suffices to look at the positive moments of the right hand side. One important step is to justify the following approximation

$$\mathbb{E}\Big[\Big(\int_{-\infty}^{\infty} Z^{\mathbf{nw}}(t,x) e^{f(x)} dx\Big)^p\Big] \approx \mathbb{E}\Big[\int_{-\infty}^{\infty} Z^{\mathbf{nw}}(t,x)^p e^{pf(x)} dx\Big]$$

This approximation should hold because the main contribution of the integral $\int_{-\infty}^{\infty} Z^{\mathbf{nw}}(t,x) e^{f(x)} dx$ is localized. Once have this approximation, using the independence of $\mathcal{Z}^{\mathbf{nw}}$ and f, together with the stationarity of $\{\mathcal{Z}^{\mathbf{nw}}(t,x) e^{\frac{x^2}{2t}}\}$ in x, we obtain

$$\mathbb{E}\big[Z^f(t,0)^p\big] \approx \mathbb{E}\Big[\int_{-\infty}^{\infty} Z^{\mathbf{nw}}(t,x)^p e^{pf(x)} dx\Big] = \mathbb{E}\big[Z^{\mathbf{nw}}(t,0)^p\big] \int_{-\infty}^{\infty} \mathbb{E}\big[e^{p(f(x)-\frac{x^2}{2t})}\big] dx$$

This implies that the Lyapunov exponent of $\mathcal{Z}^f$ equals the sum of the Lyapunov exponents of $\mathcal{Z}^{\mathbf{nw}}$ and a quantity which can be computed from the initial data. By the result of [DT19] (see Proposition 3.1.9) and (1.2.1), the first summand equals $\frac{p^3-p}{24}$ and the second summand equals $g(p)$. Hence, we get Theorem 1.2.4. For a rigorous proof, one ingredient would be the spatial short-range regularity of the KPZ equation, which has been studied in [CGH19].

In Chapter 4, we will look at the KPZ equation defined on a half-line $\mathbb{R}_+$. Similar to the full-line case, the solution can be defined as the logarithm of the half-line SHE. We show an upper tail LDP for the solution to the half-line KPZ equation by computing the Lyapunov exponent of the half-line SHE. However, the method of computation is very different from the idea presented in the last paragraph. Our computation in Chapter 4 follows the idea of [DT19] relies heavily on the integrability of the solution obtained in [BBCW18, Par19b].

1.3 Short time large deviations of the KPZ equation

This section serves as a summary of Chapter 2. We establish the first short time LDP of the KPZ equation. Moreover, we prove a Gaussian - $\frac{5}{2}$ power law crossover in the lower tail of rate function. In particular, our asymptotic result of the rate function confirms the physics prediction of [KK07, KK09, KMS16, MKV16, LDMRS16].

Theorem 1.3.1. *Recall that $\mathcal{H}^{nw}$ denotes the solution of the KPZ equation with the narrow wedge initial data.*

(a) For any $\lambda > 0$, the limits exist

$$\lim_{t\to 0} t^{\frac{1}{2}} \log \mathbb{P}\big[\mathcal{H}^{nw}(2t,0) + \log\sqrt{4\pi t} \le -\lambda\big] =: -\Phi(-\lambda),$$
$$\lim_{t\to 0} t^{\frac{1}{2}} \log \mathbb{P}\big[\mathcal{H}^{nw}(2t,0) + \log\sqrt{4\pi t} \ge \lambda\big] =: -\Phi(\lambda)$$

(b) $\lim_{\lambda\to 0} \lambda^{-2}\Phi(\lambda) = \frac{1}{\sqrt{2\pi}}$.

(c) $\lim_{\lambda\to \infty} \lambda^{-\frac{5}{2}}\Phi(-\lambda) = \frac{4}{15\pi}$.

Remark 1.3.2. Our method also works for the KPZ equation starting from the flat initial data $\mathcal{H}(0,x) = 0$. We will have a different constant on the right of (b) and (c)

$$\lim_{\lambda\to 0} \lambda^{-2}\Phi(\lambda) = \sqrt{\frac{2}{\pi}}, \qquad \lim_{\lambda\to \infty} \lambda^{-\frac{5}{2}}\Phi(-\lambda) = \frac{8}{15\pi}$$

In the following, we focus on explaining how to obtain part (c) since it is more difficult and interesting. By a scaling $\mathcal{H}_\varepsilon^{\mathbf{nw}}(t,x) := \mathcal{H}^{\mathbf{nw}}(\varepsilon t, \varepsilon^{\frac{1}{2}} x) + \log \varepsilon^{\frac{1}{2}}$. The short time LDP problem can be formulated in terms of the LDP of $\mathcal{H}_\epsilon^{\mathbf{nw}}(2,0) + \log\sqrt{4\pi}$ as $\epsilon \to 0$ where $\mathcal{H}_\varepsilon^{\mathbf{nw}}$ satisfies

$$\partial_t \mathcal{H}_\epsilon^{\mathbf{nw}}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}_\epsilon^{\mathbf{nw}}(t,x) + \frac{1}{2}\big(\partial_x \mathcal{H}_\epsilon^{\mathbf{nw}}(t,x)\big)^2 + \sqrt{\epsilon}\xi.$$

with the narrow wedge initial data. In the following, we explain how to obtain the LDP for $\mathcal{H}_\epsilon^{\mathbf{nw}}(2,0) + \log\sqrt{4\pi}$ as $\varepsilon \to 0$ and the asymptotic of the rate function. We will first illustrate the idea on a heuristic level and then explain how to make it rigorous.

The weak-noise theory is a physics heuristic that generalizes the Freidlin-Wentzell theory for the SPDEs. Since the spacetime white noise satisfies a LDP, heuristically, there is a LDP for the trajectory of $\mathcal{H}_\epsilon^{\mathbf{nw}}$. For suitable function $h : [0,2] \times \mathbb{R} \to \mathbb{R}$, we have

$$\lim_{\epsilon\to 0} -\epsilon \log \mathbb{P}\Big(\mathcal{H}_\epsilon^{\mathbf{nw}} \approx h\Big) = \frac{1}{2}\int_{[0,2]\times\mathbb{R}} \Big(\partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}(\partial_x h)^2\Big)^2 dt dx.$$

Applying the contraction principle, the large deviation rate function for $\mathcal{H}_\epsilon^{\mathbf{nw}}(2,0) = -\lambda$ corresponds to minimizing the integral on the right hand side of the above display, conditioning on $h(2,0) = -\lambda$. As stated in [KMS16], applying calculus of variation, we get a pair of PDEs

$$\begin{aligned} &\partial_t h - \frac{1}{2}\partial_{xx} h - \frac{1}{2}\big(\partial_x h\big)^2 = \rho, \\ &\partial_t \rho + \frac{1}{2}\partial_{xx}\rho = \partial_x\big(\rho \partial_x h\big). \end{aligned} \tag{1.3.1}$$

We scale ρ and h by $\widetilde{\rho}(t,x) = \lambda^{-1}\rho(t, \lambda^{\frac{1}{2}} x)$ and $\widetilde{h}(t,x) = \lambda^{-1} h(t, \lambda^{\frac{1}{2}} x)$. Then

$$\begin{aligned} &\partial_t \widetilde{h} - (2\lambda)^{-1}\partial_{xx}\widetilde{h} - \frac{1}{2}\big(\partial_x \widetilde{h}\big)^2 = \widetilde{\rho}, \\ &\partial_t \widetilde{\rho} + (2\lambda)^{-1}\partial_{xx}\widetilde{\rho} = \partial_x\big(\widetilde{\rho}\partial_x \widetilde{h}\big). \end{aligned} \tag{1.3.2}$$

Denote $\|\cdot\|_2$ the norm of $L^2([0,2]\times\mathbb{R})$, then $\|\rho\|_2^2 = \lambda^{\frac{5}{2}}\|\widetilde{\rho}\|_2^2$. As $\lambda \to \infty$, the viscosities in the above PDEs should disappear. [KMS16] solves the PDEs

$$\begin{aligned}\partial_t\widetilde{h} - \frac{1}{2}\big(\partial_x\widetilde{h}\big)^2 &= \widetilde{\rho},\\ \partial_t\widetilde{\rho} &= \partial_x\big(\widetilde{\rho}\partial_x\widetilde{h}\big).\end{aligned} \tag{1.3.3}$$

and obtain

$$\widetilde{\rho}(x,t) = \rho_*(x,t) := -\frac{1}{2\pi}r(t)\Big(1 - \frac{x^2}{\ell(t)^2}\Big)\mathbf{1}_{\{|x|\leq \ell(t)\}} \tag{1.3.4}$$

where $r(t)$ and $\ell(t)$ are explicit functions. In addition, we have $\frac{1}{2}\|\rho_*\|_2^2 = \frac{4}{15\pi}$. This number matches with the one in Theorem 1.3.1 (c).

From the mathematics point of view, there are problems for the above heuristic. The first problem is that the weak noise theory has not been established in a rigorous way, in particular for the KPZ equation which is a singular SPDE. There are previous works dealing with the LDP for some singular SPDE on a torus, see [CM97, BDM08, HW15a, CD19]. To generalize these results to the full-line equation, one may encounter extra technical challenges. On a different aspect, the equations (1.3.3) are non-linear and have a similar form as Burgers equation. So it is natural to expect that (1.3.3) have non-unique weak solution. In order to guarantee the uniqueness, one needs to impose certain entropy condition. One also need to argue that the limit of the solutions in (1.3.2) converges to the entropy solution as $\lambda \to \infty$.

For the rigorous proof of Theorem 1.3.1, we adopt a different approach by working at the level of SHE $\mathcal{Z}^{\mathbf{nw}} = \exp(\mathcal{H}^{\mathbf{nw}})$. The SHE can be expressed using the Wiener chaos expansion as an infinite sum, where each term is a multiple Ito-Wiener stochastic integral against the white noise ξ. The large deviation of a finite sum of Wiener chaos has been developed in [HW15a]. By an exponential approximation from finite to infinite sum, we obtain the a LDP for the trajectory of the SHE under weak noise scaling. By a contraction principle, we note that the rate functions Φ_- and Φ_+ in Theorem (c) exist: For $\lambda > 0$,

$$\Phi(-\lambda) = \inf\Big\{\frac{1}{2}\|\rho\|_2^2 : \log\mathbb{E}\Big[\exp\Big(\int_0^2 \rho\big(s, B_{\mathrm{b}}(s)\big)ds\Big)\Big] \leq -\lambda\Big\}$$

$$\Phi(\lambda) = \inf\Big\{\frac{1}{2}\|\rho\|_2^2 : \log\mathbb{E}\Big[\exp\Big(\int_0^2 \rho\big(s, B_{\mathrm{b}}(s)\big)ds\Big)\Big] \geq \lambda\Big\}$$

where $(B_{\mathrm{b}}(s), s \in [0,2])$ is a Brownian bridge from 0 to 0. We now explain how to prove Theorem 1.3.1 (c) using this variation formula. Apply scaling $\rho(t,x) = \lambda\widetilde{\rho}(t, \lambda^{-\frac{1}{2}}x)$,

$$\Phi(-\lambda) = \lambda^{\frac{5}{2}} \inf\Big\{\frac{1}{2}\|\widetilde{\rho}\|_2 : \lambda^{-1}\log\mathbb{E}\Big[\exp\Big(\lambda\int_0^2 \widetilde{\rho}\big(s, \lambda^{-\frac{1}{2}}B_{\mathrm{b}}(s)\big)ds\Big)\Big] \leq -1\Big\} \tag{1.3.5}$$

We only need to show that the $\lambda \to \infty$ limit of the infimum above is $\frac{4}{15\pi}$. To prove this, we want to show that ρ_* in (1.3.4) is asymptotically the minimizer when $\lambda \to \infty$. Using the LDP of Brownian bridge $\lambda^{-1}B_{\mathrm{b}}(s)$ together with Varadhan's lemma,

$$\begin{aligned}
&\lim_{\lambda\to\infty} \lambda^{-1}\log\mathbb{E}\Big[\exp\Big(\lambda\int_0^2 \rho_*\big(s, \lambda^{-\frac{1}{2}}B_{\mathrm{b}}(s)\big)ds\Big)\Big] \\
&= \sup\Big\{\int_0^2 \rho_*(s, x(s)) - \frac{1}{2}\dot{x}(s)^2 ds : x(0) = x(2) = 0, x \in H^1([0,2])\Big\}
\end{aligned}$$

Solving the Euler-Lagrange equation for the variation problem in the supremum above, we know that the geodesic solves $\ddot{x}(s) + \partial_x\rho_*(s, x(s)) = 0$. Solving this PDE, we find that the geodesic from $x(0) = 0$ to $x(2) = 0$ is non-unique and forms a lens-shape, see Figure 2.1. By utilizing this collection of geodesics, we are able to show that as $\lambda \to \infty$, ρ_* is asymptotically the minimizer for the infimum on the right hand side of (1.3.5).

1.4 KPZ limit of the stochastic higher spin six vertex model

This section serves as a summary of Chapter 6. We define the stochastic higher spin six vertex (SHS6V) model as an interacting particle system on the lattice $\mathbb{Z}$. Before that, we need to introduce the $\mathbb{L}$-matrix.

Definition 1.4.1. We define the $J = 1$ $\mathbb{L}$-matrix to be a stochastic matrix with row and column indexed by $\{0,\dots,I\}\times\{0,1\}$. The element of the $J=1$ $\mathbb{L}$-matrix is specified by

$$L_\alpha^{(J)}(m,0;m,0)=\frac{1+\alpha q^m}{1+\alpha},\qquad L_\alpha^{(J)}(m,0;m-1,1)=\frac{\alpha(1-q^m)}{1+\alpha},$$
$$L_\alpha^{(J)}(m,1;m+1,0)=\frac{1-\nu q^m}{1+\alpha},\qquad L_\alpha^{(J)}(m,1;m,1)=\frac{\alpha+\nu q^m}{1+\alpha}.$$

where $\nu=q^{-I}$ (this guarantees that when $J=1$, $L^{(J)}(I,1;I+1,0)=0$) and $L^{(J)}(i_1,j_1;i_2,j_2)=0$ for all other values of (i_1,j_1), (i_2,j_2).

For simplicity, here we only provide the expression of $\mathbb{L}$-matrix here when $J=1$. When $J>1$, we refer to $\mathbb{L}$-matrix is a stochastic matrix with row and column indexed by $\{0,\dots,I\}\times\{0,\dots,J\}$. We refer to its explicit expression to (6.1.4). One can alternatively define $L_\alpha^{(J)}$ from $L_\alpha^{(1)}$ through a procedure called fusion, see Proposition 6.1.5 or Lemma 7.1.3.

Definition 1.4.2 (SHS6V model as an interacting particle system)**.** For any state $\vec{g}=(g_x)_{x\in\mathbb{Z}}\in\mathbb{G}$, we specify the update rule from state $\vec{g}$ to $\vec{g}'$ as follows: Assume the leftmost particle in the configuration $\vec{g}$ is at x (i.e. $g_x>0$ and $g_z=0$ for all $z<x$). Starting from x, we update g_x to g'_x by setting $h_x=0$ and randomly choosing g'_x according to the probability $L_\alpha^{(J)}(g_x,h_x=0;g'_x,h_{x+1})$ where $h_{x+1}:=g_x-g'_x$. Proceeding sequentially, we update g_{x+1} to g'_{x+1} according to the probability $L_\alpha^{(J)}(g_{x+1},h_{x+1};g'_{x+1},h_{x+2})$ where $h_{x+2}:=g_{x+1}+h_{x+1}-g'_{x+1}$. Continuing for $g_{x+2},g_{x+3},\dots$, we have defined the update rule from $\vec{g}$ to $\vec{g}'=(g'_x)_{x\in\mathbb{Z}}$, see Figure 1.1 for visualization of the update procedure. We call the discrete time-homogeneous Markov process $\vec{g}(t)\in\mathbb{G}$ with the update rule defined above **the left-finite SHS6V model**.

We remark that with additional efforts, the restriction requiring the existence of a leftmost particle can be dropped. In other words, we can properly define the SHS6V model as a Markov process starting from any state of $\{0,1,\dots,I\}^{\mathbb{Z}}$. We call this the bi-infinite SHS6V model and refer to Section 6.2 for details of the definition.

For a particle configuration $\vec{g}\in\mathbb{G}$, define $N_x(\vec{g})=\sum_{y\le x}g_y$. For the left-finite SHS6V model

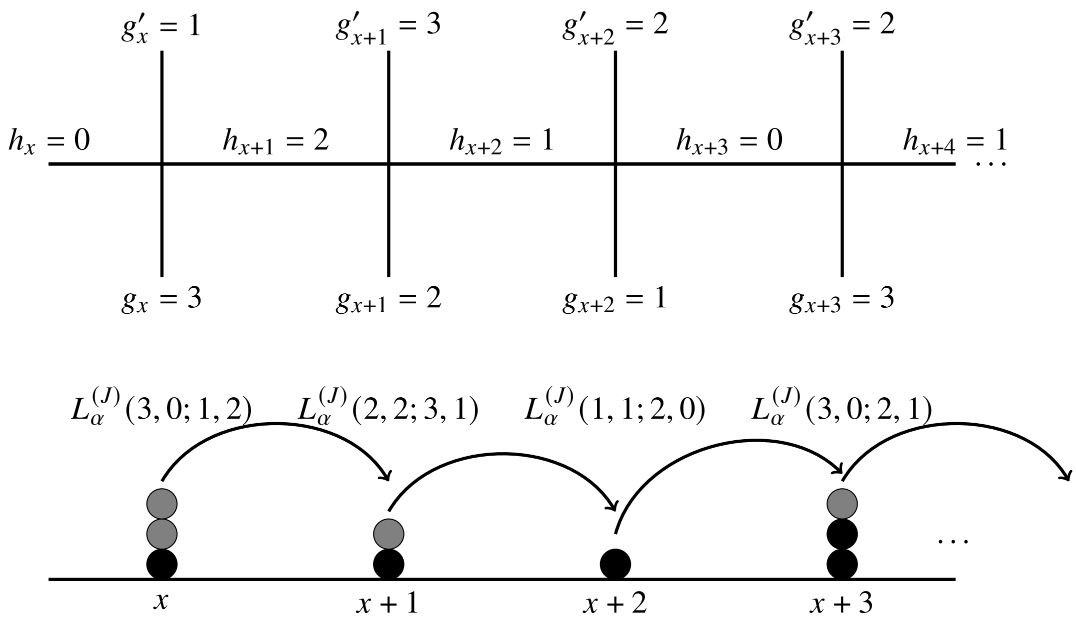


Figure 1.1: The visualization of the sequential update rule for the left-finite SHS6V model in Definition 1.4.2. Assuming x is the location of the leftmost particle, we update sequentially for positions $x, x+1, x+2, \ldots$ according to the stochastic matrix $L_\alpha^{(J)}$, the gray particles in the picture above will move one step to the right.

$\vec{g}(t) \in \mathbb{G}$, we define the *height function* as

$$N(t,x) = N_x(\vec{g}(t)) - N_0(\vec{g}(0)).$$

Having defined $N(t,x)$ on the lattice, we linearly interpolate it first in space variable x then in time variable t, which makes $N(t,x)$ a $C([0,\infty),\mathbb{R})$-valued process. For construction of height functions of the bi-infinite version of the SHS6V model, see Section 6.2. Our main result is the weak convergence from the SHS6V model to the KPZ equation.

Theorem 1.4.3. *Fix $b \in \big(\frac{I+J-2}{I+J-1}, 1\big)$, $I \geq 2$ and $J \geq 1$, for small $\epsilon > 0$, let $q = e^{\sqrt{\epsilon}}$ and define α via $b = \frac{1+\alpha q}{1+\alpha}$. We call this weakly asymmetric scaling. Assume that $\{N_\epsilon(0,x)\}_{\epsilon>0}$ is nearly stationary with density ρ (see Definition 6.5.5 for detail) and*

$$\sqrt{\epsilon}\Big(N_\epsilon(0,\epsilon^{-1}x) - \rho\epsilon^{-1}x\Big) \Rightarrow \mathcal{H}^{ic}(x) \text{ in } C(\mathbb{R}) \text{ as } \epsilon \downarrow 0,$$

then there exists μ_ε *and* λ_ε *(defined in* (6.1.9)*) such that the following weak convergence in* $C([0,\infty), C(\mathbb{R}))$ *holds*

$$\sqrt{\epsilon}\Big(N_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x + \epsilon^{-2}t\mu_\epsilon) - \rho(\epsilon^{-1}x + \epsilon^{-2}\mu_\epsilon t)\Big) - t\log\lambda_\epsilon \Rightarrow \mathcal{H}(t,x) \text{ as } \varepsilon \downarrow 0,$$

where $\mathcal{H}(t,x)$ *is the Hopf-Cole solution of the KPZ equation*

$$\partial_t\mathcal{H}(t,x) = \frac{JV_*}{2}\partial_x^2\mathcal{H}(t,x) - \frac{JV_*}{2}\big(\partial_x\mathcal{H}(t,x)\big)^2 + \sqrt{JD_*}\xi(t,x),$$

with initial condition $\mathcal{H}^{ic}(x)$*, where the coefficients* V_* *and* D_* *are given by* (6.1.12) *and* (6.1.13)

To prove Theorem 1.4.3, we show that the microscopic Hopf-Cole transform (an exponential transform) of $N(t,x)$ converges to the SHE. Once this is proved, we can take a logarithm to show that the height function converges to the KPZ equation.

To prove this convergence to the SHE, we use the Markov duality method developed in [CGST20]. We refer to Definition 5.1.2 for what Markov duality means.

[CP16] shows that the SHS6V model enjoys a self-duality. The simplest case of self-duality (duality to one particle system) guarantees the existence of the discrete SHE

$$dZ = \mathcal{L}Zdt + dM. \tag{1.4.1}$$

Here, Z is the microscopic Hopf-Cole transform of $N(t,x)$, $\mathcal{L}$ is an infinitesimal generator of a geometric random walk performed by a single particle in the system, which approximates the Laplacian, dM is a discrete martingale increment satisfying $\mathbb{E}[dM(t)|\mathcal{F}(t-1)] = 0$ (where $\mathcal{F}(t-1)$ is the sigma algebra generated by $\vec{g}(s)$ for $s = 0, 1, \ldots, t-1$). By passing through the scaling limit, we expect that $\mathcal{L}$ converges to the Laplacian, and the martingale increment dM converges to ξZ. As a consequence, the discrete SHE (1.4.1) converges to the continuum SHE.

As illustrated in [BG97], we can use the martingale problem to define the solution of the SHE. The proof of Theorem 1.4.3 is composed of the following three steps: (1) Showing the tightness of Z; (2) Identifying the limit of the linear martingale problem; (3) Identifying the limit of the quadratic martingale problem.

Steps 1) and 2) follow from a similar approach as in [BG97, CT17, CGST20]. The most difficult step is 3). To show this, one needs to prove a *self-averaging* property for the quadratic variation of the martingale M.

Let us explain what the self-averaging is and how Markov duality can be applied to prove it. Denote the discrete gradient by $\nabla f(x) := f(x+1) - f(x)$. Roughly speaking, the terminology "self-averaging" refers to the phenomena that as $\epsilon \downarrow 0$

(A) For $x_1 \neq x_2$, the average of $\epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2)$ over a long time interval of length $O(\epsilon^{-2})$ will vanish.

(B) There exists a positive constant λ such that the average of $(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2 - \lambda Z(t,x)^2$ over a long time interval of length $O(\epsilon^{-2})$ will vanish.

In general, proving such self-averaging is not easy because of the appearance of the spacial gradient and the non-linearity in Z. The power of Markov duality is that it allows us to turn the problem of proving self-averaging into a problem of estimating two particle transition probability of the SHS6V model. Thanks to the integrability of the model, the latter object admits exact formula and one can obtain a good estimate of it.

1.5 The STE limit of the SHS6V model

This section serves as a summary of Chapter 7. The *telegraph equation* (TE) is a hyperbolic PDE given by

$$\begin{cases} u_{XY}(X,Y) + \beta_1 u_Y(X,Y) + \beta_2 u_X(X,Y) = f(X,Y), \\ u(X,0) = \chi(X), \quad u(0,Y) = \psi(Y), \end{cases}$$

where the functions $\chi, \psi \in C^1$ satisfy $\chi(0) = \psi(0)$. When f is a deterministic function, the TE is a classical object, see [CH08, Chapter V]. The solution theory of the TE goes back to [CH08], we present it in the way of [BG19, Section 4]. In fact, the above equation admits a unique solution which reads

$$u(X,Y) = \psi(0)\mathcal{R}(X,Y,0,0) + \int_0^Y \mathcal{R}(X,Y;0,y)\big(\psi'(y) + \beta_2\psi(y)\big)dy$$
$$+ \int_0^X \mathcal{R}(X,Y;x,0)\big(\chi'(x) + \beta_1\chi(x)\big)dx + \int_0^X \int_0^Y \mathcal{R}(X,Y,x,y)f(x,y)dxdy. \quad (1.5.1)$$

Here, $\mathcal{R}(X,Y,x,y)$ is the *Riemann function*

$$\mathcal{R}(X,Y;x,y) = \frac{1}{2\pi\mathbf{i}}\oint_{-\beta_1} \frac{\beta_2-\beta_1}{(z+\beta_1)(z+\beta_2)} \exp\left[(\beta_1-\beta_2)\Big(-(X-x)\frac{z}{z+\beta_2} + (Y-y)\frac{z}{z+\beta_1}\Big)\right]dz.$$

The Riemann function can be viewed as a fundamental solution of the TE, see Section 4 of [BG19].

The stochastic versions of the TE were intensively studied in the last 50 years, we refer the reader to [BG19, Section 1.1] for a brief review. In this section, we consider the f given by $f(X,Y) = \sqrt{\theta(X,Y)}\eta(X,Y)$, where η is the space-time white noise with dirac delta correlation function and θ is a deterministic integrable function. By (1.5.1), the solution to the *stochastic telegraph equation* (STE) is a Gaussian field.

Having introduced the TE and STE, we proceed to define the SHS6V model on a corner. In this section, we are going to view the SHS6V model as a stochastic path ensemble.

Definition 1.5.1 (SHS6V model as a stochastic path ensemble)**.** We define the SHS6V model on a corner to be a stochastic path ensemble on $\mathbb{Z}^2_{\geq 0}$. The boundary condition specified by $\{v_{x,0}\}_{x\in\mathbb{Z}_{\geq 0}}$ and $\{h_{0,y}\}_{y\in\mathbb{Z}_{\geq 0}}$ such that $v_{x,0} \in \{0,1,\dots,I\}$, $h_{0,y} \in \{0,1,\dots,J\}$. In other words, we have $h_{0,y}$ number of lines entering into the vertex $(0,y)$ from the left boundary and $v_{x,0}$ number of lines flowing into the vertex $(x,0)$ from the bottom boundary. Sequentially taking (x,y) to be $(0,0) \to (1,0) \to (0,1) \to (2,0) \to (2,1)\dots$, for vertex at (x,y), given $v_{x,y}, h_{x,y}$ as the number of vertical and horizontal input lines, we randomly choose the number of vertical and horizontal output lines $(v_{x,y+1}, h_{x+1,y}) \in \{0,1,\dots,I\} \times \{0,1,\dots,J\}$ according to probability $L^{(J)}_\alpha(v_{x,y}, h_{x,y}; \cdot,\cdot)$. Proceeding with this sequential sampling, we get a collection of paths going to the up-right direction.

The stochastic path ensemble interpretation of the SHS6V model is closely related to the particle system viewpoint in the previous section. More precisely, the trajectory of each particle in the space-time domain is an up-right path in the path ensemble. The only difference here is that we consider the model on a corner and as time evolves, new particles can enter the system from the left boundary.

We associate a *height function* $H : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Z}$ to the path ensemble, where the paths play a role as the level lines of the height function (see Figure 1.2). Define for any $x, y \in \mathbb{Z}_{\geq 0}$,

$$H(x, y) = \sum_{j=1}^{y} h_{0,j-1} - \sum_{i=1}^{x} v_{i-1,y}.$$

Clearly, we have $H(0, 0) = 0$ and $H(x, y) - H(x-1, y) = -v_{x-1,y}$. Since the vertex is conservative, we also have

$$H(x, y) - H(x, y-1) = h_{x,y-1}.$$

Let us introduce the scaling of the SHS6V model. Fix $I, J \in \mathbb{Z}_{\geq 1}$ and positive β_1, β_2 such that

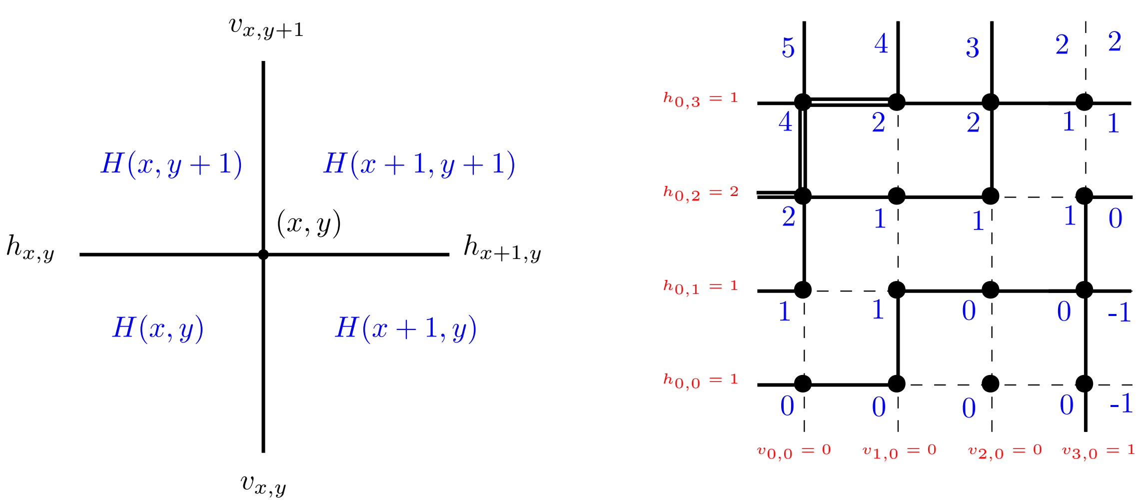


Figure 1.2: Left: Illustration of the height function around a vertex (x, y), note that $H(x, y+1) = H(x, y) + h_{x,y}$, $H(x+1, y) = H(x, y) - v_{x,y}$ and $H(x+1, y+1) = H(x, y) + h_{x,y} - v_{x,y+1} = H(x, y) - v_{x,y} + h_{x+1,y}$. Right: Sampled stochastic path ensemble on a quadrant. The red number indicates the number lines entering into the boundary, the blue number represents the height at each vertex.

$\beta_1 \neq \beta_2$, we scale the parameter q, α in the way that

$$q = e^{\frac{\beta_1-\beta_2}{L}}, \qquad \frac{1+\alpha q^J}{1+\alpha} = e^{-\frac{J\beta_2}{L}}, \quad L \to \infty.$$

Under this scaling, we prove a law of large number and functional central limit theorem of the SHS6V model. The hydrodynamic limit and fluctuation are respectively the TE and STE.

Theorem 1.5.2. *Define* $\mathfrak{q} = e^{\beta_1-\beta_2}$ *and fix* $A, B > 0$*, consider two monotone Lipschitz functions* χ *and* ψ*. Suppose that the boundary for the SHS6V model is chosen in the way that as* $L \to \infty$*,* $\frac{1}{L}H(Lx, 0) \to \chi(x)$ *and* $\frac{1}{L}H(0, Ly) \to \psi(y)$ *uniformly in probability for* $x \in [0, A]$ *and* $y \in [0, B]$*, then as* $L \to \infty$*,*

$$\frac{1}{L} \sup_{x\in[0,A]\times[0,B]} |H(Lx, Ly) - L\mathbf{h}(x, y)| \xrightarrow{p} 0,$$

where $\xrightarrow{p}$ *means the convergence in probability.* $\mathfrak{q}^{\mathbf{h}(x,y)}$ *is the unique solution to the telegraph equation*

$$\frac{\partial^2}{\partial x \partial y}\mathfrak{q}^{\mathbf{h}(x,y)} + J\beta_2 \frac{\partial}{\partial x}\mathfrak{q}^{\mathbf{h}(x,y)} + I\beta_1 \frac{\partial}{\partial y}\mathfrak{q}^{\mathbf{h}(x,y)} = 0,$$

with the boundary condition specified by $\mathfrak{q}^{\mathbf{h}(x,0)} = \mathfrak{q}^{\chi(x)}$ *and* $\mathfrak{q}^{\mathbf{h}(\mathbf{0},\mathbf{y})} = \mathfrak{q}^{\psi(y)}$.

Theorem 1.5.3. *Assuming further that* $\chi(x)$ *and* $\psi(y)$ *are piecewise* C^1*-smooth, we have the weak convergence as* $L \to \infty$*,*

$$\sqrt{L}\Big(q^{H(Lx,Ly)} - \mathbb{E}\Big[q^{H(Lx,Ly)}\Big]\Big) \Rightarrow \varphi(x, y) \quad \text{in } C(\mathbb{R}^2_{\geq 0}),$$

where $\varphi(x, y)$ *is a random continuous function which solves the stochastic telegraph equation*

$$\varphi_{xy} + I\beta_1\varphi_y + J\beta_2\varphi_x = \eta \cdot \sqrt{(\beta_1 + \beta_2)\mathfrak{q}^{\mathbf{h}}_x\mathfrak{q}^{\mathbf{h}}_y + J(\beta_2 - \beta_1)\beta_2\mathfrak{q}^{\mathbf{h}}\mathfrak{q}^{\mathbf{h}}_x + I(\beta_1 - \beta_2)\beta_1\mathfrak{q}^{\mathbf{h}}\mathfrak{q}^{\mathbf{h}}_y},$$

where $\mathfrak{q}^{\mathbf{h}}_x := \partial_x(\mathfrak{q}^{\mathbf{h}(x,y)})$ *and* $\mathfrak{q}^{\mathbf{h}}_y := \partial_y(\mathfrak{q}^{\mathbf{h}(x,y)})$*, the boundary of* φ *is given by zero.*

[BG19] shows that the stochastic six vertex model height function converges to a telegraph equation and its fluctuation field converges to a stochastic telegraph equation. The key observation is the following *four point relation*, which says that if we define

$$\xi^{\mathrm{S6V}}(x+1,y+1) = q^{H(x+1,y+1)} - b_1 q^{H(x,y+1)} - b_2 q^{H(x+1,y)} + (b_1+b_2-1)q^{H(x,y)},$$

Here b_1, b_2 are the weight of the six vertex model configuration (in our notation $b_1 = \frac{\alpha+\nu}{1+\alpha}$, $b_2 = \frac{1+\alpha q}{1+\alpha}$). Then the conditional expectation and variance of ξ read

$$\mathbb{E}\Big[\xi^{\mathrm{S6V}}(x+1,y+1)\Big|\mathcal{F}(x,y)\Big] = 0, \tag{1.5.2}$$
$$\mathbb{E}\Big[\xi^{\mathrm{S6V}}(x+1,y+1)^2\Big|\mathcal{F}(x,y)\Big] = \gamma_1\Delta_x\Delta_y + \gamma_2 q^{H(x,y)}\Delta_x + \gamma_3 q^{H(x,y)}\Delta_y,$$

where $\mathcal{F}(x,y)$ is a sigma algebra generated by $\{H(u,v) : u \le x \text{ or } v \le y\}$ and $\Delta_x := q^{H(x+1,y)} - q^{H(x,y)}$, $\Delta_y := q^{H(x,y+1)} - q^{H(x,y)}$. The parameters $\gamma_i, i = 1,2,3$ depend on b_1, b_2.

In our paper, we generalize the above relations to the SHS6V model. Define

$$\xi^{\mathrm{S6SHV}}(x+1,y+1) = q^{H(x+1,y+1)} - \frac{\alpha+\nu}{1+\alpha} q^{H(x,y+1)} - \frac{1+\alpha q^J}{1+\alpha} q^{H(x+1,y)} + \frac{\nu+\alpha q^J}{1+\alpha} q^{H(x,y)},$$

In Chapter 7, we will prove that

$$\mathbb{E}\Big[\xi^{\mathrm{SHS6V}}(x+1,y+1)\Big|\mathcal{F}(x,y)\Big] = 0, \tag{1.5.3}$$
$$\mathbb{E}\Big[\xi^{\mathrm{SHS6V}}(x+1,y+1)^2\Big|\mathcal{F}(x,y)\Big] = \gamma_1\Delta_x\Delta_y + \gamma_2 q^{H(x,y)}\Delta_x + \gamma_3 q^{H(x,y)}\Delta_y + \mathbf{R}(x,y), \tag{1.5.4}$$

where $\mathbf{R}(x,y)$ is an error term that is negligible under our scaling. From now on, we may also use ξ to denote ξ^{SHS6V}.

Why does such a generalization exist? In the context of the stochastic six vertex model, (1.5.2) is related to the self-duality discovered in [CP16, Theorem 2.21], though it is more of a local relation than the way duality is generally stated. More precisely, from the interacting particle system perspective, let $\mathcal{F}(y)$ be sigma algebra generated by the system up to time y. The one particle self-duality implies that

$$\mathbb{E}\Big[q^{H(x+1,y+1)}\Big|\mathcal{F}(y)\Big] = \sum_{z\leq x+1} p(x+1,z)q^{H(z,y)} \tag{1.5.5}$$

$$\mathbb{E}\Big[q^{H(x,y+1)}\Big|\mathcal{F}(y)\Big] = \sum_{z\leq x} p(x,z)q^{H(z,y)} \tag{1.5.6}$$

where p is the transition probability of a geometric walk performed by a single particle. It is not hard to see that for $z < x$, $p(x+1,z) = \frac{\alpha+\nu}{1+\alpha}p(x,z)$. Multiplying (1.5.6) by $(-\frac{\alpha+\nu}{1+\alpha})$ and add the result to (1.5.5), one gets

$$\mathbb{E}\Big[\xi^{\mathrm{SHS6V}}(x+1,y+1)|\mathcal{F}(y)\Big] = 0$$

Despite this argument provides convincing evidence that (1.5.3) holds, it is not a proof. The reason is that we are considering the model on a corner instead of the full line (to which the Markov duality applies). We will prove (1.5.5) using fusion.

For the quadratic relation (1.5.4), the situation is more subtle here and we refer the details to Chapter 7.

Chapter 2: Short time Large deviations of the KPZ equation

Chapter Abstract: We establish the Freidlin–Wentzell Large Deviation Principle (LDP) for the Stochastic Heat Equation with multiplicative noise in one spatial dimension. That is, we introduce a small parameter $\sqrt{\varepsilon}$ to the noise, and establish an LDP for the trajectory of the solution. Such a Freidlin–Wentzell LDP gives the short-time, one-point LDP for the KPZ equation in terms of a variational problem. Analyzing this variational problem under the narrow wedge initial data, we prove a quadratic law for the near-center tail and a $\frac{5}{2}$ law for the deep lower tail. These power laws confirm existing physics predictions [KK07, KK09, MKV16, LDMRS16, KMS16].

This paper is published at [LT21].

2.1 Introduction

In this paper we study the KPZ equation in one spatial dimension

$$\partial_t \mathcal{H} = \tfrac{1}{2}\partial_{xx}\mathcal{H} + \tfrac{1}{2}(\partial_x \mathcal{H})^2 + \xi, \tag{2.1.1}$$

where $\mathcal{H} = \mathcal{H}(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}$, and $\xi = \xi(t,x)$ denotes the spacetime white noise. The equation was introduced by [KPZ86] to describe the evolution of a randomly growing interface, and is connected to many physical systems including directed polymers in a random environment, last passage percolation, randomly stirred fluids, and interacting particle systems. The equation exhibits integrability and has statistical distributions related to random matrices. We refer to [FS10, Qua11, Cor12, QS15, CW17, CS19] and the references therein for the mathematical study of and related to the KPZ equation.

Due to the roughness of $\mathcal{H}$, the term $(\partial_x \mathcal{H})^2$ in (2.1.1) does not make literal sense, and the well posedness of the KPZ equation requires renormalization [Hai14, GIP15]. In this paper we work with the notion of Hopf–Cole solution. Informally exponentiating $Z = \exp(\mathcal{H})$ brings the KPZ equation to the Stochastic Heat Equation (SHE)

$$\partial_t Z = \tfrac{1}{2}\partial_{xx} Z + \xi Z. \tag{2.1.2}$$

It is standard to establish the well posedness of (2.1.2) by chaos expansion; see Section 2.2.1 for more discussions on Wiener chaos. For a function-valued initial data $Z(0, \cdot) \geq 0$ that is not identically zero, [Mue91] showed that $Z(t, x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$ almost surely. The Hopf–Cole solution of the KPZ equation is then defined as $\mathcal{H} := \log Z$. This notion of solution coincides with that of [Hai14, GIP15] under suitable assumptions. An often considered initial data is to start the SHE from a Dirac delta at the origin, i.e., $Z(0, \cdot) = \delta_0(\cdot)$, which is referred to as the narrow wedge initial data for $\mathcal{H}$. For such an initial data, [Flo14] established the positivity for $Z(t, x)$ so that the Hopf–Cole solution $\mathcal{H} := \log Z$ is well-defined.

Large deviations of the KPZ equation have been intensively studied in the mathematics and physics communities in recent years. Results are quite fruitful in the long time regime, $t \to \infty$. For the narrow wedge initial data, physics literature predicted that the one-point, lower-tail Large Deviation Principle (LDP) rate function should go through a crossover from a cubic power to a $\frac{5}{2}$ power [KLD18b]. (The prediction of the $\frac{5}{2}$ power actually first appeared in the short time regime; see the discussion about the short time regime below.) The work [CG20b] derived rigorous, detailed bounds on the one-point tail probabilities for the narrow wedge initial data and in particular proved the cubic-to-$\frac{5}{2}$ crossover. Similar bounds are obtained in [CG20a] for general initial data. The exact lower-tail rate function were derived in the physics works [SMP17, CGK$^+$18, KLDP18, Le 19], and was rigorously proven in [Tsa18, CC19]. Each of these works adopts a different method. In [KLD19], the four methods in [SMP17, CGK$^+$18, KLDP18, Tsa18] were shown to be closely related. As for the upper tail, the physics work [LDMS16] derived a $\frac{3}{2}$ power law for the entire rate function under the narrow wedge initial data, and [DT19] gave a rigorous proof for this upper-tail LDP. The work [GL20] extended this upper-tail LDP to general initial data.

For the finite time regime, $t \in (0, \infty)$ fixed, motivated by studying the positivity or regularity (of the one-point density) of the SHE or related equations, the works [Mue91, MN08, Flo14, CHN16, HL18] established tail probability bounds of the SHE or related equations.

In this paper we focus on *short* time large deviations of the KPZ equation. Employing the Weak Noise Theory (WNT), the physics works [KK07, KK09, MKV16, KMS16] predicted that the one-point, lower-tail rate function should crossover from a quadratic power law to a $\frac{5}{2}$ power law for the narrow wedge and flat initial data. By analyzing an exact formula, the physics work [LDMRS16] obtained the entire one-point rate function for the narrow wedge initial data; see Section 2.1.4. This was confirmed by the numerical result [HLDM$^+$18b]. From this one-point rate function [LDMRS16] also demonstrated the crossover. The quadratic power arises from the Gaussian nature of the KPZ equation in short time, while the $\frac{5}{2}$ power appears to be a persisting trait of the deep lower tail of the KPZ equation in all time regimes. Our main result gives the first proof of the short time LDP for the KPZ equation and the quadratic-to-$\frac{5}{2}$ crossover.

Theorem 2.1.1. *Let h denote the solution of the KPZ equation* (2.1.1) *with the initial data $Z(0, \bullet) = \delta_0(\bullet)$.*

(a) For any $\lambda > 0$, the limits exist

$$\lim_{t\to 0} t^{\frac{1}{2}} \log \mathbb{P}\big[\mathcal{H}(2t, 0) + \log\sqrt{4\pi t} \le -\lambda\big] =: -\Phi(-\lambda),$$
$$\lim_{t\to 0} t^{\frac{1}{2}} \log \mathbb{P}\big[\mathcal{H}(2t, 0) + \log\sqrt{4\pi t} \ge \lambda\big] =: -\Phi(\lambda).$$

(b) $\lim_{\lambda\to 0} \lambda^{-2}\Phi(\lambda) = \frac{1}{\sqrt{2\pi}}$.

(c) $\lim_{\lambda\to \infty} \lambda^{-\frac{5}{2}}\Phi(-\lambda) = \frac{4}{15\pi}$.

Remark 2.1.2. Our method works also for the flat initial data $\mathcal{H}(0, x) \equiv 0$, but we treat only the narrow wedge initial data to keep the paper at a reasonable length.

Our result generalizes immediately to $h(2t, x)$, for $x \in \mathbb{R}$. This is because, under the delta initial data, the one-point law of $Z(2t, x)/p(2t, x)$ does not depend on x. This fact can be verified from the Feynman–Kac formula for the SHE.

Remark 2.1.3. Even though LDP rate functions are model dependent, the $\frac{5}{2}$ tail seems to be somewhat ubiquitous in the KPZ class. It shows up in all time regimes for the KPZ equation, and has also been observed in the TASEP [DL98]. A very interesting question is to investigate to what extend is the $\frac{5}{2}$ tail universal, and to find a unifying approach to understand the origin of the tail.

Remark 2.1.4. The aforementioned physics works [KK09, MKV16, LDMRS16, KMS16] also derived the asymptotics of the deep upper tail. The prediction is $\lim_{\lambda\to\infty} \lambda^{-3/2}\Phi(\lambda) = \frac{4}{3}$. We leave this question for future work.

Remark 2.1.5. The short-time large deviations for the KPZ equation were also studied under other initial data or on a half-line. For the KPZ equation starting from Brownian initial data, the problem was studied in physics works [KLD17, MS17]. For the half-line KPZ equation, the same problem was studied in the physics work [KLD18a, MV18]; see also [Kra19] for a summary of these results. It is interesting to see whether our method generalizes in these situations.

Let us emphasize that, even though we follow the overarching idea of the WNT, our method *significantly differs* from existing physics heuristics. As will be explained below, the WNT amounts to establishing a Freidlin–Wentzell LDP and analyzing the corresponding variational problem. The second step — analyzing the variational problem — is the *harder* step. The physics works [KK09, MKV16, KMS16] provide convincing heuristic for this step by a formal PDEs argument. However, as will be explained in Section 2.1.1, to make this PDE argument rigorous requires elaborate treatments and seems challenging. We hence adopt a different method.

In Section 2.1.1, we will recall the physics heuristic from [KK09, MKV16, KMS16] and explain why it seems challenging to make the heuristic rigorous. In Section 2.1.2, we will explain our method for proving Theorem 2.1.1.

2.1.1 Discussions about the physics heuristics

Here we recall the method used in the physics works [KK09, MKV16, KMS16]. The first step is to perform scaling to turn the short-time LDP into a Freidlin–Wentzell LDP. One scales

$$\mathcal{H}_\varepsilon(t,x) := \mathcal{H}(\varepsilon t, \varepsilon^{1/2}x) + \log(\varepsilon^{1/2}), \tag{2.1.3}$$

which brings the KPZ equation into

$$\partial_t \mathcal{H}_\varepsilon = \tfrac{1}{2}\partial_{xx}\mathcal{H}_\varepsilon + \tfrac{1}{2}(\partial_x \mathcal{H}_\varepsilon)^2 + \sqrt{\varepsilon}\xi. \tag{2.1.4}$$

The term $\log(\varepsilon^{1/2})$ in (2.1.3) ensures that the narrow wedge initial data stays invariant. The equation (2.1.4) is in the form for studying Freidlin–Wentzell LDPs. Roughly speaking, for a generic $\rho \in L^2([0,T]\times\mathbb{R})$, we expect $\mathbb{P}[\sqrt{\varepsilon}\xi \approx \rho] \approx \exp(-\frac{1}{2}\varepsilon^{-1}\|\rho\|_{L^2}^2)$. When the event $\{\sqrt{\varepsilon}\xi \approx \rho\}$ occurs, one expects $\mathcal{H}_\varepsilon$ to approximate the solution $\mathsf{h} = \mathsf{h}(\rho; t, x)$ of

$$\partial_t \mathsf{h} = \tfrac{1}{2}\partial_{xx}\mathsf{h} + \tfrac{1}{2}(\partial_x \mathsf{h})^2 + \rho. \tag{2.1.5}$$

In more formal terms, one expects $\{\mathcal{H}_\varepsilon\}$ to satisfy an LDP with speed ε^{-1} and the rate function $J(f) = \inf\{\frac{1}{2}\|\rho\|_{L^2} : \mathsf{h}(\rho) = f\}$. Once such an LDP is established in a suitable space, by the contraction principle we should have

$$\Phi(\lambda) = -\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\big[\mathcal{H}_\varepsilon(2,0) \geq \lambda\big] = \inf\big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{h}(\rho; 2, 0) \geq \lambda\big\}, \qquad \lambda > 0, \tag{2.1.6}$$

$$\Phi(-\lambda) = -\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\big[\mathcal{H}_\varepsilon(2,0) \leq -\lambda\big] = \inf\big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{h}(\rho; 2, 0) \leq -\lambda\big\}, \quad -\lambda < 0. \tag{2.1.7}$$

To find the infimum in (2.1.7), one can perform variation of $\frac{1}{2}\|\rho\|_{L^2}^2 = \frac{1}{2}\int_0^2\int_{\mathbb{R}} \rho^2\, \mathrm{d}x\mathrm{d}t$ in ρ under the constraint $\mathsf{h}_\lambda(\rho; 2, 0) = -\lambda$, c.f., [MKV16, Sect A, Supplementary Material]. The result suggests that any minimizer ρ should solve

$$\partial_t \rho = -\tfrac{1}{2}\partial_{xx}\rho + \partial_x(\rho\, \partial_x \mathsf{h}). \tag{2.1.8}$$

With a negative Laplacian $-\frac{1}{2}\partial_{xx}\rho$, the equation (2.1.8) needs to be solved *backward* in time from the terminal data $\rho(2,x) = -c(\lambda)\delta_0(x)$, c.f., [MKV16, Sect A, Supplementary Material], where $c(\lambda) > 0$ is a constant fixed by $\mathsf{h}(\rho; 2, 0) = -\lambda$.

In the near-center regime, i.e., $\lambda \to 0$, standard perturbation arguments can be applied to analyze (2.1.5) and (2.1.8) to conclude the quadratic power law.

We will focus on the deep lower tail regime, i.e., $-\lambda \to -\infty$. We scale $\lambda^{-1}\mathsf{h}(\rho; t, \lambda^{1/2}x) \mapsto \mathsf{h}(\rho; t, x)$ and $\lambda^{-1}\rho(t, \lambda^{1/2}x) \mapsto \rho(t,x)$. To see why such scaling is relevant, note that, under the conditioning $\mathsf{h}(\rho; 2, 0) \leq -\lambda$, it is natural to scale h by λ^{-1}. Time cannot be scaled since we are probing h at $t = 2$. After scaling h by λ^{-1}, we find that the quadratic term $\frac{1}{2}(\partial_x \mathsf{h})^2$ in (2.1.5) gains an excess λ factor compared to the left hand side. To bring the quadratic term back to the same footing as the left hand side, we scale x by $\lambda^{-1/2}$. Similar considerations lead to the same scaling of ρ. Under such scaling the equations (2.1.5) and (2.1.8) become

$$\partial_t \mathsf{h} = \tfrac{1}{2}\lambda^{-1}\partial_{xx}\mathsf{h} + \tfrac{1}{2}(\partial_x \mathsf{h})^2 + \rho, \tag{2.1.9}$$

$$\partial_t \rho = -\tfrac{1}{2}\lambda^{-1}\partial_{xx}\rho + \partial_x(\rho\, \partial_x \mathsf{h}). \tag{2.1.10}$$

As $\lambda \to \infty$ it is tempting to drop the Laplacian terms in (2.1.9)–(2.1.10). Doing so produces

$$\partial_t \mathsf{h} = \tfrac{1}{2}(\partial_x \mathsf{h})^2 + \rho, \tag{2.1.11}$$

$$\partial_t \rho = \partial_x(\rho\, \partial_x \mathsf{h}), \tag{2.1.12}$$

with the initial data $\lim_{t \downarrow 0}(\mathsf{h}(t,x)t) = -\frac{1}{2}x^2$ and the terminal data $\rho(2,x) = -c(1)\delta_0(x)$.

The equations (2.1.11)–(2.1.12) can be solved by the procedure in [KK09, MKV16, KMS16]. For the completeness of presentation we briefly recall the procedure below. It begins by solving

(2.1.11)–(2.1.12) by power series expansion in x. In view of the initial data of h and the terminal data of ρ, it is natural to assume $\mathsf{h}(t,x) = \mathsf{h}(t,-x)$ and $\rho(t,x) = \rho(t,-x)$. Under such assumptions, the series terminates at the quadratic power for both h and ρ and produces the solution $\mathsf{h}(t,x) = k(t) + \frac{1}{2}a(t)x^2$ and $\rho(t,x) = -\frac{1}{2\pi}r(t) + \frac{1}{2\pi}(r(t)/\ell^2(t))x^2$. The factor $\frac{1}{2\pi}$ is just a convention we choose; the functions $a(t), k(t), r(t)$, and $\ell(t)$ can be found by inserting the series solution in (2.1.11)–(2.1.12). The only relevant property to our current discussion is that $r(t) > 0$.

The series solution, however, is nonphysical. Indeed, with $r(t) > 0$, we have $\|\rho\|_{L^2} = \infty$. This issue is rectified by observing that the minimizing ρ of the right hand side of (2.1.7) should be nonpositive. This is so because $\mathsf{h}(\rho; t, x)$ increases in ρ. Hence the positive part ρ_+ of ρ would only make $\mathsf{h}(\rho; 2, 0) = -1$ harder to achieve while costing excess L^2 norm. This observation prompts us to truncate

$$\rho_*(t,x) := -\tfrac{1}{2\pi}r(t)\big(1 - \tfrac{x^2}{\ell(t)^2}\big)_+.$$

It can be verified that such a ρ_* and a suitably truncated h solve (2.1.11)–(2.1.12).

Remark 2.1.6. It may appear that the preceding scaling applies also to the upper-tail regime $\lambda \to \infty$, but that is *not* the case. In the upper-tail regime, the analyses of the physics works [KK09, MKV16, KMS16] show that, in the *pre-scaled* coordinates, the optimal $\rho(t,x)$ concentrates in a small corridor of size $O(\lambda^{-1/2})$ around $x = 0$. This behavior is in sharp contrast with that of the lower-tail, where the optimal $\rho(t,x)$ spans across a region in x of width $O(\lambda^{1/2})$ in the pre-scaled coordinate. The distinction of behaviors in the upper- and lower-tail regimes is ubiquitous in the KPZ universality class. As a result, the preceding scaling does not apply to the upper-tail regime.

Challenge in making the PDE argument rigorous

To make this PDE analysis rigorous requires elaborate treatments and seems challenging. This is so because (2.1.11)–(2.1.12) are fully nonlinear equations. Taking derivative $u = \partial_x \mathsf{h}$ in (2.1.11)–

(2.1.12) gives

$$\partial_t u = \tfrac{1}{2}\partial_x(u^2) + \partial_x \rho,$$
$$\partial_t \rho = \partial_x(\rho u).$$

These equations do *not* have unique weak solutions, just like the inviscid Burgers equation [Eva98, Chapter 3.4].One needs to impose certain entropy conditions to ensure the uniqueness of weak solutions, and argue that in the limit $\lambda \to \infty$ the solution of (2.1.11)–(2.1.12) converges to the entropy solution.

2.1.2 Our method

Our method, which *differs* from the physics heuristic described in Section 2.1.1, operates at the level of the SHE instead of the KPZ equation. Recall that we defined the solution of the KPZ equation thorough the Hopf–Cole transformation, so the solution h_ε to (2.1.4) is given by $h_\varepsilon := \log Z_\varepsilon + \log(\varepsilon^{1/2})$, where Z_ε solves

$$\partial_t Z_\varepsilon = \tfrac{1}{2}\partial_{xx} Z_\varepsilon + \sqrt{\varepsilon}\xi Z_\varepsilon, \tag{2.1.13}$$

with the delta initial condition $Z_\varepsilon(0,\bullet) = \delta_0(x)$. We seek to establish the the Freidlin–Wentzell LDP for (2.1.13). Roughly speaking, the LDP states that $\mathbb{P}[Z_\varepsilon \approx \mathsf{Z}] \approx \exp(-\varepsilon^{-1}\frac{1}{2}\|\rho\|^2_{L^2})$, where $\mathsf{Z} = \mathsf{Z}(\rho; t, x)$ solves the PDE

$$\partial_t \mathsf{Z} = \tfrac{1}{2}\partial_{xx}\mathsf{Z} + \rho\mathsf{Z}. \tag{2.1.14}$$

The precise statement of the Freidlin–Wentzell LDP as well as the well posedness of (2.1.14) will be given in Section 2.1.2. Use the contraction principle to specialize the Freidlin–Wentzell LDP to one point. We have

$$\Phi(\lambda) = \inf \left\{ \tfrac{1}{2}\|\rho\|_{L^2}^2 : \log \mathsf{Z}(\rho; 2, 0) \geq \lambda \right\}, \tag{2.1.15}$$

$$\Phi(-\lambda) = \inf \left\{ \tfrac{1}{2}\|\rho\|_{L^2}^2 : \log \mathsf{Z}(\rho; 2, 0) \leq -\lambda \right\}. \tag{2.1.16}$$

To analyze the variational problems (2.1.15)–(2.1.16), we express Z by the Feynman–Kac formula as

$$\mathsf{Z}(\rho; t, x) = \mathbb{E}_{0\to x}\Big[\exp\Big(\int_0^t \rho(s, B_{\mathrm{b}}(s))\, \mathrm{d}s \Big) \Big] p(t, x), \tag{2.1.17}$$

where the $\mathbb{E}_{0\to x}$ is taken with respect to a Brownian bridge $B_{\mathrm{b}}(s)$ that starts from $B_{\mathrm{b}}(0) = 0$ and ends in $B_{\mathrm{b}}(t) = x$, and $p(t, x) := \exp(-x^2/2t)/\sqrt{2\pi t}$ denotes the standard heat kernel.

Given the Feynman–Kac formula, standard perturbation argument can be applied to obtained the quadratic law in the near-center regime, $\lambda \to 0$; this is done in Section 2.4.1.

Here we focus on the deep lower tail regime, i.e., analyzing (2.1.16) in the limit $-\lambda \to -\infty$. The scaling $\rho(\bullet, \bullet) \mapsto \lambda\rho(\bullet, \lambda^{-\frac{1}{2}}\bullet)$ mentioned in Section 2.1.1 gives

$$\Phi(-\lambda) = \lambda^{5/2} \inf \left\{ \tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{h}_\lambda(\rho; 2, 0) \leq -1 \right\}, \tag{2.1.18}$$

where

$$\mathsf{h}_\lambda(\rho; t, x) := (\text{lower order term}) - \tfrac{x^2}{2t} + \lambda^{-1} \log \mathbb{E}_{0\to\lambda^{1/2}x}\Big[\exp\Big(\int_0^t \lambda\rho(s, \lambda^{-\frac{1}{2}} B_{\mathrm{b}}(s))\, \mathrm{d}s \Big) \Big]. \tag{2.1.19}$$

The details of this scaling are given in Section 2.4.2, and the precise expression of (2.1.19) is given in (2.4.11).

We seek to analyze the right hand side of (2.1.19) for $(t, x) = (2, 0)$. For a suitable class of ρ, Varadhan's lemma gives, as $-\lambda \to -\infty$,

$$\lambda^{-1} \log \mathbb{E}_{0\to 0}\Big[\exp\Big(\int_0^2 \lambda\rho(s, \lambda^{-\frac{1}{2}}B_{\mathrm{b}}(s))\,\mathrm{d}s\Big)\Big] \longrightarrow -\inf_{\gamma}\Big\{\int_0^2 \tfrac{1}{2}\gamma'(s)^2 - \rho(s,\gamma(s))\,\mathrm{d}s\Big\}, \tag{2.1.20}$$

where the infimum is taken over all H^1 path $\gamma(s)$ that starts and ends in 0, i.e., $\gamma(0) = \gamma(2) = 0$. This limit transition is reminiscent of the convergence (under the zero-temperature limit) of the free energy of a directed polymer to that of a last passage percolation. Our task is hence to find the $\rho = \rho(s, y)$ with the minimal L^2 norm such that the right hand side of (2.1.20) is ≤ -1.

It is natural to *guess* that the minimizing ρ should be the ρ_* obtained in the aforementioned PDE heuristic. Taking this explicit ρ_*, we prove the convergence (2.1.20) (by Varadhan's lemma) and solve the path variational problem on the right side of (2.1.20); see Lemma 2.4.2 and Proposition 2.4.3. The explicit constant $\frac{4}{15\pi}$ in Theorem 2.1.1 (c) comes from the L^2 norm of ρ_*.

The last step is to *verify* that such a ρ_* is indeed the minimizer. This is done in Section 2.4.2. There we appeal to an identity (2.4.30) that involves ρ_*. This identity follows from the fact that for $\rho = \rho_*$, the right hand side of (2.1.20) is equal to -1. Using this identity, we show that, for any ρ that satisfies the required condition $\mathsf{h}_\lambda(\rho; 2, 0) \leq -1$, the quantity $\langle \rho_* - \rho, \rho_* \rangle$ is approximately ≤ 0; see (2.4.32). This bound then verifies that ρ_* is the minimizer.

Freidlin–Wentzell LDP for the SHE

Here we state our result on the Freidlin–Wentzell LDP for the SHE (2.1.13). For the purpose of proving Theorem 2.1.1, it suffices to just consider the narrow wedge initial data, but we also consider function-valued initial data for their independent interest.

Let us set up the notation, first for function-valued initial data. For $a \in \mathbb{R}$, define the weighted sup norm $\|g\|_a := \sup_{x\in\mathbb{R}}\{e^{-a|x|}|g(x)|\}$. Let $C_a(\mathbb{R}) := \{g \in C(\mathbb{R}) : \|g\|_a < \infty\}$, and endow this space with the norm $\|\bullet\|_a$. Slightly abusing notation, for functions that depend also on time, we use the same notation

$$\|f\|_a := \left\{e^{-a|x|}|f(t,x)| \,:\, (t,x)\in[0,T]\times\mathbb{R}\right\} \tag{2.1.21}$$

to denote the analogous norm, and let $C_a([0,T]\times\mathbb{R}) := \{f \in C([0,T]\times\mathbb{R}) : \|f\|_a < \infty\}$, endowed with the norm $\|\bullet\|_a$. Adopt the notation $C_{a_*^+}(\mathbb{R}) := \cap_{a>a_*} C_a(\mathbb{R})$ and $C_{a_*^+}([0,T]\times\mathbb{R}) := \cap_{a>a_*} C_a([0,T]\times\mathbb{R})$. Let $p(t,x) := \exp(-\frac{x^2}{2t})/\sqrt{2\pi t}$ denote the standard heat kernel. Recall that the mild solution of (2.1.13) with a deterministic initial data g_* is a process Z_ε that satisfies

$$Z_\varepsilon(t,x) = \int_{\mathbb{R}} p(t,x-y)g_*(y)\,\mathrm{d}y + \varepsilon^{\frac{1}{2}}\int_{\mathbb{R}} p(t-s,x-y)Z_\varepsilon(s,y)\xi(s,y)\,\mathrm{d}s\mathrm{d}y. \tag{2.1.22}$$

It is standard, e.g., [Qua11, Sections 2.1–2.6], to show that for any $g_* \in C_{a_*^+}(\mathbb{R})$, there exists a unique mild solution Z_ε of (2.1.13) given by the chaos expansion; see Section 2.2.1 for a discussion about chaos expansion. Further, as shown later in Corollary 2.3.6, the chaos expansion (and hence Z_ε) is $C_{a_*^+}([0,T]\times\mathbb{R})$-valued. Next we turn to the rate function. Fix $g_* \in C_{a_*+}(\mathbb{R})$. For $\rho \in L^2([0,T]\times\mathbb{R})$, consider the PDE

$$\partial_t \mathsf{Z} = \tfrac{1}{2}\partial_{xx}\mathsf{Z} + \rho\mathsf{Z}, \qquad \mathsf{Z}(\rho;0,\bullet) = g_*(\bullet),$$

where $\mathsf{Z} = \mathsf{Z}(\rho;t,x)$, $t\in[0,T]$, and $x\in\mathbb{R}$. This PDE is interpreted in the Duhamel sense as

$$\mathsf{Z}(\rho;t,x) = \int_{\mathbb{R}} p(t,x-y)g_*(y)\,\mathrm{d}y + \int_0^t\int_{\mathbb{R}} \rho(s,y)\mathsf{Z}(\rho;s,y)\,\mathrm{d}y\mathrm{d}s. \tag{2.1.23}$$

We will show in Section 2.2.1 that (2.1.23) admit a unique $C_{a_*^+}([0,T]\times\mathbb{R})$-valued solution. We will often write $\mathsf{Z}(\rho) = \mathsf{Z}(\rho;\bullet,\bullet)$ and accordingly view $\rho\mapsto\mathsf{Z}(\rho)$ as a function $L^2([0,T]\times\mathbb{R}) \to C_a([0,T]\times\mathbb{R})$, for $a > a_*$. Here ρ should be viewed as a deviation of the spacetime white noise $\sqrt{\varepsilon}\xi$. For each such deviation ρ we run the PDE (2.1.23) to obtain the corresponding deviation $\mathsf{Z}(\rho) = \mathsf{Z}(\rho;t,x)$ of Z_ε. Now, since the spacetime white noise ξ is Gaussian with the correlation

$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta_0(t-s)\delta_0(x-y)$, one expects the rate function to be the L^2 norm of ρ, more precisely

$$I(f) := \inf\left\{\tfrac{1}{2}\|\rho\|_{L^2} \ : \ \rho \in L^2([0,T]\times\mathbb{R}), \mathsf{Z}(\rho) = f\right\}, \tag{2.1.24}$$

with the convention $\inf\emptyset := +\infty$.

As for the narrow wedge initial data, we adopt the same notation as in the preceding but replace $g_* \in C_{a_*^+}(\mathbb{R})$ with $g_* = \delta_0$. More explicitly, the mild solution of the SHE (2.1.13) satisfies

$$Z_\varepsilon(t,x) = p(t,x) + \varepsilon^{\frac{1}{2}} \int_{\mathbb{R}} p(t-s,x-y) Z_\varepsilon(s,y)\xi(s,y)\,\mathrm{d}s\mathrm{d}y, \tag{2.1.22-nw}$$

and the function $\mathsf{Z}(\rho)$ now solves

$$\mathsf{Z}(\rho;t,x) = p(t,x) + \int_0^t \int_{\mathbb{R}} \rho(s,y)\mathsf{Z}(\rho;s,y)\,\mathrm{d}y\mathrm{d}s. \tag{2.1.23-nw}$$

Recall that Z_ε starts from the delta initial condition $Z_\varepsilon(0,\bullet) = \delta_0(x)$. The smoothing effect of the Laplacian in the SHE makes $Z_\varepsilon(t,\bullet)$ function-valued for all $t > 0$, but when $t \to 0$ the process $Z_\varepsilon(t,\bullet)$ becomes singular as it approaches δ_0. To avoid the singularity, we work with the space $C_a([\eta,T]\times\mathbb{R})$, $\eta > 0$, $a \in \mathbb{R}$, equipped with the norm

$$\|f\|_{a,\eta} := \left\{e^{-a|x|}|f(t,x)| \ : \ (t,x) \in [\eta,T]\times\mathbb{R}\right\}. \tag{2.1.25}$$

It is standard to show that (2.1.22-nw) admits a unique solution that is $C_a([\eta,T]\times\mathbb{R})$-valued for all $\eta > 0$ and $a \in \mathbb{R}$. The same holds for (2.1.23-nw).

Let Ω be a topological space. Recall that a function $\varphi : \Omega \to \mathbb{R}\cup\{+\infty\}$ is a **good rate function** if φ is lower semi-continuous and the set $\{f : \varphi(f) \leq r\}$ is compact for all $r < +\infty$. Recall that a sequence $\{W_\varepsilon\}$ of Ω-valued random variables **satisfies an LDP with speed ε^{-1} and the rate function φ** if for any closed $F \subset \Omega$ and open $G \subset \Omega$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\big[W_\varepsilon \in G\big] \geq -\inf_{f\in G}\varphi(f), \qquad \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\big[W_\varepsilon \in F\big] \leq -\inf_{f\in F}\varphi(f).$$

In this paper we prove the following Freidlin–Wentzell LDP for the SHE.

Proposition 2.1.7.

(a) *Fix* $a_* \in \mathbb{R}$, $g_* \in C_{a_*^+}(\mathbb{R})$, *and* $T < \infty$. *Let* Z_ε *be the solution of* (2.1.22) *and let* $\mathsf{Z}(\rho)$ *be the solution of* (2.1.23).

For any $a > a_*$, *the function* $I : C_a([0,T]\times\mathbb{R}) \to \mathbb{R}\cup\{+\infty\}$ *in* (2.1.24) *is a good rate function. Further,* $\{Z_\varepsilon\}_\varepsilon$ *satisfies an LDP in* $C_a([0,T]\times\mathbb{R})$ *with speed* ε^{-1} *and the rate function* I.

(b) *Fix* $T < \infty$. *Let* Z_ε *be the solution of* (2.1.22-nw) *and let and let* $\mathsf{Z}(\rho)$ *be the solution of* (2.1.23-nw).

For any $a \in \mathbb{R}$ *and* $\eta \in (0,T)$, *the function* $I : C_a([\eta,T]\times\mathbb{R}) \to \mathbb{R}\cup\{+\infty\}$ *in* (2.1.24) *is a good rate function. Further,* $\{Z_\varepsilon\}_\varepsilon$ *satisfies an LDP in* $C_a([\eta,T]\times\mathbb{R})$ *with speed* ε^{-1} *and the rate function* I.

2.1.3 Literature on the WNT and Freidlin-Wentzell LDPs for stochastic PDEs

The WNT, also known as the optimal fluctuation theory, dates back at least to the works [HL66, ZL66, Lif68] in condensed matter physics. In the context of stochastic PDEs, the WNT studies large deviations of the solution's trajectory when the noise is scaled to be weaker and weaker. Such scaling is often equivalent to the short time scaling of a fixed SPDE. (See (2.1.3)–(2.1.4) for the case of the KPZ equation.) In the physics literature, the WNT was carried out in [Fog98] for the noisy Burgers equation, in [KK07, KK09] for directed polymer and in [KMS16, MKV16] for the KPZ equation. The WNT is also known as the instanton method in turbulence theory [FKLM96, FGV01, GGS15], the macroscopic fluctuation theory in lattice gases [BDSG+15], and WKB methods in reaction-diffusion systems [EK04, MS11].

The Freidlin-Wentzell LDP has been established for various stochastic PDEs, including reaction-diffusion-like stochastic equations [CM97, BDM08], the stochastic Allen–Cahn equation [HW15b], and the stochastic Navier–Stokes equation [CD19].

2.1.4 Some discussions about the rate function Φ

The physics work [LDMRS16] used a different method to derive

$$\Phi(\lambda) = \begin{cases} \frac{-1}{\sqrt{4\pi}} \min\limits_{z\in[-1,+\infty)} \big\{ z e^{\lambda} + \mathrm{Li}_{\frac{5}{2}}(-z) \big\}, & \lambda \leq \lambda_c, \\ \frac{-1}{\sqrt{4\pi}} \min\limits_{z\in[-1,0)} \big\{ z e^{\lambda} + \mathrm{Li}_{\frac{5}{2}}(-z) - \frac{8\sqrt{\pi}}{3}(-\log(-z)) \big\}, & \lambda \geq \lambda_c, \end{cases}$$

where $\mathrm{Li}_{\nu}(z)$ is the poly-logarithm function and $\lambda_c = \log\zeta(\frac{3}{2})$. Though not completely mathematically rigorous, the derivation is based on convincing arguments and is backed by the numerical result [HLDM+18b]. Based on this expression, the work obtained many properties of Φ, including its analyticity on $\lambda \in \mathbb{R}$, and lower-order terms in the deep lower-tail regime $-\lambda \to -\infty$ (beyond the leading term $\frac{4}{15\pi}\lambda^{\frac{5}{2}}$). Our results do not cover these detailed properties of Φ. Rigorously proving these properties is an interesting open question.

Acknowledgments

We thank Ivan Corwin for suggesting this problem to us and for useful discussions, thank Sayan Das, Amir Dembo, Promit Ghosal, Konstantin Matetski, and Shalin Parekh for useful discussions, and thank Martin Hairer, Hao Shen, and Hendrik Weber for clarifying some points about the literature. We thank the anonymous referees for useful comments that improve the presentation of this paper. The research of YL is partially supported by the Fernholz Foundation's "Summer Minerva Fellow" program and also received summer support from Ivan Corwin's NSF grant DMS-1811143. The research of LCT is partially supported by the NSF through DMS-1953407.

Outline of the rest of the paper

In Section 2.2, we recall the formalism of Wiener chaos, recall a result from [HW15b] that gives the LDP for finitely many chaos, and prepare some properties of the function $\mathsf{Z}(\rho)$. In Section 2.3, we establish tail probability bounds on the Wiener chaos for the SHE. Based on such tail bounds, we leverage the LDP for finitely many chaos into the LDP for the SHE, thereby proving

Proposition 2.1.7. In Section 2.4, we analyze the variational problem given by the one-point LDP for the SHE and prove Theorem 2.1.1.

2.2 Wiener spaces, Wiener chaos, and the function $\mathsf{Z}(\rho)$

In this section we recall the formalism of Wiener spaces and chaos, and prepare some properties of $\mathsf{Z}(\rho)$.

2.2.1 Function-valued initial data

Throughout this subsection we fix $T < \infty$, $a_* \in \mathbb{R}$, and $g_* \in C_{a_*^+}(\mathbb{R})$, and initiate the SHE (2.1.13) from $Z_\varepsilon(0, \bullet) = g_*(\bullet)$.

Wiener spaces and chaos

We will mostly follow [HW15b, Section 3]. The basic elements of the Wiener space formalism consists of $(\mathcal{B}, \mathcal{H}, \mu)$, where $\mathcal{B}$ is a Banach space over $\mathbb{R}$ equipped with a Gaussian measure μ, and $\mathcal{H} \subset \mathcal{B}$ is the Cameron–Martin space of $\mathcal{B}$. In our setting $\mathcal{H} = L^2([0,T] \times \mathbb{R})$, and $\mathcal{B}$ can be any a Banach space such that the embedding $\mathcal{H} \subset \mathcal{B}$ is dense and Hilbert–Schmidt. To be concrete, fixing an arbitrary orthonormal basis $\{e_1, e_2, \ldots\}$ of $\mathcal{H} = L^2([0,T] \times \mathbb{R})$, we let

$$\mathcal{B} := \Big\{ \xi = \sum \xi_i e_i \ : \ \xi_1, \xi_2, \ldots \in \mathbb{R}, \ \|\xi\|_{\mathcal{B}} < \infty \Big\}, \qquad \big\| \sum \xi_i e_i \big\|_{\mathcal{B}}^2 := \sum\nolimits_{i \geq 1} \tfrac{1}{i^2} |\xi_i|^2. \tag{2.2.1}$$

Identifying $\mathcal{B}$ as a subset of $\mathbb{R}^{\mathbb{Z}_{\geq 1}}$, we set $\mu := \otimes_{\mathbb{Z}_{\geq 1}} \nu$, where ν is the standard Gaussian measure on $\mathbb{R}$. The space $\mathcal{B}$ serves as the *sample space*. For example, for $f \in L^2([0,T] \times \mathbb{R})$ with $f = \sum f_i e_i$, the function

$$W(f) : \mathcal{B} \to \mathbb{R}, \qquad W(f) := \sum\nolimits_{i \geq 1} f_i \xi_i \tag{2.2.2}$$

should be identified with the random variable $\int_0^T \int_{\mathbb{R}} f(t,x) \xi(t,x) \, \mathrm{d}t \mathrm{d}x$. This identification justifies using ξ to denote both elements of $\mathcal{B}$ and the spacetime white noise.

The Hermite polynomials $H_n(x)$ are the unique polynomials satisfying $\deg(H_n) = n$ and

$$e^{\tau x - \frac{\tau^2}{2}} = \sum_{n=0}^{\infty} \tau^n H_n(x). \tag{2.2.3}$$

The **n-th $\mathbb{R}$-valued Wiener chaos** is the closure in $L^2(\mathcal{B} \to \mathbb{R}, \mu)$ of the linear subspace spanned by $\prod_{i=1}^{\infty} H_{\alpha_i}(W(e_i))$, for $(\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \ldots$ and $\alpha_1 + \alpha_2 + \ldots = n$. Since our goal is to establish a *functional* LDP, it is natural to consider Wiener chaos at the functional level. We will follow the formalism of Banach-valued Wiener chaos from [HW15b, Section 3]. Fix $a > a_*$ and consider $E = C_a([0,T] \times \mathbb{R})$, which is a separable Banach space. The **n-th E-valued Wiener chaos** is the space

$$\Big\{\Psi \in L^2(\mathcal{B} \to E, \mu) \,:\, \int \Psi(\xi)\psi(\xi)\mu(\mathrm{d}\xi) = 0, \ \forall \psi \in (m\text{-th } \mathbb{R}\text{-valued Wiener chaos}), \text{ with } m \neq n\Big\}.$$

In probabilistic notation, the n-th E-valued Wiener chaos consists of $C_a([0,T]\times\mathbb{R})$-valued random variables Ψ such that $\mathbb{E}[\|\Psi\|_a^2] < \infty$ and that $\mathbb{E}[\Psi\psi] = 0$, for all ψ in the m-th $\mathbb{R}$-valued Wiener chaos with $m \neq n$.

We now turn to the SHE. Set

$$Y_n(t,x) := \int_{\Delta_n(t)} \int_{\mathbb{R}^{n+1}} p(s_n - s_{n+1}, y_n - y_{n+1}) g_*(y_{n+1}) \mathrm{d}y_{n+1} \prod_{i=1}^{n} p(s_{i-1} - s_i, y_{i-1} - y_i)\xi(s_i, y_i)\, \mathrm{d}s_i \mathrm{d}y_i, \tag{2.2.4}$$

where $\Delta_n(t) = \{\vec{s} = (s_0, s_1, \ldots, s_{n+1}) : 0 = s_{n+1} < s_n < \cdots < s_1 < s_0 = t\}$, with the convention $s_0 := t$ and $y_0 := x$. Iterating (2.1.22) gives

$$Z_\varepsilon(t,x) = \sum_{n=0}^{\infty} \varepsilon^{\frac{n}{2}} Y_n(t,x). \tag{2.2.5}$$

We will show later in Proposition 2.3.5 that each Y_n defines a $C_a([0,T] \times \mathbb{R})$-valued random variable, and show in Corollary 2.3.6 that the right hand side of (2.2.5) converges in $\|\cdot\|_a$ almost surely. It is standard to show that (2.2.5) gives the unique mild solution of the SHE. Further, given the n-

fold stochastic integral expression in (2.2.4), it is standard to show that, for fixed $(t,x) \in [0,T]\times\mathbb{R}$, the random variable $Y_n(t,x)$ lies in the n-th $\mathbb{R}$-valued Wiener chaos, and $Y_n \in C_a([0,T]\times\mathbb{R}) =: E$ lies in the n-th E-valued Wiener chaos. Accordingly, we refer to the series (2.2.5) as the **chaos expansion** for the SHE.

Let $Z_{N,\varepsilon} := \sum_{n=0}^{N} \varepsilon^{\frac{n}{2}} Y_n$ denote the partial sum of the chaos expansion (2.2.5). The LDPs of finitely many E-valued Wiener chaos has been established in [HW15b, Theorem 3.5]. We next apply this result to obtain an LDP for $Z_{N,\varepsilon}$. Following the notation in [HW15b], we view Y_n as a function $\mathcal{B} \to C_a([0,T]\times\mathbb{R})$, denoted $Y_n(\xi)$, and define

$$(Y_n)_{\mathrm{hom}} : L^2([0,T]\times\mathbb{R}) \to C_a([0,T]\times\mathbb{R}), \qquad (Y_n)_{\mathrm{hom}}(\rho) := \int_{\mathcal{B}} Y_n(\xi+\rho)\,\mu(\mathrm{d}\xi). \tag{2.2.6}$$

The last integral is well-defined for any $\rho \in L^2([0,T]\times\mathbb{R})$ by the Cameron–Martin theorem. Further define

$$I_N : C_a([0,T]\times\mathbb{R}) \to \mathbb{R}\cup\{+\infty\} \quad I_N(f) := \inf\Big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 \,:\, \rho \in L^2([0,T]\times\mathbb{R}),\ \sum_{n=0}^{N}(Y_n)_{\mathrm{hom}}(\rho) = f\Big\}, \tag{2.2.7}$$

with the convention $\inf\emptyset := +\infty$. We now apply [HW15b, Theorem 3.5] to obtain an LDP for $Z_{N,\varepsilon}$.

Proposition 2.2.1 (Special case of [HW15b, Theorem 3.5])**.** *For any fixed $a > a_*$, the function I_N in* (2.2.7) *is a good rate function. For fixed $N < \infty$, $\{Z_{N,\varepsilon} := \sum_{n=0}^{N} \varepsilon^{\frac{n}{2}} Y_n\}_\varepsilon$ satisfies an LDP on $C_a([0,T]\times\mathbb{R})$ with speed ε^{-1} and the rate function I_N.*

Proof. Applying [HW15b, Theorem 3.5] with $\delta(\varepsilon) = 0$ and with $\mathbf{\Psi}^{(\varepsilon)} = (Y_0, \varepsilon^{1/2}Y_1, \ldots, \varepsilon^{N/2}Y_N) \in E^{N+1}$ gives an LDP on $C_a([0,T]\times\mathbb{R})^{N+1}$ for $\mathbf{\Psi}^{(\varepsilon)}$ with speed ε^{-1} and the rate function $J(f_0,\ldots,f_N) := \inf\{\frac{1}{2}\|\rho\|_{L^2}^2 : \rho \in L^2([0,T]\times\mathbb{R}),\ (Y_n)_{\mathrm{hom}}(\rho) = f_n, n = 0,\ldots,N\}$. Since the map $C_a([0,T]\times\mathbb{R})^{N+1} \to C_a([0,T]\times\mathbb{R})$, $(f_0,\ldots,f_N) \mapsto f_0 + \ldots + f_N$ is continuous, the claimed result follows by the contraction principle. □

Properties of the function $\mathsf{Z}(\rho)$

Recall that $\mathsf{Z}(\rho)$ denotes the solution of (2.1.23). We begin by developing an series expansion for $\mathsf{Z}(\rho)$ that mimics the chaos expansion for the SHE. For fixed $\rho \in L^2([0,T] \times \mathbb{R})$, let

$$\mathsf{Y}_n(\rho; t, x) := \int_{\Delta_n(t)} \int_{\mathbb{R}^{n+1}} p(s_n - s_{n+1}, y_n - y_{n+1}) g_*(y_{n+1}) \mathrm{d}y_{n+1} \prod_{i=1}^{n} p(s_{i-1} - s_i, y_{i-1} - y_i) \rho(s_i, y_i) \mathrm{d}s_i \mathrm{d}y_i. \tag{2.2.8}$$

where $\Delta_n(t) := \{\vec{s} = (s_0, s_1, \ldots, s_{n+1}) : 0 = s_{n+1} < s_n < \cdots < s_1 < s_0 = t\}$, with the convention $s_0 := t$ and $y_0 := x$. Iterating (2.1.23) shows that the unique solution is given by

$$\mathsf{Z}(\rho; t, x) = \sum_{n=0}^{\infty} \mathsf{Y}_n(\rho; t, x), \tag{2.2.9}$$

provided that the right hand side of (2.2.9) converges in $\|\bullet\|_a$.

To verify this convergence we proceed to establish a bound on $\|\mathsf{Y}_n(\rho)\|_a$. Hereafter, we will use $C = C(a_1, a_2, \ldots)$ to denote a deterministic positive finite constant. The constant may change from line to line or even within the same line, but depends only on the designated variables $a_1, a_2, \ldots$. Recall that $p(t, x)$ denotes the standard heat kernel. The following bounds will be useful in our subsequent analysis. The proof of these bounds is standard and hence omitted.

Lemma 2.2.2. *Fix $a \in \mathbb{R}$ and $\theta \in (0, \frac{1}{2})$. There exists $C = C(a, \theta, T)$ such that for all $x, x' \in \mathbb{R}$ and $s < t \in [0, T]$,*

(a) $p(t, x) \leq C t^{-1/2} e^{a|x|}$,

(b) $\int_{\mathbb{R}} p(t, x - y) e^{a|y|} \mathrm{d}y \leq C e^{a|x|}$,

(c) $\int_{\mathbb{R}} p(t, x - y)^2 e^{a|y|} \mathrm{d}y \leq C t^{-\frac{1}{2}} e^{a|y|}$,

(d) $\int_{\mathbb{R}} \big(p(t, x - y) - p(t, x' - y)\big)^2 e^{a|y|} \mathrm{d}y \leq C |x - x'|^{2\theta} \, t^{-\frac{1}{2} - \theta} (e^{a|x|} \vee e^{a|x'|})$, *and*

(e) $\int_{\mathbb{R}} \big(p(t, x - y) - p(s, x - y)\big)^2 e^{a|y|} \mathrm{d}y \leq C |t - s|^{\theta} \, s^{-\frac{1}{2} - \theta} e^{a|x|}$.

Fix $a \in \mathbb{R}$, $\eta \in (0,T)$, and $\theta \in (0,\frac{1}{2})$. There exists $C = C(a,\theta,T,\eta)$ such that for all $s < t \in [\eta, T]$ and $x, x', y \in \mathbb{R}$,

(i) $|p(t,x-y) - p(t,x'-y)| \leq C|x-x'|^{\theta}(e^{a|x-y|} \vee e^{a|x'-y|})$*, and*

(ii) $|p(t,x) - p(s,x)| \leq C|t-s|e^{a|x|}$.

The next lemma gives a bound on $\|\mathsf{Y}_n(\rho)\|_a$ and verifies the convergence of the right hand side of (2.2.9).

Lemma 2.2.3. *Fix $a > a_*$. There exists $C = C(T,a)$ such that, for all $\rho \in L^2([0,T]\times\mathbb{R})$ and $n \in \mathbb{Z}_{\geq 0}$, we have $\|\mathsf{Y}_n(\rho)\|_a \leq \frac{C^n}{\Gamma(n/2)^{1/2}}\|\rho\|^n_{L^2}$.*

Proof. Throughout this proof we write $C = C(T,a)$. Let $F_n(t) := \sup_{x\in\mathbb{R}} e^{2a|x|}|\mathsf{Y}_n(\rho;t,x)|^2$. For $n = 0$, we have $\mathsf{Y}_0(\rho;t,x) = \int_{\mathbb{R}} p(t,x-y)g_*(y)\mathrm{d}y$. That $g_* \in C_{a^*_+}(\mathbb{R})$ implies $|g_*(y)| \leq Ce^{a|y|}$. Combining this with Lemma 2.2.2(b) gives $F_0(t) \leq C$. Next, for $n \geq 1$, referring to (2.2.8), we see that $\mathsf{Y}_n(\rho;t,x)$ can be expressed iteratively as

$$\mathsf{Y}_n(\rho;t,x) = \int_0^t \int_{\mathbb{R}} p(t-s,x-y)\mathsf{Y}_{n-1}(\rho;s,y)\rho(s,y)\mathrm{d}s\mathrm{d}y.$$

Take square on both sides and apply the Cauchy–Schwarz inequality to get $\mathsf{Y}_n(\rho;t,x)^2 \leq \int_0^t\int_{\mathbb{R}} p(t-s,x-y)^2\mathsf{Y}_{n-1}(\rho;s,y)^2\mathrm{d}s\mathrm{d}y\,\|\rho\|^2_{L^2}$. Within the last integral, use $\mathsf{Y}_{n-1}(\rho;s,y)^2 \leq F_{n-1}(s)e^{2a|y|}$ and Lemma 2.2.2(c), and divide both sides by $e^{-2a|x|}$. We obtain $F_n(t) \leq C\|\rho\|^2_{L^2}\int_0^t F_{n-1}(s)(t-s)^{-1/2}\mathrm{d}s$. Iterating this inequality and using $F_0(t) \leq C$ complete the proof. □

As it turns out, the function $(Y_n)_{\mathrm{hom}}(\rho)$ in (2.2.6) is equal to $\mathsf{Y}_n(\rho)$ in (2.2.8).

Lemma 2.2.4. *For any $\rho \in L^2([0,T]\times\mathbb{R})$ and $n \in \mathbb{Z}_{\geq 0}$, we have $(Y_n)_{\mathrm{hom}}(\rho) = \mathsf{Y}_n(\rho)$.*

Proof. Recall the notation $W(f)$ from (2.2.2). Since $\rho \in L^2([0,T]\times\mathbb{R})$, the Cameron–Martin theorem gives

$$(Y_n)_{\mathrm{hom}}(\rho) := \int_{\mathcal{B}} \mathsf{Y}_n(\rho+\xi)\mu(\mathrm{d}\xi) = \mathbb{E}\big[\exp\big(W(\rho) - \tfrac{1}{2}\|\rho\|^2_{L^2}\big)\mathsf{Y}_n\big]. \tag{2.2.10}$$

Taking $\tau = \|\rho\|_{L^2}$ and $x = W(\rho/\|\rho\|_{L^2})$ in (2.2.3) gives $\exp(W(\rho)-\frac{1}{2}\|\rho\|_{L^2}^2) = \sum_{m=0}^{\infty} \|\rho\|_{L^2}^m H_m(W(\rho/\|\rho\|_{L^2}))$. Invoke the well-known identity, c.f., [Nua06, Proposition 1.1.4],

$$\|\rho\|_{L^2}^m H_m(W(\rho/\|\rho\|_{L^2})) = \int_{\Delta_m(T)} \int_{\mathbb{R}^m} \prod_{i=1}^m \rho(s_i, y_i)\xi(s_i, y_i)\mathrm{d}s_i\mathrm{d}y_i, \tag{2.2.11}$$

insert the result into (2.2.10), and exchange the sum and expectation in the result. We have

$$(Y_n)_{\mathrm{hom}}(\rho; t, x) = \sum_{m=0}^{\infty} \mathbb{E}\Big[\Big(\int_{\Delta_m(T)} \int_{\mathbb{R}^m} \rho^{\otimes m}(\vec{s}, \vec{y}) \prod_{i=1}^m \xi(s_i, y_i)\,\mathrm{d}s_i\mathrm{d}y_i\Big)\, Y_n(t, x)\Big].$$

Within the last expression, the random variable on the right hand side of (2.2.11) belongs to the m-th $\mathbb{R}$-valued Wiener chaos. Since Y_n belongs to the n-th E-valued Wiener chaos, the expectation is nonzero only when $m = n$. Calculating this expectation from (2.2.4) concludes the desired result. □

2.2.2 The narrow wedge initial data

Throughout this subsection we fix $0 < \eta < T < \infty$ and $a \in \mathbb{R}$, and initiate the SHE (2.1.13) from $Z_\varepsilon(0, \bullet) = \delta_0(\bullet)$.

For the Wiener space formalism, the spaces $\mathcal{H} = L^2([0, T] \times \mathbb{R})$ and $\mathcal{B}$ remain the same as in Section 2.2.1, while the space E now changes to $E = C_a([\eta, T] \times \mathbb{R})$. The chaos expansion takes the same form as (2.2.5) but with

$$Y_n(t, x) := \int_{\Delta_n(t)} \int_{\mathbb{R}^{n+1}} p(s_n - s_{n+1}, y_n) \prod_{i=1}^n p(s_{i-1} - s_i, y_{i-1} - y_i)\xi(s_i, y_i)\,\mathrm{d}s_i\mathrm{d}y_i. \tag{2.2.4-nw}$$

Recall the norm $\|\bullet\|_{a,\eta}$ from (2.1.25). Proposition 2.3.5-nw in the following asserts that each Y_n defines a $C_a([\eta, T]\times\mathbb{R})$-valued random variable, and Corollary 2.3.6-nw asserts that the right hand side of (2.2.5) converges in $\|\bullet\|_{a,\eta}$ almost surely. The functions $(Y_n)_{\mathrm{hom}}(\rho)$ and I_N are defined the same way as in Section 2.2.1, but with $C_a([\eta,T]\times\mathbb{R})$ in place of $C_a([0,T]\times\mathbb{R})$. More explicitly,

$$(Y_n)_{\mathrm{hom}}: L^2([0,T]\times\mathbb{R}) \to C_a([\eta,T]\times\mathbb{R}), \qquad (Y_n)_{\mathrm{hom}}(\rho) := \int_{\mathcal{B}} Y_n(\xi+\rho)\,\mu(\mathrm{d}\xi), \tag{2.2.6-nw}$$

$$I_N : C_a([\eta,T]\times\mathbb{R}) \to \mathbb{R}\cup\{+\infty\},\ I_N(f) := \inf\Big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \rho\in L^2([0,T]\times\mathbb{R}),\ \sum_{n=0}^{N}(Y_n)_{\mathrm{hom}}(\rho) = f\Big\}, \tag{2.2.7-nw}$$

with the convention $\inf\emptyset := +\infty$.

Likewise, for the equation (2.1.23-nw), the unique solution is given by the expansion (2.2.9) but with

$$\mathsf{Y}_n(\rho;t,x) := \int_{\Delta_n(t)}\int_{\mathbb{R}^n} p(s_n - s_{n+1}, y_n)\prod_{i=1}^{n} p(s_{i-1}-s_i, y_{i-1}-y_i)\rho(s_i,y_i)\mathrm{d}s_i\mathrm{d}y_i. \tag{2.2.8-nw}$$

Similar proofs of Proposition 2.2.1 and Lemmas 2.2.3 and 2.2.4 applied in the current setting give

Proposition 2.2.1-nw. *For any fixed $a\in\mathbb{R}$ and $\eta\in(0,T)$, the function I_N in* (2.2.7-nw) *is a good rate function. For fixed $N<\infty$, $\{Z_{N,\varepsilon} := \sum_{n=0}^{N}\varepsilon^{\frac{n}{2}}Y_n\}_\varepsilon$ satisfies an LDP on $C_a([0,T]\times\mathbb{R})$ with speed ε^{-1} and the rate function I_N.*

Lemma 2.2.3-nw. *Fix $a\in\mathbb{R}$ and $\eta<T\in(0,\infty)$. There exists $C=C(T,a,\eta)$ such that, for all $\rho\in L^2([0,T]\times\mathbb{R})$ and $n\in\mathbb{Z}_{\geq 0}$, we have $\|\mathsf{Y}_n(\rho)\|_{a,\eta}\leq \frac{C^n}{\Gamma(n/2)^{1/2}}\|\rho\|_{L^2}^n$.*

Lemma 2.2.4-nw. *For any $\rho\in L^2([0,T]\times\mathbb{R})$ and $n\in\mathbb{Z}_{\geq 0}$, we have $(Y_n)_{\mathrm{hom}}(\rho)=\mathsf{Y}_n(\rho)$.*

2.3 Freidlin–Wentzell LDP for the SHE

2.3.1 Function-valued initial data

Throughout this subsection, we fix $T<\infty$, $a_*\in\mathbb{R}$, and $g_*\in C_{a_*^+}(\mathbb{R}) = \cap_{a>a_*}C_a(\mathbb{R})$, and let Z_ε denote the solution of (2.1.13) with the initial data g_*.

Recall from Proposition 2.2.1 that $Z_{N,\varepsilon} := \sum_{n=0}^{N}\varepsilon^{\frac{n}{2}}Y_n$ satisfies an LDP with the rate function

I_N given in (2.2.7). By Lemma 2.2.4, the function I_N can be expressed as

$$I_N(f) := (2.2.7) = \inf\Big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 \,:\, \rho \in L^2([0,T]\times\mathbb{R}),\ \sum_{n=0}^{N} \mathsf{Y}_n(\rho) = f\Big\}. \tag{2.3.1}$$

Recall that $\mathsf{Z}(\rho) = \sum_{n=0}^{\infty} \mathsf{Y}_n(\rho)$. Referring to the definition of I in (2.1.24), we see that formally taking $N \to \infty$ in (2.3.1) produces $I(f)$. The proof of Proposition 2.1.7 hence amounts to justifying this limit transition at the level of LDPs. Key to justifying such a limit transition is a tight enough bound on the tail probability $\mathbb{P}[\|Y_n\|_a \geq r]$, which we establish in Section 2.3.1.

Tail probability of $\|Y_n\|_a$

We will utilize the fact that, for any $(t,x) \in [0,T]\times\mathbb{R}$, the random variable $Y_n(t,x)$ belongs to the n-th $\mathbb{R}$-valued Wiener chaos. For X in the n-th $\mathbb{R}$-valued Wiener chaos, the hypercontractivity inequality asserts that higher moments of X are controlled by the second moments, c.f., [Nua06, Theorem 1.4.1],

$$\mathbb{E}\big[|X|^p\big] \leq p^{\frac{np}{2}}\big(\mathbb{E}\big[|X|^2\big]\big)^{\frac{p}{2}}, \quad \text{for all } p \geq 2. \tag{2.3.2}$$

We now use this inequality to produce a tail probability bound.

Lemma 2.3.1. *Let X be an $\mathbb{R}$-valued random variable in the n-th Wiener chaos and let $\sigma^2 := \mathbb{E}[X^2]$. There exists a universal constant $C \in (0,\infty)$ such that, for all $n \in \mathbb{Z}_{\geq 1}$ and $r \geq 0$,*

$$\mathbb{P}\big[|X| \geq r\big] \leq \exp\big(-\tfrac{n}{C}\sigma^{-\frac{2}{n}} r^{\frac{2}{n}} + n\big).$$

Proof. Assume without loss of generality $\sigma = 1$. We seek to bound $\mathbb{E}[\exp(\alpha|X|^{2/n})]$ for $\alpha > 0$. To this end, invoke Taylor expansion to get $\mathbb{E}[\exp(\alpha|X|^{2/n})] = \sum_{k=0}^{n} \frac{1}{k!}\alpha^k \mathbb{E}[|X|^{2k/n}] + \sum_{k=n+1}^{\infty} \frac{1}{k!}\alpha^k \mathbb{E}[|X|^{2k/n}]$. On the right hand side, use (2.3.2) to bound the moments for $k \geq n+1$. As for $k \leq n$, we simply bound $\mathbb{E}[|X|^{2k/n}] \leq (\mathbb{E}[|X|^2])^{k/n} = 1$. Combining these bounds gives $\mathbb{E}[\exp(\alpha|X|^{2/n})] \leq \sum_{k=0}^{n} \frac{1}{k!}\alpha^k + \sum_{k=n+1}^{\infty} \frac{1}{k!}\alpha^k (\frac{2k}{n})^k$. The first term on the right hand side is bounded by e^{α}. For the second term, using the inequality $k^k \le e^k k!$ gives $\sum_{k=n+1}^{\infty} \frac{1}{k!}\alpha^k(\frac{2k}{n})^k \le \sum_{k=n+1}^{\infty}(\frac{2e\alpha}{n})^k$. Combining these bounds and setting $\alpha = n/(4e)$ in the result gives $\mathbb{E}[\exp(\frac{n}{4e}|X|^{2/n})] \le e^{\frac{n}{4e}} + 2^{-n} \le e^n$. Now applying Markov's inequality completes the proof. □

In light of Lemma 2.3.1, bounding the tail probability of $Y_n(t,x)$ amounts to bounding its second moment, which we do next. Recall that T, $g_* \in C_{a_*^+}(\mathbb{R})$, and $a_* \in \mathbb{R}$ are fixed throughout this section.

Proposition 2.3.2. *Fix $a > a_*$, $\theta_1 \in (0,1)$, $\theta_2 \in (0,\frac{1}{2})$, and $n \in \mathbb{Z}_{\ge 1}$. There exists $C = C(T,a,\theta_1,\theta_2)$ such that for all $t,t' \in [0,T]$ and $x,x' \in \mathbb{R}$,*

(a) $\mathbb{E}\big[Y_n(t,x)^2\big] \le e^{2a|x|}\frac{C^n}{\Gamma(\frac{n}{2})}$,

(b) $\mathbb{E}\big[\big(Y_n(t,x) - Y_n(t,x')\big)^2\big] \le \frac{C^n}{\Gamma(\frac{n}{2})}(e^{2a|x|} \vee e^{2a|x'|})|x-x'|^{\theta_1}$, *and*

(c) $\mathbb{E}\big[\big(Y_n(t,x) - Y_n(t',x)\big)^2\big] \le \frac{C^n}{\Gamma(\frac{n}{2})}e^{2a|x|}|t-t'|^{\theta_2}$.

Proof. Fix $a > a_*$, $\theta_1 \in (0,1)$, $\theta_2 \in (0,\frac{1}{2})$, and $n \in \mathbb{Z}_{\ge 1}$. Throughout this proof we write $C = C(T,g_*,a,\theta_1,\theta_2)$.

(a) We begin by developing an iterative bound. It is readily verified from (2.2.4) that the chaos can be expressed as

$$Y_n(t,x) = \int_0^t \int_{\mathbb{R}} p(t-s,x-y)Y_{n-1}(s,y)\xi(s,y)\mathrm{d}s\mathrm{d}y. \tag{2.3.3}$$

Applying Itô's isometry gives $\mathbb{E}[Y_n(t,x)^2] = \int_0^t \int_{\mathbb{R}} p(t-s,x-y)^2\mathbb{E}[Y_{n-1}(s,y)^2]\mathrm{d}s\mathrm{d}y$. To streamline notation, set $F_n(s) := \sup_{x\in\mathbb{R}} e^{-2a|x|}\mathbb{E}[Y_n(s,x)^2]$. The last integral is bounded by $\int_0^t F_{n-1}(s)\int p(t-s,x-y)^2 e^{2a|y|}\mathrm{d}y$. Further using Lemma 2.2.2 (c) to bound the last integral gives $\mathbb{E}[Y_n(t,x)^2] \le C\int_0^t (t-s)^{-\frac{1}{2}}e^{2a|x|}F_{n-1}(s)\mathrm{d}s$. Multiplying both sides by $\exp(-2a|x|)$ and taking the supremum over x give

$$F_n(t) \le C\int_0^t (t-s)^{-\frac{1}{2}}F_{n-1}(s)\mathrm{d}s. \tag{2.3.4}$$

To utilize the iterative bound (2.3.4), we need to establish a bound on $F_0(t)$. By definition

$$F_0(t) := \sup_{x\in\mathbb{R}} \Big\{ e^{-2a|x|} \Big(\int p(t, x-y) g_*(y) \mathrm{d}y \Big)^2 \Big\}.$$

Note that $g_* \in C_{a_*^+}(\mathbb{R})$ implies $|g_*(y)| \leq Ce^{a|y|}$. Insert this bound into the definition of $F_0(t)$, and use Lemma 2.2.2 (b) to bound the resulting integral (over y). The result gives $|F_0(t)| \leq C$. Iterating (2.3.4) from $n = 1$ and using $|F_0(t)| \leq C$ give $F_n(t) \leq C^n (\Gamma(n/2))^{-1} t^n$, which concludes the desired result.

(b) Set $x = x$ and $x = x'$ in (2.3.3), take the difference of the result, and Apply Itô's isometry. We have

$$\mathbb{E}\big[\big(Y_n(t,x) - Y_n(t,x')\big)^2\big] = \int_0^t \int_{\mathbb{R}} \big(p(t-s, x-y) - p(t-s, x'-y)\big)^2 \mathbb{E}\big[Y_{n-1}(s,y)^2\big] \mathrm{d}s\mathrm{d}y. \quad (2.3.5)$$

Use Part (a) to bound $\mathbb{E}[Y_{n-1}(t,x)^2]$, and apply Lemma 2.2.2 (d) to bound the resulting integral. Doing so produces the desired result.

(c) Assume without loss of generality $t > t'$. Set $t = t$ and $t = t'$ in (2.3.3), take the difference, and apply Itô's isometry to the result. We have

$$\begin{aligned} \mathbb{E}\big[\big(Y_n(t,x) - Y_n(t',x)\big)^2\big] = &\int_0^{t'} \int_{\mathbb{R}} \big(p(t-s, x-y) - p(t'-s, x-y)\big)^2 \mathbb{E}\big[Y_{n-1}(s,y)^2\big] \mathrm{d}s\mathrm{d}y \\ &+ \int_{t'}^{t} \int_{\mathbb{R}} p(t-s, x-y) \mathbb{E}\big[Y_{n-1}(s,y)^2\big] \mathrm{d}s\mathrm{d}y. \end{aligned} \quad (2.3.6)$$

On the right hand side, use Part (a) to bound $\mathbb{E}[Y_{n-1}(s,y)^2]$, apply Lemma 2.2.2 (e) and Lemma 2.2.2 (c) to bound the resulting integrals, respectively. Doing so produces the desired result. □

Based on Lemmas 2.3.1 and Proposition 2.3.2, we now derive some pointwise Hölder bounds on Y_n.

Corollary 2.3.3. *Fix $a \in (a_*, \infty)$, $\alpha \in (0, \frac{1}{4})$, and $\beta \in (0, \frac{1}{2})$. There exists $C = C(T, a, \alpha, \beta)$ such* *that for all $n \in \mathbb{Z}_{\geq 1}$, $r \geq 0$, $t, t' \in [0,T]$, and $x, x' \in \mathbb{R}$,*

(a) $\mathbb{P}\Big[\,|Y_n(t,x) - Y_n(t,x')| \geq |x-x'|^{\beta}(e^{a|x|} \vee e^{a|x'|})r\Big] \leq \exp\big(-\tfrac{1}{C}n^{\frac{3}{2}}r^{\frac{2}{n}} + n\big)$*, and*

(b) $\mathbb{P}\Big[\,|Y_n(t',x) - Y_n(t,x)| \geq e^{a|x|}|t-t'|^{\alpha}r\Big] \leq \exp\big(-\tfrac{1}{C}n^{\frac{3}{2}}r^{\frac{2}{n}} + n\big)$.

Proof. Set $U := (e^{-a|x|} \wedge e^{-a|x'|})\frac{Y_n(t,x)-Y_n(t,x')}{|x-x'|^{\beta}}$, $V := (e^{-a|x|} \wedge e^{-a|x'|})\frac{Y_n(t,x)-Y_n(t',x)}{|t-t'|^{\alpha}}$, $\sigma^2 := \mathbb{E}[U^2]$, and $\eta^2 := \mathbb{E}[V^2]$. Proposition 2.3.2 (b) and (c) give $\sigma^2 \leq C^n/\Gamma(\frac{n}{2})$ and $\eta^2 \leq C^n/\Gamma(\frac{n}{2})$. Taking $\frac{1}{n}$ power on both sides and using $\Gamma(\frac{n}{2})^{-1/n} \leq Cn^{-1/2}$, we have $\sigma^{\frac{2}{n}} \leq Cn^{-1/2}$ and $\eta^{\frac{2}{n}} \leq Cn^{-1/2}$. Next, since $Y_n(t,x)$, $Y_n(t,x')$, $Y_n(t',x)$, and $Y_n(t',x')$ belong to the n-th $\mathbb{R}$-valued Wiener chaos, U and V also belong to the n-th Wiener chaos. The desired results now follow from Lemma 2.3.1. □

Our next step is to leverage the pointwise bounds in Corollary 2.3.3 to a functional bound. To this end it is convenient to first work with Hölder seminorms. For $f \in C([0,T] \times \mathbb{R})$ and $k \in \mathbb{Z}$, set

$$[f]_{a,\alpha,\beta,k} := e^{-a|k|} \sup\bigg\{\frac{|f(t_1,x_1) - f(t_2,x_2)|}{|t_1-t_2|^{\alpha} + |x_1-x_2|^{\beta}} \;:\; (t_1,x_1) \neq (t_2,x_2) \in [0,T] \times [k,k+1]\bigg\}. \tag{2.3.7}$$

This quantity measures the Hölder continuity of f on $[0,T] \times [k,k+1]$.

Proposition 2.3.4. *Fix $a \in (a_*, \infty)$, $\alpha \in (0, \frac{1}{4})$, and $\beta \in (0, \frac{1}{2})$. There exists $C = C(T, a, \alpha, \beta)$ such that, for all $r \geq (Cn^{-\frac{1}{2}})^{\frac{n}{2}}$, $n \in \mathbb{Z}_{\geq 1}$, and $k \in \mathbb{Z}$,*

$$\mathbb{P}\big[\,[Y_n]_{a,\alpha,\beta,k} \geq r\big] \leq C\, \exp\big(-\tfrac{1}{C}n^{\frac{3}{2}}r^{\frac{2}{n}}\big).$$

Proof. Throughout this proof we write $C = C(T, a_*, a, \alpha, \beta)$.

The proof follows similar argument in the proof of Kolmogorov's continuity theorem. The starting point is an inductive partition of $[0,T] \times [k,k+1]$ into nested rectangles. Let $\tau_0 := T$ and $\zeta_0 := 1$ denote the side lengths of $R^{(0)}_{11} := [0,T] \times [k,k+1]$. We proceed by induction in

$\ell = 0, 1, 2, \ldots$. Assume, for $\ell \geq 0$, we have obtained the rectangles $R^{(\ell)}_{ij}$, for $i = 1, \ldots, \prod_{\ell'=1}^{\ell-1} m_{\ell'}$ and $j = 1, \ldots, \prod_{\ell'=1}^{\ell-1} n_{\ell'}$. We partition each $R^{(\ell)}_{ij}$ into $m_\ell \times n_\ell$ rectangles of equal size. The side lengths of the resulting rectangles are therefore $\tau_{\ell+1} = \tau_\ell / m_\ell$ and $\zeta_{\ell+1} = \zeta_\ell / n_\ell$. The numbers m_ℓ and n_ℓ are chosen in such a way that

$$\tfrac{1}{2} \leq \tau_\ell^\alpha / \zeta_\ell^\beta \leq 2, \quad \text{for } \ell = 1, 2, \ldots, \tag{2.3.8}$$

$$2 \leq m_\ell, n_\ell \leq C, \quad \text{for } \ell = 0, 1, 2, \ldots. \tag{2.3.9}$$

Let $\mathcal{V}_\ell := \{(i\tau_\ell, k + j\zeta_\ell) : i = 1, \ldots, \prod_{\ell'=1}^{\ell-1} m_{\ell'}, j = 1, \ldots, \prod_{\ell'=1}^{\ell-1} n_{\ell'}\}$ denote the set of the vertices at the ℓ-th level, and let $\mathcal{E}_\ell$ denote the corresponding set of edges.

For $(t_1, x_1) \neq (t_2, x_2) \in [0, T] \times [k, k+1]$, let

$$\ell_* = \ell_*(t_1, x_1, t_2, x_2) := \min\{\ell \in \mathbb{Z}_{\geq 0} : |t_1 - t_2| \geq \tau_\ell \text{ or } |x_1 - x_2| \geq \zeta_\ell\}. \tag{2.3.10}$$

It is standard to show that, for any $f \in C([0, T] \times \mathbb{R})$,

$$|f(t_1, x_1) - f(t_2, x_2)| \leq C \sum_{\ell \geq \ell_*} \max_{\mathbf{e} \in \mathcal{E}_\ell} |f(\partial \mathbf{e})|. \tag{2.3.11}$$

Here $|f(\partial \mathbf{e})| := |f(s_1, y_1) - f(s_2, y_2)|$, where (s_1, y_1) and (s_2, y_2) are the two ends of the edge $\mathbf{e} \in \mathcal{E}_\ell$.

Below we will apply (2.3.11) for $f = e^{-a|k|} Y_n$. To prepare for this application let us first derive a bound on

$$\sum_{\ell_0 \geq 0} \mathbb{P}\Big[\sum_{\ell \geq \ell_0} \max_{\mathbf{e} \in \mathcal{E}_\ell} e^{-a|k|} |Y_n(\partial \mathbf{e})| \geq (\tau_{\ell_0}^\alpha + \zeta_{\ell_0}^\beta) r \Big]. \tag{2.3.12}$$

Set $\delta := (\frac{1}{2}(\frac{1}{4} - \alpha)) \wedge (\frac{1}{2}(\frac{1}{2} - \beta))$. Fix any edge $\mathbf{e} \in \mathcal{E}_\ell$. If $\mathbf{e}$ is in the t direction, apply Corollary 2.3.3(b) with $\{(t, x), (t', x)\} = \partial \mathbf{e}$, $\alpha \mapsto \alpha + \delta$, and $r \mapsto \tau_\ell^{-\delta} r$. If $\mathbf{e}$ is in the x direction, apply

Corollary 2.3.3(a) with $\{(t,x),(t,x')\} = \partial\mathbf{e}$, $\beta \mapsto \beta+\delta$, and $r \mapsto \zeta_\ell^{-\delta} r$. The result gives

$$\mathbb{P}\big[e^{-a|k|-|a|}|Y_n(\partial\mathbf{e})| \geq \tau_\ell^\alpha r\big] \leq \exp\big(-\tfrac{1}{C}n^{\frac{3}{2}}\tau_\ell^{-\delta} r^{\frac{2}{n}} + n\big), \quad \text{if } \mathbf{e} \text{ is in the } t \text{ direction}, \tag{2.3.13}$$

$$\mathbb{P}\big[e^{-a|k|-|a|}|Y_n(\partial\mathbf{e})| \geq \zeta_\ell^\beta r\big] \leq \exp\big(-\tfrac{1}{C}n^{\frac{3}{2}}\zeta_\ell^{-\delta} r^{\frac{2}{n}} + n\big), \quad \text{if } \mathbf{e} \text{ is in the } x \text{ direction}. \tag{2.3.14}$$

On the right hand sides of (2.3.13)–(2.3.14), use $m_\ell, n_\ell \geq 2$ to bound $\tau_\ell^{-\delta} \geq e^{\frac{\ell}{C}}$ and $\zeta_\ell^{-\delta} \geq e^{-\frac{\ell}{C}}$. Take the union bound of the result over $\mathbf{e} \in \mathcal{E}_\ell$. The condition $m_\ell, n_\ell \leq C$ gives $|\mathcal{E}_\ell| \leq C^\ell$. Hence

$$\mathbb{P}\Big[\max_{\mathbf{e}\in\mathcal{E}_\ell} e^{-a|k|}|Y_n(\partial\mathbf{e})| \geq e^{|a|}(\tau_\ell^\alpha + \zeta_\ell^\beta) r\Big] \leq C^\ell \exp\big(-\tfrac{1}{C}e^{\frac{\ell}{Cn}} n^{\frac{3}{2}} r^{\frac{2}{n}} + n\big). \tag{2.3.15}$$

Next, the condition $m_\ell, n_\ell \geq 2$ implies $\tau_\ell \leq \tau_{\ell_0} 2^{-\ell+\ell_0}$ and $\zeta_\ell \leq \zeta_{\ell_0} 2^{-\ell+\ell_0}$, and therefore $\sum_{\ell\geq\ell_0}(\tau_\ell^\alpha + \zeta_\ell^\beta) r \leq C(\tau_{\ell_0}^\alpha + \zeta_{\ell_0}^\beta) r$. Use this inequality to take the union bound of (2.3.15) over $\ell \geq \ell_0$ and absorb $e^{|a|}$ into C. We have

$$\mathbb{P}\Big[\sum_{\ell\geq\ell_0}\max_{\mathbf{e}\in\mathcal{E}_\ell} e^{-a|k|}|Y_n(\partial\mathbf{e})| \geq (\tau_{\ell_0}^\alpha + \zeta_{\ell_0}^\beta) C r\Big] \leq \sum_{\ell\geq\ell_0} C^\ell \exp\big(-\tfrac{1}{C}e^{\frac{\ell}{Cn}} n^{\frac{3}{2}} r^{\frac{2}{n}} + n\big).$$

Use $e^{\frac{\ell}{Cn}} \geq 1 + \frac{\ell}{Cn}$ on the right hand side, sum both sides over $\ell_0 \in \mathbb{Z}_{\geq 0}$, and rename $Cr \mapsto r$. Doing so gives
(2.3.12) $\leq \exp(-\frac{1}{C}n^{\frac{3}{2}} r^{\frac{2}{n}}) \sum_{\ell_0\geq 0}\sum_{\ell\geq\ell_0} \exp(-\frac{\ell}{C}n^{\frac{1}{2}} r^{\frac{2}{n}} + n + \ell C)$. For all $r \geq (C_0 n^{-\frac{1}{2}})^{\frac{n}{2}}$ and C_0 sufficiently large, the last double sum is convergent and bounded. Hence

$$\text{(2.3.12)} \leq C\exp\big(-\tfrac{1}{C}n^{\frac{3}{2}} r^{\frac{2}{n}}\big), \quad \text{for all } r \geq (Cn^{-\frac{1}{2}})^{\frac{n}{2}}. \tag{2.3.16}$$

Now, set $f = e^{-a|k|}Y_n$ in (2.3.11) and use (2.3.16). We have that, for any $r \geq (Cn^{-\frac{1}{2}})^{\frac{n}{2}}$,

$$e^{-a|k|}|Y_n(t_1,x_1) - Y_n(t_2,x_2)| \leq C\,(\tau_{\ell_*}^\alpha + \zeta_{\ell_*}^\beta) r, \quad \forall (t_1,x_1),(t_2,x_2) \in [0,T]\times[k,k+1] \tag{2.3.17}$$

holds with probability $\geq 1 - C\exp(-\frac{1}{C}n^{\frac{3}{2}} r^{\frac{2}{n}})$. Referring to the definition of ℓ_* in (2.3.10), we see that either $|t_1 - t_2| \geq \tau_{\ell_*}$ or $|x_1 - x_2| \geq \zeta_{\ell_*}$ holds. Combining this fact with the condition (2.3.8) gives $\frac{\tau_{\ell_*}^\alpha + \zeta_{\ell_*}^\beta}{|t_1-t_2|^\alpha + |x_1-x_2|^\beta} \leq 3$. Divide both sides of (2.3.17) by $|t_1 - t_2|^\alpha + |x_1 - x_2|^\beta$, use the last inequality on the right hand side, take supremum of over $(t_1, x_1) \neq (t_2, x_2) \in [0, T] \times [k, k+1]$ in the result, and rename $3Cr \mapsto r$. Doing so concludes the desired result. □

We now state and prove a bound on $\mathbb{P}[\, \|Y_n\|_a \geq r]$.

Proposition 2.3.5. *Fix $a > a_*$. There exists $C = C(T, a)$ such that, for all $r \geq (Cn^{-\frac{1}{2}})^{\frac{n}{2}}$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$\mathbb{P}\big[\, \|Y_n\|_a \geq r \big] \leq C \, \exp\big(-\tfrac{1}{C} n^{\frac{3}{2}} r^{\frac{2}{n}} \big).$$

Proof. Throughout this proof we write $C = C(T, a)$.

For $n = 0$, note that $Y_0(t, x) = \int_{\mathbb{R}} p(t, x - y) g_*(y) \, \mathrm{d}y$ is deterministic. It is straightforward to check from Lemma 2.2.2(b) and $g_* \in C_{a_*^+}(\mathbb{R})$ that $\|Y_0\|_a < \infty$. Let $b := (a + a_*)/2$. For $n \geq 1$, note from (2.2.4) that $Y_n(0, 0) = 0$. Given this property, from the definitions (2.1.21) and (2.3.7) of $\|\cdot\|_a$ and $[\cdot]_{a,\alpha,\beta,k}$ it is straightforward to check

$$\|Y_n\|_a \leq C \sum_{k \in \mathbb{Z}} [Y_n]_{a, \frac{1}{8}, \frac{1}{4}, k} \leq C \sum_{k \in \mathbb{Z}} [Y_n]_{b, \frac{1}{8}, \frac{1}{4}, k} \, e^{-\frac{1}{2}(a - a_*)|k|}.$$

Apply Proposition 2.3.4 with $r \mapsto e^{\frac{1}{2}(a-a_*)|k|} r$ and $(a, \alpha, \beta) \mapsto (b, \frac{1}{8}, \frac{1}{4})$, and take the union bound of the result over $k \in \mathbb{Z}$. We have $\mathbb{P}[\, \|Y_n\|_a \geq Cr] \leq \sum_{k \in \mathbb{Z}} C \, \exp(-\frac{1}{C} n^{\frac{3}{2}} e^{\frac{|k|}{Cn}} r^{\frac{2}{n}})$. Within the last expression, use $e^{\frac{|k|}{Cn}} \geq 1 + \frac{|k|}{Cn}$, sum the result over $k \in \mathbb{Z}$, and rename $Cr \mapsto r$ in the result. Doing so concludes the desired result. □

Proposition 2.3.5 immediately implies

Corollary 2.3.6. *Fix $a > a_*$. We have $\mathbb{E}[\, \|Y_n\|_a^k] < \infty$ for all $k, n \in \mathbb{Z}_{\geq 0}$, and $\mathbb{P}[\sum_{n=0}^\infty \|Y_n\|_a < \infty] = 1$.*

Proof of Proposition 2.1.7 (a)

Recall I from (2.1.24). We begin by show that this function is a good rate function.

Lemma 2.3.7. *For any $a > a_*$, the function $I : C_a([0,T]\times\mathbb{R}) \to \mathbb{R}\cup\{+\infty\}$ is a good rate function.*

Proof. Throughout this proof we write $\mathcal{H} = L^2([0,T]\times\mathbb{R})$ and $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_{L^2}$. Recall that $\mathcal{H} \subset \mathcal{B}$ is the Cameron–Martin subspace of $\mathcal{B}$.

We begin with a reduction. It is well-known that under μ, the random vector $\sqrt{\varepsilon}\xi$ satisfies an LDP on $\mathcal{B}$ with speed ε^{-1} and the good rate function $I_* : \mathcal{B} \to \mathbb{R}\cup\{+\infty\}$ given by $I_*(\rho) := \frac{1}{2}\|\rho\|^2_{\mathcal{H}}$ for $\rho \in \mathcal{H}$ and $I_*(\rho) := +\infty$ for $\rho \notin \mathcal{H}$, c.f. [Led96, Chapter 4]. Recall that Z maps $\mathcal{H}$ to $C_a([0,T]\times\mathbb{R})$. We extend the domain of this map to $\mathcal{B}$ by setting the function be 0 outside $\mathcal{H}$, i.e.,

$$\widetilde{\mathsf{Z}} : \mathcal{B} \to C_a([0,T]\times\mathbb{R}), \quad \widetilde{\mathsf{Z}}(\zeta) := \begin{cases} \mathsf{Z}(\zeta), & \text{when } \zeta \in \mathcal{H}, \\ 0 & \text{, otherwise.} \end{cases}$$

Referring to (2.1.24), we see that I is a pullback of I_* via $\widetilde{\mathsf{Z}}$. Let $\Omega(r) := \{\zeta \in \mathcal{B} : I_*(\zeta) \leq r\}$ denote a sub-level set of I_*. By [DS01, Lemma 2.1.4], to prove I is a good rate function, it suffices to construct a sequence of continuous functions $\varphi_N : \mathcal{B} \to C_a([0,T]\times\mathbb{R})$ such that for all $r < \infty$,

$$\lim_{N\to\infty} \sup_{\zeta\in\Omega(r)} \|\widetilde{\mathsf{Z}}(\zeta) - \varphi_N(\zeta)\|_a = 0. \tag{2.3.18'}$$

Since $I_*(\zeta) < \infty$ only when $\zeta \in \mathcal{H}$, we have $\Omega(r) = \{\rho \in \mathcal{H} : \|\rho\|^2_{\mathcal{H}} \leq 2r\}$, and (2.3.18') reduces to

$$\lim_{N\to\infty} \sup_{\zeta\in\Omega(r)} \|\mathsf{Z}(\rho) - \varphi_N(\rho)\|_a = 0. \tag{2.3.18}$$

We will construct the φ_N via truncation. First, combining (2.2.9) and Lemma 2.2.4 gives, for

$\rho \in \mathcal{H}$,

$$\mathsf{Z}(\rho) = \sum_{n=0}^{\infty} \mathsf{Y}_n(\rho) = \sum_{n=0}^{N} (Y_n)_{\mathrm{hom}}(\rho) + \sum_{n>N} \mathsf{Y}_n(\rho). \tag{2.3.19}$$

The $n > N$ terms in (2.3.19) can be bounded by Lemma 2.2.3.

Focusing on the $n \leq N$ terms in (2.3.19), we seek to approximate each $(Y_n)_{\mathrm{hom}}(\rho)$ by a continuous function. To this end we follow the argument in [HW15b, Section 3]. Recall the notation $W(f)$ from (2.2.2) and recall the orthonormal basis $\{e_1, e_2, \ldots\} \subset \mathcal{H}$ from Section 2.2.1. Regarding $W(e_i) : \mathcal{B} \to \mathbb{R}$ as a random variable, we let $\mathcal{F}_k$ be the sigma algebra generated by $W(e_1), \ldots, W(e_k)$, and set $\Psi_{n,k} := \mathbb{E}[Y_n | \mathcal{F}_k]$. Given that Y_n belongs to the n-th E-valued Wiener chaos (recall that $E = C_a([0,T] \times \mathbb{R})$), it is standard to check:

(i) $\lim_{k\to\infty} \mathbb{E}[\|Y_n - \Psi_{n,k}\|_a^2] = 0$,

(ii) $\Psi_{n,k}$ can be expressed as a finite sum of the form $\Psi_{n,k} = \sum y_\alpha \prod_{i=1}^{k} W(e_i)^{\alpha_i}$, where $y_\alpha \in C_a([0,T] \times \mathbb{R})$ and $\alpha = (\alpha_1, \alpha_2, \ldots) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \ldots$.

Now consider the function $(\Psi_{n,k})_{\mathrm{hom}} : \mathcal{B} \to C_a([0,T] \times \mathbb{R})$ defined by $(\Psi_{n,k})_{\mathrm{hom}}(\zeta) := \int_{\mathcal{B}} \Psi_{n,k}(\xi + \zeta)\mu(\mathrm{d}\xi)$. A priori, such an integral is guaranteed to be well-defined only for $\zeta \in \mathcal{H}$. Yet for the special case considered here, the integral is well-defined for all $\zeta \in \mathcal{B}$ and the result gives a continuous function $\mathcal{B} \to C_a([0,T] \times \mathbb{R})$. To see why, recall the definition of $\mathcal{B}$ from (2.2.1), and for $\zeta \in \mathcal{B}$ write $\zeta = \sum_{i\geq 1} \zeta_i e_i$. From (ii) we have $\int_{\mathcal{B}} \Psi_{n,k}(\xi + \zeta)\mu(\mathrm{d}\xi) = \sum y_\alpha \prod_{i=1}^{k} \mathbb{E}[(\zeta_i + \Xi_i)^{\alpha_i}]$, where $\Xi_1, \Xi_2, \ldots$ are independent standard $\mathbb{R}$-valued Gaussian random variables, and the sum is *finite*. From the last expression we see that the integral is well-defined and gives a continuous function $\mathcal{B} \to C_a([0,T] \times \mathbb{R})$. Next, for $\rho \in \mathcal{H}$, by the Cameron–Martin theorem, we have $\|(Y_n)_{\mathrm{hom}}(\rho) - (\Psi_{n,k})_{\mathrm{hom}}(\rho)\|_a = \| \int_{\mathcal{B}} \exp\big(W(\rho) - \frac{1}{2}\|\rho\|_{\mathcal{H}}^2\big)\big(Y_n(\xi) - \Psi_{n,k}(\xi)\big)\mu(\mathrm{d}\xi)\|_a$. Applying the Cauchy–Schwarz inequality to the last expression gives

$$\|(Y_n)_{\mathrm{hom}}(\rho) - (\Psi_{n,k})_{\mathrm{hom}}(\rho)\|_a^2 \leq \exp\big(\tfrac{1}{2}\|\rho\|_{\mathcal{H}}^2\big)\mathbb{E}\big[\|Y_n - \Psi_{n,k}\|_a^2\big]. \tag{2.3.20}$$

The right hand side converges to zero as $k \to \infty$ by (i). We have obtained an approximate of $(Y_n)_{\text{hom}}$ by the continuous function $(\Psi_{n,k})_{\text{hom}}$.

We now construct φ_N. For fixed N, invoke (i) to obtain $k_n \in \mathbb{Z}_{\geq 1}$ such that $\mathbb{E}[\|Y_n - \Psi_{n,k_n}\|_a^2] \leq (N+1)^{-2}$. Set $\varphi_N := \sum_{n=0}^N \Psi_{n,k_n}$. This is a continuous function $\mathcal{B} \to C_a([0,T] \times \mathbb{R})$ since each $\Psi_{n,k}$ is. Subtract φ_N from both sides of (2.3.19), take $\|\bullet\|_a$ on both sides, and use (2.3.20), $\mathbb{E}[\|Y_n - \Psi_{n,k_n}\|_a^2] \leq (N+1)^{-2}$, and Lemma 2.2.3 to bound the result. We have, for all $\rho \in \mathcal{H}$,

$$\| \mathsf{Z}(\rho) - \varphi_N(\rho)\|_a \leq \exp\big(\tfrac{1}{4}\|\rho\|_{\mathcal{H}}^2\big)(N+1)^{-1} + \sum_{n \geq N} \frac{1}{\Gamma(n/2)^{\frac{1}{2}}}\big(C(a,T)\,\|\rho\|_{\mathcal{H}}\big)^n.$$

Now consider $\rho \in \Omega(2r)$, whence $\|\rho\|_{\mathcal{H}}^2 \leq 2r$. We see that the desired property (2.3.18) follows.

□

Recall that $Z_{N,\varepsilon} := \sum_{n=0}^N \varepsilon^{n/2} Y_n$. Next we show that $Z_{N,\varepsilon}$ is an exponentially good approximation of Z_ε.

Proposition 2.3.8. *For any $r > 0$ and $a > a_*$, we have* $\lim_{N\to\infty} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\big[\|Z_{N,\varepsilon} - Z_\varepsilon\|_a \geq r\big] = -\infty$.

Proof. By definition, $Z_\varepsilon - Z_{N,\varepsilon} = \sum_{n>N} \varepsilon^{\frac{n}{2}} Y_n$. Fix arbitrary $N \in \mathbb{Z}_{\geq 1}$ and $r > 0$. We seek to apply Proposition 2.3.5 with $r \mapsto 2^{N-n}\varepsilon^{-n/2} r$ and $n > N$. For fixed N, r, the required condition $2^{N-n}\varepsilon^{-n/2} r \geq (Cn^{-1/2})^{n/2}$ is satisfied for all $n > N$ as long as ε is small enough. Summing the result over $N > n$ and applying the union bound gives

$$\mathbb{P}\big[\|Z_\varepsilon - Z_{N,\varepsilon}\|_a \geq r\big] \leq \sum_{n>N} \mathbb{P}\big[\|Y_n\|_a \geq 2^{N-n}\varepsilon^{-\frac{n}{2}} r\big] \leq C \sum_{n>N} \exp\big(-\tfrac{1}{C}\varepsilon^{-1} n^{\frac{3}{2}} e^{\frac{N-n}{Cn}}\big),$$

where $C = C(T,a,r)$. On the right hand side, use $e^{\frac{N-n}{Cn}} \geq 1 - \frac{N-n}{Cn}$ (which holds since $n > N$), sum the result. On both sides of the result, apply $\varepsilon \log(\,\bullet\,)$, and take the limits $\varepsilon \to 0$ and $N \to \infty$ in order. Doing so concludes the desired result. □

We seek to apply [DZ94, Theorem 4.2.16 (b)]. Doing so requires establishing a few properties of the rate functions. Let $B_r(f) := \{f' \in C_a([0,T]\times\mathbb{R}) : \|f'-f\|_a < r\}$ denote the open ball of radius r around f. Recall I from (2.1.24) and recall I_N from (2.3.1).

Lemma 2.3.9.

(a) For any closed $F \subset C_a([0,T]\times\mathbb{R})$, we have $\inf_{f\in F} I(f) \le \liminf_{N\to\infty} \inf_{f\in F} I_N(f)$.

(b) For any $f_0 \in C_a([0,T]\times\mathbb{R})$, we have $I(f_0) = \lim_{r\to 0}\liminf_{N\to\infty} \inf_{f\in B_r(f_0)} I_N(f)$.

Proof. (a) Let A denote the right hand side and assume without loss of generality $A < \infty$. Referring to the definition of I_N in (2.3.1), we let $\{(N_k,\rho_k)\}_{k=1}^{\infty} \subset \mathbb{Z}_{\ge 1}\times L^2([0,T]\times\mathbb{R})$ be such that $N_1 < N_2 < \ldots \to \infty$, $\|\rho_k\|_{L^2} \le A + \frac{1}{k}$, and $\sum_{n=0}^{N_k} \mathsf{Y}_n(\rho_k) =: f_k \in F$. Our next step is to relate (ρ_k, f_k) to I. Recall that $\mathsf{Z}(\rho) = \sum_{n=0}^{\infty}\mathsf{Y}_n(\rho)$. Letting $f'_k := f_k + \sum_{n>N_k}\mathsf{Y}_n(\rho_k) \in C_a([0,T]\times\mathbb{R})$, we have $\mathsf{Z}(\rho_k) = \widetilde{f_k}$. Referring to the definition of I in (2.1.24), we see that $I(f'_k) \le \frac{1}{2}\|\rho_k\|_{L^2} \le A + \frac{1}{k}$. Also, $\|f'_k - f_k\|_a \le \sum_{n>N_k}\|\mathsf{Y}_n(\rho_k)\|_a$. Using Lemma 2.2.3 and $\|\rho_k\|_{L^2} \le A+1$ to bound the last expression gives

$$\lim_{k\to\infty}\|f'_k - f_k\|_a = 0. \tag{2.3.21}$$

By Lemma 2.3.7, the sequence $\{f'_k\}_{k=1}^{\infty}$ is contained in a compact set. Hence, after passing to a subsequence we have $f'_k \to f_*$ in $C_a([0,T]\times\mathbb{R})$. The condition (2.3.21) remains true after passing to the subsequence. Since $f_k \in F$ and F is closed, we have $f_* \in F$. By Lemma 2.3.7, I is lower semi-continuous, whereby $I(f_*) \le \liminf_k I(f'_k)$. Lower bound the left hand side by $\inf_{f\in F} I(f)$ and upper bound the right hand side by $\liminf_k (A + \frac{1}{k}) = A$. We conclude the desired result.

(b) Apply Part (a) with $F = \overline{B_r(f_0)}$ and use the lower semicontinuity of I on the left hand side of the result. Doing so gives the inequality $\le$ for the desired result. It hence suffices to show the reverse inequality $\ge$. To this end, we assume without loss of generality $I(f_0) < \infty$, and let $\{\widetilde{\rho}_k\}_{k=1}^{\infty} \subset L^2([0,T]\times\mathbb{R})$ be such that $\|\widetilde{\rho}_k\|_{L^2} \le I(f_0) + \frac{1}{k}$ and that $\mathsf{Z}(\rho_k) = \sum_{n=0}^{\infty}\mathsf{Y}_n(\widetilde{\rho}_k) = f_0$. Let $\widetilde{f_k} := \sum_{n=0}^{n}\mathsf{Z}(\rho_k)$. Referring to the definition of I_N in (2.3.1), we see that $I_N(\widetilde{f_k}) \le \frac{1}{2}\|\rho_k\|_{L^2} \le I(f_0) + \frac{1}{k}$. Also, using Lemma 2.2.3 and $\|\rho_k\|_{L^2} \le I(f_0)+1$ gives $\lim_{k\to\infty}\|f_0 - \widetilde{f_k}\|_a = 0$. This statement implies that, for any given $r > 0$ and for all k large enough (depending on r), we have $\widetilde{f_k} \in B_r(f_0)$. From this and $I_N(\widetilde{f_k}) \le I(f_0) + \frac{1}{k}$ the desired result follows. □

We are now ready to complete the proof of Proposition 2.1.7 (a). The LDP for $\{Z_{N,\varepsilon}\}_\varepsilon$ is established in Proposition 2.2.1 with the rate function I_N. Given this, we apply [DZ94, Theorem 4.2.16 (b)] to go from the large deviations of $\{Z_{N,\varepsilon}\}_\varepsilon$ to that of $\{Z_\varepsilon\}_\varepsilon$. This theorem asserts that $\{Z_\varepsilon\}_\varepsilon$ satisfies an LDP with the rate function I contingent upon the following conditions.

1. I is a good rate function,
2. $\{Z_{N,\varepsilon}\}_\varepsilon$ is an exponentially good approximation (defined in [DZ94, Definition 4.2.14]) of $\{Z_\varepsilon\}_\varepsilon$,
3. $I(f_0) = \sup_{r>0} \liminf_{N\to\infty} \inf_{f\in B_r(f_0)} I_N(f)$, and
4. $\inf_{f\in F} I(f) \le \limsup_{N\to\infty} \inf_{f\in F} I_m(f)$, for every closed set $F \subset C_a([0,T]\times\mathbb{R})$.

These conditions are verified by Lemma 2.3.7, Proposition 2.3.8, Lemma 2.3.9 (b), and Lemma 2.3.9 (a), respectively. Applying [DZ94, Theorem 4.2.16 (b)] completes the proof of Proposition 2.1.7 (a).

2.3.2 The narrow wedge initial data, Proof of Proposition 2.1.7 (b)

Throughout this subsection, we fix $0 < \eta < T < \infty$, $a \in \mathbb{R}$, and let Z_ε denote the solution of (2.1.13) with the initial data $Z_\varepsilon(0,\cdot) = \delta_0(\cdot)$.

The proof of Proposition 2.1.7 (b) parallels that of Proposition 2.1.7 (a), starting with the analog of Proposition 2.3.2-nw:

Proposition 2.3.2-nw. *Fix $\theta_1 \in (0, \frac{1}{2})$, $\theta_2 \in (0,1)$, and $n \in \mathbb{Z}_{\ge 1}$. There exists $C = C(T,\eta,a,\theta_1,\theta_2)$ such that for all $t,t' \in [\eta,T]$ and $x,x' \in \mathbb{R}$,*

(a) $\mathbb{E}\big[\big(Y_n(t,x) - Y_n(t,x')\big)^2\big] \le \frac{C^n}{\Gamma(\frac{n}{2})}(e^{2a|x|} \vee e^{2a|x'|})|x-x'|^{\theta_2}$, *and*

(b) $\mathbb{E}\big[\big(Y_n(t,x) - Y_n(t',x)\big)^2\big] \le \frac{C^n}{\Gamma(\frac{n}{2})} e^{2a|x|}|t-t'|^{\theta_1}$.

Proof. Throughout this proof we write $C = C(T, \eta, a, \theta_1, \theta_2)$.

(a) By [Cor18, Lemma 2.4], we have

$$\mathbb{E}[Y_n(t,x)^2] = t^{\frac{n}{2}} 2^{-n} \Gamma(\tfrac{n}{2})^{-1} p(t,x)^2. \tag{2.3.22}$$

The identity (2.3.5) continues to hold here. Inserting (2.3.22) into the right hand side of (2.3.5) gives

$$\mathbb{E}\big[(Y_n(t,x) - Y_n(t,x'))^2\big] \le \frac{C^n}{\Gamma(\frac{n}{2})} \int_0^t \int_{\mathbb{R}} \big(p(t-s,x-y) - p(t-s,x'-y)\big)^2 p(s,y)^2 \mathrm{d}y\mathrm{d}s.$$

On the right hand side, divide the integral into two parts for $s > \eta/2$ and for $s < \eta/2$. For the former use Lemma 2.2.2 (a) to bound $p(s,y)^2 \le Ce^{2a|y|}$ (note that $s > \eta/2$) and use Lemma 2.2.2 (d) to bound the remaining integral; for the latter use Lemma 2.2.2 (i) to bound $(p(t-s,x-y) - p(t-s,x'-y))^2 \le C|x-x'|^{\theta_2}(e^{2a|x-y|} \vee a^{2a|x'-y|})$ (note that $t - s \ge \eta/2$) and use Lemma 2.2.2 (c) to bound the remaining integral. Doing so concludes the desired result.

(b) The identity (2.3.6) continues to hold here. Inserting (2.3.22) into the right hand side of (2.3.6) gives

$$\mathbb{E}\big[(Y_n(t,x) - Y_n(t',x))^2\big] \le \frac{C^n}{\Gamma(\frac{n}{2})}\Big(\int_0^{t'} \int_{\mathbb{R}} \big(p(t-s,x-y) - p(t'-s,x-y)\big)^2 p(s,y)^2 \mathrm{d}y\mathrm{d}s \tag{2.3.23}$$

$$+ \int_{t'}^{t} \int_{\mathbb{R}} p(t-s,x-y)^2 p(s,y)^2 \mathrm{d}y\mathrm{d}s\Big). \tag{2.3.24}$$

On the right hand side of (2.3.23), divide the integral into two parts for $s > \eta/2$ and for $s < \eta/2$. For the former use Lemma 2.2.2 (a) to bound $p(s,y)^2 \le Ce^{2a|y|}$ (note that $s > \eta/2$) and use Lemma 2.2.2 (e) to bound the remaining integral; for the latter use Lemma 2.2.2 (ii) to bound $(p(t-s,x-y) - p(t'-s,x-y))^2 \le C|t'-t|^{\theta_1} e^{2a|x-y|}$ (note that $t' - s \ge \eta/2$) and use Lemma 2.2.2 (c) to bound the remaining integral. The integral in (2.3.24) can be evaluated to be $\int_{t'}^{t} 4^{-1}\pi^{-3/2} t^{-1/2} s^{-1/2} (t-s)^{-1/2} \exp(-\frac{x^2}{2t})\mathrm{d}s$. Using $s, t \ge \eta$ to bound the last integral gives

(2.3.24) $\leq C|t-t'|^{1/2}e^{2a|x|} \leq C|t-t'|^{\theta_1}e^{2a|x|}$. From the preceding bounds we conclude the desired result. □

Given Proposition 2.3.2-nw, a similar proof of Proposition 2.3.5-nw adapted to the current setting yields

Proposition 2.3.5-nw. *There exists $C = C(T,\eta,a)$ such that, for all $r \geq (Cn^{-\frac{1}{2}})^{\frac{n}{2}}$ and $n \in \mathbb{Z}_{\geq 0}$,*

$$\mathbb{P}\big[\, \|Y_n\|_{a,\eta} \geq r \big] \leq C \, \exp\big(-\tfrac{1}{C}n^{\frac{3}{2}}r^{\frac{2}{n}}\big).$$

Corollary 2.3.6-nw. *We have $\mathbb{E}[\, \|Y_n\|_{a,\eta}^k] < \infty$ for all $k, n \in \mathbb{Z}_{\geq 0}$, and $\mathbb{P}[\sum_{n=0}^{\infty} \|Y_n\|_{a,\eta} < \infty] = 1$.*

Given Proposition 2.3.5-nw, the rest of the proof for Proposition 2.1.7 (b) follows the arguments in Sections 2.3.1 mutatis mutandis.

2.4 The quadratic and $\frac{5}{2}$ laws

Fix $Z_\varepsilon(0,\cdot) = \delta_0(\cdot)$. Our goal is to prove Theorem 2.1.1. By the scaling (2.1.3), we have

$$\mathbb{P}\big[\mathcal{H}(2\varepsilon,0) + \sqrt{4\pi\varepsilon} \geq \lambda\big] = \mathbb{P}\big[\sqrt{4\pi}Z_\varepsilon(2,0) \geq e^{\lambda}\big], \quad \mathbb{P}\big[\mathcal{H}(2\varepsilon,0) + \sqrt{4\pi\varepsilon} \leq -\lambda\big] = \mathbb{P}\big[\sqrt{4\pi}Z_\varepsilon(2,0) \leq e^{-\lambda}\big].$$

Hence Theorem 2.1.1 (a) follows from Proposition 2.1.7 (b) (for any $a \in \mathbb{R}$ and $T \geq 2$) and the contraction principle, with

$$\Phi(\lambda) = \inf\{\tfrac{1}{2}\|\rho\|_{L^2}^2 \,:\, \sqrt{4\pi}\mathsf{Z}(\rho;2,0) \geq e^{\lambda}\}, \tag{2.4.1}$$

$$\Phi(-\lambda) = \inf\{\tfrac{1}{2}\|\rho\|_{L^2}^2 \,:\, \sqrt{4\pi}\mathsf{Z}(\rho;2,0) \leq e^{-\lambda}\}. \tag{2.4.2}$$

Proving Theorem 2.1.1 (b) and (c) thus amounts to evaluating the infimums in (2.4.1) and (2.4.2), which will be carried out in Sections 2.4.1 and 2.4.2, respectively.

2.4.1 Near-center tails, proof of Theorem 2.1.1 (b)

In view of (2.4.1) – (2.4.2), our goal is to show

$$\lim_{\lambda\to 0} \lambda^{-2} \inf\{\tfrac{1}{2}\|\rho\|^2_{L^2} \,:\, \sqrt{4\pi}\mathsf{Z}(\rho;2,0) \geq e^{\lambda}\} = \tfrac{1}{\sqrt{2\pi}}, \tag{2.4.3}$$

$$\lim_{\lambda\to 0} \lambda^{-2} \inf\{\tfrac{1}{2}\|\rho\|^2_{L^2} \,:\, \sqrt{4\pi}\mathsf{Z}(\rho;2,0) \leq e^{-\lambda}\} = \tfrac{1}{\sqrt{2\pi}}. \tag{2.4.4}$$

The proofs of (2.4.3) and (2.4.4) are the same so we consider only (2.4.3). Fix $\rho \in L^2([0,2]\times\mathbb{R})$. Since our goal is to prove (2.4.3), we assume $\|\rho\|_{L^2} \leq \lambda$ and $\lambda \leq 1$. Recall that $\mathsf{Z}(\rho;t,x) = \sum_{n=0}^{\infty} \mathsf{Y}_n(\rho;t,x)$, with $\mathsf{Y}_n(\rho;t,x)$ is given (2.2.8-nw). Let $O(\lambda^k)$ denote a generic function of λ such that $|O(\lambda^k)| \leq C\lambda^k$, for all $\lambda \in (0,1]$. Specialize at $(t,x) = (2,0)$ and apply the bound in Lemma 2.2.3-nw for $n \geq 2$. We have

$$\sqrt{4\pi}\mathsf{Z}(\rho;2,0) = 1 + \sqrt{4\pi}\int_0^2 \int_{\mathbb{R}} \rho(s,y)p(2-s,y)p(s,y)\,\mathrm{d}y\mathrm{d}s + O(\lambda^2). \tag{2.4.5}$$

Now assume $\sqrt{4\pi}\mathsf{Z}(\rho;2,0) \geq e^{\lambda}$. Inserting this inequality into (2.4.5) and Taylor expanding e^{λ} gives

$$\sqrt{4\pi}\int_0^2 \int_{\mathbb{R}} \rho(s,y)p(2-s,y)p(s,y)\,\mathrm{d}y\mathrm{d}s \geq \lambda + O(\lambda^2).$$

On the left hand side, apply the Cauchy–Schwarz inequality to separate $\rho(s,y)$ and $p(2-s,y)p(s,y)$, and use

$$\int_0^2 \int_{\mathbb{R}} p(2-s,y)^2 p(s,y)^2 \mathrm{d}y\mathrm{d}s = 2^{-5/2}\pi^{-1/2} \tag{2.4.6}$$

We have $\|\rho\|_{L^2} \geq (2/\pi)^{1/4}\lambda + O(\lambda^2)$. Taking square of both sides and divide the result by $\frac{1}{2\lambda^2}$ gives the inequality '$\geq$' in (2.4.3).

To show the reverse inequality, take $\kappa > 1$ and $\rho(s,y) = \lambda\kappa 2^{3/2} p(2-s,y)p(s,y)$. Inserting this ρ into (2.4.5) and using (2.4.6) give $\sqrt{4\pi}\mathsf{Z}(\rho;2,0) \geq 1 + \kappa\lambda + O(\lambda^2)$. With $\kappa > 1$, the last expression is larger than e^{λ} for all λ small enough. On the other hand, by using (2.4.6) we have

$\frac{1}{2}\lambda^{-2}\|\rho\|_{L^2}^2 = \frac{\kappa^2}{\sqrt{2\pi}}$. Hence the left hand side of (2.4.3) is bounded by $\frac{\kappa^2}{\sqrt{2\pi}}$. Now taking $\kappa \downarrow 1$ completes the proof.

2.4.2 Deep lower tail, proof of Theorem 2.1.1 (c)

The Feynman–Kac formula and scaling

Here we consider the deep lower-tail regime, i.e., $-\lambda \to -\infty$. The first step is to express $\mathsf{Z}(\rho; t, x)$ by the Feynman–Kac formula. Namely,

$$\mathsf{Z}(\rho; t, x) = \mathbb{E}_x\Big[\exp\Big(\int_0^t \rho(s, B(t-s))\,\mathrm{d}s\Big)\,\delta_0(B(t))\Big] \tag{2.4.7}$$

$$= \mathbb{E}_{0\to x}\Big[\exp\Big(\int_0^t \rho(s, B_{\mathrm{b}}(s))\,\mathrm{d}s\Big)\Big]p(t, x). \tag{2.4.8}$$

In (2.4.7), the expectation $\mathbb{E}_x$ is taken with respect to a Brownian motion that starts from x, and in (2.4.8) the $\mathbb{E}_{0\to x}$ is taken with respect to a Brownian bridge $B_{\mathrm{b}}(s)$ that starts from $B_{\mathrm{b}}(0) = 0$ and ends in $B_{\mathrm{b}}(t) = x$. Indeed, the expression (2.4.7) is equivalent to (2.2.9) upon Taylor-expanding the exponential in (2.4.7) and exchanging the sum with the expectation. The exchange is justified by the bound in Lemma 2.2.3-nw. Set

$$\mathsf{h}(\rho; t, x) := \log(\sqrt{4\pi}\mathsf{Z}(\rho; t, x)) = \log(\sqrt{4\pi}p(t, x)) + \log\mathbb{E}_{0\to x}\Big[\exp\Big(\int_0^t \rho(s, B_{\mathrm{b}}(s))\,\mathrm{d}s\Big)\Big]. \tag{2.4.9}$$

Take log on both sides of (2.4.7) and insert the result into (2.4.2). We have

$$\Phi(-\lambda) = \inf\big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{h}(\rho; 2, 0) \le -\lambda\big\}. \tag{2.4.10}$$

We expect the right hand side of (2.4.10) to grow as $\lambda^{5/2}$ when $\lambda \to \infty$. As pointed out in [KK07, KK09, MKV16, KMS16], such a power law follows from scaling. More precisely, when $\lambda \to \infty$, it is natural to scale $\mathsf{h} \mapsto \lambda^{-1}\mathsf{h}$ and $\rho \mapsto \lambda\rho$. Accordingly, for the Brownian bridge in (2.4.9) to complete on the same footing, it is desirable to have a factor $\lambda^{-1/2}$ multiplying $B_{\mathrm{b}}(s)$.

This is so because large deviations of $\lambda^{-1/2}B_{\mathrm{b}}(s)$ occurs at rate λ, which is compatible with the scaling $\rho \mapsto \lambda\rho$. To implement these scaling, in (2.4.9) replace $\rho(t,x) \mapsto \lambda\rho(t,\lambda^{-1/2}x)$ and $x \mapsto \lambda^{1/2}x$ and divide the result by λ. Let $\mathsf{h}_\lambda(\rho;t,x) := \lambda^{-1}\mathsf{h}(\lambda\rho(\bullet,\lambda^{-1/2}\bullet);t,\lambda^{1/2}x)$ denote the resulting function on the left hand side. We have

$$\mathsf{h}_\lambda(\rho;t,x) = \lambda^{-1}\log(\sqrt{4\pi}p(t,\lambda^{\frac{1}{2}}x)) + \lambda^{-1}\log\mathbb{E}_{0\to\lambda^{1/2}x}\Big[\exp\Big(\int_0^t \lambda\rho(s,\lambda^{-\frac{1}{2}}B_{\mathrm{b}}(s))\,\mathrm{d}s\Big)\Big]. \tag{2.4.11}$$

The replacement $\rho(t,x) \mapsto \lambda\rho(t,\lambda^{-1/2}x)$ changes $\|\rho\|_{L^2}^2$ by a factor of $\lambda^{5/2}$, so (2.4.10) translates into

$$\Phi(-\lambda) = \lambda^{\frac{5}{2}}\inf\big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{h}_\lambda(\rho;2,0) \leq -1\big\}. \tag{2.4.12}$$

Proving Theorem 2.1.1 (c) hence amounts to proving

$$\lim_{\lambda\to\infty}\big(\inf\big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{h}_\lambda(\rho;2,0) \leq -1\big\}\big) = \frac{4}{15\pi}. \tag{2.4.13}$$

The optimal deviation ρ_* and its geodesics

We begin by introducing a function $\rho_* \in L^2([0,2]\times\mathbb{R})$. The definition of this function is motivated by physics argument [KK09, MKV16, KMS16]; see Section 2.1.1. In the context of Proposition 2.1.7, ρ describes possible deviations of the spacetime white noise $\sqrt{\varepsilon}\xi$. Such ρ_* is a candidate for the optimal ρ, so we refer to ρ_* as the **optimal deviation**.

To define ρ_*, consider the unique $C^1[1,2)$-valued solution $r(t)$ of the equation

$$r'(t) = 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\,r^2\sqrt{r-\pi/2}, \text{ for } t \in (1,2), \quad r(1) = \pi/2, \quad \text{and } r|_{(1,2)} > \pi/2, \tag{2.4.14}$$

and symmetrically extend it to $C^1(0,2)$ by setting $r(t) := r(2-t)$ for $t \in (0,1)$. Integrating

(2.4.14) gives

$$\frac{(r(t)-\pi/2)^{\frac{1}{2}}}{r(t)\pi/2}+(\tfrac{2}{\pi})^{\frac{3}{2}}\arctan\Big(\big(\tfrac{r(t)}{\pi/2}-1\big)^{\frac{1}{2}}\Big)=(\tfrac{2}{\pi})^{\frac{1}{2}}\,|t-1|. \tag{2.4.15}$$

Let us note a few useful properties of $r(t)$. It can be checked from (2.4.15) that $\lim_{s\downarrow 0} r(s) = \lim_{s\uparrow 2} r(s) = +\infty$. The integral $\int_0^2 r(t)\,\mathrm{d}t = 2\int_1^2 r(t)\,\mathrm{d}t$ can be evaluated with the aid of (2.4.14): perform the change of variables $2\int_1^2 r(t)\,\mathrm{d}t = 2\int_{\pi/2}^{\infty} \frac{r}{r'(t)}\mathrm{d}r$ and use (2.4.14) to substitute $r'(t)$. The result reads

$$\int_0^2 r(t)\,\mathrm{d}t = \int_0^2 |r(t)|\,\mathrm{d}t = 2\pi. \tag{2.4.16}$$

Set $\ell(t) := 1/r(t)$ for $t \in (0,2)$, and let $\ell(0) := 0$ and $\ell(2) := 0$ so that $\ell \in C[0,2]$. We define

$$\rho_*(t,x) := -\frac{r(t)}{2\pi}\Big(1-\frac{x^2}{\ell(t)^2}\Big)_+. \tag{2.4.17}$$

Next, setting $\rho = \rho_*$ in (2.4.9), we seek to characterize the $\lambda \to \infty$ limit of the resulting function:

$$\mathsf{h}_*(t,x) := \lim_{\lambda\to\infty} \mathsf{h}_\lambda(\rho_*;t,x), \tag{2.4.18}$$

for all $(t,x) \in (0,2]\times\mathbb{R}$. Even though only $\mathsf{h}_*(2,0)$ will be relevant toward the proof of (2.4.13), we treat general $(t,x) \in (0,2]\times\mathbb{R}$ for its independent interest.

Remark 2.4.1. Indeed, with ρ_* being the optimal deviation of the spacetime white noise, the function h_* should be viewed as the limit shape of $\mathcal{H}_{\varepsilon,\lambda}(t,x) := \lambda^{-1}\log Z_\varepsilon(t,\lambda^{1/2}x)$ under the conditioning $\{\mathcal{H}_{\varepsilon,\lambda}(0,2) \le -1\}$ with $\lambda \gg 1$. A explicit expression of $\mathsf{h}_*(1,x)$ is given in [HMS19]. One can show that [HMS19, Eq's (10)-(11)] coincide with the variational expression of h_* given in (2.4.22) below.

Proving that h_* is the limit shape of $h_{\varepsilon,\lambda}$ remains open, which we leave for future work.

To characterize (2.4.18), we first turn the limit into certain minimization problem over paths, by using Varadhan's lemma. To setup notation, we let $H^1_{0,x}[0,t]$ denote the space of H^1 functions on $[0,t]$ such that $\gamma(0)=0$ and $\gamma(t)=x$, and likewise for $C_{0,x}[0,t]$. For $\gamma\in H^1_{0,x}[0,t]$, set

$$U(\gamma;t,x)=\int_0^t \tfrac{1}{2}\gamma'(s)^2-\rho_*(s,\gamma(s))\,\mathrm{d}s. \tag{2.4.19}$$

Lemma 2.4.2. *For any* $(t,x)\in(0,2]\times\mathbb{R}$,

$$\lim_{\lambda\to\infty}\mathsf{h}_\lambda(\rho_*;t,x)=:\mathsf{h}_*(t,x)=-\inf\big\{U(\gamma;t,x):\gamma\in H^1_{0,x}[0,t]\big\}. \tag{2.4.20}$$

Proof. Let $F(\gamma):=\int_0^t\rho_*(s,\gamma(s))\,\mathrm{d}s$. In (2.4.11), set $\rho\mapsto\rho_*$ and let $\lambda\to\infty$ to get

$$\lim_{\lambda\to\infty}\mathsf{h}_\lambda(\rho_*;t,x)=-\tfrac{x^2}{2t}+\lim_{\lambda\to\infty}\lambda^{-1}\log\mathbb{E}_{0\to\lambda^{1/2}x}\big[\exp\big(\lambda F(\lambda^{-\frac{1}{2}}B_{\mathrm{b}}(s))\big)\big]. \tag{2.4.21}$$

We have assumed that the last limit exists. To prove the existence of the limit and to evaluate it we appeal to Varadhan's lemma. To start, let us establish the LDP for $\{\lambda^{-1/2}B_{\mathrm{b}}(s):s\in[0,t]\}$. Express B_{b} as $B_{\mathrm{b}}(s)=B(s)+(x-B(t))s/t$, where B denotes a standard Brownian motion. Since the map $\gamma\mapsto\gamma+(x-\gamma(t))s/t$ from $\{\gamma\in C[0,t]:\gamma(0)=0\}$ to $C_{0,x}[0,t]$ is continuous, we can use the contraction principle to push forward the LDP for $\lambda^{-1/2}B$. The result asserts that $\lambda^{-1/2}B_{\mathrm{b}}$ enjoys an LDP with speed λ and the rate function $I_{\mathrm{bb}}(\gamma):=\inf\{\frac{1}{2}\int_0^t(\gamma'(s)-v-\frac{x}{t})^2\mathrm{d}s:v\in\mathbb{R}\}$ for $\gamma\in H^1_{0,x}[0,t]$ and $I_{\mathrm{bb}}(\gamma)=+\infty$ otherwise. Optimizing over $v\in\mathbb{R}$ gives

$$I_{\mathrm{bb}}(\gamma)=\begin{cases}\int_0^t\frac{1}{2}\gamma'(s)^2\mathrm{d}s-\frac{x^2}{2t}, & \text{for } \gamma\in H^1_{0,x}[0,t],\\ +\infty, & \text{for } \gamma\in C_{0,x}[0,t]\setminus H^1_{0,x}[0,t].\end{cases}$$

To apply Varadhan's lemma we need to check, for $F(\gamma):=\int_0^t\rho_*(s,\gamma(s))\,\mathrm{d}s$:

(i) $F:C_{0,x}[0,t]\to\mathbb{R}$ is continuous.

This statement would follow if ρ_* were uniformly continuous on $[0,t]\times\mathbb{R}$. The function $\rho_*(s,y)$ however is discontinuous at $(0,0)$ and $(2,0)$. To circumvent this issue, for small $\delta>0$, we consider the truncation $\rho_*^\delta(s,y) := \mathbf{1}_{\{|s-1|<1-\delta\}}\rho_*(s,y)$. The truncated functional $F_\delta(\gamma) := \int \rho_*^\delta(t,\gamma(t))\,\mathrm{d}t$ is continuous on $C_{0,x}[0,t]$. The difference $F - F_\delta$ is bounded by $|(F-F_\delta)(\gamma)| \leq \int_{|s-1|>1-\delta} |\rho_*(s,\gamma(s))|\,\mathrm{d}s \leq \frac{1}{2\pi}\int_{|s-1|>1-\delta} |r(s)|\mathrm{d}s$. By (2.4.16), the last expression converges to zero as $\delta \to 0$, *uniformly* in $\gamma \in C_{0,x}[0,t]$. From these properties we conclude that $F : C_{0,x}[0,t] \to \mathbb{R}$ is continuous.

(ii) $\lim_{M\to\infty} \limsup_{\lambda\to\infty} \lambda^{-1} \log \mathbb{E}_{0\to x}\big[\exp\big(\lambda F(\lambda^{-1/2}B_{\mathrm{b}})\big)\mathbf{1}\{F(\lambda^{-1/2}B_{\mathrm{b}}) > M\}\big] = -\infty$
This holds since $\rho_* \leq 0$, which implies $F \leq 0$.

Varadhan's lemma applied to the last term in (2.4.21) completes the proof. □

Lemma 2.4.2 expresses $\mathsf{h}_*(t,x)$ in terms of a variational problem over paths. We refer to the minimizing path(s) in (2.4.20) (if exists) as a **geodesic**. The next step is to identify the geodesic. Let

$$\Omega := \{(s,y) : s \in [0,2], |y| \leq \ell(s)\}$$

denote the support of ρ_*, with the boundary $\partial\Omega = \{(s,y) : t \in [0,2], |y| = \ell(s)\}$.

Proposition 2.4.3.

(a) For any $(t,x) \in (0,2] \times \mathbb{R}$, the infimum

$$\mathsf{h}_*(t,x) = -\inf\left\{U(\gamma;t,x) : \gamma \in H^1_{0,x}[0,t]\right\} \tag{2.4.22}$$

is attended in $H^1_{0,x}[0,t]$.

(b) When $(t,x) = (2,0)$, the geodesics are $\alpha\ell(\bullet)$, $|\alpha| \leq 1$.

(c) When $(t,x) \in \Omega \cap \{t \in (0,2)\}$, the unique geodesic is $(x/\ell(t))\ell(\bullet)$.

(d) When $(t,x) \in \Omega^{\mathrm{c}} \cap \{t \in (0,2]\}$, is the geodesic is the unique $C^1_{0,x}[0,t]$ path such that $\gamma|_{[0,t_*]} = \ell|_{[0,t_*]}$ and $\gamma|_{[t_*,t]}$ is linear, for some $t_* \in (0,t)$.

See Figure 2.1 for an illustration for these geodesics.

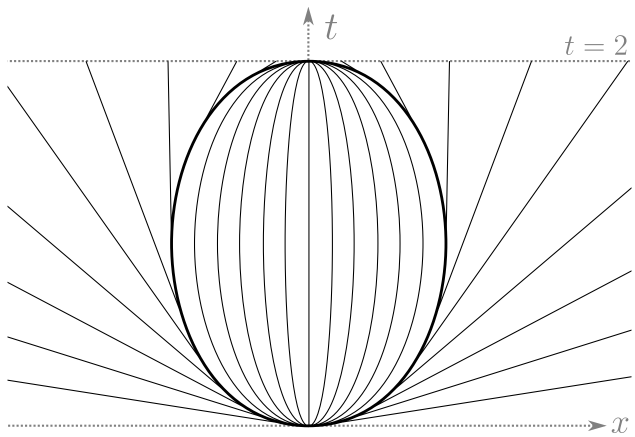


Figure 2.1: The solid curves are the geodesics for (2.4.22), with the thick ones being $\pm\ell(\bullet)$. Those geodesics outside $\pm\ell(\bullet)$ are linear, and touch $\pm\ell(\bullet)$ at tangent.

Remark 2.4.4. An intriguing feature of Proposition 2.4.3(b) is the *nonuniqueness* of the geodesics between $(0,0)$ and $(2,0)$. For any $|\alpha| \leq 1$, $\gamma = \alpha\ell$ is one such geodesic, so the paths span a lens-shaped region Ω. For the exponential Last Passage Percolation (LPP), [BGS19] proved that the point-to-point geodesic (in the context of LPP) does not concentrate around any given path under a lower-tail conditioning. Though the setups differ, the result of [BGS19] and Proposition 2.4.3(b) are consistent. It is an intriguing question to explore deeper connection between these two phenomena. For example, is it true that for LPP under lower-tail conditioning, the distribution of the geodesic spans a lens-like region?

To streamline the proof of Proposition 2.4.3, let us prepare a few technical tools. The Euler–Lagrangian equation for (2.4.19) is

$$\gamma'' = -\partial_x \rho_*(s,\gamma(s)) = \begin{cases} -\frac{r(s)}{\pi\ell(s)^2}\gamma, & \text{when } (s,\gamma(s)) \in \Omega^\circ, \\ 0 & \text{, when } (s,\gamma(s)) \in \Omega^{\mathrm{c}}. \end{cases} \tag{2.4.23}$$

The equation (2.4.23) is ambiguous when $(s,\gamma(s)) \in \partial\Omega$ because $\partial_x \rho_*$ is not continuous there. We will avoid referencing (2.4.23) when $(s,\gamma(s)) \in \partial\Omega$. It will be convenient to also consider

$$\gamma'' = -\frac{r(s)}{\pi\ell(s)^2}\gamma, \tag{2.4.24}$$

which coincides with (2.4.23) in Ω°.

Lemma 2.4.5.

(a) The function ℓ is strictly concave and $\lim_{s\downarrow 0}|\ell'(s)| = +\infty$.

(b) For any $\alpha \in \mathbb{R}$, the function $\alpha\ell(s)$ solves (2.4.24) *for $s \in (0,2)$.*

(c) For any for any $|\alpha| \le 1$, $U(\alpha\ell; 2, 0) = -1$.

(d) In $(\partial\Omega)^c$, any geodesic of (2.4.22) *is C^2 and solves* (2.4.23).

(e) When $(t,x) \in \Omega$, any geodesic of (2.4.22) *lies entirely in Ω.*

(f) Let $\gamma \in H^1_{0,x}[0,t]$ be a geodesic of (2.4.22)*, and consider $(t_*, \gamma(t_*)) \in \partial\Omega$ with $t_* \in (0,t)$. Then*

$$\lim_{\beta\downarrow 0}\Big(\frac{1}{\beta}\int_{t_*}^{t_*+\beta}\gamma'(s)\mathrm{d}s - \frac{1}{\beta}\int_{t_*-\beta}^{t_*}\gamma'(s)\mathrm{d}s\Big) = 0.$$

Proof. Parts (a)–(c) follow by straightforward calculations from $\ell(s) = 1/r(s)$, (2.4.14), and (2.4.16). Part (d) follows by standard variation procedure.

(e) The geodesic γ starts and ends within Ω, i.e., $(0,\gamma(0)) = (0,0) \in \Omega$ and $(t,\gamma(t)) = (t,x) \in \Omega$. If the geodesic ever leaves Ω, then there exists $t_1 < t_2 \in [0,t]$ such that $\gamma|_{(t_1,t_2)}$ lies outside Ω and $(t_i,\gamma(t_i)) \in \partial\Omega$ for $i = 1,2$. See Figure 2.2 for an illustration. Let us compare the functional $U(\,\bullet\,; t, x)$ (c.f., (2.4.19)) restricted onto the segments $\gamma|_{[t_1,t_2]}$ and $\pm\ell|_{[t_1,t_2]}$, where the $\pm$ sign depends on which side of the boundary $(t_1,\gamma(t_1))$ and $(t_2,\gamma(t_2))$ belong to, c.f., Figure 2.2. First ρ_* vanishes along both segments. Next, the strict concavity of ℓ from Part (a) implies $\int_{t_1}^{t_2}\gamma'(s)^2\mathrm{d}s > \int_{t_1}^{t_2}\ell'(s)^2\mathrm{d}s$. Therefore, we can modify γ by replacing the segment $\gamma|_{[t_1,t_2]}$ with $\pm\ell|_{[t_1,t_2]}$ to decreases the value of $U(\gamma; 2, 0)$. This contradicts with assumption that γ is a geodesic. Hence the geodesic must stay completely within Ω.

(f) The idea is to perform variation. Fix a neighborhood O of t_* with $\overline{O} \subset (0,2)$. For $f \in$

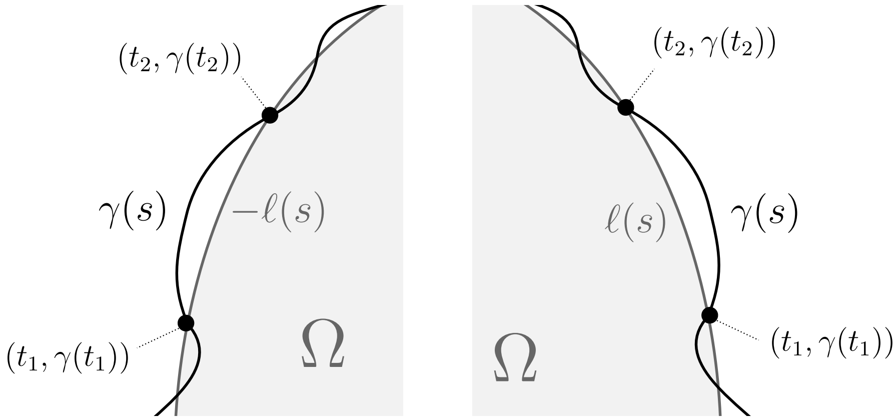


Figure 2.2: Illustration of Part (e) of the proof of Lemma 2.4.5

$C_c^\infty(O)$ consider

$$F(\alpha) := \int_0^t \tfrac{1}{2}(\gamma' + \alpha f')^2 - \rho_*(s, \gamma + \alpha f)\, \mathrm{d}s.$$

The derivative $\partial_x \rho_*$ is bounded on $\overline{O} \times \mathbb{R}$ (even though not continuous). Taylor expanding F around $\alpha = 0$ then gives $\int \gamma'(s) f'(s) \mathrm{d}s \leq c \int |f(s)| \mathrm{d}s$, for some constant $c < \infty$. Within the last inequality, substitute $f(s) \mapsto f(s+u)$, integrate the result over $u \in [-\frac{1}{2}\beta, \frac{1}{2}\beta]$, and divide both sides by β. This gives

$$\frac{1}{\beta} \int \gamma'(s)(f(s + \tfrac{1}{2}\beta) - f(s - \tfrac{1}{2}\beta)) \mathrm{d}s = \frac{1}{\beta} \int (\gamma'(s - \tfrac{1}{2}\beta) - \gamma'(s + \tfrac{1}{2}\beta)) f(s) \mathrm{d}s \leq c \int |f(s)| \mathrm{d}s.$$

This inequality holds for smooth $f(s)$ supported in $\{s : s \pm \frac{1}{2}\beta \in O\}$. Since $\gamma' \in L^2[0,t]$, the equality extends to $f \in L^2$. Specializing $f = \pm \mathbf{1}_{(t_* - \frac{1}{2}\beta, t_* + \frac{1}{2}\beta)}$ and taking $\beta \downarrow 0$ gives the desired result. □

Proof of Proposition 2.4.3. (a) The proof follows from standard argument of the direct method. Take any minimizing sequence $\{\gamma_n\}$. For such a sequence, $\{\gamma_n'\}$ is bounded in $L^2[0,t]$. By the Banach–Alaoglu theorem, after passing to a subsequence we have $\gamma_n' \to \eta \in L^2[0,t]$ weakly in $L^2[0,t]$. Let $\gamma(\overline{s}) := \int_0^{\overline{s}} \eta(s) \mathrm{d}s$. We then have $\gamma_n \to \gamma$ in $C_{0,x}[0,t]$ and $\int_0^t \gamma'(s)^2 \mathrm{d}s = \|\eta\|_{L^2}^2 \leq \lim_n \|\gamma_n'\|_{L^2}^2$. Also, by Property (i) in the proof of Lemma 2.4.2, $\int_0^t \rho_*(s, \gamma_n(s)) \mathrm{d}s \to$

$\int_0^t \rho_*(s,\gamma(s))\mathrm{d}s$. We have verified that $\gamma \in H^1_{0,x}[0,t]$ a geodesic.

(b) The proof amounts to showing that any geodesic must be of the form $\alpha\ell$, for some $|\alpha| \le 1$. Once this is done, Lemma 2.4.5(c) guarantees that any such path is a geodesic.

We begin with a reduction. For a geodesic $\gamma \in H^1_{0,0}[0,2]$, consider its first and second halves $\gamma_1 := \gamma|_{[0,1]}$ and $\gamma_2(s) := \gamma(2-s)|_{s\in[0,1]}$. Joining each half with itself end-to-end gives the symmetric paths $\overline{\gamma}_i(s) := \gamma_i(s)\mathbf{1}_{[0,1]}(s) + \gamma_i(s-1)\mathbf{1}_{(1,2]}(s)$, for $s \in [0,2]$ and $i = 1,2$. These symmetrized paths are also geodesics. To see why, note that since $\rho_*(s,y)$ is symmetric around $s = 1$, we have $U(\overline{\gamma}_i;2,0) = 2U(\gamma_i;1,\gamma(1))$, for $i = 1,2$, and $U(\gamma;2,0) = U(\gamma_1;1,\gamma(1)) + U(\gamma_2;1,\gamma(1))$. On the other hand, γ being a geodesic implies $U(\gamma;2,0) \le U(\overline{\gamma}_i;2,0)$, for $i = 1,2$. From the these relations we infer that $U(\overline{\gamma}_1;2,0) = U(\overline{\gamma}_2;2,0) = U(\gamma;2,0)$, namely, the symmetrized paths $\overline{\gamma}_1$ and $\overline{\gamma}_2$ are also geodesics. Recall that our goal is to show any geodesic must be of the form $\alpha\ell$, for some $|\alpha| \le 1$. If we can establish the statement for $\overline{\gamma}_1$ and $\overline{\gamma}_2$, the same immediately follows for γ. Hence, without loss of generality, hereafter we consider only symmetric geodesics.

Fix a geodesic $\gamma \in H^1_{0,0}[0,2]$. As argued in the preceding paragraph, we can and shall assume $\gamma(s)$ is symmetric around $s = 1$, and by Lemma 2.4.5(e) the path lies entirely in Ω. The last condition implies $|\gamma(1)| \le \ell(1)$. Consider first the case $|\gamma(1)| < \ell(1)$. By Lemma 2.4.5(d), within a neighborhood of $s = 1$ the path $\gamma(s)$ is C^2 and solves (2.4.23) and therefore (2.4.24). The symmetry of γ gives $\gamma'(1) = 0$. The uniqueness of the ODE (2.4.24) and Lemma 2.4.5(b) now imply $\gamma(s) = \alpha\ell(s)$, for $\alpha = \gamma(1)/\ell(1)$ and for all s in a neighborhood of $s = 1$. This matching $\gamma(s) = \alpha\ell(s)$ extends to $s \in (0,2)$ by standard continuity argument. This concludes the desired result for the case $|\gamma(1)| < \ell(1)$.

Turning to the case $|\gamma(1)| = \ell(1)$, we need to show $\gamma = \pm\ell$. Let us argue by contradiction. Assuming the contrary, we can find $t_2 \in (0,1)\cup(1,2)$ such that $(t_2,\gamma(t_2)) \in \Omega^\circ$. By the symmetry of γ around $s = 1$ we can and shall assume $t_2 \in (1,2)$. Tracking along γ backward in time from t_2, we let $t_* := \inf\{s \in [0,t_*] : |\gamma(s)| < \ell(s)\}$ be the first hitting time of $\partial\Omega$. Indeed $t_* \in [1,t_2)$ and $\gamma(t_*) = \pm\ell(t_*)$. Let us take '+' for simplicity of notation; see Figure 2.3 for an illustration. The case for '−' can be treated by the same argument. By Lemma 2.4.5(d), $\gamma|_{(t_*,t_2)}$ solves (2.4.23) and therefore (2.4.24). On the other hand, ℓ also solves (2.4.24) by Lemma 2.4.5(b). These facts along with the well-posedness of (2.4.24) at $(t_*, \ell(t_*))$ imply that $\gamma|_{[t_*,t_2)} \in C^2[t_*, t_2)$ and $\lim_{\beta\downarrow 0} \gamma'(t_* + \beta) \neq \ell'(t_*)$. Either '<' or '>' holds between these two quantities. The property $\{(t, \gamma(t))\}_{t\in(t_*,t_2)} \subset \Omega^\circ$ tells us that it is '<', namely $\lim_{\beta\downarrow 0} \gamma'(t_* + \beta) < \ell'(t_*)$. Combining this inequality with Lemma 2.4.5(f) gives $\lim_{\beta\downarrow 0} \frac{1}{\beta} \int_{t_*-\beta}^{t_*} \gamma'(s)\mathrm{d}s = \lim_{\beta\downarrow 0} \frac{1}{\beta}(\ell(t_*) - \gamma(t_* - \beta)) < \ell'(t_*)$. Recall from Lemma 2.4.5(a) that ℓ is concave. The last inequality then forces $\gamma(t_* - \beta) > \ell(t_* - \beta)$ for all small enough $\beta > 0$. This statement contradicts with the fact that γ lies within Ω. We have reached a contradiction and hence completed the proof for the case $|\gamma(1)| = \ell(1)$.

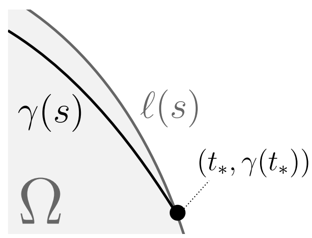


Figure 2.3: Illustration of Part (b) of the proof of Proposition 2.4.3. Only the portion $s \geq t_*$ of the curve $\gamma(s)$ is shown.

(c) Our goal is to characterize the geodesic between $(0, 0)$ and (t, x). The idea is to 'embed' such a minimization problem into a minimization problem between $(0, 0)$ and $(2, 0)$. More precisely consider

$$\inf \left\{ U(\gamma; 2, 0) : \gamma \in H^1_{0,x}[0, 2],\ \gamma(t) = x \right\}. \tag{2.4.25}$$

The infimum is taken over all H^1 path that joins $(0, 0)$ and $(2, 0)$ *and* passes through (t, x). Such an infimum can be divided into two parts as

$$(2.4.25) = \inf \left\{ U(\gamma; t, x) : \gamma \in H^1_{0,x}[0, t] \right\} + \inf \left\{ \int_t^2 \tfrac{1}{2}\gamma'(s)^2 - \rho_*(s, \gamma(s))\,\mathrm{d}s : \gamma \in H^1_{x,0}[t, 2] \right\}. \tag{2.4.26}$$

Take any geodesic $\gamma \in H^1_{0,x}[0, t]$ for the first infimum in (2.4.26) and any geodesic $\overline{\gamma} \in H^1_{x,0}[t, 2]$

for the second infimum in (2.4.26). (The existence of such geodesics can be established by the same argument in Part (a).) The concatenated path $\gamma_c(s) := \gamma(s)\mathbf{1}_{s\in[0,t]} + \overline{\gamma}(s)\mathbf{1}_{s\in(t,2]}$ is a geodesic for (2.4.25). Hence $U(\gamma_c; 2, 0) \geq U(\widetilde{\gamma}; 2, 0)$, for any $\widetilde{\gamma} \in H^1_{0,0}[0, 2]$ that passes through (t, x). Set $\alpha = x/\ell(t)$. The last inequality holds in particular for $\widetilde{\gamma} = \alpha\ell$. On the other hand, under current assumption $(t, x) \in \Omega$, we have $|\alpha| \leq 1$, so Part (b) asserts that $\alpha\ell$ minimizes (2.4.25) even *without* the constraint $\gamma(t) = x$. Therefore, $U(\gamma_c; 2, 0) = U(\alpha\ell; 2, 0)$, and γ_c itself is a geodesic for $\inf\{U(\bullet; 0, 2) : \widetilde{\gamma} \in H^1_{0,0}[0, 2]\}$. The last statement and Part (b) force $\gamma_c = \alpha\ell$, which concludes the desired result.

(d) Fix a geodesic $\gamma \in H^1_{0,x}[0, t]$. By Lemma 2.4.5(d) and the fact that $(\partial_x \rho_*)|_{\Omega^c} = 0$, the path γ is linear outside Ω. Tracking along γ backward in time from t, we let $t_* := \inf\{s \in [0, t] : |\gamma(s)| > \ell(s)\} > 0$ be the first hitting time of the boundary. By Lemma 2.4.5(a) must have $t_* > 0$. The segment $\gamma|_{[0,t_*]}$ is itself is a geodesic for $U(\bullet; t_*, \gamma(t_*))$. Since $(t_*, \gamma(t_*)) = (t_*, \pm\ell(t_*)) \in \Omega$, Part (c) implies that $\gamma|_{[0,t_*]} = \pm\ell|_{[0,t_*]}$. The path γ is C^1 except possibly at $s = t_*$, but Lemma 2.4.5(f) guarantees that $\gamma(s)$ is also C^1 at $s = t_*$. For the given $(t, x) \in \Omega^c$, there is exactly one $t_* \in (0, t)$ that satisfies all the prescribed properties, so we have identified the unique geodesic γ. □

Given Lemma 2.4.2 and Proposition 2.4.3, it is possible to evaluate $\mathsf{h}_*(t, x)$ by calculating $U(\gamma; t, x)$ along the geodesic(s) given in Proposition 2.4.3. In particular, Proposition 2.4.3(b) and Lemma 2.4.5(c) gives

$$\mathsf{h}_*(2, 0) := \lim_{\lambda\to\infty} \mathsf{h}_\lambda(\rho_*; 2, 0) = -1. \tag{2.4.27}$$

Also, straightforward calculations from (2.4.17) (with the help of (2.4.16)) gives $\frac{1}{2}\|\rho_*\|^2_{L^2} = \frac{4}{15\pi}$.

We are now ready to prove one side of the inequalities in (2.4.13), namely

$$\limsup_{\lambda\to\infty} \big(\inf\big\{\tfrac{1}{2}\|\rho\|^2_{L^2} : \mathsf{h}_\lambda(\rho; 2, 0) \leq -1\big\}\big) \leq \tfrac{1}{2}\|\rho_*\|^2_{L^2} = \tfrac{4}{15\pi}. \tag{2.4.28}$$

To show (2.4.28) we would like to have $\mathsf{h}_\lambda(\rho_*; 2, 0) \leq -1$ for all large enough λ, but (2.4.27) only gives the inequality for $\lambda = +\infty$. We circumvent this issue by scaling. Fix $\kappa > 1$ and let $(\rho_*)_\kappa(t,x) := \kappa\rho_*(t,\kappa^{1/2}x)$. Referring to the scaling from (2.4.9) to (2.4.11), we see that $\mathsf{h}_\lambda((\rho_*)_\kappa;2,0) = \kappa\mathsf{h}_\lambda(\rho_*;2,0)$. This identity together with (2.4.27) implies $\mathsf{h}_\lambda((\rho_*)_\kappa;2,0) < -1$ for all large enough λ. On the other hand, $\frac{1}{2}\|(\rho_*)_\kappa\|^2_{L^2} = \frac{\kappa^{5/2}}{2}\|\rho_*\|^2_{L^2}$, so the left hand side of (2.4.28) is at most $\frac{\kappa^{5/2}}{2}\|\rho_*\|^2_{L^2}$. Letting $\kappa \downarrow 1$ concludes (2.4.28).

The reverse inequality

To prove (2.4.13), it now remains only to show the reverse inequality. Fix any $\rho \in L^2([0,2]\times\mathbb{R})$ with $\mathsf{h}_\lambda(\rho;2,0) \le -1$.

The first step is to relate $\mathsf{h}_\lambda(\rho;2,0)$ to the functional $U(\gamma;2,0)$, c.f., (2.4.19). Within (2.4.11), set $(t,x) \mapsto (2,0)$, express the Brownian bridge as $B_\mathrm{b}(t) = B(t) - tB(2)/2$, where B_b denotes a standard Brownian motion, and apply the Cameron–Martin–Girsanov theorem with $\lambda^{1/2}\gamma \in H^1_{0,0}[0,2]$ being the drift/shift. The result gives

$$\mathsf{h}_\lambda(\rho;2,0) = -\int_0^2 \tfrac{1}{2}\gamma'(t)^2 \mathrm{d}t + \lambda^{-1}\log\mathbb{E}_{0\to 0}\Big[\exp\Big(\int_0^2 \Big(\lambda\rho(t,\gamma+\lambda^{-\frac{1}{2}}B_\mathrm{b})\,\mathrm{d}t + \lambda^{\frac{1}{2}}\gamma'(t)\mathrm{d}B(t)\Big)\Big)\Big].$$

Applying Jensen's inequality to the last term yields, for any $\gamma \in H^1_{0,0}[0,2]$,

$$-1 \ge \mathsf{h}_\lambda(\rho;2,0) \ge -\lambda^{-1}\log\sqrt{4\pi} - \int_0^2 \tfrac{1}{2}\gamma'(t)^2 - \mathbb{E}_{0\to 0}\big[\rho(t,\gamma+\lambda^{-\frac{1}{2}}B_\mathrm{b})\big]\,\mathrm{d}t. \tag{2.4.29}$$

On the right hand side, the first term vanishes as $\lambda \to \infty$, and the second term resemble the functional $U(\gamma;2,0)$. The difference are that ρ replaces ρ_*, and there is an additional expectation over $\lambda^{-\frac{1}{2}}B_\mathrm{b}$.

We next use (2.4.29) to derive a useful inequality. First, recall from Lemma 2.4.5(c) that, for all $|\alpha| \le 1$,

$$-1 = -U(\alpha\ell;2,0) = -\int_0^2 \tfrac{1}{2}(\alpha\ell')^2 - \rho_*(t,\alpha\ell)\,\mathrm{d}t. \tag{2.4.30}$$

Substitute $\gamma \mapsto \alpha\ell$ in (2.4.29) and subtract (2.4.30) from the result. This gives, for all $|\alpha| \le 1$,

$$\int_0^2 \big(\rho_*(t,\alpha\ell) - \mathbb{E}_{0\to 0}\big[\rho(t,\alpha\ell + \lambda^{-\frac{1}{2}}B_{\mathrm{b}})\big]\big)\, \mathrm{d}t \ge -\lambda^{-1}\log\sqrt{4\pi}.$$

Multiply both sides by $-\frac{1}{2\pi}(1-\alpha^2)_+$ and integrate the result over $\alpha \in \mathbb{R}$. On the left hand side of the result, swap the integrals, multiply the integrand by $1 = r(t)\ell(t)$, and recognize $-\frac{r(t)}{2\pi}(1 - x^2/\ell(t)^2)_+ = \rho_*(t,x)$. We have

$$\int_0^2 \int_{\mathbb{R}} \rho_*(t,\alpha\ell)\Big(\rho_*(t,\alpha\ell) - \mathbb{E}_{0\to 0}\big[\rho(t,\alpha\ell + \lambda^{-\frac{1}{2}}B_{\mathrm{b}})\big]\Big)\, \ell(t)\mathrm{d}\alpha\mathrm{d}t \le \lambda^{-1}\tfrac{15}{16}\log\sqrt{4\pi}. \qquad (2.4.31)$$

To see why (2.4.31) is useful, let us *pretend* for a moment that $\lambda = +\infty$ in (2.4.31). The discussion in this paragraph is informal, and serves merely as a *motivation* for the rest of the proof. Informally set $\lambda = +\infty$ in (2.4.31), and perform the change of variables $x = \alpha\ell(t)$ on the left hand side. The result gives $\langle\rho_*, \rho_* - \rho\rangle \le 0$ and hence $\|\rho_*\|_{L^2}^2 + \|\rho - \rho_*\|_{L^2}^2 \le \|\rho\|_{L^2}^2$. The last inequality implies $\|\rho_*\|_{L^2}^2 \le \|\rho\|_{L^2}^2$, which is the desired result.

In light of the preceding discussion, we seek to develop an estimate of $\langle\rho_*, \rho_* - \rho\rangle$. To alleviate heavy notation we will often abbreviate $\lambda^{-1/2}B_{\mathrm{b}} =: \mathrm{bb}$. Write $\langle\rho_*, \rho_* - \rho\rangle = \int(\rho_*^2 - \rho_*\rho)(t,x)\mathrm{d}x\mathrm{d}t$. Within the integral add and subtract $\mathbb{E}[\rho_*^2(t, x - \mathrm{bb})]$ and $\mathbb{E}[\rho_*(t, x - \mathrm{bb})\rho(t,x)]$. This gives $\langle\rho_*, \rho_* - \rho\rangle = A_1 + A_2 + A_3$, where

$$\begin{aligned}
A_1 &:= \mathbb{E}\int_0^2 \int_{\mathbb{R}} \rho_*(t, x - \mathrm{bb})\big(\rho_*(t, x - \mathrm{bb}) - \rho(t,x)\big)\, \mathrm{d}x\mathrm{d}t,\\
A_2 &:= \mathbb{E}\int_0^2 \int_{\mathbb{R}} \rho_*^2(t,x) - \rho_*^2(t, x - \mathrm{bb})\, \mathrm{d}x\mathrm{d}t,\\
A_3 &:= \mathbb{E}\int_0^2 \int_{\mathbb{R}} \big(\rho_*(t, x - \mathrm{bb}) - \rho_*(t,x)\big)\rho(t,x)\, \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

For A_1, the change of variables $x = \alpha\ell(t) + \mathrm{bb} = \alpha\ell(t) + \lambda^{-1/2}B_{\mathrm{b}}(t)$ reveals that A_1 is equal to the left hand side of (2.4.31). Hence $A_1 \le \lambda^{-1}\frac{16}{15}\log\sqrt{4\pi}$. The term A_2 does not depend on ρ, and it is readily checked from (2.4.17) that $\lim_{\lambda\to\infty}|A_2| = 0$. As for A_3, the Cauchy–Schwarz inequality gives $|A_3| \leq A_{31}^{1/2}\|\rho\|_{L^2}$, where $A_{31} := \mathbb{E}\int(\rho_*(t,x-\mathrm{bb}) - \rho_*(t,x))^2\mathrm{d}t\mathrm{d}x$. The term A_{31} does not depend on ρ, and it is readily checked from (2.4.17) that $\lim_{\lambda\to\infty}|A_{31}| = 0$. Adopt the notation $o_\lambda(1)$ for a generic quantity that depends only on λ such that $\lim_{\lambda\to\infty}|o_\lambda(1)| = 0$. Collecting the preceding results on A_1, A_2, and A_3 now gives

$$\langle\rho_*, \rho_* - \rho\rangle \leq o_\lambda(1)(1 + \|\rho\|_{L^2}). \tag{2.4.32}$$

Since $\|\rho\|_{L^2}^2 = \|\rho_*\|_{L^2}^2 + \|\rho - \rho_*\|_{L^2}^2 - 2\langle\rho_*, \rho_* - \rho\rangle$, the bound (2.4.32) implies $\|\rho_*\|_{L^2}^2 \leq (1+o_\lambda(1))\|\rho\|_{L^2}^2 + o_\lambda(1)$. This inequality holds for all $\rho \in L^2$ with $\mathsf{h}_\lambda(\rho; 0, 2) \leq -1$, and $o_\lambda(1) \to 0$ does not depend on ρ. The desired result hence follows:

$$\liminf_{\lambda\to\infty}\Big(\inf\big\{\tfrac{1}{2}\|\rho\|_{L^2}^2 : \mathsf{h}_\lambda(\rho; 2, 0) \leq -1\big\}\Big) \geq \tfrac{1}{2}\|\rho_*\|_{L^2}^2 = \tfrac{4}{15\pi}.$$

Chapter 3: Lyapunov exponents of the SHE for general initial data

Chapter Abstract: We consider the $(1+1)$-dimensional stochastic heat equation (SHE) with multiplicative white noise and the Cole-Hopf solution of the Kardar-Parisi-Zhang (KPZ) equation. We show an exact way of computing the Lyapunov exponents of the SHE for a large class of initial data which includes any bounded deterministic positive initial data and the stationary initial data. As a consequence, we derive exact formulas for the upper tail large deviation rate functions of the KPZ equation for general initial data.

This chapter is available on arxiv [GL20].

3.1 Background and Main result

In this paper, we consider the solution of the $(1+1)$-dimensional SHE under general initial condition and ask how any positive moment of that solution grows as time goes to ∞. In particular, we take the logarithm of p-th moment of that solution for any $p \in \mathbb{R}_{>0}$ and show that the when scaled by time, those converge. Those limits are known as *Lyapunov exponents* which are tied to the large deviation problem of the KPZ equation. Namely, the upper tail large deviation of the Cole-Hopf solution of the KPZ equation (centered by time/24) is the Legendre-Fenchel dual of the Lyapunov exponent of the stochastic heat equation. To the best of our knowledge, our result is the first to provide exact computation of the Lyapunov exponents of the SHE and the upper tail large deviation of the KPZ for general initial data.

Let us recall the KPZ equation, written formally as

$$\partial_t \mathcal{H}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}(t,x) + \frac{1}{2}(\partial_x \mathcal{H}(t,x))^2 + \xi, \qquad \mathcal{H}(0,x) = \mathcal{H}_0(x). \tag{3.1.1}$$

The KPZ equation governs growth of the interface $\mathcal{H}(t,x)$ which is subjected to a roughening by the space-time white noise ξ. Due to the presence of ξ, the solution of the KPZ equation is ill-posed. A formal solution of the KPZ equation comes from the Cole-Hopf transform given as

$$\mathcal{H}(t,x) := \log(\mathcal{Z}(t,x)) \tag{3.1.2}$$

where $\mathcal{Z}(t,x)$ is the solution of the SHE:

$$\partial_t \mathcal{Z}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}(t,x) + \mathcal{Z}(t,x)\xi(t,x), \qquad \mathcal{Z}(0,x) = \exp(\mathcal{H}_0(x)). \tag{3.1.3}$$

The SHE is pervasive in the diffusion theory of particles in random environment [Mol96, Kho14], continuous directed random polymers [HHF85, Com17] and many other fields. The solution theory of the SHE is well known [Cor18, BC95, Wal86] via Ito integral theory or, martingale problem. The logarithm in (3.1.2) is well defined due to the strict positivity of the solution of the SHE [Mue91]. The Cole-Hopf solution correctly approximates discrete growth processes [BG97, CT17, CGST20, Lin20a] and has shown to appear naturally in various renormalization and regularization schemes [Hai13, GIP15, GP17].

In this paper, we consider a class of initial data of the KPZ equation which satisfies few technical conditions. Those conditions are designed as a minimal requirement to study the upper tail large deviation problem of the KPZ. Below, we introduce a class of functions **Hyp** whose members typifies the conditions that we need. Our main result, Theorem 3.1.2 is anchored in to the study of the solutions corresponding to the initial profiles in **Hyp**. As we show later in Corollary 3.1.4 and 3.1.6, a wide range of interesting initial profiles of the KPZ equation falls inside this class. Since the definition of **Hyp** instigates those technical conditions which might seem less enlightening at the first sight, we recommend to look at Corollary 3.1.4, 3.1.6 and Remarks 3.1.5, 3.1.7 before proceeding towards Theorem 3.1.2.

Definition 3.1.1. A set of measurable functions $(g, \{f_t\}_{t>0})$ with deterministic $g : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ and $f_t : \mathbb{R} \to \mathbb{R}$ (possibly random) for all $t \in \mathbb{R}_{>0}$ belongs to the class **Hyp** if the following conditions are satisfied.

1. (Coherence conditions:) For $x \in \mathbb{R}$ and $t \in \mathbb{R}_{\geq 0}$, define $M_p^{f_t}(t,x) := \mathbb{E}[e^{pf_t(x)}]$ when $f_t(x)$ is random and $M_p^{f_t}(t,x) := e^{pf_t(x)}$ when $f_t(x)$ is deterministic. For all $p \in \mathbb{R}_{>0}$,

$$g(p) = \lim_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}} \Big\{\frac{-px^2}{2t} + \log M_p^{f_t}(x,t)\Big\}. \tag{3.1.4}$$

Furthermore, for every $p \in \mathbb{R}_{>0}$,

$$\liminf_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log\bigg(\int e^{-\frac{p(1-\epsilon)x^2}{2t}} M_{p(1+\epsilon)}^{f_t}(t,x)dx\bigg) \leq g(p). \tag{3.1.5}$$

2. (Growth and lower bound conditions:) For each $p \in \mathbb{R}_{>0}$, there exist constants $C, K, L > 0$ and $0 < \alpha < 1$ depending on p such that for all $t > 0$,

$$M_p^{f_t}(t,x) \leq C(e^{C|x|} + e^{\frac{\alpha p x^2}{2t}}), \tag{3.1.6}$$

$$\sup_{x\in[-L,L]} \log M_p^{f_t}(t,x) > -K. \tag{3.1.7}$$

3. (Pseudo-stationarity:) We call $\{\theta_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ a sequence of grid points if it satisfies

$$\ldots < \theta_{-1} < \theta_0 = 0 < \theta_1 < \ldots, \qquad \lim_{n\to\infty} \theta_n = \infty, \qquad \lim_{n\to-\infty} \theta_n = -\infty,$$
$$(\max\{c|n|, 1\})^{-\beta} \leq |\theta_n - \theta_{n+1}| \leq 1$$

for some $c > 0$, $\beta \in (0,1)$ and all $n \in \mathbb{Z}$. There exist constants $C, t_0, s_0 > 0$ and a sequence of grid points $\{\theta_n\}_{n\in\mathbb{Z}}$ such that for all $t > t_0$, $n \in \mathbb{Z}$, $\sup_{x\in[\theta_n,\theta_{n+1}]} |f_t(x) - f_t(\theta_n)| \leq s_0$ when f_t is deterministic or,

$$\mathbb{P}\Big(\sup_{x\in[\theta_n,\theta_{n+1}]} |f_t(x) - f_t(\theta_n)| \geq s\Big) \leq e^{-Cs^{1+\delta}}, \qquad \forall s \geq s_0 \tag{3.1.8}$$

when f_t is random.

We will often call $(g, \{f_t\}_{t>0}) \in \mathbf{Hyp}$ as *KPZ data*.

Fix a KPZ data $(g, \{f_t\}_{t>0})$. We assume that there exists unique solution of the SHE started from initial data e^{f_t} (at least) upto time t. We denote the SHE solutions started from f_t by $\mathcal{Z}^{f_t}$ and the corresponding Cole-Hopf solution of the KPZ equation by $\mathcal{H}^{f_t}$. For any $p \in \mathbb{R}_{>0}$, we define the p-th moment Lyapunov exponent as

$$\mathrm{Lya}_p(\{f_t\}_{t>0}) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\big(\mathcal{Z}^{f_t}(t,0)\big)^p\Big]. \tag{3.1.9}$$

For the KPZ equation, we consider the upper tail probability $\mathbb{P}(\mathcal{H}^{f_t}(t,0) + \frac{t}{24} \geq st)$ where s is a positive real number. We ask what is the upper tail large deviation rate function, namely, what is the limit of $t^{-1}\log \mathbb{P}(\mathcal{H}^{f_t}(t,0) + \frac{t}{24} \geq st)$ as t goes to ∞. Our main result which we state as follows computes the p-th moment Lyapunov exponent of the SHE solutions $\{\mathcal{Z}^{f_t}\}_{t>0}$ and the upper tail large deviation rate function of the Cole-Hopf solution $\{\mathcal{H}^{f_t}\}_{t>0}$. We defer its proof to Section 3.2.

Theorem 3.1.2. *Let $\big(g, \{f_t\}_{t\geq 0}\big)$ be a set of functions in the class* **Hyp**. *Then, we have the following:*

(a) For any $p \in \mathbb{R}_{>0}$,

$$\mathrm{Lya}_p(\{f_t\}_{t>0}) = \frac{p^3 - p}{24} + g(p). \tag{3.1.10}$$

(b) Suppose $g(p) \in C^1(\mathbb{R}_{>0})$ and $\zeta := \lim_{p\to 0} g'(p)$ is finite. Then, for $s > \zeta$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{f_t}(t,0) + \frac{t}{24} > st\Big) = -\max_{p\geq 0}\big\{sp - \frac{p^3}{24} - g(p)\big\}. \tag{3.1.11}$$

Remark 3.1.3. The class **Hyp** in Theorem 3.1.2 contains a large collection of interesting initial profiles for the KPZ equation. Corollary 3.1.4 will show the application of Theorem 3.1.2 to a wide variety of deterministic initial data with moderate growth whereas Corollary 3.1.6 will show the same for the Brownian initial data. It is only bounded deterministic initial data and the delta initial data of the SHE for which all integer moment Lyapunov exponents were known (see

[BC95, Che15, BC14b, CG20a]) before. We would like to stress that the *narrow wedge initial data* of the KPZ equation which corresponds to taking $\mathcal{Z}(0,x) = \delta_{x=0}$ (i.e., the delta initial data of the SHE) is not covered by Theorem 3.1.2. However, the fractional moment Lyapunov exponents in the narrow wedge case are recently found in [DT19] and those are one of the key inputs to our proof of Theorem 3.1.2.

Corollary 3.1.4 (Deterministic initial data)**.** *Consider a class of measurable functions $f_t : \mathbb{R} \to \mathbb{R}$ with the following properties:*

(i). There exist $\delta, \alpha \in (0,1)$ and constant $C, t_0 > 0$ such that for $t > t_0$, $|f_t(x)| \leq C\big(1+|x|^{\delta}\big)+\frac{\alpha x^2}{2t}$

(ii). There exist $\beta \geq 0$, $t_0, s_0 > 0$ and a sequence of grid points (see pseudo-stationarity condition of Definition 3.1.1) $\{\theta_n\}_{n\in\mathbb{Z}}$ such that $|f_t(y) - f_t(\theta_n)| \leq s_0$ for all $t \geq t_0$, $y \in [\theta_n, \theta_{n+1}]$ and $n \in \mathbb{Z}$.

We assume that there exists unique solution of the SHE started from initial data e^{f_t} (at least) up to time t. Recall the notations $\mathcal{Z}^{f_t}$ and $\mathcal{H}^{f_t}$ which denote the unique solution of the SHE (started from e^{f_t}) and the corresponding Cole-Hopf solution of the KPZ equation respectively. Then, we have the following:

a) For any $p \in \mathbb{R}_{>0}$,

$$\mathrm{Lya}_p(\{f_t\}_{t>0}) = \frac{p^3 - p}{24}. \tag{3.1.12}$$

b) For all $s \in \mathbb{R}_{>0}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{f_t}(t,0) + \frac{t}{24} > st\Big) = -\frac{4\sqrt{2}}{3} s^{\frac{3}{2}}. \tag{3.1.13}$$

Remark 3.1.5. Note that Corollary 3.1.4 applies to any bounded deterministic initial data. To see this, we set $f_t := f$ for all $t > 0$ where f is any given bounded deterministic function. For this choice of the sequence $\{f_t\}_{t>0}$, the conditions of Corollary 3.1.4 will be trivially satisfied with $\theta_n = n$ for all $n \in \mathbb{Z}$. Thus, Corollary 3.1.4 gives Lyapunov exponents for any moments of the $(1 + 1)$-dimensional SHE started from any bounded deterministic positive initial data. This solves an open problem mentioned in [Che15, pg. 1489, (1.12)]. Moreover, Corollary 3.1.4 is applicable to a far larger class of deterministic initial data. For instance, consider the class of function $\phi_\delta(x) = |x|^\delta$ indexed by $\delta \in (0,1)$. By taking $\theta_n = n$ for all $n \in \mathbb{Z}$ and setting $f_t := \phi_\delta$ for all $t > 0$, it is straightforward to see that ϕ_δ satisfies both conditions of Corollary 3.1.4. Thus, ϕ_δ has the same value of the Lyapunov exponents as in the case of constant initial data. The same conclusion also holds for the class of functions $\psi_\alpha(x) = \alpha x^2/2t$ indexed $\alpha \in (0,1)$. To see this, its suffices to verify both conditions of Corollary 3.1.4 for ψ_α. It is easy to check that ψ_α satisfies condition (i). To check the other condition, we define θ_n to be $\mathrm{sign}(n) \times |n|^{1/2}$ for any $n \in \mathbb{Z}$. Note that $||n+1|^{1/2} - |n|^{1/2}| \geq \min\{1, 4^{-1}|n|^{-1/2}\}$ for any $n \in \mathbb{Z}$. Thus, $\{\mathrm{sign}(n) \times |n|^{1/2}\}_{n\in\mathbb{Z}}$ is indeed a sequence of grid point. Since $\sup_{x\in[\theta_n,\theta_{n+1}]} |\psi_\alpha(x) - \psi_\alpha(\theta_n)|$ is bounded above by $1/2t$, ψ_α also satisfies the second condition of Corollary 3.1.4 for all large t.

Corollary 3.1.6 (Brownian initial data)**.** *Let $B(x)$ be a two-sided Brownian motion. For any $t > 0, x \in \mathbb{R}$, define $f_t(x) := B(x) + a_+ x \mathbb{1}_{\{x>0\}} - a_- x \mathbb{1}_{\{x<0\}}$ where a_+ and a_- are the drift parameters for $x > 0$ and $x < 0$ respectively. Denote by $a = \max\{a_+, a_-\}$. Then, we have*

a) For any $p > 0$,

$$\mathrm{Lya}_p(\{f_t\}_{t>0}) = \frac{p^3}{24} - \frac{p}{24} + \frac{p}{2}\Big(\max\big\{(\frac{p}{2}+a), 0\big\}\Big)^2 \tag{3.1.14}$$

b) If $a \geq 0$, then for all $s > \frac{a^2}{2}$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{f_t}(t,0) + \frac{t}{24} > st\Big) = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}} + sa - \frac{a^3}{6}. \tag{3.1.15}$$

If $a < 0$, then,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{f_t}(t,0) + \frac{t}{24} > st\Big) = \begin{cases} -\frac{4}{3}\sqrt{2} s^{\frac{3}{2}} & 0 < s \leq \frac{a^2}{2} \\ -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}} + sa - \frac{a^3}{6} & s \geq \frac{a^2}{2}. \end{cases}$$

The proofs of Corollary 3.1.4 and 3.1.6 will be given in Section 3.3.

Remark 3.1.7. It is worthwhile to note the contrast between the upper tail large deviation probability (LDP) of the KPZ equation under constant or, narrow wedge initial data and the Brownian initial data. The large deviation rate function in (3.1.15) is $-\frac{2\sqrt{2}}{3}s^{3/2}$ when both the drift parameters a_+ and a_- are equal to 0. Comparing this rate function with (3.1.13) or, [DT19, Theorem 1.1] shows a difference by a factor of 2. This difference between the upper tail LDPs is consistent with the difference of the upper tail asymptotics of the KPZ equation under KPZ scaling. In the physics literature, the contrast between the LDPs under different initial data of the KPZ equation is echoed in [LDMRS16, LDMS16, MS17]. Our result rigorously confirms those predictions.

3.1.1 Proof Ideas

In this section, we will present a sketch of the proof of Theorem 3.1.2 and review the relevant tools that we use. We start with setting some necessary notations. The *narrow wedge* solution $\mathcal{H}^{\mathbf{nw}}$ of the KPZ is the Cole-Hopf transform of the fundamental solution of SHE $\mathcal{Z}^{\mathbf{nw}}$ which is associated to the the delta initial data $\mathcal{Z}^{\mathbf{nw}}(0,x) = \delta_{x=0}$. As t goes to 0, $\log \mathcal{Z}^{\mathbf{nw}}(t,x)$ is well approximated by the heat kernel whose logarithm is given by a thin parabola $\frac{x^2}{2t}$, rendering $\mathcal{H}^{\mathbf{nw}}$ to have *narrow wedge* like structure.

The proof of our main results consists of following tools: (1) a composition law which connects the SHE under general initial data with its fundamental solution, (2) Lyapunov exponents of the fundamental solution of the SHE, and (3) tails bounds on the spatial regularity of the narrow wedge solution of the KPZ.

The following identity gives a convolution formula of the one point distribution of the KPZ equation in terms of the spatial process $\mathcal{Z}^{\mathbf{nw}}(t,\cdot)$ and the initial data f of the KPZ equation. The proof of this formula is associated to the linearity and time reversal property of the SHE. For details, we refer to Lemma 1.18 of [CH16].

Proposition 3.1.8 (Convolution Formula)**.** *Let $\mathcal{Z}^f$ be the unique solution of the SHE started from the initial condition e^f for a measurable function $f : \mathbb{R} \to \mathbb{R}$. Then for any $t > 0$,*

$$\mathcal{Z}^f(t,0) \stackrel{d}{=} \int \mathcal{Z}^{\mathbf{nw}}(t,y) e^{f(y)} dy$$

To complement Proposition 3.1.8, we will make use of the exact expressions of any real positive moment Lyapunov exponents of the fundamental solution of the SHE from [DT19] and the tail bounds on the spatial regularity of the narrow wedge solution of the KPZ equation from [CGH19]. The following result describes the first of these two tools.

Proposition 3.1.9 (Lyapunov exponents of fundamental solution, [DT19], Theorem 1.1)**.** *For every $p > 0$,*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\Big] = \frac{p^3 - p}{24}.$$

The third of our main tools is (super)-exponential tail bounds on the spatial regularity of the narrow wedge solution of the KPZ equation. This is given in the following result.

Proposition 3.1.10 (Tail bounds of increments, [CGH19], Prop. 4.4)**.** *For any $t_0 > 1$, $\nu > 0$ and $\epsilon \in (0,1)$, there exist $s_0 = s_0(t_0,\nu,\epsilon)$ and $c = c(t_0,\nu,\epsilon)$ such that, for $t \geq t_0$ and $s \geq s_0$,*

$$\mathbb{P}\Big(\sup_{x\in[0,t^{\frac{1}{3}}]} \Big\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) - \frac{\nu x^2}{2}\Big\} \geq s\Big) \leq \exp(-cs^{\frac{9}{8}-\epsilon}). \tag{3.1.16}$$

$$\mathbb{P}\Big(\inf_{x\in[0,t^{\frac{1}{3}}]} \Big\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2}\Big\} \leq -s\Big) \leq \exp(-cs^{\frac{9}{8}-\epsilon}). \tag{3.1.17}$$

In order to prove (3.1.10) of Theorem 3.1.2, we first use Proposition 3.1.8 to note

$$\mathrm{Lya}_p(\{f_t\}_{t>0}) = \lim_{t\to\infty} \frac{1}{t} \log\Big(\mathbb{E}\Big[\Big(\int_{\mathbb{R}} \mathcal{Z}^{\mathbf{nw}}(t,y) e^{f_t(y)} dy\Big)^p\Big]\Big)$$

and thereafter, focus our effort to analyze $\mathbb{E}[(\int_{\mathbb{R}} \mathcal{Z}^{\mathbf{nw}}(t,y)\exp(f_t(y))dy)^p]$. Recall $M_p^{f_t}(t,x)$ from Definition 3.1.1. Our main technical achievement is to justify the following heuristic approximation

$$\frac{1}{t}\log \mathbb{E}\Big[\Big(\int_{\mathbb{R}} \mathcal{Z}^{\mathbf{nw}}(t,y) e^{f_t(y)} dy\Big)^p\Big] \approx \frac{1}{t}\log \int_{\mathbb{R}} \mathbb{E}\big[(\mathcal{Z}^{\mathbf{nw}}(t,y))^p\big] M_p^{f_t}(t,x) dx \tag{3.1.18}$$

for all large $t \in \mathbb{R}_{>0}$ where $\approx$ signs indicates the equality upto some additive constant which decays as $t \to \infty$. For this, we observe that the main contributions of the left hand side of (3.1.18) comes from $\mathcal{Z}^{\mathbf{nw}}(t,y_0)e^{f_t(y_0)}$ where y_0 is a point in $\mathbb{R}$ such that the function $\phi(y) := -\frac{py^2}{2} + \log M_p^{f_t}(t,y)$ when evaluated at y_0 attains a close proximity to the supremum value $\sup_{y\in R}\phi(y)$. Similarly, we find that the main contribution of the right hand side of (3.1.18) comes from $\mathbb{E}\big[(\mathcal{Z}^{\mathbf{nw}}(t,y_0))^p\big]M_p^{f_t}(t,y_0)$.

For showing there are indeed such local representatives of both sides of (3.1.18), we require to demonstrate that the contributions of $\mathbb{E}\big[(\int_{\mathbb{R}\setminus B}\mathcal{Z}^{\mathbf{nw}}(t,y)e^{f_t(y)}dy)^p\big]$ and $\int_{\mathbb{R}\setminus B}\mathbb{E}\big[(\mathcal{Z}^{\mathbf{nw}}(t,y))^p\big]M_p^{f_t}(t,x)dx$ cannot grow significantly higher than their local counterparts where B is small interval around y_0. This is done by controlling fluctuation of the spatial process $\mathcal{Z}^{\mathbf{nw}}(t,\cdot)$ and the growth and regularity of the initial data f_t. The fluctuation of $\mathcal{Z}^{\mathbf{nw}}(t,\cdot)$ is controlled by the tail probability bounds (3.1.16) and (3.1.17) on the spatial regularity of $\mathcal{H}^{\mathbf{nw}}(t,\cdot)$. The growth and the regularity estimates of the initial data are provided by the growth and lower bound conditions (3.1.6), (3.1.7) and pseudo-stationarity condition (3.1.8) of Definition 3.1.1.

To analyze the integral on the right hand side of the above display, one may first ask how we deal with $\mathbb{E}\big[(\mathcal{Z}^{\mathbf{nw}}(t,y))^p\big]$ for all $y\in\mathbb{R}$. This will be done by combining Proposition 3.1.9 with the following result.

Proposition 3.1.11 (Stationarity, [ACQ11], Prop. 1.4)**.** *For any fixed $t>0$, the random process $\mathcal{H}^{nw}(t,x)+\frac{x^2}{2t}$ is stationary in x.*

Proposition 3.1.11 allows to write the right hand side of (3.1.18) as sum of two terms, namely, $t^{-1}\log(\mathbb{E}[(\mathcal{Z}^{\mathbf{nw}}(t,0))^p])$ and $t^{-1}\log\big(\int_{\mathbb{R}}\exp(-\frac{px^2}{2t})M_p^{f_t}(t,x)dx\big)$. To conclude the proof of (3.1.10), it suffices to show that the limits of those two summands of the right hand side of (3.1.18) coincide with $(p^3-p)/24$ and $g(p)$ respectively as $t\to\infty$. The first limit is given in Proposition 3.1.9 and the second limit will be proved using the coherence conditions (3.1.4) and (3.1.5).

For showing (3.1.11) of Theorem 3.1.2, our main tool is the following proposition which relates the upper tail large deviation rate function of $\mathcal{H}^{f_t}$ in terms of the Lyapunov exponents.

Proposition 3.1.12. *Let $X(t)$ be a stochastic process indexed by $t\in\mathbb{R}_{>0}$. Fix $h\in C^1(\mathbb{R}_{>0})$ such that $h':(0,\infty)\to(\zeta,\infty)$ is continuous, bijective and increasing for some $\zeta\in\mathbb{R}$. Assume that*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\Big[e^{pX(t)}\Big] = h(p), \quad \forall p\in\mathbb{R}_{>0}. \tag{3.1.19}$$

Then, we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{P}\Big(X(t)>st\Big) = -\sup_{p>0}\{ps-h(p)\},\quad \forall s>\zeta. \tag{3.1.20}$$

We defer the proof of this proposition to Section 3.4.1. Allying this proposition with (3.1.10) yields the proof of (3.1.11). It is worthwhile to note that the use of Proposition 3.1.12 necessitates $g(\cdot)$ (in Definition 3.1.1) to be a convex function and few other technical properties. Under the assumption that $\big(g,\{f_t\}_{t\geq 0}\big)$ belong to a class **Hyp**, the following lemma shows that $g(\cdot)$ indeed satisfies those properties. The proof of this lemma is deferred to Section 3.4.2.

Lemma 3.1.13. *For any set of functions $\big(g,\{f_t\}_{t\geq 0}\big)$ in the class* **Hyp**, *we have the following:*

(i) g is convex and non-negative.

(ii) For every $p>0$ and $\omega>0$, define

$$\mathrm{MAX}^{f}_{p,\omega}(t) := \Big\{x : -\frac{px^2}{2t}+\log M^{f_t}_{p}(t,x) \geq \sup_{y\in\mathbb{R}}\big\{-\frac{py^2}{2t}+\log M^{f_t}_{p}(t,y)\big\}-\omega\Big\}. \tag{3.1.21}$$

Then there exists $T_0>0$ such that for $t>T_0$, $\mathrm{MAX}^{f}_{p,\omega}(t)$ is nonempty for all $p,\omega>0$. Define $x_{p,\omega}(t) := \mathrm{argmax}\big\{|x| : x\in\mathrm{MAX}^{f}_{p,\omega}(t)\big\}$. There exists a constant $C=C(p,\omega)>0$ such that for all $t>T_0$, $|x_{q,\omega}(t)|\leq Ct$ for all $\frac{p}{2}<q<2p$.

3.1.2 Previous works

Our main result on the Lyapunov exponent of the SHE and the upper tail large deviation of the KPZ equation fits into the broader endeavor of studying the intermittency phenomenon and large deviation problems of the random field solution of stochastic partial differential equations. Intermittency, an universal phenomenon for random fields of mutiplicative type is characterized by enormous moment growth rate of the random field. The nature of the intermittency is captured through the Lyapunov exponents. In last few decades, there were extensive amount of works on studying the growth rate of Lyapunov exponents under variation in structure of the noise [GM90, CM94, BC95, HHNT15, FK09, CJKS13, CD15, BC16] and the partial differential operators [Che17, CHN19]. Large deviation of the stochastic partial differential equations [HW15a, CD19] is an active area of research in recent years. Upper and lower tail large deviation of the KPZ equation behold special interests in theoretical as well as in experimental side and have been recently investigated in a vast amount of works. For detailed history along this line of works, we refer to [LDMRS16, HLDM+18a, CGK+18, Tsa18, DT19, KLD19] and the references therein. Below, we compare our results with few of those previous works.

Based on the replica Bethe ansatz techniques, Kardar [Kar87, Section 2.2] predicted the integer moment Lyapunov exponents of the fundamental solution of the SHE. Bertini and Cancrini [BC95, Section 2.4] made a rigorous attempt to show the exact match between the integer moment Lyapunov exponents of the SHE under constant initial data and Kardar's prediction. Unfortunately, the computation of [BC95] was incorrect beyond the second moment Lyapunov exponent. This was later fixed by [Che15] who computed all integer moments Lyapunov exponents for any deterministic bounded positive initial data of the SHE. The main tool of [Che15] was the moment formulas of the SHE in terms of integral of local time of Brownian bridges derived from the Feynman-Kac representation of the solution.

Alternatively, the integer moments of the fundamental solution of the SHE which are widely believed to be same as the solution of the attractive delta-Bose gas have formulas in terms of contour integrals. We refer to [Gho18] and the reference therein for a comprehensive discussion on this. Similar formulas are known for the moments of the parabolic Anderson model, semi-discrete directed polymers, q-Whittaker process (see [BC14b, BC14a]) etc. By analyzing the contour integrals, [CG20a] derived a sharp upper and lower bound to the integer moments of the fundamental solution of SHE which positively confirms Kardar's prediction. Recently, [DT19] were able to obtain similar tight upper and lower bound to the fractional moments. Using sharp bounds on the moments, [DT19] computed any positive fractional moment Lyapunov exponent of the fundamental solution. As an application of their result, [DT19] also found the one point upper tail large deviation of the narrow wedge solution of the KPZ equation. We refer to [HHNT15, CHKN18] for tight bounds on the moments of the SHE when the noise is colored in space/time and the initial data is a continuous bounded function.

In spite of a substantial amount of works on the fractional moments Lyapunov exponents of the SHE with colored noise, the case of general initial data for the SHE with white noise was largely being untouched. The same conclusion applies to the status of the upper tail large deviation result for the KPZ equation started from general initial data. However, tight bounds on the upper tail probabilities of the KPZ are available. For instance, [CG20a] obtained the following result: for any $t_0 > 0$, there exists $s_0 = s_0(t_0), c_1 = c_1(t_0), c_2 = c_2(t_0) > 0$ such that for all $s > s_0$ and $t > t_0$,

$$e^{-c_1 s^{3/2}} \leq \mathbb{P}\big(\mathcal{H}(t,0) + \frac{t}{24} \geq st^{1/3}\big) \leq e^{-c_2 s^{3/2}} \tag{3.1.22}$$

where the initial data of the KPZ solution $\mathcal{H}$ belongs to a large class of functions including the narrow wedge and the stationary initial data. We refer to Section of [CG20a] and the references therein for more information. In the physics literature, the upper tail large deviation of the KPZ equation has been studied recently using optimal fluctuation theory which corresponds to Freidlin-Wentzell type large deviation theory of stochastic PDEs with small noise. By formal computations, [MKV16, JKM16, MS17] (see also [LDMRS16, LDMS16]) demonstrated the upper tail LDP of the KPZ started from a large class of initial data including the flat and stationary data. Corollary 3.1.4 and 3.1.6 rigorously confirms those results from physics literature. In a way, Theorem 3.1.2 is the first result which provides a concrete pathway to compute the Lyapunov exponent of the SHE started from general initial data and the upper tail large deviation rate function of the associated Cole-Hopf solution of the KPZ equation.

The probability of the KPZ equation being smaller than its typical value is captured through its lower tail probability. Like the upper tail, one point lower tail probabilities of the KPZ equation are equally important. The first tight estimates of the lower tail probabilities of the narrow wedge solution is obtained in [CG20b] and the lower tail large deviation is rigorously proved in [Tsa18, CC19] (see also [CGK+18, KLD19] and the reference therein). The case of general initial data was considered in [CG20a] where the authors provided an upper bound to the lower tail probability of the KPZ equation. However, there are only very few things known about the lower tail large deviation under general initial data. In the physics literature, recently [Le 19] found a connection between the latter and the Kadomtsev-Petviashvili (KP) equation. It is unclear to us how much of the techniques of the present paper will help to get the lower tail large deviation of the KPZ equation started from general initial data.

Outline

Section 3.2 will prove the Theorem 3.1.2. Applying Theorem 3.1.2, Corollary 3.1.4 and Corollary 3.1.6 will be shown in Subsections 3.3.1 and 3.3.2 of Section 3.3. Proofs of Proposition 3.1.12 and Lemma 3.1.13 are given in Subsection 3.4.1 and 3.4.2 of Section 3.4.

Acknowledgment: PG and YL would like to thank Ivan Corwin, Sayan Das, Shalin Parekh for helpful conversations. YL was partially supported by the Fernholz Foundation's "Summer Minerva Fellow" program and also received summer support from Ivan Corwin's NSF grant DMS-1811143, DMS-1664650.

3.2 Lyapunov exponents and large deviation: Proof of Theorem 3.1.2

The main goal of this section is to prove Theorem 3.1.2. The part (a) of Theorem 3.1.2 is to compute the Lyapunov exponents $\lim_{t\to\infty} t^{-1}\log\mathbb{E}[\mathcal{Z}^{f_t}(t,0)^p]$ for all $p\in\mathbb{R}_{>0}$. The part (b) involves showing the upper tail large deviation rate function of the KPZ equation. Both of these two results are proved for general initial data. The part (b) is a straightforward consequence of part (a). This is shown in Section 3.2.3 using Proposition 3.1.12. We prove part (a) as follows.

Note that (3.1.10) follows once we show for all $p\in\mathbb{R}_{>0}$,

$$\underbrace{\frac{p^3-p}{24}+g(p)\leq \liminf_{t\to\infty} t^{-1}\log\mathbb{E}[\mathcal{Z}^{f_t}(t,0)^p]}_{\mathsf{LimInf}_p}\leq \underbrace{\limsup_{t\to\infty} t^{-1}\log\mathbb{E}[\mathcal{Z}^{f_t}(t,0)^p]\leq \frac{p^3-p}{24}+g(p)}_{\mathsf{LimSup}_p}. \tag{3.2.1}$$

We denote the left and right inequality by LimInf_p and LimSup_p and the proof of these inequalities will be shown in Section 3.2.1 and 3.2.2 respectively.

3.2.1 Proof of LimSup_p for all $p \in \mathbb{R}_{>0}$

We divide the proof in two stages. In *Stage 1*, we prove LimSup_p inequality when $p > 1$ and *Stage 2* will cover the case when $p \in (0, 1]$.

Stage 1:

There are two main steps in the proof of this stage. The first step is to obtain the following upper bound

$$\mathbb{E}\Big[\mathcal{Z}^{f_t}(t,0)^p\Big] \leq \Big(\frac{2\pi t}{\epsilon q}\Big)^{\frac{p}{2q}}\mathbb{E}\Big[\int_{-\infty}^{\infty} e^{\frac{\epsilon p x^2}{2t}}\mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{pf_t(x)}dx\Big]. \tag{3.2.2}$$

by applying Hölder's inequality in the convolution formula of Proposition 3.1.8. The second step is to bound the expectation of the right hand side of the above display. For this, we first distribute the expectation over $\mathcal{Z}^{\mathbf{nw}}(t,x)^p$ and $e^{pf_t(x)}$ as x varies in $\mathbb{R}$. The computation of the expectation of $\mathcal{Z}^{\mathbf{nw}}(t,x)^p$ for $x \in \mathbb{R}$ will be carried out using the spatial stationarity of $\mathcal{Z}^{\mathbf{nw}}(t,x)$ from Proposition 3.1.11 and the narrow wedge LDP from Proposition 3.1.9. For the upper bound on the part involving $e^{pf_t(x)}$, we use the property (3.1.5). Below, we give details of each step.

By the convolution formula of Proposition 3.1.8, $\mathbb{E}[(\mathcal{Z}^{f_t}(t,0))^p]$ is equal to $\mathbb{E}[(\int_{-\infty}^{\infty}\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx)^p]$. In what follows, we bound $\int_{-\infty}^{\infty}\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx$ in order to show (3.2.2). Denote by $q = \frac{p}{p-1}$. We write $\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}$ as a product of $e^{-\frac{\epsilon x^2}{2t}}$ and $e^{\frac{\epsilon x^2}{2t}}\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}$. By applying Hölder's inequality

$$\int_{-\infty}^{\infty}\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx \leq \Big(\int_{-\infty}^{\infty} e^{-\frac{\epsilon q x^2}{2t}}dx\Big)^{\frac{1}{q}}\Big(\int_{-\infty}^{\infty} e^{\frac{\epsilon p x^2}{2t}}\mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{pf_t(x)}dx\Big)^{\frac{1}{p}},$$

The last inequality in conjunction with the fact that $\int_{-\infty}^{\infty} e^{-\frac{\epsilon q x^2}{2t}} dx$ is equal to $\sqrt{2\pi t/\epsilon q}$ yields

$$\mathbb{E}\Big[\Big(\int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x) e^{f_t(x)} dx\Big)^p\Big] \leq \Big(\frac{2\pi t}{\epsilon q}\Big)^{\frac{p}{2q}} \mathbb{E}\Big[\int_{-\infty}^{\infty} e^{\frac{\epsilon p x^2}{2t}} \mathcal{Z}^{\mathbf{nw}}(t,x)^p e^{p f_t(x)} dx\Big].$$

Note that the above inequality shows the upper bound in (3.2.2). We apply Fubini's theorem to interchange the expectation and the integral in the above display. Using the stationarity of $\mathcal{Z}^{\mathbf{nw}}(t,x)e^{x^2/2t}$ (see Proposition 3.1.11), one can write the expectation in the right hand side of the above display as the product of $\mathbb{E}\big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p\big]$ and $\int_{-\infty}^{\infty} e^{\frac{-(1-\epsilon)p x^2}{2t}} M_p^{f_t}(t,x) dx$ where $M_p^{f_t}(t,x)$ is defined in the coherence conditions (see Definition 3.1.1) for the KPZ data $(g, \{f_t\}_{t>0})$. Taking logarithm on both sides of the inequality, dividing by t and letting $t \to \infty$, $\epsilon \to 0$ shows

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[(\mathcal{Z}^{f_t}(t,0))^p\Big] \leq \frac{p^3-p}{24} + \liminf_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log\Big(\int_{-\infty}^{\infty} e^{-\frac{(1-\epsilon)p x^2}{2t}} M_p^{f_t}(t,x) dx\Big)$$

where the factor $(p^3-p)/24$ in the right hand side is obtained by applying Proposition 3.1.9. To get the desired upper bound in LimSup_p, it suffices to show that

$$\liminf_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log\Big(\int_{-\infty}^{\infty} e^{-\frac{(1-\epsilon)p x^2}{2t}} M_p^{f_t}(t,x) dx\Big) \leq g(p). \tag{3.2.3}$$

For showing (3.2.3), we use the property (3.1.5) of the KPZ data $\big(g, \{f_t\}_{t\geq 0}\big)$. By Hölder's inequality, $\mathbb{E}\big[e^{p f_t(x)}\big]$ is bounded above by $(\mathbb{E}[e^{p(1+\epsilon) f_t(x)}])^{1/(1+\epsilon)}$ which we can bound by $1+\mathbb{E}[e^{p(1+\epsilon) f_t(x)}]$. Applying this upper bound into the left hand side of (3.2.3),

$$\begin{aligned}
\text{l.h.s. of (3.2.3)} &\leq \liminf_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log\Big(\int_{-\infty}^{\infty} e^{-\frac{(1-\epsilon)p x^2}{2t}} \big(1 + M_{p(1+\epsilon)}^{f_t}(t,x)\big) dx\Big) \\
&\leq \liminf_{\epsilon\to 0} \limsup_{t\to\infty} \frac{1}{t} \log\Big(\sqrt{\frac{2\pi t}{(1-\epsilon)p}} + \int_{-\infty}^{\infty} e^{-\frac{(1-\epsilon)p x^2}{2t}} M_{p(1+\epsilon)}^{f_t}(t,x) dx\Big) \\
&\leq g(p).
\end{aligned}$$

We have used the property $g(p) \geq 0$ from Lemma 3.1.13 (i) and (3.1.5) in the last line. This completes the proof when $p > 1$.

Stage 2:

The derivation of LimSup_p for $p \in (0,1]$ depends on Proposition 3.2.1 and 3.2.2. We first state them; use them to prove LimSup_p for $p \in (0,1]$; and then, prove them in turn.

Proposition 3.2.1. *Fix any $\nu \in (0,1)$. For any $u, v \in \mathbb{R}$, define a random function $S_{[u,v]} : (0,\infty) \to \mathbb{R}$ as*

$$S_{[u,v]}(t) := \sup_{x\in[u,v]} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,u) + \frac{(x-u)u}{t} - \frac{\nu(x-u)^2}{2}\Big). \tag{3.2.4}$$

Let $\{\theta_n\}_{n\in\mathbb{Z}}$ be a sequence of grid points such that the sequence $\{f_t\}_{t>0}$ satisfies (3.1.8). *Then, we have the following:*

(i) For all $n \in \mathbb{Z}$,

$$\Big(\mathcal{H}^{\mathbf{nw}}(t,\theta_n) + \frac{\theta_n^2}{2t}, S_{[\theta_n,\theta_{n+1}]}(t)\Big) \stackrel{d}{=} \Big(\mathcal{H}^{\mathbf{nw}}(t,0), S_{[0,\theta_{n+1}-\theta_n]}(t)\Big), \tag{3.2.5}$$

(ii) For all $p \in \mathbb{R}_{>0}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}\Big] \le \frac{p^3-p}{24}. \tag{3.2.6}$$

Proposition 3.2.2. *For any $n \in \mathbb{Z}$, we define $\mathbf{E}^{(n)}_{t,p} := \mathbb{E}[(\int_{\theta_n}^{\theta_{n+1}} e^{f_t(x)}dx)^p]$. Then,*

$$\limsup_{t\to\infty} \frac{1}{t} \log \left(\sum_{n\in\mathbb{Z}_{\ge 0}} e^{-\frac{p\theta_n^2}{2t}} \mathbf{E}^{(n)}_{t,p} + \sum_{n\in\mathbb{Z}_{<0}} e^{-\frac{p\theta_{n+1}^2}{2t}} \mathbf{E}^{(n)}_{t,p}\right) \le g(p). \tag{3.2.7}$$

Proof of LimSup_p for $p \in (0,1]$: Fix $p \in (0,1]$. We show that there exists $C = C(p) > 0$ such that for all $t > 0$,

$$\mathbb{E}\Big[\mathcal{Z}^{f_t}(t,0)^p\Big] \le C\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}\Big]\left(\sum_{n\in\mathbb{Z}_{\ge 0}} e^{-\frac{p\theta_n^2}{2t}} \mathbf{E}^{(n)}_{t,p} + \sum_{n\in\mathbb{Z}_{<0}} e^{-\frac{p\theta_{n+1}^2}{2t}} \mathbf{E}^{(n)}_{t,p}\right). \tag{3.2.8}$$

From the above inequality, we first show how LimSup_p follows. By taking logarithms of both sides of the above inequality, dividing them by t and letting $t \to \infty$, we get LimSup_p once the following inequalities are satisfied

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}\Big] \le \frac{p^3-p}{24},$$
$$\limsup_{t\to\infty} \frac{1}{t} \log \left(\sum_{n\in\mathbb{Z}_{\ge 0}} e^{-\frac{p\theta_n^2}{2t}} \mathbf{E}^{(n)}_{t,p} + \sum_{n\in\mathbb{Z}_{<0}} e^{-\frac{p\theta_{n+1}^2}{2t}} \mathbf{E}^{(n)}_{t,p} \right) \le g(p).$$

But, these two inequalities are given by (3.2.6) and (3.2.7) of Proposition 3.2.1 and 3.2.2 respectively. This completes the proof of LimSup_p when $p \in (0,1]$ modulo (3.2.8) which we prove as follows.

By the convolutional formula of Proposition 3.1.8, it suffices to show (3.2.8) with $\mathbb{E}[(\int \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx)^p]$ in place of $\mathbb{E}[\mathcal{Z}^{f_t}(t,0)^p]$. Owing to the subadditivity of function $g(x) = x^p$ for $x > 0$ and $p \in (0,1]$,

$$\begin{aligned}\mathbb{E}\Big[\big(\int \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\big)^p\Big] &= \mathbb{E}\Big[\Big(\sum_{n\in\mathbb{Z}} \int_{\theta_n}^{\theta_{n+1}} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\Big)^p\Big] \\ &\le \sum_{n\in\mathbb{Z}} \mathbb{E}\Big[\Big(\int_{\theta_n}^{\theta_{n+1}} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\Big)^p\Big] \end{aligned} \tag{3.2.9}$$

Note that (3.2.8) follows from the above inequality if there exists $C = C(p) > 0$ such that

$$\mathbb{E}\Big[\Big(\int_{\theta_n}^{\theta_{n+1}} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\Big)^p\Big] \le C\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}\Big]\mathbf{E}^{(n)}_{t,p} \times \begin{cases} e^{-\frac{p\theta_n^2}{2t}} & n \ge 0, \\ e^{-\frac{p\theta_{n+1}^2}{2t}} & n < 0. \end{cases} \tag{3.2.10}$$

holds for all $n \in \mathbb{Z}$. We show this bound below.

We first show (3.2.10) for $n \ge 0$. Recall the definition of $S_n(t)$ from (3.2.4). Since $S_{[\theta_n,\theta_{n+1}]}(t)$ is greater than $\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,\theta_n) + (x-\theta_n)\theta_n/t - \nu(x-\theta_n)^2/2$ for any $x \in [\theta_n, \theta_{n+1}]$, we may write

$$\mathcal{H}^{\mathbf{nw}}(t,x) \leq \mathcal{H}^{\mathbf{nw}}(t,\theta_n) + S_{[\theta_n,\theta_{n+1}]}(t) - \frac{(x-\theta_n)\theta_n}{t} + \frac{\nu(x-\theta_n)^2}{2}. \tag{3.2.11}$$

Exponentiating both sides of the inequality yields

$$\mathcal{Z}^{\mathbf{nw}}(t,x) \leq \mathcal{Z}^{\mathbf{nw}}(t,\theta_n)e^{S_{[\theta_n,\theta_{n+1}]}(t)}e^{\frac{\nu(x-\theta_n)^2}{2}}e^{-\frac{(x-\theta_n)\theta_n}{t}} \leq C\mathcal{Z}^{\mathbf{nw}}(t,\theta_n)e^{S_{[\theta_n,\theta_{n+1}]}(t)} \tag{3.2.12}$$

where the last inequality follows since $\exp(2^{-1}\nu(x-\theta_n)^2 - t^{-1}(x-\theta_n)\theta_n)$is upper bounded by a constant over $x \in [\theta_n, \theta_{n+1}]$. Bounding $\mathcal{Z}^{\mathbf{nw}}(t,x)$ with $C\mathcal{Z}^{\mathbf{nw}}(t,\theta_n)e^{S_{[\theta_n,\theta_{n+1}]}(t)}$ yields

$$\Big(\int_{\theta_n}^{\theta_{n+1}} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\Big)^p \leq C\mathcal{Z}^{\mathbf{nw}}(t,\theta_n)^p e^{pS_{[\theta_n,\theta_{n+1}]}(t)}\Big(\int_{\theta_n}^{\theta_{n+1}} e^{f_t(x)}dx\Big)^p.$$

Taking the expectation for both sides in the above display and using the independence between $\mathcal{Z}^{\mathbf{nw}}(t,\cdot)$ and $f_t(\cdot)$ shows

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_{\theta_n}^{\theta_{n+1}} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\Big)^p\Big] &\leq C\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,\theta_n)^p e^{pS_{[\theta_n,\theta_{n+1}]}(t)}\Big]\mathbf{E}_{t,p}^{(n)} \\ &= C\mathbb{E}\Big[\big(\mathcal{Z}^{\mathbf{nw}}(t,\theta_n)e^{\frac{\theta_n^2}{2t}}\big)^p e^{pS_{[\theta_n,\theta_{n+1}]}(t)}\Big]\mathbf{E}_{t,p}^{(n)}e^{-\frac{p\theta_n^2}{2t}} \end{aligned}\tag{3.2.13}$$

By (3.2.5) of Proposition 3.2.1, $(\mathcal{Z}^{\mathbf{nw}}(t,\theta_n)e^{\frac{\theta_n^2}{2t}}, S_{[\theta_n,\theta_{n+1}]}(t))$ is same in distribution with

$$(\mathcal{Z}^{\mathbf{nw}}(t,0), S_{[0,\theta_{n+1}-\theta_n]}(t)).$$

Note that $e^{pS_{[0,\theta_{n+1}-\theta_n]}(t)}$ is bounded above by $e^{pS_{[0,1]}(t)}$ since $|\theta_{n+1}-\theta_n| \leq 1$. Thus, the right hand side of the above display is less than $C\mathbb{E}[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}]\mathbf{E}_{t,p}^{(n)}e^{-\frac{p\theta_n^2}{2t}}$. This shows (3.2.10) for $n \geq 0$.

We turn to prove (3.2.10) for $n < 0$. The key part of the proof relies on the fact that the law of $\mathcal{Z}^{\mathbf{nw}}(t,\cdot)$ is invariant under the reflection w.r.t. 0, i.e., $\{\mathcal{Z}^{\mathbf{nw}}(t,x) : x \geq 0\}$ is same in distribution with $\{\mathcal{Z}^{\mathbf{nw}}(t,x) : x \leq 0\}$. By this reflection invariance of the law of $\mathcal{Z}^{\mathbf{nw}}(t,\cdot)$, it suffices to bound $\mathbb{E}[(\int_{-\theta_{n+1}}^{-\theta_n} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(-x)}dx)^p]$ instead of $\mathbb{E}[(\int_{\theta_n}^{\theta_{n+1}} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx)^p]$. Note that $-\theta_{n+1} \geq 0$ for any $n \in \mathbb{Z}_{<0}$. By (3.2.12), we can bound $\mathcal{Z}^{\mathbf{nw}}(t,x)$ by $C\mathcal{Z}^{\mathbf{nw}}(t,-\theta_{n+1})e^{S_{[-\theta_{n+1},-\theta_n]}(t)}$ for some constant $C = C(p,\nu) > 0$ for any $x \in [-\theta_{n+1}, -\theta_n]$. This allows us to write

$$\mathbb{E}\Big[\Big(\int_{-\theta_{n+1}}^{-\theta_n} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(-x)}dx\Big)^p\Big]$$
$$\leq C\mathbb{E}\Big[\big(\mathcal{Z}^{\mathbf{nw}}(t,-\theta_{n+1})e^{\frac{\theta_{n+1}^2}{2t}}\big)^p e^{pS_{[-\theta_{n+1},-\theta_n]}(t)}\Big]\mathbb{E}\Big[\big(\int_{-\theta_{n+1}}^{-\theta_n} e^{f_t(-x)}dx\big)^p\Big]e^{-\frac{p\theta_{n+1}^2}{2t}} \tag{3.2.14}$$

in the same way as in (3.2.13). In what follows, we explain how to obtain (3.2.10) for $n < 0$ from the above inequality. We first bound $\mathbb{E}[(\mathcal{Z}^{\mathbf{nw}}(t,-\theta_{n+1})e^{\frac{\theta_{n+1}^2}{2t}})^p e^{pS_{[-\theta_{n+1},-\theta_n]}(t)}]$ by $\mathbb{E}[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}]$ in the right side of (3.2.14) and this substitution is justified by (3.2.5) of Proposition 3.2.1. Next, we identify $\mathbb{E}[(\int_{-\theta_{n+1}}^{-\theta_n} e^{f_t(-x)}dx)^p]$ with $\mathbf{E}_{t,p}^{(n)}$ in (3.2.14) by change of variable inside the integral. Combining the outcomes of these two steps with the fact that left side of (3.2.14) is equal to $\mathbb{E}[(\int_{\theta_n}^{\theta_{n+1}} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx)^p]$ shows (3.2.10) for $n < 0$. This completes the proof of the desired result.

Proof of Proposition 3.2.1. (i) Recall the definition of $S_{[\theta_n,\theta_{n+1}]}(t)$ from (3.2.4). Rewriting $\frac{(x-\theta_n)\theta_n}{t}$ into $\frac{x^2}{2t} - \frac{\theta_n^2}{2t} - \frac{(x-\theta_n)^2}{2t}$, we get

$$\begin{aligned} S_{[\theta_n,\theta_{n+1}]}(t) &= \sup_{x\in[\theta_n,\theta_{n+1}]} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t} - \mathcal{H}^{\mathbf{nw}}(t,\theta_n) - \frac{\theta_n^2}{2t} - \frac{(x-\theta_n)^2}{2t} - \frac{\nu(x-\theta_n)^2}{2}\Big), \\ &= \sup_{x\in[0,\theta_{n+1}-\theta_n]} \Big(\mathcal{H}^{\mathbf{nw}}(t,x+\theta_n) + \frac{(x+\theta_n)^2}{2t} - \mathcal{H}^{\mathbf{nw}}(t,\theta_n) - \frac{\theta_n^2}{2t} - \frac{x^2}{2t} - \frac{\nu x^2}{2}\Big) \end{aligned} \tag{3.2.15}$$

where second line is due to a change of variable $x \to x + \theta_n$. By Proposition 3.1.11, for any fixed $t > 0$, the process $\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t}$ is stationary in x. This implies $\{\mathcal{H}^{\mathbf{nw}}(t,x+\theta_n) + \frac{(x+\theta_n)^2}{2t} : x \in [0,\theta_{n+1}-\theta_n]\}$ is same in distribution with $\{\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t} \in [0,\theta_n]\}$ for any $n \in \mathbb{Z}$. Note that

$$S_{[0,\theta_{n+1}-\theta_n]}(t) = \sup_{x\in[0,\theta_{n+1}-\theta_n]} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) - \frac{\nu x^2}{2}\Big)$$

$$= \sup_{x\in[0,\theta_{n+1}-\theta_n]} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t} - \mathcal{H}^{\mathbf{nw}}(t,0) - \frac{x^2}{2t} - \frac{\nu x^2}{2}\Big) \tag{3.2.16}$$

Now, (3.2.5) follows by comparing (3.2.16) with (3.2.15) and using the stationarity of $\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t}$, which implies the equivalence of the law of $\{\mathcal{H}^{\mathbf{nw}}(t,x+\theta_n) + \frac{(x+\theta_n)^2}{2t} : x \in [0,\theta_{n+1}-\theta_n]\}$ with $\{\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t} : x \in [0,\theta_{n+1}-\theta_n]\}$.

(ii) For any $\epsilon > 0$, we seek to show that there exists $C = C(p,\nu,\epsilon) > 0$ such that

$$\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}\Big] \le C\Big(\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^{p+\epsilon}\Big]\Big)^{\frac{p}{p+\epsilon}}. \tag{3.2.17}$$

Before proceeding to its proof, we first explain how the above inequality implies (3.2.7). Taking the logarithm and then dividing both side of above display by t and letting $t \to \infty$, we get

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}\Big] \le \frac{p}{p+\epsilon} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^{p+\epsilon}\Big] = \frac{p(p+\epsilon)^2 - p}{24},$$

where the last equality follows from Proposition 3.1.9. Letting $\epsilon \to 0$ in the last display, we get the desired (3.2.6).

It remains to show (3.2.17) which is proved as follows. By Hölder's inequality, for arbitrary $\epsilon > 0$,

$$\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^p e^{pS_{[0,1]}(t)}\Big] \le \Big(\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}(t,0)^{p+\epsilon}\Big]\Big)^{\frac{p}{p+\epsilon}} \Big(\mathbb{E}\Big[e^{\frac{p(p+\epsilon)}{\epsilon} S_{[0,1]}(t)}\Big]\Big)^{\frac{\epsilon}{p+\epsilon}} \tag{3.2.18}$$

From the last inequality, (3.2.17) follows if we can bound $\mathbb{E}[e^{\frac{p(p+\epsilon)}{\epsilon} S_{[0,1]}(t)}]$ by some constant $C = C(p,\nu,\epsilon) > 0$. We will now accomplish this using the tail probability bound of $S_{[0,1]}(t)$. By (3.1.16) of Proposition 3.1.10, we know that for any fixed $\delta > 0$, there exist $s_0 = s_0(\delta,\nu) > 0$ and $c = c(\delta,\nu) > 0$ such that $\mathbb{P}(S_{[0,1]}(t) \ge s) \le \exp(-cs^{9/8-\delta})$ for all $s \ge s_0$ and $t > 1$. We choose $\delta = \frac{1}{17}$. One may notice that $\frac{9}{8} - \frac{1}{17} > 1 + \frac{1}{17}$. With this computation and tail bound of $S_{[0,1]}(t)$ in hand, we write

$$\mathbb{E}[e^{\frac{p(p+\epsilon)}{\epsilon}S_{[0,1]}(t)}] \leq e^{\frac{p(p+\epsilon)}{\epsilon}s_0} + \int_{s_0}^{\infty} e^{\frac{p(p+\epsilon)}{\epsilon}s - cs^{1+\frac{1}{17}}} ds. \tag{3.2.19}$$

The right hand side of the above inequality is a finite constant whose value would depend on p, ν, ϵ. Combining this with (3.2.18) yields the proof of (3.2.17).

□

Proof of Proposition 3.2.2. Recall the notation $M_p^{f_t}(t,x)$ from Definition 3.1.1. We will prove (3.2.7) using the following claim: there exist $C_1 = C_1(p,\epsilon)$ and $C_2 = C_2(p,\epsilon) > 0$ such that for all $t > 1$,

$$\sum_{n\in\mathbb{Z}_{\geq 0}} e^{-\frac{p\theta_n^2}{2t}} \mathbf{E}_{t,p}^{(n)} + \sum_{n\in\mathbb{Z}_{<0}} e^{-\frac{p\theta_{n+1}^2}{2t}} \mathbf{E}_{t,p}^{(n)} \leq C_1 t^{\frac{1}{2(1-\beta)}} + C_2 \int_{\mathbb{R}} e^{-\frac{p(1-\epsilon)x^2}{2t}} M_{p(1+\epsilon)}^{f_t}(t,x)dx \tag{3.2.20}$$

where $\beta \in (0,1)$ is the same constant as in the *pseudo-stationarity* condition of Definition 3.1.1. Recall $\mathbf{E}_{t,p}^{(n)} = \mathbb{E}[(\int_{\theta_n}^{\theta_{n+1}} e^{f_t(x)}dx)^p]$. After proving (3.2.7) which we do as follows, we will proceed to prove the above inequality. Taking the logarithm of both sides of (3.2.20) and noting that $\log(c_1 a + c_2 b) \leq \log(\max\{c_1, c_2\}) + \log 2a + \max\{\log a, \log b\}$ for any $a \geq 1, b > 0$, $c_1, c_2 > 0$, we get

$$\begin{aligned} \log\big(\text{r.h.s. of (3.2.20)}\big) \leq{}& \log(\max\{C_1, C_2\}) + \log 2t^{\frac{1}{2(1-\beta)}} \\ &+ \max\Big\{\log t^{\frac{1}{2(1-\beta)}}, \log\Big(\int_{\mathbb{R}} e^{-\frac{p(1-\epsilon)x^2}{2t}} M_{p(1+\epsilon)}^{f_t}(t,x)\Big)\Big\}. \end{aligned} \tag{3.2.21}$$

Now, we divide both sides by t and let $t \to \infty$. On doing so, we claim that the limit of the right hand side is less than $g(p)$. To see this, we first write

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}\max\Big\{\log t^{\frac{1}{2(1-\beta)}}, \log(\int_{\mathbb{R}} \exp(-p(1-\epsilon)x^2/2t) M_{p(1+\epsilon)}^{f_t}(t,x))dx)\Big\} \\ &\quad \leq \max\Big\{\limsup_{t\to\infty}\frac{1}{t}\log t^{\frac{1}{2(1-\beta)}}, \limsup_{t\to\infty}\frac{1}{t}\log(\int_{\mathbb{R}} \exp(-p(1-\epsilon)x^2/2t) M_{p(1+\epsilon)}^{f_t}(t,x))dx)\Big\} \end{aligned}$$

Then, we note

$$\lim_{t\to\infty}\frac{1}{t}\log(\max\{C_1,C_2\})=0,\quad \lim_{t\to\infty}\frac{1}{t}\log t^{\frac{1}{2(1-\beta)}}=0,$$
$$\liminf_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\Big(\int_{\mathbb{R}}e^{-\frac{p(1-\epsilon)x^2}{2t}}M^{f_t}_{p(1+\epsilon)}(t,x)\Big)\leq g(p)$$

where the last inequality follows by applying (3.1.5). Substituting these limiting results into the right hand side of (3.2.21) and using the non-negativitiy of $g(p)$ (Lemma 3.1.13 (i)) conclude (3.2.7). This proves Proposition 3.2.2 modulo (3.2.20) which is proved as follows.

Note that (3.2.20) bounds a discrete sum by an integral. This passage from discrete to continuum requires a locally uniform control on the discrete summands of (3.2.21) which we seek to extract from the tail bounds of (3.1.8). To this aim, for any $n\in\mathbb{Z}$,

$$\mathrm{TV}_{f_t}(n):=\sup_{y\in[\theta_n,\theta_{n+1}]}|f_t(x)-f_t(\theta_n)|. \tag{3.2.22}$$

Since $\mathrm{TV}_{f_t}(n)$ is the supremum of $|f_t(x)-f_t(\theta_n)|$ as x varies in $[\theta_n,\theta_{n+1}]$, we may bound $f_t(x)$ by $\mathrm{TV}_{f_t}(n)+f_t(\theta_n)$ for all $x\in[\theta_n,\theta_{n+1}]$. This allows us to bound $\mathbb{E}[(\int_{\theta_n}^{\theta_{n+1}}e^{f_t(x)}dx)^p]$ by $\mathbb{E}[e^{pf_t(\theta_n)}e^{p\mathrm{TV}_{f_t}(n)}]$. Hereafter, we prove (3.2.20) in two *steps*. *Step 1* will show that there exist $c_1=c_1(p,\epsilon)>0$ and $c_2=c_2(p,\epsilon)>0$ such that the following inequality

$$\mathbb{E}\big[e^{pf_t(\theta_n)}e^{p\mathrm{TV}_{f_t}(n)}\big]\leq c_1\big(1+\mathbb{E}\big[e^{p(1+\epsilon/2)f_t(\theta_n)}\big]\big)\leq c_2\big(1+\int_{\theta_n}^{\theta_{n+1}}M^{f_t}_{p(1+\epsilon)}(t,x)dx\big) \tag{3.2.23}$$

holds for all $n\in\mathbb{Z}$. In *Step 2*, we will prove the following: there exist $c'_1=c'_1(p,\epsilon)>0$ and $c'_2=c'_2(p,\epsilon)>0$ such that for all $t>1$

$$\sum_{n\in\mathbb{Z}_{\geq 0}}e^{-\frac{p\theta_n^2}{2t}}\Big(1+\int_{\theta_n}^{\theta_{n+1}}M^{f_t}_{p(1+\epsilon)}(t,x)dx\Big)+\sum_{n\in\mathbb{Z}_{<0}}e^{-\frac{p\theta_{n+1}^2}{2t}}\Big(1+\int_{\theta_n}^{\theta_{n+1}}M^{f_t}_{p(1+\epsilon)}(t,x)dx\Big)$$
$$\leq c'_1t^{\frac{1}{2(1-\beta)}}+c'_2\int_{\mathbb{R}}e^{-\frac{p(1-\epsilon)x^2}{2t}}M^{f_t}_{p(1+\epsilon)}(t,x)dx \tag{3.2.24}$$

where $\beta\in(0,1]$ is the same constant as in the *pseudo-stationarity* condition of Definition 3.1.1.

Combining (3.2.23) with (3.2.24) yields (3.2.20).

Step 1: We start with showing the first inequality of (3.2.23). By denoting $X := \exp(p f_t(\theta_n))$ and $W := \exp(p\mathrm{TV}_{f_t}(n))$, we apply Hölder's inequality to bound $\mathbb{E}[XW]$ by

$$(\mathbb{E}[X^{(1+\epsilon/2)}])^{1/(1+\epsilon/2)}(\mathbb{E}[W^{(2+\epsilon)/\epsilon}])^{\epsilon/(2+\epsilon)}.$$

The first inequality of (3.2.23) will follow from this upper bound once we show

$$(\mathbb{E}[X^{(1+\epsilon/2)}])^{1/(1+\epsilon/2)} \leq 1 + \mathbb{E}[X^{(1+\epsilon/2)}], \qquad (\mathbb{E}[W^{(2+\epsilon)/\epsilon}])^{\epsilon/(2+\epsilon)} \leq c_1 \tag{3.2.25}$$

for some constant $c_1 = c_1(p, \epsilon) > 0$. The left hand side inequality is straightforward since $x^a \leq \max\{1, x\}$ for any $x > 0$ and $a \in (0, 1)$. To prove the right hand side inequality, we use the tail bound $\mathrm{TV}_{f_t}(n)$. By (3.1.8), we know that for any $\delta > 0$, there exist $s_0 = s_0(\delta) > 0$ and $c = c(\delta) > 0$ such that $\mathbb{P}(\mathrm{TV}_{f_t}(n) > s) \leq \exp(-cs^{1+\delta})$ for all $s \geq s_0$ and $t > 0$. With this tail estimate, we can bound $\mathbb{E}[W^{(2+\epsilon)/\epsilon}]$ by $\exp(ps_0(2+\epsilon)/\epsilon) + \int_{s_0}^{\infty} \exp(ps(2+\epsilon)/\epsilon - cs^{1+\delta})ds$ from above. Since this upper bound is a constant which only depends on p, δ, ϵ, we get the right hand side inequality of the above display. Combining both proofs shows the first inequality of (3.2.23).

Now, we show the second inequality of (3.2.23). Since $f_t(\theta_n)$ is bounded above by $f_t(x) + \mathrm{TV}_{f_t}(n)$ for all $n \in \mathbb{Z}$ and $x \in [\theta_n, \theta_{n+1}]$, we get

$$\mathbb{E}\Big[e^{p(1+\epsilon/2)f_t(\theta_n)}\Big] \leq \int_{\theta_n}^{\theta_{n+1}} \mathbb{E}\Big[e^{p(1+\epsilon/2)f_t(x)}e^{p(1+\epsilon/2)\mathrm{TV}_{f_t}(n)}\Big]dx$$

From this upper bound, the second inequality of (3.2.23) follows if we can show that $\mathbb{E}\big[\exp((1 + \epsilon/2)f_t(x)) \exp\big(p(1+\epsilon/2)\mathrm{TV}_{f_t}(n)\big)\big]$ is bounded by $c_1 + c_2 M^{f_t}_{p(1+\epsilon)}(t, x)$ for some constants $c_1 = c_1(p, \epsilon) > 0$ and $c_2 = c_2(p, \epsilon) > 0$. The proof of this upper bound is similar in spirit to the argument in the previous paragraph. We claim and prove this bound as follows. By denoting $X' := \exp(pf_t(x)(1 + \epsilon/2))$ and $W' := \exp\big(p(1 + \epsilon/2)\mathrm{TV}_{f_t}(n)\big)$, we use Hölder's inequality to bound $\mathbb{E}[X'W'] \le (E[(X')^u])^{1/u}(\mathbb{E}[(W')^v]^{1/v})$ where $u = (1+\epsilon)/(1+\epsilon/2)$ and $u^{-1}+v^{-1} = 1$. Using similar argument as in the proof of (3.2.25), we bound $(E[(X')^u])^{1/u}$ by $1+E[(X')^u]$ and $(\mathbb{E}[(W')^v]^{1/v})$ by some constant which only depends on p,ϵ. Combining these shows that $\mathbb{E}[X'W']$ is bounded above by $c(1+E[X')^u])$. This proves our claim since $E[(X')^u] = \mathbb{E}[\exp(p(1+\epsilon)f_t(x))] = M^{f_t}_{p(1+\epsilon)}(t,x)$. As a consequence, we get the second inequality of (3.2.23).

Step 2: To prove (3.2.24), we first claim $\sum_{n\in\mathbb{Z}_{\ge 0}}\exp(-\frac{p\theta_n^2}{2t})$ and $\sum_{n\in\mathbb{Z}_{<0}}\exp(-\frac{p\theta_{n+1}^2}{2t})$ can be bounded by $Ct^{1/(2(1-\beta))}$ for all $t>1$ where β is the same constant as in the *pseudo-stationarity condition* of Definition 3.1.1 and the constant $C>0$ depends on p and β. Note that $1 \ge |\theta_{n+1}-\theta_n| \ge \min\{1, c|n|^{-\beta}\}$ for some $c>0$, $\beta\in(0,1)$. Therefore, there exists $c_1,c_2>0$ such that $c_1 n \ge |\theta_n| \ge c_2|n|^{1-\beta}$ for all $n\in\mathbb{Z}$. Due to the last inequality, we may write

$$|\theta_{n+1}-\theta_n| \ge D\max\{1, |\theta_n|^{-\frac{\beta}{(1-\beta)}}\} \ge D\max\{1, (|x|+1)^{-\frac{\beta}{(1-\beta)}}\}, \quad \forall x\in[\theta_n,\theta_{n+1}]$$

for some constant $D>0$. Since $\exp(-\frac{p\theta_n^2}{2t})$ and $\exp(-\frac{p\theta_{m+1}^2}{2t})$ decreases as $n\uparrow\infty$ and $m\downarrow-\infty$ bounding the Riemann sum with its integral approximation yields

$$\begin{aligned}\max\Big\{\sum_{n\in\mathbb{Z}_{\ge 0}}\exp(-\frac{p\theta_n^2}{2t}), \sum_{n\in\mathbb{Z}_{<0}}\exp(-\frac{p\theta_{n+1}^2}{2t})\Big\} &\le 1 + D^{-1}\int_{\mathbb{R}}(|x|+1)^{\frac{\beta}{1-\beta}}\exp(-\frac{px^2}{2t})dx \\ &\le 1 + 2^{\frac{\beta}{1-\beta}}D^{-1}\int_{\mathbb{R}}(|x|^{\frac{\beta}{1-\beta}}+1)\exp(-\frac{px^2}{2t})dx.\end{aligned}$$

where the last inequality follows since $(|x|+1)^{\beta/(1-\beta)}$ is bounded by $2^{\beta/(1-\beta)}(|x|^{\beta/(1-\beta)}+1)$. The integral on the right hand side of the above display is bounded by $Ct^{1/(2(1-\beta))}$ when $t>1$ for some constant $C=C(p,\beta)>0$. This proves our claim. To complete the proof of (3.2.24), it remains to show the following: there exists $t_0=t_0(\epsilon)>0$ such that for all $t>t_0$,

$$\sum_{n\in\mathbb{Z}_{\ge 0}} e^{-\frac{p\theta_n^2}{2t}}\int_{\theta_n}^{\theta_{n+1}} M^{f_t}_{p(1+\epsilon)}(t,x)dx \le C_1 + C_2\int_{\mathbb{R}} e^{-\frac{p(1-\epsilon)x^2}{2t}} M^{f_t}_{p(1+\epsilon)}(t,x)dx, \tag{3.2.26}$$

$$\sum_{n\in\mathbb{Z}_{<0}} e^{-\frac{p\theta_{n+1}^2}{2t}} \int_{\theta_n}^{\theta_{n+1}} M^{f_t}_{p(1+\epsilon)}(t,x)dx \le C_1 + C_2 \int_{\mathbb{R}} e^{-\frac{p(1-\epsilon)x^2}{2t}} M^{f_t}_{p(1+\epsilon)}(t,x)dx. \tag{3.2.27}$$

for some constants $C_1 = C_1(p,\epsilon) > 0$ and $C_2 = C_2(p,\epsilon)$. We only prove (3.2.26). The proof of the other inequality is similar and details are skipped.

For any given $\epsilon > 0$, there exists $n_0 = n_0(\epsilon) \in \mathbb{Z}_{\ge 0}$ such that $\theta_n^2 \ge (1-\epsilon)x^2$ for all $x \in [\theta_n, \theta_{n+1}]$ and $n \ge n_0$. We write left side of (3.2.26) as

$$\text{l.h.s. of (3.2.26)} = \sum_{0\le n<n_0(\epsilon)} e^{-\frac{p\theta_n^2}{2t}} \int_{\theta_n}^{\theta_{n+1}} M^{f_t}_{p(1+\epsilon)}(t,x)dx + \sum_{n\ge n_0(\epsilon)} e^{-\frac{p\theta_n^2}{2t}} \int_{\theta_n}^{\theta_{n+1}} M^{f_t}_{p(1+\epsilon)}(t,x)dx. \tag{3.2.28}$$

We can bound the last term on the right side of the above display by $\int_{\mathbb{R}} \exp(-\frac{p(1-\epsilon)x^2}{2t}) M^{f_t}_{p(1+\epsilon)}(t,x)dx$ since $\theta_n^2 \ge (1-\epsilon)x^2$ for all $x \in [\theta_n, \theta_{n+1}]$ and $n \ge n_0$. Using the pointwise upper bound on $M^{f_t}_{p(1+\epsilon)}(t,x)$ from (3.1.6), we can write

$$\sum_{0\le n<n_0(\epsilon)} e^{-\frac{p\theta_n^2}{2t}} \int_{\theta_n}^{\theta_{n+1}} M^{f_t}_{p(1+\epsilon)}(t,x)dx \le \sum_{0\le n<n_0(\epsilon)} e^{-\frac{p\theta_n^2}{2t}} e^{Cp(1+\epsilon)(1+\theta_{n_0}^\delta)+\frac{\alpha\theta_{n_0}^2}{2t}} \le n_0 e^{Cp(1+\epsilon)(1+\theta_{n_0}^\delta)+\alpha p\theta_{n_0}^2}.$$

where the last inequality follows by bounding $e^{-\frac{p\theta_{n_0}^2}{2t}}$ by 1 for all $0 \le n < n_0$ and taking $t > 1$. Due to the above bound, the first term in the right side of (3.2.28) is bounded by some constant $C = C(p,\epsilon) > 0$. Combining the upper bounds on both summands of (3.2.28) yields (3.2.26). This completes the proof of (3.2.24) and Proposition 3.2.2.

□

3.2.2 Proof of LimInf_p for all $p \in \mathbb{R}_{>0}$

Fix any $p, \nu > 0$. Recall the notation $x_{p,\omega}(t)$ of Lemma 3.1.13. For any $0 < \epsilon < p/2$ and $\omega > 0$, let $n_{p,\epsilon,\omega}(t) \in \mathbb{Z}$ be such that $x_{p-\epsilon,\omega}(t) \in [\theta_{n_{p,\epsilon,\omega}(t)}, \theta_{n_{p,\epsilon,\omega}(t)+1}]$ where $\{\theta_n\}_{n\in\mathbb{Z}}$ is a sequence of grid points (see Definition 3.1.1) such that f_t satisfies (3.1.8) for all $t > 0$. For notational convenience, we will denote $n_{p,\epsilon,\omega}(t)$ by $n(t)$ and the interval $[\theta_{n(t)}, \theta_{n(t)+1}]$ by $I(t)$.

For convenience, we use the following shorthand notations:

$$\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t) := \mathcal{Z}^{\mathbf{nw}}(t, x_{p-\epsilon,\omega}(t))e^{\frac{x^2_{p-\epsilon,\omega}(t)}{2t}}, \tag{3.2.29}$$

$$Y_{p,\epsilon}(t) := \inf_{x\in I(t)} \Big\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big) + \frac{\big(x - x_{p-\epsilon,\omega}(t)\big)x_{p-\epsilon,\omega}(t)}{t} + \frac{\nu\big(x - x_{p-\epsilon,\omega}(t)\big)^2}{2}\Big\} \tag{3.2.30}$$

As in Section 3.2.1, we rely on the convolution formula of Proposition 3.1.8 to express the moments of $\mathcal{Z}^{f_t}(t,0)$ in terms the moment of a integral involving $\mathcal{Z}^{\mathbf{nw}}(t,\cdot)$ and $e^{f_t(\cdot)}$. To prove LimInf_p, we analyze the expected value of p-th moment of this integral over the interval $I(t)$. After localization of the integral, as we show, proving LimInf_p requires lower bound on the p-th moment of $\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t)e^{Y_{p,\epsilon}(t)}$ and $\int_{I(t)} e^{f_t(x)}dx$. Proposition 3.2.3 and 3.2.4 will provide such lower bound. In what follows, we first state those propositions; prove LimInf_p and then, proceed to prove those ensuing propositions.

Proposition 3.2.3. *We have* $\liminf_{\epsilon\to 0}\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^p e^{pY_{p,\epsilon}(t)}] \geq \frac{p^3-p}{24}$.

Proposition 3.2.4. *We have*

$$\liminf_{\epsilon\to 0}\liminf_{t\to\infty}\frac{1}{t}\log\Big(e^{-\frac{px_{p-\epsilon,\omega}(t)^2}{2t}}\mathbb{E}\Big[\Big(\int_{I(t)} e^{f_t(x)}dx\Big)^p\Big]\Big) \geq g(p). \tag{3.2.31}$$

Proof of LimInf_p***:*** Due to Proposition 3.1.8, it suffices to show the liminf of $t^{-1}\log\mathbb{E}[(\int_{-\infty}^{\infty}\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)})^p]$ as $t\to\infty$ is bounded below by $(p^3-p)/24 + g(p)$. Since $\mathcal{Z}^{\mathbf{nw}}(t,x)$ and the exponential of $f_t(x)$ are both almost surely non-negative, $\int_{-\infty}^{\infty}\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx$ is lower bounded by the integral of $\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}$ over the interval $I(t)$. We claim and prove that there exists a constant $C = C(p) > 0$ such that

$$\mathbb{E}\Big[\Big(\int_{I(t)}\mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\Big)^p\Big] \geq C\mathbb{E}\Big[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^p e^{pY_{p,\epsilon}(t)}\Big]\cdot e^{-\frac{px_{p-\epsilon,w}(t)^2}{2t}}\mathbb{E}\Big[\Big(\int_{I(t)} e^{f_t(x)}dx\Big)^p\Big]. \tag{3.2.32}$$

By assuming this inequality, we first prove LimInf_p. We take logarithm of both sides of (3.2.32), divide them by t and let $t \to \infty$. After these set of operations, the liminf of the right hand side as $t \to \infty, \epsilon \to 0$ will be bounded below by $(p^3-p)/24+g(p)$ via the inequalities in Proposition 3.2.3 and 3.2.4. From this, the desired inequality of LimInf_p follows since $\mathbb{E}[(\int_{-\infty}^{\infty} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx)^p]$ exceeds $\mathbb{E}[(\int_{I(t)} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx)^p]$. In the rest of the proof, we focus on showing (3.2.32). We first derive it from the following inequality: there exists $C = C(p) > 0$ such that for all $x \in I(t)$,

$$\mathcal{Z}^{\mathbf{nw}}(t,x) \geq C\mathcal{Z}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big)e^{Y_{p,\epsilon}(t)} \tag{3.2.33}$$

Owing to this, $\int_{I(t)} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx$ can be bounded below by the product of $C\mathcal{Z}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big)e^{Y_{p,\epsilon}(t)}$ and $\int_{I(t)} e^{f_t(x)}dx$. This readily implies

$$\mathbb{E}\Big[\Big(\int_{I(t)} \mathcal{Z}^{\mathbf{nw}}(t,x)e^{f_t(x)}dx\Big)^p\Big] \geq C\,\mathbb{E}\Big[\mathcal{Z}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big)^p e^{pY_{p,\epsilon}(t)}\Big]\mathbb{E}\Big[\Big(\int_{I(t)} e^{f_t(x)}dx\Big)^p\Big] \tag{3.2.34}$$

From the above inequality, (3.2.32) follows by multiplying and dividing the right hand side of (3.2.34) by $\exp(px^2_{p-\epsilon,\omega}(t)/2t)$ and recalling that $\mathcal{Z}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big)^p \exp(px^2_{p-\epsilon,\omega}(t)/2t)$ is equal to $(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^p$ (which is defined in (3.2.29)). It remains to show (3.2.33) which we show as follows.

Recall that $Y_{p,\epsilon}(t)$ is defined as an infimum of the right hand side of (3.2.30) over $I(t)$. So for all $x \in I(t)$,

$$\mathcal{H}^{\mathbf{nw}}(t,x) \geq \mathcal{H}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big) + Y_{p,\epsilon}(t) - \frac{\big(x - x_{p-\epsilon,\omega}(t)\big)x_{p-\epsilon,\omega}(t)}{t} - \frac{\nu\big(x - x_{p-\epsilon,\omega}(t)\big)^2}{2}.$$

Taking the exponential on the both sides and recalling $\mathcal{Z}^{\mathbf{nw}}(t,x) = e^{\mathcal{H}^{\mathbf{nw}}(t,x)}$, we get

$$\mathcal{Z}^{\mathbf{nw}}(t,x) \geq \mathcal{Z}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big)e^{Y_{p,\epsilon}(t)}e^{-\frac{\nu(x-x_{p-\epsilon,\omega}(t))^2}{2}}e^{-\frac{(x-x_{p-\epsilon,\omega}(t))x_{p-\epsilon,\omega}(t)}{t}}$$

By Lemma 3.1.13, for any fixed $p \in \mathbb{R}_{>0}$, there exists $C' = C'(p) >$ such that for all t and $0 < \epsilon < \frac{p}{2}$, $|x_{p-\epsilon,\omega}(t)| \leq C't$. Invoking this bound on the absolute value of $x_{p-\epsilon,\omega}(t)$, we may lower bound the infimum value of

$$\exp(-\nu(x - x_{p-\epsilon,\omega}(t))^2/2 - (x - x_{p-\epsilon,\omega}(t))x_{p-\epsilon,\omega}(t)/t)$$

as x varies in $I(t)$ in the right hand side of the above display by some constant $C = C(p) > 0$ (recall $I(t) = [\theta_{n(t)}, \theta_{n(t)+1}]$, whose length is no bigger than 1). This yields (3.2.33) and hence, completes the proof of LimInf_p.

□

Proof of Proposition 3.2.3. Our main goal is to show there exists $C = C(p, \epsilon) > 0$ such that

$$\mathbb{E}\Big[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^p e^{pY_{p,\epsilon}(t)}\Big] \geq C\Big(\mathbb{E}\big[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^{p-\epsilon}\big]\Big)^{\frac{p}{p-\epsilon}}. \tag{3.2.35}$$

Before proceeding to the proof of the above inequality, we demonstrate how this implies Proposition 3.2.3. Taking the logarithm of both sides of (3.2.35), then dividing them by t and letting $t \to \infty$ yields that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}\Big[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^p e^{pY_{p,\epsilon}(t)}\Big] \geq \liminf_{t\to\infty} \frac{1}{t} \log \Big(\mathbb{E}\Big[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^{p-\epsilon}\Big]\Big)^{\frac{p}{p-\epsilon}} = \frac{p}{p-\epsilon}\cdot\frac{(p-\epsilon)^3 - (p-\epsilon)}{24}, \tag{3.2.36}$$

To see the last equality, we first note that $\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t)$ is same in distribution with $\mathcal{Z}^{\mathbf{nw}}(t, 0)$ by Proposition 3.1.11. Combining this with Proposition 3.1.9 shows that the limit of $t^{-1} \log \mathbb{E}[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^{p-\epsilon}]$ is equal to $((p-\epsilon)^3 - (p-\epsilon))/24$ as t goes to ∞. As a consequence, we get the above equality. Letting $\epsilon \to 0$ in the above display, we obtain the desired result of Proposition 3.2.3. Thus, completing the proof of Proposition 3.2.3 boils down to showing (3.2.35) which we prove as follows.

We write $(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^{p-\epsilon}$ as a product of $X := (\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^{p-\epsilon}e^{(p-\epsilon)Y_{p,\epsilon}(t)}$ and $W := e^{-(p-\epsilon)Y_{p,\epsilon}(t)}$. Applying the Hölder's inequality, we have $\mathbb{E}[XW] \leq \mathbb{E}[X^{p/(p-\epsilon)}]^{(p-\epsilon)/p}\mathbb{E}[W^{p/\epsilon}]^{\epsilon/p}$. Multiplying both sides of this inequality by $\mathbb{E}[W^{p/\epsilon}]^{-\epsilon/p}$ and raising both sides to the power $p/(p-\epsilon)$ yields

$$\mathbb{E}\Big[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^p e^{pY_{p,\epsilon}(t)}\Big] \geq \Big(\mathbb{E}\Big[(\mathcal{Z}^{\mathbf{nw}}_{p,\epsilon}(t))^{p-\epsilon}\Big]\Big)^{\frac{p}{p-\epsilon}}\Big(\mathbb{E}\Big[e^{-\frac{p(p-\epsilon)}{\epsilon}Y_{p,\epsilon}(t)}\Big]\Big)^{-\frac{\epsilon}{p-\epsilon}}. \tag{3.2.37}$$

From the above inequality, (3.2.35) follows once we show that $\mathbb{E}[\exp(-p(p-\epsilon)Y_{p,\epsilon}(t)/\epsilon)]$ is uniformly upper bounded by a constant $C' = C'(p,\epsilon)$ for all $t > 1$. This will be shown hereafter. For proving this bound, our main tools are the spatial stationarity $\mathcal{H}^{\mathbf{nw}}(t,x) + x^2/2t$ and the tail bounds of Proposition 3.1.10. By expressing $(x - x_{p-\epsilon,\omega}(t))x_{p-\epsilon,\omega}(t)$ in the definition of $Y_{p,\epsilon}(t)$ as $2^{-1}\big(x^2 - (x_{p-\epsilon,\omega}(t))^2 - (x - x_{p-\epsilon,\omega}(t))^2\big)$, we may rewrite $Y_{p,\epsilon}(t)$ as

$$Y_{p,\epsilon}(t) = \inf_{x\in I(t)} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t} - \mathcal{H}^{\mathbf{nw}}\big(t, x_{p-\epsilon,\omega}(t)\big) - \frac{x_{p-\epsilon,\omega}(t)^2}{2t} - \frac{\big(x - x_{p-\epsilon,\omega}(t)\big)^2}{2t} + \frac{\nu\big(x - x_{p-\epsilon,\omega}(t)\big)^2}{2}\Big)$$

Using stationarity of $\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t}$ in Proposition 3.1.11, we can shift the spatial variable x of the above display to the left by $x_{p-\epsilon,\omega}(t)$ and obtain the distributional identity

$$Y_{p,\epsilon}(t) \stackrel{d}{=} \inf_{x\in I^0(t)} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) + \frac{x^2}{2t} - \mathcal{H}^{\mathbf{nw}}(t,0) - \frac{x^2}{2t} + \frac{\nu x^2}{2}\Big) = \inf_{x\in I^0(t)} \Big(\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2}\Big) \tag{3.2.38}$$

where $I^0(t) := [\theta_{n(t)} - x_{p-\epsilon,\omega}(t), \theta_{n(t)+1} - x_{p-\epsilon,\omega}(t)] \subseteq [-1,1]$. Recall that (3.1.17) of Proposition 3.1.10 provides a lower tail bound of the random variable $\inf_{x\in[0,1]}\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2t}\}$. Since the law of the process $\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2t} : x \in [0,1]\}$ is same as $\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2t} : x \in [0,-1]\}$, there exist $c = c(\delta,\nu) > 0$, $s_0 = s_0(\delta,\nu) > 0$ such that for all $s > s_0$

$$\mathbb{P}\big(\inf_{x\in[-1,0]}\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2t}\} \le -s\big) \le \exp(-cs^{\frac{9}{8}-\delta}).$$

Owing to the lower tail bound of $\inf_{x\in[0,1]}\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2t}\}$ and $\inf_{x\in[-1,0]}\{\mathcal{H}^{\mathbf{nw}}(t,x) - \mathcal{H}^{\mathbf{nw}}(t,0) + \frac{\nu x^2}{2t}\}$ and the distributional identity (3.2.38), for any $\delta \in (0,1)$, there exist $c = c(\delta,\nu) > 0$ and $s_0 = s_0(\delta,\nu)$ such that $\mathbb{P}(Y_{p,\epsilon}(t) \le -s) \le \exp(-cs^{9/8-\delta})$. We chose $\delta = \frac{1}{17}$. It is straightforward to see that $\frac{9}{8} - \frac{1}{17} > 1 + \frac{1}{17}$. As a consequence, we may write

$$\mathbb{E}[e^{-\frac{p(p-\epsilon)}{\epsilon}Y_{p,\epsilon}(t)}] \le e^{\frac{p(p-\epsilon)}{\epsilon}s_0}\mathbb{P}(Y_{p,\epsilon}(t) \ge -s_0) + \frac{p(p-\epsilon)}{\epsilon}\int_{s_0}^{\infty} e^{\frac{p(p-\epsilon)}{\epsilon}s} e^{-cs^{1+\frac{1}{17}}}\, ds. \tag{3.2.39}$$

The integral on the right hand side of the above inequality is finite and its value is equal to some constant $C'' = C''(\nu, p, \epsilon) > 0$. This demonstrates why $\mathbb{E}[\exp(-p(p-\epsilon)Y_{p,\epsilon}(t)/\epsilon)]$ is bounded by some constant which only depends on ν, p and ϵ. Substituting this bound into (3.2.37) yields (3.2.35). This completes the proof of Proposition 3.2.3.

□

Proof of Proposition 3.2.4. To prove (3.2.31), we show the following inequality: there exists constant $C = C(p, \epsilon) > 0$ such that

$$\mathbb{E}\Big[\Big(\int_{I(t)} e^{f_t(x)}dx\Big)^p\Big] \geq C|I(t)|^p\Big(\mathbb{E}\Big[e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}\Big]\Big)^{\frac{p}{p-\epsilon}} \tag{3.2.40}$$

where $|I(t)|$ is the length of the interval $I(t) = [\theta_{n(t)}, \theta_{n(t)+1}]$. Let us explain why the above inequality is sufficient for proving (3.2.31). Owing to (3.2.40), we may write

$$\log\Big(e^{-\frac{px^2_{p-\epsilon,\omega}(t)}{2t}}\mathbb{E}\Big[\Big(\int_{I(t)} e^{f_t(x)}dx\Big)^p\Big]\Big) \geq -\frac{px^2_{p-\epsilon,\omega}(t)}{2t} + \frac{p}{p-\epsilon}\log\mathbb{E}\Big[e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}\Big] + p\log|I(t)| + \log C$$

Recall that $1 \geq |\theta_n - \theta_{n+1}| \geq \max\{1, c|n|^{-\beta}\}$ for some $c > 0, \beta \in (0,1)$ and all $n \in \mathbb{Z}$ by the *pseudo-stationarity* condition of Definition 3.1.1 and $|\theta_{n(t)}| \leq Ct$ for some constant $C = C(p,\epsilon) > 0$ by Lemma 3.1.13-(ii). From the inequality $|\theta_n - \theta_{n+1}| \geq \max\{1, c|n|^{-\beta}\}$, we get $|\theta_n| \geq cn^{1-\beta}$ for some constant $c = c(\beta) > 0$. Combining this with the upper bound $|\theta_{n(t)}| \leq Ct$ yields $n(t) \leq |Ct|^{1/(1-\beta)}$ and hence, shows $|I(t)| \geq |Ct|^{-\beta/(1-\beta)}$. Conjugating this last inequality with the upper bound $|I(t)| \leq 1$ implies that $t^{-1}\log|I(t)|$ converges to 0 as $t \to \infty$. Now, dividing both sides of the above display by t and letting $t \to \infty$ followed by $\epsilon \to 0$, $\omega \to 0$ shows (3.2.31) if the following inequality is satisfied

$$\liminf_{\epsilon\to 0}\liminf_{t\to\infty}\frac{1}{t}\Big(-\frac{px^2_{p-\epsilon,\omega}(t)}{2t} + \frac{p}{p-\epsilon}\log\mathbb{E}\Big[e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}\Big]\Big) \geq g(p). \tag{3.2.41}$$

We prove this inequality as follows. By taking a factor $p/(p-\epsilon)$ out of the parantheses of the left hand side of the above display and recalling the definition of $g(\cdot)$ from (3.1.4) of Definition 3.1.1, we may write

$$\text{l.h.s. of (3.2.41)} \geq \liminf_{\omega\to 0} \liminf_{\epsilon\to 0} \frac{p}{p-\epsilon} g(p-\epsilon).$$

Since g is a convex function, g is continuous at p. This shows that $\liminf_{\epsilon\to 0} pg(p-\epsilon)/(p-\epsilon)$ is equal to $g(p)$ and indeed, (3.2.41) holds. Consequently, we get (3.2.31) modulo (3.2.40). The rest of the proof will show (3.2.40).

Recall the definition of $\mathrm{TV}_{f_t}(\cdot)$ from the proof of Proposition 3.2.2. Since $\mathrm{TV}_{f_t}(n(t))$ is the supremum of $|f_t(x) - f_t(\theta_{n(t)})|$ over $x \in I(t)$, we know that $f_t(x) \geq f_t(x_{p-\epsilon,\omega}(t)) - 2\mathrm{TV}_{f_t}(n(t))$ for all $x \in I(t)$. Taking the exponential on both sides of this inequality and then integrating on $I(t)$ shows $\int_{I(t)} \exp(f_t(x))dx \geq |I(t)| \exp(f_t(x_{p-\epsilon,\omega}(t)) - 2\mathrm{TV}_{f_t}(n(t)))$ which after raising to p-th power and taking expectation yields

$$\mathbb{E}\Big[\Big(\int_{I(t)} e^{f_t(x)}dx\Big)^p\Big] \geq |I(t)|^p \mathbb{E}\Big[e^{pf_t(x_{p-\epsilon,\omega}(t))} e^{-2p\mathrm{TV}_{f_t}(n(t))}\Big] \tag{3.2.42}$$

It will suffice to show that the right hand side of the above display is bounded below by a constant multiple of $(\mathbb{E}[e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}])^{p/(p-\epsilon)}$. To get this lower bound, we write $e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}$ as a product of two random variables $\mathcal{X} := e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}e^{-(p-\epsilon)\mathrm{TV}_{f_t}(n(t))}$ and $\mathcal{W} := e^{(p-\epsilon)\mathrm{TV}_{f_t}(n(t))}$. Using the Hölder inequality, we get $\mathbb{E}[\mathcal{X}\mathcal{W}] \leq (\mathbb{E}[\mathcal{X}^{p/(p-\epsilon)}])^{(p-\epsilon)/p}(\mathbb{E}[\mathcal{W}^{p/\epsilon}])^{\epsilon/p}$. Multiplying both sides of this inequality by $(\mathbb{E}[\mathcal{W}^{p/\epsilon}])^{-\epsilon/p}$ and raising both sides to the power $p/(p-\epsilon)$ results in

$$\mathbb{E}\Big[e^{pf_t(x_{p-\epsilon,\omega}(t))}e^{-p\mathrm{TV}_{f_t}(n(t))}\Big] \geq \Big(\mathbb{E}\Big[e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}\Big]\Big)^{\frac{p-\epsilon}{p}}\Big(\mathbb{E}\Big[e^{\frac{p(p-\epsilon)}{\epsilon}\mathrm{TV}_{f_t}(n(t))}\Big]\Big)^{-\frac{\epsilon}{p-\epsilon}}$$

By the super-exponential tail bounds for $\mathrm{TV}_{f_t}(\cdot)$ specified in (3.1.8) of Definition 3.1.1, we know that $\mathbb{E}[\exp(p(p-\epsilon)\mathrm{TV}_{f_t}(n(t))/\epsilon)]$ is upper bounded by a constant. Combining this observation with the inequality of the above display yields

$$\mathbb{E}\Big[e^{pf_t(x_{p-\epsilon,\omega}(t))}e^{-2p\mathrm{TV}_{f_t}(n(t))}\Big] \geq C\Big(\mathbb{E}\Big[e^{(p-\epsilon)f_t(x_{p-\epsilon,\omega}(t))}\Big]\Big)^{\frac{p}{p-\epsilon}}$$

for some $C > 0$. Substituting this into the right hand side of (3.2.42) gives (3.2.40). This completes the proof.

□

3.2.3 Proof of (3.1.11)

We take $X(t) = \mathcal{H}^{f_t}(t,0) + \frac{t}{24}$, by Theorem 3.1.2 part (a), we see that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\exp(pX(t))\big] = \frac{p^3}{24} + g(p).$$

We conclude (3.1.11) via applying Proposition 3.1.12. It suffices to verify $h(p) := g(p)+\frac{p^3}{24}$ indeed satisfies the condition in Proposition 3.1.12. By Lemma 3.1.13, $g(p)$ is convex, since $g \in C^1(\mathbb{R}_{>0})$ as we assume, thus $g'(p)$ is increasing. Consequently, $h'(p) = g'(p)+\frac{p^2}{8}$ is continuous and strictly increasing on $(0,\infty)$. Moreover, $\lim_{p\to 0} h'(p) = \lim_{p\to 0} g'(p) + \frac{p^2}{8} = \zeta$ and

$$\lim_{p\to\infty} h'(p) = \lim_{p\to\infty} g'(p) + \frac{p^2}{8} = \infty.$$

This implies that $h'(p)$ is a continuous bijection from (ζ,∞) to $(0,\infty)$, so it satisfies the condition in Proposition 3.1.12. Applying this proposition completes the proof of (3.1.11).

3.3 Proof of Corollary 3.1.4 & Corollary 3.1.6

3.3.1 Proof of Corollary 3.1.4

We consider the following KPZ data $(g, \{f_t\}_{t>0})$ with $g \equiv 0$ and $\{f_t\}_{t>0}$ satisfies the conditions (i) and (ii) of Corollary 3.1.4. We claim that $(g, \{f_t\}_{t>0})$ belongs to the class **Hyp** (see Definition 3.1.1). Modulo this claim, by Theorem 3.1.2, we have (3.1.12). Furthermore, since $g \in C^1(\mathbb{R}_{>0})$ with $\zeta = \lim_{p\to 0} g'(p) = 0$, by (3.1.11), we get

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{f_t}(t,0) + \frac{t}{24} > ts\Big) = -\sup_{s>0}\big(ps - p^3/24\big) = -\frac{4\sqrt{2}}{3} s^{\frac{3}{2}}. \tag{3.3.1}$$

This shows (3.1.13). To complete the proof of Corollary 3.1.4, it suffices to verify our claim that $(g, \{f_t\}_{t>0})$ with $g \equiv 0$ belongs to the class **Hyp**, i.e., $(g, \{f_t\}_{t>0})$ has to satisfy (3.1.4),(3.1.5), (3.1.6), (3.1.7) and (3.1.8). Note that (3.1.6) and (3.1.7) follow immediately from the property (i) of f_t which says that there exist $\delta, \alpha \in (0,1)$ and constant $C, T_0 > 0$ such that $|f_t(x)| \leq C(1+|x|^{\delta}) + \frac{\alpha x^2}{2t}$. In what follows, we successively prove (3.1.4), (3.1.5) and (3.1.8) for $(g, \{f_t\}_{t>0})$.

PROOF OF (3.1.4): Since $\{f_t(\cdot)\}_{t\geq 0}$ is a sequence of deterministic initial data, the following limit

$$\lim_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbb{R}} \Big(-\frac{px^2}{2t} + \log e^{pf_t(x)}\Big) = 0. \tag{3.3.2}$$

will show (3.1.4). Our main objective is to prove (3.3.2). For all large $t > 0$, owing to the growth condition $|f_t(x)| \leq C(1+|x|^{\delta}) + \frac{\alpha x^2}{2t}$ for some constant $C > 0$ and $\delta, \alpha \in (0,1)$,

$$\underbrace{\sup_{x\in\mathbb{R}} \Big\{ -\frac{px^2}{2t} - Cp(1+|x|^{\delta}) - \frac{p\alpha x^2}{2t}\Big\}}_{\mathsf{Sup}_t^{(1)}} \leq \sup_{x\in\mathbb{R}} \Big\{ -\frac{px^2}{2t} + \log e^{pf_t(x)}\Big\} \leq \underbrace{\sup_{x\in\mathbb{R}} \Big\{ -\frac{px^2}{2t} + Cp(1+|x|^{\delta}) + \frac{p\alpha x^2}{2t}\Big\}}_{\mathsf{Sup}_t^{(2)}}$$

In order to prove (3.3.2), it suffices to show $t^{-1}\mathsf{Sup}_t^{(1)}$ and $t^{-1}\mathsf{Sup}_t^{(2)}$ converge to 0 as t tends to ∞. We only show $t^{-1}\mathsf{Sup}_t^{(2)} \to 0$ as $t \to \infty$. The other convergence follows verbatim. We rewrite $\mathsf{Sup}_t^{(2)}$ as $Cp + \sup_{x\in\mathbb{R}}\{-p(1-\alpha)x^2/2t + Cp|x|^{\delta}\}$. We do a change of variable $x \to t^{1/(2-\delta)}x$ in this new form of $\mathsf{Sup}_t^{(2)}$. As a consequence, we can further rewrite $\mathsf{Sup}_t^{(2)}$ as $Cp + pt^{\delta/(2-\delta)} \sup_{x\in\mathbb{R}}\{-(1-\alpha)x^2/2 + C|x|^{\delta}\}$. Note that the function $\phi(x) = -(1-\alpha)x^2/2 + C|x|^{\delta}$ satisfies $\phi(0) = 0$ and $\phi(+\infty) = \phi(-\infty) = -\infty$. Thus, the supremum value of $\phi(x)$ as x varies in $\mathbb{R}$ is finite. This shows we may upper bound $\mathsf{Sup}_t^{(2)}$ by $Cp + C'pt^{\delta/(2-\delta)}$ for some constant $C' > 0$ and lower bound it by Cp. These upper and lower bound when divided by t with letting $t \to \infty$ converge to 0. This proves the claim that $t^{-1}\mathsf{Sup}_t^{(2)} \to 0$ as $t \to \infty$ and hence, shows (3.3.2).

PROOF OF (3.1.5): Note that (3.1.5) will follow if the following limit holds

$$\lim_{t\to\infty} \frac{1}{t} \log\Big(\int e^{-\frac{p(1-\epsilon)x^2}{2t}} \cdot e^{p(1+\epsilon)f_t(x)} dx\Big) = 0 \tag{3.3.3}$$

for all small $\epsilon > 0$. Throughout the rest of the proof, we show (3.3.3). Since there exist $C > 0$, $\delta, \alpha \in (0,1)$ such that $|f_t(x)| \leq C(1+|x|^\delta) + \frac{\alpha x^2}{2t}$ for all large $t > 0$, we may write

$$\int e^{-\frac{p((1-\epsilon)+\alpha(1+\epsilon))x^2}{2t} - Cp(1+\epsilon)|x|^\delta} dx \leq \int e^{-\frac{p(1-\epsilon)x^2}{2t}} e^{p(1+\epsilon)f_t(x)} dx \leq e^{Cp(1+\epsilon)} \int e^{-\frac{p((1-\epsilon)-\alpha(1+\epsilon))x^2}{2t} + Cp(1+\epsilon)|x|^\delta} dx \tag{3.3.4}$$

We choose ϵ small such that $(1-\epsilon) - \alpha(1+\epsilon) > 0$. For proving (3.3.3), one needs to show that the logarithm of the left and right hand side of (3.3.4) when divided by t with $t \to \infty$ converge to 0. We only show this for the right hand side and the other convergence follows from similar argument. For convenience, we denote the right hand of (3.3.4) by RHS_t. By a change of variable $x \to t^{1/(2-\delta)}x$ inside the integral of RHS_t, we may write

$$\mathsf{RHS}_t = e^{Cp(1+\epsilon)} t^{1/(2-\delta)} \int e^{t^{\frac{\delta}{2-\delta}}\left(-p((1-\epsilon)-\alpha(1+\epsilon))x^2/2 + Cp(1+\epsilon)|x|^\delta\right)} dx \tag{3.3.5}$$

By splitting the domain of the above integral into two parts $\{x : |x| \leq 1\}$ and $\{x : |x| > 1\}$, we write RHS_t as sum of $\exp(Cp(1+\epsilon))t^{1/(2-\delta)}\mathcal{A}_1$ and $\exp(Cp(1+\epsilon))t^{1/(2-\delta)}\mathcal{A}_2$ where $\mathcal{A}_1$ and $\mathcal{A}_2$ denote the integral in (3.3.5) computed over the region $\{x : |x| \leq 1\}$ and $\{x : |x| > 1\}$ respectively. To show $t^{-1} \log \mathsf{RHS}_t \to 0$ as $t \to \infty$, we first find upper bound to $\mathcal{A}_1$ and $\mathcal{A}_2$. Since $-p\big((1-\epsilon) - \alpha(1+\epsilon)\big)x^2/2 + Cp|x|^\delta$ is bounded by some constant $C' = C'(p, \epsilon, \alpha) > 0$ for all $|x| \leq 1$ and all large t, we can bound $\mathcal{A}_1$ by $\exp(C' t^{\delta/(2-\delta)})$. By using the inequality $|x|^\delta < |x|$ for all $|x| > 1$ (holds since $\delta < 1$), we may write

$$\mathcal{A}_2 \leq \int_{|x|>1} e^{t^{\frac{\delta}{2-\delta}}\left(-p((1-\epsilon)-\alpha(1+\epsilon))x^2/2 + Cp(1+\epsilon)|x|\right)} dx \leq \int e^{t^{\frac{\delta}{2-\delta}}\left((-p(1-\epsilon)-\alpha(1+\epsilon))x^2/2 + Cp(1+\epsilon)|x|\right)} dx$$

where the last inequality is obtained by leveraging the positivity of the integrand. Note that the integral on the right hand side of the above display is a Gaussian integral. It is straightforward to see that this Gaussian integral can be bounded above by $C_1 t^{-\delta/(2(2-\delta))} \exp(C_2 t^{\delta/(2-\delta)})$ for some $C_1 = C_1(p, \alpha, \epsilon) > 0$ and $C_2 = C_2(p, \alpha, \epsilon) > 0$. Combining the upper bounds on $\mathcal{A}_1$ and $\mathcal{A}_2$ and substituting those into the right hand side of (3.3.5) yields

$$\mathsf{RHS}_t \leq e^{Cp(1+\epsilon)t^{1/(2-\delta)}}\Big(e^{C't^{\delta/(2-\delta)}} + C_1 t^{-\delta/(2(2-\delta))} e^{C_2 t^{\delta/(2-\delta)}}\Big) \tag{3.3.6}$$

where C', C_1, C_2 are some positive constants depending on p, α and ϵ. Taking logarithm on both sides, divising them by t and letting $t \to \infty$ shows $t^{-1} \log \mathsf{RHS}_t$ converge to 0. This completes the proof of (3.1.5).

PROOF OF (3.1.8): This trivially follows from the condition (*ii*) for the sequence of initial data $\{f_t\}_{t>0}$.

3.3.2 Proof of Corollary 3.1.6

For all $t > 0$, define $f_t : \mathbb{R} \to \mathbb{R}$ as $f_t(x) := B(x) + a_+ x \mathbb{1}_{\{x \geq 0\}} - a_- x \mathbb{1}_{\{x \leq 0\}}$ and define $g : (0, \infty) \to \mathbb{R}$ as $g(p) = \frac{p}{2} \max\left\{(\frac{p}{2} + a), 0\right\}^2$ where $a := \max\{a_+, a_-\}$. We claim and prove that $(g, \{f_t\}_{t \geq 0})$ belongs to the class **Hyp**. For now, we assume this claim and show how this implies (3.1.14) and (3.1.15).

Note that (3.1.14) follows immediately from (3.1.10) of Theorem 3.1.2 since $(g, \{f_t\}_{t \geq 0}) \in$ **Hyp** by our assumption. We turn now to show (3.1.15). Suppose that $a > 0$. Then, $g(p) = \frac{p}{2}(\frac{p}{2} + a)^2$ and hence, $g \in C^1(\mathbb{R}_{>0})$ with $\zeta = \lim_{p \to 0} g'(p) = \frac{a^2}{2}$ and it is striaghtforward to compute that $\sup_{p>0}\left\{sp - \frac{p^3}{24} - g(p)\right\} = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}} + sa - \frac{a^3}{6}$ for $s > \frac{a^2}{2}$. By (3.1.11) of Theorem 3.1.2, for $s > a^2/2$,

$$\lim_{t \to 0} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}_t^{f_t}(0) + \frac{t}{24} > s\Big) = \sup_{p>0}\Big\{ps - \frac{p^3}{24} - (\frac{p^3}{8} + \frac{p^2 a}{2} + \frac{pa^2}{2})\Big\} = -\frac{2\sqrt{2}}{3} s^{\frac{3}{2}} + sa - \frac{a^3}{6}.$$

This verifies (3.1.15) when $a > 0$. Now, we suppose $a < 0$. Notice that $g(p) = 0$ if $p < -2a$ and $g(p) = \frac{p}{2}(\frac{p}{2} + a)^2$ if $p \geq -2a$. In this case, we also have $g \in C^1(R_{>0})$ and $\zeta = \lim_{p \to 0} g'(p) = 0$.

By a direct computation, we get

$$\sup_{p>0}\left\{sp-\frac{p^3}{24}-g(p)\right\}=\begin{cases}-\frac{4\sqrt{2}}{3}s^{\frac{3}{2}} & 0<s\le\frac{a^2}{2}\\ -\frac{2\sqrt{2}}{3}s^{\frac{3}{2}}+sa-\frac{a^3}{6} & s\ge\frac{a^2}{2}.\end{cases}$$

This shows (3.1.15) when $a<0$.

We now turn to prove our claim $(g,\{f_t\}_{t\ge0})\in\mathbf{Hyp}$. For this, we serially show that $(g,\{f_t\}_{t\ge0})$ satisfies (3.1.4), (3.1.5), (3.1.6), (3.1.7) and (3.1.8).

PROOF OF (3.1.4): Since $\mathbb{E}[e^{pB(x)}]=\exp(p^2|x|/2)$ for any $x\in\mathbb{R}$, we have $\log\mathbb{E}\Big[e^{pf_t(x)}\Big]=p^2|x|/2+pa_+x\mathbb{1}_{\{x\ge0\}}-pa_-x\mathbb{1}_{\{x\le0\}}$. By a direct computation, we get that the maximum value of $-px^2/2t+\log\mathbb{E}\Big[e^{pf_t(x)}\Big]$ over $x\in\mathbb{R}$ is given by $\frac{pt}{2}\big(\max\{(\frac{p}{2}+a),0\}\big)^2$ where $a=\max\{a_+,a_-\}$. Consequently,

$$\lim_{t\to\infty}\frac{1}{t}\sup_{x\in\mathbb{R}}\Big(-\frac{px^2}{2t}+\log\mathbb{E}\Big[e^{pf_t(x)}\Big]\Big)=\frac{p}{2}\big(\max\{(\frac{p}{2}+a),0\}\big)^2=g(p).$$

This verifies (3.1.4).

PROOF OF (3.1.5): By using the inequality $pa_+x\mathbb{1}_{\{x\ge0\}}-pa_-x\mathbb{1}_{\{x\le0\}}\le pa|x|$ and the identity $\mathbb{E}[e^{pB(x)}]=e^{p^2|x|/2}$, we get $\mathbb{E}\Big[e^{p(1+\epsilon)f(x)}\Big]\le e^{|x|p^2(1+\epsilon)^2/2+p(1+\epsilon)a|x|}$. Owing to this inequality, we may write

$$\int e^{-\frac{p(1-\epsilon)x^2}{2t}}\mathbb{E}\Big[e^{p(1+\epsilon)f_t(x)}\Big]dx\le\int e^{-\frac{p(1-\epsilon)x^2}{2t}}e^{|x|p^2(1+\epsilon)^2/2+p(1+\epsilon)a|x|}dx. \tag{3.3.7}$$

We will first consider the case $a<-p/2$ and then, will move onto the case $a\ge-p/2$. Note that $g(p)=0$ when $a<-p/2$. For any $a<0$ and $p>0$ satisfying $a<-p/2$, there exists $\epsilon_0=\epsilon_0(a,p)>0$ such that

$$\frac{p^2}{2}(1+\epsilon)^2+ap(1+\epsilon)<0,\quad\forall 0<\epsilon<\epsilon_0.$$

Therefore, we can bound the right hand side of (3.3.7) by $\int \exp(-p(1-\epsilon)x^2/2t)dt$ for all $0 < \epsilon < \epsilon_0$. Since the limit of $t^{-1}\log(\int \exp(-p(1-\epsilon)x^2/2t)dt)$ is equal to 0 as $t \to \infty$ for all small $\epsilon > 0$, we get

$$\limsup_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\left(\int e^{-\frac{p(1-\epsilon)x^2}{2t}}\mathbb{E}\left[e^{p(1+\epsilon)f_t(x)}\right]dx\right) \leq 0, \quad \text{when } a < -\frac{p}{2}.$$

Now, we turn to the case $a \geq -p/2$. We start by noting that the right hand side of (3.3.7) can be bounded by sum of two Gaussian integrals $\int e^{-\phi(x)}dx$ and $\int e^{-\phi(-x)}dx$ where $\phi(x) := p(1-\epsilon)x^2/(2t) - xp^2(1+\epsilon)^2/2 - xp(1+\epsilon)a$. By a direct computation, one can show that $\int e^{-\phi(x)}dx$ and $\int e^{-\phi(-x)}dx$ are both equal to $\sqrt{\frac{2\pi t}{p(1-\epsilon)}}\exp\left(\frac{(p^2(1+\epsilon)/2+p(1+\epsilon)a)^2 t}{2p(1-\epsilon)}\right)$. With this exact formula, it is straightforward to check that

$$\liminf_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log\left(\sqrt{\frac{2\pi t}{p(1-\epsilon)}}\exp\left(\frac{(p^2(1+\epsilon)/2+p(1+\epsilon)a)^2 t}{2p(1-\epsilon)}\right)\right) = \frac{(p^2/2+pa)^2}{2p} = g(p). \tag{3.3.8}$$

Taking logarithm of both sides of (3.3.7), dividing them by t and letting $t \to \infty, \epsilon \to 0$ yields

$$\begin{aligned}\liminf_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\text{l.h.s. of (3.3.7)}) &\leq \liminf_{\epsilon\to 0}\limsup_{t\to\infty}\frac{1}{t}\log(\text{r.h.s. of (3.3.7)})\\ &\leq \liminf_{\epsilon\to 0}\limsup_{t\to\infty}\text{l.h.s. of (3.3.8)} = g(p).\end{aligned}$$

This completes the proof of (3.1.5).

PROOF OF (3.1.6) & (3.1.7): We know $M_p^{f_t}(t,x) = \mathbb{E}\left[e^{pf_t(x)}\right] = e^{p^2|x|/2+a_+x}$ if $x > 0$ and is equal to $e^{p^2|x|/2-a_-x}$ if $x \leq 0$. From this exact formula of $M_p^{f_t}(t,x)$, it is clear that the growth condition (3.1.6) holds for $\{f_t\}_{t>0}$. Since f_t is same for all $t > 0$, so (3.1.7) is true.

PROOF OF (3.1.8): From the definition of $f_t(x)$, we know

$$\frac{|f_t(x)-f_t(y)|}{|x-y|^{\frac{1}{2}}} = \begin{cases} \frac{|B(x)-B(y)+a_+(x-y)|}{|x-y|^{\frac{1}{2}}} & x,y>0, \\ \frac{|B(x)-B(y)-a_-(x-y)|}{|x-y|^{\frac{1}{2}}} & x,y\le 0 \\ \frac{|B(x)-B(y)+a_+x-a_-y|}{|x-y|^{\frac{1}{2}}} & x>0, y\le 0 \\ \frac{|B(x)-B(y)+a_+y-a_-x|}{|x-y|^{\frac{1}{2}}} & y>0, x\le 0 \end{cases} \tag{3.3.9}$$

From the above relations, we intend to show that the following inequality

$$\frac{|f_t(x)-f_t(y)|}{|x-y|^{\frac{1}{2}}} \le \frac{|B(x)-B(y)|}{|x-y|^{\frac{1}{2}}} + \max(|a_+|,|a_-|). \tag{3.3.10}$$

holds for all $x, y \in \mathbb{R}$ such that $|x-y| \le 1$. Note that the above inequality trivially holds from (3.3.9) when $x, y > 0$ or, $x, y \le 0$. It remains to show this inequality when x and y have different sign and $|x-y| \le 1$. We only show the above inequality when $x > 0$, $y \le 0$ and $|x-y| \le 1$. The other case follows from similar argument and is skipped for brevity. We can bound $|B(x)-B(y)+a_+x-a_-y|$ by $|B(x)-B(y)|+|a_+x-a_-y|$ and $|a_+x-a_-y|$ by $\max\{a_+,a_-\}(|x|+|y|)$ using triangle inequality. In the case when $x > 0, y \le 0$ and $|x-y| \le 1$, $|x-y|^{1/2}$ is in fact equal to $(|x|+|y|)^{1/2}$. Combining these observations shows

$$\frac{|B(x)-B(y)+a_+x-a_-y|}{|x-y|^{\frac{1}{2}}} \le \frac{|B(x)-B(y)|+\max\{a_+,a_-\}(|x|+|y|)}{(|x|+|y|)^{1/2}} \le \frac{|B(x)-B(y)|}{|x-y|^{\frac{1}{2}}} + \max\{|a_+|,|a_-|\}.$$

The last inequality holds since $(|x|+|y|)^{1/2} = |x-y|^{1/2} \le 1$. This shows (3.3.10).

Now, we show how (3.3.10) implies (3.1.8). We define $\theta_n = n$ for all $n \in \mathbb{Z}$. Fix any $n \in \mathbb{Z}$. By (3.3.10), we may bound $|f_t(x)-f_t(\theta_n)|$ by $|B(x)-B(\theta_n)|+\max\{|a_+|,|a_-|\}$ for any $x \in [\theta_n, \theta_{n+1}]$. As a consequence, for all $s > \max\{|a_+|,|a_-|\}$,

$$\mathbb{P}\big(\sup_{x\in[\theta_n,\theta_{n+1}]} |f_t(x)-f_t(\theta_n)| \ge s\big) \le \mathbb{P}\big(\sup_{x\in[\theta_n,\theta_{n+1}]} |B(x)-B(\theta_n)| \ge s-\max\{|a_+|,|a_-|\}\big) \le e^{-c(s-\max\{|a_+|,|a_-|\})^2}$$

where c is a constant which does not depend on n or t. The last inequality follows by applying reflection principle and tail decay of a Gaussian random variable. This shows (3.1.8).

3.4 Auxiliary Results

3.4.1 Proof of Proposition 3.1.12

To prove (3.1.20), it suffices to show that for $s > \zeta$,

$$\underbrace{\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(X(t) > st\big) \leq -\max_{p\in\mathbb{R}_{>0}} \{ps - h(p)\},}_{\mathfrak{LimSup}} \qquad \underbrace{\liminf_{t\to\infty} \frac{1}{t} \mathbb{P}\big(X(t) > st\big) \geq -\max_{p\in\mathbb{R}_{>0}} \{ps - h(p)\}.}_{\mathfrak{LimInf}} \tag{3.4.1}$$

We first show $\mathfrak{LimSup}$. Recall the definition of h. Note that h' is strictly increasing and has a continuous inverse. Let us define $\mathfrak{q} : (0,\infty) \to (\zeta,\infty)$ as $\mathfrak{q}(s) := (h')^{-1}(s)$. Note that the supremum of $ps - h(p)$ is attained when p is equal to $\mathfrak{q}(s)$ and therefore, $\sup_{p>0}\{ps - h(p)\} = \mathfrak{q}(s)s - h(\mathfrak{q}(s))$. By using the Markov's inequality, we get $\mathbb{P}(X(t) > st) \leq e^{-\mathfrak{q}(s)st}\mathbb{E}[e^{\mathfrak{q}(s)X(t)}]$. We take the logarithm of both sides of this inequality, divide them by t and let $t \to \infty$. Consequently,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(X(t) > st\Big) \leq \limsup_{t\to\infty} \frac{1}{t}\Big(- \mathfrak{q}(s)st + \log \mathbb{E}[e^{\mathfrak{q}(s)X(t)}]\Big) = -(s\mathfrak{q}(s) - h(\mathfrak{q}(s))).$$

where the last equality follows from (3.1.19). This proves $\mathfrak{LimSup}$.

We turn to show $\mathfrak{LimInf}$. To this aim, we define $\mathfrak{q}_\epsilon : (0,\infty) \to (\zeta,\infty)$ as $\mathfrak{q}_\epsilon(s) = (h')^{-1}(s+\epsilon)$. For convenience of notation, we will use $\mathfrak{q}_\epsilon$ to denote $\mathfrak{q}_\epsilon(s)$. Fix any $s,t > 0$. We define a exponentially tilted probability measure $\widetilde{\mathbb{P}}_{t,s}$ as

$$\widetilde{\mathbb{P}}_{t,s}\big(X(t) \in A\big) := \frac{1}{\mathbb{E}\big[e^{\mathfrak{q}_\epsilon X(t)}\big]}\mathbb{E}\big[e^{\mathfrak{q}_\epsilon X(t)}\mathbb{1}_{\{X(t)\in A\}}\big]$$

where A is a Borel set in $\mathbb{R}$. We denote the expectation with respect to $\widetilde{\mathbb{P}}_{t,s}$ by $\widetilde{\mathbb{E}}_{t,s}$. We claim that for showing $\mathfrak{LimInf}$, it suffices to verify for any fixed $s > 0$,

$$\lim_{t\to\infty} \widetilde{\mathbb{P}}_{t,s}\Big(X(t) \in [ts, t(s+2\epsilon)]\Big) = 1. \tag{3.4.2}$$

Let us first explain how $\mathfrak{LimInf}$ follows from (3.4.2). From the definition of $\widetilde{\mathbb{P}}_{t,s}$, we know the following change of measure formula-

$$\mathbb{P}(X(t) \geq ts) = \widetilde{\mathbb{E}}_{t,s}[e^{-\mathfrak{q}_\epsilon X(t)} \mathbb{1}_{\{X(t)\geq ts\}}] \cdot \mathbb{E}\Big[e^{\mathfrak{q}_\epsilon X(t)}\Big]. \tag{3.4.3}$$

Since $\{ts \leq X(t) \leq t(s+2\epsilon)\}$ is contained in $\{X(t) \geq ts\}$, we get the following inequality

$$\widetilde{\mathbb{E}}_{t,s}\Big[e^{-\mathfrak{q}_\epsilon X(t)} \mathbb{1}_{\{X(t)\geq ts\}}\Big] \geq \widetilde{\mathbb{E}}_{t,s}\Big[e^{-\mathfrak{q}_\epsilon X(t)} \mathbb{1}_{\{ts\leq X(t)\leq t(s+2\epsilon)\}}\Big] \geq e^{-(s+2\epsilon)t\mathfrak{q}_\epsilon} \widetilde{\mathbb{P}}_{t,s}\Big(ts \leq X(t) \leq t(s+2\epsilon)\Big). \tag{3.4.4}$$

Substituting this inequality into the right hand side of (3.4.3) yields

$$\mathbb{P}\Big(X(t) \geq ts\Big) \geq e^{-(s+2\epsilon)t\mathfrak{q}_\epsilon} \mathbb{E}\Big[e^{\mathfrak{q}_\epsilon X(t)}\Big] \widetilde{\mathbb{P}}_{t,s}\Big(st \leq X(t) \leq (s+2\epsilon)t\Big). \tag{3.4.5}$$

We take the logarithm of both sides of the above inequality and divide them by t for both sides of (3.4.5). Letting $t \to \infty$, we conclude that

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(X(t) \geq ts\Big) \geq -(s+2\epsilon)\mathfrak{q}_\epsilon + \liminf_{t\to\infty} \frac{1}{t}\mathbb{E}\Big[e^{\mathfrak{q}_\epsilon X(t)}\Big] = -(s+2\epsilon)\mathfrak{q}_\epsilon + h(\mathfrak{q}_\epsilon)$$

where the first inequality follows from (3.4.2) and the second equality follows from (3.1.19). Recall that $\lim_{\epsilon\to 0} \mathfrak{q}_\epsilon = \mathfrak{q}(s)$. By the continuity of h, as $\epsilon \to 0$, the right hand side in the above display converges to $-s\mathfrak{q}(s) + h(\mathfrak{q}(s))$. Recall that $-s\mathfrak{q}(s) + h(\mathfrak{q}(s))$ is equal to $-\max_{p\in\mathbb{R}_{>0}}\{sp - h(p)\}$. This completes demonstrating how $\mathfrak{LimInf}$ follows from (3.4.2). Throughout the rest, we prove (3.4.2).

In order to prove (3.4.2), it is enough to demonstrate $\lim_{t\to\infty} \widetilde{\mathbb{P}}_{t,s}\big(X(t) \notin [ts, t(s+2\epsilon)]\big) = 0$.

This follows from the combination of following results:

$$\limsup_{t\to\infty} \frac{1}{t} \log \widetilde{\mathbb{P}}_{t,s}(X(t) < ts) < 0, \qquad \limsup_{t\to\infty} \frac{1}{t} \log \widetilde{\mathbb{P}}_{t,s}\big(X(t) > t(s+2\epsilon)\big) < 0. \tag{3.4.6}$$

We proceed to prove these below. We first show $\lim_{t\to\infty} t^{-1} \log \widetilde{\mathbb{P}}_{t,s}(X(t) < ts) < 0$. By Markov's inequality, for $\lambda > 0$,

$$\widetilde{\mathbb{P}}_{t,s}(X(t) < ts) \leq e^{\lambda st} \widetilde{\mathbb{E}}_{t,s}[e^{-\lambda X(t)}] = e^{\lambda st} \frac{\mathbb{E}[e^{(\mathfrak{q}_\epsilon - \lambda)X(t)}]}{\mathbb{E}[e^{\mathfrak{q}_\epsilon X(t)}]},$$

We take the logarithm of both sides and divide them by t. Letting $t \to \infty$ and utilizing (3.1.19), we get

$$\limsup_{t\to\infty} \frac{1}{t} \log \widetilde{\mathbb{P}}_{t,s}\Big(X(t) < ts\Big) \leq \lambda s + h(\mathfrak{q}_\epsilon - \lambda) - h(\mathfrak{q}_\epsilon)$$

The desired result will follow from the above inequality if we can find a positive λ such that the right hand side above is negative. To find such λ, we consider $H : (0, \infty) \to \mathbb{R}$ as $H(\lambda) := \lambda s + h(\mathfrak{q}_\epsilon - \lambda) - h(\mathfrak{q}_\epsilon)$. It is straightforward that $H(0) = 0$ and $H'(0) = s - h'(\mathfrak{q}_\epsilon) = -\epsilon < 0$ since $\mathfrak{q}_\epsilon := (h')^{-1}(s + \epsilon)$. Since H has continuous derivative, there exists $\lambda^* > 0$ such that $H(\lambda^*) < 0$. This implies $\limsup_{t\to\infty} t^{-1} \log \widetilde{\mathbb{P}}(X(t) < ts) < 0$ which concludes the desired result.

Now we show $\limsup_{t\to\infty} t^{-1} \log \widetilde{\mathbb{P}}_{t,s}(X(t) > t(s + 2\epsilon)) < 0$. By Markov's inequality, for $\lambda > 0$,

$$\widetilde{\mathbb{P}}_{t,s}\big(X(t) > t(s+2\epsilon)\big) \leq e^{-\lambda(s+2\epsilon)t} \widetilde{\mathbb{E}}_{t,s}\big[e^{\lambda X(t)}\big] = e^{-\lambda t(s+2\epsilon)} \frac{\mathbb{E}[e^{(\mathfrak{q}_\epsilon + \lambda)X(t)}]}{\mathbb{E}[e^{\mathfrak{q}_\epsilon X(t)}]}.$$

In the same way as in the previous case, we get

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(X(t) > t(s+2\epsilon)\Big) \leq -\lambda(s+2\epsilon) + h(\mathfrak{q}_\epsilon + \lambda) - h(\mathfrak{q}_\epsilon).$$

To conclude the desired result, we will find $\lambda > 0$ such that the right hand side is less than 0. Like as in before, we consider $\widetilde{H} : (0, \infty) \to \mathbb{R}$ as $\widetilde{H}(\lambda) := -\lambda(s + 2\epsilon) + h(\mathfrak{q}_\epsilon + \lambda) - h(\mathfrak{q}_\epsilon)$ for which we know that $\widetilde{H}(0) = 0$, $\widetilde{H}'(0) = -s - 2\epsilon + h'(\mathfrak{q}_\epsilon) = -\epsilon$. Combination of these observations with the continuity of $\widetilde{H}'$ yields the existence $\lambda > 0$ such that the right hand side of the above inequality is less than 0. This proves the second limiting result of (3.4.6) and hence, completes the proof.

3.4.2 Proof of Lemma 3.1.13

For proving (i), we first note that the logarithm of $M_p^{f_t}(t, x)$ is a convex function of $p \in (0, \infty)$ which can be checked by verifying that the second derivative of $\log M_p^{f_t}(t, x)$ w.r.t. p stays positive for all $p \in (0, \infty)$. Since the convexity is preserved under taking pointwise supremum and/or, limit of a sequence of convex functions, the convexity of $g(p)$ for $p \in (0, \infty)$ now follows from its definition and the fact that $\log M_p^{f_t}(t, x)$ is convex in p. To see the non-negativity of $g(p)$, for $t > T_0$, we write

$$\sup_{x \in \mathbb{R}} \Big(\frac{-px^2}{2t} + \log M_p^{f_t}(t, x) \Big) \geq -\frac{pL^2}{2t} + \sup_{x \in [-L, L]} \log M_p^{f_t}(t, x) \geq -\frac{pL^2}{2t} - K$$

where the first inequality follows noting that the function $\frac{-px^2}{2t}$ takes its minimum value in the interval $[-L, L]$ at $\pm L$ and the second inequality is obtained by applying the lower bound condition (3.1.7) on f_t. By dividing both sides of the above inequality by t and letting t go to ∞, the limit of the left hand side yields $g(p)$ whereas the right hand side goes to 0. This proves that $g(p) \geq 0$ for all $p > 0$.

We turn to show (ii). We first prove that for every $p > 0$ and $\omega > 0$, the set $\mathrm{MAX}^f_{p,\omega}(t)$ is nonempty. By the definition of supremum, it suffices to prove $\sup_{y \in \mathbb{R}} \{-\frac{py^2}{2t} + \log M_p^{f_t}(t, y)\}$ is finite. By the growth condition (3.1.6), we know that for all $t > 0$,

$$-\frac{px^2}{2t} + \log M_p^{f_t}(t, x) \leq -\frac{px^2}{2t} + C|x| + \frac{\alpha p x^2}{2t} + C = \frac{p(\alpha - 1)x^2}{2t} + C|x| + C.$$

Since $\alpha < 1$, the supremum of the right hand side over $x \in \mathbb{R}$ is finite, which implies $\sup_{y\in\mathbb{R}}\{-\frac{py^2}{2t} + \log M_p^{f_t}(t, y)\}$ is finite. This shows $\mathrm{MAX}^f_{p,\omega}(t)$ is nonempty.

It remains to show that for fixed $p, \omega > 0$, there exists T_0 and $C = C(p, \omega)$, $|x_{q,\omega}(t)| \leq Ct$ for all $t > T_0$ and $\frac{p}{2} < q < 2p$. We prove this by contradiction. Suppose that $|x_{q,\omega}(t)|$ exceeds $\frac{4Ct}{q(1-\alpha)}$ for some $q \in [p/2, 2p]$ where C is the constant in the growth condition (3.1.6) of f_t. In this occasion, we will show that the lower bound to $-qx^2_{q,\omega}(t)/2t + M_q^{f_t}(t, x_{q,\omega}(t))$ has to exceeds its upper bound. Below, we separately compute an upper bound and a lower bound to $-qx^2_{q,\omega}(t)/2t + M_q^{f_t}(t, x_{q,\omega}(t))$. Before proceeding to those computations, we note that for any $\frac{p}{2} \leq q \leq 2p$, $x \in \mathbb{R}$ and $t \in \mathbb{R}_{>0}$ (using Hölder's inequality when $f_t(x)$ is random),

$$-\frac{qx^2}{2t} + \frac{2q}{p} \log M_{p/2}^{f_t}(t, x) \leq -\frac{qx^2}{2t} + \log M_q^{f_t}(t, x) \leq -\frac{qx^2}{2t} + \frac{q}{2p} \log M_{2p}^{f_t}(t, x). \tag{3.4.7}$$

UPPER BOUND TO $-qx^2_{q,\omega}(t)/2t + M_q^{f_t}(t, x_{q,\omega}(t))$: By the growth condition (3.1.6), the right hand side of the second inequality in (3.4.7) is bounded above by $-q(1-\alpha)x^2/2t + C|x|$. Plugging the bound on $|x_{q,\omega}(t)|$ into this bound shows that $-qx^2_{q,\omega}(t)/2t + \log M_q^{f_t}(t, x_{q,\omega}(t))$ is bounded above by the maximum of $-q(1-\alpha)x^2/2t + C|x|$ over $x \in \mathbb{R}$, which equals $-\frac{2C^2t}{q(1-\alpha)}$.

LOWER BOUND TO $-qx^2_{q,\omega}(t)/2t + M_q^{f_t}(t, x_{q,\omega}(t))$: We claim and prove that $-qx^2_{q,\omega}(t)/2t + M_q^{f_t}(t, x_{q,\omega}(t))$ is bounded below by $-\frac{qL^2}{2t} - 4K - \delta$ where K is the same constant as in the lower bound condition (3.1.7) of f_t.

Recall that $x_{q,\omega}(t) \in \mathrm{MAX}^f_{q,\omega}(t)$. Referring to (3.1.21),

$$-\frac{qx_{q,\omega}(t)^2}{2t} + \log M_q^{f_t}(t, x_{q,\omega}(t)) \geq \sup_{y\in\mathbb{R}} \Big\{ -\frac{qy^2}{2t} + \log M_q^{f_t}(t, y) \Big\} - \omega$$

Due to the first inequality of (3.4.7), $-\frac{qy^2}{2t} + \log M_q^{f_t}(t, y)$ is bounded below by $-qy^2/2 + 2p^{-1}q \log M_{p/2}^{f_t}(t, y)$ for all $y \in \mathbb{R}$. Substituting this inequality into the right hand side of the above display and restricting the supremum over the interval $[-L, L]$, we get

$$-\frac{qx_{q,\omega}(t)^2}{2t} + \log \mathbb{E}\Big[e^{qf_t(x_{q,\omega}(t))}\Big] \geq \max_{y\in[-L,L]} \Big\{ - \frac{qy^2}{2t} + \frac{2q}{p} \log M^{f_t}_{p/2}(t, y)\Big\} - \omega$$

where the constant L is same as in the lower bound condition (3.1.7) for $\log M^{f_t}_{p/2}(t, y)$. One may bound the right hand side of the above display from below by $-qL^2/2t+2p^{-1}q \max_{y\in[-L,L]}\{\log M_{p/2}(t, y)\}-\omega$. Since $2p^{-1}q \max_{y\in[-L,L]}\{\log M_{p/2}(t, y)\}$ bounded below by $-4K$ due to (3.1.7) and $q < 2p$, we find the right hand side in the above display is lower bounded by $-\frac{qL^2}{2t} - 4K - \omega$.

As we have shown above, if $|x_{q,\omega}(t)| > \frac{4Ct}{q(1-\alpha)}$, our lower bound to $-\frac{qx_{q,\omega}(t)^2}{2t} + \log M^{f_t}_q(t, z)$ (which is $-\frac{qL^2}{2t} - 4K - \delta$) exceeds the upper bound (which is $-\frac{2C^2t}{q(1-\alpha)}$) for all large t. This is a contradiction. Hence, the result follows.

Chapter 4: Lyapunov exponents of the half-line SHE

Chapter Abstract: We consider the half-line stochastic heat equation (SHE) with Robin boundary parameter $A = -\frac{1}{2}$. Under narrow wedge initial condition, we compute every positive (including non-integer) Lyapunov exponents of the half-line SHE. As a consequence, we prove a large deviation principle for the upper tail of the half-line KPZ equation under Neumann boundary parameter $A = -\frac{1}{2}$ with rate function $\Phi_{+}^{\mathrm{hf}}(s) = \frac{2}{3}s^{\frac{3}{2}}$. This confirms the prediction of [KLD18a, MV18] for the upper tail exponent of the half-line KPZ equation.

This chapter is available on arxiv [Lin20b].

4.1 Introduction

In this paper, we study the half-line KPZ equation, namely the KPZ equation on $\mathbb{R}_{\geq 0}$, with Neumann boundary parameter A. Introduced in [CS18], the equation is formally written as

$$\begin{cases} \partial_t \mathcal{H}^{\mathrm{hf}}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{H}^{\mathrm{hf}}(t,x) + \frac{1}{2}(\partial_x \mathcal{H}^{\mathrm{hf}}(t,x))^2 + \xi(t,x), \\ \partial_x \mathcal{H}^{\mathrm{hf}}(t,x)\Big|_{x=0} = A, \end{cases} \tag{4.1.1}$$

where $\xi(t,x)$ is the Gaussian space time white noise. The solution theory of (4.1.1) is ill-posed due to the non-linearity and the space-time white noise. One way to properly define the solution is to consider the *Hopf-Cole solution* $\mathcal{H}^{\mathrm{hf}}(t,x) := \log \mathcal{Z}^{\mathrm{hf}}(t,x)$ where $\mathcal{Z}^{\mathrm{hf}}$ solves the half-line stochastic heat equation (SHE) with Robin boundary parameter A, i.e.

$$\begin{cases} \partial_t \mathcal{Z}^{\mathrm{hf}}(t,x) = \frac{1}{2}\partial_{xx}\mathcal{Z}^{\mathrm{hf}}(t,x) + \frac{1}{2}\mathcal{Z}^{\mathrm{hf}}(t,x)\xi(t,x), \\ \partial_x \mathcal{Z}^{\mathrm{hf}}(t,x)\Big|_{x=0} = A\mathcal{Z}^{\mathrm{hf}}(t,0). \end{cases}$$

We say $\mathcal{Z}^{\mathrm{hf}}$ is a solution to the half-line SHE if for every $t > 0$, $\mathcal{Z}^{\mathrm{hf}}(t, \cdot)$ is adapted to the sigma algebra generated by $\mathcal{Z}^{\mathrm{hf}}(0, \cdot)$ and the space-time white noise up to time t, and satisfies the mild formulation

$$\mathcal{Z}^{\mathrm{hf}}(t,x) = \int_{\mathbb{R}_{\geq 0}} p_t^{\mathrm{hf}}(x,y)\mathcal{Z}^{\mathrm{hf}}(0,y)dy + \int_0^t \int_{\mathbb{R}_{\geq 0}} p_{t-s}^{\mathrm{hf}}(x,y)\mathcal{Z}^{\mathrm{hf}}(s,y)\xi(s,y)dyds,$$

For fixed $x \in \mathbb{R}_{\geq 0}$, $p_t^{\mathrm{hf}}(x,y)$ satisfies the half-line heat equation $\partial_t p_t^{\mathrm{hf}}(x,y) = \frac{1}{2}\partial_{yy} p_t^{\mathrm{hf}}(x,y)$ for all $y > 0$ with boundary condition $p_0^{\mathrm{hf}}(x,y) = \delta_x(y)$ and $\partial_x p^{\mathrm{hf}}(t,0) = -Ap^{\mathrm{hf}}(t,0)$. [CS18] proves the existence, uniqueness and positivity of $\mathcal{Z}^{\mathrm{hf}}$ for non-negative boundary parameter A and later [Par19b] extends these results to the scope of all $A \in \mathbb{R}$. As a consequence, the Hopf-Cole solution $\mathcal{H}^{\mathrm{hf}}(t,x) = \log \mathcal{Z}^{\mathrm{hf}}(t,x)$ is well-defined. Note that the solution to (4.1.1) can also be formulated in other different but equivalent ways, see [GPS20, GH19].

The half-line KPZ equation plays an important role characterizing how the surface grows subject to a boundary. In addition, it is the (weak) scaling limit of various half space models lying in the half-space KPZ universality [Wu18, CS18, Par19a]. Interestingly, such half-space random growth models usually exhibit a phase transition depending on the strength of repulsion/attraction at the boundary, which is characterized by the boundary parameter. Such phase transition is related to wetting/depinning transition which goes back to [Kar85, Kar87] and was proved for various discrete half-space models [BR01, SI04, BBCS18].

For the half-line KPZ equation, we restrict ourselves in a particular initial condition called *narrow wedge initial condition*, which corresponds to setting $\mathcal{Z}^{\mathrm{hf}}(0,x)$ to be a Dirac-delta function at zero. It is widely believed that the fluctuation of $\mathcal{H}^{\mathrm{hf}}(2t,0)$ at late time exhibits a phase transition at $A = -\frac{1}{2}$ [Par19b, Conjecture 1.2]. More precisely, the fluctuation of $\mathcal{H}^{\mathrm{hf}}(2t,0)$ will be Gaussian/Tracy-Widom GOE/GSE [TW96] depending on whether the boundary parameter A is smaller than, equal to or larger than $-\frac{1}{2}$.

[Par19b, Theorem 1.1] (also see [BBCW18, Remark 1.1]) shows that when $A = -\frac{1}{2}$, for all $s \in \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{P}\Big(\frac{\mathcal{H}^{\mathrm{hf}}(2t,0) + \frac{t}{12}}{t^{\frac{1}{3}}} \le s\Big) = F_{\mathrm{GOE}}(s), \tag{4.1.2}$$

where $F_{\mathrm{GOE}}(s)$ is distribution function of the Tracy-Widom GOE distribution [TW96]. The key ingredient to arriving at (4.1.2) is the exact formula for the Laplace transform of $\mathcal{Z}^{\mathrm{hf}}(2t,0) + \frac{t}{12}$ developed in [BBCW18, Par19b], see Theorem 4.1.3. The conjectured Gaussian/GSE fluctuation for $A < -\frac{1}{2}$ and $A > -\frac{1}{2}$ are supported in a non-rigorous way by the works [GLD12, BBC16, DNKDT19, KLD20]. In the GSE region $A > -\frac{1}{2}$, an exact formula of the Laplace transform of $\mathcal{Z}^{\mathrm{hf}}(2t,0)$ is also conjectured in the aforementioned works.

In this paper, we focus on the critical regime $A = -\frac{1}{2}$.

Having considered the limit theorem (4.1.2), it is natural to think about the large deviation principle (LDP), i.e. the probability that $\mathcal{H}^{\mathrm{hf}}(2t,0) + \frac{t}{12}$ deviates from zero in a size of t, as $t \to \infty$. It is expected that for $s > 0$,

$$-\lim_{t\to\infty} \frac{1}{t^2} \log \mathbb{P}\Big(\mathcal{H}^{\mathrm{hf}}(2t,0) + \frac{t}{12} < -st\Big) = \Phi_-^{\mathrm{hf}}(s) \qquad \text{(lower tail)}$$

$$-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\Big(\mathcal{H}^{\mathrm{hf}}(2t,0) + \frac{t}{12} > st\Big) = \Phi_+^{\mathrm{hf}}(s). \qquad \text{(upper tail)}$$

Note that the upper and lower tail LDP have different speeds (t vs t^2). One way to explain such phenomenon is to view $\mathcal{H}^{\mathrm{hf}}(2t,0)$ as the free energy of a half-space continuum directed random polymer with a wall at $x = 0$. For various discrete/continuum polymers, the t vs t^2 phenomenon is observed and explained in [LDMS16, BGS17, DT19]. Here, let we provide a different explanation. If we replace s with $t^{\frac{2}{3}}s$ in (4.1.2), the right hand side of (4.1.2) becomes $F_{\mathrm{GOE}}(t^{\frac{2}{3}}s)$. Since Tracy-Widom GOE distribution has left and right tail: as $s \to \infty$ $F_{\mathrm{GOE}}(-s) \sim \exp(-\frac{s^3}{24})$, $1 - F_{\mathrm{GOE}}(s) \sim \exp(-\frac{2}{3}s^{\frac{3}{2}})$, see [TW09]. Hence, we recover the t^2 and t speed of LDP for the lower and upper tail. [Tsa18, Corollary 1.3] proves the LDP for the lower tail and identifies the rate function $\Phi_-^{\mathrm{hf}}(s)$.

In this paper, we prove that the upper tail LDP holds with $\Phi_+^{\mathrm{hf}}(s) = \frac{2}{3}s^{\frac{3}{2}}$. Note that this is the first rigorous result concerning the upper tail LDP of the *half-space models* in the KPZ universality class. The $\frac{3}{2}$-exponent for the upper tail also arises in the work of [KLD18a, MV18], where the LDP for half-line KPZ equation at short time was studied.

The upper tail LDP of the half-line KPZ equation is closely related to the *Lyapunov exponent* of the half-line SHE. More precisely, for $p \in \mathbb{R}_{>0}$ we call the p-th Lyapunov exponent of the SHE to be limit of $t^{-1} \log \mathbb{E}\big[\mathcal{Z}^{\mathrm{hf}}(2t, 0)^p\big]$ as $t \to \infty$. If such limit exists for every p, in the spirit of Gärtner-Ellis theorem, $\mathcal{H}^{\mathrm{hf}}(2t, 0) = \log \mathcal{Z}^{\mathrm{hf}}(2t, 0)$ satisfies a LDP with rate function to be the Legendre-Fenchel transform of the Lyapunov exponents (as a function of p). We remark that the Lyapunov exponents also capture the nature of intermittency, which is a universal property for the random fields with multiplicative noise and has been studied extensive in the literature [GM90, CM94, GKM07, FK09, CJK13, CJKS13, CD15, Che15, BC16, KKX17].

4.1.1 Main result and proof idea

From now on, we use $\mathcal{Z}^{\mathrm{hf}}(2t, 0)$ to denote the solution to the half-line SHE with Robin boundary parameter $A = -\frac{1}{2}$ and Dirac-delta initial data $\mathcal{Z}^{\mathrm{hf}}(0, x) = \delta_{x=0}$. Our main contribution is rigorously computing the Lyapunov exponents of the half-line SHE.

Theorem 4.1.1 (Lyapunov exponents and upper tail LDP)**.** *We have*

(i) For every $p \in \mathbb{R}_{>0}$, one has $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\big[\mathcal{Z}^{hf}(2t, 0)^p \exp(\frac{pt}{12})\big] = \frac{p^3}{3}$.

(ii) For every $s \in \mathbb{R}_{>0}$, one has the upper tail LDP: $-\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(\mathcal{H}^{hf}(2t, 0) + \frac{t}{12} > st\big) = \Phi_+^{hf}(s) = \frac{2}{3} s^{\frac{3}{2}}$.

Remark 4.1.2. The above upper tail LDP rate function matches with the right tail of the GOE, which is the limiting distribution in (4.1.2). Such matching between the upper tail LDP rate function of KPZ equation and the right tail of the limiting Tracy GUE/GOE/Baik-Rains distribution has been predicted in [LDMS16, LDMRS16, MS17] and has been confirmed in various situations [DT19, GL20].

Let us briefly explain the idea for the proof of Theorem 4.1.1. A more detailed discussion will be given in Section 4.2. First of all, it is not hard to get Theorem 4.1.1 (ii) once we obtain (i), see Proposition 1.12 of [GL20]. Hence, we focus on the proof of (i). One crucial input is the following exact formula obtained in [Par19b, Theorem 1.3] (also see [BBCW18, Theorem 7.6]), which is stated as follows.

Theorem 4.1.3 (Theorem 1.3 of [Par19b], Theorem 7.6 of [BBCW18]). *When $A = -\frac{1}{2}$, we have for all $s \geq 0$,*

$$\mathbb{E}\Big[\exp\big(-s\mathcal{Z}^{hf}(2t,0)e^{\frac{t}{12}}\big)\Big] = \mathbb{E}\Big[\prod_{i=1}^{\infty} \frac{1}{\sqrt{1+4s\exp(t^{\frac{1}{3}}\mathbf{a_k})}}\Big], \tag{4.1.3}$$

where $\mathbf{a}_1 > \mathbf{a}_2 > \cdots$ is the GOE-Airy point process defined in Definition 4.2.1.

Our argument for proving Theorem 4.1.1 (i) follows [DT19] at the beginning. Firstly, we write the p-th moment of $\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}$ in terms of the Laplace transform (Lemma 4.2.5)

$$\mathbb{E}\Big[\big(\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)^p\Big] = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^\infty s^{-\alpha}\partial_s^n\Big(\mathbb{E}\big[\exp\big(-s\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)\big]\Big)ds$$

where $n = \lfloor p \rfloor + 1, \alpha = p + 1 - n \in [0,1)$. Decompose the integral region into $(0,1]$ and $(1,\infty)$ and denote the latter integral by $\mathcal{R}_p(t)$, we get

$$\mathbb{E}\Big[\Big(\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)^p\Big] = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^1 s^{-\alpha}\partial_s^n\Big(\mathbb{E}\big[\exp\big(-s\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)\big]\Big)ds + \mathcal{R}_p(t) \tag{4.1.4}$$

It turns out $\mathcal{R}_p(t)$ is uniformly bounded in t so we only need to focus on the first term on the right hand side (4.1.4). The Laplace transform $\mathbb{E}\big[\exp\big(-s\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)\big]$ in the above integral admits an explicit formula in terms of the GOE point process given by the right hand side of (4.1.3). Rewrite the expectation of products of the GOE point process into a Fredholm Pfaffian (Lemma 4.2.4), we get

$$\mathbb{E}\big[\exp\big(-s\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)\big] = 1 + \sum_{L=1}^{\infty}\frac{1}{L!}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}(x_i)dx_i,$$

where the Pfaffian kernel K is 2×2 matrix defined in Definition 4.2.1 and the function $\phi_{s,t}$ is specified in (4.2.5). Inserting the above expression to the first term on the right hand side of (4.1.4), bringing the infinite summation over L outside the derivative over s and integral from 0 to 1 (which will be justified in Lemma 4.2.7), we get

$$\mathbb{E}\Big[\Big(\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\Big)^p\Big] = \sum_{L=1}^{\infty} \frac{(-1)^n}{\Gamma(1-\alpha)L!} \int_0^1 s^{-\alpha} \int_{\mathbb{R}^L} \mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L} \partial_s^n \Big(\prod_{i=1}^{L} \phi_{s,t}(x_i)\Big) dx_1 \dots dx_L + \mathcal{R}_p(t).$$

The next step is to decompose the infinite summation in the above display into $L = 1$ and $L \geq 2$. In particular, we denote the first term in the summation by $\mathcal{A}_p(t)$ and the L-th term ($L \geq 2$) by $\mathcal{B}_{p,L}(t)$, then

$$\mathbb{E}\Big[\big(\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)^p\Big] = \mathcal{A}_p(t) + \sum_{L=2}^{\infty} \mathcal{B}_{p,L}(t) + \mathcal{R}_p(t)$$

We call $\mathcal{A}_p(t)$ the *leading order term* which will be shown to hold the dominating $t \to \infty$ asymptotic. It equals an integral of the $(1,2)$ entry of the 2×2 matrix $K(x,x)$ (the Pfaffian of a 2×2 matrix equals its $(1,2)$ entry). The second term on the right hand side of the above display is composed of *the higher order terms*. Each $\mathcal{B}_{p,L}(t)$ is related to an integral of L-th correlation function $\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}$ of the GOE point process. In our proof, we show in Proposition 4.2.8 and 4.2.9 that for fixed $p > 0$, as $t \to \infty$, 1) : $\mathcal{A}_p(t)$ grows asymptotically as $\exp(p^3 t/3)$. 2) : $\sum_{L=2}^{\infty} \big|\mathcal{B}_{p,L}(t)\big|$ is asymptotically upper bounded by $\exp\big((p^3 - \delta_p)t/3\big)$ for some $\delta_p > 0$. This demonstrates Theorem 4.1.1 (i).

So far, we have followed the idea in [DT19]. The analysis of the leading order term $\mathcal{A}_p(t)$ in Proposition 4.2.8 involves a steepest descent type analysis of the integral of $K_{12}(x,x)$. The harder problem is to control the higher order terms. In the situation of [DT19], the authors deal with a Fredholm determinant $\det(I + A) = 1 + \sum_{L=1}^{\infty} \mathrm{Tr}(A^{\wedge L})$,[1] where A is a positive, trace class operator. [DT19] upper bounds the higher order term $\mathrm{Tr}(A^{\wedge L})$ by $\big(\mathrm{Tr}(A)\big)^L / L!$. This can be understood by

[1] There is a misstatement in page 4 of [DT19] where an extra $L!$ appears in the definition of Fredholm determinant.

setting $\{\lambda_i\}_{i=1}^{\infty}$ to be the eigenvalues of A, since λ_i are all non-negative,

$$\mathrm{Tr}(A^{\wedge L}) = \sum_{1\le i_1<\cdots<i_L} \lambda_{i_1}\dots\lambda_{i_L} \le \frac{1}{L!}\Big(\sum_{i=1}^{\infty}\lambda_i\Big)^L = \frac{1}{L!}\big(\mathrm{Tr}(A)\big)^L.$$

Unfortunately, we are unaware of an analogue for such bound in the case of Fredholm Pfaffian. Instead of mimicking [DT19], we adopt a more direct approach. Using Hadamard's inequality and a determinantal analysis, we obtain two upper bounds of $\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}$ (Proposition 4.4.1). These upper bounds will be applied to control various terms in $\sum_{L=2}^{\infty}\mathcal{B}_{p,L}(t)$ depending on whether L is greater than a fixed threshold. We want to highlight that we did not pursue to get the sharp bound of the p-th moment of $\mathcal{Z}^{\mathrm{hf}}(2t,0)$ which holds uniformly for large p and t, as shown in [DT19, Theorem 1.1 (a)*]. However, it is sufficient to apply our method to obtain the Lyapunov exponents and LDP for our problem as well as for [DT19, Theorem 1.1].

4.1.2 Previous results

Recently, there has been significant progress in understanding the Lyapunov exponents and tails of various stochastic PDEs, see [GL20, Section 1.2] and reference therein. Here, we restrict our discussion to the scope of the KPZ equation and the SHE.

Full line KPZ equation/SHE

The KPZ equation was introduced in [KPZ86] as a paradigmatic model for the random surface growth. It is a representative of the *KPZ universality class* [ACQ11, Cor12], a collection of models sharing the universal scaling exponent and large time scaling behavior. Recently, the upper/lower tail LDP of the KPZ equation receives plenty of attention from the mathematics and physics community. In fact, there are two regimes for the LDP of the KPZ equation, long time and short time (Freidlin-Wentzell regime). We will focus on discussing the long time regime and for the latter situation, see the physics literature [KK07, KK09, KMS16, MKV16, LDMRS16].

The upper tail of the KPZ equation is closely connected to the Lyapunov exponents of SHE.

[BC95] first computes the integer Lyapunov exponents of the SHE. However, due to an incorrect use of Skorokhod's lemma, their result is only valid for the second moment. By analyzing the Brownian local time from Feynman-Kac representation of the SHE, [Che15] obtained the integer Lyapunov exponents for the SHE under deterministic bounded initial data. For the SHE under narrow wedge initial condition, the integer moment of the solution admits a contour integral formula [BC14a, Gho18]. By a residue calculus, [CG20a] obtains the integer Lyapunov exponents, from which they obtain a bound for the upper tail of the KPZ equation, showing the correct exponent $3/2$. [DT19] improves their result by identifying all positive real Lyapunov exponents of the SHE. As a consequence, they obtain the upper tail LDP of the KPZ equation with rate function $\frac{4}{3}s^{\frac{3}{2}}$. Recently, [GL20] is capable of computing all positive Lyapunov exponents for the SHE starting with a class of general (including random) initial data and obtain the corresponding upper tail LDP of the KPZ equation.

Unlike the upper tail, the lower tail of the KPZ equation does not have a strong connection to the moment of SHE. For the narrow wedge initial condition, via a delicate analysis of the exact formula in [BG16], [CG20b] derives a tight bound which detects the crossover of the tail exponent from 3 to $5/2$ depending on the depth of the tail, which was first observed in the physics work [SMP17]. The LDP for the lower tail of the KPZ equation is obtained by [Tsa18, CC19]. For the KPZ equation with general initial data, [CG20b] obtains an upper bound for the lower tail probability. Besides that, very few things are known at present.

Half-line KPZ equation/SHE

Compared with the knowledge for full-line equation, smaller amount of results are known for the half-line KPZ equation/SHE. [CS18] proves that on a closed interval or a half line, the open ASEP weakly converges to the KPZ equation with Neumann boundary parameter $A \geq 0$. Such convergence was extended later by [Par19b] to all $A \in \mathbb{R}$. [BBCW18, Par19b] obtain the Laplace transform formula for the half-line SHE under narrow wedge initial condition when $A = -\frac{1}{2}$, which helps to capture the Tracy-Widom GOE fluctuation of the KPZ equation. As discussed before, there is a conjectured Gaussian-GOE-GSE phase transition the for half-line KPZ equation, which is only proved at the critical parameter $A = -\frac{1}{2}$. Recently, there are progress identifying new limiting distribution for the half-space KPZ equation under stationary initial data [BKD20]. Such distribution is believed to be universal and arises in other half-space model starting from stationary initial data [BFO20].

Regarding the tail of half-space KPZ equation, let us focus on $A = -\frac{1}{2}$ and narrow wedge initial condition. Results for other boundary parameters and initial conditions are fairly untouched for now. Under the aforementioned boundary condition, a tight estimate of the lower tail was obtained in [Kim19], which detects the similar crossover of the tail exponent that appears in the full-line situation. The LDP for the lower tail was obtained by [Tsa18]. Few things were rigorously proved for the upper tail aside from the current work. [BBC16] (also see [BBC20]) obtains a moment formula of the half-line SHE by solving the delta-Bose gas (the result is not rigorous, since the uniqueness of the solution to the delta-Bose gas is unknown). It is also unclear whether one can extract the integer Lyapunov exponents for the half-line SHE (thus obtaining tail bounds of the half-line KPZ equation) from a similar residue calculus of the integral formula as carried out in [CG20a], due to the extra complexity.

On a different aspect, it is worth to mention the works of [KLD18a, MV18] in which the authors consider the LDP for the half-line KPZ equation in short time. The same exponent $3/2$ in the rate function is obtained therein. In addition, [KLD18a] predicted the upper tail exponent to be $\frac{2}{3}x^{\frac{3}{2}}$ when $A = -\frac{1}{2}, 0$ and $\frac{4}{3}x^{\frac{3}{2}}$ when $A = +\infty$. For the future work, it is appealing to prove a LDP for the upper tail for general boundary parameter A and see how the LDP rate function changes when A belongs to the Gaussian/GSE regime.

Outline. The rest of the paper is organized as follows. In section 4.2, we give an overview for the proof of the main theorem and provide more details for what is discussed in Section 4.1.1. In particular, we transform our problem into proving Proposition 4.2.8 and 4.2.9. Section 4.3 was devoted to prove Proposition 4.2.8. In Section 4.4, we provide two different upper bound for the pfaffian $\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}$ which is crucial to the proof of Proposition 4.2.9. We also justify Lemma 4.2.7 in that section. Section 4.5 completes the proof of Proposition 4.2.9.

Acknowledgment. The author thanks Guillaume Barraquand, Ivan Corwin, Sayan Das, Yujin Kim and Li-Cheng Tsai for helpful discussions. The author was partially supported by the Fernholz Foundation's "Summer Minerva Fellow" program and also received summer support from Ivan Corwin's NSF grant DMS-1811143, DMS-1664650.

4.2 Proof of Theorem 4.1.1: A detailed overview

In this section, we explain with more details how we prove Theorem 4.1.1. We begin with the definition of GOE point process that was mentioned in the introduction. As formulated in [AGZ10, Section 4.2.1], a point process on $\mathbb{R}$ is a random point configuration $\mathcal{X}$. The L-th correlation function ρ_L w.r.t. the measure μ associated to $\mathcal{X}$ is defined in the way that for arbitrary distinct Borel sets $B_1,\dots,B_L$,

$$\int_{B_1\times\cdots\times B_L}\rho_L(x_1,\dots,x_L)d\mu^{\otimes L}=\mathbb{E}\Big[\#\big\{(x_1,\dots,x_L),\text{ such that } x_i\in\mathcal{X}\cap B_i, i=1,\dots,L\big\}\Big].$$

We say a point process is a Pfaffian, if there exists a matrix kernel $K:\mathbb{R}\times\mathbb{R}\to\mathbb{C}^{2\times 2}$ such that the correlation function $\rho_L(x_1,\dots,x_L)=\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}$ for arbitrary $L\in\mathbb{Z}_{\geq 1}$.

Definition 4.2.1 (GOE point process)**.** We say $\mathcal{X}=\{\mathbf{a}_1>\mathbf{a}_2>\cdots\}$ is the GOE-Airy point process, if it is a Pfaffian point process on $\mathbb{R}$ with kernel

$$K(x,y)=\begin{bmatrix}K_{11}(x,y), & K_{12}(x,y)\\ K_{21}(x,y), & K_{22}(x,y)\end{bmatrix}$$

with the entries

$$K_{11}(x,y)=\int_0^\infty Ai(x+\lambda)Ai'(y+\lambda)-Ai(y+\lambda)Ai'(x+\lambda)d\lambda, \tag{4.2.1}$$

$$K_{12}(x,y) = -K_{21}(y,x) = \frac{1}{2}\int_0^\infty Ai(x+\lambda)Ai(y+\lambda)d\lambda + \frac{1}{2}Ai(x)\int_{-\infty}^{y} Ai(\lambda)d\lambda, \tag{4.2.2}$$

$$\begin{aligned} K_{22}(x,y) &= \frac{1}{4}\int_0^\infty \Big(\int_\lambda^\infty Ai(y+\mu)d\mu\Big)Ai(x+\lambda)d\lambda - \frac{1}{4}\int_0^\infty \Big(\int_\lambda^\infty Ai(x+\mu)d\mu\Big)Ai(y+\lambda)d\lambda \\ &\quad - \frac{1}{4}\int_0^\infty Ai(x+\lambda)d\lambda + \frac{1}{4}\int_0^\infty Ai(y+\lambda)d\lambda - \frac{sgn(x-y)}{4}. \end{aligned} \tag{4.2.3}$$

Here, $sgn(x)$ is defined as the sign function $\mathbf{1}_{\{x>0\}} - \mathbf{1}_{\{x<0\}}$. Furthermore, we set $K_{21}(x,y) = -K_{12}(y,x)$.

Remark 4.2.2. Note that our expression of the kernel (4.2.1), (4.2.2), (4.2.3) is different from that in Eq. (6.1a), (6.1b), (6.1c) of [BBCW18]. However, they are demonstrated to be the same, see (2.9) and (6.17) of [Fer04] or [BBCS18, Lemma 2.6].

It turns out that the right hand side of (4.1.3) can be rewritten as a *Fredholm Pfaffian*, which has been first defined in [Rai00]. We reproduce the definition of the Fredholm Pfaffian from [BBCS18, Definition 2.3].

Definition 4.2.3 (Fredholm Pfaffian)**.** Let $K(x,y)$ be an asymmetric 2×2 matrix and μ to be a measure on $\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{C}$ be a measurable function. We define the Fredholm Pfaffian by the series expansion

$$Pf\big[J+K\big]_{L^2(\mathbb{R},f\mu)} = \sum_{L=0}^{\infty}\frac{1}{L!}\int_{\mathbb{R}^L} Pf\big[K(x_i,x_j)\big]_{i,j=1}^{L}\Big(\prod_{i=1}^{L} f(x_i)\Big)d\mu^{\otimes L}(x_1,\dots,x_L),$$

where

$$J(x,y) = \mathbf{1}_{\{x=y\}}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Lemma 4.2.4 (Pfaffian point process and Fredholm Pfaffian [Rai00])**.** *Let* $\mathbf{a_1} > \mathbf{a}_2 > \dots$ *be a Pfaffian point process with kernel* K *and* $f:\mathbb{R}\to\mathbb{C}$ *be a measurable function. We have*

$$\mathbb{E}\Big[\prod_{i=1}^{\infty}\big(1+f(\mathbf{a}_i)\big)\Big] = \mathrm{Pf}\big[J+K\big]_{L^2(\mathbb{R},f\mu)}, \tag{4.2.4}$$

as long as both sides of the above equation converge absolutely.

We take $\mathbf{a}_1 > \mathbf{a_2} > \dots$ to be the GOE point process defined in Definition 4.2.1 and the function f in the above lemma to be

$$\phi_{s,t}(x) := \frac{1}{\sqrt{1+4s\exp(t^{1/3}x)}} - 1, \tag{4.2.5}$$

Theorem 7.6 of [BBCW18] has already justified the convergence of both sides of (4.2.4). As a result,

$$\mathbb{E}\Big[\prod_{i=1}^{\infty} \frac{1}{\sqrt{1+4s\exp(t^{\frac{1}{3}}\mathbf{a}_i)}}\Big] = \sum_{L=0}^{\infty} \frac{1}{L!}\int_{\mathbb{R}^L} \mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L} \prod_{i=1}^{L} \phi_{s,t}(x_i)dx_i.$$

By Theorem 4.1.3, the left hand side in the above display equals the Laplace transform of $\mathcal{Z}^{\mathrm{hf}}(2t,0)\exp(\frac{t}{12})$, thus

$$\mathbb{E}\Big[\exp\big(-s\mathcal{Z}^{\mathrm{hf}}(2t,0)e^{\frac{t}{12}}\big)\Big] = \sum_{L=0}^{\infty} \frac{1}{L!}\int_{\mathbb{R}^L} \mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L} \prod_{i=1}^{L} \phi_{s,t}(x_i)dx_i \tag{4.2.6}$$

To prove Theorem 4.1.1, the next step is to link the Laplace transform of $\mathcal{Z}^{\mathrm{hf}}(2t,0)\exp(\frac{t}{12})$ with its fractional moment. The following lemma was stated as [DT19, Lemma 1.2], which can be verified via Fubini's theorem.

Lemma 4.2.5. *For arbitrary non-negative random variable X, $0 \le \alpha < 1$ and $n \in \mathbb{Z}_{\ge 1}$,*

$$\mathbb{E}\Big[X^{n-1+\alpha}\Big] = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^{\infty} s^{-\alpha}\partial_s^n\Big(\mathbb{E}\big[e^{-sX}\big]\Big)ds.$$

As a convention, we use ∂_s^n to denote the n-th partial derivative with respect to s.

For fixed $p > 0$, we set $n = \lfloor p \rfloor + 1$ and $\alpha = p - n + 1$. It is clear that $n \in \mathbb{Z}_{\ge 1}$, $\alpha \in [0,1)$. Applying Lemma 4.2.5 with $X = \mathcal{Z}^{\mathrm{hf}}(2t,0)\exp\big(\frac{t}{12}\big)$ (note that $\mathcal{Z}^{\mathrm{hf}}(2t,0)$ is almost surely positive), we find that

$$\mathbb{E}\Big[\mathcal{Z}^{\mathrm{hf}}(2t,0)^p e^{\frac{pt}{12}}\Big] = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^{\infty} s^{-\alpha}\partial_s^n\Big(\mathbb{E}\Big[e^{-s(\mathcal{Z}^{\mathrm{hf}}(2t,0)+\frac{t}{12})}\Big]\Big)ds.$$

Splitting the interval of integration into $[0,1]$ and $[1,\infty)$ yields

$$\mathbb{E}\Big[\mathcal{Z}^{\mathrm{hf}}(2t,0)^p e^{\frac{pt}{12}}\Big] = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^1 s^{-\alpha}\partial_s^n\Big(\mathbb{E}\Big[e^{-s(\mathcal{Z}^{\mathrm{hf}}(2t,0)+\frac{t}{12})}\Big]\Big)ds + \mathcal{R}_p(t), \tag{4.2.7}$$

where $\mathcal{R}_p(t) := \frac{(-1)^n}{\Gamma(1-\alpha)}\int_1^\infty s^{-\alpha}\partial_s^n\big(\mathbb{E}\big[e^{-s(\mathcal{Z}^{\mathrm{hf}}(2t,0)+\frac{t}{12})}\big]\big)ds$.

Lemma 4.2.6. *For fixed $p>0$, $|\mathcal{R}_p(t)|$ is uniformly bounded by a constant for every $t>0$.*

Proof. Since $\mathcal{R}_p(t) = \frac{(-1)^n}{\Gamma(1-\alpha)}\int_1^\infty s^{-\alpha}\mathbb{E}\big[e^{-sX}X^n\big]ds$ with $X = \mathcal{Z}^{\mathrm{hf}}(2t,0)\exp\big(\frac{t}{12}\big)$. Note that

$$\mathbb{E}\Big[e^{-sX}X^n\Big] \le \sup_{x\ge 0}\big(e^{-sx}x^n\big) = s^{-n}n^n e^{-n}.$$

Replacing $\mathbb{E}\big[e^{-sX}X^n\big]$ with this upper bound inside the integral yields

$$0 \le (-1)^n\mathcal{R}_p(t) \le \frac{1}{\Gamma(1-\alpha)}\int_1^\infty s^{-n-\alpha}n^n e^{-n}ds = \frac{n^n e^{-n}}{\Gamma(1-\alpha)(n+\alpha)}.$$

Since α and n are determined by p, so the right hand side is a constant that only depends on p, this completes our proof. □

By (4.2.6), we see that the first term on the RHS of (4.2.7) can be written as

$$\int_0^1 s^{-\alpha}\partial_s^n\mathbb{E}\big[e^{-s(\mathcal{Z}^{\mathrm{hf}}(2t,0)+\frac{t}{12})}\big]ds = \int_0^1 s^{-\alpha}\partial_s^n\Big(\sum_{L=1}^\infty\frac{1}{L!}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^L\prod_{i=1}^L\phi_{s,t}(x_i)dx_i\Big)ds. \tag{4.2.8}$$

Note that we throw out the $L=0$ term in the summation since it is always 1 and has s-derivative to be 0. It turns out that we can interchange the order of derivative, integration and summation for the right hand side of the above display, for which we formulate as a lemma. The proof of it is deferred to Section 4.4.2.

Lemma 4.2.7. *We have*

$$\int_0^1 s^{-\alpha}\partial_s^n\Big(\sum_{L=1}^{\infty}\frac{1}{L!}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^L\prod_{i=1}^L\phi_{s,t}(x_i)dx_i\Big)ds$$
$$=\sum_{L=1}^{\infty}\frac{1}{L!}\int_0^1 s^{-\alpha}ds\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^L\partial_s^n\Big(\prod_{i=1}^L\phi_{s,t}(x_i)\Big)dx_1\dots dx_L. \tag{4.2.9}$$

Consequently, it follows from (4.2.8) *and the above display that*

$$\frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^1 s^{-\alpha}\partial_s^n\Big(\mathbb{E}\Big[\exp\big(-s\mathcal{Z}^{hf}(2t,0)e^{\frac{t}{12}}\big)\Big]\Big)ds$$
$$=\sum_{L=1}^{\infty}\frac{(-1)^n}{\Gamma(1-\alpha)}\frac{1}{L!}\int_0^1 s^{-\alpha}ds\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^L\partial_s^n\Big(\prod_{i=1}^L\phi_{s,t}(x_i)\Big)dx_1\dots dx_L. \tag{4.2.10}$$

Set the first term in the right hand side summation above as $\mathcal{A}_p(t)$ and the higher order terms as $\mathcal{B}_{p,L}(t)$ ($L\geq 2$). Since the Pfaffian of a 2×2 matrix equals its $(1,2)$ entry, when $L=1$, $\mathrm{Pf}\big[K(x,x)\big]=K_{12}(x,x)$. Hence,

$$\mathcal{A}_p(t)=\frac{(-1)^n}{\Gamma(1-\alpha)}\int_0^1 s^{-\alpha}\int_{\mathbb{R}}K_{12}(x,x)\big(\partial_s^n\phi_{s,t}(x)\big)dx \tag{4.2.11}$$
$$\mathcal{B}_{p,L}(t)=\frac{(-1)^n}{\Gamma(1-\alpha)L!}\int_0^1 s^{-\alpha}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^L\partial_s^n\Big(\prod_{i=1}^L\phi_{s,t}(x_i)\Big)dx_1\dots dx_L,\quad L\geq 2. \tag{4.2.12}$$

Under this notation, the left hand side of (4.2.10) equals $\mathcal{A}_p(t)+\sum_{p=2}^{\infty}\mathcal{B}_{p,L}(t)$. Referring to (4.2.7), we obtain

$$\mathbb{E}\Big[\mathcal{Z}^{\mathrm{hf}}(2t,0)^p e^{\frac{pt}{12}}\Big]=\mathcal{A}_p(t)+\sum_{L=2}^{\infty}\mathcal{B}_{p,L}(t)+\mathcal{R}_p(t). \tag{4.2.13}$$

We want to show that the logarithm of the left hand side in the above display, after divided by t and letting $t\to\infty$, converges to $p^3/3$. By Lemma 4.2.6, $|\mathcal{R}_p(t)|$ is uniformly upper bounded by a constant for all $t>0$. Therefore, to prove Theorem 4.1.1, it suffices to demonstrate that the following facts for $\mathcal{A}_p(t)$ and $\sum_{L=2}^{\infty}|\mathcal{B}_{p,L}(t)|$.

Proposition 4.2.8. *For fixed* $p\in\mathbb{R}_{>0}$, $\lim_{t\to\infty}\frac{1}{t}\log\mathcal{A}_p(t)=e^{\frac{p^3}{3}}$.

Proposition 4.2.9. *For fixed* $p \in \mathbb{R}_{>0}$, $\limsup_{t\to\infty} \frac{1}{t} \log\left(\sum_{L=2}^{\infty} |\mathcal{B}_{p,L}(t)|\right) \leq e^{\frac{p^3}{3}-\delta_p}$ *with* $\delta_p = \min(\frac{2}{3}, \frac{p^3}{4})$.

We will prove the two propositions in Section 4.3 and Section 4.5. Let us first conclude the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. For part (i), from Proposition 4.2.8 and 4.2.9, we know that $\mathcal{A}_p(t)$ grows exponentially faster than $\sum_{L=2}^{\infty} |\mathcal{B}_{p,L}(t)|$ as $t \to \infty$. Along with the fact that $|\mathcal{R}_p(t)|$ is upper bounded by a constant for all t, there exists $t_0 > 0$ such that for all $t > t_0$,

$$\sum_{L=2}^{\infty} |\mathcal{B}_{p,L}(t)| + \left|\mathcal{R}_p(t)\right| \leq \frac{1}{2}\mathcal{A}_p(t).$$

Referring to the decomposition (4.2.13) and using triangle inequality, we see that for $t > t_0$,

$$\log\left(\frac{1}{2}\mathcal{A}_p(t)\right) \leq \log \mathbb{E}\left[\mathcal{Z}^{\mathrm{hf}}(2t,0)^p\right] \leq \log\left(\frac{3}{2}\mathcal{A}_p(t)\right).$$

Dividing very term in the above display by t, Theorem 4.1.1 (i) follows easily from Proposition 4.2.8 as we take $t \to \infty$. Since we know that $t^{-1}\lim_{t\to\infty} \log \mathbb{E}\left[\mathcal{Z}^{\mathrm{hf}}(2t,0)^p \exp(\frac{pt}{12})\right] = p^3/3$, this completes the proof of part (i). Applying [GL20, Proposition 1.12] by setting $h(p) = \frac{p^3}{3}$ therein, we obtain the upper tail LDP with rate function to be $\sup_{p>0}(ps - p^3/3) = \frac{2}{3}s^{\frac{3}{2}}$, thus we obtain Theorem 4.1.1 (ii). □

4.3 Asymptotic of $\mathcal{A}_p(t)$: Proof of Proposition 4.2.8

In this section, we prove Proposition 4.2.8. One crucial step is Lemma 4.3.2, whose proof relies on Lemma B.0.1 and a steepest descent type analysis. Throughout the rest of the paper, we use C, C_1, C_2 to denote a constant, which may vary from line to line. We might not generally specify when irrelevant terms are being absorbed into the constants. We might also write $C(a), C(a,b)$ when we want to specify which parameters the constant depends on.

Lemma 4.3.1. *Denote* $\mathsf{B}(u,v)$ *to be the beta function* $\int_0^1 x^{u-1}(1-x)^{v-1}dx$*. For* $\gamma > 0, \alpha < 1$ *and* $\alpha + \beta > 1$,

$$\int_0^\infty \frac{s^{-\alpha}}{(1+\gamma s)^\beta} ds = \gamma^{\alpha-1}\mathsf{B}(1-\alpha, \beta+\alpha-1)$$

Proof. Via a change of variable $s = \frac{t}{\gamma(1-t)}$, we get

$$\int_0^\infty \frac{s^{-\alpha}ds}{(1+\gamma s)^\beta} = \gamma^{\alpha-1}\int_0^1 t^{-\alpha}(1-t)^{\alpha+\beta-2}dt = \gamma^{\alpha-1}\mathsf{B}(1-\alpha, \beta+\alpha-1). \qquad \square$$

Lemma 4.3.2. *For fixed* $p, t_0 > 0$*, there exists constant* $C = C(p, t_0)$ *such that for all* $t > t_0$*,*

$$\frac{1}{C} t^{-\frac{2}{3}} e^{\frac{1}{3}p^3 t} \le \int_0^\infty K_{12}(t^{\frac{2}{3}}x, t^{\frac{2}{3}}x)e^{ptx}dx \le C t^{-\frac{2}{3}} e^{\frac{1}{3}p^3 t}.$$

Proof. Throughout the proof we write $C = C(p, t_0)$ and denote by $U_p(x) = -\frac{2}{3}x^{\frac{3}{2}} + px$. Using the inequality in Lemma B.0.1 (i),

$$\frac{1}{C}\int_0^\infty \frac{e^{tU_p(x)}}{(1+t^{\frac{2}{3}}x)^{\frac{1}{4}}}dx \le \int_0^\infty K_{12}(t^{\frac{2}{3}}x, t^{\frac{2}{3}}x)e^{ptx}dx \le C\int_0^\infty \frac{e^{tU_p(x)}}{(1+t^{\frac{2}{3}}x)^{\frac{1}{4}}}dx$$

The proof is completed if we can show there exists a constant C such that for all $t > t_0$,

$$\frac{1}{C} t^{-\frac{2}{3}} e^{\frac{p^3 t}{3}} \le \int_0^\infty \frac{e^{tU_p(x)}}{(1+t^{\frac{2}{3}}x)^{\frac{1}{4}}}dx \le C t^{-\frac{2}{3}} e^{\frac{p^3 t}{3}} \tag{4.3.1}$$

An elementary calculus tells that the maximum of $U_p(x) = -\frac{2}{3}x^{\frac{3}{2}} + px$ on $[0, \infty)$ is reached at $x = p$, with $U_p(p) = \frac{1}{3}p^3$. So it is natural to expect that the main contribution of the integral in the above display comes around a small region around $x = p$. Having this intuition in mind, we let $\mathsf{q} = \frac{p}{4}$ decompose

$$\int_0^\infty \frac{e^{tU_p(x)}}{(1+t^{\frac{2}{3}}x)^{\frac{1}{4}}}dx = \Big(\int_{[(p-\mathsf{q})^2,(p+\mathsf{q})^2]} + \int_{\mathbb{R}_{>0}\setminus[(p-\mathsf{q})^2,(p+\mathsf{q})^2]}\Big)\frac{e^{tU_p(x)}}{(1+t^{\frac{2}{3}}x)^{\frac{1}{4}}}dx = \mathcal{K}_1 + \mathcal{K}_2. \tag{4.3.2}$$

It suffices to analyze $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. For $\mathcal{K}_1$, we make a change of variable $x = (p+r)^2$.

Noting that $U_p\big((p+r)^2\big) = \frac{p^3}{3} - (\frac{2}{3}r + p)r^2$, we get

$$\mathcal{K}_1 = \int_{-\mathsf{q}}^{\mathsf{q}} \frac{2(p+r)e^{tU_p((p+r)^2)}}{(1+t^{\frac{2}{3}}(p+r)^2)^{\frac{1}{4}}} dr = e^{\frac{p^3 t}{3}} \int_{-\mathsf{q}}^{\mathsf{q}} \frac{2(p+r)e^{-t(\frac{2}{3}r+p)r^2}}{(1+t^{\frac{2}{3}}(p+r)^2)^{\frac{1}{4}}} dr \tag{4.3.3}$$

Recall that $\mathsf{q} = \frac{p}{4}$, so there exists a constant $C = C(p, t_0)$ such that for all $r \in [-\frac{p}{4}, \frac{p}{4}]$ and $t > t_0$,

$$\frac{e^{-Ctr^2}}{Ct^{\frac{1}{6}}} \leq \frac{2(p+r)e^{-t(\frac{2}{3}r+p)r^2}}{(1+t^{\frac{2}{3}}(p+r)^2)^{\frac{1}{4}}} \leq \frac{Ce^{-\frac{1}{C}tr^2}}{t^{\frac{1}{6}}}. \tag{4.3.4}$$

By a change of variable $r \to t^{-\frac{1}{2}}r$, there exists constant C_1 such that for $t > t_0$

$$C_1^{-1}t^{-\frac{2}{3}} \leq \int_{-\mathsf{q}}^{\mathsf{q}} \frac{e^{-Ctr^2}}{Ct^{\frac{1}{6}}} \leq \int_{-\mathsf{q}}^{\mathsf{q}} \frac{Ce^{-\frac{1}{C}tr^2}}{t^{\frac{1}{6}}} \leq C_1 t^{-\frac{2}{3}}$$

Integrating the terms in (4.3.4) from $-\mathsf{q}$ to q and utilizing the displayed inequality above and (4.3.3), we conclude that $\frac{1}{C}t^{-\frac{2}{3}}e^{\frac{p^3 t}{3}} \leq \mathcal{K}_1 \leq Ct^{-\frac{2}{3}}e^{\frac{p^3 t}{3}}$ for $t > t_0$.

For $\mathcal{K}_2$, by a change of variable $x = r^2$ and noting $U_p(r^2) = \frac{p^3}{3} - (r-p)^2(\frac{2}{3}r + \frac{1}{3}p)$, we have

$$\mathcal{K}_2 = e^{\frac{p^3 t}{3}} \int_{\mathbb{R}_{>0}\setminus[p-\mathsf{q},p+\mathsf{q}]} \frac{e^{t(r-p)^2(-\frac{2}{3}r-\frac{1}{3}p)}}{(1+t^{\frac{2}{3}}r^2)^{\frac{1}{4}}} dr \leq e^{\frac{t(p^3-p\mathsf{q}^2)}{3}} \int_{\mathbb{R}_{>0}\setminus[p-\mathsf{q},p+\mathsf{q}]} \frac{e^{-\frac{2}{3}tr(r-p)^2}}{(1+t^{\frac{2}{3}}r^2)^{\frac{1}{4}}} dr \tag{4.3.5}$$

The inequality in the above display follows from noticing $\frac{1}{3}(r-p)^2 p \geq \frac{p\mathsf{q}^2}{3}$ when $r \notin [p-\mathsf{q}, p+\mathsf{q}]$. For the integral on the right hand side of the above display, we find that $\int_{\mathbb{R}_{>0}\setminus[p-\mathsf{q},p+\mathsf{q}]} \frac{e^{-\frac{2}{3}tr(r-p)^2}}{(1+t^{\frac{2}{3}}r^2)^{\frac{1}{4}}} dr \leq \int_{\mathbb{R}_{>0}} e^{-\frac{2}{3}t\mathsf{q}^2 r} dr = \frac{3}{2\mathsf{q}^2 t}$. Since we assume $t \geq t_0$, by taking $C = \frac{3}{2\mathsf{q}^2 t_0}$, we know that

$$0 \leq \mathcal{K}_2 \leq \frac{3}{2\mathsf{q}^2 t} e^{\frac{t(p^3-p\mathsf{q}^2)}{3}} \leq Ce^{\frac{t(p^3-p\mathsf{q}^2)}{3}}.$$

Combining this with (4.3.3) and recall from (4.3.2) that $\int_0^\infty \frac{e^{tU_p(x)}}{(1+t^{\frac{2}{3}}x)^{\frac{1}{4}}} dx = \mathcal{K}_1 + \mathcal{K}_2$, we see that $\mathcal{K}_1$ is the dominating term. This completes the proof of (4.3.1). $\square$

We are now ready to prove Proposition 4.2.8.

Proof of Proposition 4.2.8. Recall from (4.2.5) that $\phi_{s,t}(x) = \frac{1}{\sqrt{1+4s\exp(t^{1/3}x)}} - 1$, so

$$\partial_s^n \phi_{s,t}(x) = (-2)^n (2n-1)!!(1+4s\exp(t^{\frac{1}{3}}x))^{-\frac{2n+1}{2}}.$$

By Fubini's theorem, we switch the order of integration on the right hand side of (4.2.11), hence

$$\mathcal{A}_p(t) = \frac{2^n(2n-1)!!}{\Gamma(1-\alpha)} \int_{\mathbb{R}} K_{12}(x,x) e^{nt^{\frac{1}{3}}x} dx \int_0^1 \frac{s^{-\alpha}}{(1+4s\exp(t^{\frac{1}{3}}x))^{\frac{2n+1}{2}}} ds,$$

Writing the integral w.r.t s in the above display as $\int_0^1 = \int_0^\infty - \int_1^\infty$, we get $\mathcal{A}_p(t) = \mathcal{A}'_p(t) - \mathcal{A}''_p(t)$, where

$$\mathcal{A}'_p(t) = \frac{2^n(2n-1)!!}{\Gamma(1-\alpha)} \int_{\mathbb{R}} K_{12}(x,x) e^{nt^{\frac{1}{3}}x} dx \int_0^\infty \frac{s^{-\alpha}}{(1+4s\exp(t^{\frac{1}{3}}x))^{\frac{2n+1}{2}}} ds, \tag{4.3.6}$$

$$\mathcal{A}''_p(t) = \frac{2^n(2n-1)!!}{\Gamma(1-\alpha)} \int_{\mathbb{R}} K_{12}(x,x) e^{nt^{\frac{1}{3}}x} dx \int_1^\infty \frac{s^{-\alpha}}{(1+4s\exp(t^{\frac{1}{3}}x))^{\frac{2n+1}{2}}} ds. \tag{4.3.7}$$

To conclude our proof of Proposition 4.2.8, it suffices to show the following propositions.

Proposition 4.3.3. *For fixed $p, t_0 > 0$ there exists $C = C(p, t_0)$ such that for all $t > t_0$, $\frac{1}{C}e^{\frac{1}{3}p^3 t} \leq \mathcal{A}'_p(t) \leq Ce^{\frac{1}{3}p^3 t}$.*

Proposition 4.3.4. *For fixed $p, t_0 > 0$, there exists $C = C(p, t_0)$ such that for all $t > t_0$, $|\mathcal{A}''_p(t)| \leq C$.*

Let us first complete our proof of Proposition 4.2.8 using Proposition 4.3.3 and 4.3.4. Recall $\mathcal{A}_p(t) = \mathcal{A}'_p(t) - \mathcal{A}''_p(t)$. With the help of these lemmas, it is clear that $\mathcal{A}'_p(t)$ is the dominating term for large enough t. Hence,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathcal{A}_p(t) = \lim_{t\to\infty} t^{-1} \log \mathcal{A}'_p(t) = \frac{p^3}{3}.$$

This completes our proof of Proposition 4.2.8. □

For the rest of this section, we prove Proposition 4.3.3 and 4.3.4 respectively.

Proof of Proposition 4.3.3. Applying Lemma 4.3.1 to the second integral on the right hand side of (4.3.6) (with $\beta = \frac{2n+1}{2}$ and $\gamma = 4\exp(t^{\frac{1}{3}}x)$), we see that (recall $p = n - 1 + \alpha$)

$$\mathcal{A}'_p(t) = c_p \int_{-\infty}^{\infty} K_{12}(x,x) \exp\left(pt^{\frac{1}{3}}x\right) dx.$$

where c_p is a constant that equals $\frac{2^n(2n-1)!!\,4^{\alpha-1}}{\Gamma(1-\alpha)} \mathsf{B}\left(1-\alpha, \frac{2n-1}{2}+\alpha\right)$. We will not use this explicit expression of c_p and later we just write it as a generic constant C. By a change of variable $x \to t^{\frac{2}{3}}x$, we have $\mathcal{A}'_p(t) = Ct^{\frac{2}{3}} \int_{-\infty}^{\infty} K_{12}(t^{2/3}x, t^{2/3}x) \exp(ptx) dx$. Decompose the integral region into $(-\infty, 0) \cup [0, \infty)$, we obtain

$$\mathcal{A}'_p(t) = Ct^{\frac{2}{3}} \left(\int_0^{\infty} K_{12}(t^{\frac{2}{3}}x, t^{\frac{2}{3}}x) e^{ptx} dx + \int_{-\infty}^{0} K_{12}(t^{\frac{2}{3}}x, t^{\frac{2}{3}}x) e^{ptx} dx \right) \tag{4.3.8}$$

For the first integral on the right hand side of the (4.3.8), referring to Lemma 4.3.2, we have

$$\frac{1}{C_1} t^{-\frac{2}{3}} e^{\frac{p^3 t}{3}} \le \int_0^{\infty} K_{12}(t^{\frac{2}{3}}x, t^{\frac{2}{3}}x) e^{ptx} dx \le C_1 t^{-\frac{2}{3}} e^{\frac{p^3 t}{3}}. \tag{4.3.9}$$

For the second integral on the right hand side of (4.3.8), we apply Lemma B.0.1 (ii) and get $t > t_0$

$$0 \le \int_{-\infty}^{0} K_{12}(t^{\frac{2}{3}}x, t^{\frac{2}{3}}x) e^{ptx} dx \le C_2 \int_{-\infty}^{0} \left(1 - t^{\frac{2}{3}}x\right)^{\frac{1}{2}} e^{ptx} dx \le C_3. \tag{4.3.10}$$

where C_1, C_2, C_3 only depends on p, t_0. Combining (4.3.8), (4.3.9) and (4.3.10), we know that $CC_1^{-1} e^{\frac{p^3 t}{3}} \le \mathcal{A}'_p(t) \le CC_1 e^{\frac{p^3 t}{3}} + CC_3 t^{\frac{2}{3}}$. Note that $t^{\frac{2}{3}}$ can be upper bounded by a constant times $e^{\frac{p^3}{3}t}$ when $t > t_0$, we conclude Proposition 4.3.3. ◻

Proof of Proposition 4.3.4. Recall the expression of $\mathcal{A}''_p(t)$ from (4.3.7). Since $K_{12}(x,x)$ is non-negative for all x, $\mathcal{A}''_p(t)$ is lower bounded by 0. To get the upper bound, we decompose $\mathcal{A}''_p(t) = \frac{2^n(2n-1)!!}{\Gamma(1-\alpha)}(\mathsf{A}_1 + \mathsf{A}_2)$ where

$$\mathsf{A}_1 = \int_0^{\infty} K_{12}(x,x) \exp\left(nt^{\frac{1}{3}}x\right) dx \int_1^{\infty} \frac{s^{-\alpha}}{\left(1 + 4s\exp(t^{\frac{1}{3}}x)\right)^{\frac{2n+1}{2}}} ds, \tag{4.3.11}$$

$$\mathsf{A}_2 = \int_{-\infty}^{0} K_{12}(x,x) \exp\left(nt^{\frac{1}{3}}x\right) dx \int_{1}^{\infty} \frac{s^{-\alpha}}{\left(1+4s\exp(t^{\frac{1}{3}}x)\right)^{\frac{2n+1}{2}}} ds.$$

Let us upper bound A_1 and A_2 respectively. We start with A_1, using $1+4s\exp\left(t^{\frac{1}{3}}x\right) \geq 4s\exp\left(t^{\frac{1}{3}}x\right)$,

$$\int_{1}^{\infty} \frac{s^{-\alpha}}{\left(1+4s\exp(t^{\frac{1}{3}}x)\right)^{\frac{2n+1}{2}}} \leq \int_{1}^{\infty} s^{-\alpha}\left(4s\exp\left(t^{\frac{1}{3}}x\right)\right)^{-\frac{2n+1}{2}} ds = \frac{\exp(-\frac{2n+1}{2}t^{\frac{1}{3}}x)}{2^{2n+1}\left(\frac{2n-1}{2}+\alpha\right)}.$$

Applying this inequality to the right hand side of (4.3.11), we have $\mathsf{A}_1 \leq C\int_0^\infty K_{12}(x,x)\exp(-\frac{1}{2}t^{\frac{1}{3}}x)dx$. Using Lemma B.0.1 (i), for all $t > 0$, there exists a constant C_1 such that

$$\mathsf{A}_1 \leq C\int_0^\infty \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{(1+x)^{\frac{1}{4}}} e^{-\frac{1}{2}t^{\frac{1}{3}}x} dx \leq C\int_0^\infty \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{(1+x)^{\frac{1}{4}}} dx = C_1.$$

We continue to upper bound A_2. Relaxing the integral region from $[1,\infty)$ to $[0,\infty)$, we get

$$\int_{1}^{\infty} \frac{s^{-\alpha}}{\left(1+4s\exp(t^{\frac{1}{3}}x)\right)^{\frac{2n+1}{2}}} \leq \int_{0}^{\infty} \frac{s^{-\alpha}}{\left(1+4s\exp(t^{\frac{1}{3}}x)\right)^{\frac{2n+1}{2}}} ds = 4^{\alpha-1}\mathsf{B}\Big(1-\alpha, \frac{2n-1+2\alpha}{2}\Big) e^{(\alpha-1)t^{\frac{1}{3}}x}.$$

The equality above follows from a change of variable $s \to \frac{1}{4}\exp(-t^{\frac{1}{3}}x)s$ and Lemma 4.3.1. Due to the above display (set the product of $4^{\alpha-1}$ and the beta function to be a constant C)

$$\mathsf{A}_2 \leq C\int_{-\infty}^{0} K_{12}(x,x)\exp\left((n+\alpha-1)t^{\frac{1}{3}}x\right)dx = C\int_{-\infty}^{0} K_{12}(x,x)\exp\left(pt^{\frac{1}{3}}x\right)dx.$$

Using Lemma B.0.1 to upper bound $K_{12}(x,x)$ for negative x, there exists a constant C_2 such that for all $t > t_0$,

$$\mathsf{A}_2 \leq C\int_{-\infty}^{0} \sqrt{1-x}\, e^{pt^{\frac{1}{3}}x} dx = C\int_{-\infty}^{0} \sqrt{1-x}\, e^{pt_0^{\frac{1}{3}}x} dx = C_2.$$

Having $\mathsf{A}_1, \mathsf{A}_2$ upper bounded by a constant uniformly for $t > t_0$, we conclude our lemma by recalling that $\mathcal{A}_p''(t)$ is a constant multiple of $\mathsf{A}_1 + \mathsf{A}_2$. □

4.4 Controlling the Pfaffian and Proof of Lemma 4.2.7

In this section, we give two upper bounds of the L-th Pfaffian $\mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L}$ on the right hand side of (4.2.12) uniformly for all $L \in \mathbb{Z}_{\geq 1}$. This is the main technical contribution of our paper. The purpose is two folded. First, these upper bounds are the crucial inputs to the proof of Proposition 4.2.9 presented in Section 4.5. Secondly, they can be used to validate the interchange of derivative, integration and summation in Lemma 4.2.7.

4.4.1 Controlling the Pfaffian

We obtain two upper bounds for $\mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L}$ for all $L \in Z_{\geq 1}$ and $x_1, \dots, x_L$. Each upper bound has its advantage. The first upper bound has a slower growth in L and a slower exponential decay in x_i as $x_i \to \infty$. The second upper bound has faster exponential decay in x_i but also a more rapid growth in L. For the proof of Proposition 4.2.9, we will use both of the upper bounds to control various terms of $\mathcal{B}_{p,L}$ depending on how large our L is. To prove these bounds, we utilize upper and lower bounds for K_{ij}, $i, j \in \{1, 2\}$ that are established in Lemma B.0.2.

Define $F_{\alpha,\beta}(x) = e^{-\alpha x^{\frac{3}{2}}} \mathbf{1}_{\{x \geq 0\}} + (1-x)^{\beta} \mathbf{1}_{\{x<0\}}$. It is clear that $F_{\alpha_1,\beta_1}(x) F_{\alpha_2,\beta_2}(x) = F_{\alpha_1+\alpha_2,\beta_1+\beta_2}(x)$. In addition, for $\beta_1 \leq \beta_2$, we have $F_{\alpha,\beta_1}(x) \leq F_{\alpha,\beta_2}(x)$.

Proposition 4.4.1. *There exists constant C such that for all $L \in \mathbb{Z}_{\geq 1}$ and $(x_1, \dots, x_L) \in \mathbb{R}^L$,*

(i) $\left|\mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L}\right| \leq (2L)^{L/2} C^L \prod_{i=1}^{L} F_{\frac{1}{3},2}(x_i)$

(ii) $\left|\mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L}\right| \leq \sqrt{(2L)!}\, C^L \prod_{i=1}^{L} F_{\frac{2}{3},2}(x_i)$

Proof of Proposition 4.4.1 (i). The idea for proving Proposition 4.4.1 (i) is as follows. Up to a sign, the Pfaffian of a matrix equals the square root of its determinant. We apply Hadamard's inequality to upper bound the determinant in terms of the product of ℓ_∞-norms of each row of the matrix. Finally, we apply Lemma B.0.2 to control these ℓ_∞-norms.

Now we start our proof. It is well-known that

$$\left|\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\right| = \sqrt{\det\big[K(x_i,x_j)\big]_{i,j=1}^{L}}. \tag{4.4.1}$$

Notice that each entry $K(x_i,x_j)$ is a 2×2 matrix, so $\big[K(x_i,x_j)\big]_{i,j=1}^{L}$ is a $2L\times 2L$ matrix. Denote r_i to be the i-th row vector of this matrix, $i = 1,\dots 2L$. Applying Hadamard's inequality to the determinant on the right hand side above, we see that

$$\left|\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\right| \le (2L)^{\frac{L}{2}}\sqrt{\prod_{i=1}^{2L}\|\mathsf{r}_i\|_\infty}, \tag{4.4.2}$$

where $\|\cdot\|_\infty$ denotes the ℓ^∞-norm of a vector. It suffices to upper bound each $\|\mathsf{r}_i\|_\infty$. We do that according to whether i is odd or even. For each $k = 1,\dots,L$, the vector r_{2k-1} is composed of the elements $K_{11}(x_k,x_j)$ and $K_{12}(x_k,x_j)$, $j = 1,\dots,L$. Thus,

$$\|\mathsf{r}_{2k-1}\|_\infty = \max_{j=1,\dots,L}\Big(\max\big(|K_{11}(x_k,x_j)|,|K_{12}(x_k,x_j)|\big)\Big) \tag{4.4.3}$$

Using Lemma B.0.2 (a) and (b) for K_{11} and K_{12} respectively, there exists constant C such that $|K_{11}(x_k,x_j)| \le CF_{\frac{2}{3},\frac{5}{4}}(x_k)$ and $|K_{12}(x_k,x_j)| \le CF_{\frac{2}{3},\frac{3}{4}}(x_k)$. Since $F_{\frac{2}{3},\frac{3}{4}}(x_k) \le F_{\frac{2}{3},\frac{5}{4}}(x_k)$, referring to (4.4.3) implies $\|\mathsf{r}_{2k-1}\|_\infty \le CF_{\frac{2}{3},\frac{5}{4}}(x_k)$. Similarly, the row vector r_{2k} is composed of $K_{21}(x_k,x_j)$ and $K_{22}(x_k,x_j)$, using Lemma B.0.2 (b) and (c) for K_{12} and K_{22} respectively (note that $|K_{21}(x_k,x_j)| = |K_{12}(x_j,x_k)| \le CF_{0,\frac{3}{4}}(x_k)$), we get

$$\|\mathsf{r}_{2k}\|_\infty = \max_{j=1,\dots,L}\Big(\max\big(|K_{21}(x_k,x_j)|,|K_{22}(x_k,x_j)|\big)\Big) \le CF_{0,\frac{3}{4}}(x_k).$$

Inserting the upper bounds for $\|r_{2k-1}\|$ and $\|r_{2k}\|$ into the right hand side of (4.4.2), we have

$$\left|\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\right| \le (2L)^{\frac{L}{2}}C^L\sqrt{\prod_{k=1}^{L}F_{\frac{2}{3},\frac{5}{4}}(x_k)}\cdot\sqrt{\prod_{k=1}^{L}F_{0,\frac{3}{4}}(x_k)} \le (2L)^{\frac{L}{2}}C^L\prod_{k=1}^{L}F_{\frac{1}{3},2}(x_k).$$

The last equality follows from $\sqrt{F_{\frac{2}{3},\frac{5}{4}}(x_k)F_{0,\frac{3}{4}}(x_k)} = F_{\frac{1}{3},1}(x_k) \leq F_{\frac{1}{3},2}(x_k)$. This completes our proof. □

To prove Proposition 4.4.1 (ii), we upper bound the determinant on the right hand side of (4.4.1) in a different way. Instead of using Hadamard's inequality, we work with the permutation expansion of the determinant and seek to upper bound each term therein. We introduce some notations. Rewrite the $2L \times 2L$ matrix $\left[K(x_i, x_j)\right]_{i,j=1}^{L}$ as $\left[\mathsf{D}(i,j)\right]_{i,j=1}^{2L}$ in a way that for all $i, j \in \{1, \dots, L\}$,

$$\mathsf{D}(2i-1, 2j-1) = K_{11}(x_i, x_j), \qquad \mathsf{D}(2i-1, 2j) = K_{12}(x_i, x_j),$$
$$\mathsf{D}(2i, 2j-1) = K_{21}(x_i, x_j), \qquad \mathsf{D}(2i, 2j) = K_{22}(x_i, x_j).$$

Note that we suppress the dependence on $x_1, \dots, x_L$ in the notation of D. Define maps $\theta : \{1, \dots, 2L\} \to \{1, \dots, L\}$ and $\tau : \{1, \dots, 2L\} \to \{0, 1\}$ such that $\theta(n) = \lfloor \frac{n}{2} \rfloor$ and $\tau(n) = n - 2\lfloor n/2 \rfloor$. It is clear that

$$(\theta, \tau) : \{1, \dots, 2L\} \to \{1, \dots, L\} \times \{0, 1\}$$

is a bijection.

Lemma 4.4.2. *There exists a constant C such that for arbitrary $L \in \mathbb{Z}_{\geq 1}$ and $i, j \in \{1, \dots, 2L\}$,*

$$|\mathsf{D}(i,j)| \leq C F_{\frac{2}{3}\tau(i),\frac{3}{4}}(x_{\theta(i)}) F_{\frac{2}{3}\tau(j),\frac{3}{4}}(x_{\theta(j)}).$$

Proof. We divide our proof of the above inequality into four cases. *Case* 1: i, j are both odd. *Case* 2: i, j are both even. *Case* 3: i is odd and j is even. *Case* 4: i is even and j is odd.

Case 1. i, j are both odd. Then $\tau(i) = \tau(j) = 1$ and $\mathsf{D}(i,j) = K_{11}(x_{\theta(i)}, x_{\theta(j)})$. By Lemma B.0.2 (a),

$$|\mathsf{D}(i,j)| \leq C F_{\frac{2}{3},\frac{3}{4}}(x_{\theta(i)}) F_{\frac{2}{3},\frac{3}{4}}(x_{\theta(j)}) = C F_{\frac{2}{3}\tau(i),\frac{3}{4}}(x_{\theta(i)}) F_{\frac{2}{3}\tau(j),\frac{3}{4}}(x_{\theta(j)})$$

Case 2. i, j are both even. Then $\tau(i) = \tau(j) = 0$ and $\mathsf{D}(i, j) = K_{22}(x_{\theta(i)}, x_{\theta(j)})$. By Lemma B.0.2 (c),

$$|\mathsf{D}(i, j)| \leq CF_{0,\frac{3}{4}}(x_{\theta(i)}) \leq CF_{0,\frac{3}{4}}(x_{\theta(i)})F_{0,\frac{3}{4}}(x_{\theta(j)}) = CF_{\frac{2}{3}\tau(i),\frac{3}{4}}(x_{\theta(i)})F_{\frac{2}{3}\tau(j),\frac{3}{4}}(x_{\theta(j)}).$$

where the second inequality above follows from $F_{0,\frac{3}{4}}(x) = 1 + (1-x)^{\frac{3}{4}}\mathbf{1}_{\{x\leq 0\}} \geq 1$.

Case 3. i is odd and j is even. Then $\tau(i) = 1$, $\tau(j) = 0$ and $\mathsf{D}(i, j) = K_{12}(x_{\theta(i)}, x_{\theta(j)})$. Using Lemma B.0.2 (b),

$$|\mathsf{D}(i, j)| \leq CF_{\frac{2}{3},\frac{3}{4}}(x_{\theta(i)}) \leq CF_{\frac{2}{3},\frac{3}{4}}(x_{\theta(i)})F_{0,\frac{3}{4}}(x_{\theta(j)}) = CF_{\frac{2}{3}\tau(i),\frac{3}{4}}(x_{\theta(i)})F_{\frac{2}{3}\tau(j),\frac{3}{4}}(x_{\theta(j)}).$$

Case 4. i is even and j is odd. Then $\tau(i) = 0$, $\tau(j) = 1$ and $\mathsf{D}(i, j) = K_{21}(x_{\theta(i)}, x_{\theta(j)})$. The desired inequality follows from **Case 3** and the fact $K_{21}(x, y) = -K_{12}(y, x)$.

Since our discussion has covered all the cases, we conclude the proof of our lemma. □

Proof of Proposition 4.4.1 (ii). Referring to (4.4.1) and permutation expansion of the determinant,

$$\left(\mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L}\right)^2 = \det\big[\mathsf{D}(x_i, x_j)\big]_{i,j=1}^{2L} = \sum_{\sigma\in S_{2L}}\prod_{i=1}^{L}\mathsf{D}(i, \sigma(i)) \tag{4.4.4}$$

where S_{2L} is the permutation group of $\{1, \ldots, 2L\}$. Applying Lemma 4.4.2, there exists a constant C such that for any permutation $\sigma \in S_{2L}$,

$$\Big|\prod_{i=1}^{2L}\mathsf{D}\big(i, \sigma(i)\big)\Big| \leq C^{2L}\prod_{i=1}^{2L}\big(F_{\frac{2}{3}\tau(i),\frac{3}{4}}(x_{\theta(i)})F_{\frac{2}{3}\tau(\sigma(i)),\frac{3}{4}}(x_{\theta(\sigma(i))})\big) = C^{2L}\Big(\prod_{i=1}^{2L}F_{\frac{2}{3}\tau(i),\frac{3}{4}}(x_{\theta(i)})\Big)^2$$

The first equality above is due to $\{\sigma(1), \ldots, \sigma(2L)\} = \{1, \ldots, 2L\}$. Using the bijectivity of $(\tau, \theta) : \{1, \ldots, 2L\} \to \{1, \ldots, L\} \times \{0, 1\}$, we see that

$$\prod_{i=1}^{2L}F_{\frac{2}{3}\tau(i),\frac{3}{4}}(x_{\theta(i)}) = \prod_{i=1}^{L}\big(F_{\frac{2}{3},\frac{3}{4}}(x_i)F_{0,\frac{3}{4}}(x_i)\big) = \prod_{i=1}^{L}F_{\frac{2}{3},\frac{3}{2}}(x_i).$$

This implies that for all permutation $\sigma \in S_{2L}$, the absolute value of $\prod_{i=1}^{2L} \mathsf{D}\big(i, \sigma(i)\big)$ can be upper bounded by $C^{2L}\big(\prod_{i=1}^{L} F_{\frac{2}{3},\frac{3}{2}}(x_i)\big)^2$. Referring to (4.4.4), since there are $(2L)!$ terms of $\prod_{i=1}^{2L} \mathsf{D}\big(i, \sigma(i)\big)$ in the summation on the right hand side,

$$\Big(\mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L}\Big)^2 \leq (2L)! \max_{\sigma \in S_{2L}} \Big(\prod_{i=1}^{2L} \mathsf{D}(i, \sigma(i))\Big) \leq (2L)! C^{2L}\Big(\prod_{i=1}^{L} F_{\frac{2}{3},\frac{3}{2}}(x_i)\Big)^2$$

Taking the square root for both sides above and using $F_{\frac{2}{3},\frac{3}{2}}(x_i) \leq F_{\frac{2}{3},2}(x_i)$, this completes the proof. □

4.4.2 Proof of Lemma 4.2.7

In this subsection, we devote to justify that we can interchange the order of derivative, integration and summation for the right hand side (4.2.8) and provide a proof of Lemma 4.2.7. We will only rely on Proposition 4.4.1 (i). For the ensuing discussion, we denote the k-th partial derivative $\partial_s^k \phi_{s,t}(x)$ by $\phi_{s,t}^{(k)}(x)$. In particular, when $k = 0$, $\phi_{s,t}^{(k)}(x)$ coincides with $\phi_{s,t}(x)$.

Lemma 4.4.3. *Fix $k \in \mathbb{Z}_{\geq 0}$. Recall from* (4.2.5) *that $\phi_{s,t}(x) = \frac{1}{\sqrt{1+4s\exp(t^{1/3}x)}} - 1$, there exists $C = C(k)$ such that for all $s \geq 0$,*

$$|\phi_{s,t}^{(k)}(x)| \leq 1 \wedge 2se^{t^{\frac{1}{3}}x} \quad \text{if } k = 0; \qquad |\phi_{s,t}^{(k)}(x)| \leq C\big(e^{kt^{\frac{1}{3}}x} \wedge s^{-k}\big) \quad \text{if } k \in \mathbb{Z}_{\geq 1}, \tag{4.4.5}$$

Proof. It is easy to verify that for all $y \geq 0$, $0 \leq 1 - \frac{1}{\sqrt{1+y}} \leq 1 \wedge \frac{1}{2}y$. Taking $y = 4s\exp(t^{\frac{1}{3}}x)$ implies the first inequality in (4.4.5). For the second inequality, when $k \in \mathbb{Z}_{\geq 1}$, we compute $\phi_{s,t}^{(k)}(x) = (-2)^k(2k-1)!!\frac{\exp(kt^{\frac{1}{3}}x)}{(1+4s\exp(t^{\frac{1}{3}}x))^{\frac{2k+1}{2}}}$. Lower bounding $(1+4s\exp(t^{\frac{1}{3}}x))^{\frac{2k+1}{2}}$ in the denominator by 1, we get $|\phi_{s,t}^{(k)}(x)| \leq 2^k(2k-1)!!\exp(kt^{\frac{1}{3}}x)$. On the other hand, we have

$$|\phi_{s,t}^{(k)}(x)| \leq 2^k(2k-1)!!\frac{\exp(kt^{\frac{1}{3}}x)}{\big(1+4s\exp(t^{\frac{1}{3}}x)\big)^k} \leq s^{-k}2^k(2k-1)!!,$$

Combining these two upper bounds completes our proof. □

Let us introduce a few notations. Define $\mathfrak{M}(L, n) := \big\{\vec{m} = (m_1, \ldots, m_L) \in \mathbb{Z}_{\geq 0}^L, \sum_{i=1}^{L} m_i = n\big\}$

and for $\vec{m} \in \mathfrak{M}(L, n)$, we set $\binom{n}{\vec{m}} := \frac{n!}{\prod_{i=1}^{L} m_i!}$.

Lemma 4.4.4. *Fix $n \in \mathbb{Z}_{\geq 0}$ and $t > 0$, there exists a constant $C = C(n, t)$ such that for all $L \in \mathbb{Z}_{\geq 1}$ and $s \in [0, 1]$,*

$$\Big| \int_{\mathbb{R}^L} \mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L} \partial_s^n \Big(\prod_{i=1}^{L} \phi_{s,t}(x_i) \Big) dx_1 \dots dx_L \Big| \leq (2L)^{\frac{L}{2}} C^L.$$

Proof. By Leibniz's rule,

$$\partial_s^n \Big(\prod_{i=1}^{L} \phi_{s,t}(x_i) \Big) = \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \prod_{i=1}^{L} \phi_{s,t}^{(m_i)}(x_i). \tag{4.4.6}$$

According to Lemma 4.4.3, there exists constant $C_1 = C_1(n) > 2$ such that for each $0 \leq m_i \leq n$ and $s \in [0, 1]$

$$|\phi_{s,t}^{(m_i)}(x_i)| \leq 2se^{t^{\frac{1}{3}}x} \leq 2e^{t^{\frac{1}{3}}x} \quad \text{if } m_i = 0; \qquad |\phi_{s,t}^{(m_i)}(x_i)| \leq C_1 e^{m_i t^{\frac{1}{3}}x} \quad \text{if } 1 \leq m_i \leq n.$$

Hence, we have $|\phi_{s,t}^{(m_i)}(x_i)| \leq C_1 \big(\exp(t^{\frac{1}{3}}x) \vee \exp(t^{\frac{1}{3}}nx) \big)$. Taking the absolute value for both sides of (4.4.6) and applying triangle inequality,

$$\Big| \partial_s^n \Big(\prod_{i=1}^{L} \phi_{s,t}(x_i) \Big) \Big| \leq C_1^L \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \prod_{i=1}^{L} \Big(e^{t^{\frac{1}{3}}x_i} \vee e^{nt^{\frac{1}{3}}x_i} \Big) = C_1^L \prod_{i=1}^{L} \Big(e^{t^{\frac{1}{3}}x_i} \vee e^{nt^{\frac{1}{3}}x_i} \Big) \Big(\sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \Big) \tag{4.4.7}$$

It suffices to show that there exists a constant $C_2 = C_2(n)$ such that for all $L \in \mathbb{Z}_{\geq 1}$

$$\sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \leq C_2^L. \tag{4.4.8}$$

Once this is shown, by (4.4.7) we see that $\big| \partial_s^n \big(\prod_{i=1}^{L} \phi_{s,t}(x_i) \big) \big|$ is upper bounded by $(C_1 C_2)^L \prod_{i=1}^{L} \big(e^{t^{\frac{1}{3}}x_i} \vee$

$e^{nt^{\frac{1}{3}}x_i})$. Applying Proposition 4.4.1 (i), there exists a constant C such that

$$\Big|\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\partial_s^n\Big(\prod_{i=1}^{L}\phi_{s,t}(x_i)\Big)\Big| \le (2L)^{\frac{L}{2}}C^L\prod_{i=1}^{L}\Big(F_{\frac{1}{3},2}(x_i)\big(e^{t^{\frac{1}{3}}x_i}\vee e^{nt^{\frac{1}{3}}x_i}\big)\Big) \tag{4.4.9}$$

Integrating both sides on $\mathbb{R}^L$ implies that

$$\Big|\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\partial_s^n\Big(\prod_{i=1}^{L}\phi_{s,t}(x_i)\Big)dx_1\dots dx_L\Big| \le (2L)^{\frac{L}{2}}C^L\Big(\int_{\mathbb{R}}F_{\frac{1}{3},2}(x)\big(e^{t^{\frac{1}{3}}x}\vee e^{nt^{\frac{1}{3}}x}\big)dx\Big)^L.$$

Since t is fixed, the integrand on the right hand side above decays super-exponentially as $x\to+\infty$ and exponentially as $x\to-\infty$, hence is integrable. The value of the integration above is a constant that only depends on n,t. This completes our proof of the lemma.

It remains to prove (4.4.8). Let $\#A$ be the number of elements in A. Note that

$$\#\mathfrak{M}(L,n)=\#\{\vec{m}=(m_1,\dots,m_L)\in\mathbb{Z}_{\ge0}^L,\sum_{i=1}^{L}m_i=n\}\le L\#\mathfrak{M}(L,n-1). \tag{4.4.10}$$

Iterating this inequality yields $\#\mathfrak{M}(L,n)\le L^n$. In addition, for each $\vec{m}\in\mathfrak{M}(L,n)$, $\binom{n}{\vec{m}}$ is upper bounded by $n!$. We can find a large constant $C_2=C_2(n)$ such that for all $L\ge1$,

$$\sum_{\vec{m}\in\mathfrak{M}(L,n)}\binom{n}{\vec{m}}\le n!\#\mathfrak{M}(L,n)\le n!L^n\le C_2^L.$$ □

The next two propositions validate that we can interchange the order of derivative, summation and integral.

Proposition 4.4.5. *For every fixed $n,L\in\mathbb{Z}_{\ge1}$, $s\in[0,1]$ and $t>0$, we have*

$$\partial_s^n\Big(\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}(x_i)dx_i\Big)=\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\partial_s^n\Big(\prod_{i=1}^{L}\phi_{s,t}(x_i)\Big)dx_1\dots dx_L$$

Proof. The proof follows from the dominated convergence theorem. It suffices to show that we can find an integrable function $G(x_1,\dots,x_L)$ such that for all $s \in [0,1]$,

$$\Big|\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\partial_s^n\Big(\prod_{i=1}^{L}\phi_{s,t}(x_i)\Big)\Big| \le G(x_1,\dots,x_L) \tag{4.4.11}$$

By (4.4.9), we can take

$$G(x_1,\dots,x_L) = (2L)^{\frac{L}{2}}C^L\prod_{i=1}^{L}F_{\frac{1}{3},2}(x_i)\big(e^{t^{\frac{1}{3}}x_i}\vee e^{nt^{\frac{1}{3}}x_i}\big)$$

Since G is integrable and satisfy (4.4.11), this completes our proof. □

Proposition 4.4.6. *For fixed $n \in \mathbb{Z}_{\ge 1}, t > 0$ and $s \in [0,1]$ we have the following interchange of differentiation and summation holds*

$$\partial_s^n\Big(\sum_{L=1}^{\infty}\frac{1}{L!}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}(x_i)dx_i\Big) = \sum_{L=1}^{\infty}\frac{1}{L!}\partial_s^n\Big(\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}(x_i)dx_i\Big).$$

Proof. A sufficient condition for the interchange of the order of derivative and infinite summation is that (see [DT19, Proposition 4.2]),

(i) $\sum_{L=1}^{\infty}\frac{1}{L!}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}(x_i)dx_i$ converges pointwisely for $s \in [0,1]$.

(ii) For all $n \in \mathbb{Z}_{\ge 1}$, $\sum_{L=1}^{\infty}\frac{1}{L!}\partial_s^n\big(\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}(x_i)dx_i\big)$ converges uniformly for $s \in [0,1]$.

Applying Proposition 4.4.5, for (ii), we can place ∂_s^n inside the integral. Then both (i) and (ii) follow from Lemma 4.4.4 and the convergence of $\sum_{L=1}^{\infty}\frac{C^L(2L)^{\frac{L}{2}}}{L!}$. □

Proof of Lemma 4.2.7. It is enough to show (4.2.9). Combining Proposition 4.4.5 and 4.4.6, we know that

$$\begin{aligned}&\int_0^1 s^{-\alpha}\partial_s^n\Big(\sum_{L=1}^{\infty}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}(x_i)dx_i\Big)ds\\&=\int_0^1 s^{-\alpha}\Big(\sum_{L=1}^{\infty}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\partial_s^n\Big(\prod_{i=1}^{L}\phi_{s,t}(x_i)\Big)dx_1\dots dx_L\Big)ds\end{aligned}$$

Thus, what remains to prove is the interchange of integral and summation:

$$\int_0^1 s^{-\alpha}\Big(\sum_{L=1}^{\infty}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\partial_s^n\Big(\prod_{i=1}^{L}\phi_{s,t}(x_i)\Big)dx_1\dots dx_L\Big)ds$$
$$=\sum_{L=1}^{\infty}\int_0^1 s^{-\alpha}\Big(\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\partial_s^n\Big(\prod_{i=1}^{L}\phi_{s,t}(x_i)\Big)dx_1\dots dx_L\Big)ds$$

This can be justified via the dominated convergence theorem, using again Lemma 4.4.4 and the convergence of $\sum_{L=1}^{\infty}\frac{C^L(2L)^{\frac{L}{2}}}{L!}$. $\square$

4.5 Proof of Proposition 4.2.9

In this section, we prove Proposition 4.2.9. The main inputs will be Proposition 4.4.1 and Proposition 4.5.2 that we will show in a moment.

Define $V_n(x)=nx-\frac{1}{3}x^{\frac{3}{2}}$ and $U_n(x)=nx-\frac{2}{3}x^{3/2}$. By straightforward calculus, $V_n(\sigma\wedge 4n^2)$ (resp. $U_n(\sigma\wedge n^2)$) is the maximum of V_n (resp. U_n) over $x\in[0,\sigma]$, with $\sigma\wedge 4n^2$ (resp. $\sigma\wedge n^2$) to be the unique maximizer. Recall from the beginning of Section 4.4 that $F_{\alpha,\beta}(x)=e^{-\alpha x^{\frac{3}{2}}}\mathbf{1}_{\{x\geq 0\}}+(1-x)^{\beta}\mathbf{1}_{\{x<0\}}$.

Lemma 4.5.1. *Fix $n\in\mathbb{Z}_{\geq 1}$ and $t_0>0$, there exists $C=C(n,t_0)$ such that for all $\sigma\geq 0$ and $t>t_0$,*

(i) $\int_{-\infty}^{\sigma}\exp(tnx)F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx\leq Ct^{-\frac{1}{2}}\exp(tV_n(\sigma\wedge 4n^2))$

(ii) $\int_{-\infty}^{\sigma}\exp(tnx)F_{\frac{2}{3},2}(t^{\frac{2}{3}}x)\leq Ct^{-\frac{1}{2}}\exp(tU_n(\sigma\wedge n^2))$

Proof. Let us first demonstrate (i) and (ii) will follow in a similar way. Decompose

$$\int_{-\infty}^{\sigma}e^{tnx}F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx=\int_{-\infty}^{0}e^{tnx}F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx+\int_{0}^{\sigma}e^{tnx}F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx \tag{4.5.1}$$

Since $F_{\alpha,\beta}(x)$ equals $(1-x)^{\beta}$ when x is negative, we can rewrite the first term on the right hand side in the above display into $\int_{-\infty}^{0}\exp(tnx)(1-t^{\frac{2}{3}}x)^2dx$. We have

$$\int_{-\infty}^{0}e^{tnx}(1-t^{\frac{2}{3}}x)^2dx=t^{-1}\int_0^{\infty}e^{-nx}(1+t^{-\frac{1}{3}}x)^2dx\leq t^{-1}\int_0^{\infty}e^{-nx}(1+t_0^{-1/3}x)^2dx\leq Ct^{-1},$$

where the first equality is due to a change of variable $x \to -t^{-1}x$, the second equality follows our condition $t \geq t_0$ and the third is due to $\int_0^\infty e^{-nx}(1+t_0^{-\frac{1}{3}x})^2 dx$ is finite constant only depending on n, t_0. We have shown that the first term on the right hand side of (4.5.1) is upper bounded by Ct^{-1} for some $C = C(n, t_0)$. Since $V_n(\sigma \wedge 4n^2)$ is non-negative, which implies that $t^{-1} \leq Ct^{-\frac{1}{2}} \exp(tV_n(\sigma \wedge 4n^2))$ for $t \geq t_0$. To prove (i), it suffices to prove the second term on the right hand side of (4.5.1) is also upper bounded by $Ct^{-\frac{1}{2}} \exp(tV_n(\sigma \wedge 4n^2))$. Note that when $x \geq 0$, $F_{\frac{1}{3},2}(t^{\frac{2}{3}}x) = \exp(-\frac{1}{3}tx^{\frac{3}{2}})$ which yields $\int_0^\sigma \exp(tnx)F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx = \int_0^\sigma \exp(tV_n(x))dx$. Thus we only need to show that there exists a constant $C = C(n, t_0)$ such that for $t > t_0$

$$\int_0^\sigma e^{tV_n(x)} dx \leq Ct^{-\frac{1}{2}} e^{tV_n(\sigma \wedge 4n^2)}. \tag{4.5.2}$$

To this aim, we split our discussion into the following three cases.

Case 1. $\sigma \in [0, n^2]$. Since $V_n''(x) = -\frac{1}{4\sqrt{x}} < 0$, $V_n(x)$ is concave on $\sigma \in [0, n^2]$. Hence, for all $x \in [0, \sigma]$,

$$V_n(x) \leq V_n(\sigma) + V_n'(\sigma)(x-\sigma) = V_n(\sigma) + (n - \frac{1}{2}\sqrt{\sigma})(x-\sigma) \leq V_n(\sigma) + \frac{n}{2}(x-\sigma).$$

The last inequality above follows since $x \leq \sigma$ and $n - \frac{1}{2}\sqrt{\sigma} \geq n/2$. Using the displayed inequality above,

$$\int_0^\sigma e^{tV_n(x)} dx \leq \int_0^\sigma e^{t(V_n(\sigma) + \frac{n}{2}(x-\sigma))} dx = e^{tV_n(\sigma)} \int_0^\sigma e^{\frac{nt}{2}(x-\sigma)} dx \leq \frac{2}{nt} e^{tV_n(\sigma)}.$$

Since $\sigma \in [0, n^2]$, $V_n(\sigma) = V_n(\sigma \wedge 4n^2)$. Moreover, for $t \geq t_0$, $\frac{2}{nt} e^{tV_n(\sigma)} \leq \frac{2}{n\sqrt{t_0}} t^{-\frac{1}{2}} e^{tV_n(\sigma)}$, which implies (4.5.2).

Case 2. $\sigma \in [n^2, 4n^2]$. First, via a change of variable $x = r^2$, $\int_0^\sigma e^{tV_n(x)} dx = \int_0^{\sqrt{\sigma}} 2re^{tV_n(r^2)} dr$. Therefore, we only need to prove (4.5.2) with $\int_0^{\sqrt{\sigma}} 2re^{tV_n(r^2)} dr$ in place of $\int_0^\sigma e^{tV_n(x)} dx$. Since

$r \leq \sqrt{\sigma} \leq 2n$,

$$V_n(r^2) - V_n(\sigma) = n(r^2 - \sigma) - \frac{1}{3}(r^3 - \sigma^{\frac{3}{2}}) \leq \frac{\sqrt{\sigma}}{2}(r^2 - \sigma) - \frac{1}{3}(r^3 - \sigma^{\frac{3}{2}}) = -\frac{1}{3}(r - \sqrt{\sigma})^2(r + \frac{1}{2}\sqrt{\sigma}).$$

Since $\sigma \geq n^2$ and $r \geq 0$, from the above displayed inequality, $V_n(r^2) \leq V_n(\sigma) - \frac{n}{6}(r - \sqrt{\sigma})^2$. Consequently,

$$\int_0^{\sqrt{\sigma}} 2re^{tV_n(r^2)}dr \leq \int_0^{\sqrt{\sigma}} 2re^{tV_n(\sigma)}e^{-\frac{tn}{6}(r-\sqrt{\sigma})^2}dr \leq e^{tV_n(\sigma)}\int_0^{\sqrt{\sigma}} 2re^{-\frac{t}{6}(r-\sqrt{\sigma})^2}dr \leq \frac{C}{\sqrt{t}}e^{tV_n(\sigma)}.$$

This completes the proof of (4.5.2). In the last inequality above, we used $\sqrt{\sigma} \leq 2n$ and $t \geq t_0$, which implies that

$$\int_0^{\sqrt{\sigma}} 2re^{-\frac{t}{6}(r-\sqrt{\sigma})^2}dr \leq \int_{-\infty}^{\infty} 2(r+\sqrt{\sigma})e^{-\frac{t}{6}r^2}dr \leq \int_{-\infty}^{\infty} 2(r+2n)e^{-\frac{t}{6}r^2}dr = C_1t^{-1}+C_2t^{-1/2} \leq Ct^{-1/2}.$$

Case 3. $\sigma > 4n^2$. As illustrated in **Case 2**, we only need to show that for $t > t_0$,

$$\int_0^{\sqrt{\sigma}} 2re^{tV_n(r^2)}dr \leq Ct^{-\frac{1}{2}}e^{tV_n(\sigma\wedge 4n^2)}$$

Note that $V_n(r^2) - \frac{4}{3}n^3 = nr^2 - \frac{1}{3}r^3 - \frac{4}{3}n^3 = -\frac{1}{3}(r - 2n)^2(n + r) \leq -\frac{n}{3}(r - 2n)^2$. This implies

$$\int_0^{\sqrt{\sigma}} 2re^{tV_n(r^2)}dr \leq \int_0^{\sqrt{\sigma}} 2re^{\frac{4}{3}tn^3}e^{-\frac{n}{3}t(r-2n)^2}dr \leq e^{\frac{4}{3}n^3t}\int_0^{\sqrt{\sigma}} 2re^{-\frac{n}{3}t(r-2n)^2}dr \leq \frac{C}{\sqrt{t}}e^{\frac{4}{3}n^3t}.$$

The last inequality is due to a similar argument as in **Case 2** above. Since $\sigma > 4n^2$, we have $V_n(\sigma \wedge 4n^2) = V_n(4n^2) = \frac{4}{3}n^3$, thus we showed (4.5.2). So far, we complete the proof of (i).

The proof of (ii) will be rather similar to (i), instead of showing (4.5.2), one needs to show that there exists $C = C(n, t_0)$ such that for $t \geq t_0$, $\int_0^{\sigma} \exp\big(tU_n(x)\big)dx \leq Ct^{-\frac{1}{2}} \exp\big(tU_n(\sigma \wedge n^2)\big)$. We skip the details. □

Proposition 4.5.2. *For fixed $n \in \mathbb{Z}_{\geq 1}$ and $t_0 > 0$, there exists constant $C = C(n, t_0)$ such that for every $\sigma \geq 0$ and $t > t_0$,*

(a) $\int_{-\infty}^{\infty} |\phi_{e^{-t\sigma},t}(x)| F_{\frac{1}{3},2}(x) dx \leq C t^{\frac{1}{6}} \exp\big(tV_1(\sigma \wedge 4) - t\sigma\big)$

(b) $\int_{-\infty}^{\infty} |\phi^{(n)}_{e^{-t\sigma},t}(x)| F_{\frac{1}{3},2}(x) dx \leq C t^{\frac{1}{6}} \exp\big(tV_n(\sigma \wedge 4n^2)\big)$

(c) $\int_{-\infty}^{\infty} |\phi_{e^{-t\sigma},t}(x)| F_{\frac{2}{3},2}(x) dx \leq C t^{\frac{1}{6}} \exp\big(tU_1(\sigma \wedge 1) - t\sigma\big)$

(d) $\int_{-\infty}^{\infty} |\phi^{(n)}_{e^{-t\sigma},t}(x)| F_{\frac{2}{3},2}(x) dx \leq C t^{\frac{1}{6}} \exp\big(tU_n(\sigma \wedge n^2)\big)$.

Proof. We first prove (a). Via a change of variable $x \to t^{\frac{2}{3}} x$,

$$\int_{-\infty}^{\infty} |\phi_{e^{-t\sigma},t}(x)| F_{\frac{1}{3},2}(x) dx = t^{\frac{2}{3}} \int_{-\infty}^{\infty} |\phi_{e^{-t\sigma},t}(t^{\frac{2}{3}} x)| F_{\frac{1}{3},2}(t^{\frac{2}{3}} x) dx. \tag{4.5.3}$$

We decompose the integral region on the right hand side above into $(-\infty, \sigma) \cup (\sigma, \infty)$ and write

$$\int_{-\infty}^{\infty} |\phi_{e^{-t\sigma},t}(t^{\frac{2}{3}} x)| F_{\frac{1}{3},2}(t^{\frac{2}{3}} x) dx = \Big(\int_{-\infty}^{\sigma} + \int_{\sigma}^{\infty}\Big) |\phi_{e^{-t\sigma},t}(t^{\frac{2}{3}} x)| F_{\frac{1}{3},2}(t^{\frac{2}{3}} x) dx = \mathsf{E}_1 + \mathsf{E}_2$$

So the left hand side of (4.5.3) equals $t^{\frac{2}{3}}(\mathsf{E}_1 + \mathsf{E}_2)$. We provide upper bounds for E_1 and E_2 respectively. By Lemma 4.4.3, $|\phi_{e^{-t\sigma},t}(t^{\frac{2}{3}} x)| \leq 2 \exp\big(t(x - \sigma)\big)$, so

$$\mathsf{E}_1 \leq 2 e^{-t\sigma} \int_{-\infty}^{\sigma} e^{tx} F_{\frac{1}{3},2}(t^{\frac{2}{3}} x) dx \leq C t^{-\frac{1}{2}} e^{tV_1(\sigma \wedge 4) - t\sigma}. \tag{4.5.4}$$

where the last inequality above is due to Lemma 4.5.1 (i) (setting $n = 1$ therein). On the other hand, since $F_{\frac{1}{3},2}(t^{\frac{2}{3}} x) = \exp(-\frac{1}{3} t x^{\frac{3}{2}})$ when $x \geq 0$,

$$\mathsf{E}_2 = \int_{\sigma}^{\infty} |\phi_{e^{-t\sigma},t}(t^{\frac{2}{3}} x)| e^{-\frac{1}{3} t x^{3/2}} dx \leq \int_{\sigma}^{\infty} e^{-\frac{1}{3} t x^{3/2}} dx = t^{-\frac{2}{3}} \int_{t^{\frac{2}{3}}\sigma}^{\infty} e^{-\frac{1}{3} x^{3/2}} dx \leq C t^{-\frac{2}{3}} e^{-\frac{1}{3} t \sigma^{\frac{3}{2}}} \tag{4.5.5}$$

The first inequality above is due to Lemma 4.4.3, which yields $|\phi_{e^{-t\sigma},t}(t^{\frac{2}{3}} x)| \leq 1$. The second inequality follows from a change of variable $x \to t^{-\frac{2}{3}} \sigma$, and the last inequality is due to the fact $\int_y^{\infty} \exp(-\frac{1}{3} x^{\frac{3}{2}}) dx \leq C \exp(-\frac{1}{3} y^{\frac{3}{2}})$, which holds for all $y \geq 0$. Combining (4.5.4) and (4.5.5) and recall that $\int_{-\infty}^{\infty} \phi_{e^{-t\sigma},t}(x) F_{\frac{1}{3},2}(x)dx = t^{\frac{2}{3}}(\mathsf{E}_1 + \mathsf{E}_2)$, we obtain

$$\int_{-\infty}^{\infty} |\phi_{e^{-t\sigma},t}(x)| F_{\frac{1}{3},2}(x)dx \leq Ct^{\frac{2}{3}}\big(t^{-\frac{1}{2}}\exp(tV_1(\sigma\wedge 4) - t\sigma) + t^{-\frac{2}{3}}e^{-\frac{1}{3}t\sigma^{\frac{3}{2}}}\big).$$

Since $V_1(\sigma\wedge 4)$ is the maximum of $V_1(x)$ for $x \in [0,\sigma]$, $V_1(\sigma\wedge 4) - \sigma \geq V_1(\sigma) - \sigma = -\frac{1}{3}\sigma^{\frac{3}{2}}$, so the first term on the right hand side above dominates, this completes the proof of (a).

For the proof of (b), via a change of variable $x \to t^{\frac{2}{3}}x$,

$$\int_{-\infty}^{\infty} |\phi^{(n)}_{e^{-t\sigma},t}(x)| F_{\frac{1}{3},2}(x)dx = t^{\frac{2}{3}} \int_{-\infty}^{\infty} |\phi^{(n)}_{e^{-t\sigma},t}(t^{\frac{2}{3}}x)| F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx$$

Decompose the integral on the right hand side in the above display as

$$\int_{-\infty}^{\infty} |\phi^{(n)}_{e^{-t\sigma},t}(t^{\frac{2}{3}}x)| F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx = \Big(\int_{-\infty}^{\sigma} + \int_{\sigma}^{\infty}\Big) |\phi^{(n)}_{e^{-t\sigma},t}(t^{\frac{2}{3}}x)| F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx = \mathsf{E}'_1 + \mathsf{E}'_2.$$

Let us upper bound E'_1 and E'_2 respectively. By Lemma 4.4.3, we know that $|\phi^{(n)}_{e^{-t\sigma},t}(t^{\frac{2}{3}}x)| \leq C\exp(ntx)$. Using this together with Lemma 4.5.1 (i), we get

$$\mathsf{E}'_1 \leq C\int_{-\infty}^{\sigma} e^{ntx} F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)dx \leq Ct^{-\frac{1}{2}}\exp\big(tV_n(\sigma\wedge 4n^2)\big). \tag{4.5.6}$$

For E'_2, note that $F_{\frac{1}{3},2}(t^{\frac{2}{3}}x)$ simplifies to $\exp(-\frac{1}{3}tx^{\frac{3}{2}})$ for $x \geq 0$. By Lemma 4.4.3, $|\phi^{(n)}_{e^{-t\sigma},t}(t^{\frac{2}{3}}x)| \leq C\exp(nt\sigma)$ (note that $s = e^{-t\sigma}$). Using this inequality implies

$$\mathsf{E}'_2 = \int_{\sigma}^{\infty} \phi^{(n)}_{e^{-t\sigma},t}(t^{\frac{2}{3}}x)e^{-\frac{1}{3}tx^{\frac{3}{2}}}dx \leq Ce^{tn\sigma}\int_{\sigma}^{\infty} e^{-\frac{1}{3}tx^{\frac{3}{2}}}dx \leq Ct^{-\frac{2}{3}}e^{tn\sigma - \frac{1}{3}t\sigma^{\frac{3}{2}}}. \tag{4.5.7}$$

Recall that $\int_{-\infty}^{\infty} |\phi^{(n)}_{e^{-t\sigma},t}(x)| F_{\frac{1}{3},2}(x)dx = t^{\frac{2}{3}}(\mathsf{E}'_1 + \mathsf{E}'_2)$, combining (4.5.6) and (4.5.7) yields

$$\int_{-\infty}^{\infty} |\phi^{(n)}_{e^{-t\sigma},t}(x)| F_{\frac{1}{3},2}(x)dx \leq Ct^{\frac{2}{3}}\Big(t^{-\frac{1}{2}}e^{tV_n(\sigma\wedge 4n^2)} + t^{-\frac{2}{3}}e^{t(n\sigma - \frac{1}{3}\sigma^{\frac{3}{2}})}\Big) \leq Ct^{\frac{1}{6}}\exp\big(tV_n(\sigma\wedge 4n^2)\big).$$

The last inequality above is due to $V_n(\sigma\wedge 4n^2) \geq V_n(\sigma) = n\sigma - \frac{1}{3}\sigma^{\frac{3}{2}}$. This completes the proof of (b).

The proof for (c), (d) follows a rather similar argument as for (a), (b). Instead of using Lemma 4.5.1 (i), one needs to apply Lemma 4.5.1 (ii). We omit the details here. □

Fix $p \in \mathbb{R} > 0$, referring to (4.2.12) and applying Leibniz's rule (4.4.6), we know that

$$\mathcal{B}_{p,L}(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)L!} \sum_{\vec{m} \in \mathfrak{M}(L,n)} \binom{n}{\vec{m}} \mathcal{I}_{\vec{m}}, \tag{4.5.8}$$

where

$$\mathcal{I}_{\vec{m}} = \int_0^1 s^{-\alpha} ds \int_{\mathbb{R}^L} \mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L} \prod_{i=1}^{L} \phi_{s,t}^{(m_i)}(x_i) dx_i. \tag{4.5.9}$$

Note that n and α are determined by p in a way that $n = \lfloor p \rfloor + 1$ and $\alpha = p + 1 - n \in [0, 1)$. We want to upper bound $\mathcal{B}_{p,L}(t)$, it suffices to study $\mathcal{I}_{\vec{m}}$ For every $\vec{m} \in \mathfrak{M}(L, n)$.

Proposition 4.5.3. *Fix $p \in \mathbb{R}_{>0}$ and $t_0 > 0$. Define $\delta_p := \min(\frac{2}{3}, \frac{p^3}{4})$. There exists $C = C(p, t_0)$ such that for all $L \geq 2$, $t \geq t_0$ and $\vec{m} \in \mathfrak{M}(L, n)$, we have $|\mathcal{I}_{\vec{m}}| \leq C^L (2L)^{\frac{L}{2}} t^{\frac{L}{6}} e^{\frac{p^3}{3}t - \delta_p t}$.*

Proof. Without loss of generality, we assume that $m_1, \ldots, m_r > 0$ and $m_{r+1} = \cdots = m_L = 0$ for some integer r satisfying $1 \leq r \leq n \wedge L$. Referring to (4.5.9), by a change of variable $s = e^{-t\sigma}$, we obtain

$$\mathcal{I}_{\vec{m}} = \int_0^{\infty} e^{t(\alpha-1)\sigma} d\sigma \int_{\mathbb{R}^L} \mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L} \prod_{i=1}^{L} \phi_{e^{-t\sigma},t}^{(m_i)}(x_i) dx_i.$$

It suffice to show that the right hand side of the above display is upper bounded by $C^L (2L)^{\frac{L}{2}} t^{\frac{L}{6}} e^{\frac{p^3 t}{3} - \delta_p t}$. We divide our argument into two stages. We prove the inequality for $L \geq 4n^3$ in *Stage 1* and *Stage 2* will cover the case $2 \leq L < 4n^3$.

Stage 1. $L \geq 2n^3$. Via Proposition 4.4.1 (i), $\big|\mathrm{Pf}\big[K(x_i, x_j)\big]_{i,j=1}^{L}\big|$ is upper bounded by $C^L (2L)^{\frac{L}{2}} \prod_{i=1}^{L} F_{\frac{1}{3},2}(x_i)$, thus

$$|\mathcal{I}_{\vec{m}}| \leq C^L (2L)^{\frac{L}{2}} \int_0^{\infty} e^{t(\alpha-1)\sigma} \prod_{i=1}^{L} \Big(\int_{\mathbb{R}} F_{\frac{1}{3},2}(x) |\phi_{e^{-t\sigma},t}^{(m_i)}(x)| dx \Big) d\sigma. \tag{4.5.10}$$

Applying Proposition 4.5.2 (a) and (b). Since $m_i \in \mathbb{Z}_{\geq 1}$ for $1 \leq i \leq r$ and $m_i = 0$ for $i \geq r+1$, there exists a constant $C = C(n, t_0)$ such that for $t > t_0$,

$$\int_{\mathbb{R}} F_{\frac{1}{3},2}(x) |\phi^{(m_i)}_{e^{-t\sigma},t}(x)| dx \leq \begin{cases} Ct^{\frac{1}{6}} \exp(tV_{m_i}(\sigma \wedge 4m_i^2)) & i \leq r \\ Ct^{\frac{1}{6}} \exp(tV_1(\sigma \wedge 4) - t\sigma) & i > r. \end{cases}$$

Applying this inequality to the right hand side of (4.5.10), we find that $|\mathcal{I}_{\vec{m}}| \leq C^L (2L)^{\frac{L}{2}} t^{\frac{L}{6}} \int_0^\infty e^{t\mathsf{M}_1(\sigma)} d\sigma$, where

$$\mathsf{M}_1(\sigma) := (\alpha - 1)\sigma + \sum_{i=1}^{r} V_{m_i}(\sigma \wedge 4m_i^2) + (L - r)(V_1(\sigma \wedge 4) - \sigma). \tag{4.5.11}$$

To prove Proposition 4.5.3, it suffices to show that for there exists $C = C(n, t_0)$, such that for all $t \geq t_0$ and $\vec{m} \in \mathfrak{M}(L, n)$,

$$\int_0^\infty e^{t\mathsf{M}_1(\sigma)} d\sigma \leq Ce^{\frac{p^3}{3}t - \delta_p t}. \tag{4.5.12}$$

where $\delta_p = \min(\frac{2}{3}, \frac{p^3}{4})$. To this aim, we decompose

$$\int_0^\infty e^{t\mathsf{M}_1(\sigma)} d\sigma = \int_0^4 e^{t\mathsf{M}_1(\sigma)} d\sigma + \int_4^\infty e^{t\mathsf{M}_1(\sigma)} d\sigma = \mathcal{J}_1 + \mathcal{J}_2.$$

For $\mathcal{J}_1$, since $\sigma \leq 4$, M_1 simplifies to

$$\begin{aligned} \mathsf{M}_1(\sigma) &= (\alpha - 1)\sigma + \sum_{i=1}^{r} V_{m_i}(\sigma) + (L - r)(V_1(\sigma) - \sigma) \\ &= (\alpha - 1)\sigma + \sum_{i=1}^{r} (m_i \sigma - \frac{1}{3}\sigma^{\frac{3}{2}}) - \frac{1}{3}(L - r)\sigma^{\frac{3}{2}} = p\sigma - \frac{L}{3}\sigma^{\frac{3}{2}}. \end{aligned}$$

The last equality is due to $\sum_{i=1}^{r} m_i + \alpha - 1 = n + \alpha - 1 = p$. Since $L \geq 4n^3 \geq 4$, $\mathsf{M}_1(\sigma) = p\sigma - \frac{L}{3}\sigma^{\frac{3}{2}} \leq p\sigma - \frac{4}{3}\sigma^{\frac{3}{2}} \leq \frac{p^3}{12}$. We find that

$$\mathcal{J}_1 = \int_0^4 e^{t\mathsf{M}_1(\sigma)} d\sigma \leq \int_0^4 e^{\frac{p^3 t}{12}} d\sigma \leq 4e^{\frac{p^3 t}{12}}. \tag{4.5.13}$$

For $\mathcal{J}_2$, referring to (4.5.11), since $\sigma \geq 4$, $V_1(\sigma \wedge 4) = V_1(4) = \frac{4}{3}$. Moreover, the maximum of $V_{m_i}(\sigma) = m_i\sigma - \frac{1}{3}\sigma^{\frac{3}{2}}$ equals $\frac{4}{3}m_i^3$, hence $V_{m_i}(\sigma \wedge 4m_i^2) \leq \frac{4}{3}m_i^3$. As a result,

$$\mathsf{M}_1(\sigma) \leq (\alpha - 1)\sigma + \frac{4}{3}\sum_{i=1}^{r} m_i^3 + (L - r)(\frac{4}{3} - \sigma) \leq (\alpha - 1)\sigma + \frac{4}{3}n^3 - (L - r)\frac{8}{3}.$$

The last inequality follows from the fact that $\sum_{i=1}^{r} m_i^3 \leq (\sum_{i=1}^{r} m_i)^3 = n^3$ and $\frac{4}{3} - \sigma \leq -\frac{8}{3}$. Note that r is the number of m_i which is non-zero, so $r \leq n$. Moreover, since $L \geq 4n^3$,

$$\mathsf{M}_1(\sigma) \leq \frac{4}{3}n^3 - (4n^3 - n)\frac{8}{3} + (\alpha - 1)\sigma \leq (\alpha - 1)\sigma$$

Consequently, we have $\mathcal{J}_2 = \int_4^\infty e^{t\mathsf{M}_1(\sigma)}d\sigma \leq \int_0^\infty e^{(\alpha-1)\sigma t}d\sigma = \frac{t^{-1}}{1-\alpha}$. Combining this with (4.5.13) yields that there exists $C = C(p, t_0)$ such that for all $t > t_0$ and $\vec{m} \in \mathfrak{M}(L, n)$ and $L \geq 4n^3$ (note that $\frac{p^3}{12} \leq \frac{p^3}{3} - \delta_p$),

$$\int_0^\infty e^{t\mathsf{M}_1(\sigma)}d\sigma = \mathcal{J}_1 + \mathcal{J}_2 \leq 4e^{\frac{p^3t}{12}} + \frac{t^{-1}}{1-\alpha} \leq Ce^{\frac{p^3}{3}t - \delta_p t}.$$

We prove the desired (4.5.12) and conclude our proof for *Stage 1*.

Stage 2. $2 \leq L \leq 4n^3$. Via Proposition 4.4.1 (ii), $\left|\mathrm{Pf}\left[K(x_i, x_j)\right]_{i,j=1}^{L}\right|$ is bounded by $C^L \prod_{i=1}^{L} F_{\frac{2}{3},2}(x_i)$ for all $x_1, \ldots, x_L$ and $2 \leq L \leq 4n^3$. Note that we throw out the multiplier $\sqrt{2L}!$ in the upper bound since it is bounded by a constant that only depends on n when $L \leq 4n^3$. Thus

$$|\mathcal{I}_{\vec{m}}| = C^L \int_0^\infty e^{t(\alpha-1)\sigma} \prod_{i=1}^{L} \Big(\int_{\mathbb{R}} F_{\frac{2}{3},2}(x)|\phi_{e^{-t\sigma},t}^{(m_i)}(x)|dx \Big) d\sigma$$

As before, we assume $m_1, \ldots, m_r \geq 1$ and $m_{r+1} = \cdots = m_L = 0$. Applying Proposition 4.5.2 (c), (d). For each $\int_{\mathbb{R}} F_{\frac{2}{3},2}(x)|\phi_{e^{-t\sigma},t}^{(m_i)}(x)|dx$, $i = 1, \ldots, r$, since $m_i \in \mathbb{Z}_{\geq 1}$, this integral can be upper bounded by $Ct^{\frac{1}{6}} \exp\big(tU_{m_i}(\sigma \wedge m_i^2)\big)$. When $i \geq r+1$, $m_i = 0$, the integral can be upper bounded by $Ct^{\frac{1}{6}} \exp\big(tU_1(\sigma \wedge 1) - t\sigma\big)$. Therefore, there exists a constant $C = C(n, t_0)$ such that for all $t > t_0$,

$2 \leq L \leq 4n^3$ and $\vec{m} \in \mathfrak{M}(L,n)$,

$$|\mathcal{I}_{\vec{m}}| \leq C^L t^{\frac{L}{6}} \int_0^\infty e^{t\mathsf{M}_2(\sigma)} d\sigma \tag{4.5.14}$$

where

$$\mathsf{M}_2(\sigma) := (\alpha - 1)\sigma + \sum_{i=1}^r U_{m_i}(\sigma \wedge m_i^2) + (L-r)\big(U_1(\sigma \wedge 1) - \sigma\big). \tag{4.5.15}$$

To conclude the proof of Proposition 4.5.3, it suffices to show that there exists $C = C(n, t_0)$ such that for all $t > t_0$ and $L \geq 2$ and $\vec{m} \in \mathfrak{M}(L,n)$,

$$\int_0^\infty e^{t\mathsf{M}_2(\sigma)} d\sigma \leq C e^{\frac{p^3}{3}t - \delta_p t}. \tag{4.5.16}$$

Once this is shown, applying (4.5.14) completes the proof of the Proposition 4.5.3.

We are left to show (4.5.16). To this aim, we divide our argument into two cases, depending on $r = 1$ or not.

Case 1. $r = 1$. In this case, $m_1 = n$ and $m_i = 0$ for $i > 1$. As a result,

$$\mathsf{M}_2(\sigma) = (\alpha - 1)\sigma + U_n(\sigma \wedge n^2) + (L-1)\big(U_1(\sigma \wedge 1) - \sigma\big). \tag{4.5.17}$$

We decompose

$$\int_0^\infty e^{t\mathsf{M}_2(\sigma)} d\sigma = \Big(\int_0^1 + \int_1^n + \int_n^\infty\Big) e^{t\mathsf{M}_2(\sigma)} d\sigma = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,$$

and we are going to upper bound $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_3$ respectively.

For $\mathcal{I}_1$, when $\sigma \leq 1$, the right hand side of (4.5.17) can be simplified as

$$\mathsf{M}_2(\sigma) = (\alpha - L)\sigma + U_n(\sigma) + (L-1)U_1(\sigma) = p\sigma - \frac{2L}{3}\sigma^{\frac{3}{2}}.$$

Since $L \geq 2$, similar to the discussion in *Stage 1*, $\mathsf{M}_2(\sigma) = p\sigma - \frac{2L}{3}\sigma^{\frac{3}{2}} \leq p\sigma - \frac{4}{3}\sigma^{\frac{3}{2}} \leq \frac{p^3}{12}$.

Thereby,

$$\mathcal{I}_1 = \int_0^1 e^{t\mathsf{M}_2(\sigma)} d\sigma \le \int_0^1 e^{\frac{p^3 t}{12}} d\sigma = e^{\frac{p^3 t}{12}}. \tag{4.5.18}$$

For $\mathcal{I}_2$, when $1 \le \sigma \le n^2$, referring to (4.5.17), we can simplify $\mathsf{M}_2(\sigma) = p\sigma - \frac{2}{3}\sigma^{\frac{3}{2}} + (L-1)\big(\frac{1}{3} - \sigma\big)$. Note that the maximum of $p\sigma - \frac{2}{3}\sigma^{\frac{3}{2}}$ under the condition $\sigma \ge 0$ equals $\frac{1}{3}p^3$. Using this in conjunction with $L \ge 2$, $\mathsf{M}_2(\sigma) \le \frac{1}{3}p^3 + \frac{1}{3} - \sigma$. As a result,

$$\mathcal{I}_2 = \int_1^{n^2} e^{t\mathsf{M}_2(\sigma)} d\sigma \le e^{\frac{p^3}{3}t} \int_1^\infty e^{(\frac{1}{3} - \sigma)t} d\sigma = t^{-1} e^{\frac{p^3 - 2}{3}t}. \tag{4.5.19}$$

For $\mathcal{I}_3$, the right hand side of (4.5.17) simplifies to

$$\mathsf{M}_2(\sigma) = (\alpha - 1)\sigma + U_n(n^2) + (L-1)(U_1(1) - \sigma) = (\alpha - 1)\sigma + \frac{1}{3}n^3 + (L-1)(\frac{1}{3} - \sigma).$$

Since $\alpha < 1$, $\sigma \ge n^2$ and $L \ge 2$,

$$\mathsf{M}_2(\sigma) \le n^2(\alpha - 1) + \frac{1}{3}n^3 + (L-1)(\frac{1}{3} - \sigma) \le n^2(\alpha - 1) + \frac{1}{3}n^3 + (\frac{1}{3} - \sigma)$$

Note that $n^2(\alpha - 1) + \frac{1}{3}n^3 \le \frac{1}{3}(n + \alpha - 1)^3 = \frac{1}{3}p^3$, hence $\mathsf{M}_1(\sigma) \le \frac{1}{3}p^3 + \frac{1}{3} - \sigma$. Thereby,

$$\int_{n^2}^\infty e^{t\mathsf{M}_2(\sigma)} d\sigma \le e^{\frac{p^3 t}{3}} \int_{n^2}^\infty e^{(\frac{1}{3} - \sigma)t} d\sigma = t^{-1} e^{(\frac{p^3 + 1}{3} - n^2)t} \le t^{-1} e^{\frac{p^3 - 2}{3}t}. \tag{4.5.20}$$

Combining (4.5.18), (4.5.19) and (4.5.20), we conclude that for $t \ge t_0$,

$$\int_0^\infty e^{t\mathsf{M}_2(\sigma)} d\sigma \le \left(e^{\frac{1}{12}p^3 t} + t^{-1} e^{\frac{p^3 - 2}{3}t} + t^{-1} e^{\frac{p^3 - 2}{3}t}\right) \le C e^{\frac{1}{3}p^3 t - \delta_p t},$$

The last inequality follows since $\delta_p = \min(\frac{2}{3}, \frac{1}{4}p^3)$. So far we have shown (4.5.16) when $r = 1$.

Case 2. $r \ge 2$. This implies $n \ge 2$. We write

$$\int_0^\infty e^{t\mathsf{M}_2(\sigma)} d\sigma = \Big(\int_0^1 + \int_1^\infty\Big) e^{t\mathsf{M}_2(\sigma)} d\sigma = \mathcal{I}t_1 + \mathcal{I}t_2$$

For $\mathcal{I}t_1$, by (4.5.15), when $\sigma \le 1$, $\mathsf{M}_2(\sigma) = p\sigma - \frac{2}{3}L\sigma^3$. Via the same argument as in **Case 1**, we conclude that $\mathcal{I}t_1 \le e^{\frac{1}{12}p^3 t}$. For $\mathcal{I}t_2$, using the inequality $U_{m_i}(\sigma \wedge m_i^2) \le \frac{1}{3}m_i^3$ and $U_1(\sigma \wedge 1) = U_1(1) = \frac{1}{3}$, we get

$$\mathsf{M}_2(\sigma) \le \frac{1}{3}\sum_{i=1}^{r} m_i^3 + \frac{1}{3}(L-r) + (\alpha - L + r - 1)\sigma \le \frac{1}{3}\sum_{i=1}^{r} m_i^3 + (\alpha-1)\sigma \tag{4.5.21}$$

Since we assume $r \ge 2$, it is convincible that $\sum_{i=1}^{r} m_i^3$ is at most $(n-1)^3+1$, since the cubic sum will increase if we let mass concentrate on fewer terms. To justify this, note that $\sum_{i=2}^{r} m_i^3 \le \big(\sum_{i=2}^{r} m_i\big)^3 = (n-m_1)^3$. Thus

$$\sum_{i=1}^{r} m_i^3 \le m_1^3 + (n-m_1)^3 = (n-1)^3 + 1 + 3n(m_1-1)(m_1-(n-1)) \le (n-1)^3+1.$$

Applying this inequality to the right hand side of (4.5.21), we see that

$$\mathsf{M}_2(\sigma) \le \frac{1}{3}\big((n-1)^3+1\big)+(\alpha-1)\sigma \le \frac{1}{3}(n-1)^3+\alpha-\frac{2}{3}+(\alpha-1)(\sigma-1) \le \frac{1}{3}(\alpha+n-1)^3-\frac{2}{3}+(\alpha-1)(\sigma-1).$$

The second inequality above follows from $\sigma \ge 1$ and the third equality is due to $\frac{1}{3}(n-1)^3 + \alpha \le \frac{1}{3}(n-1)^3 + (n-1)^2\alpha \le \frac{1}{3}(n-1+\alpha)^3$. Recall that $p = \alpha + n - 1$, we obtain $\mathsf{M}_2(\sigma) \le \frac{1}{3}p^3 - \frac{2}{3} + (\alpha-1)(\sigma-1)$, and thus $\mathcal{I}t_2 \le \int_1^\infty \exp\big(t(\frac{1}{3}p^3 - \frac{2}{3} + (\alpha-1)(\sigma-1))\big)d\sigma = \frac{1}{(1-\alpha)t}e^{(\frac{1}{3}p^3-\frac{2}{3})t}$. So there exists a constant C such that for $t > t_0$,

$$\int_0^\infty e^{t\mathsf{M}_2(\sigma)}d\sigma = \mathcal{I}t_1 + \mathcal{I}t_2 \le e^{\frac{p^3 t}{12}} + (1-\alpha)^{-1}t^{-1}e^{\frac{p^3-2}{3}t} \le Ce^{\frac{p^3}{3}t-\delta_p t}.$$

This implies (4.5.16) and completes the proof of *Stage 2*. □

Proof of Proposition 4.2.9. It suffices to prove that for fixed $p > 0$, there exists a constant $C = C(p)$ such that for all $L \ge 2$ and $t > 1$,

$$|\mathcal{B}_{p,L}(t)| \le \frac{C^L(2L)^{\frac{L}{2}}}{L!}t^{\frac{L}{6}}e^{\frac{p^3}{3}t-\delta_p t}. \tag{4.5.22}$$

One this is shown, we conclude our proof by observing

$$\Big|\sum_{L=2}^{\infty}\mathcal{B}_{p,L}(t)\Big| \leq \sum_{L=2}^{\infty}|\mathcal{B}_{p,L}(t)| \leq e^{\frac{p^3}{3}t-\delta_p t}\sum_{L=2}^{\infty}\frac{C^L(2L)^{\frac{L}{2}}t^{\frac{L}{6}}}{L!} \tag{4.5.23}$$

Using the inequality of Stirling's formula, we know that $L^L \leq e^L L!$ for all $L \in \mathbb{Z}_{\geq 1}$. Consequently, $(2L)^{\frac{L}{2}} = 2^L(L^L)^{\frac{1}{2}} \leq 2^L e^{\frac{L}{2}}\sqrt{L!}$. So there exist constants C_1, C_2 such that for all $t > 1$,

$$\sum_{L=2}^{\infty}\frac{C^L(2L)^{\frac{L}{2}}t^{\frac{L}{6}}}{L!} \leq \sum_{L=2}^{\infty}\frac{C_1^L t^{\frac{L}{6}}}{\sqrt{L!}} \leq e^{C_2 t^{\frac{1}{3}}}.$$

Combining the above inequality with (4.5.23), the left hand side of (4.5.23) is upper bounded by $\exp(p^3 t/3 - \delta_p t + C_2 t^{\frac{1}{3}})$. Taking the logarithm and dividing by t for both sides and letting $t \to \infty$ completes the proof of Proposition 4.2.9.

We are left to show (4.5.22). referring to (4.2.12) and using Leibniz's rule,

$$\begin{aligned}\mathcal{B}_{p,L}(t) &= \frac{(-1)^n}{\Gamma(1-\alpha)}\sum_{\vec{m}\in\mathfrak{M}(L,n)}\int_0^1 s^{-\alpha}\frac{1}{L!}\int_{\mathbb{R}^L}\mathrm{Pf}\big[K(x_i,x_j)\big]_{i,j=1}^{L}\prod_{i=1}^{L}\phi_{s,t}^{(m_i)}(x_i)dx_i \\ &= \frac{(-1)^n}{\Gamma(1-\alpha)L!}\sum_{\vec{m}\in\mathfrak{M}(L,n)}\binom{n}{\vec{m}}\mathcal{I}_{\vec{m}} \end{aligned} \tag{4.5.24}$$

where $\mathcal{I}_{\vec{m}}$ is defined in (4.5.9). Using Proposition 4.5.3 and the above display, recalling that $\#\mathfrak{M}(L,n)$ represents the number of elements that lie in $\mathfrak{M}(L,n)$, we get for all $L \geq 2$,

$$|\mathcal{B}_{p,L}(t)| \leq \frac{1}{\Gamma(1-\alpha)L!}\sum_{\vec{m}\in\mathfrak{M}(L,n)}\binom{n}{\vec{m}}|\mathcal{I}_{\vec{m}}| \leq \frac{n!}{\Gamma(1-\alpha)}(\#\mathfrak{M}(L,n))\max_{\vec{m}\in\mathfrak{M}(L,n)}|\mathcal{I}_{\vec{m}}|$$

where the first inequality follows from taking the absolute value of both sides of (4.5.24) and applying triangle inequality to the right hand side. The second inequality follows from upper bounding $\binom{n!}{\vec{m}}$ by $n!$. Recall from (4.4.10) that $\#\mathfrak{M}(L,n) \leq L^n$. To prove (4.5.22), applying

Proposition 4.5.3 to upper bound each $|\mathcal{I}_{\vec{m}}|$, we obtain

$$|\mathcal{B}_{p,L}(t)| \le \frac{n!L^n}{\Gamma(1-\alpha)L!} C^L (2L)^{\frac{L}{6}} t^{\frac{L}{6}} e^{\frac{p^3 t}{3} - \delta_p t}.$$

Note that L^n grows slower than C_1^L for $C_1 > 1$, as $L \to \infty$. So there exists a constant C_1 such that $\frac{n!L^n}{\Gamma(1-\alpha)} C^L \le C_1^L$. Applying this inequality to the right hand side of the above display completes the proof of (4.5.22). □

Chapter 5: Markov duality for stochastic six vertex model

Chapter Abstract: We prove that Schütz's ASEP Markov duality functional is also a Markov duality functional for the stochastic six vertex model. We introduce a new method that uses induction on the number of particles to prove the Markov duality.

This chapter is published at [Lin19].

5.1 Introduction

5.1.1 Stochastic six vertex model

The stochastic six vertex model (S6V model) is a classical model in 2d statistical physics first introduced by Gwa and Spohn [GS92], as a special case of the six vertex (ice) model (see for example [Lie74] and [Bax16]). We associate each vertex in $\mathbb{Z}^2$ with six types of configurations with weights parametrized by $0 < b_1, b_2 < 1$, see Figure 5.1. The configurations chosen for two neighboring vertices need to be compatible in the sense that the lines keep flowing. We consider the lines from the south and the west as the inputs and the lines to the north and the east as the outputs. Each vertex is conservative in the sense that the number of input lines equals the number of output lines. The model is stochastic in the sense that when we fix the inputs, the weights of possible configurations sum up to 1.

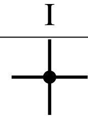
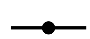
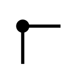
Type	I	II	III	IV	V	VI
Configuration						
Weight	1	1	b_2	$1 - b_2$	b_1	$1 - b_1$

Figure 5.1: Six types of configurations for the vertex.

The S6V model is a member of the KPZ universality class (see [Cor12, Qua11] for a nice survey).

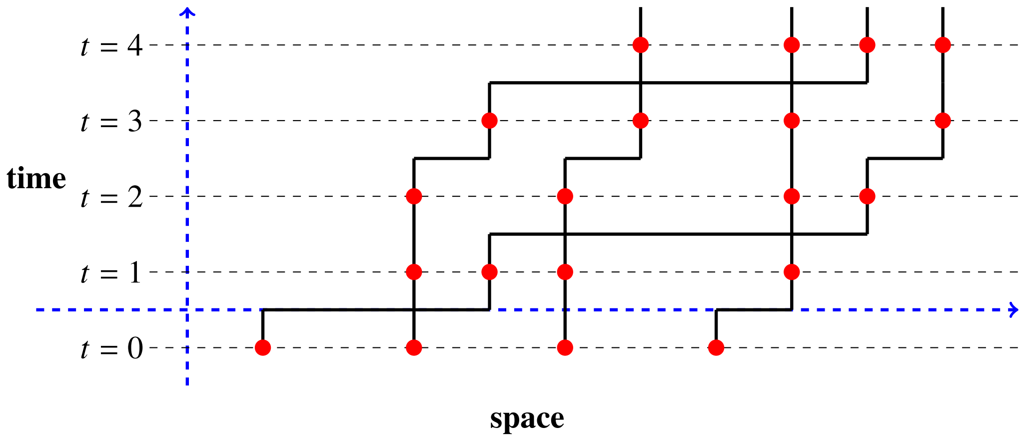


Figure 5.2: S6V model viewed as an interacting particle system.

We briefly review a few results that have been obtained for the S6V model. [BCG16] proves that under step initial condition, the one point fluctuation of the S6V model height function is asymptotically Tracy-Widom GUE. One point fluctuations of the S6V model under more general initial condition including stationary is obtained in [AB16, Agg18a]. In a slightly different direction, [CGST20] shows that under scaling $\frac{b_1}{b_2} \to 1$ with b_1 fixed, the fluctuation of the S6V model height function converges weakly to the solution of KPZ equation. More recently, [BG18, ST19] showed under a different scaling, the height fluctuation of the S6V model converges in finite dimensional distribution to the solution of stochastic telegraph equation.

In this paper, we consider the S6V model as an interacting particle system (see [GS92] or Section 2.2 of [BCG16]). Consider vertex configurations in the upper half-plane, we restrict ourselves to the boundary condition that there are no lines coming from the left boundary. Then the input lines coming from the bottom of the horizontal axis can be viewed as the trajectories of an exclusion-type particle system. We see the vertical axis as time variable and horizontal axis as space variable. The vertex configurations compose several paths, which can be viewed as trajectories of particles with the vertical lines denoting the particle location. As illustrated by Figure 5.2, we cut our plane by the line $y = t - \frac{1}{2}$ and the particle location at time t is give by the intersections (red points in the figure) of these trajectories with $y = t - \frac{1}{2}$. To rigorously define our interacting particle system, we first introduce the following state spaces.

Definition 5.1.1. We define the space of left-finite particle occupation configuration $\mathbb{G}$ to be

$$\mathbb{G} = \left\{\vec{g} = (\cdots, g_{-1}, g_0, g_1, \cdots) \in \{0,1\}^{\mathbb{Z}} : \ \exists\, i \in \mathbb{Z} \text{ so that } g_x = 0, \ \forall x \leq i\right\},$$

where g_x is understood as the number of particles (either zero or one) at location x. We also define the space of left-finite particle location configuration to be

$$\mathbb{X} = \{\vec{x} = (x_1 < x_2 < \cdots) : x_i \in \mathbb{Z} \cup \{+\infty\} \text{ for every } i \in \mathbb{N}\},$$

where x_i stands for the location of i-th particle counting from the left. Note that there might be infinite or finite number of particles in our particle configuration. For the latter case, there exists some $m \in \mathbb{N}$ so that $x_i = +\infty$ for $i \geq m$. It is straightforward that there is a bijection $\varphi : \mathbb{X} \to \mathbb{G}$ defined by

$$\vec{g} = \varphi(\vec{x}) \text{ such that } g_i = \mathbb{1}_{\{\text{there exists } n \in \mathbb{N} \text{ so that } x_n = i\}} \text{ for every } i \in \mathbb{Z}.$$

Having specified our state space, we proceed to define the particle interpretation of the S6V model as the following discrete-time Markov processes. The following definition is similar as the one that appears in Section 2.1 of [CGST20].

Definition 5.1.2. We define the S6V location process, which is a discrete-time $\mathbb{X}$-valued Markov process $\vec{x}(t) = (x_1(t) < x_2(t) < \cdots)$ with the update rule (transition probability) from $\vec{x}(t)$ to $\vec{x}(t+1)$ specified as follows:

We denote $x_0(t) = -\infty$ for any $t \in \mathbb{Z}_{\geq 0}$, this is just a convention to simplify the notation. We sequentially consider $i = 1, 2, \ldots$ and update as following independent probabilities:

(a) When $x_i(t) > x_{i-1}(t+1)$, we update $x_i(t)$ to $x_i(t+1)$ via

$$\mathbb{P}\left(x_i(t+1) - x_i(t) = n\right) = \begin{cases} b_1, & \text{if } n = 0; \\ (1-b_1)(1-b_2)b_2^{n-1} & \text{if } 1 \le n \le x_{i+1}(t) - x_i(t) - 1; \\ (1-b_1)b_2^{n-1} & \text{if } n = x_{i+1}(t) - x_i(t); \\ 0 & \text{else;} \end{cases}$$

(b) When $x_i(t) = x_{i-1}(t+1)$, we update $x_i(t)$ to $x_i(t+1)$ via

$$\mathbb{P}\left(x_i(t+1) - x_i(t) = n\right) = \begin{cases} (1-b_2)b_2^{n-1} & \text{if } 1 \le n \le x_{i+1}(t) - x_i(t) - 1; \\ b_2^{n-1} & \text{if } n = x_{i+1}(t) - x_i(t); \\ 0 & \text{else.} \end{cases}$$

We also define the S6V occupation process $\vec{g}(t) = (g_x(t))_{x\in\mathbb{Z}} \in \mathbb{G}$ by setting $\vec{g}(t) = \varphi(\vec{x}(t))$ i.e.

$$g_x(t) = \mathbb{1}_{\{\text{there exists } n\in\mathbb{N} \text{ so that } x_n(t)=x\}} \text{ for every } x \in \mathbb{Z}.$$

Clearly, $\vec{g}(t)$ is a discrete-time $\mathbb{G}$-valued Markov process. We remark that the occupation process $\vec{g}(t)$ and location process $\vec{x}(t)$ are just two ways to describe the particle interpretation of the S6V model.

For a left-finite particle configuration $\vec{g} \in \mathbb{G}$, we define the height function $N_x(\vec{g})$ to be the total number of particles in this particle configuration that is on the left or at location x, i.e.

$$N_x(\vec{g}) = \sum_{i\le x} g_i.$$

Our result is a Markov duality between the S6V occupation process and its space reversal, which we define below:

Definition 5.1.3. Define the space of reversed k-particle location configuration

$$\mathbb{Y}^k = \left\{\vec{y} = (y_1 > \cdots > y_k) : \vec{y} \in \mathbb{Z}^k\right\}.$$

The reversed k-particle S6V location process $\vec{y}(t) = (y_1(t) > \cdots > y_k(t))$ is a $\mathbb{Y}^k$-valued Markov process so that $(-y_k(t) < \cdots < -y_1(t))$ has the same update rule as the S6V location process.

5.1.2 Markov Duality

Definition 5.1.4. Given two discrete (continuous) time Markov processes $X(t) \in U$ and $Y(t) \in V$ and a function $H : U \times V \to \mathbb{R}$, we say that $X(t)$ and $Y(t)$ are dual with respect to H if for any $x \in U, y \in V$ and $t \in \mathbb{Z}_{\geq 0}$ ($t \in \mathbb{R}_{\geq 0}$ for continuous time), we have

$$\mathbb{E}^x\left[H(X(t), y)\right] = \mathbb{E}^y\left[H(x, Y(t))\right].$$

Here we use $\mathbb{E}^x$ to denote that we take the expectation under initial condition $X(0) = x$. Likewise, $\mathbb{E}^y$ represents the expectation with initial condition $Y(0) = y$.

Markov duality has been found for different interacting particle systems including the contact process, voter model and symmetric simple exclusion process, see [Lig12, Lig13]. It also plays an important role in the analysis of models in the KPZ universality class. The first such example is the asymmetric simple exclusion process (ASEP), which is an interacting particle system on $\mathbb{Z}$ with at most one particle at each site. Each particle jumps to the left with rate ℓ and jumps to the right with rate r. If the site is already occupied by another particle, the jump is excluded.

We consider ASEP as a process $\vec{g}(t) = (g_x(t))_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$, where $g_x(t)$ is an indicator for the event that at time t, a particle is at site x. We call $\vec{g}(t)$ the ASEP occupation process. When the ASEP has finite k-particles, in terms of particle location, we also consider the k-particle ASEP location process $\vec{y}(t) = (y_1(t) > \cdots > y_k(t)) \in \mathbb{Y}^k$ where $y_i(t)$ denotes the location of i-th particle counting from the right at time t.

Schütz [Sch97] derived the following ASEP duality using a spin chain representation: For any fixed $k \in \mathbb{N}$, the ASEP occupation process $\vec{g}(t)$ and the k-particle ASEP location process $\vec{y}(t)$ with the jump rate r and ℓ reversed are dual with respect to the duality functional

$$H(\vec{g}, \vec{y}) = \prod_{i=1}^{k} g_{y_i} q^{-N_{y_i}(\vec{g})}, \tag{5.1.1}$$

where $q = \ell / r$. This generalizes the the Markov duality satisfied by the symmetric simple exclusion process [Lig12] where ℓ and r are set to be equal. We call (5.1.1) Schütz's ASEP Markov duality functional.

[BCS14] uses a different approach to prove Schütz's result by directly applying the Markov generator on the duality functional. Further, they use this method to show that the processes $\vec{g}(t)$ and $\vec{y}(t)$ are also dual with respect to the functional

$$G(\vec{g}, \vec{y}) = \prod_{i=1}^{k} q^{-N_{y_i}(\vec{g})}. \tag{5.1.2}$$

The ASEP is a continous time limit of the S6V model if we scale the parameter by $b_1 = \epsilon \ell, b_2 = \epsilon r$ and scale time by $\epsilon^{-1} t$ and shift the space to the right by $\epsilon^{-1} t$, see [BCG16, Agg17]. Given the ASEP is the limit of the S6V model and enjoys the Markov duality with respect to the functionals in (5.1.1) and (5.1.2), one might wonder if these functionals are the Markov duality functionals for the S6V model as well. Indeed, by setting $q = \frac{b_1}{b_2}$, [CP16, Theorem 2.21] justifies that the S6V occupation process and the reversed k-particle S6V location process are dual with respect to the functional in (5.1.2)[1].

Our main result shows that the S6V model also enjoys a Markov duality with respect to the functional in (5.1.1).

Theorem 5.1.5. *Consider the S6V model with parameter b_1, b_2 and set $q = \frac{b_1}{b_2}$. For any $k \in \mathbb{N}$,*

[1]In fact, [CP16, Theorem 2.21] proves the duality for a higher spin generalization of S6V model called stochastic higher spin vertex model.

the S6V occupation process $\vec{g}(t) \in \mathbb{G}$ and the reversed k-particle S6V location process $\vec{y}(t) \in \mathbb{Y}^k$ are dual with respect to the function H given in (5.1.1).

Remark 5.1.6. We remark that the Markov duality in Theorem 5.1.5 appeared first in [CP16, Theorem 2.23] and was later used in proving [CGST20, Proposition 5.3]. In fact, [CP16, Theorem 2.23] claims a Markov duality for the stochastic higher spin vertex model (see [CP16, Section 2] for definition), which is a higher spin generalization of the S6V model (the stochastic higher spin vertex model has vertical and horizontal spin $\frac{I}{2}, \frac{J}{2}$, where $I, J \in \mathbb{N}$. When $I = J = 1$, it degenerates to the S6V model). However, this Markov duality is false when $I > 1$. In fact, the author of the present paper found a counterexample which is recorded in the erratum [CP19]. For the S6V model, the Markov duality holds but the proof of [CP16, Theorem 2.23] still breaks down[2]. In this paper, we offer the first correct proof for this Markov duality.

Remark 5.1.7. It appears that the proof of Theorem 5.1.5 also adapts to the space inhomogeneous stochastic six vertex model, where we allow the parameters b_1, b_2 in Figure 5.1 to vary at different locations $x \in \mathbb{Z}$ and are expressed by $b_{1,x}$ and $b_{2,x}$. Under the condition that there exists $q > 0$ such that $b_{1,x} = qb_{2,x}$ for all $x \in \mathbb{Z}$, Theorem 5.1.5 holds for this space inhomogeneous stochastic six vertex model as well. To avoid extra notation, we have opted not to state and prove this more general result here.

Remark 5.1.8. It is natural to ask whether our method produces duality for stochastic higher spin vertex model with vertical and horizontal spin $\frac{I}{2}, \frac{J}{2}$. Using fusion, one can prove the same duality in Theorem 5.1.5 holds if we take $I = 1$ and $J \in \mathbb{N}$ (the proof is similar to that of [CP16, Corollary 3.2]). For $I > 1$, as presented in Section 5.2, our proof relies heavily on the structure of the duality functional (5.1.1) and the particle exclusion property (i.e. at most one particle allows to stay in each location) of the S6V model. It is unclear how to adapt our method proving duality for stochastic higher spin vertex model such as [Kua18, Theorem 4.10], since both of the duality functional and the model become more complicated.

[2]The proof of [CP16, Theorem 2.23] claims that the S6V duality (5.1.1) can be deduced by taking the discrete gradient of the Markov duality functional in (5.1.2), which is not true when the number of particles k in Theorem 5.1.5 is larger than 1 [CP].

Duality has been obtained for generalization of the ASEP and S6V model using algebraic methods, see [BS15a, BS15b, CGRS16, Kua16, Kua17, Kua18]. In particular, [CGRS16] proves two ASEP(q, j) (which is a higher spin generalization of ASEP) dualities based on the higher spin representations of $U_q[\mathfrak{sl}_2]$. In the spirit of [CGRS16], duality has also been proved for multi-species version of ASEP [BS15b, Kua17]. [Kua18] obtains a duality for the multi-species version of the stochastic higher spin vertex model via an algebraic construction. Instead of using algebraic tools to prove duality, our proof of Theorem 5.1.5 follows a straightforward induction approach.

We remark that the duality functional from [Kua18, Theorem 4.10] has a degeneration to S6V model. We state this degeneration here as a lemma.

Proposition 5.1.9. *Consider the S6V model with parameter b_1, b_2 and set $q = \frac{b_1}{b_2}$. For any $k \in \mathbb{N}$, the S6V occupation process $\vec{g}(t) \in \mathbb{G}$ and the reversed k-particle S6V location process $\vec{y}(t) \in \mathbb{Y}^k$ are dual with respect to the functional*

$$D(\vec{g}, \vec{y}) = \prod_{i=1}^{k} (1 - g_{y_i}) q^{-N_{y_i}(\vec{g})}. \tag{5.1.3}$$

We remark that there is a misstatement in [Kua18, Theorem 4.10]. The particles in the process $\mathcal{Z}$ and $\mathcal{Z}_{rev}$ were stated to jump to the left and to the right respectively (see pp.164 of [Kua18]). However, after discussing with the author, we realize that the correct statement is that the particles in $\mathcal{Z}$ jump to the right and those in $\mathcal{Z}_{rev}$ jump to the left [Kua].

Proof. Taking the spin parameter $m_x = 1$ for all $x \in \mathbb{Z}$ and species number $n = 1$ and substituting q by $q^{1/2}$, the multi-species higher spin vertex model considered in [Kua18, Theorem 4.10] degenerates to the stochastic six vertex model (see [Kua18, Section 2.6.2] for detail). Referring to the duality functional $\langle \xi | D(u_0) | \eta \rangle$ considered in [Kua18, Theorem 4.10] (note that ξ is the configuration for the process $\mathcal{Z}$ and η is the configuration for the process $\mathcal{Z}_{rev}$). By substituting the configuration η by $\vec{g} = (g_x)_{x \in \mathbb{Z}}$ and the configuration ξ by the k-particle location configuration $\vec{y} = (y_1, \cdots, y_k)$, we obtain that the reversed S6V occupation process $\widetilde{g}(t)$ (with particles jumping to the left) is dual to the k-particle S6V location process $\widetilde{y}(t)$ (with particles jumping to the right) with respect to the functional

$$\widetilde{D}(\vec{g}, \vec{y}) = \prod_{i=1}^{k} (1 - g_{y_i}) q^{-\sum_{z>y_i} g_z - \frac{1}{2} g_{y_i}} = \prod_{i=1}^{k} (1 - g_{y_i}) q^{-\sum_{z \geq y_i} g_z}.$$

In the last equality we used the fact that $g_{y_i} \in \{0, 1\}$. Since $\widetilde{g}(t)$ and $\widetilde{y}(t)$ are nothing but the space reversal of $\vec{g}(t)$ and $\vec{y}(t)$ in the lemma. After swapping the role of left and right (then $\sum_{z \geq y_i} g_z$ is exactly the height function $N_{y_i}(\vec{g})$), we readily obtain the duality in (5.1.3). □

When we take $k = 1$ in Theorem 5.1.5, our duality can be simply derived by subtracting the functional G in (5.1.2) by the functional D in (5.1.3) (see Lemma 5.2.1). However, it appears that when $k > 1$, there is no easy way to obtain our duality by combining the duality functionals in (5.1.2) and (5.1.3).

Finally, we explain several applications of our duality. Theorem 5.1.5 combined with the other S6V duality (5.1.2) are the main tools for proving the self-averaging property of the specific quadratic function of the S6V height function in [CGST20, Proposition 5.3], which is the crux in proving the convergence of stochastic six vertex model to KPZ equation. In a different direction, by using duality, we can compute the exact moment formula of certain observables of our model. [BCS14] uses the ASEP duality (5.1.1) to derive the moment generating function of the ASEP height function under Bernoulli step initial data. Applying a similar approach, we expect by using our duality and the S6V model Bethe ansatz eigenfunction given by [CP16, Proposition 2.12], we can reprove the moment formula appearing in [BCG16, Theorem 4.12] and [AB16, Theorem 4.4]. Since this application of the duality is not related to our paper, we do not pursue to give the proof here.

5.1.3 Acknowledgment

The author wants to thank Ivan Corwin for his helpful discussions and comments pertaining to the earlier draft of this paper and also for his advice on the aspect of paper writing. We wish to thank Promit Ghosal for his useful comments and Jeffrey Kuan for his helpful discussion about the result in his paper. We are also grateful to the anonymous referees for their valuable suggestions. The author was partially supported by the Fernholz Foundation's "Summer Minerva Fellow" program and also received summer support from Ivan Corwin's NSF grant DMS:1811143.

5.2 Proof of Theorem 5.1.5

In this section, we prove Theorem 5.1.5. We first introduce several notations for our proof. Define the space of ℓ-particle location configuration

$$\mathbb{X}^{\ell} = \left\{\vec{x} = (x_1 < \cdots < x_{\ell}) : \vec{x} \in \mathbb{Z}^{\ell}\right\}.$$

We also denote by $|\vec{g}|$, $|\vec{x}|$ and $|\vec{y}|$ the number of particles in the particle configuration $\vec{g} \in \mathbb{G}$, $\vec{x} \in \mathbb{X}$ and $\vec{y} \in \mathbb{Y}^k$ (obviously $|\vec{y}| = k$ when $\vec{y} \in \mathbb{Y}^k$) respectively.

Referring to the Definition 5.1.2 of Markov duality, we need to show that for any $k \in \mathbb{N}$ and under any initial states $\vec{g} \in \mathbb{G}$, $\vec{y} \in \mathbb{Y}^k$, we have

$$\mathbb{E}^{\vec{g}}\big[H(\vec{g}(t), \vec{y})\big] = \mathbb{E}^{\vec{y}}\big[H(\vec{g}, \vec{y}(t))\big].$$

By Markov property, it suffices to prove that the preceding equation holds for $t = 1$, namely, for any $k \in \mathbb{N}$ and any $\vec{g} \in \mathbb{G}$, $\vec{y} \in \mathbb{Y}^k$, we have

$$\mathbb{E}^{\vec{g}}\big[H(\vec{g}(1), \vec{y})\big] = \mathbb{E}^{\vec{y}}\big[H(\vec{g}, \vec{y}(1))\big]. \tag{5.2.1}$$

Observing that $|\vec{y}|$ is finite (since $\vec{y} \in \mathbb{Y}^k$) whereas $|\vec{g}|$ can either be finite or infinite. We claim that it suffices to prove (5.2.1) for all $\vec{g} \in \mathbb{G}$ such that $|\vec{g}|$ is finite, here is the reason: Suppose we have proved (5.2.1) for every $\vec{g} \in \mathbb{G}$ with $|\vec{g}| < \infty$. For $\vec{g} \in \mathbb{G}$ and $\vec{y} = (y_1 > \cdots > y_k) \in \mathbb{Y}^k$ such that $|\vec{g}| = \infty$, we consider a particle configuration $\vec{g'} \in \mathbb{G}$ that corresponds with $\vec{g} \in \mathbb{G}$ in the following way:

$$g'_i = \begin{cases} g_i & \text{if } i \le y_1; \\ 0 & \text{if } i > y_1. \end{cases}$$

Clearly, $|\vec{g'}| < \infty$ and hence $\mathbb{E}^{\vec{g'}}\big[H(\vec{g}(1), \vec{y})\big] = \mathbb{E}^{\vec{y}}\big[H(\vec{g'}, \vec{y}(1))\big]$. Additionally, observing that the particles in the configuration $\vec{g}$ which are on the right of y_1 have no contribution to both of the expectations $\mathbb{E}^{\vec{g}}\big[H(\vec{g}(1), \vec{y})\big]$ and $\mathbb{E}^{\vec{y}}\big[H(\vec{g}, \vec{y}(1))\big]$ (see Figure 5.3), thus

$$\mathbb{E}^{\vec{g}}\big[H(\vec{g}(1), \vec{y})\big] = \mathbb{E}^{\vec{g'}}\big[H(\vec{g}(1), \vec{y})\big], \qquad \mathbb{E}^{\vec{y}}\big[H(\vec{g}, \vec{y}(1))\big] = \mathbb{E}^{\vec{y}}\big[H(\vec{g'}, \vec{y}(1))\big].$$

We conclude that $\mathbb{E}^{\vec{g}}\big[H(\vec{g}(1), \vec{y})\big] = \mathbb{E}^{\vec{y}}\big[H(\vec{g}, \vec{y}(1))\big]$ also holds for all $\vec{g} \in \mathbb{G}$ with $|\vec{g}| = \infty$.

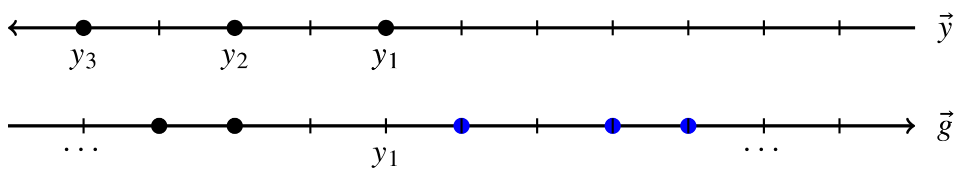


Figure 5.3: The picture above shows an example for the initial states $\vec{g} \in \mathbb{G}$ and $\vec{y} \in \mathbb{Y}^3$. Since the particles in $\vec{g}$ jumps to the right and the particles in $\vec{y}$ jumps to the left (as illustrated by the arrows), the blue particles in $\vec{g}$ (that are on the right of y_1) do not contribute to the computation of $\mathbb{E}^{\vec{g}}\big[H(\vec{g}(1), \vec{y})\big]$ and $\mathbb{E}^{\vec{y}}\big[H(\vec{g}, \vec{y}(1))\big]$.

It remains to prove (5.2.1) when $|\vec{g}|$ is finite. In other words, we need to prove (5.2.1) for all $\ell, k \in \mathbb{N}$ and all $\vec{y} \in \mathbb{Y}^k$, $\vec{g} \in \mathbb{G}$ satisfying $|\vec{g}| = \ell$. We apply induction according to $\ell + k$. The first thing is to show that (5.2.1) holds when $\min(\ell, k) = 1$, as the induction basis.

Lemma 5.2.1. *When* $\min(\ell, k) = 1$, (5.2.1) *holds.*

Proof of Lemma 5.2.1. Since $\min(\ell, k) = 1$, we have either $k > 1$ and $\ell = 1$, or $k = 1$. Note that

$$H(\vec{g}(1), \vec{y}) = \prod_{i=1}^{k} g_{y_i}(1) q^{-N_{y_i}(\vec{g}(1))}.$$

If $k > 1$ and $\ell = 1$, since $\vec{g}(1)$ has only one non-zero component and $k > 1$, we have $\prod_{i=1}^{k} g_{y_i}(1) = 0$ for any $\vec{g}(1)$ and thus $\mathbb{E}^{\vec{g}}\left[H(\vec{g}(1), \vec{y})\right] = 0$. Similarly, we have $\mathbb{E}^{\vec{y}}\left[H(\vec{g}, \vec{y}(1))\right] = 0$ hence the desired equality holds.

If $k = 1$, we note that $G(\vec{g}, \vec{y}) = H(\vec{g}, \vec{y}) - D(\vec{g}, \vec{y})$, where H and D are given by (5.1.2) and (5.1.3). Since the subtraction of two duality functionals is still a duality functional (which follows from Definition 5.1.2), we obtain the desired (5.2.1). □

Before explaining how the induction works, we slightly reformulate (5.2.1). In order to keep track of the location of the particles, we utilize the S6V location process $\vec{x}(t)$ in Definition 5.1.2. Via the bijection $\varphi : \mathbb{X} \to \mathbb{G}$ (see Definition 5.1.1), we identify a configuration $\vec{g} \in \mathbb{G}$ with $\vec{x} \in \mathbb{X}$ and define the function $\widetilde{H}$ as

$$\widetilde{H}(\vec{x}, \vec{y}) = H(\varphi(\vec{x}), \vec{y}).$$

By the relation $\vec{g}(t) = \varphi(\vec{x}(t))$ between the S6V occupation process $\vec{g}(t)$ and the S6V location process $\vec{x}(t)$, (5.2.1) can be paraphrased into the following:

For any $\ell, k \in \mathbb{N}$ and any initial states $\vec{x} \in \mathbb{X}^{\ell}$ and $\vec{y} \in \mathbb{Y}^{k}$, we have

$$\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1), \vec{y})\right] = \mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x}, \vec{y}(1))\right]. \tag{5.2.2}$$

When $\min(\ell, k) = 1$, (5.2.2) is established via Lemma 5.2.1. When $\min(\ell, k) \geq 2$, we define our induction hypothesis as

$$\text{(5.2.2) holds for any } \vec{x} \in \mathbb{X}^{n} \text{ and } \vec{y} \in \mathbb{Y}^{m} \text{ with } n + m < \ell + k. \tag{HYP$_{\ell,\mathrm{k}}$}$$

It suffices to prove (5.2.2) for any $\vec{x} \in \mathbb{X}^{\ell}$ and $\vec{y} \in \mathbb{Y}^{k}$ under (HYP$_{\ell,\mathrm{k}}$). We briefly explain our strategy: We decompose the LHS expectation of (5.2.2) into a combination of the expectations which are in the form of $\mathbb{E}^{\vec{x}'}\left[\widetilde{H}(\vec{x}'(1), \vec{y}')\right]$ with $|\vec{x}'| + |\vec{y}'| < \ell + k$. A similar decomposition occurs in the RHS expectation. By applying ($\mathrm{HYP}_{\ell,\mathrm{k}}$), we get the desired (5.2.2).

In the sequel, when there is only one particle in the S6V location process, we denote by $\mathbf{p}(x,y)$ the one particle transition probability from location x to y. Similarly $\overleftarrow{\mathbf{p}}(x,y)$ denotes the one particle transition probability from x to y for the reversed S6V location process. Clearly, $\mathbf{p}(x,y) = \overleftarrow{\mathbf{p}}(y,x)$ and

$$\mathbf{p}(x,y) = \begin{cases} b_1 & \text{if } y = x; \\ (1-b_1)(1-b_2)b_2^{y-x-1} & \text{if } y > x; \\ 0 & \text{else.} \end{cases}$$

In the sequel, we will frequently use the following elementary fact.

Lemma 5.2.2. *Consider S6V location processes*

$$\vec{x}(t) = \big(x_1(t) < \cdots < x_\ell(t)\big) \textit{ with initial state } \vec{x} = (x_1 < \cdots < x_\ell),$$
$$\vec{x'}(t) = \big(x'_1(t), \ldots, x'_{\ell-1}(t)\big) \textit{ with initial state } \vec{x'} = (x_2 < \cdots < x_\ell),$$

and $\vec{y} = (y_1 > \cdots > y_k) \in \mathbb{Y}^k$, *if* $x_1 < y_k$, *then*

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_1\}}\big] = q^{-k} b_1 \mathbb{E}^{\vec{x'}}\big[\widetilde{H}(\vec{x'}(1),\vec{y})\big]. \tag{5.2.3}$$

If $x_1 = y_k$, *then*

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_1\}}\big] = q^{-k} b_1 \mathbb{E}^{\vec{x'}}\big[\widetilde{H}(\vec{x'}(1),\vec{y'})\big] \tag{5.2.4}$$

where $\vec{y'} = (y_2 > \cdots > y_k)$

Proof. We only prove for (5.2.3), the proof of (5.2.4) is similar. Knowing $x_1(1) = y_k$, the particles at $\vec{x'} = (x_2, \ldots, x_\ell)$ update as an independent S6V model (see Figure 5.4). Note that the probability of $x_1(1) = x_1$ is b_1, furthermore, knowing $x_1(1) = x_1$, we have

$$\widetilde{H}(\vec{x}(1),\vec{y}) = q^{-k}\widetilde{H}\big((x_2(1),\ldots,x_k(1)),\vec{y}\big).$$

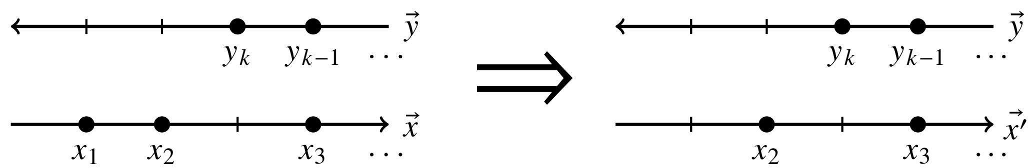


Figure 5.4: Given $x_1(1) = x_1$, in one step update procedure, there is no interaction between the particle at x_1 and the remaining particles in $\vec{x}$. This allows us to treat the remaining particles as an independent S6V location process starting from $\vec{x'} = (x_2 < \cdots < x_\ell)$. This provides the intuition of (5.2.3).

Therefore,

$$\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1), \vec{y})\mathbb{1}_{\{x_1(1)=x_1\}}\right] = q^{-k} b_1 \mathbb{E}^{\vec{x'}}\left[\widetilde{H}(\vec{x'}(1), \vec{y})\right],$$

which is the desired (5.2.3). □

Proof of Theorem 5.1.5. We denote the processes in (5.2.2) by $\vec{x}(t) = \big(x_1(t) < \cdots < x_\ell(t)\big)$, $\vec{y}(t) = (y_1(t) > \cdots > y_k(t))$ and the initial states by $\vec{x} = (x_1 < \cdots < x_\ell)$, $\vec{y} = (y_1 > \cdots > y_k)$. We split our proof into different cases depending on the relation fo $\vec{x}$ and $\vec{y}$.

Case (1): $y_k \notin \{x_1, \cdots, x_\ell\}$

Denote by s the positive integer satisfying $x_s < y_k < x_{s+1}$. We consider the S6V location processes

$$\vec{x'}(t) = \big(x'_1(t) < \cdots < x'_s(t)\big) \quad \text{with initial state } \vec{x'} = (x_1 < \cdots < x_s),$$
$$\vec{x''}(t) = \big(x''_1(t) < \cdots < x''_{\ell-s}(t)\big) \text{ with initial state } \vec{x''} = (x_{s+1} < \cdots < x_\ell),$$

and the reversed S6V location processes, see Figure 5.5 (here we do not put arrow on y' since it only has one particle).

$$y'(t) \text{ with initial state } y' = y_k,$$
$$\vec{y''}(t) = \big(y''_1(t) > \cdots > y''_{k-1}(t)\big) \text{ with initial state } \vec{y''} = (y_1 > \cdots > y_{k-1}).$$

Observing $|\vec{x'}| + |y'|$ and $|\vec{x''}| + |\vec{y''}|$ are both less than $\ell + k$, hence via (HYP$_{\ell,\mathrm{k}}$)

$$\mathbb{E}^{\vec{x'}}\big[\widetilde{H}(\vec{x'}(1), y')\big] = \mathbb{E}^{y'}\big[\widetilde{H}(\vec{x'}, y'(1))\big], \qquad \mathbb{E}^{\vec{x''}}\big[\widetilde{H}(\vec{x''}(1), \vec{y''})\big] = \mathbb{E}^{\vec{y''}}\big[\widetilde{H}(\vec{x''}, \vec{y''}(1))\big].$$

To prove (5.2.2), it suffices to show

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y})\big] = q^{-s(k-1)}\mathbb{E}^{\vec{x'}}\big[\widetilde{H}(\vec{x'}(1), y')\big], \mathbb{E}^{\vec{x''}}\big[\widetilde{H}(\vec{x''}(1), \vec{y''})\big], \tag{5.2.5}$$

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x}, \vec{y}(1))\big] = q^{-s(k-1)}\mathbb{E}^{y'}\big[\widetilde{H}(\vec{x'}, y'(1))\big]\mathbb{E}^{\vec{y''}}\big[\widetilde{H}(\vec{x''}, \vec{y''}(1))\big]. \tag{5.2.6}$$

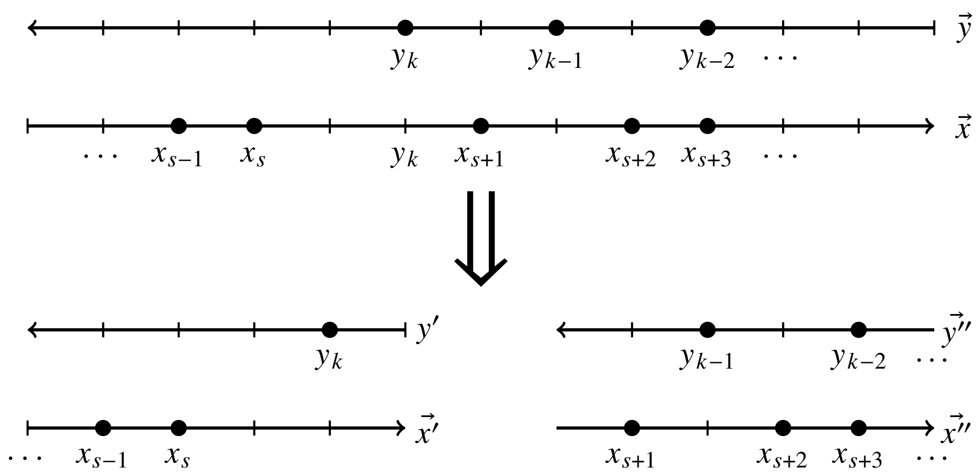


Figure 5.5: By the form of duality functional (5.1.1), we see that $H(\vec{x}(1), \vec{y})$ is zero unless $x_s(1) = y_k$. Knowing $x_s(1) = y_k$, in one step update procedure there is no interaction between the particles at $\vec{x'} = (x_1, \ldots, x_s)$ and those at $\vec{x''} = (x_{s+1}, \ldots, x_\ell)$ (since $x_s(1) < x_{s+1}$). Hence, we treat the particles at $\vec{x'} = (x_{s+1}, \ldots, x_\ell)$ as an independent S6V location process. This provides the intuition of (5.2.7).

We first show (5.2.5). Observing that by the update rule of $\vec{x}(t)$ defined in Definition 5.1.2, $H(\vec{x}(1), \vec{y}) = 0$ if $x_s(1) \neq y_k$, thus

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y})\big] = \mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y})\mathbb{1}_{\{x_s(1)=y_k\}}\big].$$

Knowing $x_s(1) = y_k$, we can treat the particles at $\vec{x''} = (x_{s+1} < \cdots < x_\ell)$ as an independent S6V model. Therefore, it is straightforward that

$$\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_s(1)=y_k\}}\right] = q^{-s(k-1)}\mathbb{E}^{\vec{x'}}\left[\widetilde{H}(\vec{x'}(1),y')\mathbb{1}_{\{x'_s(1)=y_k\}}\right]\mathbb{E}^{\vec{x''}}\left[\widetilde{H}(\vec{x''}(1),\vec{y''})\right]. \tag{5.2.7}$$

The factor $q^{-s(k-1)}$ comes from knowing $x_s(1) = y_k$,

$$\widetilde{H}(\vec{x}(1),\vec{y}) = q^{-s(k-1)}\widetilde{H}\big((x_1(1),\dots,x_s(1)),y'\big)\widetilde{H}\big((x_{s+1}(1),\dots,x_\ell(1)),\vec{y''}\big)$$

Using the fact that $\mathbb{E}^{\vec{x'}}\left[\widetilde{H}(\vec{x'}(1),y')\mathbb{1}_{\{x_s(1)=y_k\}}\right] = \mathbb{E}^{\vec{x'}}\left[\widetilde{H}(\vec{x'}(1),y')\right]$ in (5.2.7), we conclude (5.2.5). Likewise, to show (5.2.6), we have

$$\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\right] = \mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_{k-1}(1)\geq x_{s+1}\}}\right] = q^{-s(k-1)}\mathbb{E}^{y'}\left[\widetilde{H}(\vec{x'},y'(1))\right]\mathbb{E}^{\vec{y''}}\left[\widetilde{H}(\vec{x''},\vec{y''}(1))\mathbb{1}_{\{y''_{k-1}(1)\geq x_{s+1}\}}\right].$$

Using the fact that

$$\mathbb{E}^{\vec{y''}}\left[\widetilde{H}(\vec{x''},\vec{y''}(1))\mathbb{1}_{\{y''_{k-1}(1)\geq x_{s+1}\}}\right] = \mathbb{E}^{\vec{y''}}\left[\widetilde{H}(\vec{x''},\vec{y''}(1))\right],$$

we conclude (5.2.6).

Case (2): $y_k \in \{x_1,\cdots,x_\ell\}$

We divide our discussion into three sub-cases.

Case (2a): $y_k = x_1$.

In this case, let us consider the S6V location process and the reversed S6V location process

$$\vec{x'}(t) = (x'_1(t) < \cdots < x'_{\ell-1}(t)) \text{ with initial state } \vec{x'} = (x_2 < \cdots < x_\ell),$$
$$\vec{y'}(t) = (y'_1(t) > \cdots > y'_{k-1}(t)) \text{ with initial state } \vec{y'} = (y_1 > \cdots > y_{k-1}).$$

Since $|\vec{x'}|+|\vec{y'}| < \ell+k$, the induction hypothesis ($\mathrm{HYP}_{\ell,\mathrm{k}}$) gives $\mathbb{E}^{\vec{x'}}\left[\widetilde{H}(\vec{x'}(1),\vec{y'})\right] = \mathbb{E}^{\vec{y'}}\left[\widetilde{H}(\vec{x'},\vec{y'}(1))\right]$.

To prove (5.2.2), it suffices to show

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y})\big] = q^{-k} b_1 \mathbb{E}^{\vec{x}'}\big[\widetilde{H}(\vec{x}'(1), \vec{y}')\big]. \tag{5.2.8}$$

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x}, \vec{y}(1))\big] = q^{-k} b_1 \mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}', \vec{y}'(1))\big]. \tag{5.2.9}$$

We first justify (5.2.8). Since $x_1 = y_k$ and $\vec{x}(t)$ starts from $\vec{x}$, it follows from the update rule that $H(\vec{x}(1), \vec{y}) = 0$ unless $x_1(1) = x_1$. Thus,

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y})\big] = \mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y})\mathbb{1}_{\{x_1(1)=x_1\}}\big]$$

Using Lemma 5.2.2, we conclude (5.2.8). Under the same reasoning,

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x}, \vec{y}(1))\big] = \mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x}, \vec{y}(1))\mathbb{1}_{\{y_k(1)=y_k\}}\big] = q^{-k} b_1 \mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}', \vec{y}'(1))\big],$$

which concludes (5.2.9).

Case (2b): $y_k = x_2 > x_1$

The proof for this case is more involved than the previous ones. We consider the S6V location processes and reversed S6V location process (see Figure 5.6)

$$\vec{x}'(t) = (x'_1(t) < \cdots < x'_{\ell-1}(t)) \text{ with initial state } \vec{x}' = (x_2 < \cdots < x_\ell),$$

$$\vec{x}''(t) = \big(x''_1(t) < \cdots < x''_{\ell-2}(t)\big) \text{ with initial state } \vec{x}'' = \big(x_3 < \cdots < x_\ell\big),$$

$$\vec{y}'(t) = (y'_1(t) > \cdots > y'_{k-1}(t)) \text{ with initial state } \vec{y}' = (y_1 > \cdots > y_{k-1}).$$

To simplify our notation, we denote

$$L_1 = \mathbb{E}^{\vec{x}'}\big[\widetilde{H}(\vec{x}'(1), \vec{y})\big], \qquad L_2 = \mathbb{E}^{\vec{x}'}\big[\widetilde{H}(\vec{x}'(1), \vec{y}')\big], \qquad L_3 = \mathbb{E}^{\vec{x}''}\big[\widetilde{H}(\vec{x}''(1), \vec{y}')\big],$$

$$R_1 = \mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x}', \vec{y}(1))\big], \qquad R_2 = \mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}', \vec{y}'(1))\big], \qquad R_3 = \mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}'', \vec{y}'(1))\big].$$

Since $|\vec{x'}|+|\vec{y}|$, $|\vec{x'}|+|\vec{y'}|$, $|\vec{x''}|+|\vec{y'}|$ are all less than $\ell+k$, we have by induction hypothesis ($\mathrm{HYP}_{\ell,\mathrm{k}}$)

$$L_1 = R_1, \qquad L_2 = R_2, \qquad L_3 = R_3. \tag{5.2.10}$$

To prove (5.2.2), it suffices to show that

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\big] = q^{-k}L_1 + b_2^{x_2-x_1-1}\big(q^{-k}L_2 + q^{-(2k-1)}(b_1b_2 - b_1 - b_2)L_3\big), \tag{5.2.11}$$

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\big] = q^{-k}R_1 + b_2^{x_2-x_1-1}\big(q^{-k}R_2 + q^{-(2k-1)}(b_1b_2 - b_1 - b_2)R_3\big). \tag{5.2.12}$$

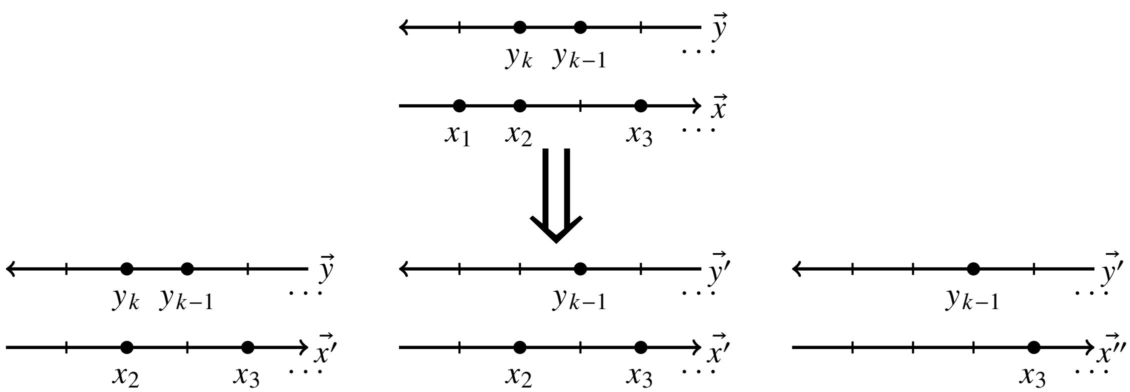


Figure 5.6: When $x_2 = y_k$, we consider three pairs of S6V models starting with less than $|\vec{x}|+|\vec{y}| = \ell + k$ number of particles, we prove (5.2.2) via expressing $\mathbb{E}\big[\widetilde{H}(\vec{x}(1),\vec{y})\big]$ (resp. $\mathbb{E}\big[\widetilde{H}(\vec{x},\vec{y}(1))\big]$) in terms of L_1, L_2, L_3 (resp. R_1, R_2, R_3) and using the induction hypothesis (5.2.10).

Let us show (5.2.11) first. Since $x_2 = y_k$, by Lemma 5.2.2,

$$L_1 = b_1 q^{-k} L_3. \tag{5.2.13}$$

Expanding the LHS expectation of (5.2.2) as following (according to the update rule, $x_1(1)$ can not exceed x_2)

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\big] = \mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)<x_2\}}\big] + \mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_2\}}\big]. \tag{5.2.14}$$

For the first term on the RHS of (5.2.14), given $x_1(1) < x_2$, the particles at $x_2, \dots, x_\ell$ update as an independent S6V model. Using the fact that

$$\mathbb{P}(x_1(1) < x_2) = \sum_{y=x_1}^{x_2-1} \mathbf{p}(x, y) = 1 - (1 - b_1) b_2^{x_2 - x_1 - 1}$$

and knowing $x_1(1) < x_2$

$$\widetilde{H}(\vec{x}(1).\vec{y}) = q^{-k} \widetilde{H}\big((x_2(1), \dots, x_\ell(1)), \vec{y}\big),$$

one has

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y}) \mathbb{1}_{\{x_1(1) < x_2\}}\big] = (1 - (1 - b_1) b_2^{x_2 - x_1 - 1}) q^{-k} L_1. \tag{5.2.15}$$

Let us compute the second term on the RHS of (5.2.14),

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y}) \mathbb{1}_{\{x_1(1) = x_2\}}\big] = \sum_{\substack{\vec{z} = (z_1 < \dots < z_\ell) \\ z_1 = x_2}} \mathbb{P}^{\vec{x}}\big(\vec{x}(1) = \vec{z}\big) \widetilde{H}(\vec{z}, \vec{y}). \tag{5.2.16}$$

For $\vec{z}' = (z_2 < \dots < z_\ell)$ and $\vec{z} = (z_1 < \dots < z_\ell)$ with $z_1 = x_2$, we have

$$\begin{gathered} \mathbb{P}^{\vec{x}}\big(\vec{x}(1) = \vec{z}\big) = b_2^{x_2 - x_1 - 1} \mathbb{P}^{\vec{x}'}\big(\vec{x}'(1) = \vec{z}'\big), \\ \widetilde{H}(\vec{z}, \vec{y}) = q^{-k} \widetilde{H}(\vec{z}', \vec{y}'). \end{gathered} \tag{5.2.17}$$

Plugging the expression on the RHS of (5.2.17) into (5.2.16) gives

$$\begin{aligned} \mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1), \vec{y}) \mathbb{1}_{\{x_1(1) = x_2\}}\big] &= q^{-k} b_2^{x_2 - x_1 - 1} \sum_{\substack{\vec{z}' = (z_2 < \dots < z_\ell) \\ z_2 > x_2}} \mathbb{P}^{\vec{x}'}\big(\vec{x}'(1) = \vec{z}'\big) \widetilde{H}(\vec{z}', \vec{y}'), \\ &= q^{-k} b_2^{x_2 - x_1 - 1} \mathbb{E}^{\vec{x}'}\big[\widetilde{H}(\vec{x}'(1), \vec{y}') \mathbb{1}_{\{x_1'(1) \neq x_2\}}\big]. \end{aligned}$$

Consequently,

$$\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_2\}}\right] = q^{-k}b_2^{x_2-x_1-1}\left(L_2 - \mathbb{E}^{\vec{x}'}\left[\widetilde{H}(\vec{x}'(1),\vec{y}')\mathbb{1}_{\{x_1'(1)=x_2\}}\right]\right). \tag{5.2.18}$$

It is straightforward that

$$\mathbb{E}^{\vec{x}'}\left[\widetilde{H}(\vec{x}'(1),\vec{y}')\mathbb{1}_{\{x_1'(1)=x_2\}}\right] = q^{-(k-1)}b_1\mathbb{E}^{\vec{x}''}\left[\widetilde{H}(\vec{x}''(1),\vec{y}')\right] = q^{-(k-1)}b_1L_3.$$

Substituting this back to (5.2.18) gives

$$\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_2\}}\right] = q^{-k}b_2^{x_2-x_1-1}L_2 - q^{-(2k-1)}b_1b_2^{x_2-x_1-1}L_3. \tag{5.2.19}$$

Note that the LHS of (5.2.15) and (5.2.19) are the first and second terms on the RHS of (5.2.14), one has

$$\begin{aligned}\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1),\vec{y})\right] &= (1-(1-b_1)b_2^{x_2-x_1-1})q^{-k}L_1 + (q^{-k}b_2^{x_2-x_1-1}L_2 - q^{-(2k-1)}b_1b_2^{x_2-x_1-1}L_3),\\ &= q^{-k}L_1 + b_2^{x_2-x_1-1}(q^{-k}L_2 + q^{-(2k-1)}(b_1b_2-b_1-b_2)L_3).\end{aligned}$$

Therefore, we conclude (5.2.11). Note that in the last line above we used the $L_1 = b_1q^{-k}L_3$ provided by (5.2.13) and the relation $b_1 = qb_2$,

We turn our attention to demonstrate (5.2.12). Since $y_k = x_2$, according to the update rule of $\vec{y}(t) = (y_1(t) > \cdots > y_k(t))$ with initial state $\vec{y}$, the only possible case for $\widetilde{H}(\vec{x},\vec{y}(1)) \neq 0$ is either $y_k(1) = x_1$ or $y_k(1) = x_2$. Therefore, we have

$$\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\right] = \mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_2\}}\right] + \mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\right]. \tag{5.2.20}$$

For the first term on the RHS of (5.2.20), we readily have

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_2\}}\big] = q^{-k}\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x'},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_2\}}\big] = q^{-k}\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x'},\vec{y}(1))\big] = q^{-k}R_1. \tag{5.2.21}$$

The first equality above is due to the fact that the condition $y_k(1) = x_2$ implies $\widetilde{H}(\vec{x},\vec{y}(1)) = q^{-k}\widetilde{H}(\vec{x'},\vec{y}(1))$.

For the second term on the RHS of (5.2.20), we expand the expectation (using the condition $x_k = y_2$)

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\big] = \mathbf{A}+\mathbf{B}, \tag{5.2.22}$$

where

$$\mathbf{A} = \sum_{\substack{\vec{w}=(w_1>\cdots>w_k)\\ w_k=x_1, w_{k-1}>x_2}} \mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w}\big)\widetilde{H}(\vec{x},\vec{w}), \qquad \mathbf{B} = \sum_{\substack{\vec{w}=(w_1>\cdots>w_k)\\ w_k=x_1, w_{k-1}=x_2}} \mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w}\big)\widetilde{H}(\vec{x},\vec{w})$$

It is easy to check that given $\vec{w} = (w_1 > \cdots > w_k)$ and $\vec{w'} = (w_1 > \cdots > w_{k-1})$ with condition $w_k = x_1$ and $w_{k-1} > x_2$ implies

$$\mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w}\big) = \overleftarrow{\mathbf{p}}(x_2,x_1)\mathbb{P}^{\vec{y'}}\big(\vec{y'}(1)=\vec{w'}\big), \qquad \widetilde{H}(\vec{x},\vec{w}) = q^{-(2k-1)}\widetilde{H}(\vec{x''},\vec{w'}).$$

Therefore,

$$\begin{aligned}\mathbf{A} = \mathbb{P}\big(\vec{y}(1)=\vec{w}\big)\widetilde{H}(\vec{x},\vec{w}) &= q^{-(2k-1)}\overleftarrow{\mathbf{p}}(x_2,x_1)\sum_{\substack{\vec{w'}=(w_1>\cdots>w_{k-1})\\ w_{k-1}>x_2}} \mathbb{P}^{\vec{y'}}\big(\vec{y'}(1)=\vec{w'}\big)\widetilde{H}(\vec{x''},\vec{w'}),\\ &= q^{-(2k-1)}(1-b_1)(1-b_2)b_2^{x_2-x_1-1}\mathbb{E}^{\vec{y'}}\big[\widetilde{H}(\vec{x''},\vec{y'}(1))\big],\end{aligned}$$

Using $\mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}'', \vec{y}'(1))\big] = R_3$,

$$\mathbf{A} = q^{-(2k-1)}(1-b_1)(1-b_2)b_2^{x_2-x_1-1}R_3. \tag{5.2.23}$$

Similarly, given $\vec{w} = (w_1 > \cdots > w_k)$ and $\vec{w}' = (w_1 > \cdots > w_{k-1})$ with $w_k = x_1$ and $w_{k-1} = x_2$, we have

$$\mathbb{P}^{\vec{y}}\big(\vec{y}(1) = \vec{w}\big) = b_2^{x_2-x_1-1}\mathbb{P}^{\vec{y}'}\big(\vec{y}'(1) = \vec{w}'\big), \qquad \widetilde{H}(\vec{x}, \vec{w}) = q^{-k}\widetilde{H}(\vec{x}', \vec{w}').$$

Thus,

$$\begin{aligned}\mathbf{B} = \sum_{\substack{\vec{w}=(w_1>\cdots>w_k)\\ w_k=x_1, w_{k-1}=x_2}} \mathbb{P}^{\vec{y}}\big(\vec{y}(1) = \vec{w}\big)\widetilde{H}(\vec{x}, \vec{w}) &= q^{-k}b_2^{x_2-x_1-1} \sum_{\substack{\vec{w}'=(w_1>\cdots>w_{k-1})\\ w_{k-1}=x_2}} \mathbb{P}^{\vec{y}'}\big(\vec{y}'(1) = \vec{w}'\big)\widetilde{H}(\vec{x}', \vec{w}'),\\ &= q^{-k}b_2^{x_2-x_1-1}\mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}', \vec{y}'(1))\mathbb{1}_{\{y'_{k-1}(1)=x_2\}}\big].\end{aligned}$$

Consequently, one has

$$\mathbf{B} = q^{-k}b_2^{x_2-x_1-1}\bigg(R_2 - \mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}', \vec{y}'(1))\mathbb{1}_{\{y'_{k-1}(1)>x_2\}}\big]\bigg). \tag{5.2.24}$$

Under event $\{y'_{k-1}(1) > x_2\}$, we have $\widetilde{H}(\vec{x}', \vec{y}'(1)) = q^{-(k-1)}\widetilde{H}(\vec{x}'', \vec{y}'(1))$ and hence

$$\mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}', \vec{y}'(1))\mathbb{1}_{\{y'_{k-1}(1)>x_2\}}\big] = q^{-(k-1)}\mathbb{E}^{\vec{y}'}\big[\widetilde{H}(\vec{x}'', \vec{y}'(1))\mathbb{1}_{\{y'_{k-1}(1)>x_2\}}\big] = q^{-(k-1)}R_3.$$

We obtain from (5.2.24) that

$$\mathbf{B} = q^{-k}b_2^{x_2-x_1-1}(R_2 - q^{-(k-1)}R_3). \tag{5.2.25}$$

Recall (5.2.22) that $\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\big] = \mathbf{A}+\mathbf{B}$, using (5.2.23) and (5.2.25), we get

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\big] = q^{-(2k-1)}(1-b_1)(1-b_2)b_2^{x_2-x_1-1}R_3 + q^{-k}b_2^{x_2-x_1-1}(R_2 - q^{-(k-1)}R_3). \tag{5.2.26}$$

Note that the LHS of (5.2.21) and (5.2.26) are the first and second term on the RHS of (5.2.20), hence

$$\begin{aligned}\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\big] &= q^{-k}R_1 + q^{-(2k-1)}(1-b_1)(1-b_2)b_2^{x_2-x_1-1}R_3 + q^{-k}b_2^{x_2-x_1-1}(R_2 - q^{-(k-1)}R_3),\\ &= q^{-k}R_1 + b_2^{x_2-x_1-1}(q^{-k}R_2 + q^{-(2k-1)}(b_1b_2 - b_1 - b_2)R_3).\end{aligned}$$

We have proved the desired (5.2.12), thus concluding (5.2.2) for the case $x_2 = y_k$.

It only remains to prove (5.2.2) for the following case:

Case (2c): $y_k > x_2 > x_1$.

The computation for this case is similar to Case (2b). Let us consider the S6V location processes

$$\begin{aligned}&\vec{x}'(t) = (x'_1(t) < \cdots < x'_{\ell-1}(t)) \text{ with initial state } \vec{x}' = (x_2 < \cdots < x_\ell),\\ &\vec{x}''(t) = \big(x''_1(t) < \cdots < x''_{\ell-2}(t)\big) \text{ with initial state } \vec{x}'' = \big(x_3 < \cdots < x_\ell\big),\end{aligned}$$

and denote

$$\begin{aligned}L_1 &= \mathbb{E}^{\vec{x}'}\big[\widetilde{H}(\vec{x}'(1),\vec{y})\big], \qquad L_2 = \mathbb{E}^{\vec{x}''}\big[\widetilde{H}(\vec{x}''(1),\vec{y})\big],\\ R_1 &= \mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x}',\vec{y}(1))\big], \qquad R_2 = \mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x}'',\vec{y}(1))\big].\end{aligned}$$

By ($\mathrm{HYP}_{\ell,\mathrm{k}}$), we have

$$L_1 = R_1, \qquad L_2 = R_2. \tag{5.2.27}$$

To conclude (5.2.2), it suffices to show that

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\big] = q^{-k}L_1 + q^{-(k-1)}b_2^{x_2-x_1}\big(L_1 - q^{-k}L_2\big), \tag{5.2.28}$$

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\big] = q^{-k}R_1 + q^{-(k-1)}b_2^{x_2-x_1}\big(R_1 - q^{-k}R_2\big). \tag{5.2.29}$$

To prove (5.2.28), we first write

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\big] = \mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)<x_2\}}\big] + \mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_2\}}\big]. \tag{5.2.30}$$

Similar as (5.2.15), the first term on the RHS of (5.2.30) can be expressed as

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)<x_2\}}\big] = q^{-k}(1-(1-b_1)b_2^{x_2-x_1-1})L_1, \tag{5.2.31}$$

while the second term on the RHS of (5.2.30) equals

$$\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_2\}}\big] = \sum_{\substack{\vec{z}=(z_1<\cdots<z_\ell)\\ z_1=x_2}} \mathbb{P}^{\vec{x}}\big(\vec{x}(1)=\vec{z}\big)\widetilde{H}(\vec{z},\vec{y}). \tag{5.2.32}$$

Given $\vec{z} = \big(z_1 < \cdots < z_\ell\big)$ and $\vec{z'} = \big(z_2 < \cdots < z_\ell\big)$ with $z_1 = x_2$, we have

$$\mathbb{P}^{\vec{x}}\big(\vec{x}(1)=\vec{z}\big) = b_2^{x_2-x_1-1}\mathbb{P}^{\vec{x'}}\big(\vec{x'}(1)=\vec{z'}\big), \qquad \widetilde{H}(\vec{z},\vec{y}) = q^{-k}\widetilde{H}(\vec{z'},\vec{y}).$$

Substituting back to (5.2.32) yields

$$\begin{aligned}\mathbb{E}^{\vec{x}}\big[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_2\}}\big] &= q^{-k}b_2^{x_2-x_1-1}\sum_{\substack{\vec{z'}=(z_2<\cdots<z_\ell)\\ z_2>x_2}} \mathbb{P}^{\vec{x'}}\big(\vec{x'}(1)=\vec{z'}\big)\widetilde{H}(\vec{z'},\vec{y}),\\ &= q^{-k}b_2^{x_2-x_1-1}\mathbb{E}^{\vec{x'}}\big[\widetilde{H}(\vec{x'}(1),\vec{y})\mathbb{1}_{\{x'_1(1)>x_2\}}\big].\end{aligned}$$

Consequently,

$$\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1),\vec{y})\right] = q^{-k} b_2^{x_2-x_1-1}\left(L_1 - \mathbb{E}^{\vec{x'}}\left[\widetilde{H}(\vec{x'}(1),\vec{y})\mathbb{1}_{\{x_1'(1)=x_2\}}\right]\right). \tag{5.2.33}$$

By Lemma 5.2.2, we have

$$\mathbb{E}^{\vec{x'}}\left[\widetilde{H}(\vec{x'}(1),\vec{y})\mathbb{1}_{\{x_1'(1)=x_2\}}\right] = q^{-k} b_1 \mathbb{E}^{\vec{x''}}\left[\widetilde{H}(\vec{x''}(1),\vec{y})\right] = q^{-k} b_1 L_2.$$

Using (5.2.33),

$$\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1),\vec{y})\mathbb{1}_{\{x_1(1)=x_2\}}\right] = q^{-k} b_2^{x_2-x_1-1}\left(L_1 - q^{-k} b_1 L_2\right). \tag{5.2.34}$$

Plugging (5.2.31) and (5.2.34) into the RHS of (5.2.30) yields

$$\begin{aligned}\mathbb{E}^{\vec{x}}\left[\widetilde{H}(\vec{x}(1),\vec{y})\right] &= q^{-k}(1-(1-b_1)b_2^{x_2-x_1-1})L_1 + q^{-k} b_2^{x_2-x_1-1}\left(L_1 - q^{-k} b_1 L_2\right),\\ &= q^{-k} L_1 + q^{-(k-1)} b_2^{x_2-x_1}\left(L_1 - q^{-k} L_2\right),\end{aligned}$$

We conclude (5.2.28). Here, in the last line above we used again the relation $b_1 = qb_2$.

We turn to demonstrate (5.2.29). As $y_k(1) < x_1$ implies $\widetilde{H}(\vec{x},\vec{y}(1)) = 0$, we have

$$\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\right] = \mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)>x_1\}}\right] + \mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\right].$$

Since $y_k(1) > x_1$ implies $\widetilde{H}(\vec{x},\vec{y}(1)) = q^{-k}\widetilde{H}(\vec{x'},\vec{y}(1))$,

$$\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)>x_1\}}\right] = q^{-k}\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x'},\vec{y}(1))\mathbb{1}_{\{y_k(1)>x_1\}}\right] = q^{-k}\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x'},\vec{y}(1))\right] = q^{-k} R_1.$$

Consequently,

$$\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\right] = q^{-k} R_1 + \mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\right]. \tag{5.2.35}$$

For the second term on the RHS of (5.2.35), we have

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\big] = \sum_{\substack{\vec{w}=(w_1>\cdots>w_k)\\ w_k=x_1}} \mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w}\big)\widetilde{H}(\vec{x},\vec{w}) = \sum_{\substack{\vec{w}=(w_1>\cdots>w_k)\\ w_k=x_1, w_{k-1}\geq y_k}} \mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w}\big)\widetilde{H}(\vec{x},\vec{w}). \tag{5.2.36}$$

Since $y_k > x_2$, given $\vec{w} = (w_1 > \cdots > w_{k-1} > x_1)$ and $\vec{w'} = (w_1 > \cdots > w_{k-1} > x_2)$ satisfying $w_{k-1} \geq y_k$, we have

$$\mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w}\big) = b_2^{x_2-x_1}\mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w'}\big), \qquad \widetilde{H}(\vec{x},\vec{w}) = q^{-(k-1)}\widetilde{H}(\vec{x'},\vec{w'}).$$

Substituting back to the RHS of (5.2.36) yields

$$\begin{aligned}\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\big] &= q^{-(k-1)}b_2^{x_2-x_1} \sum_{\substack{\vec{w'}=(w_1>\cdots>w_k)\\ w_k=x_2, w_{k-1}\geq y_k}} \mathbb{P}^{\vec{y}}\big(\vec{y}(1)=\vec{w'}\big)\widetilde{H}(\vec{x'},\vec{w'}),\\ &= q^{-(k-1)}b_2^{x_2-x_1}\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x'},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_2\}}\big].\end{aligned}$$

Consequently,

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\big] = q^{-(k-1)}b_2^{x_2-x_1}\big(R_1 - \mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x'},\vec{y}(1))\mathbb{1}_{\{y_k(1)>x_2\}}\big]\big). \tag{5.2.37}$$

Under event $\{y_k(1) > x_2\}$, we have $\widetilde{H}(\vec{x'},\vec{y}(1)) = q^{-k}\widetilde{H}(\vec{x''},\vec{y}(1))$ and accordingly

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x'},\vec{y}(1))\mathbb{1}_{\{y_k(1)>x_2\}}\big] = q^{-k}\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x''},\vec{y}(1)\mathbb{1}_{\{y_k(1)>x_2\}})\big] = q^{-k}\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x''},\vec{y}(1)\big] = q^{-k}R_2.$$

Therefore, we have by (5.2.37)

$$\mathbb{E}^{\vec{y}}\big[\widetilde{H}(\vec{x},\vec{y}(1))\mathbb{1}_{\{y_k(1)=x_1\}}\big] = q^{-(k-1)}b_2^{x_2-x_1}R_1 - q^{-(2k-1)}b_2^{x_2-x_1}R_2. \tag{5.2.38}$$

Substituting (5.2.38) back to (5.2.35) entails

$$\begin{aligned}\mathbb{E}^{\vec{y}}\left[\widetilde{H}(\vec{x},\vec{y}(1))\right] &= q^{-k}R_1 + q^{-(k-1)}b_2^{x_2-x_1}R_1 - q^{-(2k-1)}b_2^{x_2-x_1}R_2,\\ &= q^{-k}R_1 + q^{-(k-1)}b_2^{x_2-x_1}\left(R_1 - q^{-k}R_2\right).\end{aligned}$$

which concludes the desired (5.2.29).

As all the possible cases for $x \in \mathbb{X}^{\ell}$ and $\vec{y} \in \mathbb{Y}^{k}$ were discussed, we justify (5.2.2) and conclude Theorem 5.1.5. □

Chapter 6: KPZ equation limit of stochastic higher spin six vertex model

Chapter Abstract: We consider the stochastic higher spin six vertex (SHS6V) model introduced in [CP16] with general integer spin parameters I,J . Starting from near stationary initial condition, we prove that the SHS6V model converges to the KPZ equation under weakly asymmetric scaling.

This chapter is published at [Lin20a].

6.1 Introduction

6.1.1 KPZ equation and weak KPZ universality

The KPZ equation is the following non-linear stochastic partial differential equation (SPDE) introduced in the seminal work [KPZ86], which describes the random evolution of an interface that has the property of relaxation and lateral growth

$$\mathcal{H}(t,x) = \frac{\delta}{2}\partial_x^2 \mathcal{H}(t,x) + \frac{\kappa}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \sqrt{D}\xi(t,x). \tag{6.1.1}$$

Here $\xi(t,x)$ is the *space time white noise*, which could be formally understood as a Gaussian field with covariance function $\mathbb{E}\big[\xi(t,x)\xi(s,y)\big] = \delta(t-s)\delta(x-y)$, where δ is the Dirac delta function.

Care is needed to make sense of (6.1.1) due to the nonlinearity $(\partial_x \mathcal{H}(t,x))^2$. The Hopf-Cole solution to the KPZ equation is defined by

$$\mathcal{H}(t,x) = \frac{\delta}{\kappa}\log \mathcal{Z}(t,x), \tag{6.1.2}$$

where $\mathcal{Z}(t,x)$ is the *mild solution* of the SHE

$$\mathcal{Z}(t,x) = \frac{\delta}{2}\partial_x^2 \mathcal{Z}(t,x) + \frac{\kappa\sqrt{D}}{\delta}\mathcal{Z}(t,x)\xi(t,x).$$

So long as $\mathcal{Z}(0,x)$ is (almost surely) positive, [Mue91] proved that $\mathcal{Z}(t,x)$ remains positive for all $t > 0$ and x. This justifies the well-definedness of (6.1.2). Other equivalent definitions of the solution are given by regularity structure [Hai14], paracontrolled distribution [GP17] or the notion of energy solution [GJ14, GP18].

It is well-known that there is no non-trivial scaling under which the KPZ equation is invariant in law. More precisely, if we define $\mathcal{H}_\epsilon(t,x) = \epsilon^z \mathcal{H}(\epsilon^{-b}t, \epsilon^{-1}x)$, using the scaling of space-time white noise $\xi(\epsilon^{-b}t, \epsilon^{-1}x) = \epsilon^{\frac{b+1}{2}}\xi(t,x)$ (in law), then

$$\partial_t \mathcal{H}_\epsilon(t,x) = \frac{\delta}{2}\epsilon^{2-b}\partial_x^2 \mathcal{H}_\epsilon(t,x) + \frac{\kappa}{2}\epsilon^{-z+2-b}(\partial_x \mathcal{H}_\epsilon(t,x))^2 + \epsilon^{z+\frac{1}{2}-\frac{b}{2}}\sqrt{D}\xi(t,x). \tag{6.1.3}$$

It is clear that there is no b, z such that the coefficients in the above equation match with those in (6.1.1). However, if we simultaneously scale some of the parameters δ, κ, D, it is possible that the KPZ equation remains unchanged: such scaling is called *weak scaling*. It is thus natural to believe that the KPZ equation is the weak scaling limit of microscopic models with similar properties such as relaxation and lateral growth. Roughly speaking, this is the *weak universality of the KPZ equation*, see [Cor12, Qua11] for an extensive survey. We emphasize that the weak universality of the KPZ equation should be distinguished from *KPZ universality*, which says that without tuning of the parameter of the model, the microscopic system converges to a universal limit called *KPZ fixed point* under $[1:2:3]$ scaling, see [MQR16, DOV18, BL19] for some recent progress and breakthroughs in identifying the KPZ fixed point.

The weak universality of the KPZ equation has been verified for a number of interacting particle systems. The first result was given in the work of [BG97], for Asymmetric Simple Exclusion Process (ASEP). For more results of the weak universality of KPZ equation, see Section 1.5.3 of

[CGST20] for a brief review.

Recently [CGST20, Theorem 1.1] proved that under weak asymmetric scaling (which corresponds to taking $b = 2, z = \frac{1}{2}$ and $\kappa \to \sqrt{\epsilon}\kappa$ in (6.1.3)), the stochastic six vertex model converges to the KPZ equation. In this paper, we consider stochastic higher spin six vertex model (SHS6V) model introduced in [CP16][1]. We prove that under similar weak asymmetric scaling, the SHS6V model converges to the KPZ equation. This extends the result of [CGST20, Theorem 1.1] to the full generality. We like to emphasize that there are some significant new complications in our case compared with [CGST20], see Section 6.1.4 for discussion.

Before ending this section, we remark that there might be other SPDEs (besides the KPZ equation) arising from the vertex model. For instance, it was shown in [BG18, ST19] that under a different scaling, the stochastic six vertex model converges to the solution of the stochastic telegraph equation. It is interesting to ask whether the SHS6V model converges to other SPDEs, this question is left for future work.

6.1.2 The SHS6V model

The SHS6V model introduced in [CP16] (also see [Bor17]) belongs to the family of vertex models which themselves are examples of quantum integrable systems. In general, the R-matrix (which can be thought of as the weights associated to the vertex) are not stochastic. [GS92, BCG16] studied the stochastic six vertex model, which is a stochastic version of the six vertex model introduced by [Pau35]. The authors of [CP16] worked with the L-matrices, which is a stochastic version of the R-matrices[2] and they defined the SHS6V model. The stochasticity allows us to define the vertex model on the entire line as an interacting particle system which follows sequential Markov update rule. Moreover, the L-matrices in [CP16] satisfy the Yang-Baxter equation which implies the integrability of the model. In particular, the transfer matrices are diagonalizable by a complete set of Bethe ansatz eigenfunctions [BCPS15, CP16]. The model also

[1] The SHS6V model has vertical and horizontal spin parameters $I, J \in \mathbb{Z}_{\geq 1}$. The stochastic six vertex model is a degeneration of it by taking $I = J = 1$.

[2] See [CP16, Remark 2.2] for more discussion of the relation between L-matrices and R-matrices.

enjoys Markov duality. The stochastic R-matrices of the SHS6V model have four parameters, by specifying which the SHS6V model degenerates to known integrable systems such as stochastic six vertex model, ASEP, q-Hahn TASEP, q-TASEP. Indeed, it is on top of a hierarchy of KPZ class integrable probabilistic systems. Recent studies of the SHS6V model and its dynamical version include [OP17, Agg18b, Bor18, BP18, IMS20].

Let us recall the definition of the SHS6V model from [CP16]. Fix $I, J \in \mathbb{Z}_{\geq 1}, \alpha, q \in \mathbb{R}$, we define the L-matrix $L_\alpha^{(J)} : \mathbb{Z}_{\geq 0}^4 \to \mathbb{R}$ via

$$\begin{aligned} L_\alpha^{(J)}(i_1, j_1; i_2, j_2) =& \mathbf{1}_{\{i_1+j_1=i_2+j_2\}} q^{\frac{2j_1-j_1^2}{4} - \frac{2j_2-j_2^2}{4} + \frac{i_2^2+i_1^2}{4} + \frac{i_2(j_2-1)+i_1 j_1}{2}} \\ & \times \frac{\nu^{j_1-i_2} \alpha^{j_2-j_1+i_2} (-\alpha\nu^{-1}; q)_{j_2-i_1}}{(q;q)_{i_2} (-\alpha; q)_{i_2+j_2} (q^{J+1-j_1}; q)_{j_1-j_2}} {}_4\overline{\phi}_3 \left(\begin{matrix} q^{-i_2}; q^{-i_1}, -\alpha q^J, -q\nu\alpha^{-1} \\ \nu, q^{1+j_2-i_1}, q^{J+1-i_2-j_2} \end{matrix} \middle| q, q \right). \end{aligned} \tag{6.1.4}$$

Here, $\nu = q^{-I}$ and ${}_4\overline{\phi}_3$ is the regularized terminating basic hyper-geometric series defined by

$${}_{r+1}\overline{\phi}_r \left(\begin{matrix} q^{-n}, a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{matrix} \middle| q, z \right) = \sum_{k=0}^{n} z^k \frac{(q^{-n}; q)_k}{(q;q)_k} \prod_{i=1}^{r} (a_i; q)_k (b_i q^k; q)_{n-k},$$

where we recall the q-Pochhammer symbols $(a, q)_n$ (here n is allowed to be negative) are defined by

$$(a;q)_n := \begin{cases} \prod_{i=1}^{n} (1 - aq^{i-1}), & n > 0, \\ 1, & n = 0, \\ \prod_{k=0}^{-n-1} (1 - aq^{n+k})^{-1}, & n < 0. \end{cases}$$

We view $L_\alpha^{(J)}$ as a matrix with row indexed by $(i_1, j_1) \in \mathbb{Z}_{\geq 0}^2$ and column indexed by $(i_2, j_2) \in \mathbb{Z}_{\geq 0}^2$. Note that the L-matrix in (6.1.4) actually depends on four generic parameters α, q, I, J, we suppress the dependence on q, I in the notation of $L_\alpha^{(J)}$ to simplify the notation.

It is straightforward by definition that for $(i_1, j_1) \in \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$ (using $\nu = q^{-I}$)

$$L_\alpha^{(J)}(i_1, j_1; i_2, j_2) = 0, \qquad \text{for all } (i_2, j_2) \in \mathbb{Z}_{\geq 0}^2 \backslash \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\},$$

which means there is no way to transition out of $\{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$ from itself. Therefore, in the following we restrict ourselves to the block with $(i_1, j_1), (i_2, j_2) \in \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$.

When $J = 1$, by straightforward calculation, the L-matrix defined above simplifies to

$$\begin{aligned} L_\alpha^{(1)}(m, 0; m, 0) &= \frac{1 + \alpha q^m}{1 + \alpha}, \qquad & L_\alpha^{(1)}(m, 0; m-1, 1) &= \frac{\alpha(1 - q^m)}{1 + \alpha}, \\ L_\alpha^{(1)}(m, 1; m+1, 0) &= \frac{1 - \nu q^m}{1 + \alpha}, \qquad & L_\alpha^{(1)}(m, 1; m, 1) &= \frac{\alpha + \nu q^m}{1 + \alpha}. \end{aligned} \tag{6.1.5}$$

For the history of the expression (6.1.4), we remark that more intricate expressions for a quantity similar to the $L_\alpha^{(J)}$ had been known in the context of quantum integrable systems since the work of [KR87]. Relatively compact expressions of $L_\alpha^{(J)}$ became available only in more recent times after the work of [Man14]. [CP16] also provides a probabilistic proof for this expression.

From our perspective, we will think of $L_\alpha^{(J)}(i_1, j_1; i_2, j_2)$ as the weight associated to a vertex configuration with i_1 input lines from south, j_1 input lines from west, i_2 output lines to the north and j_2 output lines to the east see Figure 6.1. Since we have restricted $L_\alpha^{(J)}(i_1, j_1; i_2, j_2)$ to $(i_1, j_1), (i_2, j_2) \in \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$, we can have at most I vertical lines and J horizontal lines in the vertex configuration. Note that due to the indicator in (6.1.4), all non-zero vertex weights $L_\alpha^{(J)}(i_1, j_1; i_2, j_2)$ satisfy $i_1 + j_1 = i_2 + j_2$, a property that we consider as conservation of lines.

In this paper, we always assume the following condition.

Condition 6.1.1. *We take $q > 1, \alpha < -q^{-(I+J-1)}$ and as we noted before, $\nu = q^{-I}$.*

It follows from [CP16] that under Condition 6.1.1, $L_\alpha^{(J)}$ is a stochastic matrix on $\{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$. In other words, for any fixed $(i_1, j_1) \in \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$, $L_\alpha^{(J)}(i_1, j_1; \cdot, \cdot)$

defines a probability measure on $\{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$. Although in this paper we will not investigate the range of parameters out of Condition 6.1.1, it is worth remarking that there are other choices of parameters which make $L_\alpha^{(J)}$ stochastic, a few of them are provided in [CP16, Proposition 2.3].

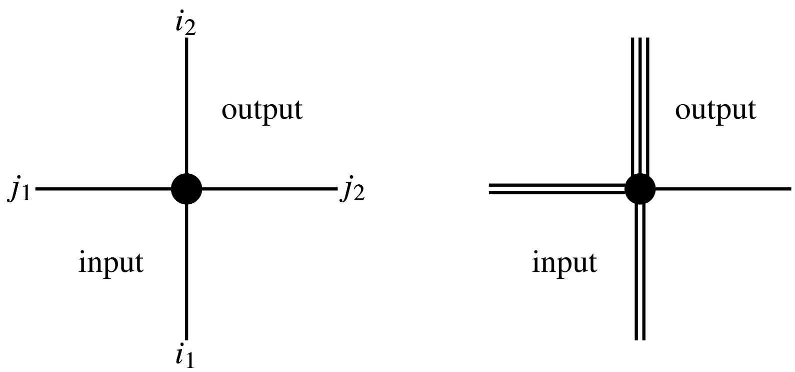


Figure 6.1: **Left:** The vertex configuration labeled by four tuples of integer $(i_1, j_1; i_2, j_2) \in \mathbb{Z}^4_{\geq 0}$ (from bottom and then in the clockwise order) has weight $L_\alpha^{(J)}(i_1, j_1; i_2, j_2)$, which takes i_1 vertical input lines and j_1 horizontal input lines, and produce i_2 vertical output lines and j_2 horizontal output lines. **Right:** The representation of the vertex configuration $(i_1, j_1; i_2, j_2) = (2, 2; 3, 1)$ in terms of lines.

There are several equivalent ways to define the SHS6V model. In this paper, we view the SHS6V model as a one-dimensional interacting particle system, which follows a sequential update rule. We proceed to give a precise definition of it. Denote by the space of left-finite particle configuration

$$\mathbb{G} = \{\vec{g} = (\ldots, g_{-1}, g_0, g_1 \ldots) : \text{ all } g_i \in \{0, 1, \ldots, I\} \text{ and there exists } x \in \mathbb{Z} \text{ such that } g_i = 0 \text{ for all } i < x.\}, \tag{6.1.6}$$

where g_x should be understood as the number of particles at position x. We define a discrete time Markov process $\vec{g}(t) = (g_x(t))_{x\in\mathbb{Z}} \in \mathbb{G}$ as follows.

Definition 6.1.2 (left-finite fused SHS6V model)**.** For any state $\vec{g} = (g_x)_{x\in\mathbb{Z}} \in \mathbb{G}$, we specify the update rule from state $\vec{g}$ to $\vec{g}'$ as follows: Assume the leftmost particle in the configuration $\vec{g}$ is at x (i.e. $g_x > 0$ and $g_z = 0$ for all $z < x$). Starting from x, we update g_x to g'_x by setting $h_x = 0$ and randomly choosing g'_x according to the probability $L_\alpha^{(J)}(g_x, h_x = 0; g'_x, h_{x+1})$ where

$h_{x+1} := g_x - g'_x$. Proceeding sequentially, we update g_{x+1} to g'_{x+1} according to the probability $L_\alpha^{(J)}(g_{x+1}, h_{x+1}; g'_{x+1}, h_{x+2})$ where $h_{x+2} := g_{x+1} + h_{x+1} - g'_{x+1}$. Continuing for $g_{x+2}, g_{x+3}, \dots$, we have defined the update rule from $\vec{g}$ to $\vec{g}' = (g'_x)_{x\in\mathbb{Z}}$, see Figure 6.2 for visualization of the update procedure. We call the discrete **time-homogeneous** Markov process $\vec{g}(t) \in \mathbb{G}$ with the update rule defined above **the left-finite fused SHS6V model**.[3]

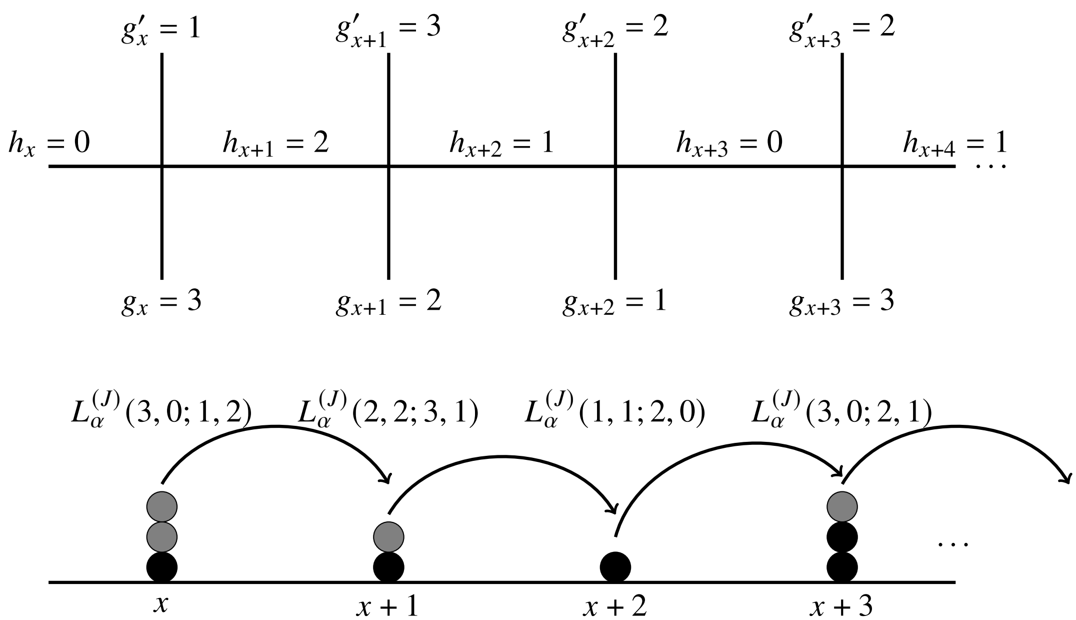


Figure 6.2: The visualization of the sequential update rule for the left-finite fused SHS6V model in Definition 6.1.2. Assuming x is the location of the leftmost particle, we update sequentially for positions $x, x+1, x+2, \dots$ according to the stochastic matrix $L_\alpha^{(J)}$, the gray particles in the picture above will move one step to the right.

For $s \in \mathbb{Z}_{\geq 0}$, we define $\mathrm{mod}_J(s) := s - J\lfloor s/J \rfloor$. For instance,

$$\big(\mathrm{mod}_J(0), \mathrm{mod}_J(1), \dots, \mathrm{mod}_J(J-1), \mathrm{mod}_J(J), \mathrm{mod}_J(J+1), \dots\big) = \big(0, 1, \dots, J-1, 0, 1, \dots\big).$$

We further define $\alpha(t) = \alpha q^{\mathrm{mod}_J(t)}$ for $t \in \mathbb{Z}_{\geq 0}$.

Definition 6.1.3 (left-finite unfused SHS6V model)**.** For all state $\vec{\eta} \in \mathbb{G}$, we specify the update rule at time t from state $\vec{\eta}$ to $\vec{\eta}' \in \mathbb{G}$ as follows. Assume the leftmost particle in the configuration $\vec{\eta}$ is

[3]Note that in Definition 6.1.2, although the update from $\vec{g}$ to $\vec{g}'$ may never stop as it goes to the right, the process is well-defined since we only care about the sigma algebra generated by $(g_x)_{x\leq z, x\in\mathbb{Z}}$ for all $z \in \mathbb{Z}$.

at x. Starting from x, we update η_x to η'_x by setting $h_x = 0$ and randomly choosing η'_x according to the probability $L^{(1)}_{\alpha(t)}(\eta_x, h_x; \eta'_x, h_{x+1})$ where $h_{x+1} := \eta_x + h_x - \eta'_x$. Proceeding sequentially, we update η_{x+1} to η_{x+1} according to the probability $L^{(1)}_{\alpha(t)}(\eta_{x+1}, h_{x+1}; \eta'_{x+1}, h_{x+2})$ where $h_{x+2} := \eta_{x+1} + h_{x+1} - \eta'_{x+1}$. Continuing for $\eta_{x+2}, \eta_{x+3}, \dots$, we have defined the update rule from $\vec{\eta}$ to $\vec{\eta}' = (\eta'_x)_{x\in\mathbb{Z}}$. We call the discrete **time-inhomogeneous** Markov process $\vec{\eta}(t) \in \mathbb{G}$ with the update rule defined above **the left-finite unfused SHS6V model**.

Remark 6.1.4. It is straightforward to check that under Condition 6.1.1, for all $t \in \mathbb{Z}_{\geq 0}$, $L^{(1)}_{\alpha(t)}$ in (6.1.5) is a stochastic matrix which transfers $\{0, 1, \dots, I\} \times \{0, 1\}$ to itself.

In this paper, as a notational convention, we always use $\vec{g}(t)$ to denote the fused SHS6V model and $\vec{\eta}(t)$ to denote the unfused one. The connection between them is specified in the following proposition.

Proposition 6.1.5 ([CP16], Theorem 3.15)**.** *Consider the left-finite fused SHS6V model $\vec{g}(t)$ and the left-finite unfused SHS6V model $\vec{\eta}(t)$. If $\vec{g}(0) = \vec{\eta}(0)$ in law, then*

$$(\vec{g}(t), t \geq 0) = (\vec{\eta}(Jt), t \geq 0) \quad \textit{in law} .$$

By Proposition 6.1.5, we can construct the SHS6V model with higher horizontal spin ($J \in \mathbb{Z}_{\geq 1}$) from those with horizontal spin $J = 1$. This procedure is called *fusion*, which goes back to the work of [KR87]. Thanks to Proposition 6.1.5, for any left-finite SHS6V model $\vec{g}(t)$, we can couple it with a left-finite unfused SHS6V model $\vec{\eta}(t)$ so that $\vec{g}(t) = \vec{\eta}(Jt)$. We will extend the definition of unfused SHS6V model $\vec{\eta}(t)$ in Lemma 6.2.1 so that it takes value in a larger space of bi-infinite particle configuration $\{0, 1, \dots, I\}^{\mathbb{Z}}$ (thus extend as well the definition of the fused SHS6V model using the relation $\vec{g}(t) = \vec{\eta}(Jt)$).

For the particle configuration $\vec{g} \in \mathbb{G}$, define

$$N_x(\vec{g}) = \sum_{y\leq x} g_y. \tag{6.1.7}$$

For the left-finite unfused SHS6V model $\vec{\eta}(t) \in \mathbb{G}$, we define the *unfused height function* as

$$N^{\mathsf{uf}}(t, x) = N_x(\vec{\eta}(t)) - N_0(\vec{\eta}(0)). \tag{6.1.8}$$

Note that in the notation of unfused height function, we suppress the underlying process $\vec{\eta}(t)$. Similarly, we define the *fused height function* $N^{\mathsf{f}}(t, x)$ for the left-finite fused SHS6V model $\vec{g}(t) \in \mathbb{G}$ as

$$N^{\mathsf{f}}(t, x) = N_x(\vec{g}(t)) - N_0(\vec{g}(0)).$$

Since $\vec{g}(t) = \vec{\eta}(Jt)$, certainly one has for all $t \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{Z}$, $N^{\mathsf{f}}(t, x) = N^{\mathsf{uf}}(Jt, x)$.

We will state our result for the fused height function $N^{\mathsf{f}}(t, x)$ though we will mainly work with the unfused height function $N^{\mathsf{uf}}(t, x)$ in our proof. In the future, the notation of $N^{\mathsf{uf}}(t, x)$ will often be shortened to $N(t, x)$.

Having defined $N^{\mathsf{f}}(t, x)$ (respectively, $N^{\mathsf{uf}}(t, x)$) on the lattice, we linearly interpolate it first in space variable x then in time variable t, which makes $N^{\mathsf{f}}(t, x)$ (respectively, $N^{\mathsf{uf}}(t, x)$) a $C([0, \infty), C(\mathbb{R}))$-valued process. For construction of height functions of the bi-infinite version of the fused or unfused SHS6V model, see Lemma 6.2.1.

6.1.3 Result

The main result of our paper shows that the fluctuation of the fused height function $N^{\mathsf{f}}(t, x)$ converges weakly to the solution of the KPZ equation. Fix $\rho \in (0, I)$, define

$$\lambda = \frac{1 + \alpha - q^{\rho}(\alpha + \nu)}{1 + \alpha q^J - q^{\rho}(\alpha q^J + \nu)}, \qquad \mu = \frac{\alpha q^{\rho}(1 - q^J)(1 - \nu)}{(1 + \alpha q^J - q^{\rho}(\alpha q^J + \nu))(1 + \alpha - q^{\rho}(\alpha + \nu))}. \tag{6.1.9}$$

As a matter of convention, we endow the space $C(\mathbb{R})$ and $C([0, \infty), C(\mathbb{R}))$ with the topology of uniform convergence over compact subsets, and write " $\Rightarrow$ " for the weak convergence of probability laws. We present our main theorem.

Theorem 6.1.6. *Fix $b \in \left(\frac{I+J-2}{I+J-1}, 1\right)$, $I \geq 2$ and $J \geq 1$, for small $\epsilon > 0$, wet $q = e^{\sqrt{\epsilon}}$ and define α via $b = \frac{1+\alpha q}{1+\alpha}$. We call this weakly asymmetric scaling. Assume that $\{N^{\mathsf{f}}_\epsilon(0,x)\}_{\epsilon>0}$ is nearly stationary with density ρ (see Definition 6.5.5) and*

$$\sqrt{\epsilon}\Big(N^{\mathsf{f}}_\epsilon(0, \epsilon^{-1}x) - \rho\epsilon^{-1}x\Big) \Rightarrow \mathcal{H}^{ic}(x) \text{ in } C(\mathbb{R}) \text{ as } \epsilon \downarrow 0,$$

then

$$\sqrt{\epsilon}\Big(N^{\mathsf{f}}_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x + \epsilon^{-2}t\mu_\epsilon) - \rho(\epsilon^{-1}x + \epsilon^{-2}\mu_\epsilon t)\Big) - t\log\lambda_\epsilon \Rightarrow \mathcal{H}(t,x) \tag{6.1.10}$$

$$\text{in } C([0,\infty), C(\mathbb{R})) \text{ as } \epsilon \downarrow 0,$$

where $\mathcal{H}(t,x)$ is the Hopf-Cole solution of the KPZ equation

$$\partial_t \mathcal{H}(t,x) = \frac{JV_*}{2}\partial_x^2 \mathcal{H}(t,x) - \frac{JV_*}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \sqrt{JD_*}\xi(t,x), \tag{6.1.11}$$

with initial condition $\mathcal{H}^{ic}(x)$, where the coefficients are given by

$$V_* = \frac{(I+J)b - (I+J-2)}{I^2(1-b)}, \tag{6.1.12}$$

$$D_* = \frac{\rho(I-\rho)}{I}\frac{(I+J)b - (I+J-2)}{I^2(1-b)}. \tag{6.1.13}$$

Note that the restriction of $b \in (\frac{I+J-2}{I+J-1}, 1)$ in Theorem 6.1.6 is necessary and sufficient to ensure Condition 6.1.1 holds for ϵ small enough. In Appendix D, we will demonstrate how our theorem agrees with the non-rigorous KPZ scaling theory used in physics[4].

Remark 6.1.7. In a different setting where $0 < q, \nu < 1$ (in contrast to Condition 6.1.1, there is no $I \in \mathbb{Z}_{\geq 1}$ such that $\nu = q^{-I}$) and $\alpha \geq 0$, one can show that $L^{(J)}_\alpha$ is a stochastic matrix on $\mathbb{Z}_{\geq 0} \times \{0, 1, \ldots, J\}$ (instead of $\{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$ for our case). In this regime, the SHS6V

[4] The KPZ scaling theory is a non-rigorous physics method used to compute the constants (the coefficients of the KPZ equation (6.1.11) as in our case) arising in limit theorems for the models in the KPZ universality class [KMHH92, Spo12], which has been confirmed in a few cases such as [DT16, Gho17].

model allows arbitrary number of particles at each site (instead of at most I particles for our case). [CT17] proves the weak universality of the SHS6V model[5] under a different type of weak scaling that corresponds to taking $b = 3, z = 1, \delta \to \epsilon\delta, \kappa \to \epsilon^2\kappa$ in (6.1.3). Under this scaling, the number of particles at each site diverges to infinity with rate ϵ^{-1}. This simplifies considerably the control of the quadratic variation of the martingale in the discrete SHE (6.1.14), which is the main complexity in our analysis.

Remark 6.1.8. Taking $I = J = 1$, Theorem 6.1.6 recovers [CGST20, Theorem 1.1]. We assume $I \geq 2$ in Theorem 6.1.6 merely due to some technical subtleties we met in Section 6.7. The proof for $I = 1$ needs particular modification and we do not pursue it here.

The proof of Theorem 6.1.6 will be given in the end of Section 6.5, as a corollary of Theorem 6.5.6.

6.1.4 Method

In this section, we explain the method used in proving Theorem 6.1.6. Although initially our methods follow [CGST20], rather quickly, we encounter novel complexities that are not present in [CGST20] which require new ideas.

As illustrated in Section 6.1.2, via fusion, to study the fused SHS6V model, it suffices to work with the unfused version. Similar to [CGST20], the first step is to perform a microscopic Hopf-Cole transform of the SHS6V model (6.5.6). The existence of the microscopic Hopf-Cole transform is guaranteed by one particle version of the duality (6.3.8) (which goes back to [CP16, Theorem 2.21]). The microscopic Hopf-Cole transform $Z(t, x)$, which is essentially an exponential version of the unfused height function $N(t, x)$, satisfies a discrete version of SHE

$$dZ = \mathcal{L}Z dt + dM. \tag{6.1.14}$$

[5] In the context of [CT17], the authors prove the weak universality for the higher spin exclusion process defined in [CP16, Definition 2.10], which is equivalent to the SHS6V model after a gap-particle transform. We describe their result in the language of the SHS6V model here.

Here $\mathcal{L}$ is an operator which approximates the Laplacian and M is a martingale. Owing to the definition of the Hopf-Cole solution to the KPZ equation, Theorem 6.1.6 is equivalent to showing that the above discrete SHE converges to its continuum version (Theorem 6.5.6). The proof of Theorem 6.5.6 reduces to three steps:

1). Showing tightness.

2). Identifying the limit of the linear martingale problem.

3). Identifying the limit of the quadratic martingale problem.

Steps 1) and 2) follow from a similar approach as in [CGST20]. Step 3) is the difficult part; Proposition 6.6.8 does this by proving a form of self-averaging for the quadratic variation of the martingale M. We will focus on discussing the method for proving this self-averaging result in the rest of the section. We remark that other recent KPZ equation convergence results using the Hopf-Cole transform include ASEP-(q, j) [CST18], Hall-Littlewood PushTASEP [Gho17], weakly asymmetric bridges [Lab17], open ASEP [CS18, Par19b].

We will explain what is self-averaging in a moment, but first introduce two tools used in proving it. The first tool is the *Markov duality* and the second is the *exact formula of two particle transition probability* of the SHS6V model.

The stochastic six vertex model enjoys two Markov dualities [CP16, Theorem 2.21] and [Lin19, Theorem 1.5][6], which are exploited in proving the self-averaging [CGST20, Proposition 5.6]. The Markov duality in [CP16, Theorem 2.21] also works for the SHS6V model (Proposition 6.3.6 in our paper), yet it is unknown whether there exists a generalization of [Lin19, Theorem 1.5] for the SHS6V model. [Kua18, Theorem 4.10] discovers a general duality for the multi-species SHS6V model using the algebraic machinery[7]. At first glance, the duality functional written in [Kua18, Theorem 4.10] takes a rather complicated form, but we only need a two particle version of this

[6] The Markov duality proved in [Lin19] first appears in [CP16, Theorem 2.23]. In fact [CP16, Theorem 2.23] claims a more general Markov duality for the SHS6V model. In discussions with the authors of [CP16], we recognized a gap in that proof as well as a counter-example to the result when $I > 1$, see [CP19] for detail.

[7] As a remark, the functional in [Kua18, Theorem 4.10] also serves as the duality functional for a multi-species version of ASEP(q, j), see [CGRS16, Kua17].

duality, in which case the duality functional simplifies greatly (Proposition 6.3.7 in our paper) and is applicable for proving the desired self-averaging. We remark that this is the first application of [Kua18, Theorem 4.10] as far as we know.

In [BCG16, Theorem 3.6], an integral formula was obtained for general k particle transition probability of the stochastic six vertex model via a generalized Fourier theory (Bethe ansatz), using a complete set of eigenfunction of the stochastic six vertex model transition matrix obtained in [BCG16, Theorem 3.4] together with the Plancherel identity [TW08, Theorem 2.1]. [CGST20] applies the steepest descent analysis to a two particle version of this formula to extract a space-time bound, which is the key to control the quadratic variation of the martingale in (6.1.14).

For the SHS6V model, it is natural to expect that the similar method should apply, since we also have a set of eigenfunctions from [CP16, Proposiiton 2.12] and a generalized Plancherel identity from [BCPS15, Corollary 3.13]. However, the Plancherel identity was originally designed only for $0 < q, \nu < 1$ and there is a technical issue in extending this identity to $q > 1, \nu = q^{-I}$ which has not been addressed in the existing literatures[8] (see Remark 6.4.5). Fortunately, we find that when $I \geq 2$ and there are only two particles, such analytic continuation does work, which produces an integral formula for the two particle SHS6V model transition probability (Theorem 6.4.4). In terms of large contours, the integral formula consists of two double contour integrals and one single contour integral. We find that the single contour integral can be expressed as a residue of one of the double contour integrals. This simplifies our analysis since via certain contour deformation, the single contour integral will be canceled out.

We will analyze (a tilted version of) this integral formula (Corollary 6.5.3) in Section 6.7 using steepest descent analysis and obtain a very precise estimate of the (tilted) two particle transition probability $\mathbf{V}$ defined in (6.5.20). Compared with the analysis for stochastic six vertex model in [CGST20, Section 6], one difficulty is to find (and justify) the contours for different I, J such that

[8] [CP16, Proposition A.3] claims the Plancherel identity for $\nu = q^{-I}$ can be obtained by analytic continuation of [BCPS15, Corollary 3.13]. After discussions with the authors of [CP16], they agree that there is an issue in this analytic continuation (and the resulting identity) due to poles encountered along the way [CP19].

the steepest descent analysis applies. Also in certain cases (Section 6.7.5) the steepest descent contour can only be implicitly defined (compared with [CGST20, Section 6] where all the steepest descent contour are circles), which complicates our analysis.

Now let us explain what is self-averaging and how these two tools can be applied to prove it. Denote the discrete gradient by $\nabla f(x) := f(x+1) - f(x)$. Roughly speaking, the terminology “self-averaging" refers to the phenomena that as $\epsilon \downarrow 0$

$(\mathbf{A})$ For $x_1 \neq x_2$, the average of $\epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2)$ over a long time interval of length $O(\epsilon^{-2})$ will vanish.

$(\mathbf{B})$ There exists a positive constant λ such that the average of $(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2 - \lambda Z(t,x)^2$ over a long time interval of length $O(\epsilon^{-2})$ will vanish.

The proofs of $(\mathbf{A})$ and $(\mathbf{B})$ are given in Lemma 6.8.2 and Lemma 6.8.3 respectively, let us make a brief discussion about our strategy here. As we will see in (6.8.15), under weakly asymmetric scaling,

$$\epsilon^{-\frac{1}{2}}\nabla Z(t,x) = (\rho - \widetilde{\eta}_{x+1}(t))Z(t,x) + \text{ error term }. \tag{6.1.15}$$

where $\rho \in (0,I)$ is the density, $\widetilde{\eta}_x(t) = \eta_{x+\hat{\mu}(t)}(t)$ and $\hat{\mu}(t)$ is some constant defined in (6.5.4). Pointwisely, $\epsilon^{-\frac{1}{2}}\nabla Z(t,x)$ is of the same order as $Z(t,x)$. But $(\mathbf{A})$ tells that after averaging over a long time interval (we will just say "averaging" afterwards for short), $\epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2)$ vanishes for $x_1 \neq x_2$, this explains the terminology of “self-averaging". To prove $(\mathbf{A})$, by the first duality in Lemma 6.5.2 (which goes back to Proposition 6.3.6), one is able to write down the conditional quadratic variation in terms of the summation of (a tilted version of) two particle transition probability $\mathbf{V}$, i.e. for $x_1 \leq x_2$

$$\mathbb{E}\big[Z(t,x_1)Z(t,x_2)\big|\mathcal{F}(s)\big] = \sum_{y_1 \leq y_2} \mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2) \tag{6.1.16}$$

This allows us to move the gradients from $Z(t,x_1)$ and $Z(t,x_2)$ to $\mathbf{V}$. We proceed by using a very precise estimate of $\mathbf{V}$ from Proposition 6.7.1 (which is proved by making use of the steepest descent analysis of the integral formula of $\mathbf{V}$). Referring to Proposition 6.7.1, each gradient on $\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)$ gives an extra decay of $\frac{1}{\sqrt{t-s+1}}$, which helps us to conclude $(\mathbf{A})$. We remark that for demonstrating $(\mathbf{A})$, our argument is actually simpler than that of [CGST20]. Since we assume $I \geq 2$, (6.1.16) holds for all $x_1 \leq x_2$, while in the situation of the stochastic six vertex model ($I = 1$), (6.1.16) holds only for $x_1 < x_2$, due to the exclusion restriction (i.e. two particles can not stay in the same site). In fact, [CGST20] needs both of the duality [CP16, Theorem 2.21] and [Lin19, Theorem 1.5] to prove $(\mathbf{A})$.

For $(\mathbf{B})$, there are two tasks: Identifying λ and proving the self-averaging. These were done simultaneously for the stochastic six vertex model [CGST20]: Note that by (6.1.15),

$$(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2 = (\widetilde{\eta}_{x+1}(t) - \rho)^2 Z(t,x)^2 + \text{ error term }. \tag{6.1.17}$$

For the stochastic six vertex model, $\widetilde{\eta}_x(t) \in \{0,1\}$ for all t,x, hence $\widetilde{\eta}_x(t)^2 = \widetilde{\eta}_x(t)$. [CGST20, Lemma 7.1] uses this crucial observation to obtain

$$\begin{aligned}(\widetilde{\eta}_{x+1}(t) - \rho)^2 Z(t,x)^2 &= \rho^2 Z(t,x)^2 + (1-2\rho)\widetilde{\eta}_{x+1}(t) Z(t,x) \\ &= \rho(1-\rho)Z(t,x)^2 + \epsilon^{-\frac{1}{2}}(2\rho - 1)\nabla Z(t,x) Z(t,x) + \text{ error term }.\end{aligned}$$

By similar method used in demonstrating $(\mathbf{A})$, it is not hard to prove that $\epsilon^{-\frac{1}{2}}\nabla Z(t,x)Z(t,x)$ vanishes after averaging, implying that $\lambda = \rho(1-\rho)$.

For our case, first we note that $\widetilde{\eta}_x(t) \in \{0,1,\dots,I\}$ with $I \geq 2$, so the $\widetilde{\eta}_x(t)^2 = \widetilde{\eta}_x(t)$ identity obviously fails. We need to find another way to determine λ and prove the self-averaging. We proceed by first guessing the λ. Via (6.1.17), the average of $\epsilon^{-1}(\nabla Z(t,x))^2$ over a long time interval can be approximated by the average of $(\widetilde{\eta}_x(t) - \rho)^2 Z(t,x)^2$. In Appendix C, we derive a family of stationary distribution of the SHS6V model, which is a product measure $\bigotimes \pi_\rho$, where π_ρ is a probability measure on $\{0,1,\dots,I\}$ indexed by its mean $\rho \in (0,I)$. Starting the SHS6V model $\vec{\eta}(t)$ from $\vec{\eta}(0) \sim \bigotimes \pi_\rho$, it is clear that $\widetilde{\eta}_x(t) \sim \pi_\rho$ for all t,x. In a heuristic level, one can approximate the average of $(\widetilde{\eta}_{x+1}(t)-\rho)^2 Z(t,x)^2$ by that of the $\mathbb{E}_{\pi_\rho}\big[(\widetilde{\eta}_{x+1}(t)-\rho)^2\big]Z(t,x)^2$. Under weakly asymmetric scaling, one computes that

$$\lim_{\epsilon\downarrow 0}\mathbb{E}_{\pi_\rho}\big[(\widetilde{\eta}_{x+1}(t)-\rho)^2\big]=\frac{\rho(I-\rho)}{I},$$

which suggests $\lambda=\frac{\rho(I-\rho)}{I}$.

To prove $(\mathbf{B})$ with $\lambda=\frac{\rho(I-\rho)}{I}$, note that the second duality in Lemma 6.5.2 (which goes back to Proposition 6.3.7) implies

$$\mathbb{E}\big[D(t,x,x)\big|\mathcal{F}(s)\big]=\sum_{y_1\le y_2}D(s,y_1,y_2)\mathbf{V}\big((x,x),(y_1,y_2),t,s\big)\tag{6.1.18}$$

where approximately[9]

$$D(s,y_1,y_2)=\begin{cases}Z(s,y_1)^2\big(I-\widetilde{\eta}_{y_1}(s)\big)\big(I-1-\widetilde{\eta}_{y_1}(s)\big) & \text{if } y_1=y_2,\\ \frac{I-1}{I}Z(s,y_1)Z(s,y_2)\big(I-\widetilde{\eta}_{y_1}(s)\big)\big(I-\widetilde{\eta}_{y_2}(s)\big) & \text{if } y_1<y_2\end{cases}\tag{6.1.19}$$

Note that the expression of $D(s,y_1,y_2)$ is different depending on whether $y_1=y_2$, which is crucial to our proof. Rewriting $(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2-\frac{\rho(I-\rho)}{I}Z(t,x)^2$ in terms of the two duality functionals in (6.1.16) and (6.1.19)

$$\begin{aligned}(\epsilon^{-\frac{1}{2}}\nabla Z(t,x))^2-\frac{\rho(I-\rho)}{I}Z(t,x)^2&=\Big((\widetilde{\eta}_{x+1}(t)-\rho)^2-\frac{\rho(I-\rho)}{I}\Big)Z(t,x)^2+\text{ error term}\\ &=\Big((I-\widetilde{\eta}_{x+1}(t))(I-1-\widetilde{\eta}_{x+1}(t))-\frac{(I-1)(I-\rho)^2}{I}\Big)Z(t,x+1)^2-(2\rho+1-2I)\epsilon^{-\frac{1}{2}}\nabla Z(t,x)Z(t,x)\\ &\quad+\text{ error term },\\ &=\Big(D(t,x+1,x+1)-\frac{(I-1)(I-\rho)^2}{I}Z(t,x+1)^2\Big)-(2\rho+1-2I)\epsilon^{-\frac{1}{2}}\nabla Z(t,x)Z(t,x)+\text{ error term }.\end{aligned}$$

[9] In fact, the functional $D(s,y_1,y_2)$ below is only an approximate version of the duality functional defined in (6.5.19), we use this approximate version here to avoid extra notations and make our argument more intuitive.

It is not hard to show that the second term $\epsilon^{-\frac{1}{2}}\nabla Z(t,x)Z(t,x)$ vanishes after averaging. For the first term above, we combine both of the dualities (6.1.16), (6.1.18) and get

$$\begin{aligned}
&\mathbb{E}\Big[D(t,x+1,x+1) - \frac{(I-1)(I-\rho)^2}{I} Z(t,x+1)^2 \Big| \mathcal{F}(s)\Big] \\
&= \sum_{y_1 \le y_2} \mathbf{V}\big(x+1,x+1,y_1,y_2,t,s\big)\Big(D(s,y_1,y_2) - \frac{(I-1)(I-\rho)^2}{I} Z(s,y_1)Z(s,y_2)\Big). \qquad (6.1.20)
\end{aligned}$$

The number of pairs (y_1, y_2) such that $y_1 = y_2$ compared with $y_1 < y_2$ is negligible in the summation above so it suffices to study for $y_1 < y_2$

$$\begin{aligned}
&D(s,y_1,y_2) - \frac{(I-1)(I-\rho)^2}{I} Z(s,y_1)Z(s,y_2) \\
&= \Big(\frac{I-1}{I}(I-\widetilde{\eta}_{y_1}(s))(I-\widetilde{\eta}_{y_2}(s)) - \frac{(I-1)(I-\rho)^2}{I}\Big) Z(s,y_1)Z(s,y_2) \\
&= (I-\widetilde{\eta}_{y_2}(s))(\epsilon^{-\frac{1}{2}}\nabla Z(s,y_1))Z(s,y_2) + (I-\rho)(\epsilon^{-\frac{1}{2}}\nabla Z(s,y_2))Z(s,y_1) + \text{ error term }.
\end{aligned}$$

Inserting this expression into the RHS of (6.1.20) and using the summation by part formula (see (6.8.39)), we can move the gradient from Z to $\mathbf{V}$. Similar to the argument for $(\mathbf{A})$, applying the estimate in Proposition 6.7.1 completes the proof of $(\mathbf{B})$.

6.1.5 Outline

The paper will be organized as follows. In Section 6.2 we give an equivalent definition of SHS6V model through fusion. At the beginning, we require the existence of a leftmost particle. After that we extend the definition to a bi-infinite version of the SHS6V model (Lemma 6.2.1), which is the object that we study for the rest of the paper. In Section 6.3, we introduce two Markov dualities enjoyed by the model. The first one is taken directly from the [CP16, Theorem 2.21]. The second one is a certain degeneration from a general duality in [Kua18, Theorem 4.10]. Section 6.4 contains the derivation of integral formula for the two point transition probability of the SHS6V model. In Section 5, we define the microscopic Hopf-Cole transform and prove that it satisfies a discrete version of SHE. Due to the definition of the Hopf-Cole solution to the KPZ equation, it suffices to prove that the discrete SHE converges to its continuum version. In Section 6.6, we prove this result in two steps. First, we establish the tightness of the discrete SHE. Second, we show that any limit point is the solution to the SHE in continuum, assuming the self-averaging property (Proposition 6.6.8). The last two sections are devoted to the proof of Proposition 6.6.8. In Section 6.7, we obtain a very precise estimate for the two point transition probability by applying steepest descent analysis to the integral formula that we obtain in Section 6.4. In Section 6.8, we prove Proposition 6.6.8 using the Markov duality and our estimate of the two point transition probability.

6.1.6 Notation

In this paper, we denote $\mathbb{Z}_{\geq i} = \{n \in \mathbb{Z} : n \geq i\}$. $\mathbf{1}_E$ denotes the indicator function of an event E. We use $\mathbb{E}$ (respectively, $\mathbb{P}$) to denote the expectation (respectively, probability) with respect to the process or random variable that follow. The symbol C_r represents a circular contour centered at the origin with radius r. All contours, unless otherwise specified, are counterclockwise.

6.1.7 Acknowledgment

The author heartily thanks his advisor Ivan Corwin for suggesting this problem, his encouragement and helpful discussion, as well as reading part of the manuscript. We thank Jeffrey Kuan for the helpful discussion over the degeneration of [Kua18, Theorem 4.10] to the stochastic higher spin six vertex model. We also wish to thank Promit Ghosal and Li-Cheng Tsai, who provided very useful discussions about the technical part of the paper. We are also grateful to the anonymous referees for their valuable suggestions. The author was partially supported by the Fernholz Foundation's "Summer Minerva Fellow" program and also received summer support from Ivan Corwin's NSF grant DMS:1811143.

6.2 The bi-infinite SHS6V model

The main goal of this section is to extend the definition of the left-finite unfused (fused) SHS6V model in Definition 6.1.3 (Definition 6.1.2) to the space of bi-infinite configurations $\{0, 1, \dots, I\}^{\mathbb{Z}}$. The motivation of such extension is to include one important class of initial condition called *near stationary initial condition* as in [BG97]. We will proceed following the idea of [CGST20], which goes back to [CT17]. By fusion (Proposition 6.1.5), it suffices to extend the left-finite unfused SHS6V model $\vec{\eta}(t)$, the extension of the fused version $\vec{g}(t)$ follows readily by taking $\vec{g}(t) = \vec{\eta}(Jt)$.

For the extension, the first step is to restate the SHS6V model in a parallel update rule. To this end, we equip the probability space with a family of independent Bernoulli random variables $B(t, y, \eta), B'(t, y, \eta)$ such that

$$B(t, y, \eta) \sim \mathrm{Ber}\left(\frac{\alpha(t)(1 - q^{\eta})}{1 + \alpha(t)}\right), \qquad B'(t, y, \eta) \sim \mathrm{Ber}\left(\frac{\alpha(t) + \nu q^{\eta}}{1 + \alpha(t)}\right), \tag{6.2.1}$$

for $t \in \mathbb{Z}_{\geq 0}$, $y \in \mathbb{Z}$ and $\eta \in \{0, 1, \dots, I\}$, recall that $\alpha(t) = \alpha q^{\mathrm{mod}_J(t)}$.

Treating these Bernoulli random variables as a random environment, we find an equivalent way to define the left-finite unfused SHS6V model, through recursion. Given initial state $\vec{\eta}(0) \in \mathbb{G}$, define $N(0, x) := N_x(\vec{\eta}(0)) - N_0(\vec{\eta}(0))$ (recall the notation from (6.1.7)) and recursively for $t = 0, 1, \dots,$

$$N(t+1, y) := \begin{cases} N(t, y) - B(t, y, \eta_y(t)) & \text{if } N(t, y-1) - N(t+1, y-1) = 0, \\ N(t, y) - B'(t, y, \eta_y(t)) & \text{if } N(t, y-1) - N(t+1, y-1) = 1. \end{cases} \tag{6.2.2}$$

$$\eta_y(t+1) := N(t+1, y) - N(t+1, y-1).$$

It is straightforward to see that $\vec{\eta}(t) = (\eta_y(t))_{y \in \mathbb{Z}}$ is a left-finite unfused SHS6V model and $N(t, x)$ is indeed its height function defined by (6.1.8).

The recursion (6.2.2) is equivalent to

$$N(t,y)-N(t+1,y) = \Big(N(t,y-1)-N(t+1,y-1)\Big)\Big(B'(t,y,\eta_y(t))-B(t,y,\eta_y(t))\Big)+B(t,y,\eta_y(t)). \tag{6.2.3}$$

Iterating (6.2.3) implies

$$N(t,y) - N(t+1,y) = \sum_{y'=-\infty}^{y} \prod_{z=y'+1}^{y} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big)B(t,z,\eta_z(t)). \tag{6.2.4}$$

Note that the summation above is finite. The reason is that since $\vec{\eta}(t) \in \mathbb{G}$, there exists w such that $\eta_z(t) = 0$ for all $z < w$, which implies $B(t,z,\eta_z(t)) = 0$ for all $z < w$.

In light of (6.2.4), we extend the Definition 6.1.3 to the space of bi-infinite particle configuration $\{0,1,\dots,I\}^{\mathbb{Z}}$.

Lemma 6.2.1. *For any bi-infinite particle configuration $\vec{\eta}(0) \in \{0,1,\dots,I\}^{\mathbb{Z}}$, define the initial height function*

$$N(0,x) = \mathbf{1}_{\{x>0\}} \sum_{i=1}^{x} \eta_i(0) - \mathbf{1}_{\{x<0\}} \sum_{i=1}^{-x} \eta_{-i}(0).$$

Note that if $\vec{\eta}(0) \in \mathbb{G}$, $N(0,x)$ defined above coincides with that defined in (6.1.8). *We inductively define the $\vec{\eta}(t)$ and $N(t,x)$ for $t = 0,1,\dots$ via the recursion*

$$N(t,y) - N(t+1,y) := \sum_{y'=-\infty}^{y} \prod_{z=y'+1}^{y} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big)B(t,z,\eta_z(t)), \tag{6.2.5}$$

$$\eta_y(t+1) := N(t+1,y) - N(t+1,y-1). \tag{6.2.6}$$

For $p \geq 1$, the infinite sum in (6.2.5) *converges almost surely and in L^p to a $\{0,1\}$-valued random variable. Furthermore, consider left-finite initial configuration $\vec{\eta}^w(0) = (\eta_i(0)\mathbf{1}_{\{i\geq w\}})_{i\in\mathbb{Z}}$ and the* *height function* $N^w(t,y)$ *inductively defined by* (6.2.5) *and* (6.2.6)*, then for all* $t \in \mathbb{Z}_{\geq 0}$ *and* $y \in \mathbb{Z}$

$$\lim_{w \to -\infty} N^w(t,y) = N(t,y) \text{ in } L^p.$$

Remark 6.2.2. It is clear that via (6.2.5), one can recover the recursion (6.2.2) since

$$\begin{aligned}
&N(t,y) - N(t+1,y) \\
&= \sum_{y'=-\infty}^{y} \prod_{z=y'+1}^{y} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big) B(t,z,\eta_z(t)) \\
&= B(t,y,\eta_y(t)) + \Big(B'(t,y,\eta_y(t)) - B'(t,y,\eta_y(t))\Big) \sum_{y'=-\infty}^{y-1} \prod_{z=y'+1}^{y-1} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big) \\
&= B(t,y,\eta_y(t)) + \Big(B'(t,y,\eta_y(t)) - B'(t,y,\eta_y(t))\Big)\Big(N(t,y-1) - N(t,y)\Big).
\end{aligned}$$

In particular, if $\vec{\eta}(0) \in \mathbb{G}$, the $\vec{\eta}(t)$ defined in Lemma 6.2.1 is a left-finite unfused SHS6V model. Therefore, Lemma 6.2.1 truly extends the scope of Definition 6.1.3.

Proof of Lemma 6.2.1. Define the canonical filtration

$$\mathcal{F}(t) = \sigma\Big(\vec{\eta}(0), B(s,z,\eta), B'(s,z,\eta), 0 \leq s \leq t-1\Big).$$

It is not hard to see (via (6.2.5) and (6.2.6)) that $N(t,y)$ and $\vec{\eta}(t)$ are adapted to this filtration.

Let us first justify the convergence of the infinite summation (6.2.5). To simplify notation, we denote by $\mathbb{E}'\big[\,\cdot\,\big] = \mathbb{E}\big[\,\cdot\,\big|\mathcal{F}(t)\big]$. For $x < y \in \mathbb{Z}$, denote by

$$K_{x,y}(t) := \sum_{y'=x}^{y} \prod_{z=y'+1}^{y} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big) B(t,y',\eta_{y'}(t))$$

Observing that $K_{x,y}(t) \in \{0,1\}$ for all realization of $B, B' \in \{0,1\}$. Therefore, as $x \to -\infty$, the L^p convergence of $K_{x,y}(t)$ implies the almost sure convergence. Note that B, B' are independent Bernoulli random variables with mean given in (6.2.1). As a consequence, there exists constant

$\delta > 0$ such that

$$\mathbb{P}\big(B'(t,z,\eta) - B(t,z,\eta) = 0\big) > \delta, \quad \forall\, (t,z,\eta) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z} \times \{0,1,\dots,I\}.$$

Since $|B'(t,z,\eta) - B(t,z,\eta)| \leq 1$,

$$\mathbb{E}'\big[\big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\big)^p\big] \leq 1-\delta.$$

Furthermore, note that conditioning on $\mathcal{F}(t)$, $B(t,z,\eta_z(t)), B'(t,z,\eta_z(t))$ are all independent, which yields

$$\begin{aligned}&\mathbb{E}'\Bigg[\Bigg(B(t,y',\eta_{y'}(t)) \prod_{z=y'+1}^{y} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big)\Bigg)^p\Bigg]\\ &= \mathbb{E}'\big[B(t,y',\eta_{y'}(t))^p\big] \prod_{z=y'+1}^{y} \mathbb{E}'\big[\big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\big)^p\big] \leq (1-\delta)^{y-y'}. \qquad (6.2.7)\end{aligned}$$

Taking expectation on both side of (6.2.7), by tower property

$$\Bigg\|\Bigg(\prod_{z=y'+1}^{y} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big) B(t,y',\eta_{y'}(t))\Bigg)\Bigg\|_p \leq (1-\delta)^{\frac{y-y'}{p}},$$

which implies the convergence of $K_{x,y}(t)$ in L^p as $x \to -\infty$.

We proceed to justify

$$\lim_{w \to -\infty} N^w(t,y) = N(t,y) \quad \text{ in } L^p. \qquad (6.2.8)$$

We prove this by applying induction on t. The $t = 0$ case is immediately checked. Assuming that we have a proof for $t = s$, we show that (6.2.8) also holds for $t = s+1$. Note that for all $y \in \mathbb{Z}$,

$$\eta_y^w(s) = N^w(s,y) - N^w(s,y-1) \to N(s,y) - N(s,y-1) = \eta_y(s) \text{ in } L^p \quad \text{ as } w \to -\infty.$$

Since both $\eta_y^w(s), \eta_y(s)$ take value in $\{0, 1, \ldots, I\}$, we obtain

$$\lim_{w\to-\infty} \mathbb{P}\big(\eta_y^w(s) = \eta_y(s)\big) = 1.$$

Taking $w \to -\infty$, one achieves

$$N^w(s, y) - N^w(s+1, y) = \sum_{y'=-\infty}^{y} \prod_{z=y'+1}^{y} \Big(B'(s, z, \eta_z^w(s)) - B(s, z, \eta_z^w(s))\Big) B(s, z, \eta_z^w(s)),$$

we find that $\lim_{w\to-\infty} N^w(s, y) - N^w(s+1, y) = N(s, y) - N(s, y+1)$ in L^p. Since we have assumed (6.2.8) for $t = s$, we have

$$N^w(s+1, y) \to N(s+1, y) \quad \text{in } L^p,$$

which completes the induction. □

Definition 6.2.3. We call the $\vec{\eta}(t) \in \{0, 1, \ldots, I\}^{\mathbb{Z}}$ defined in Lemma 6.2.1 **the bi-infinite unfused SHS6V model** and associate it with the height function $N(t, x)$ defined in Lemma 6.2.1. We simply define **the bi-infinite fused SHS6V model** $\vec{g}(t)$ and its height function $N^{\mathsf{f}}(t, x)$ via

$$\vec{g}(t) := \vec{\eta}(Jt), \qquad N^{\mathsf{f}}(t, x) := N(Jt, x).$$

It is clear that to prove Theorem 6.1.6, it suffices to work with the bi-infinite unfused SHS6V model. Unless specified otherwise, the SHS6V model now means the bi-infinite unfused SHS6V model $\vec{\eta}(t)$. We associate it with the canonical filtration $\mathcal{F}(t) = \sigma\Big(\vec{\eta}(0), B(s, z, \eta), B'(s, z, \eta), 0 \leq s \leq t-1\Big)$.

6.3 Markov duality

One main tool that we rely on to prove Theorem 6.1.6 is the Markov duality. It is a powerful property which has been found for different interacting particle systems including the contact process, voter model and symmetric simple exclusion process [Lig12, Lig13]. Using Markov duality, Spitzer and Liggett showed that the only extreme translation invariant measures for the SSEP on $\mathbb{Z}^d$ are the Bernoulli product measure.

In this section, we first state two Markov dualities for the $J = 1$ version of left-finite SHS6V model, which comes form [CP16, Theorem 2.21] and [Kua18, Theorem 4.10] respectively. The extension of them to the unfused left-finite SHS6V model is immediate since the transition operators of the model are commute. Finally we explain how to extend these dualities to the bi-infinite unfused SHS6V model constructed in the previous section.

Let us recall the definition of Markov duality in the first place.

Definition 6.3.1. Given two discrete time Markov processes $X(t) \in U$ and $Y(t) \in V$ (might be time inhomogeneous) and a function $H : U \times V \to \mathbb{R}$, we say that $X(t)$ and $Y(t)$ are dual with respect to H if for any $x \in U$, $y \in V$ and $s \leq t \in \mathbb{Z}_{\geq 0}$, we have

$$\mathbb{E}\big[H(X(t), y)\big|X(s) = x\big] = \mathbb{E}\big[H(x, Y(t))\big|Y(s) = y\big].$$

The Markov dualities that we are going to present are between the unfused SHS6V model and the k-particle reversed unfused SHS6V model location process. To define the latter process, let us first introduce several state spaces.

Definition 6.3.2. Recall the space of left-finite particle configuration $\mathbb{G}$ from (6.1.6). We likewise define the space of right-finite particle configuration

$$\mathbb{M} = \{\vec{m} = (\dots, m_{-1}, m_0, m_1, \dots) : \text{ all } m_i \in \{0, 1, \dots, I\}, \exists\, x \in \mathbb{Z} \text{ such that } m_i = 0 \text{ for all } i > x\}.$$

When there are finite number of k particles, we restrict $\mathbb{G}$ and $\mathbb{M}$ to

$$\mathbb{G}^k = \{\vec{g} \in \mathbb{G} : \sum_i g_i = k\}, \qquad \mathbb{M}^k = \{\vec{m} \in \mathbb{M} : \sum_i m_i = k\}.$$

In terms of particle positions, the spaces $\mathbb{G}^k$ and $\mathbb{M}^k$ are in bijection with

$$\mathbb{W}_I^k = \left\{ \vec{y} = (y_1 \le \cdots \le y_k) : \vec{y} \in \mathbb{Z}^k, \max_{1 \le i \le M(\vec{y})} c_i \le I \right\},$$

where $(c_1, \dots, c_{M(\vec{y})})$ denotes the cluster number in $\vec{y}$, i.e. $\vec{y} = (y_1 = \cdots = y_{c_1} < y_{c_1+1} = \cdots = y_{c_1+c_2} < \dots)$. $(y_1 \le \cdots \le y_k)$ should be understood as the location of k particles in a non-decreasing order. In particular, we denote by $\varphi : \mathbb{W}_I^k \to \mathbb{G}^k$ and $\phi : \mathbb{W}_I^k \to \mathbb{M}^k$ to be the bijective maps respectively.

Definition 6.3.3. When $J = 1$, it is clear that Definition 6.1.2 and Definition 6.1.3 define the same Markov process. We call it the left-finite $J = 1$ SHS6V model. In addition, we call $\vec{\xi}(t) = (\xi_x(t))_{x \in \mathbb{Z}} \in \mathbb{M}$ the reversed $J = 1$ SHS6V model if $\vec{\xi}'(t) = (\xi_{-x}(t))_{x \in \mathbb{Z}} \in \mathbb{G}$ is a left-finite $J = 1$ SHS6V model.

Since the SHS6V model preserves the number of particles, we can consider SHS6V model with k particles as a process on the particle locations.

Definition 6.3.4. We define the k particle $J = 1$ SHS6V model location process $\vec{x}(t) = \big(x_1(t) \le \cdots \le x_k(t)\big) \in \mathbb{W}_I^k$ if $\varphi(\vec{x}(t))$ (recall the bijective map $\varphi : \mathbb{W}_I^k \to \mathbb{G}^k$ from Definition 6.3.2) is the $J = 1$ left-finite SHS6V model. We say that $\vec{y}(t) = (y_1(t) \le \cdots \le y_k(t)) \in \mathbb{W}_I^k$ is a k-particle reversed $J = 1$ SHS6V model location process if $-\vec{y}(t) = (-y_k(t) \le \cdots \le -y_1(t))$ is a k-particle $J = 1$ SHS6V model location process. In addition, for $\vec{y}, \vec{y}' \in \mathbb{W}_I^k$, we denote by $\widetilde{\mathscr{B}}_\alpha(\vec{y}, \vec{y}')$ to be the transition probability from $\vec{y}$ to $\vec{y}'$ of the k-particle reversed $J = 1$ SHS6V model location process. As a matter of convention, $\widetilde{\mathscr{B}}_\alpha$ could be seen as an operator acting on function $f : \mathbb{W}_I^k \to \mathbb{R}$ in the manner that

$$(\widetilde{\mathscr{B}}_\alpha f)(\vec{y}) := \sum_{\vec{y}' \in \mathbb{W}_I^k} \widetilde{\mathscr{B}}_\alpha(\vec{y}, \vec{y}') f(\vec{y}').$$

Definition 6.3.5. We define the k-particle unfused SHS6V model location process $\vec{x}(t) = (x_1(t) \le \cdots \le x_k(t))$ so that $\varphi(\vec{x}(t))$ is the left-finite unfused SHS6V model. We say $\vec{y}(t) = (y_1(t) \le \cdots \le y_k(t))$ is a k-particle reversed unfused SHS6V model location process if $-\vec{y}(t) = (-y_k(t) \le \cdots \le$

$-y_1(t))$ is a k-particle unfused SHS6V model location process.

Note that for the reversed k-particle SHS6V model $\vec{y}(t)$, we denote by $\mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{x}, \vec{y}, t, s)$ the transition probability from $\vec{y}(s) = \vec{x}$ to $\vec{y}(t) = \vec{y}$. Apparently, one has

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{x}, \vec{y}, t, s) = \big(\widetilde{\mathscr{B}}_{\alpha(s)} \cdots \widetilde{\mathscr{B}}_{\alpha(t-1)}\big)(\vec{x}, \vec{y}).$$

It follows from [CP16, Corollary 2.14] (or the Yang-Baxter equation [BP18, Section 3]) that $\widetilde{\mathscr{B}}_{\alpha(i)}$ commutes with itself for different values of i (i.e. $\widetilde{\mathscr{B}}_{\alpha(i)}\widetilde{\mathscr{B}}_{\alpha(j)} = \widetilde{\mathscr{B}}_{\alpha(j)}\widetilde{\mathscr{B}}_{\alpha(i)}$). Consequently,

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{x}, \vec{y}, t, s) = \big(\widetilde{\mathscr{B}}_{\alpha(t-1)} \cdots \widetilde{\mathscr{B}}_{\alpha(s)}\big)(\vec{x}, \vec{y}). \tag{6.3.1}$$

Let us first state the $J = 1$ version of Markov duality.

Proposition 6.3.6 ([CP16], Proposition 2.21)**.** *For all $k \in \mathbb{Z}_{\geq 1}$, the $J = 1$ left-finite SHS6V model $\vec{\eta}(t) \in \mathbb{G}$ (Definition 6.3.3) and k-particle $J = 1$ reversed SHS6V model location process $\vec{y}(t)$ (Definition 6.3.4) are dual with respect to the functional $H : \mathbb{G} \times \mathbb{Y}^k \to \mathbb{R}$*

$$H(\vec{\eta}, \vec{y}) = \prod_{i=1}^{k} q^{-N_{y_i}(\vec{\eta})}, \tag{6.3.2}$$

recall $N_y(\vec{\eta}) = \sum_{i \leq y} \eta_i$.

In [Kua18], the author discovers a Markov duality for a multi-species version of the SHS6V model. For our application, we explain how to degenerate this result to a two particle SHS6V model duality. Before stating the proposition, let us recall the notation of q-deformed quantity

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}, \quad [n]_q^! := \prod_{i=1}^{n} [i]_q, \quad \binom{n}{k}_q := \frac{[n]_q^!}{[k]_q^! [n-k]_q^!}.$$

Proposition 6.3.7. *The $J = 1$ left-finite SHS6V model $\vec{\eta}(t)$ and the two particle $J = 1$ reversed*

SHS6V model location process $\vec{y}(t)$ are dual with respect to

$$G(\vec{\eta},(y_1,y_2)) = \begin{cases} q^{-2N_{y_1}(\vec{\eta})}[I-\eta_{y_1}]_{q^{\frac{1}{2}}}[I-1-\eta_{y_1}]_{q^{\frac{1}{2}}} q^{\eta_{y_1}} & \textit{if } y_1 = y_2; \\ \frac{[I-1]_{q^{\frac{1}{2}}}}{[I]_{q^{\frac{1}{2}}}} q^{-N_{y_1}(\vec{\eta})} q^{-N_{y_2}(\vec{\eta})}[I-\eta_{y_1}]_{q^{\frac{1}{2}}}[I-\eta_{y_2}]_{q^{\frac{1}{2}}} q^{\frac{1}{2}\eta_{y_1}} q^{\frac{1}{2}\eta_{y_2}} & \textit{if } y_1 < y_2. \end{cases} \quad (6.3.3)$$

We remark that there is a misstatement in [Kua18, Theorem 4.10]. The particles in the process $\mathbb{Z}Z$ and $\mathbb{Z}Z_{rev}$ were stated to jump to the left and to the right respectively. However, after discussing with the author, we realize that the right statement is that the particles in $\mathbb{Z}Z$ jump to the right and those in $\mathbb{Z}Z_{rev}$ jump to the left.

Proof. This is a degeneration from [Kua18, Theorem 4.10]. By taking the species number $n = 1$, the spin parameter $m_x = I$ for all $x \in \mathbb{Z}$ as well as replacing q by $q^{\frac{1}{2}}$, the multi-species SHS6V model considered in [Kua18] degenerates to the $J = 1$ SHS6V model (see Section 2.6.2 of [Kua18] for detail). Then Theorem 4.10 of [Kua18] reduces to: The $J = 1$ left-finite SHS6V model $\vec{\xi}(t)$ and the $J = 1$ reversed SHS6V model $\vec{\eta}(t)$ are dual with respect to the functional

$$G_1(\vec{\xi},\vec{\eta}) = \prod_{x\in\mathbb{Z}} [\eta_x]^!_{q^{\frac{1}{2}}} [I-\eta_x]^!_{q^{\frac{1}{2}}} \binom{I-\xi_x}{\eta_x}_{q^{\frac{1}{2}}} q^{-\frac{1}{2}\xi_x(\sum_{z>x} 2\eta_z+\eta_x)}.$$

Swapping the role of left and right, which makes the particles in $\vec{\xi}(t)$ jump to the left and those in $\vec{\eta}(t)$ jump to the right. Then $\vec{\eta}(t)$ becomes the $J = 1$ left-finite SHS6V model and $\vec{\xi}(t)$ becomes the $J = 1$ reversed SHS6V model. They are dual with respect to the functional

$$\begin{aligned} G_2(\vec{\eta},\vec{\xi}) &= \prod_{x\in\mathbb{Z}} [\eta_x]^!_{q^{\frac{1}{2}}} [I-\eta_x]^!_{q^{\frac{1}{2}}} \binom{I-\xi_x}{\eta_x}_{q^{\frac{1}{2}}} q^{-\frac{1}{2}\xi_x(\sum_{z<x} 2\eta_z+\eta_x)}, \\ &= \prod_{x\in\mathbb{Z}} [\eta_x]^!_{q^{\frac{1}{2}}} [I-\eta_x]^!_{q^{\frac{1}{2}}} \binom{I-\xi_x}{\eta_x}_{q^{\frac{1}{2}}} q^{-\xi_x N_x(\vec{\eta})+\frac{1}{2}\xi_x\eta_x}. \end{aligned} \quad (6.3.4)$$

Assuming $\vec{\xi}(t)$ has two particles, recall the bijective map $\phi : \mathbb{W}^2_I \to \mathbb{M}^2$ (take $k = 2$) in Definition 6.3.2, then $\vec{y}(t) = \phi^{-1}(\vec{\xi}(t))$ is the $J = 1$ reversed two particle location process. The $J = 1$ left-finite SHS6V model $\vec{\eta}(t)$ and the two particle $J = 1$ reversed SHS6V model location process

$\vec{y}(t) = (y_1(t) \le y_2(t))$ are dual with respect to $G_2(\vec{\eta}, \phi^{-1}(y_1, y_2))$, where $\vec{\xi} = \phi(y_1, y_2)$ is given by

$$\xi_x = \begin{cases} 2\mathbf{1}_{\{x=y_1\}} & \text{if } y_1 = y_2 \\ \mathbf{1}_{\{x=y_1\}} + \mathbf{1}_{\{x=y_2\}} & \text{if } y_1 < y_2 \end{cases}, \quad \text{for all } x \in \mathbb{Z}.$$

In addition, note that

$$[\eta_x]^!_{q^{\frac{1}{2}}} [I - \eta_x]^!_{q^{\frac{1}{2}}} \binom{I - \xi_x}{\eta_x}_{q^{\frac{1}{2}}} = \begin{cases} [I]_{q^{\frac{1}{2}}} & \text{if } \xi_x = 0, \\ [I - \eta_x]_{q^{\frac{1}{2}}} & \text{if } \xi_x = 1, \\ \frac{[I-\eta_x]_{q^{\frac{1}{2}}} [I-1-\eta_x]_{q^{\frac{1}{2}}}}{[I-1]_{q^{\frac{1}{2}}}} & \text{if } \xi_x = 2. \end{cases} \tag{6.3.5}$$

When $\vec{\xi} = \phi(y_1, y_2)$, there are at most two values for $x \in \mathbb{Z}$ so that $\xi_x \neq 0$. To make sense of the infinite product in (6.3.4), one needs to normalize $G_2(\vec{\eta}, \vec{\xi})$ by dividing each factor in the product (6.3.4) by $[I]_{q^{\frac{1}{2}}}$. After such normalization, it is straightforward via (6.3.5) that $G_2(\vec{\eta}, \phi(y_1, y_2))$ equals the functional $G(\vec{\eta}, (y_1, y_2))$ in (6.3.3) up to a constant factor. □

We note that the duality functional in (6.3.2) and (6.3.3) does not depend on parameter α. By Markov property and the commutative property between $\widetilde{\mathcal{B}}_{\alpha(i)}$ for different value of i, it is clear that the same Markov dualities in Proposition 6.3.6 and Proposition 6.3.7 apply for the left-finite unfused SHS6V model.

Corollary 6.3.8. *For all $k \in \mathbb{Z}_{\ge 1}$, the left-finite unfused SHS6V model $\vec{\eta}(t) \in \mathbb{G}$ (Definition 6.1.3) and the reversed k-particle unfused SHS6V model location process $\vec{y}(t) \in \mathbb{W}_I^k$ (Definition 6.3.5) are dual with respect to the functional H in* (6.3.2). *The left-finite SHS6V model $\vec{\eta}(t)$ and the two particle reversed unfused SHS6V model location process $\vec{y}(t)$ are dual with respect to the functional G in* (6.3.3).

Proof. Due to Proposition 6.3.6, we see that for all $\vec{\eta} \in \mathbb{G}$ and $\vec{y} \in \mathbb{W}_I^k$,

$$\mathbb{E}\big[H(\vec{\eta}(t), \vec{y}) \big| \vec{\eta}(t-1) = \vec{\eta}\big] = \sum_{\vec{x} \in \mathbb{W}_I^k} \widetilde{\mathcal{B}}_{\alpha(t-1)}(\vec{y}, \vec{x}) H(\vec{\eta}, \vec{x}). \tag{6.3.6}$$

Using Markov property and applying (6.3.6) repetitively, we see that

$$\begin{aligned}\mathbb{E}\big[H(\vec{\eta}(t),\vec{y})\big|\vec{\eta}(s)=\vec{\eta}\big] &= \sum_{\vec{x}\in\mathbb{W}_I^k}\big(\widetilde{\mathscr{B}}_{\alpha(s)}\cdots\widetilde{\mathscr{B}}_{\alpha(t-1)}\big)(\vec{y},\vec{x})H(\vec{\eta},\vec{x})\\ &= \sum_{\vec{x}\in\mathbb{W}_I^k}\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big(\vec{y},\vec{x},t,s\big)H(\vec{\eta},\vec{x})\\ &= \mathbb{E}\big[H(\vec{\eta},\vec{y}(t))\big|\vec{y}(s)=\vec{y}\big]\end{aligned}$$

Here, the second equality follows from (6.3.1). This proves the desired duality with respect to the functional H. The duality with respect to the functional G follows by a similar argument. □

For our application, we like to extend the dualities stated in Proposition 6.3.6 and Proposition 6.3.7 to the bi-infinite SHS6V model. Denote by

$$\widetilde{D}(t,y_1,y_2)=\begin{cases} q^{-2N(t,y_1)}[I-\eta_{y_1}(t)]_{q^{\frac12}}[I-1-\eta_{y_1}(t)]_{q^{\frac12}}q^{\eta_{y_1}(t)} & \text{if } y_1=y_2;\\ \frac{[I-1]_{q^{\frac12}}}{[I]_{q^{\frac12}}}q^{-N(t,y_1)}q^{-N(t,y_2)}[I-\eta_{y_1}(t)]_{q^{\frac12}}[I-\eta_{y_2}(t)]_{q^{\frac12}}q^{\frac12\eta_{y_1}(t)}q^{\frac12\eta_{y_2}(t)} & \text{if } y_1<y_2.\end{cases} \tag{6.3.7}$$

Here $\vec{\eta}(t)=(\eta_x(t))_{x\in\mathbb{Z}}$ is the bi-infinite unfused SHS6V model defined in Definition 6.2.3 and $N(t,y)$ is the associated height function.

Corollary 6.3.9. *For the bi-infinite unfused SHS6V model $\vec{\eta}(t)$, for $\vec{y}=(y_1\le\cdots\le y_k)\in\mathbb{W}_I^k$ one has*

$$\mathbb{E}\Big[\prod_{i=1}^k q^{-N(t,y_i)}\Big|\mathcal{F}(s)\Big]=\sum_{\vec{x}\in\mathbb{W}_I^k}\mathbf{P}_{\overleftarrow{SHS6V}}\big(\vec{y},\vec{x},t,s\big)\prod_{i=1}^k q^{-N(s,x_i)}. \tag{6.3.8}$$

For $y_1\le y_2\in\mathbb{Z}$ (Since $I\ge 2$, this is equivalent to $(y_1,y_2)\in\mathbb{W}_I^2$)

$$\mathbb{E}\big[\widetilde{D}(t,y_1,y_2)\big|\mathcal{F}(s)\big]=\sum_{x_1\le x_2\in\mathbb{Z}^2}\mathbf{P}_{\overleftarrow{SHS6V}}\big((y_1,y_2),(x_1,x_2),t,s\big)\widetilde{D}(s,x_1,x_2). \tag{6.3.9}$$

Proof. Let us prove (6.3.8) in the first place. Given initial condition of the bi-infinite unfused SHS6V model $\vec{\eta}(0)$, we construct a sequence of left-finite SHS6V model $\vec{\eta}^w(t)$ with initial condition $\vec{\eta}^w(0) := (\eta_i(0)\mathbf{1}_{\{i\geq w\}})_{i\in\mathbb{Z}}$. We denote by $N^w(t,y)$ the associated height function. The first duality in Corollary 6.3.8 implies that for any $w \in \mathbb{Z}$

$$\mathbb{E}\Big[\prod_{i=1}^{k} q^{-N^w(t,y_i)}\Big|\mathcal{F}(s)\Big] = \sum_{\vec{x}\in\mathbb{W}_I^k} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big(\vec{y},\vec{x},t,s\big)\prod_{i=1}^{k} q^{-N^w(s,x_i)}. \tag{6.3.10}$$

Let us show the LHS and RHS of (6.3.10) approximates those of (6.3.8) as $w \to -\infty$.

For the approximation of the LHS, as $|\eta_x(0)| \leq I$ for all $x \in \mathbb{Z}$, we have $|N^w(0,y_i)| \leq I|y_i|$. Moreover, in a single time step, $N^w(t,y_i)$ may change by at most one, hence for all $w \in \mathbb{Z}$

$$\begin{aligned} |N^w(t,y_i)| &\leq |N^w(0,y_i)| + t \\ &\leq y_i I + t. \end{aligned} \tag{6.3.11}$$

Therefore, for fixed $t \in \mathbb{Z}_{\geq 0}$ and $q > 1$, $\prod_{i=1}^k q^{-N^w(t,y_i)}$ is uniformly bounded. Via Lemma 6.2.1, we know that $N^w(t,y_i) \to N(t,y_i)$ in probability, by conditional dominated convergence theorem, one has

$$\lim_{w\to-\infty}\mathbb{E}\Big[\prod_{i=1}^{k} q^{-N^w(t,y_i)}\Big|\mathcal{F}(s)\Big] = \mathbb{E}\Big[\prod_{i=1}^{k} q^{-N(t,y_i)}\Big|\mathcal{F}(s)\Big].$$

For the RHS approximation, according to Definition 6.3.5, when there is only one particle in the reversed SHS6V model location process, it jumps to the left (at time t) as a geometric random variables with parameter $\frac{\nu+\alpha(t)}{1+\alpha(t)}$. When there are k particles, they jump to the left (at time t) as k independent geometric random variables with parameter $\frac{\nu+\alpha(t)}{1+\alpha(t)}$ except when they hit each other. So there exists constant C such that for all $t, \vec{x}, \vec{y}$

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big(\vec{y},\vec{x},t+1,t\big) \leq C\prod_{i=1}^{k}\left(\frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{|y_i-x_i|}.$$

Denote by $\theta = \sup_{t \in \mathbb{Z}_{\geq 0}} \frac{\nu+\alpha(t)}{1+\alpha(t)}$, one has

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{y}, \vec{x}, t+1, t) \leq C \prod_{i=1}^{k} \theta^{|y_i - x_i|}. \tag{6.3.12}$$

For fixed $s \leq t$, observing that $\mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{y}, \vec{x}, t, s)$ can be written as a $(t-s)$-fold convolution of one-step transition probability. The convolution can be expanded into a sum over all trajectories from $\vec{y} = (y_1, \dots, y_k)$ to $\vec{x} = (x_1, \dots, x_k)$. The contribution of each trajectories can be bounded by the product of $t-s$ one-step transition probability, which is upper bounded by the RHS of (6.3.12). As the particles in the reversed SHS6V model can only jump to the left, the number of the trajectories can be upper bounded by $\prod_{i=1}^{k} \binom{|x_i - y_i| + t - s}{t-s}$. We obtain

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{y}, \vec{x}, t, s) \leq C \prod_{i=1}^{k} \binom{|x_i - y_i| + t - s}{t - s} \theta^{|y_i - x_i|} \tag{6.3.13}$$

Furthermore, it is readily verified that under Condition 6.1.1

$$q^I \theta = \sup_{t \in \mathbb{Z}_{\geq 0}} \frac{1 + q^I \alpha(t)}{1 + \alpha(t)} < 1.$$

Using the bounds in (6.3.11) and (6.3.13), fix $s \leq t \in \mathbb{Z}_{\geq 0}$ and $\vec{y} \in \mathbb{W}_I^k$, we have for all $\vec{x} \in \mathbb{W}_I^k$

$$\begin{aligned}
\mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{y}, \vec{x}, t, s) q^{-N^w(s, x_i)} &\leq C \prod_{i=1}^{k} \binom{|x_i - y_i| + t - s}{t - s} \theta^{|y_i - x_i|} q^{I|x_i|}, \\
&\leq C \prod_{i=1}^{k} \binom{|x_i - y_i| + t - s}{t - s} (q^I \theta)^{|y_i - x_i|}, \\
&\leq C \prod_{i=1}^{k} \delta^{|y_i - x_i|}
\end{aligned}$$

for some constant $0 < \delta < 1$. Since $N^w(s, x_i) \to N(s, x_i)$ in probability, we find that

$$\sum_{x \in \mathbb{W}_I^I} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{y}, \vec{x}, t, s) \prod_{i=1}^{k} q^{-N^w(s, x_i)} \longrightarrow \sum_{x \in \mathbb{W}_I^k} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}(\vec{y}, \vec{x}, t, s) \prod_{i=1}^{k} q^{-N(s, x_i)} \quad \text{in probability.}$$

Therefore, We conclude (6.3.8). The proof of (6.3.9) is similar to (6.3.8), where we consider instead

$$\widetilde{D}^w(t,y_1,y_2) = \begin{cases} q^{-2N^w(t,y_1)}[I-\eta^w_{y_1}(t)]_{q^{\frac{1}{2}}}[I-1-\eta^w_{y_1}(t)]_{q^{\frac{1}{2}}}q^{\eta^w_{y_1}(t)} & \text{if } y_1 = y_2; \\ \frac{[I-1]_{q^{\frac{1}{2}}}}{[I]_{q^{\frac{1}{2}}}}q^{-N^w(t,y_1)}q^{-N^w(t,y_2)}[I-\eta^w_{y_1}(t)]_{q^{\frac{1}{2}}}[I-\eta^w_{y_2}(t)]_{q^{\frac{1}{2}}}q^{\frac{1}{2}\eta^w_{y_1}(t)}q^{\frac{1}{2}\eta^w_{y_2}(t)} & \text{if } y_1 < y_2. \end{cases}$$

Applying the second duality in Corollary 6.3.8, we find that

$$\mathbb{E}\big[\widetilde{D}^w(t,y_1,y_2)\big|\mathcal{F}(s)\big] = \sum_{x_1\le x_2\in\mathbb{Z}^2} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((y_1,y_2),(x_1,x_2),t,s\big)\widetilde{D}^w(s,x_1,x_2).$$

By taking $w\to-\infty$ and using similar approximation, we conclude (6.3.9). □

6.4 Integral formula for the two particle transition probability

In this section, we give an explicit integral formula for $\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1,x_2),(y_1,y_2),t,s\big)$ (note that for the rest of the paper, we prefer to swap the order of (x_1,x_2) and (y_1,y_2) in the notation compared with the RHS of (6.3.9)). Our approach is to utilize the generalized Fourier theory (Bethe ansatz) developed in [BCPS15]. Let us review a few results obtained in [BCPS15] and [CP16] on which we rely to derive the integral formula.

Definition 6.4.1. For $\vec{y}\in(y_1\le\cdots\le y_k)\in\mathbb{Z}^k$, we define the left and right Bethe ansatz eigenfunction[10]

$$\Psi^\ell_{\vec{w}}(\vec{y}) = \sum_{\sigma\in S_k}\prod_{1\le B<A\le k}\frac{w_{\sigma(A)}-qw_{\sigma(B)}}{w_{\sigma(A)}-w_{\sigma(B)}}\prod_{i=1}^k\left(\frac{1-w_{\sigma(j)}}{1-\nu w_{\sigma(j)}}\right)^{-x_{k+1-j}},$$

$$\Psi^r_{\vec{w}}(\vec{y}) = (-1)^k(1-q)^k q^{\frac{k(k-1)}{2}}m_{q,\nu}(\vec{y})\sum_{\sigma\in S_k}\prod_{1\le B<A\le k}\frac{w_{\sigma(A)}-q^{-1}w_{\sigma(B)}}{w_{\sigma(A)}-w_{\sigma(B)}}\prod_{i=1}^k\left(\frac{1-w_{\sigma(j)}}{1-\nu w_{\sigma(j)}}\right)^{x_{k+1-j}},$$

[10]Comparing with the original definition for Bethe ansatz function defined in (2.11) and (2.14) of [BCPS15], we reverse the order of components in the vector: We prefer to write $\vec{y}=(y_1\le\cdots\le y_k)$ instead of $\vec{y}=(y_1\ge\cdots\ge y_k)$.

where S_k is the permutation group of $\{1, \dots, k\}$ and

$$m_{q,\nu}(\vec{y}) := \prod_{i=1}^{M(\vec{y})} \frac{(\nu; q)_{c_i}}{(q; q)_{c_i}}, \tag{6.4.1}$$

where $(c_1, \dots, c_{M(\vec{y})})$ denotes the cluster number in $\vec{y}$, i.e. $\vec{y} = (y_1 = \dots = y_{c_1} < y_{c_1+1} = \dots = y_{c_1+c_2} < \dots)$.

It turns out that $\Psi^{\ell}_{\vec{w}}$ are the eigenfunctions of the operator $\widetilde{\mathscr{B}}_\alpha$ defined in Definition 6.3.4.

Lemma 6.4.2 (Proposition 2.12 of [CP16])**.** *For all $k \in \mathbb{Z}_{\geq 1}$ and $\vec{w} = (w_1, \dots, w_k) \in \mathbb{C}^k$ such that $\left|\frac{1-w_i}{1-\nu w_i} \frac{\alpha+\nu}{1+\alpha}\right| < 1$ for all $i \in \{1, \dots, k\}$,*

$$\big(\widetilde{\mathscr{B}}_\alpha \Psi^{\ell}_{\vec{w}}\big)(\vec{y}) = \left(\prod_{i=1}^{k} \frac{1 + \alpha q w_i}{1 + \alpha w_i}\right) \Psi^{\ell}_{\vec{w}}(\vec{y})$$

[BCPS15] shows that the left and right Bethe ansatz eigenfunctions enjoy the following bi-orthogonal relation.

Lemma 6.4.3 (Corollary 3.13 of [BCPS15])**.** *For $0 < q, \nu < 1$ and $k \in \mathbb{Z}_{\geq 1}$ $\vec{x} = (x_1 \leq \dots \leq x_k) \in \mathbb{Z}^k$ and $\vec{y} = (y_1 \leq \dots \leq y_k) \in \mathbb{Z}^k$,*

$$\sum_{\lambda \vdash k} \oint_\gamma \cdots \oint_\gamma dm^q_\lambda(\vec{w}) \prod_{i=1}^{\ell(\lambda)} \frac{1}{(w_i, q)_{\lambda_j} (\nu w_i, q)_{\lambda_j}} \Psi^{\ell}_{\vec{w} \circ \lambda}(\vec{x}) \Psi^{r}_{\vec{w} \circ \lambda}(\vec{y}) = \mathbf{1}_{\{\vec{x} = \vec{y}\}} \tag{6.4.2}$$

Some notations must be specified here. γ is a very small circular contour around 1 so as to exclude all the poles of the integrand except 1. The Plancherel measure is defined as

$$dm^q_\lambda(\vec{w}) = \frac{(-1)^k (1-q)^k q^{-k(k-1)/2}}{m_1! m_2! \dots} det \left[\frac{1}{w_i q^{\lambda_i} - w_i}\right]_{i,j=1}^{\ell(\lambda)} \prod_{i=1}^{k} q^{\lambda_i(\lambda_i - 1)/2} w_i^{\lambda_j} \frac{dw_i}{2\pi \mathbf{i}}, \tag{6.4.3}$$

where the sum in (6.4.3) *is taken over the partition λ of k, that is to say, $\lambda = (\lambda_1 \geq \dots \geq \lambda_s) \in \mathbb{Z}^s_{\geq 1}$ with $\sum_{i=1}^{s} \lambda_i = k$, $\ell(\lambda) = s$ is the* length *of the partition λ. For instance, the partitions of $k = 3$ are given by $(2, 1)$ and $(1, 1, 1)$. We denote by m_j to be number of components that equal j in λ so* *that* $\lambda = 1^{m_1}2^{m_2}\dots$. *Furthermore, we define*

$$\vec{w}\circ\lambda := (w_1,\dots,q^{\lambda_1-1}w_1, w_2,\dots,q^{\lambda_2-1}w_2,\dots,w_s,\dots,q^{\lambda_s-1}w_s).$$

We are in a position to present our formula.

Theorem 6.4.4. *Assume* $I \geq 2$, *for any* $x_1 \leq x_2 \in \mathbb{Z}$ *and* $y_1 \leq y_2 \in \mathbb{Z}$, *the two point transition probability of reversed SHS6V model admits the following integral formula*

$$\begin{aligned}
&\mathbf{P}_{\overleftarrow{SHS6V}}\big((x_1,x_2),(y_1,y_2),t,s\big)\\
&= c(y_1,y_2)\Bigg[\oint_{C_R}\oint_{C_R}\prod_{i=1}^{2}\widetilde{\mathfrak{D}}(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\widetilde{\mathfrak{R}}(z_i,t,s)z_i^{x_i-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\\
&\quad-\oint_{C_R}\oint_{C_R}\widetilde{\mathfrak{F}}(z_1,z_2)\prod_{i=1}^{2}\widetilde{\mathfrak{D}}(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\widetilde{\mathfrak{R}}(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\\
&\quad+\mathsf{Res}_{z_1=\widetilde{\mathfrak{s}}(z_2)}\oint_{C_R}\oint_{C_R}\widetilde{\mathfrak{F}}(z_1,z_2)\prod_{i=1}^{2}\widetilde{\mathfrak{D}}(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\widetilde{\mathfrak{R}}(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\Bigg],
\end{aligned}\tag{6.4.4}$$

where C_R *is a circle centered at zero with a large enough radius* R *so as to include all the poles of all the integrands. In addition,*

$$\begin{aligned}
c(y_1,y_2) &:= \mathbf{1}_{\{y_1<y_2\}} + \frac{1-q\nu}{(1+q)(1-\nu)}\mathbf{1}_{\{y_1=y_2\}},\\
\widetilde{\mathfrak{D}}(z) &:= \frac{(1+\alpha q^J)z-(\nu+\alpha q^J)}{(1+\alpha)z-(\nu+\alpha)},\\
\widetilde{\mathfrak{R}}(z,t,s) &:= \prod_{k=s+J\lfloor\frac{t-s}{J}\rfloor}^{t-1}\frac{(1+\alpha(k)q)z-(\nu+\alpha(k)q)}{(1+\alpha(k))z-(\nu+\alpha(k))},\\
\widetilde{\mathfrak{F}}(z_1,z_2) &:= \frac{q\nu-\nu+(\nu-q)z_2+(1-q\nu)z_1+(q-1)z_1z_2}{q\nu-\nu+(\nu-q)z_1+(1-q\nu)z_2+(q-1)z_1z_2},\\
\widetilde{\mathfrak{s}}(z) &:= \frac{(1-q\nu)z-\nu(1-q)}{(q-\nu)+(1-q)z}.
\end{aligned}\tag{6.4.5}$$

Note that $z_1 = \widetilde{\mathfrak{s}}(z_2)$ corresponds to the pole produced by the denominator of $\widetilde{\mathfrak{F}}(z_1, z_2)$ and

$$\mathsf{Res}_{z_1=\widetilde{\mathfrak{s}}(z_2)} \oint_{C_R} \oint_{C_R} \widetilde{\mathfrak{F}}(z_1, z_2) \prod_{i=1}^{2} \widetilde{\mathfrak{D}}(z_i, t, s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}$$

denotes the residue of the double contour integral above at the pole $z_1 = \widetilde{\mathfrak{s}}(z_2)$.

Proof of Theorem 6.4.4. The first step to prove Theorem 6.4.4 is utilizing the bi-orthogonality of the Bethe ansatz function. Taking $k = 2$ in the previous lemma, since the possible partition is either $\lambda = (1, 1)$ or $\lambda = (2)$, we obtain

$$\begin{aligned}\mathbf{1}_{\{(x_1,x_2)=(y_1,y_2)\}} &= \oint_\gamma \oint_\gamma dm^q_{(1,1)}(w_1, w_2) \prod_{i=1}^{2} \frac{1}{(1-w_i)(1-\nu w_i)} \Psi^\ell_{(w_1,w_2)}(x_1, x_2) \Psi^r_{(w_1,w_2)}(y_1, y_2) \\ &+ \oint_\gamma dm^q_{(2)}(w) \frac{1}{(w, q)_2 (\nu w, q)_2} \Psi^\ell_{(w,qw)}(x_1, x_2) \Psi^r_{(w,qw)}(y_1, y_2). \qquad (6.4.6)\end{aligned}$$

Note that according to the previous lemma, (6.4.6) holds only for $0 < q, \nu < 1$, we want to extend this identity to $q > 1$ and $\nu = q^{-I}$. This extension can be justified by analytic continuation. Note that the RHS of (6.4.6) is an analytic function of q, ν in a suitable domain which connects $\{(q, \nu) : (q, \nu) \in (0, 1)^2\}$ and $\{(q, \nu) : q > 1, \nu = q^{-I}\}$. The reason behind is that after plugging in $\nu = q^{-I}$, there is no new pole of integrand generated inside γ (Here we use the assumption $I \geq 2$, this analytic continuation argument is not valid when $I = 1$, see Remark 6.4.5).

Let us now fix $y_1 \leq y_2 \in \mathbb{Z}$ on both side of (6.4.6) and treat both sides as functions of (x_1, x_2). We denote by the operator

$$\widetilde{\mathscr{B}}_\alpha(s, t) := \widetilde{\mathscr{B}}_\alpha(s) \cdots \widetilde{\mathscr{B}}_\alpha(t-1).$$

Acting the operator $\widetilde{\mathscr{B}}_\alpha(s, t)$ on both side of (6.4.6). For the LHS, it is clear that

$$\big(\widetilde{\mathscr{B}}_\alpha(s, t) 1_{\{\cdot=(y_1,y_2)\}}\big)(x_1, x_2) = \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1, x_2), (y_1, y_2), t, s\big).$$

For the RHS, we move $\widetilde{\mathscr{B}}_\alpha(s,t)$ inside the integrand, which yields

$$\begin{aligned}
&\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1,x_2),(y_1,y_2),t,s\big)\\
&=\oint_\gamma\oint_\gamma dm^q_{(1,1)}(w_1,w_2)\prod_{i=1}^2\frac{1}{(1-w_i)(1-\nu w_i)}\big(\widetilde{\mathscr{B}}_\alpha(s,t)\Psi^\ell_{(w_1,w_2)}\big)(x_1,x_2)\Psi^r_{(w_1,w_2)}(y_1,y_2)\\
&+\oint_\gamma dm^q_{(2)}(w)\frac{1}{(w,q)_2(\nu w,q)_2}\big((\widetilde{\mathscr{B}}_\alpha(s,t)\Psi^\ell_{(w,qw)}\big)(x_1,x_2)\Psi^r_{(w,qw)}(y_1,y_2). \qquad (6.4.7)
\end{aligned}$$

Due to Lemma 6.4.2 (note that γ is a small circle around 1, hence w_1,w_2 satisfy the condition of Lemma 6.4.2),

$$\begin{aligned}
\big(\widetilde{\mathscr{B}}_\alpha(s,t)\Psi^\ell_{(w_1,w_2)}\big)(x_1,x_2)&=\prod_{i=1}^2\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\Psi^\ell_{(w_1,w_2)}(x_1,x_2),\\
\big(\widetilde{\mathscr{B}}_\alpha(s,t)\Psi^\ell_{(w,qw)}\big)(x_1,x_2)&=\prod_{k=s}^{t-1}\left(\frac{1+\alpha(k)qw}{1+\alpha(k)w}\cdot\frac{1+\alpha(k)q^2w}{1+\alpha(k)qw}\right)\Psi^\ell_{(w_1,w_2)}(x_1,x_2),\\
&=\prod_{k=s}^{t-1}\left(\frac{1+\alpha(k)q^2w}{1+\alpha(k)w}\right)\Psi^\ell_{(w_1,w_2)}(x_1,x_2).
\end{aligned}$$

We name the first term on the RHS of (6.4.7) I_1 and the second term I_2,

$$I_1=\oint_\gamma\oint_\gamma dm^q_{(1,1)}(w_1,w_2)\prod_{i=1}^2\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\Psi^\ell_{(w_1,w_2)}(x_1,x_2)\Psi^r_{(w_1,w_2)}(y_1,y_2), \qquad (6.4.8)$$

$$I_2=\oint_\gamma dm^q_{(2)}(w)\frac{1}{(w,q)_2(\nu w,q)_2}\prod_{k=s}^{t-1}\left(\frac{1+\alpha(k)q^2w}{1+\alpha(k)w}\right)\Psi^\ell_{(w,qw)}(x_1,x_2)\Psi^r_{(w,qw)}(y_1,y_2). \qquad (6.4.9)$$

We compute I_1 in the first place. In the integrand of (6.4.8), the function $\Psi^\ell_{(w_1,w_2)}(x_1,x_2)$ is a symmetrization of

$$\frac{w_2-qw_1}{w_2-w_1}\prod_{i=1}^2\left(\frac{1-w_i}{1-\nu w_i}\right)^{-x_{3-i}}$$

Furthermore, all other terms of the integrand (6.4.8) are symmetric function of w_1,w_2. In addition, we are integrating w_1, w_2 along the same contour, this allows us to desymmetrize the integrand

$$I_1 = 2\oint_\gamma\oint_\gamma dm^q_{(1,1)}(w_1,w_2)\prod_{i=1}^2\left(\frac{1}{(1-w_i)(1-\nu w_i)}\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\frac{w_2-qw_1}{w_2-w_1}$$
$$\times\prod_{i=1}^2\left(\frac{1-w_i}{1-\nu w_i}\right)^{-x_{3-i}}\Psi^r_{(w_1,w_2)}(y_1,y_2). \tag{6.4.10}$$

We readily calculate

$$dm^q_{(1,1)}(w_1,w_2) = \frac{(1-q)^2q^{-1}}{2}\det\left[\frac{1}{w_iq-w_j}\right]^2_{i,j=1}\prod_{i=1}^2\frac{w_i dw_i}{2\pi\mathbf{i}} = \frac{(w_1-w_2)^2}{2(w_2-qw_1)(qw_2-w_1)}\prod_{i=1}^2\frac{dw_i}{2\pi\mathbf{i}} \tag{6.4.11}$$

$$\Psi^r_{\vec{w}}(y_1,y_2) = q(1-q)^2m_{q,\nu}(y)\sum_{\sigma\in S_2}\prod_{1\le B<A\le 2}\frac{w_{\sigma(A)}-q^{-1}w_{\sigma(B)}}{w_{\sigma(A)}-w_{\sigma(B)}}\prod_{i=1}^2\left(\frac{1-w_{\sigma(i)}}{1-\nu w_{\sigma(i)}}\right)^{y_{3-i}}$$
$$= (1-q)^2m_{q,\nu}(y)\left(\frac{qw_2-w_1}{w_2-w_1}\prod_{i=1}^2\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_{3-i}} + \frac{qw_1-w_2}{w_1-w_2}\prod_{i=1}^2\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i}\right) \tag{6.4.12}$$

Replacing the terms $dm^q_{(1,1)}(w_1,w_2)$ and $\Psi^r_{\vec{w}}(y_1,y_2)$ in the integrand of (6.4.10) by the RHS of (6.4.11) and (6.4.12), one sees that

$$I_1 = (1-q)^2m_{q,\nu}(y_1,y_2)\Bigg[\oint_\gamma\oint_\gamma\prod_{i=1}^2\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_{3-i}-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}$$
$$-\oint_\gamma\oint_\gamma\frac{qw_1-w_2}{qw_2-w_1}\prod_{i=1}^2\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}\Bigg],$$
$$= (1-q)^2m_{q,\nu}(y_1,y_2)\Bigg[\oint_\gamma\oint_\gamma\prod_{i=1}^2\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_i}\frac{dw_i}{2\pi\mathbf{i}}$$
$$-\oint_\gamma\oint_\gamma\frac{qw_1-w_2}{qw_2-w_1}\prod_{i=1}^2\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}\Bigg]. \tag{6.4.13}$$

For the second equality above, we changed $\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_{3-i}-x_{3-i}}$ to $\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_i}$, due to the symmetry of

w_1, w_2.

We proceed to compute I_2, by a straightforward calculation

$$m^q_{(2)}(w) = \frac{(q-1)w}{q+1}\frac{dw}{2\pi i}, \qquad \Psi^{\ell}_{w,qw}(x_1,x_2) = (1+q)\left(\frac{1-w}{1-\nu w}\right)^{-x_1}\left(\frac{1-qw}{1-\nu qw}\right)^{-x_2},$$
$$\Psi^r_{w,qw}(y_1,y_2) = (1-q)^2 m_{q,\nu}(y)(1+q)\left(\frac{1-w}{1-\nu w}\right)^{y_2}\left(\frac{1-qw}{1-q\nu w}\right)^{y_1}.$$

Inserting these expressions into the integrand of (6.4.9) gives

$$I_2 = (1-q)^2 m_{q,\nu}(y_1,y_2)\oint_\gamma \frac{(q^2-1)w}{(w,q)_2(\nu w,q)_2}\prod_{k=s}^{t-1}\left(\frac{1+\alpha(k)q^2w}{1+\alpha(k)w}\right)\left(\frac{1-w}{1-\nu w}\right)^{y_2-x_1}\left(\frac{1-qw}{1-q\nu w}\right)^{y_1-x_2}\frac{dw}{2\pi\mathbf{i}}.$$

A crucial observation is that one can verify directly

$$I_2 = -(1-q)^2 m_{q,\nu}(y_1,y_2)\mathrm{Res}_{w_1=qw_2}\oint_\gamma\oint_\gamma \frac{qw_1-w_2}{qw_2-w_1}\prod_{i=1}^{2}\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)$$
$$\times\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}, \tag{6.4.14}$$

Note that $\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1,x_2),(y_1,y_2),t,s\big) = I_1+I_2$, using (6.4.13) and (6.4.14) one has

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1,x_2),(y_1,y_2),t,s\big)$$
$$= (1-q)^2 m_{q,\nu}(y_1,y_2)\Bigg[\oint_\gamma\oint_\gamma\prod_{i=1}^{2}\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_i}\frac{dw_i}{2\pi\mathbf{i}}$$
$$-\oint_\gamma\oint_\gamma\frac{qw_1-w_2}{qw_2-w_1}\prod_{i=1}^{2}\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}$$
$$-\mathrm{Res}_{w_1=qw_2}\oint_\gamma\oint_\gamma\frac{qw_1-w_2}{qw_2-w_1}\prod_{i=1}^{2}\frac{1}{(1-w_i)(1-\nu w_i)}\left(\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}\Bigg].$$

Recall that $\alpha(k) = \alpha q^{\mathrm{mod}_J(k)}$ for all k, we can simplify the telescoping product in the integrand via

$$\prod_{k=s}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i} = \left(\frac{1+\alpha q^J w_i}{1+\alpha w_i}\right)^{\lfloor\frac{t-s}{J}\rfloor}\prod_{k=s+J\lfloor\frac{t-s}{J}\rfloor}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}.$$

Furthermore, referring to the expression (6.4.1) and (6.4.5), we notice that $(1-q)^2 m_{q,\nu}(y_1,y_2) = c(y_1,y_2)$. Thereby,

$$
\begin{aligned}
&\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1,x_2),(y_1,y_2),t,s\big)\\
&= c(y_1,y_2)\Bigg[\oint_\gamma\oint_\gamma \prod_{i=1}^{2} \frac{1}{(1-w_i)(1-\nu w_i)}\left(\frac{1+\alpha q^J w_i}{1+\alpha w_i}\right)^{\lfloor\frac{t-s}{J}\rfloor}\left(\prod_{k=s+J\lfloor\frac{t-s}{J}\rfloor}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_i}\frac{dw_i}{2\pi\mathbf{i}}\\
&\quad-\oint_\gamma\oint_\gamma \frac{qw_1-w_2}{qw_2-w_1}\prod_{i=1}^{2} \frac{1}{(1-w_i)(1-\nu w_i)}\left(\frac{1+\alpha q^J w_i}{1+\alpha w_i}\right)^{\lfloor\frac{t-s}{J}\rfloor}\left(\prod_{k=s+J\lfloor\frac{t-s}{J}\rfloor}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}\\
&\quad-\mathsf{Res}_{w_1=qw_2}\oint_\gamma\oint_\gamma \frac{qw_1-w_2}{qw_2-w_1}\prod_{i=1}^{2} \frac{1}{(1-w_i)(1-\nu w_i)}\left(\frac{1+\alpha q^J w_i}{1+\alpha w_i}\right)^{\lfloor\frac{t-s}{J}\rfloor}\left(\prod_{k=s+J\lfloor\frac{t-s}{J}\rfloor}^{t-1}\frac{1+\alpha(k)qw_i}{1+\alpha(k)w_i}\right)\times\\
&\quad\times\left(\frac{1-w_i}{1-\nu w_i}\right)^{y_i-x_{3-i}}\frac{dw_i}{2\pi\mathbf{i}}\Bigg].
\end{aligned}
\tag{6.4.15}
$$

Lastly, we transform the small circle γ surrounding 1 into the big circle C_R via a change of variable

$$w_i = \Xi(z_i) = \frac{1-z_i}{\nu-z_i} \quad (\text{equivalently } z_i = \frac{1-\nu w_i}{1-w_i}), \qquad i=1,2.$$

By the following relations

$$
\begin{aligned}
&\frac{q\Xi(z_1)-\Xi(z_2)}{q\Xi(z_2)-\Xi(z_1)} = \widetilde{\mathfrak{F}}(z_1,z_2), && \frac{1-\Xi(z_i)}{1-\nu\Xi(z_i)} = z_i^{-1},\\
&\frac{1+\alpha q^J\Xi(z_i)}{1+\alpha\Xi(z_i)} = \widetilde{\mathfrak{D}}(z_i), && \prod_{k=s+J\lfloor\frac{t-s}{J}\rfloor}^{t-1}\frac{1+\alpha(k)q\Xi(z_i)}{1+\alpha(k)\Xi(z_i)} = \widetilde{\mathfrak{R}}(z_i,t,s),\\
&\frac{d\Xi(z_i)}{(1-\Xi(z_i))(1-\nu\Xi(z_i))} = \frac{dz_i}{(1-\nu)z_i},
\end{aligned}
$$

we obtain

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1,x_2),(y_1,y_2),t,s\big)$$

$$
\begin{aligned}
&= c(y_1, y_2)\bigg[\oint_{C_R}\oint_{C_R}\prod_{i=1}^{2}\widetilde{\mathfrak{D}}(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\widetilde{\mathfrak{R}}(z_i,t,s)z_i^{x_i-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\\
&\quad-\oint_{C_R}\oint_{C_R}\widetilde{\mathfrak{F}}(z_1,z_2)\prod_{i=1}^{2}\widetilde{\mathfrak{D}}(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\widetilde{\mathfrak{R}}(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\\
&\quad+\mathrm{Res}_{z_1=\widetilde{\mathfrak{s}}(z_2)}\oint_{C_R}\oint_{C_R}\widetilde{\mathfrak{F}}(z_1,z_2)\prod_{i=1}^{2}\widetilde{\mathfrak{D}}(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\widetilde{\mathfrak{R}}(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\bigg]. \qquad (6.4.16)
\end{aligned}
$$

This concludes the proof of Theorem 6.4.4. Note that we change the sign in front of the residue from (6.4.15) to (6.4.16). This is due to the fact that, before employing the change of variable, the set of the poles $\{qw_1 : w_1 \in \gamma\}$ lies outside the w_2-contour γ, while after the change of variable, the set of the pole $\{\widetilde{\mathfrak{s}}(z_1) : z_1 \in C_r\}$ lies inside the z_2-contour C_r, since R is chosen to be sufficiently large. □

Remark 6.4.5. We remark that our argument in proving that (6.4.6) holds for $q > 1$ and $\nu = q^{-I}$ does not work when $I = 1$. The reason is as follows: Note that the factor $\frac{1}{(\nu z_1, q)_2}$ in the integrand of (6.4.6) gives a pole for the z_1-contour at $z_1 = \nu^{-1}q$. Before the substitution of $\nu = q^{-1}$, this pole lies outside the contour γ. Yet after substituting $\nu = q^{-1}$, the pole becomes $z_1 = 1$, which runs inside the contour γ, hence the argument of analytic continuation fails. This issue is also addressed in [BCPS19], when the authors try to reproduce the integral formula for the k particle ASEP transition probability (which first appears in [TW08, Theorem 2.1]) via analytic continuation of (6.4.2). For a similar reason, our method does not yield the general k particle transition probability formula of the SHS6V model with any fixed parameter I.

6.5 Microscopic Hopf-Cole transform and SHE

In this section, we first define the microscopic Hopf-Cole transform $Z(t, x)$, which is an exponential transform of the height function $N(t, x)$. Using $k = 1$ version of duality of (6.3.8), it turns out that $Z(t, x)$ satisfies a discrete version of SHE. As the Hopf-Cole solution to the KPZ equation is the logarithm of the mild solution of the SHE, this reduces the proof of Theorem 6.1.6 to proving that $Z(t, x)$ converges to the solution of SHE. We will also derive the two dualities for $Z(t, x)$ in

Lemma 6.5.2, as a tilted version of (6.3.8). This will be used in the proof of self-averaging property Proposition 6.6.8.

6.5.1 Microscopic Hopf-Cole Transform

We first study a one particle version of the unfused SHS6V model location process (Definition 6.3.5). When there is only one particle, it performs a random walk $X'(t) = \sum_{k=0}^{t-1} R'(k)$ where $R'(k)$ are independent (but not same distributed) $\mathbb{Z}_{\geq 0}$-valued random variables with distribution

$$\mathbb{P}\big(R'(k) = n\big) = \begin{cases} \frac{1+q\alpha(k)}{1+\alpha(k)} & \text{if } n = 0; \\ \frac{\alpha(k)(1-q)}{1+\alpha(k)}\big(1 - \frac{\nu+\alpha(k)}{1+\alpha(k)}\big)\big(\frac{\nu+\alpha(k)}{1+\alpha(k)}\big)^{n-1} & \text{if } n \in \mathbb{Z} \\ 0 & \text{else.} \end{cases}$$

By tilting and centering $R'(k)$ with respect to $\mathbb{E}\big[q^{\rho R'(k)}\mathbf{1}_{\{R'(k)=\cdot\}}\big]$, we define a tilted random walk $X(t) = \sum_{k=0}^{t-1} R(k)$, where $R(k)$ are independent $\mathbb{Z}_{\geq 0} - \mu(k)$ valued with distribution[11]

$$\mathbb{P}\big(R(k) = n - \mu(k)\big) = \begin{cases} \lambda(k)\frac{1+q\alpha(k)}{1+\alpha(k)} & \text{if } n = 0; \\ \lambda(k)\frac{\alpha(k)(1-q)}{1+\alpha(k)}\big(1 - \frac{\nu+\alpha(k)}{1+\alpha(k)}\big)\big(\frac{\nu+\alpha(k)}{1+\alpha(k)}\big)^{n-1} q^{\rho n} & \text{if } n \in \mathbb{Z}_{\geq 1} \\ 0 & \text{else.} \end{cases} \tag{6.5.1}$$

Here, $\lambda(k) = \big(\mathbb{E}\big[q^{\rho R(k)}\big]\big)^{-1}$ is the normalizing parameter and $\mu(k)$ is the centering parameter which makes $\mathbb{E}\big[R(k)\big] = 0$. Under straightforward calculation, we see that

$$\lambda(k) = \frac{1 + \alpha(k) - q^{\rho}(\alpha(k) + \nu)}{1 + a(k)q - q^{\rho}(\alpha(k)q + \nu)}, \tag{6.5.2}$$

$$\mu(k) = \frac{\alpha(k)(1-q)(1-\nu)q^{\rho}}{(1 + \alpha(k)q - q^{\rho}(\alpha(k)q + \nu))(1 + \alpha(k) - q^{\rho}(\alpha(k) + \nu))}. \tag{6.5.3}$$

[11] The tilted and centered random walk $X(t)$ provides the heat kernel $\mathsf{p}(t+1, t)$ for the discrete SHE (6.5.7) satisfied by the microscopic Hopf-Cole transform (6.5.6), which is an exponential transform of the LHS of (6.1.10).

We remark that $\lambda(k)$ (respectively $\mu(k)$) are J periodic in the sense that $\lambda(k) = \lambda(J + k)$ (respectively $\mu(k) = \mu(J + k)$). Denote by

$$\hat{\lambda}(t) := \prod_{k=0}^{t-1} \lambda(k), \qquad \hat{\mu}(t) := \sum_{k=0}^{t-1} \mu(k), \qquad \Xi(t, s) := \mathbb{Z} - \hat{\mu}(t) + \hat{\mu}(s), \qquad \Xi(t) := \Xi(t, 0). \tag{6.5.4}$$

It can be verified that the parameter λ, μ defined in (6.1.9) satisfies

$$\lambda = \hat{\lambda}(J), \qquad \mu = \hat{\mu}(J),$$

hence, one has

$$\hat{\lambda}(Jt) = \lambda^t, \qquad \hat{\mu}(Jt) = \mu t. \tag{6.5.5}$$

We define the *microscopic Hopf-Cole transform* for $x \in \Xi(t)$ as

$$Z(t, x) := \hat{\lambda}(t) q^{-(N(t,x+\hat{\mu}(t)) - \rho(x+\hat{\mu}(t)))}. \tag{6.5.6}$$

For $x \in \Xi(t, s)$, we set $\mathsf{p}(t, s, x) := \mathbb{P}\big(X(t) - X(s) = x\big)$. Denote by the convolution

$$(\mathsf{p}(t, s) * f(s))(x) := \sum_{y \in \Xi(s)} \mathsf{p}(t, s, x - y) f(s, y).$$

We set

$$K(t, x) := N(t, x) - N(t + 1, x), \qquad \overline{K}(t, x) := K(t, x) - \mathbb{E}\big[K(t, x) \big| \mathcal{F}(t)\big].$$

We sometimes call $K(t, x)$ the *flux*, since it records the number of particles (either zero or one) that move across the position x between time t and $t + 1$. Now we present the discrete SHE satisfied by the microscopic Hopf-Cole transform of the unfused SHS6V model.

Proposition 6.5.1. *For $t \in \mathbb{Z}_{\geq 0}$ and $x \in \Xi(t)$, $Z(t,x)$ satisfies the following discrete SHE*

$$Z(t+1, x-\mu(t)) = (\mathsf{p}(t+1,t) * Z(t))(x-\mu(t)) + M(t,x), \tag{6.5.7}$$

where

$$M(t,x) = \lambda(t)(q-1)Z(t, x+\hat{\mu}(t))\overline{K}(t, x+\hat{\mu}(t)). \tag{6.5.8}$$

Furthermore, $M(t,x)$ is a martingale increment, i.e. $\mathbb{E}\big[M(t,x)\big|\mathcal{F}(t)\big] = 0$. The conditional quadratic variation of $M(t,x)$ equals

$$\mathbb{E}\big[M(t,x_1)M(t,x_2)\big|\mathcal{F}(t)\big] = \left(q^{\rho}\frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{|x_1-x_2|}\Theta_1(t, x_1\wedge x_2)\Theta_2(t, x_1 \wedge x_2), \quad x_1, x_2 \in \Xi(t), \tag{6.5.9}$$

where

$$\Theta_1(t,x) := q\lambda(t)Z(t,x) - \big(\mathsf{p}(t+1,t) * Z(t)\big)(x-\mu(t)), \tag{6.5.10}$$

$$\Theta_2(t,x) := -\lambda(t)Z(t,x) + \big(\mathsf{p}(t+1,t) * Z(t)\big)(x-\mu(t)). \tag{6.5.11}$$

Proof. We first show that $M(t,x)$ is a martingale increment. Note by (6.5.7),

$$M(t,x) = Z(t+1, x-\mu(t)) = (\mathsf{p}(t+1,t) * Z(t))(x-\mu(t)).$$

Taking $k = 1$ in the duality (6.3.8), one has

$$\mathbb{E}\big[Z(t+1, x-\mu(t))\big|\mathcal{F}(t)\big] = (\mathsf{p}(t+1,t) * Z(t))(x-\mu(t)).$$

Hence,

$$M(t,x) = Z(t+1, x-\mu(t)) - \mathbb{E}\big[Z(t+1, x-\mu(t))\big|\mathcal{F}(t)\big], \tag{6.5.12}$$

which implies $\mathbb{E}\big[M(t,x)\big|\mathcal{F}(t)\big] = 0$.

We turn to justify (6.5.8). Note that by (6.5.6)

$$Z(t+1, x-\mu(t)) = \lambda(t)Z(t,x)q^{N(t,x+\hat{\mu}(t))-N(t+1,x+\hat{\mu}(t))} = \lambda(t)Z(t,x)q^{K(t,x+\hat{\mu}(t))}.$$

Since $K(t, x+\hat{\mu}(t)) \in \{0,1\}$,

$$Z(t+1, x-\mu(t)) = \lambda(t)Z(t,x) + \lambda(t)(q-1)Z(t,x)K(t,x+\hat{\mu}(t)). \tag{6.5.13}$$

Combining with (6.5.12) gives

$$\begin{aligned} M(t,x) &= \lambda(t)(q-1)Z(t,x)\big(K(t,x+\hat{\mu}(t)) - \mathbb{E}\big[K(t,x+\hat{\mu}(t))\big|\mathcal{F}(t)\big]\big), \\ &= \lambda(t)(q-1)Z(t,x)\overline{K}(t,x+\hat{\mu}(t)), \end{aligned} \tag{6.5.14}$$

which gives the desired equality.

We turn our attention to (6.5.9). Define the short notation $E'\big[\,\cdot\,\big] := \mathbb{E}\big[\,\cdot\,\big|\mathcal{F}(t)\big]$ and write Var′, Cov′ to be the corresponding conditional variance and covariance. We assume without loss of generosity $x_1 \leq x_2$ and use shorthand notation $x'_i := x_i + \hat{\mu}(t) \in \mathbb{Z}, i = 1,2$. Owing to (6.5.14),

$$\begin{aligned} \mathbb{E}'\big[M(t,x'_1)M(t,x'_2)\big] &= \lambda(t)^2(q-1)^2 Z(t,x_1)Z(t,x_2)\mathbb{E}'\big[\overline{K}(t,x'_1)\overline{K}(t,x'_2)\big], \\ &= \lambda(t)^2(q-1)^2 Z(t,x_1)Z(t,x_2)\mathrm{Cov}'\big(K(t,x'_1), K(t,x'_2)\big). \end{aligned} \tag{6.5.15}$$

Define

$$L_{x'_1,x'_2}(t) = \prod_{z=x'_1+1}^{x'_2} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big), \tag{6.5.16}$$

$$K_{x'_1,x'_2}(t) = \sum_{y'=x'_1+1}^{x'_2} \prod_{z=y'+1}^{x'_2} \Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t)\Big)B(t,z,\eta_z(t)),$$

where B, B' are defined in (6.2.1). Since B, B' are all independent, due to the expression (6.2.5) of $K(t,x_1') = N(t,x_1') - N(t+1,x_1')$ provided by (6.2.5), it is straightforward that conditioning on $\mathcal{F}(t)$, $(K_{x_1',x_2'}(t), L_{x_1',x_2'}(t))$ are independent with $K(t,x_1')$. Furthermore, (6.2.5) implies

$$K(t,x_2') = K_{x_1',x_2'}(t) + L_{x_1',x_2'}(t)K(t,x_1').$$

By the independence, we see that

$$\mathrm{Cov}'\big(K(t,x_1'), K(t,x_2')\big) = \mathbb{E}'\big[L_{x_1',x_2'}(t)\big]\mathrm{Var}'\big(K(t,x_1')\big) \tag{6.5.17}$$

Referring to (6.5.16),

$$\mathbb{E}'\big[L_{x_1',x_2'}(t)\big] = \prod_{z=x_1'+1}^{x_2'} \mathbb{E}'\big[B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\big] = \left(\frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{x_2'-x_1'} \prod_{z=x_1'+1}^{x_2'} q^{\eta_z(t)}.$$

Inserting this into the RHS of (6.5.17), we find that

$$\begin{aligned}\mathrm{Cov}'\big(K(t,x_1), K(t,x_2)\big) &= \left(\frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{x_2'-x_1'} \prod_{z=x_1'+1}^{x_2'} q^{\eta_z(t)}\big(\mathbb{E}'\big[K^2(t,x_1')\big] - \mathbb{E}'\big[K(t,x_1')\big]^2\big),\\ &= \left(\frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{x_2-x_1} \prod_{z=x_1'+1}^{x_2'} q^{\eta_z(t)}\mathbb{E}'\big[K(t,x_1')\big]\big(1-\mathbb{E}'\big[K(t,x_1')\big]\big). \end{aligned} \tag{6.5.18}$$

Here, the last equality follows from the fact $K(t,x_1')^2 = K(t,x_1')$. Furthermore, due to (6.5.13),

$$\mathbb{E}'\big[K(t,x_1')\big] = \frac{\mathbb{E}\big[Z(t+1,x_1-\mu(t)) - \lambda(t)Z(t,x_1)\big|\mathcal{F}(t)\big]}{\lambda(t)(q-1)Z(t,x_1)} = \frac{(\mathsf{p}(t+1,t)*Z(t))(x_1-\mu(t)) - \lambda(t)Z(t,x_1)}{\lambda(t)(q-1)Z(t,x_1)}.$$

Inserting this into the RHS of (6.5.18) yields

$$\mathrm{Cov}'\big(K(t,x_1), K(t,x_2)\big) = \left(\frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{x_2-x_1} \frac{(\mathsf{p}(t+1,t)*Z(t))(x_1-\mu(t)) - \lambda(t)Z(t,x_1)}{\lambda(t)(q-1)Z(t,x_1)}$$

$$\times \left(1 - \frac{(\mathsf{p}(t+1,t) * Z(t))(x_1 - \mu(t)) - \lambda(t) Z(t,x_1)}{\lambda(t)(q-1)Z(t,x_1)}\right) \prod_{z=x_1'+1}^{x_2'} q^{\eta_z(t)},$$

$$= \left(\frac{\nu + \alpha(t)}{1+\alpha(t)}\right)^{x_2 - x_1} \frac{\Theta_2(t,x_1)}{\lambda(t)(q-1)Z(t,x_1)} \cdot \frac{\Theta_1(t,x_1)}{\lambda(t)(q-1)Z(t,x_1)} \prod_{z=x_1'+1}^{x_2'} q^{\eta_z(t)}.$$

Using the fact $Z(t,x_2) = q^{\rho(x_2-x_1)} Z(t,x_1) \prod_{z=x_1'+1}^{x_2'} q^{-\eta_z(t)}$, we obtain

$$\mathrm{Cov}'\big(K(t,x_1), K(t,x_2)\big) = \left(q^\rho \frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{x_2-x_1} \frac{\Theta_1(t,x_1)}{\lambda(t)(q-1)Z(t,x_1)} \cdot \frac{\Theta_2(t,x_1)}{\lambda(t)(q-1)Z(t,x_2)}.$$

Combining with (6.5.15), we arrive at the desired (6.5.9). □

For $x \in \Xi(t)$, define

$$\widetilde{\eta}_x(t) := \eta_{x+\hat{\mu}(t)}(t).$$

We consider a tilted version of the duality functional $\widetilde{D}$ in (6.3.7), for $y_1 \le y_2 \in \Xi(t)$, define

$$D(t,y_1,y_2) := \begin{cases} Z(t,y_1)^2 \left[I - \widetilde{\eta}_{y_1}(t)\right]_{q^{\frac{1}{2}}} \left[I - 1 - \widetilde{\eta}_{y_1}(t)\right]_{q^{\frac{1}{2}}} q^{\widetilde{\eta}_{y_1}(t)} & \text{if } y_1 = y_2, \\ \frac{[I-1]_{q^{\frac{1}{2}}}}{[I]_{q^{\frac{1}{2}}}} Z(t,y_1) Z(t,y_2) [I - \widetilde{\eta}_{y_1}(t)]_{q^{\frac{1}{2}}} [I - \widetilde{\eta}_{y_2}(t)]_{q^{\frac{1}{2}}} q^{\frac{1}{2}\widetilde{\eta}_{y_1}(t)} q^{\frac{1}{2}\widetilde{\eta}_{y_2}(t)} & \text{if } y_1 < y_2. \end{cases} \tag{6.5.19}$$

We further define for $x_1, x_2 \in \Xi(t)$ and $y_1, y_2 \in \Xi(s)$,

$$\begin{aligned} &\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big) \\ &:= \left(\frac{\hat{\lambda}(t)}{\hat{\lambda}(s)}\right)^2 q^{\rho(x_1+x_2-y_1-y_2+2(\hat{\mu}(t)-\hat{\mu}(s)))} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big(x_1+\hat{\mu}(t), x_2+\hat{\mu}(t), y_1+\hat{\mu}(s), y_2+\hat{\mu}(s), t, s\big). \end{aligned} \tag{6.5.20}$$

Observe that $Z(t,x)$ is a tilted version of $q^{-N(t,x)}$, thus it is clear that it inherits the two dualities stated in Corollary 6.3.9.

Lemma 6.5.2. *For $s \le t \in \mathbb{Z}_{\ge 0}$ and $x_1 \le x_2 \in \Xi(t)$,*

$$\mathbb{E}\big[Z(t,x_1)Z(t,x_2)\big|\mathcal{F}(s)\big] = \sum_{y_1 \le y_2 \in \Xi(s)} \mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2), \tag{6.5.21}$$

$$\mathbb{E}\big[D(t,x_1,x_2)\big|\mathcal{F}(s)\big] = \sum_{y_1 \le y_2 \in \Xi(s)} \mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)D(s,y_1,y_2). \tag{6.5.22}$$

Proof. We use the shorthand notation $x_i' := x_i + \hat{\mu}(t)$. Referring to (6.5.6),

$$\mathbb{E}\big[Z(t,x_1)Z(t,x_2)\big|\mathcal{F}(s)\big] = \hat{\lambda}(t)^2 q^{\rho(x_1'+x_2')}\mathbb{E}\big[q^{-N(t,x_1')}q^{-N(t,x_2')}\big|\mathcal{F}(s)\big] \tag{6.5.23}$$

Using Corollary 6.3.9, we have

$$\mathbb{E}\big[q^{-N(t,x_1')}q^{-N(t,x_2')}\big|\mathcal{F}(s)\big] = \sum_{y_1' \le y_2' \in \mathbb{Z}^2} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1',x_2'),(y_1',y_2'),t,s\big)q^{-N(s,y_1')}q^{-N(s,y_2')},$$

$$= \sum_{y_1 \le y_2 \in \Xi(s)^2} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1+\hat{\mu}(t),x_2+\hat{\mu}(t),(y_1+\hat{\mu}(s),y_2+\hat{\mu}(s),t,s\big)q^{-N(s,y_1+\hat{\mu}(s))}q^{-N(s,y_2+\hat{\mu}(s))},$$

$$= \sum_{y_1 \le y_2 \in \Xi(s)^2} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1+\hat{\mu}(t),x_2+\hat{\mu}(t),(y_1+\hat{\mu}(s),y_2+\hat{\mu}(s),t,s\big)\frac{Z(s,y_1)Z(s,y_2)}{\hat{\lambda}(s)^2}q^{-2\hat{\mu}(s)}.$$

Inserting this into the RHS of (6.5.23), via a straightforward computation, we conclude (6.5.21). The second duality (6.5.22) follows from a similar argument, we do not repeat here. □

The following corollary follows from Theorem 6.4.4.

Corollary 6.5.3. *For all $x_1 \le x_2 \in \Xi(t)$ and $y_1 \le y_2 \in \Xi(s)$, we have*

$$\begin{aligned}
&\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big) \\
&= c(\vec{y})\Big[\oint_{C_R}\oint_{C_R}\prod_{i=1}^{2}\mathfrak{D}(z_i,t,s)z_i^{x_i-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i} - \oint_{C_R}\oint_{C_R}\mathfrak{F}(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i} \\
&\quad + \mathsf{Res}_{z_1=\mathfrak{s}(z_2)}\oint_{C_R}\oint_{C_R}\mathfrak{F}(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\Big].
\end{aligned} \tag{6.5.24}$$

where C_R is a circle centered at zero with a large enough radius R so as to include all the poles of *the integrands,* $c(\vec{y})$ *is defined in* (6.4.5) *and*

$$\mathfrak{D}(z) := \lambda z^{\mu} \frac{(1+\alpha q^J)q^{-\rho}z - (\nu + \alpha q^J)}{(1+\alpha)q^{-\rho}z - (\nu+\alpha)}, \tag{6.5.25}$$

$$\mathfrak{R}(z,t,s) := \prod_{k=s+J\lfloor \frac{t-s}{J} \rfloor}^{t-1} \lambda(k) z^{\mu(k)} \frac{(1+\alpha(k)q)q^{-\rho}z - (\nu+\alpha(k)q)}{(1+\alpha(k))q^{-\rho}z - (\nu+\alpha(k))}, \tag{6.5.26}$$

$$\mathfrak{F}(z_1,z_2) := \frac{q\nu - \nu + (\nu - q)q^{-\rho}z_2 + (1-q\nu)q^{-\rho}z_1 + (q-1)q^{-2\rho}z_1z_2}{q\nu - \nu + (\nu - q)q^{-\rho}z_1 + (1-q\nu)q^{-\rho}z_2 + (q-1)q^{-2\rho}z_1z_2}, \tag{6.5.27}$$

$$\mathfrak{s}(z) := \frac{(1-q\nu)q^{-\rho}z - \nu(1-q)}{(q-\nu)q^{-\rho} + (1-q)q^{-2\rho}z}. \tag{6.5.28}$$

Proof. Note that the integral formula for $\mathbf{P}_{\overleftarrow{\text{SHS6V}}}$ is given by (6.4.4), referring to (6.5.20), we find that

$$\mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big) = \left(\frac{\hat{\lambda}(t)}{\hat{\lambda}(s)}\right)^2 q^{\rho(x_1+x_2-y_1-y_2+2\hat{\mu}(t)-2\hat{\mu}(s))} \mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big(x_1+\hat{\mu}(t), x_2+\hat{\mu}(t), y_1+\hat{\mu}(s), y_2+\hat{\mu}(s), t, s$$

$$= c(\vec{y}) \cdot \left(\frac{\hat{\lambda}(t)}{\hat{\lambda}(s)}\right)^2 q^{\rho(x_1+x_2-y_1-y_2+2\hat{\mu}(t)-2\hat{\mu}(s))} \bigg[\oint_{C_R}\oint_{C_R} \prod_{i=1}^{2} \widetilde{\mathfrak{D}}(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \widetilde{\mathfrak{R}}(z_i,t,s) z_i^{x_i-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}$$

$$- \oint_{C_R}\oint_{C_R} \widetilde{\mathfrak{F}}(z_1,z_2) \prod_{i=1}^{2} \widetilde{\mathfrak{D}}(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \widetilde{\mathfrak{R}}(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}$$

$$+ \mathrm{Res}_{z_1=\widetilde{\mathfrak{s}}(z_2)} \oint_{C_R}\oint_{C_R} \widetilde{\mathfrak{F}}(z_1,z_2) \prod_{i=1}^{2} \widetilde{\mathfrak{D}}(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \widetilde{\mathfrak{R}}(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i} \bigg].$$

We refer to the context of Theorem 6.4.4 for the notation. Multiplying the constant $\big(\frac{\hat{\lambda}(t)}{\hat{\lambda}(s)}\big)^2 q^{\rho(x_1+x_2-y_1-y_2+2\hat{\mu}(t)-2\hat{\mu}(s))}$ to each term inside the square bracket above and applying change of variable $z_i \to q^{-\rho} z_i$ readily yield the desired formula. □

6.5.2 The SHE

Consider the KPZ equation with parameter V_* and D_* given in (6.1.12) and (6.1.13),

$$\mathcal{H}(t,x) = \frac{V_*}{2}\partial_x^2 \mathcal{H}(t,x) - \frac{V_*}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \sqrt{D_*}\xi(t,x), \tag{6.5.29}$$

As mentioned in Section 6.1.1, via formally applying Hopf-Cole transform, we say that $\mathcal{H}(t,x)$ is a Hopf-Cole solution of (6.5.29) if

$$\mathcal{H}(t,x) = -\log \mathcal{Z}(t,x),$$

where $\mathcal{Z}(t,x)$ is a *mild solution* of the SHE

$$\partial_t \mathcal{Z}(t,x) = \frac{V_*}{2}\partial_x^2 \mathcal{Z}(t,x) + \sqrt{D_*}\xi(t,x)\mathcal{Z}(t,x)$$

in the sense that it satisfies the following Duhamel form

$$\mathcal{Z}(t,x) = \int_{\mathbb{R}} p(V_* t, x-y)\mathcal{Z}^{\mathrm{ic}}(y)dy + \int_0^t \int_{\mathbb{R}} p(V_*(t-s), x-y)\mathcal{Z}(s,y)\sqrt{D_*}\xi(s,y)dsdy,$$

where $p(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ is the heat kernel. The stochastic heat equation has a unique mild solution $\mathcal{Z}(t,x)$, see [Cor12] and references therein.

We recall the *weakly asymmetric scaling* for the SHS6V model stated in Theorem 6.1.6

$$\text{For } \epsilon > 0, \text{ fix } I \in \mathbb{Z}_{\geq 2}, J \in \mathbb{Z}_{\geq 1} \text{ and } b \in \left(\frac{I+J-2}{I+J-1}, 1\right), \text{ set } q = e^{\sqrt{\epsilon}} \text{ and define } \alpha \text{ via } b = \frac{1+\alpha q}{1+\alpha}. \tag{6.5.30}$$

Such scaling corresponds to taking $b = 2, z = \frac{1}{2}, \kappa \to \sqrt{\epsilon}\kappa$ and keeping δ, D unchanged in (6.1.3). Note that all parameters in the SHS6V model rely on the generic parameters q, b, I, J, ρ, since under weakly asymmetry scaling, b, I, J, ρ are all fixed and $q = e^{\sqrt{\epsilon}}$, the evolution of the entire model depends on ϵ. As we will let ϵ go to zero, it suffices to consider all $\epsilon > 0$ *small enough*, which means that we only consider $\epsilon \in (0, \epsilon_0)$ for some generic but fixed threshold $\epsilon_0 > 0$.

Lemma 6.5.4. *Under weakly asymmetric scaling* (6.5.30)*, we have the following asymptotics near* $\epsilon = 0$

$$\frac{\nu + \alpha(t)}{1+\alpha(t)} = \frac{b(I + \mathrm{mod}_J(t)) - (I + \mathrm{mod}_J(t) - 1)}{b\,\mathrm{mod}_J(t) - (\mathrm{mod}_J(t) - 1)} + O(\epsilon^{\frac{1}{2}}),$$

$$\frac{\nu + q\alpha(t)}{1+\alpha(t)} = \frac{b(I+1+\mathrm{mod}_J(t)) - (I+\mathrm{mod}_J(t))}{b\mathrm{mod}_J(t) - (\mathrm{mod}_J(t)-1)} + O(\epsilon^{\frac{1}{2}}),$$
$$\frac{1 + q\alpha(t)}{1+\alpha(t)} = \frac{b(1+\mathrm{mod}_J(t)) - \mathrm{mod}_J(t)}{b\mathrm{mod}_J(t) - (\mathrm{mod}_J(t)-1)} + O(\epsilon^{\frac{1}{2}}),$$
$$\mu(t) = \frac{1}{I} + O(\epsilon^{\frac{1}{2}}), \qquad \lambda(t) = 1 - \frac{\rho\epsilon^{\frac{1}{2}}}{I} + O(\epsilon).$$

As notational convention, we denote $O(a)$ to be a generic quantity such that $\sup_{0<a<1} |O(a)|a^{-1} < \infty$.

Proof. For every $\epsilon > 0$, we have $q = e^{\sqrt{\epsilon}}$, $\nu = e^{-I\sqrt{\epsilon}}$ and $\alpha(t) = \alpha q^{\mathrm{mod}_J(t)} = \frac{1-b}{b-e^{\sqrt{\epsilon}}} e^{\sqrt{\epsilon}\mathrm{mod}_J(t)}$, where b, I, J, ρ are fixed. The relation of $\lambda(t)$ and $\mu(t)$ with ϵ is implied by (6.5.2) and (6.5.3) The verification of the above asymptotic is then straightforward. □

To highlight the dependence on ϵ under weakly asymmetric scaling, we denote by the microscopic Hopf-Cole transform $Z_\epsilon(t,x) := Z(t,x)$. Note that presently $Z_\epsilon(t,x)$ is only defined for $t \in \mathbb{Z}_{\geq 0}$ and $x \in \Xi(t)$, we extend $Z_\epsilon(t,x)$ to be a $C([0,\infty), C(\mathbb{R}))$-valued process by first linearly interpolating in $x \in \mathbb{Z}$, then in $t \in \mathbb{Z}_{\geq 0}$. This is slightly different from exponentiating the interpolated height function $N(t,x)$. Nevertheless, under the weak asymmetric scaling $q = e^{\sqrt{\epsilon}}$, it is straightforward to see that the difference between these two interpolation schemes is negligible as $\epsilon \downarrow 0$.

As a notational convention, we write $\|X\|_p := (\mathbb{E}|X|^p)^{\frac{1}{p}}$ for $p \geq 1$. Following the work of [BG97], we define the *near stationary initial data* for the unfused/fused SHS6V model.

Definition 6.5.5. Fix $\rho \in (0, I)$, we call the initial data $N_\epsilon(0,x)$ (equivalently $N^{\mathsf{f}}_\epsilon(0,x)$) **near stationary with density** ρ if for any $n \in \mathbb{Z}_{\geq 1}$ and $a \in (0, \frac{1}{2})$, there exists constant $u := u(n,a)$ and $C := C(n,a)$ such that for all $x, x' \in \mathbb{Z}$

$$\|Z_\epsilon(0,x)\|_n \leq Ce^{u\epsilon|x|}, \qquad \|Z_\epsilon(0,x) - Z_\epsilon(0,x')\|_n \leq C(\epsilon|x-x'|)^a e^{u\epsilon(|x|+|x'|)},$$

holds for $\epsilon > 0$ small enough.

Theorem 6.5.6. *Under weakly asymmetric scaling, assuming that $N_\epsilon(0,x)$ is near stationary with density ρ and for some $C(\mathbb{R})$-valued process $\mathcal{Z}^{ic}(x)$*

$$Z_\epsilon(0,x) \Rightarrow \mathcal{Z}^{ic}(0,x) \text{ in } C(\mathbb{R}) \text{ as } \epsilon \downarrow 0,$$

then

$$Z_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x) \Rightarrow \mathcal{Z}(t,x) \text{ in } C\big([0,\infty), C(\mathbb{R})\big) \text{ as } \epsilon \downarrow 0,$$

where $\mathcal{Z}(t,x)$ is the mild solution to the SHE

$$\partial_t \mathcal{Z}(t,x) = \frac{V_*}{2}\partial_x^2 \mathcal{Z}(t,x) + \sqrt{D_*}\xi(t,x)\mathcal{Z}(t,x), \tag{6.5.31}$$

with initial condition $\mathcal{Z}^{ic}(x)$.

As a consequence of the preceding theorem, we prove Theorem 6.1.6.

Proof of Theorem 6.1.6. Via the discussion in Section 6.5.2, $\mathcal{H}(t,x) = -\log \mathcal{Z}(t,x)$ solves the KPZ equation

$$\mathcal{H}(t,x) = \frac{V_*}{2}\partial_x^2 \mathcal{H}(t,x) - \frac{V_*}{2}\big(\partial_x \mathcal{H}(t,x)\big)^2 + \sqrt{D_*}\xi(t,x).$$

One has by (6.5.7),

$$\begin{aligned} Z_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x) &= \hat{\lambda}_\epsilon(t) e^{-\sqrt{\epsilon}\left(N_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x + \epsilon^{-2}\hat{\mu}_\epsilon(t)) - \rho(\epsilon^{-1}x + \epsilon^{-2}\hat{\mu}_\epsilon(t))\right)} \\ &= e^{-\sqrt{\epsilon}\left(N_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x + \epsilon^{-2}\hat{\mu}_\epsilon(t)) - \rho(\epsilon^{-1}x + \epsilon^{-2}\hat{\mu}_\epsilon(t))\right) + \log \hat{\lambda}_\epsilon(t)}. \end{aligned}$$

By Theorem 6.5.6 and continuous mapping theorem, we obtain

$$-\log Z_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x) \Rightarrow \mathcal{H}(t,x) \text{ in } C([0,\infty), C(\mathbb{R})).$$

In other words,

$$\sqrt{\epsilon}\big(N_\epsilon(\epsilon^{-2}t,\epsilon^{-1}x+\epsilon^{-2}\hat{\mu}_\epsilon(t))-\rho(\epsilon^{-1}x+\epsilon^{-2}\hat{\mu}_\epsilon(t))\big)-\log\hat{\lambda}_\epsilon(t)\Rightarrow\mathcal{H}(t,x)\text{ in }C([0,\infty),C(\mathbb{R})). \tag{6.5.32}$$

Note that we have $N^{\mathsf{f}}_\epsilon(t,x)=N_\epsilon(Jt,x)$ (although in fact, they only equal on the lattice due to different linear interpolation scheme, but it is obvious that the difference between them is negligible). Moreover, via (6.5.5)

$$\hat{\lambda}_\epsilon(Jt)=\lambda^t_\epsilon,\qquad \hat{\mu}_\epsilon(Jt)=\mu^t_\epsilon.$$

Therefore, replacing the time variable t with Jt in (6.5.32),

$$\sqrt{\epsilon}\big(N^{\mathsf{f}}_\epsilon(\epsilon^{-2}t,\epsilon^{-1}x+\epsilon^{-2}\mu_\epsilon t)-\rho(\epsilon^{-1}x+\epsilon^{-2}\mu_\epsilon t)\big)-t\log\lambda_\epsilon\Rightarrow\widetilde{\mathcal{H}}(t,x)\text{ in }C([0,\infty),C(\mathbb{R})),$$

where $\widetilde{\mathcal{H}}(t,x):=\mathcal{H}(Jt,x)$. It is straightforward to check that $\widetilde{\mathcal{H}}(t,x)$ satisfies the KPZ equation

$$\widetilde{\mathcal{H}}(t,x)=\frac{JV_*}{2}\partial_x^2\widetilde{\mathcal{H}}(t,x)-\frac{JV_*}{2}\big(\partial_x\widetilde{\mathcal{H}}(t,x)\big)^2+\sqrt{JD_*}\xi(t,x),$$

which concludes the proof of Theorem 6.1.6. □

6.6 Tightness and proof of Theorem 6.5.6

In this section, we prove Theorem 6.5.6 assuming Proposition 6.6.8, whose proof is postponed to Section 6.8. First of all, we prove the tightness of $\{Z_\epsilon(\epsilon^{-2}\cdot,\epsilon^{-1}\cdot)\}_{0<\epsilon<1}$, which indicates that as $\epsilon\downarrow 0$, $Z_\epsilon(\epsilon^{-2}\cdot,\epsilon^{-1}\cdot)$ converges weakly along a subsequence. To identify the limit as well as proving the convergence of the entire sequence, we appeal to the martingale problem of SHE that was first introduced in the work of [BG97]. Using approximation from the microscopic SHE (6.5.7) to the SHE in continuum, we show that any subsequential limit of $Z_\epsilon(\epsilon^{-2}\cdot,\epsilon^{-1}\cdot)$ satisfies the same martingale problem, hence is the mild solution of SHE.

Hereafter, we always assume that we are under weakly asymmetric scaling (6.5.30). In general, we will not specify the dependence of parameters on ϵ. We will also write q_ϵ,ν_ϵ, etc. when we do want to emphasize the dependence. The dependence on $I \in \mathbb{Z}_{\geq 2}, J \in \mathbb{Z}_{\geq 1}$, $b = \frac{1+\alpha q}{1+\alpha} \in (\frac{I+J-2}{I+J-1}, 1)$, $\rho \in (0, I)$ will not be indicated as they are fixed.

For the ensuing discussion, we will usually write C for constants. We might not generally specify when irrelevant terms are being absorbed into the constants. We might also write $C(T), C(\beta, T), \ldots$ when we want to specify which parameters the constant depends on. We say "for all $\epsilon > 0$ small enough" if the referred statement holds for all $0 < \epsilon < \epsilon_0$ for some generic but fixed threshold $\epsilon_0 > 0$ that may change from line to line.

6.6.1 Moment bounds and tightness

The goal of this section is to prove the following Kolmogorov-Chentsov type bound for the microscopic Hopf-Cole transform.

Proposition 6.6.1. *Assume that we start the SHS6V model with near stationary initial data with density $\rho \in (0, I)$. Given $n \in \mathbb{Z}_{\geq 1}$, $a \in (0, \frac{1}{2})$ and $T > 0$, there exists positive constants $C := C(n, a, T)$, $u := u(n, a)$ such that*

$$\|Z(t,x)\|_{2n} \leq Ce^{u\epsilon|x|}, \tag{6.6.1}$$

$$\|Z(t,x) - Z(t,x')\|_{2n} \leq C|\epsilon(x-x')|^a e^{u\epsilon(|x|+|x'|)}, \tag{6.6.2}$$

$$\|Z(t,x) - Z(t',x)\|_{2n} \leq C|\epsilon^2(t-t')|^{\frac{a}{2}} e^{2u\epsilon|x|}, \tag{6.6.3}$$

for all $t, t' \in [0, \epsilon^{-2}T]$ and $x, x' \in \mathbb{R}$.

We immediately deduce the tightness of $Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)$ once we have the moment bound above.

Corollary 6.6.2. *The law of $C([0,\infty), C(\mathbb{R}))$-valued process $\{Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)\}_{0<\epsilon<1}$ is tight.*

Proof. (6.6.1), (6.6.2) and (6.6.3) indicate that with large probability $\{Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)\}_{0<\epsilon<1}$ is uniformly bounded, uniformly spatially and uniformly temporally Hölder continuous. Applying Arzela-Ascoli theorem together with Prokhorov's theorem [Bil13] yields the desired result. □

For the proof of Proposition 6.6.1, we will basically follow the framework developed in [CGST20]. Let us begin with a technical lemma which will be frequently used for the rest of the paper.

Lemma 6.6.3. *Fix $T > 0$, for any $u > 0$, there exists $\beta_0 > 0$ such that for all $\beta > \beta_0$ and $C(\beta) > 0$, there exists ϵ_0 such that for all positive $\epsilon < \epsilon_0$, $t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$ and $x \in \Xi(t)$, the following inequality holds*[12]

$$\sum_{y \in \Xi(t)} e^{-\frac{\beta|x-y|}{\sqrt{t+1}+C(\beta)}} e^{u\epsilon|y|} \leq 2\sqrt{t+1}e^{u\epsilon|x|}.$$

Proof. Take $\beta_0 = 4\sqrt{T}u$, for $\beta > \beta_0$ and arbitrary $C(\beta) > 0$, due to $t \in [0, \epsilon^{-2}T]$, one has

$$\frac{\beta|x|}{\sqrt{t+1}+C(\beta)} \geq \frac{\beta\epsilon|x|}{\sqrt{T+\epsilon^2}+C(\beta)\epsilon} \geq 2u\epsilon|x|$$

holds for $\epsilon < \epsilon_0$, where is ϵ_0 is to be chosen small enough. Thereby,

$$\begin{aligned}
\sum_{y \in \Xi(t)} e^{-\frac{\beta|x-y|}{\sqrt{t+1}+C(\beta)}} e^{u\epsilon|y|} &\leq e^{u\epsilon|x|} \sum_{y \in \Xi(t)} e^{-\frac{\beta|x-y|}{\sqrt{t+1}+C(\beta)}} e^{u\epsilon|x-y|}, \\
&\leq e^{u\epsilon|x|} \sum_{y \in \mathbb{Z}} e^{-\frac{\beta|y|}{\sqrt{t+1}+C(\beta)}} e^{u\epsilon|y|} \\
&\leq e^{u\epsilon|x|} \sum_{y \in \mathbb{Z}} e^{-\frac{\beta|y|}{2(\sqrt{t+1}+C(\beta))}} \\
&\leq 2\sqrt{t+1}e^{u\epsilon|x|}.
\end{aligned}$$

Here, the last inequality follows from

$$\sum_{x \in \Xi(t)} e^{-\frac{\beta|y|}{2(\sqrt{t+1}+C(\beta))}} \leq \frac{2}{1-e^{-\frac{\beta}{2(\sqrt{t+1}+C(\beta))}}} \leq 2\sqrt{t+1}.$$

Thus, we conclude the lemma. □

The following estimate for the one particle transition probability will be useful in proving

[12] Here, $C(\beta)$ can be any positive constant, though for application it usually depends on the value of β.

Proposition 6.6.1.

Lemma 6.6.4. *For any $u, T \in (0,\infty)$ and $a \in (0,1)$, there exists constant C (depending on a, u, T) such that*

$$(i)\ \mathsf{p}(t,s,x) \leq C(t-s+1)^{-\frac{1}{2}}, \qquad (ii) \sum_{x\in\Xi(t,s)} \mathsf{p}(t,s,x)e^{u\epsilon|x|} \leq C,$$

$$(iii) \sum_{x\in\Xi(t,s)} |x|^a \mathsf{p}(t,s,x)e^{u\epsilon|x|} \leq C(t-s+1)^{\frac{a}{2}}, \qquad (iv)\ |\mathsf{p}(t,s,x)-\mathsf{p}(t,s,x')| \leq C|x-x'|^a(t-s+1)^{-\frac{a+1}{2}}.$$

for $\epsilon > 0$ small enough and $s \leq t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$.

Proof. The proof is more or less analogous to [CGST20, Lemma 5.1]. We first claim that $\mathsf{p}(t,s,x)$ admits the following integral formula

$$\mathsf{p}(t,s,x) = \oint_{C_R} \big(\mathfrak{D}(z)\big)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}(z,t,s) z^x \frac{dz}{2\pi \mathbf{i} z}, \tag{6.6.4}$$

where $\mathfrak{D}(z), \mathfrak{R}(z,t,s)$ are defined in (6.5.25) and (6.5.26) respectively and R is large enough so that the circle C_r includes all the singularities of the integrand. This claim can be proved by observing

$$\begin{aligned}
\mathbb{E}\big[z^{-R(k)}\big] &= \sum_{n=0}^{\infty} \mathbb{P}\big(R(k) = n - \mu(k)\big) z^{\mu(k)-n},\\
&= \lambda(k)\frac{1+q\alpha(k)}{1+\alpha(k)} z^{\mu(k)} + \sum_{n=1}^{\infty} \lambda(k)\Big(1 - \frac{1+q\alpha(k)}{1+\alpha(k)}\Big)\Big(1 - \frac{\nu+\alpha(k)}{1+\alpha(k)}\Big)\Big(\frac{\nu+\alpha(k)}{1+\alpha(k)}\Big)^{n-1} q^{\rho n} z^{\mu(k)-n},\\
&= \lambda(k) z^{\mu(k)} \frac{1+\alpha(k)q - (\nu+\alpha(k)q)q^{\rho} z^{-1}}{1+\alpha(k) - (\nu+\alpha(k))q^{\rho} z^{-1}}. \qquad (6.6.5)
\end{aligned}$$

This implies

$$\mathbb{E}\big[z^{-(X(t)-X(s))}\big] = \prod_{k=s}^{t-1} \mathbb{E}\big[z^{-R(k)}\big] = \big(\mathfrak{D}(z)\big)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}(z,t,s).$$

Via Fourier inversion formula, we have

$$\mathsf{p}(t,s,x) = \mathbb{P}\big(X(t)-X(s) = x\big) \oint_{C_r} \mathbb{E}\big[z^{-(X(t)-X(s)}\big] z^x \frac{dz}{2\pi\mathbf{i}z} = \oint_{C_R} \big(\mathfrak{D}(z)\big)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}(z,t,s) \frac{dz}{2\pi\mathbf{i}z},$$

In Section 6.7, we will obtain an upper bound of $\mathsf{p}(t,s,x)$ by applying steepest descent analysis to the integral formula above and we use this upper bound here in advance. Referring to (6.7.21), by taking $x_i - y_i \to x$, we obtain for all $\beta, T > 0$, there exists positive constant $C(\beta), C(\beta, T)$ such that for $\epsilon > 0$ small enough

$$\mathsf{p}(t,s,x) \le \frac{C(\beta,T)}{\sqrt{t-s+1}} e^{-\frac{\beta|x|}{\sqrt{t-s+1}+C(\beta)}}, \qquad t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}. \tag{6.6.6}$$

which gives (i). Using (6.6.6) together with Lemma 6.6.3 gives (ii)

$$\sum_{x \in \Xi(t,s)} \mathsf{p}(t,s,x) e^{u\epsilon|x|} \le \sum_{x \in \Xi(t,s)} \frac{C(\beta,T)}{\sqrt{t-s+1}} e^{-\frac{\beta|x|}{\sqrt{t-s+1}+C(\beta)}} e^{u\epsilon|x|} \le C.$$

For (iii), we see that

$$\sum_{x \in \Xi(t,s)} |x|^a \mathsf{p}(t,s,x) e^{u\epsilon|x|} \le \sum_{x \in \Xi(t,s)} C(\beta,T)|x|^a e^{-\frac{\beta|x|}{2(\sqrt{t-s+1}+C(\beta))}} \le C\big(\sqrt{t-s+1}+C(\beta)\big)^{a+1} \le C(t-s+1)^{\frac{a+1}{2}}.$$

For the second inequality above, we used the inequality

$$\sum_{x \in \Xi(t,s)} |x|^a e^{-b|x|} \le C \int_0^\infty x^a e^{-bx} dx \le C b^{-a-1}.$$

Finally, to prove (iv), one has by (6.7.24) (taking $\beta = 1$)

$$|\nabla p(t,s,x)| = |\mathsf{p}(t,s,x+1) - \mathsf{p}(t,s,x)| \le \frac{C(T)}{t-s+1} e^{-\frac{|x|}{\sqrt{t-s+1}+C}}.$$

Summing the above equation over $[x, x'-1]$ (assuming with out loss of generosity that $x < x'$), we obtain

$$\big|\mathsf{p}(t,s,x) - \mathsf{p}(t,s,x')\big| \le \frac{C(T)}{t-s+1} \sum_{y=x}^{x'-1} e^{-\frac{|y|}{\sqrt{t-s+1}+C}}$$

If we bound each term in the geometric sum by 1, we have $\big|\mathsf{p}(t,x) - \mathsf{p}(t,x')\big| \le \frac{C}{t+1}|x'-x|$. In addition, we can bound the geometric sum by

$$\sum_{y=x}^{x'-1} e^{-\frac{|y|}{\sqrt{t-s+1}+C}} \leq 2\sum_{y=0}^{\infty} e^{-\frac{|y|}{\sqrt{t-s+1}+C}} = \frac{2}{1-e^{-\frac{1}{\sqrt{t-s+1}+C}}} \leq C\sqrt{t-s+1},$$

which implies

$$\left|\mathsf{p}(t,s,x) - \mathsf{p}(t,s,x')\right| \leq \frac{C}{\sqrt{t-s+1}}.$$

Thereby,

$$\left|\mathsf{p}(t,s,x) \; - \mathsf{p}(t,s,x')\right| \leq \min\left(\frac{C}{t-s+1}|x-x'|, \frac{C}{\sqrt{t-s+1}}\right) \leq C|x-x'|^{a}(t-s+1)^{-\frac{a+1}{2}},$$

which concludes the proof of (iv). □

Recall the discrete SHE in Proposition 6.5.1

$$Z(t,x) = (\mathsf{p}(t,t-1) * Z(t-1))(x) + M(t-1, x+\mu(t-1)). \tag{6.6.7}$$

Iterating (6.6.7) for t times yields

$$Z(t,x) = (\mathsf{p}(t,0) * Z(0))(x) + Z_{mg}(t), \tag{6.6.8}$$

where the martingale $Z_{mg}(t)$ equals

$$Z_{mg}(t) = \sum_{s=0}^{t-1} \big(\mathsf{p}(t,s+1) * M(s)\big)(x+\mu(s)). \tag{6.6.9}$$

To estimate $Z(t,x)$, it suffices to estimate $(\mathsf{p}(t,0) * Z(0))(x)$ and $Z_{mg}(t)$ respectively. In general, the former one is easier to bound due to Lemma 6.6.4, while controlling the latter one is much harder. Following the style of [CGST20], to estimate $Z_{mg}(t)$, we need to establish the following two lemmas, which are in analogy with Lemma 5.2 and Lemma 5.3 of [CGST20].

Let $\mathcal{P}_{23}(n)$ denote the set of the partitions into intervals of 2 or 3 elements. Here, the interval refers to the set of form $U = [a, b] := [a, b] \cap \mathbb{Z}$, $a \le b \in \mathbb{Z}$. For example,

$$\mathcal{P}_{23}(6) = \{\{[1,2],[3,4],[5,6],\{[1,2],[3,6]\},\{[1,4],[5,6]\},\{[1,3],[4,6]\}\}.$$

For $\vec{y} = (y_1 \le \cdots \le y_n)$ and $U = [a, b]$, we define $|\vec{y}|_U = y_b - y_a$.

Lemma 6.6.5. *Fix $n \in \mathbb{Z}_{>0}$, for all $t \in \mathbb{Z}_{\ge 0}$ and $y_1 \le \cdots \le y_n \in \mathbb{Z}$, we have*

$$\left|\mathbb{E}\left[\prod_{i=1}^{n} \overline{K}(t, y_i)\middle|\mathcal{F}(t)\right]\right| \le C(n) \sum_{\pi \in \mathcal{P}_{23}(n)} \prod_{U \in \pi} e^{-\frac{1}{C(n)}|\vec{y}|_U}.$$

Proof. [CGST20, Lemma 5.2] proved this inequality for $I = 1$. When $I \ge 2$, the proof is almost the same. Let us denote by $\mathbb{E}'[\,\cdot\,] = \mathbb{E}[\,\cdot\,|\mathcal{F}(t)]$ and

$$I(y', y) = \prod_{z=y'+1}^{y} \big(B'(t, z, \eta_z(t)) - B(t, z, \eta_z(t))\big) B(t, y', \eta_{y'}(t)).$$

Due to (6.2.7), there exists $C > 0$ such that

$$\big|\mathbb{E}'\big[I(y', y)^\ell\big]\big| \le C e^{-\frac{1}{C}|y-y'|}, \qquad \ell \in \mathbb{Z}_{\ge 1}.$$

This gives bound similar to (5.10) in [CGST20, Lemma 5.2]. The rest of the proof is the same as in [CGST20, Lemma 5.2], we do not repeat it here. □

Lemma 6.6.6. *Fix $n \in \mathbb{Z}_{\ge 1}$, recall the martingale increment $M(t, x)$ from* (6.5.7) *and let $f(t, x)$ be a deterministic function defined on $t \in [t_1, t_2] \cap \mathbb{Z}$ and $x \in \Xi(t)$. Write $f_\infty(t) := \sup_{x \in \Xi(t)} |f(t, x)|$, we have*

$$\left\|\sum_{t=t_1}^{t_2-1} \sum_{x \in \Xi(t)} f(t, x) M(t, x)\right\|_{2n}^2 \le \epsilon C(n) \sum_{t=t_1}^{t_2-1} \sum_{x \in \Xi(t)} \big|f_\infty(t) f(t, x)\big| \|Z(t, x)\|_{2n}^2.$$

Proof. Using the previous lemma, the proof is the same as the one appeared in [CGST20, Lemma

5.3]. □

Have prepared the preceding lemmas, we proceed to prove Proposition 6.6.1. Here we use a slightly different approach compared with the proof of the moment bounds in [CGST20, Proposition 5.4].

Proof of Proposition 6.6.1. Recall that $Z(t,x)$ is defined on $[0,\infty)\times\mathbb{R}$ through linear interpolation. It suffices to prove the theorem for the lattice $t \in \mathbb{Z}_{\geq 0}$ and $x, x' \in \Xi(t)$. Generalization to continuum t, x follows easily.

Let us begin with proving (6.6.1). We have by (6.6.8)

$$\|Z(t,x)\|_{2n} \leq \|\big(\mathsf{p}(t,0) * Z(0)\big)(x)\|_{2n} + \|Z_{mg}(t)\|_{2n}.$$

Using $(x+y)^2 \leq 2(x^2+y^2)$, we get

$$\|Z(t,x)\|_{2n}^2 \leq 2\|\big(\mathsf{p}(t,0) * Z(0)\big)(x)\|_{2n}^2 + 2\|Z_{mg}(t)\|_{2n}^2. \tag{6.6.10}$$

For the first term on RHS of (6.6.10), by Cauchy-Schwarz inequality,

$$\|\big(\mathsf{p}(t,0) * Z(0)\big)(x)\|_{2n}^2 \leq \big(\mathsf{p}(t,0) * \|Z(0)\|_{2n}^2\big)(x). \tag{6.6.11}$$

For the second term $\|Z_{mg}(t)\|_{2n}^2$, by (6.6.9)

$$Z_{mg}(t) = \sum_{s=0}^{t-1} \big(\mathsf{p}(t,s+1) * M(s)\big)(x+\mu(s)) = \sum_{s=0}^{t-1} \sum_{y\in\Xi(s)} \mathsf{p}\big(t,s+1,x+\mu(s)-y\big) M(s,y).$$

Applying Lemma 6.6.6, there exists a constant C_* so that

$$\|Z_{mg}(t)\|_{2n}^2 \leq C_*\epsilon \sum_{s=0}^{t-1} \sum_{y\in\Xi(s)} \Big(\sup_{y\in\Xi(s)} \mathsf{p}(t,s+1,x+\mu(s)-y)\Big)\mathsf{p}(t,s+1,x+\mu(s)-y)\|Z(s,y)\|_{2n}^2,$$

$$\leq \sum_{s=0}^{t-1} \sum_{y\in\Xi(s)} \frac{C_*\epsilon}{\sqrt{t-s}} \mathsf{p}\big(t, s+1, x+\mu(s)-y\big) \|Z(s,y)\|_{2n}^2, \tag{6.6.12}$$

where the last inequality follows from Theorem 6.6.4 (i).

Replacing the RHS of (6.6.10) by upper bound obtained in (6.6.11) and (6.6.12), we obtain

$$\|Z(t,x)\|_{2n}^2 \leq (\mathsf{p}(t,0) * \|Z(0)\|_{2n}^2)(x) + \sum_{s=0}^{t-1} \frac{C_*\epsilon}{\sqrt{t-s}} \big(\mathsf{p}(t,s+1) * \|Z(s)\|_{2n}^2\big)(x+\mu(s)). \tag{6.6.13}$$

Define the set $\Delta_n^+ = \{(s_1,\ldots,s_n) \in \mathbb{Z}_{\geq 0}^n : 0 \leq s_n < \cdots < s_1 < t\}$ for $n \in \mathbb{Z}_{\geq 1}$. Iterating (6.6.13) yields

$$\begin{aligned} &\|Z(t,x)\|_{2n}^2 \leq (\mathsf{p}(t,0) * \|Z(0)\|_{2n}^2)(x) \\ &+ \sum_{n=1}^{\infty} \sum_{(s_1,\ldots s_n)\in\Delta_n^+} \frac{(C_*\epsilon)^n}{\sqrt{t-s_1}\sqrt{s_1-s_2}\ldots\sqrt{s_{n-1}-s_n}} (\mathsf{p}(t,s_1,\ldots,s_n) * \|Z(0)\|_{2n}^2)(x+\sum_{i=1}^n \mu(s_i)). \end{aligned} \tag{6.6.14}$$

where $\mathsf{p}(t,s_1,\ldots,s_n) = \mathsf{p}(t,s_1+1) * \mathsf{p}(s_1,s_2+1) * \cdots * \mathsf{p}(s_{n-1}+1,s_n)$. Following Lemma 6.6.4, we bound

$$\begin{aligned} &(\mathsf{p}(t,0) * \|Z(0)\|_{2n}^2)(x) \leq Ce^{2u\epsilon|x|}, \\ &(\mathsf{p}(t,s_1,\ldots,s_n) * \|Z(0)\|_{2n}^2)(x+\sum_{i=1}^n \mu(s_i)) \leq Ce^{2u\epsilon(|x|+n)}. \end{aligned} \tag{6.6.15}$$

For the second term on the RHS of (6.6.14), note that via integral approximation, we readily see that

$$\begin{aligned} &\sum_{(s_1,\ldots,s_n)\in\Delta_n^+} \frac{(C_*\epsilon)^n}{\sqrt{t-s}\sqrt{s_1-s_2}\ldots\sqrt{s_{n-1}-s_n}} \leq \int_{0\leq s_1\leq\cdots\leq s_n\leq t} \frac{(C_*\epsilon)^n ds_1\ldots ds_n}{\sqrt{t-s_1}\sqrt{s_1-s_2}\ldots\sqrt{s_{n-1}-s_n}} \\ &= (C_*\epsilon t^{\frac{1}{2}})^n \int_{\tau_1+\cdots+\tau_n\leq 1} \frac{1}{\sqrt{\tau_1}\ldots\sqrt{\tau_n}} d\tau_1\ldots d\tau_n = \frac{(\Gamma(\frac{1}{2})C_*\epsilon t^{\frac{1}{2}})^n}{\Gamma(n/2)} \end{aligned} \tag{6.6.16}$$

where $\Gamma(z)$ is the Gamma function. Combining (6.6.15) and (6.6.16) yields

$$\|Z(t,x)\|_2^2 \le Ce^{2u\epsilon|x|} + \sum_{n=1}^{\infty} \frac{(\Gamma(\frac{1}{2})C_*\epsilon t^{\frac{1}{2}})^n}{\Gamma(n/2)} e^{2u\epsilon(|x|+n)} = e^{2u\epsilon|x|}\left(C + \sum_{n=1}^{\infty} \frac{(\Gamma(\frac{1}{2})C_*\epsilon t^{\frac{1}{2}} e^{2u\epsilon})^n}{\Gamma(n/2)}\right)$$

Note that $\epsilon t^{\frac{1}{2}} \le \sqrt{T}$ (since $t \in [0, \epsilon^{-2}T]$), as the growth rate of $\Gamma(\frac{n}{2})$ is much faster than that of x^n, the infinite series in the parentheses above converge, which concludes (6.6.1).

The proof for (6.6.2) and (6.6.3) relies on (6.6.1). We proceed to prove (6.6.2), denote by

$$Z^{\nabla}(t,x,x') := Z(t,x) - Z(t,x'), \qquad \mathsf{p}^{\nabla}(t,s,x,x') := \mathsf{p}(t,s,x) - \mathsf{p}(t,s,x').$$

Using (6.6.8) (subtract $Z(t,x')$ from $Z(t,x)$) , we have

$$Z^{\nabla}(t,x,x') = \sum_{y\in\Xi(t)} \mathsf{p}(t,0,y) Z^{\nabla}(0,x-y,x'-y) + Z^{\nabla}_{mg}(t),$$

where

$$Z^{\nabla}_{mg}(t) = \sum_{s=0}^{t-1} \sum_{y\in\Xi(s)} \mathsf{p}^{\nabla}\big(t,s+1,x+\mu(s)-y,x'+\mu(s)-y\big) M(s,y). \tag{6.6.17}$$

It is straightforward that

$$\|Z^{\nabla}(t,x,x')\|_{2n}^2 \le 2\sum_{y\in\Xi(t)} \mathsf{p}(t,0,y)\|Z^{\nabla}(0,x-y,x'-y)\|_{2n}^2 + 2\|Z^{\nabla}_{mg}(t)\|_{2n}^2.$$

By the definition of the near stationary initial data (Definition 6.5.5), for $a \in (0, \frac{1}{2})$, there exists C such that

$$\begin{aligned}\sum_{y\in\Xi(t)} \mathsf{p}(t,0,y)\|Z^{\nabla}(0,x-y,x'-y)\|_{2n}^2 &\le C\sum_{y\in\Xi(t)} \mathsf{p}(t,0,y)(\epsilon|x-x'|)^{2a} e^{2u\epsilon(|x-y|+|x'-y|)} \\ &\le C(\epsilon|x-x'|)^{2a} e^{2u\epsilon(|x|+|x'|)} \sum_{y\in\Xi(t)} \mathsf{p}(t,0,y) e^{4u\epsilon|y|}\end{aligned}$$

Further applying Theorem 6.6.4 (ii), one has

$$\sum_{y\in\Xi(t)} \mathsf{p}(t,0,y)e^{4u\epsilon|y|} \le C.$$

We conclude that

$$\sum_{y\in\Xi(t)} \mathsf{p}(t,0,y)\|Z^{\nabla}(0,x-y,x'-y)\|_{2n}^2 \le C(\epsilon|x-x'|)^{2a}e^{2u\epsilon(|x|+|x'|)}. \tag{6.6.18}$$

To bound $\|Z^{\nabla}_{mg}(t)\|_{2n}$, we appeal to Lemma 6.6.6. Note that due to Lemma 6.6.4 (iv),

$$\sup_{y\in\Xi(s)} \left|\mathsf{p}^{\nabla}(t,s+1,x+\mu(t-1)-y,x'+\mu(t-1)-y)\right| \le C|x-x'|^{2a}(t-s)^{-\frac{2a+1}{2}},$$

Applying Lemma 6.6.6 to (6.6.17) implies

$$\|Z^{\nabla}_{mg}(t)\|_{2n}^2 \le C\epsilon|x-x'|^{2a}\sum_{s=0}^{t-1}(t-s)^{-\frac{a+1}{2}}\sum_{y\in\Xi(s)}\mathsf{p}^{\nabla}(t-s-1,x+\mu(s)-y,x'+\mu(s)-y)\|Z(s,y)\|_{2n}^2.$$

Owing to Theorem 6.6.4 (i), we observe that

$$\begin{aligned}&\sum_{y\in\Xi(s)} \mathsf{p}^{\nabla}(t-s-1,x+\mu(s)-y,x'+\mu(s)-y)\|Z(s,y)\|_2^2\\ &\le C\sum_{y\in\Xi(s)} \mathsf{p}^{\nabla}(t-s-1,x+\mu(s)-y,x'+\mu(s)-y)e^{2u\epsilon|y|} \le Ce^{2u\epsilon(|x|+|x'|)}.\end{aligned}$$

Consequently,

$$\begin{aligned}\|Z^{\nabla}_{mg}(t)\|_{2n}^2 \le C\epsilon|x'-x|^{2a}e^{2u\epsilon(|x|+|x'|)}\sum_{s=0}^{t-1}(t-s)^{-\frac{2a+1}{2}} &\le C(\epsilon|x-x'|)^{2a}(\epsilon^2 t)^{\frac{1-2a}{2}}e^{2u\epsilon(|x|+|x'|)},\\ &\le C(\epsilon|x-x'|)^{2a}e^{2u\epsilon(|x|+|x'|)}. \qquad (6.6.19)\end{aligned}$$

We conclude (6.6.2) via combining (6.6.18) and (6.6.19).

Finally, we justify (6.6.3), we have

$$Z(t,x) - Z(t',x) = \sum_{y \in \Xi(t')} \mathsf{p}(t,t',x-y)(Z(t',y) - Z(t',x)) + Z_{mg}(t,t'),$$

where $Z_{mg}(t,t') = \sum_{s=t'}^{t-1} \sum_{y \in \Xi(s)} \mathsf{p}(t-s-1, x+\mu(s)-y) M(s,y)$. Similar to the previous proof, we have

$$\|Z(t,x) - Z(t',x)\|_{2n}^2 \leq 2 \sum_{y \in \Xi(t')} \mathsf{p}(t,t',x-y) \|Z(t',y) - Z(t',x)\|_{2n}^2 + 2\|Z_{mg}(t,t')\|_{2n}^2. \quad (6.6.20)$$

For the first term on the RHS of (6.6.20), we apply (6.6.2) and Lemma 6.6.4 (iii), for any $a \in (0, \frac{1}{2})$,

$$\begin{aligned} \sum_{y \in \Xi(t')} \mathsf{p}(t,t',x-y) \|Z(t',y) - Z(t',x)\|_{2n}^2 &\leq C\epsilon^{2a} \sum_{y \in \Xi(t')} \mathsf{p}(t,t',x-y) |x-y|^{2a} e^{u\epsilon(|x|+|y|)} \\ &\leq C\epsilon^{2a} (t-t'+1)^a e^{2u\epsilon|x|}. \end{aligned}$$

For the second term, invoking Lemma 6.6.6 gives

$$\begin{aligned} \|Z_{mg}(t,t')\|_{2n}^2 &\leq C\epsilon \sum_{s=t'}^{t-1} \frac{1}{\sqrt{t-s}} \sum_{y \in \Xi(s)} \mathsf{p}(t-s-1, x+\mu(s)-y) \|Z(s,y)\|_{2n}^2 \\ &\leq C\epsilon e^{2u\epsilon|x|} \sum_{s=t'}^{t-1} \frac{1}{\sqrt{t-s}} \leq C(\epsilon^2(t-t'))^{\frac{1}{2}} e^{2u\epsilon|x|}. \end{aligned} \quad (6.6.21)$$

Combining (6.6.20)-(6.6.21), we obtain $\|Z(t,x) - Z(t',x)\|_{2n} \leq C(\epsilon^2(t-t'))^{\frac{a}{2}} e^{u\epsilon|x|}$. We complete the proof of Proposition 6.6.1. □

Having shown the tightness of $Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)$, to prove Theorem 5.6, it suffices to show that any limit point $\mathcal{Z}$ of $Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)$ is the mild solution to the SHE (6.5.31). This is the goal of the Section 6.6.2 and Section 6.6.3, where we will formulate the notion of "solution to the martingale problem" (which is equivalent to the mild solution) and prove that any limit point of $Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)$ satisfies the martingale problem.

6.6.2 The martingale problem

We recall the martingale problem of the SHE from [BG97].

Definition 6.6.7. We say that a $C([0,\infty), C(\mathbb{R}))$-valued process $\mathcal{Z}(t,x)$ is a solution of martingale problem of the SHE (6.5.31)

$$\partial_t \mathcal{Z}(t,x) = \frac{V_*}{2}\partial_x^2 \mathcal{Z}(t,x) + \sqrt{D_*}\xi(t,x)\mathcal{Z}(t,x)$$

with initial condition $\mathcal{Z}^{ic} \in C(\mathbb{R})$ if $\mathcal{Z}(0,x) = \mathcal{Z}^{ic}(x)$ in distribution and

(i) Given any $T > 0$, there exists $u < \infty$ such that

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}} e^{-u|x|}\mathbb{E}\big[\mathcal{Z}(t,x)^2\big] < \infty.$$

(ii) For any test function $\psi \in C_c^\infty(\mathbb{R})$,

$$\mathcal{M}_\psi(t) = \int_{\mathbb{R}} \mathcal{Z}(t,x)\psi(x)dx - \int_{\mathbb{R}} \mathcal{Z}(0,x)\psi(x)dx - \frac{V_*}{2}\int_0^t \int_{\mathbb{R}} \mathcal{Z}(s,x)\psi''(x)dxds$$

is a local martingale.

(iii) For any test function $\psi \in C_c^\infty(\mathbb{R})$,

$$\mathcal{Q}_\psi(t) = \mathcal{M}_\psi(t)^2 - D_* \int_0^t \int_{\mathbb{R}} \mathcal{Z}(s,x)^2\psi(x)^2 dxds$$

is a local martingale.

[BG97, Proposition 4.11] proves the the solution $\mathcal{Z}$ to the martingale problem is also the weak solution (equivalently, the mild solution) to the SHE. Moreover, they show that there is a unique such solution.

To prove Theorem 6.5.6, it suffices to prove that any limit point of $Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)$ satisfies (i), (ii), (iii). We will do it in the next section. The main difficulty arises for justifying the quadratic martingale problem (iii), we need the following proposition, whose proof is postponed to Section 6.8.

Proposition 6.6.8. *For $s \in \mathbb{Z}_{\geq 0}$, define*

$$\tau(s) = \frac{\rho(I-\rho)}{I^2} \cdot \frac{b(I + 2\mathrm{mod}_J(s) + 1) - (I + 2\mathrm{mod}_J(s) - 1)}{b(I + 2\mathrm{mod}_J(s)) - (I + 2\mathrm{mod}_J(s) - 2)}. \tag{6.6.22}$$

Start the unfused SHS6V model from near stationary initial condition, for given $T > 0$, there exists constant C and u such that (recall the expressions Θ_1 and Θ_2 from (6.5.10)*)*

$$\left\| \epsilon^2 \sum_{s=0}^{t} \left(\epsilon^{-1}\Theta_1\Theta_2 - \tau(s) Z^2 \right)(s, x^\star - \hat{\mu}(s) + \lfloor \hat{\mu}(s) \rfloor) \right\|_2 \leq C\epsilon^{\frac{1}{4}} e^{u\epsilon|x^\star|} \tag{6.6.23}$$

for all $t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$, $x^\star \in \mathbb{Z}$ and $\epsilon > 0$ small enough.

Remark 6.6.9. In (6.6.23), we compensate the space variable $x^\star \in \mathbb{Z}$ by $\hat{\mu}(s) - \lfloor \hat{\mu}(s) \rfloor \in [0, 1)$ to ensure that $x^\star - \hat{\mu}(s) + \lfloor \hat{\mu}(s) \rfloor \in \Xi(s)$.

6.6.3 Proof of Theorem 6.5.6

The entire section is devoted to the proof of Theorem 6.5.6. As we mentioned earlier, due to the tightness obtained in Proposition 6.6.1, if suffices to prove that for any limit point $\mathbb{Z}Z$ of $Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)$ satisfies the martingale problem. The proof is accomplished once we verify (i), (ii), (iii) for $\mathcal{Z}$.

For the ensuing discussion, we denote by $\mathcal{E}_\epsilon(t)$ to be a generic process (which may differ from line to line) satisfying for all fixed $T > 0$

$$\lim_{\epsilon \downarrow 0} \sup_{t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}} \|\mathcal{E}_\epsilon(t)\|_2 = 0.$$

We start by verifying (i). Due to (6.6.1) and $Z_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x) \Rightarrow \mathbb{Z}Z(t,x)$, by Skorohod representation theorem and Fatou's lemma, (i) holds.

We continue to prove (ii). To show that $\mathcal{M}_\psi(t)$ is a local martingale, we consider a discrete analogue. Define

$$M_\psi(t) := \epsilon \sum_{s=0}^{t-1} \sum_{x\in\Xi(s)} M(s,x)\psi(\epsilon(x-\mu(s))). \tag{6.6.24}$$

Due to Proposition 6.5.1, $M(t,x)$ is a $\mathcal{F}(t)$-martingale increment, which implies $M_\psi(t)$ is a $\mathcal{F}(t)$-martingale.

Define $\langle Z(t), \psi\rangle_\epsilon := \sum_{x\in\Xi(t)} \epsilon\psi(\epsilon x) Z(t,x)$. By (6.5.7),

$$Z(t,x) = \sum_{y\in\Xi(t-1)} \mathsf{p}_\epsilon(t,t-1,x-y)Z(t-1,y) + M(t-1, x+\mu(t-1)), \quad x \in \Xi(t),$$

we obtain

$$\begin{aligned}
&\langle Z(s),\psi\rangle_\epsilon - \langle Z(s-1),\psi\rangle_\epsilon = \sum_{x\in\Xi(t)} \epsilon\psi(\epsilon x)Z(t,x) - \sum_{y\in\Xi(t-1)} \epsilon\psi(\epsilon y)Z(t-1,y)\\
&= \sum_{x\in\Xi(s)} \epsilon\psi(\epsilon x)\Big(\sum_{y\in\Xi(s-1)} \mathsf{p}_\epsilon(s,s-1,x-y)Z(s-1,y) + M(s-1,x+\mu(s-1))\Big) - \sum_{y\in\Xi(s-1)} \epsilon\psi(\epsilon y)Z(s-1,y)\\
&= \sum_{y\in\Xi(s-1)} \epsilon Z(s-1,y)\Big(\sum_{x\in\Xi(s)} \mathsf{p}_\epsilon(s,s-1,x-y)\big(\psi(\epsilon x)-\psi(\epsilon y)\big)\Big) + \sum_{x\in\Xi(s)} \epsilon\psi(\epsilon x)M(s-1,x+\mu(s-1))
\end{aligned} \tag{6.6.25}$$

Summing (6.6.25) over $s \in [1,t]\cap\mathbb{Z}$ yields

$$M_\psi(t) = \langle Z(t),\psi\rangle_\epsilon - \langle Z(0),\psi\rangle_\epsilon - \sum_{s=0}^{t-1}\epsilon \sum_{y\in\Xi(s)} Z(s,y)\Big(\sum_{x\in\Xi(s+1)} \mathsf{p}_\epsilon(s+1,s,x-y)(\psi(\epsilon x)-\psi(\epsilon y))\Big) \tag{6.6.26}$$

Recall that $R_\epsilon(s)$ is the random variable defined in (6.5.1), as usual we put on the subscript ϵ to emphasize the dependence. Note that,

$$\mathbb{E}\big[R_\epsilon(s)\big] = \sum_{x\in\Xi(1)} \mathsf{p}_\epsilon(s+1,s,x)x = 0, \qquad \mathrm{Var}\big[R_\epsilon(s)\big] = \sum_{x\in\Xi(1)} \mathsf{p}_\epsilon(s+1,s,x)x^2.$$

By Taylor expansion

$$\psi(\epsilon x) = \psi(\epsilon y) + \epsilon\psi'(\epsilon y)(x-y) + \frac{1}{2}\epsilon^2\psi''(\epsilon y)(x-y)^2 + \epsilon^3 O(|x-y|^3),$$

whereby (6.6.26) becomes

$$M_\psi(t) = \langle Z(t),\psi\rangle_\epsilon - \langle Z(0),\psi\rangle_\epsilon - \frac{1}{2}\epsilon^2\sum_{s=0}^{t-1}\mathrm{Var}\big[R_\epsilon(s)\big]\langle Z(s),\psi''\rangle_\epsilon + \mathcal{E}_\epsilon(t).$$

Furthermore, we have

$$\begin{aligned}\mathrm{Var}\big[R_\epsilon(s)\big] &= \lambda(s)\sum_{n=1}^{\infty}\frac{\alpha(s)(1-q)}{1+\alpha(s)}\left(1-\frac{\nu+\alpha(s)}{1+\alpha(s)}\right)\left(\frac{\nu+\alpha(s)}{1+\alpha(s)}\right)^{n-1}q^{\rho n}n^2 \\ &\quad - \left(\lambda(s)\sum_{n=1}^{\infty}\frac{\alpha(s)(1-q)}{1+\alpha(s)}\left(1-\frac{\nu+\alpha(s)}{1+\alpha(s)}\right)\left(\frac{\nu+\alpha(s)}{1+\alpha(s)}\right)^{n-1}q^{\rho n}n\right)^2 \\ &= \frac{(I+1+2\mathrm{mod}_J(s))b - (I+2\mathrm{mod}_J(s)-1)}{I^2(1-b)} + O(\epsilon^{\frac{1}{2}}). \qquad (6.6.27)\end{aligned}$$

In the last line, we used Lemma 6.5.4 to get asymptotics. Denote by

$$V(s) = \frac{(I+1+2\mathrm{mod}_J(s))b - (I+2\mathrm{mod}_J(s)-1)}{I^2(1-b)}$$

Then

$$M_\psi(t) = \langle Z(t),\psi\rangle_\epsilon - \langle Z(0),\psi\rangle_\epsilon - \frac{1}{2}\epsilon^2\sum_{s=0}^{t-1}V(s)\langle Z(s),\psi''\rangle_\epsilon + \mathcal{E}_\epsilon(t).$$

Note that $\{V(s)\}_{s=0}^{\infty}$ is a periodic sequence with period J, by the time regularity of $Z(t,x)$ in (6.6.3), we can replace $V(s)$ by

$$V_* = \frac{1}{J}\sum_{s=0}^{J-1} V(s) = \frac{(I+J)b - (I+J-2)}{I^2(1-b)}$$

as defined in (6.1.12). Consequently,

$$M_\psi(t) = \langle Z(t), \psi\rangle_\epsilon - \langle Z(0), \psi\rangle_\epsilon - \frac{1}{2}\epsilon^2 V_* \sum_{s=0}^{t-1} \langle Z(s), \psi''\rangle_\epsilon + \mathcal{E}_\epsilon(t).$$

Since $\lim_{\epsilon\downarrow 0}\sup_{t\in[0,\epsilon^{-2}T]\cap\mathbb{Z}} \|\mathcal{E}_\epsilon(t)\|_2 = 0$, by a standard discrete to continuous argument from the martingale $M_\psi(t)$ to $\mathcal{M}_\psi(t)$, we conclude that $\mathcal{M}_\psi(t)$ is a local martingale.

We finish the proof of (iii) based on Proposition 6.6.8. Similar to what we did in proving (ii), we want to find a discrete approximation of $Q_\psi(t)$. This is given by $M_\psi - \langle M_\psi\rangle(t)$. Referring to (6.6.24), the martingale $M_\psi(t)$ possesses the quadratic variation

$$\begin{aligned}\langle M_\psi\rangle(t) &= \epsilon^2 \sum_{s=0}^{t-1} \sum_{x,x'\in\Xi(s)} \psi(\epsilon(x-\mu(s)))\psi(\epsilon(x'-\mu(s)))\mathbb{E}\big[M(s,x)M(s,x')\big|\mathcal{F}(s)\big] \\ &= \epsilon^2 \sum_{s=0}^{t-1} \sum_{x,x'\in\Xi(s)} \psi(\epsilon(x-\mu(s)))\psi(\epsilon(x'-\mu(s)))\left(\frac{\nu+\alpha(s)}{1+\alpha(s)}q^\rho\right)^{|x-x'|} \Theta_1(s, x\wedge x')\Theta_2(s, x\wedge x')\end{aligned} \tag{6.6.28}$$

where the last equality follows from Proposition 6.5.1. Since $\psi \in C_c^\infty(\mathbb{R})$, there exists a constant C such that

$$\left|\psi(\epsilon(x-\mu(s)))\psi(\epsilon(x'-\mu(s))) - \psi(\epsilon(x\wedge x'))^2\right| \le C\epsilon(|x-x'|+1)$$

Consequently, the expression (6.6.28) is well-approximated with the corresponding term $\psi(\epsilon(x-\mu(s)))\psi(\epsilon(x'-\mu(s)))$ replaced by $\psi(\epsilon(x\wedge x'))\psi(\epsilon(x'\wedge x'))$, which yields

$$\begin{aligned}
\langle M_\psi\rangle(t) &= \epsilon^2\sum_{s=0}^{t-1}\sum_{x,x'\in\Xi(s)}\psi(\epsilon(x\wedge x'))^2\left(\frac{\nu+\alpha(s)}{1+\alpha(s)}q^\rho\right)^{|x-x'|}\Theta_1(s,x\wedge x')\Theta_2(s,x\wedge x')+\mathcal{E}_\epsilon(t),\\
&= \epsilon^2\sum_{s=0}^{t-1}\sum_{x\in\Xi(s)}\sum_{n=-\infty}^{\infty}\left(\frac{\nu+\alpha(s)}{1+\alpha(s)}q^\rho\right)^{|n|}\psi(\epsilon x)^2\Theta_1(s,x)\Theta_2(s,x)+\mathcal{E}_\epsilon(t),\\
&= \epsilon^2\sum_{s=0}^{t-1}\sum_{x\in\Xi(s)}\frac{1+\alpha(s)+(\nu+\alpha(s))q^\rho}{1+\alpha(s)-(\nu+\alpha(s))q^\rho}\psi(\epsilon x)^2\Theta_1(s,x)\Theta_2(s,x)+\mathcal{E}_\epsilon(t),\\
&= \epsilon^2\sum_{s=0}^{t-1}\frac{b(I+2\mathrm{mod}_J(s))-(I+2\mathrm{mod}_J(s)-2)}{I(1-b)}\sum_{x\in\Xi(s)}\epsilon\psi(\epsilon x)^2\big(\epsilon^{-1}\Theta_1(s,x)\Theta_2(s,x)\big)+\mathcal{E}_\epsilon(t).
\end{aligned} \tag{6.6.29}$$

Here, in the third equality we used $\sum_{n=-\infty}^{\infty}x^{-|n|}=\frac{1+x}{1-x}$. In the last equality, using Lemma 6.5.4 for asymptotics expansion of $\frac{\nu+\alpha(s)}{1+\alpha(s)}$, one has

$$\frac{1+\alpha(s)+(\nu+\alpha(s))q^\rho}{1+\alpha(s)-(\nu+\alpha(s))q^\rho}=\frac{1+\frac{\nu+\alpha(s)}{1+\alpha(s)}q^\rho}{1-\frac{\nu+\alpha(s)}{1+\alpha(s)}q^\rho}=\frac{b(I+2\mathrm{mod}_J(s))-(I+2\mathrm{mod}_J(s)-2)}{I(1-b)}+O(\epsilon^{\frac{1}{2}}).$$

Using Proposition 6.6.8, we replace the term $\epsilon^{-1}\Theta_1(s,x)\Theta_2(s,x)$ in (6.6.29) with $\tau(s)Z(s,x)^2$,

$$\begin{aligned}
\langle M_\psi\rangle(t) &= \epsilon^2\sum_{s=0}^{t-1}\frac{b(I+2\mathrm{mod}_J(s))-(I+2\mathrm{mod}_J(s)-2)}{I(1-b)}\sum_{x\in\Xi(s)}\epsilon\psi(\epsilon x)^2\tau(s)Z(s,x)^2+\mathcal{E}_\epsilon(t),\\
&= \epsilon^2\sum_{s=0}^{t-1}\frac{\rho(I-\rho)}{I^2}\cdot\frac{b(I+2\mathrm{mod}_J(s)+1)-(I+2\mathrm{mod}_J(s)-1)}{I(1-b)}\sum_{x\in\Xi(s)}\epsilon\psi(\epsilon x)^2Z(s,x)^2+\mathcal{E}_\epsilon(t).
\end{aligned}$$

Using again the time regularity of $Z(t,x)$ in (6.6.3), we conclude that

$$\langle M_\psi\rangle(t)=D_*\sum_{s=0}^{t-1}\sum_{x\in\Xi(s)}\epsilon\psi(\epsilon x)^2Z(s,x)^2+\mathcal{E}_\epsilon(t),$$

where

$$D_* = \frac{1}{J}\sum_{s=0}^{J-1}\frac{\rho(I-\rho)}{I^2}\cdot\frac{b(I+2\mathrm{mod}_J(s)+1)-(I+2\mathrm{mod}_J(s)-1)}{I(1-b)} = \frac{\rho(I-\rho)}{I}\frac{(I+J)b-(I+J-2)}{I^2(1-b)}$$

as defined in (6.1.13). Via a standard discrete to continuous argument from the martingale $M_\psi(t) - \langle M_\psi\rangle(t)$ to $Q_\psi(t)$, we conclude that $Q_\psi(t)$ is a local martingale. Since we have proved that for any limit point $\mathcal{Z}$ of $Z_\epsilon(\epsilon^{-2}\cdot, \epsilon^{-1}\cdot)$, it satisfies (i), (ii), (iii) in Definition 6.6.7, this concludes the proof of Theorem 6.5.6.

6.7 Estimate of the two particle transition probability

In this section, we prove a space-time estimate for the (tilted) two particle transition probability $\mathbf{V}_\epsilon$, using the integral formula provided in Corollary 6.5.3. This technical result is crucial to the proof of Proposition 6.6.8.

Recall from Corollary 6.5.3 that

$$\begin{aligned}
&\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\\
&= c(y_1,y_2)\Bigg[\oint_{C_R}\oint_{C_R}\prod_{i=1}^{2}\big(\mathfrak{D}_\epsilon(z_i)\big)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_i-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\\
&\quad-\oint_{C_R}\oint_{C_R}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\big(\mathfrak{D}_\epsilon(z_i)\big)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\\
&\quad+\mathrm{Res}_{z_1=\mathfrak{s}_\epsilon(z_2)}\oint_{C_R}\oint_{C_R}\mathfrak{F}_\epsilon(z_1,z_2)\big(\mathfrak{D}_\epsilon(z_i)\big)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\Bigg],
\end{aligned}\tag{6.7.1}$$

where C_r is a circle centered at zero with a large enough radius R so as to include all the poles of the integrand, $c(y_1, y_2)$ is defined in (6.4.5) and the functions in the integrand above are defined respectively in (6.5.25) - (6.5.28). We put ϵ in the notation of $\mathbf{V}_\epsilon$ and other functions to emphasize the dependence on ϵ under the weakly asymmetry scaling.

We define the discrete gradients $\nabla_{x_1}, \nabla_{x_2}, \nabla_{y_1}, \nabla_{y_2}$

$$
\begin{aligned}
\nabla_{x_1}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big) &= \mathbf{V}_\epsilon\big((x_1+1,x_2),(y_1,y_2),t,s\big) - \mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big),\\
\nabla_{x_2}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t\big) &= \mathbf{V}_\epsilon\big((x_1,x_2+1),(y_1,y_2),t,s\big) - \mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big),\\
\nabla_{y_1}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big) &= \mathbf{V}_\epsilon\big((x_1,x_2),(y_1+1,y_2),t,s\big) - \mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big),\\
\nabla_{y_2}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big) &= \mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2+1),t,s\big) - \mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big).
\end{aligned}
$$

Furthermore, we define the mixed discrete gradient

$$
\begin{aligned}
\nabla_{x_1,x_2}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big) &= \nabla_{x_2}\Big(\nabla_{x_1}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\Big)\\
&= \mathbf{V}_\epsilon\big((x_1+1,x_2+1),(y_1,y_2),t,s\big) - \mathbf{V}_\epsilon\big((x_1+1,x_2),(y_1,y_2),t,s\big)\\
&\quad - \mathbf{V}_\epsilon\big((x_1,x_2+1),(y_1,y_2),t,s\big) + \mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)
\end{aligned}
$$

We define the ∇-Weyl chamber (which is understood with respect to whichever gradient is taken) to be

$$
\begin{aligned}
&\{(x_1,x_2,y_1,y_2) : x_1+1 \le x_2 \in \Xi(t), y_1 \le y_2 \in \Xi(s)\} && \text{if } \nabla = \nabla_{x_1},\\
&\{(x_1,x_2,y_1,y_2) : x_1 \le x_2 \in \Xi(t), y_1 \le y_2 \in \Xi(s)\} && \text{if } \nabla = \nabla_{x_2},\\
&\{(x_1,x_2,y_1,y_2) : x_1 \le x_2 \in \Xi(t), y_1+1 < y_2 \in \Xi(s)\} && \text{if } \nabla = \nabla_{y_1},\\
&\{(x_1,x_2,y_1,y_2) : x_1 \le x_2 \in \Xi(t), y_1 \le y_2 \in \Xi(s)\} && \text{if } \nabla = \nabla_{y_2}.
\end{aligned}
\tag{6.7.2}
$$

We remark that $\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)$ is defined only for $x_1 \le x_2 \in \Xi(t)$ and $y_1 \le y_2 \in \Xi(s)$. In the definition of ∇-Weyl chamber, when $\nabla = \nabla_{x_1}, \nabla_{x_2}, \nabla_{y_2}$, the corresponding ∇-Weyl chamber is exactly where the quantities $\nabla_{x_1}\mathbf{V}_\epsilon$, $\nabla_{x_2}\mathbf{V}_\epsilon$ or $\nabla_{y_2}\mathbf{V}_\epsilon$ are well defined. But for $\nabla = \nabla_{y_1}$, we require $y_1 + 1 < y_2$, which is stronger than $y_1 + 1 \le y_2$ (where $\nabla_{y_1}\mathbf{V}_\epsilon$ is well defined). The motivation of this requirement is to ensure that (6.7.9) holds.

The following result is the main technical contribution of our paper.

Proposition 6.7.1. *For all fixed $\beta, T > 0$, there exists positive constant $C(\beta), C(\beta,T)$ such that for $\epsilon > 0$ small enough and $s \le t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$*

(a) For all $x_1 \le x_2 \in \Xi(t)$ and $y_1 \le y_2 \in \Xi(s)$,

$$\left|\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{t-s+1} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}}. \tag{6.7.3}$$

(b) For all (x_1,x_2,y_1,y_2) in the ∇-Weyl chamber,

$$\left|\nabla_{x_i}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}}, \quad i=1,2,$$
$$\left|\nabla_{y_i}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}}. \quad i=1,2.$$

(c) For all $x_1 < x_2 \in \Xi(t)$ and $y_1 \le y_2 \in \Xi(s)$,

$$\left|\nabla_{x_1,x_2}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^2} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}}.$$

It is helpful to divide the proof of Proposition 6.7.1 depending on whether the time increment $t-s$ is large enough. More precisely, we use the phrase $t-s$ is *large enough* if the referred statement holds for all $t-s \ge t_0$, where t_0 is some generic time threshold which may change from line to line (depend on β and T, but does not depend on ϵ). Note that this is not to be confused with the global assumption $0 \le s \le t \le \epsilon^{-2}T$, which implies $t-s \le \epsilon^{-2}T$.

Given arbitrary fixed $t_0 > 0$, let us first prove the proposition for $t-s \le t_0$.

Proof of Proposition 6.7.1 for $t-s \le t_0$. According to Lemma 6.5.4,

$$\lim_{\epsilon\downarrow 0} \sup_{t\in\mathbb{Z}_{\ge 0}} \frac{\nu+\alpha(t)}{1+\alpha(t)} = \sup_{t\in\mathbb{Z}_{\ge 0}} \frac{(I+\mathrm{mod}_J(t))b-(I+\mathrm{mod}_J(t)-1)}{\mathrm{mod}_J(t)b-(\mathrm{mod}_J(t)-1)} < 1, \tag{6.7.4}$$

here we used the condition $\frac{I+J-2}{I+J-1} < b < 1$ in (6.5.30). Taking $k = 2$ in (6.3.13) yields

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1, x_2), (y_1, y_2), t, s\big) \leq C \prod_{i=1}^{2} \binom{|x_i - y_i| + t - s}{t - s} \theta^{|x_i - y_i|} \tag{6.7.5}$$

where $\theta = \sup_{t \in \mathbb{Z}_{\geq 0}} \frac{\nu + \alpha(t)}{1 + \alpha(t)}$. So there exists $0 < \delta < 1$ such that for ϵ small enough and all $s \leq t$ such that $t - s \leq t_0$

$$\mathbf{P}_{\overleftarrow{\text{SHS6V}}}\big((x_1, x_2), (y_1, y_2), t, s\big) \leq C\delta^{|x_i - y_i|}, \tag{6.7.6}$$

Referring to the relation (6.5.20) between $\mathbf{V}$ and $\mathbf{P}_{\overleftarrow{\text{SHS6V}}}$. By $\lim_{\epsilon \downarrow 0} e^{\sqrt{\epsilon}} = 1$ along with (6.7.6), there exists $0 < \delta' < 1$ s.t.

$$\mathbf{V}_\epsilon\big((x_1, x_2), (y_1, y_2), t, s\big) \leq C\delta'^{|x_1 - y_1| + |x_2 - y_2|}.$$

Consequently, we can take $C(\beta, T)$ and $C(\beta)$ in (6.7.3) large enough such that for $t - s \leq t_0$,

$$\begin{aligned}\mathbf{V}_\epsilon\big((x_1, x_2), (y_1, y_2), t, s\big) \leq C\delta'^{|x_1 - y_1| + |x_2 - y_2|} &\leq \frac{C(\beta, T)}{t_0 + 1} e^{-\frac{\beta(|x_1 - y_1| + |x_2 - y_2|)}{\sqrt{t_0 + 1} + C(\beta)}} \\ &\leq \frac{C(\beta, T)}{t - s + 1} e^{-\frac{\beta(|x_1 - y_1| + |x_2 - y_2|)}{\sqrt{t - s + 1} + C(\beta)}}\end{aligned}$$

For the gradients, let us consider $\nabla_{x_1} \mathbf{V}_\epsilon$ for example. Note that

$$\nabla_{x_1} \mathbf{V}_\epsilon\big((x_1, x_2), (y_1, y_2), t, s\big) = \mathbf{V}_\epsilon\big((x_1 + 1, x_2), (y_1, y_2), t, s\big) - \mathbf{V}_\epsilon\big((x_1, x_2), (y_1, y_2), t, s\big)$$

Using the same argument as above, there exists constant $C(\beta, T)$ and $C(\beta)$ such that for all $s \leq t$ satisfying $t - s \leq t_0$,

$$\mathbf{V}_\epsilon\big((x_1, x_2), (y_1, y_2), t, s\big), \mathbf{V}_\epsilon\big((x_1 + 1, x_2), (y_1, y_2), t, s\big) \leq \frac{C(\beta, T)}{(t - s + 1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_1 - y_1| + |x_2 - y_2|)}{\sqrt{t - s + 1} + C(\beta)}},$$

which gives the desired bound for $\nabla_{x_1} \mathbf{V}_\epsilon\big((x_1, x_2), (y_1, y_2), t, s\big)$. The argument for the gradient $\nabla_{x_2} \mathbf{V}_\epsilon, \nabla_{y_1} \mathbf{V}_\epsilon, \nabla_{y_2} \mathbf{V}_\epsilon$ and $\nabla_{x_1, x_2} \mathbf{V}_\epsilon$ is similar. $\square$

Having proved Proposition 6.7.1 for $t-s \le t_0$, it suffices to prove the same proposition for $t-s$ large enough. In other words, we need to show that there exists $t_0 > 0$ such that the proposition holds for $t - s \ge t_0$. We decompose $\mathbf{V}_\epsilon$ (6.7.1) by

$$\mathbf{V}_\epsilon = c(y_1, y_2)\big(\mathbf{V}_\epsilon^{\mathrm{fr}} - \mathbf{V}_\epsilon^{\mathrm{in}}\big),$$

where

$$\mathbf{V}_\epsilon^{\mathrm{fr}}\big((x_1,x_2),(y_1,y_2),t,s\big) := \oint_{C_R}\oint_{C_R}\prod_{i=1}^{2}\big(\mathfrak{D}_\epsilon(z_i)\big)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_i-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}, \tag{6.7.7}$$

$$\begin{aligned}\mathbf{V}_\epsilon^{\mathrm{in}}\big((x_1,x_2),(y_1,y_2),t,s\big) := &\oint_{C_R}\oint_{C_R}\big(\mathfrak{D}_\epsilon(z_i)\big)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\\ &- \mathrm{Res}_{z_1=\mathfrak{s}_\epsilon(z_2)}\oint_{C_R}\oint_{C_R}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\big(\mathfrak{D}_\epsilon(z_i)\big)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}.\end{aligned} \tag{6.7.8}$$

Referring to (6.4.5), $c(y_1, y_2)$ equals 1 as long as $y_1 < y_2$. It is straightforward that for (x_1, x_2, y_1, y_2) in the ∇-Weyl chamber (6.7.2),

$$\begin{aligned}\nabla_{x_i}\mathbf{V}_\epsilon &= c(y_1,y_2)\big(\nabla_{x_i}\mathbf{V}_\epsilon^{\mathrm{fr}} - \nabla_{x_i}\mathbf{V}_\epsilon^{\mathrm{in}}\big),\\ \nabla_{y_i}\mathbf{V}_\epsilon &= c(y_1,y_2)\big(\nabla_{y_i}\mathbf{V}_\epsilon^{\mathrm{fr}} - \nabla_{y_i}\mathbf{V}_\epsilon^{\mathrm{in}}\big).\end{aligned} \tag{6.7.9}$$

In addition, for $x_1 + 1 \le x_2 \in \Xi(t)$ and $y_1 \le y_2 \in \Xi(s)$,

$$\nabla_{x_1,x_2}\mathbf{V}_\epsilon = c(y_1,y_2)\big(\nabla_{x_1,x_2}\mathbf{V}_\epsilon^{\mathrm{fr}} - \nabla_{x_1,x_2}\mathbf{V}_\epsilon^{\mathrm{in}}\big).$$

Note that under weakly asymmetric scaling,

$$\lim_{\epsilon\downarrow 0} c(y_1,y_2) = \mathbf{1}_{\{y_1<y_2\}} + \frac{I-1}{2I}\mathbf{1}_{\{y_1=y_2\}},$$

which implies that $c(y_1, y_2)$ is uniformly bounded for ϵ small enough, This being the case, to prove Proposition 6.7.1 for $t-s$ large enough, it suffices to prove the same result for $\mathbf{V}_\epsilon^{\mathrm{fr}}$ and $\mathbf{V}_\epsilon^{\mathrm{in}}$ respectively.

Proposition 6.7.2. *For all $\beta, T > 0$, there exists positive constant $t_0 := t_0(\beta,T)$ and $C(\beta,T)$ such that for $\epsilon > 0$ small enough and $0 \le s \le t \in [0,\epsilon^{-2}T] \cap \mathbb{Z}$ satisfying $|t-s| \ge t_0$*

(a) for all $x_1 \le x_2 \in \Xi(t)$, $y_1 \le y_2 \in \Xi(s)$

$$\left|\mathbf{V}_\epsilon^{fr}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{t-s+1} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}}}$$

(b) For all (x_1,x_2,y_1,y_2) in the ∇-Weyl chamber,

$$\left|\nabla_{x_i}\mathbf{V}_\epsilon^{fr}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}}}, \quad i=1,2,$$
$$\left|\nabla_{y_i}\mathbf{V}_\epsilon^{fr}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}}}, \quad i=1,2.$$

(c) For all $x_1+1 \le x_2 \in \Xi(t)$ and $y_1 \le y_2 \in \Xi(s)$,

$$\left|\nabla_{x_1,x_2}\mathbf{V}_\epsilon^{fr}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^2} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}}}.$$

Proposition 6.7.3. *For all $\beta, T > 0$, there exists positive constant $t_0 := t_0(\beta,T)$ and $C(\beta,T)$ such that for $\epsilon > 0$ small enough $0 \le s \le t \in [0,\epsilon^{-2}T] \cap \mathbb{Z}$ such that $|t-s| \ge t_0$,*

(a) for all $x_1 \le x_2 \in \Xi(t)$ and $y_1 \le y_2 \in \Xi(s)$,

$$\left|\mathbf{V}_\epsilon^{in}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{t-s+1} e^{-\frac{\beta(|x_2-y_1|+|x_1-y_2|)}{\sqrt{t-s+1}}}.$$

(b) For all (x_1,x_2,y_1,y_2) in the ∇-Weyl chamber,

$$\left|\nabla_{x_i}\mathbf{V}_\epsilon^{in}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_2-y_1|+|x_1-y_2|)}{\sqrt{t-s+1}}}, \quad i=1,2,$$
$$\left|\nabla_{y_i}\mathbf{V}_\epsilon^{in}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_2-y_1|+|x_1-y_2|)}{\sqrt{t-s+1}}}, \quad i=1,2.$$

(c) For all $x_1 + 1 \le x_2 \in \Xi(t)$ *and* $y_1 \le y_2 \in \Xi(s)$*,*

$$\left|\nabla_{x_1,x_2}\mathbf{V}_\epsilon^{in}((x_1,x_2),(y_1,y_2),t)\right| \le \frac{C(\beta,T)}{(t-s+1)^2}e^{-\frac{\beta(|x_2-y_1|+|x_1-y_2|)}{\sqrt{t+1}}}.$$

The reader might notice that in Proposition 6.7.3, we write $|x_2 - y_1| + |x_1 - y_2|$ on the RHS exponents (compared with $|x_1 - y_1| + |x_2 - y_2|$ in Proposition 6.7.1). This in fact yields a stronger upper bound since by $x_1 \le x_2$ and $y_1 \le y_2$, one always has

$$|x_1 - y_1| + |x_2 - y_2| \le |x_2 - y_1| + |x_1 - y_2|.$$

Hence, combining Proposition 6.7.2 and Proposition 6.7.3, we conclude Proposition 6.7.1.

6.7.1 Estimate of $\mathbf{V}_\epsilon^{\mathrm{fr}}$

In this section, we will prove Proposition 6.7.2. Referring to (6.6.4),

$$\mathsf{p}_\epsilon(t,s,x_i-y_i) = \oint_{C_R}\left(\mathfrak{D}_\epsilon(z_i)\right)^{\lfloor (t-s)/J\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_i-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i} \tag{6.7.10}$$

where R is large enough so that C_r encircles all the poles of the integrand. Therefore, from (6.7.7) we have

$$\mathbf{V}_\epsilon^{\mathrm{fr}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \mathsf{p}_\epsilon\big(t,s,x_1-y_1\big)\mathsf{p}_\epsilon\big(t,s,x_2-y_2\big). \tag{6.7.11}$$

To estimate $\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)$, it suffices to analyze $\mathsf{p}_\epsilon(t,s,x_i-y_i)$. Referring to the expression (6.5.25) and (6.5.26),

$$\mathfrak{D}_\epsilon(z) := \lambda z^\mu\frac{(1+\alpha q^J)q^{-\rho}z - (\nu+\alpha q^J)}{(1+\alpha)q^{-\rho}z-(\nu+\alpha)}, \tag{6.7.12}$$

$$\mathfrak{R}_\epsilon(z,t,s) := \prod_{k=s+J\lfloor\frac{t-s}{J}\rfloor}^{t-1}\lambda(k)z^{\mu(k)}\frac{(1+\alpha(k)q)q^{-\rho}z-(\nu+\alpha(k)q)}{(1+\alpha(k))q^{-\rho}z-(\nu+\alpha(k))}. \tag{6.7.13}$$

Define the set of poles of the integrand in (6.7.10) to be $\mathcal{P}$, it is clear that

$$\mathcal{P} \subseteq \bigcup_{k=0}^{\infty}\{q^{\rho}\frac{\nu+\alpha(k)}{1+\alpha(k)}\} \cup \{0\} = \bigcup_{k=0}^{J-1}\{q^{\rho}\frac{\nu+\alpha(k)}{1+\alpha(k)}\} \cup \{0\}.$$

Due to Lemma 6.5.4,

$$\lim_{\epsilon\downarrow 0}\frac{q^{\rho}(\alpha(k)+\nu)}{1+\alpha(k)} = \frac{(I+\mathrm{mod}_J(k))b-(I+\mathrm{mod}_J(k)-1)}{b\mathrm{mod}_J(k)-(\mathrm{mod}_J(k)-1)} \in (0,1).$$

Therefore, there exists $0<\Theta<1$ such that for ϵ small enough

$$\mathcal{P} \subseteq [0,\Theta]. \tag{6.7.14}$$

To extract the spatial decay of $\mathsf{p}_\epsilon(t,s,x_i-y_i)$, we deform the contour of z_i from C_R to C_{r_i} where

$$r_i = \mathsf{u}(t-s, -\mathrm{sgn}(x_i-y_i)\beta). \tag{6.7.15}$$

Note that when $t-s$ is large enough, r_i is close to 1, thus deforming the contour from C_R to C_{r_i}, we do not cross the poles in the integrand. We parametrize C_{r_i} by $z_i(\theta_i) = r_i e^{\mathbf{i}\theta_i}, \theta \in (-\pi,\pi]$ and get

$$\mathsf{p}_\epsilon(t,s,x_i-y_i) = \frac{1}{2\pi}\oint_{C_{r_i}} \left(\mathfrak{D}_\epsilon(z_i(\theta_i))\right)^{\lfloor (t-s)/J\rfloor}\mathfrak{R}_\epsilon(z_i(\theta_i),t,s) z_i(\theta_i)^{x_i-y_i} d\theta_i$$

We want to bound each terms that appear in the integrand above. Note that by (6.7.15), $|z_i(\theta_i)|^{x_i-y_i} = e^{-\frac{\beta}{\sqrt{t-s+1}}|x_i-y_i|}$.

To estimate $\mathfrak{R}_\epsilon(z_i,t,s)$, referring to (6.7.13), $\mathfrak{R}_\epsilon(z,t,s)$ is a product of up to J terms (since $t-s-J\lfloor\frac{t-s}{J}\rfloor \leq J$). For each term, by Lemma 6.5.4

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\left|\lambda(k)z^{\mu(k)}\frac{(1+\alpha(k)q)q^{-\rho}z-(\nu+\alpha(k)q)}{(1+\alpha(k))q^{-\rho}z-(\nu+\alpha(k))}\right| \\ &= |z|^{\frac{1}{I}}\frac{(b(1+\mathrm{mod}_J(k))-\mathrm{mod}_J(k))z-(b(I+\mathrm{mod}_J(k)+1)-(I+\mathrm{mod}_J(k))}{(b\mathrm{mod}_J(k)-(\mathrm{mod}_J(k)-1))z-((I+\mathrm{mod}_J(k))b-(I+\mathrm{mod}_J(k)-1))}.\end{aligned} \tag{6.7.16}$$

The singularities in (6.7.16) lie strictly inside the unit disk. Since r_i is close to 1 when $t-s$ is large, for ϵ small enough and $t-s$ large enough, there exists constant C such that for $z \in C_{r_i}$ and $k \in \mathbb{Z}_{\geq 0}$

$$\left| \lambda(k) z^{\mu(k)} \frac{(1+\alpha(k)q)q^{-\rho}z - (\nu + \alpha(k)q)}{(1+\alpha(k))q^{-\rho}z - (\nu+\alpha(k))} \right| \leq C,$$

which implies

$$|\mathfrak{R}_\epsilon(z_i, t, s)| \leq C. \tag{6.7.17}$$

Consequently,

$$\begin{aligned} \mathsf{p}_\epsilon(t, s, x_i - y_i) &\leq \int_{-\pi}^{\pi} |\mathfrak{D}_\epsilon(z_i)|^{\lfloor (t-s)/J \rfloor} |\mathfrak{R}_\epsilon(z_i(\theta), t, s)| |z_i(\theta)|^{x_i - y_i} d\theta \\ &\leq C e^{-\frac{\beta}{\sqrt{t-s+1}}|x_i - y_i|} \int_{-\pi}^{\pi} \left| \mathfrak{D}_\epsilon(z_i(\theta)) \right|^{\lfloor (t-s)/J \rfloor} d\theta \end{aligned} \tag{6.7.18}$$

We expect to extract the temporal decay $\frac{1}{\sqrt{t-s+1}}$ from the integral above. To this end, we need to the following lemma.

Lemma 6.7.4. *There exists positive constants $C(\beta, T)$, C such that for $\theta \in (-\pi, \pi]$*

$$\left| \mathfrak{D}_\epsilon(z(\theta)) \right|^{t-s} \leq C(\beta, T) e^{-C(t-s+1)\theta^2}, \qquad z(\theta) = \mathsf{u}(t-s, \pm\beta) e^{\mathbf{i}\theta}$$

holds for $\epsilon > 0$ small enough and large enough $t - s \leq \epsilon^{-2} T$.

As a remark, we see from (6.7.12) that the function $\mathfrak{D}_\epsilon(z)$ is not globally analytic due to the factor z^μ (μ is not an integer), but it is analytic in a neighborhood of 1. Furthermore, $\left|\mathfrak{D}_\epsilon(z)\right|$ is a continuous function in a neighborhood of the unit circle.

Proof of Lemma 6.7.4. We only prove Lemma 6.7.4 for $z(\theta) = \mathsf{u}(t-s, \beta)e^{\mathbf{i}\theta}$, the argument for $z(\theta) = \mathsf{u}(t-s, -\beta)e^{\mathbf{i}\theta}$ is similar. By writing $\left|\mathfrak{D}_\epsilon(z(\theta))\right|^{t-s} = e^{(t-s)\mathrm{Re} \log \mathfrak{D}_\epsilon(z(\theta))}$, it suffices to show that there exists positive constants $C(\beta, T), C$ such that for $\epsilon > 0$ small enough and $t - s \leq \epsilon^{-2}T$ large enough

$$\mathrm{Re} \log \mathfrak{D}_\epsilon(\mathsf{u}(t-s, \beta)e^{\mathbf{i}\theta}) \leq \frac{C(\beta, T)}{t-s+1} - C\theta^2,$$

where $\operatorname{Re} z$ denotes the real part of a complex number z.

We divide our proof into three cases. It suffices to show

- $(\theta = 0) : \log \mathfrak{D}_\epsilon(\mathsf{u}(t-s,\beta)) \leq \frac{C(\beta,T)}{t-s+1}$
- (θ small): There exists $\zeta > 0$ s.t.

$$\operatorname{Re}\, \log \mathfrak{D}_\epsilon(\mathsf{u}(t-s,\beta)e^{\mathbf{i}\theta}) \leq \frac{C(\beta,T)}{t-s+1} - C\theta^2 \qquad \text{for } |\theta| \leq \zeta.$$

- (θ large): There exists $\delta > 0$ such that $\left|\mathfrak{D}_\epsilon(\mathsf{u}(t-s,\beta)e^{\mathbf{i}\theta})\right| < 1-\delta$ for $|\theta| > \zeta$.

The proof for the first and second bullet point are done by using the local property of $\mathfrak{D}_\epsilon(z)$ near 1 (applying Taylor expansion). Let O be a small neighborhood around 1 such that $\mathfrak{D}_\epsilon(z)$ is analytic inside O.

($\theta = 0$): We write $\mathfrak{D}_\epsilon(z)$ into terms of a telescoping product

$$\mathfrak{D}_\epsilon(z) = \prod_{k=0}^{J-1} \lambda(k) z^{\mu(k)} \frac{1+\alpha(k)q - (\nu+\alpha(k)q)q^\rho z^{-1}}{1+\alpha(k) - (\alpha(k)+\nu)q^\rho}.$$

By (6.6.5), we see that

$$\mathfrak{D}_\epsilon(z) = \prod_{k=0}^{J-1} \mathbb{E}\big[z^{-R_\epsilon(k)}\big] = \mathbb{E}\big[z^{-\sum_{k=0}^{J-1} R_\epsilon(k)}\big],$$

thus

$$\mathfrak{D}_\epsilon{}'(1) = -\mathbb{E}\Big[\sum_{k=0}^{J-1} R_\epsilon(k)\Big] = 0, \qquad \mathfrak{D}_\epsilon{}''(1) = \operatorname{Var}\Big[\sum_{k=0}^{J-1} R_\epsilon(k)\Big] = \sum_{k=0}^{J-1} \operatorname{Var}\big[R_\epsilon(k)\big].$$

Referring to (6.6.27),

$$\lim_{\epsilon\downarrow 0} \sum_{k=0}^{J-1} \operatorname{Var}\big[R_\epsilon(k)\big] = \sum_{k=0}^{J-1} \frac{(I+1+2k)b - (I+2k-1)}{I^2(1-b)} = JV_*,$$

where V_* is given by (6.1.12). The above discussion implies that

$$\log \mathfrak{D}_\epsilon(1) = 0, \qquad (\log \mathfrak{D}_\epsilon)'(1) = 0.$$

Moreover, there exists constant C such that uniformly for $z \in O$ and ϵ small enough,

$$|(\log \mathfrak{D}_\epsilon)''(z)| \leq C.$$

Since $\lim_{t-s\to\infty} \mathsf{u}(t-s,\beta) = 1$, we see that $\mathsf{u}(t-s,\beta) \in O$ for $t-s$ large enough. Thus, we taylor expand $\mathfrak{D}_\epsilon(z)$ around $z = 1$ and get

$$\log \mathfrak{D}_\epsilon(\mathsf{u}(t-s,\beta)) \leq C\left|\mathsf{u}(t-s,\beta) - 1\right|^2 \leq \frac{C(\beta,T)}{t-s+1}, \tag{6.7.19}$$

which justifies the first bullet point.

(θ small): Consider the function $\mathfrak{D}_\epsilon(z(\theta))$, we calculate for $z(\theta) \in O$

$$\begin{aligned}
\partial_\theta(\log \mathfrak{D}_\epsilon(z(\theta)))\big|_{\theta=0} &\in \mathbf{i}\mathbb{R}, \\
\lim_{\epsilon\downarrow 0, t-s\to\infty} \partial_\theta^2(\log \mathfrak{D}_\epsilon(z(\theta)))\big|_{\theta=0} &= -JV_*, \\
\left|\partial_\theta^3(\log \mathfrak{D}_\epsilon(z(\theta)))\right| &\leq C.
\end{aligned}$$

Given these properties, we taylor expand $\log \mathfrak{D}_\epsilon(z(\theta))$ at $\theta = 0$, there exists $\zeta > 0$ such that

$$\text{Re } \log \mathfrak{D}_\epsilon(z(\theta)) \leq \text{Re } \log \mathfrak{D}_\epsilon(z(0)) - \frac{JV_*}{2}\theta^2 \qquad |\theta| \leq \zeta$$

In conjunction with Re $\log \mathfrak{D}_\epsilon(z(0)) \leq \frac{C(\beta,T)}{t-s+1}$ (which is shown by (6.7.19)), we conclude the second bullet point.

(θ large): We set

$$\mathfrak{D}_*(z) := z^{\frac{J}{I}} \frac{(bJ-(J-1))z-((I+J)b-(I+J-1))}{z-(Ib-(I-1))} \tag{6.7.20}$$

Referring to the expression of $\mathfrak{D}_\epsilon$ in (6.7.12) and using Lemma 6.5.4, one has

$$\lim_{\epsilon\downarrow 0}\left|\mathfrak{D}_\epsilon(z)\right| = \left|\mathfrak{D}_*(z)\right|.$$

The convergence is uniform in an open neighborhood of unit circle. Thereby,

$$\lim_{\epsilon\downarrow 0, t-s\to\infty}\left|\mathfrak{D}_\epsilon(\mathsf{u}(t-s,\beta)e^{\mathbf{i}\theta})\right| = \left|\mathfrak{D}_*(e^{\mathbf{i}\theta})\right| \quad \text{uniformly over } (-\pi,\pi].$$

As a result, we conclude the third bullet point as long as we verify the following *steepest descent condition*

$$\left|\mathfrak{D}_*(z)\right| < 1 \quad \text{for } z \in C1\backslash\{1\}. \tag{SD.$C1$}$$

To prove (SD.$C1$), we compute

$$\begin{aligned}\left|\mathfrak{D}_*(e^{\mathbf{i}\theta})\right|^2 &= \left|\frac{(bJ-(J-1))e^{\mathbf{i}\theta}-((I+J)b-(I+J-1))}{e^{\mathbf{i}\theta}-(Ib-(I-1))}\right|^2\\ &= \frac{(bJ-(J-1))^2+((I+J)b-(I+J-1))^2-2(bJ-(J-1))((I+J)b-(I+J-1))\cos\theta}{1+(Ib-(I-1))^2-2(Ib-(I-1))\cos\theta}\\ &= 1-\frac{2J(1-b)(1-\cos\theta)((I+J)b-(I+J-2))}{1+(Ib-(I-1))^2-2(Ib-(I-1))\cos\theta} < 1, \qquad \theta\in(-\pi,\pi]\backslash\{0\}.\end{aligned}$$

In the last step, we used the condition $\frac{I+J-2}{I+J-1} < b < 1$. □

Having proved Lemma 6.7.4, we proceed to finish the proof of Theorem 6.7.2.

Proof of Theorem 6.7.2. Due to Lemma 6.7.4,

$$\int_{-\pi}^{\pi}\left|\mathfrak{D}_\epsilon(z_i(\theta))\right|^{\lfloor\frac{t-s}{J}\rfloor}d\theta \le \int_{-\pi}^{\pi} C(\beta,T)e^{-C(\lfloor\frac{t-s}{J}\rfloor+1)\theta^2}d\theta \le \frac{C(\beta,T)}{\sqrt{t-s+1}}.$$

This being the case, by (6.7.18) we readily see that

$$\mathsf{p}_\epsilon(t,s,x_i-y_i) \le \frac{C(\beta,T)}{\sqrt{t-s+1}} e^{-\frac{\beta}{\sqrt{t-s+1}}|x_i-y_i|}. \tag{6.7.21}$$

Incorporating this bound into (6.7.11) concludes Theorem 6.7.2 part (a).

For the gradient, notice that one has

$$\nabla_{x_1}\mathbf{V}_\epsilon^{\mathrm{fr}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \nabla\mathsf{p}(t,s,x_1-y_1)\mathsf{p}(t,s,x_2-y_2), \tag{6.7.22}$$

$$\nabla_{y_1}\mathbf{V}_\epsilon^{\mathrm{fr}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \mathsf{p}(t,s,x_1-y_1)\nabla\mathsf{p}(t,s,x_2-y_2-1),$$

$$\nabla_{x_1,x_2}\mathbf{V}_\epsilon^{\mathrm{fr}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \nabla\mathsf{p}(t,s,x_1-y_1)\nabla\mathsf{p}(t,s,x_2-y_2). \tag{6.7.23}$$

The proof for gradients $\nabla_{x_2}, \nabla_{y_2}$ is similar to that for $\nabla_{x_1}, \nabla_{y_1}$ by symmetry. It suffices to analyze

$$\nabla\mathsf{p}(t,x_1-y_1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathfrak{D}(z_1(\theta_1))^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_1(\theta_1),t,s)z_1(\theta_1)^{x_1-y_1}(z_1(\theta_1)-1)d\theta_1$$

By the fact $\big|z_1(\theta_1)-1\big| = \big|e^{\pm\frac{\beta}{\sqrt{t-s+1}}+\mathbf{i}\theta_1}-1\big| \le C(\frac{1}{\sqrt{t-s+1}}+|\theta_1|)$, we conclude

$$\big|\nabla\mathsf{p}(t,x_i-y_i)\big| \le C(\beta,T)e^{-\frac{\beta}{\sqrt{t-s+1}}|x_i-y_i|}\int_{-\pi}^{\pi} e^{-C\lfloor\frac{t-s}{J}\rfloor\theta_1^2}(\frac{1}{\sqrt{t-s+1}}+|\theta_1|)d\theta_1 \le \frac{C(\beta,T)}{t-s+1}e^{-\frac{\beta}{\sqrt{t-s+1}}|x_i-y_i|}, \tag{6.7.24}$$

where the last inequality follows by a change of variable $\theta_1 \to \frac{\theta_1}{\sqrt{t-s+1}}$. Incorporating this bound into (6.7.22) and (6.7.23), we conclude the Theorem 6.7.2 (b), (c). □

6.7.2 Estimate of $\mathbf{V}_\epsilon^{\mathrm{in}}$, an overview.

Recall from (6.7.8) that

$$\mathbf{V}_\epsilon^{\mathrm{in}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_{C_R}\oint_{C_R}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}$$

$$
- \mathsf{Res}_{z_1=\mathfrak{s}_\epsilon(z_2)}\left[\oint_{C_R}\oint_{C_R}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}\right].
$$
(6.7.25)

We study the double contour integral in (6.7.25). Recall from (6.5.27) and (6.5.28) that

$$
\mathfrak{F}_\epsilon(z_1,z_2)=\frac{q\nu-\nu+(\nu-q)q^{-\rho}z_2+(1-q\nu)q^{-\rho}z_1+(q-1)q^{-2\rho}z_1z_2}{q\nu-\nu+(\nu-q)q^{-\rho}z_1+(1-q\nu)q^{-\rho}z_2+(q-1)q^{-2\rho}z_1z_2}, \tag{6.7.26}
$$

which produces a pole at $z_1=\mathfrak{s}_\epsilon(z_2)$ where

$$
\mathfrak{s}_\epsilon(z)=\frac{(1-q\nu)q^{-\rho}z-\nu(1-q)}{(q-\nu)q^{-\rho}+(1-q)q^{-2\rho}z}.
$$

Referring to (6.7.14), the other poles of the integrand belong to $[0,\Theta]$ for some $0<\Theta<1$.

We say the contour Γ is *admissible* if

$$
(1): \Gamma \text{ contains } [0,\Theta] \text{ but does not contain } 1-I, \quad (2): d(1-I,\Gamma)>\frac{1}{2I}, \tag{6.7.27}
$$

where the distance between a point $z\in\mathbb{C}$ and a set A is define by $d(z,A):=\inf\{|z-y|: y\in A\}$. Figure 6.3 below gives several graphical examples of admissible and not admissible contours.

Define

$$
\mathfrak{s}_*(z):=\lim_{\epsilon\downarrow 0}\mathfrak{s}_\epsilon(z)=\frac{(I-1)z+1}{I+1-z}.
$$

Note that

$$
\lim_{|z|\to\infty}\mathfrak{s}_*(z)=1-I.
$$

Note that $z_2\in C_r$, from above we have: For R large enough and ϵ small enough, if Γ is admissible, deforming the z_1-contour from C_r to Γ will cross the pole $s_\epsilon(z_2)$ for all $z_2\in C_r$. Moreover, such deformation does not cross any other poles in $\mathcal{P}$. Therefore,

$$\mathbf{V}_\epsilon^{\text{in}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_\Gamma \oint_{C_R} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}.$$

In practice, we deform the z_1-contour to some contour $\Gamma(t-s,\epsilon)$ which depends on both $t-s$ and ϵ so that it is admissible for $t-s$ large enough and ϵ small enough.

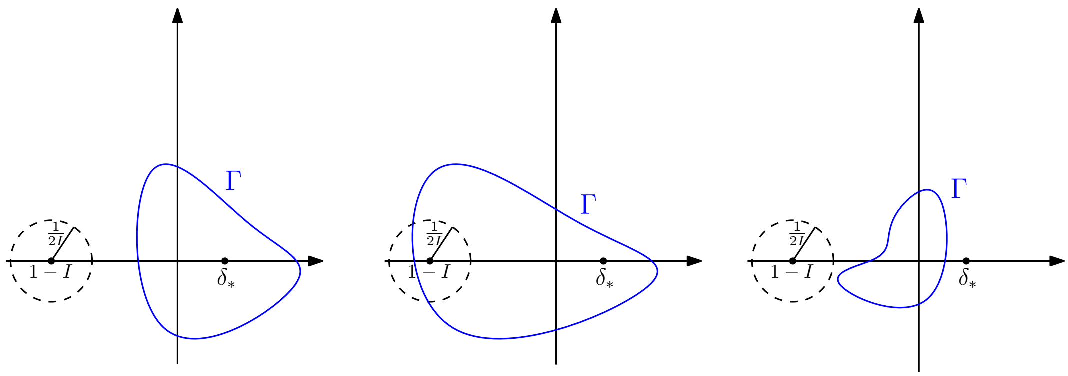


Figure 6.3: Graphical examples of admissible and not admissible contour Γ.

Assuming that we have deformed z_1-contour to $\Gamma(t-s,\epsilon)$, which is admissible. The next step is to deform the z_2-contour. Note that given $z_1 \in \Gamma(t-s,\epsilon)$, $\mathfrak{F}_\epsilon(z_1,z_2)$ generates a pole at $z_2 = \mathfrak{p}_\epsilon(z_1)$ ($\mathfrak{p}_\epsilon$ is the inverse of $\mathfrak{s}_\epsilon$)

$$\mathfrak{p}_\epsilon(z_1) = \frac{(1-q)\nu + (q-\nu)q^{-\rho} z_1}{(q-1)q^{-2\rho} z_1 + (1-q\nu)q^{-\rho}}. \tag{6.7.28}$$

We consider three potential radius

$$r_2 := \mathsf{u}(t-s, \operatorname{sgn}(x_1-y_2)k_2\beta), \quad r_2' := \mathsf{u}(t-s, \operatorname{sgn}(x_1-y_2)2k_2\beta), \quad r_2'' := \mathsf{u}(t-s, \operatorname{sgn}(x_1-y_2)3k_2\beta), \tag{6.7.29}$$

where $k_2 \geq 1$ is a constant which is irrelevant with the current discussion. We deform z_2-contour from C_R to $C_{r_2^*(z_1)}$, where

$$r_2^*(z_1) = r_2 1_{\{\mathfrak{p}_\epsilon(z_1) > r_2'\}} + r_2'' 1_{\{\mathfrak{p}_\epsilon(z_1) \leq r_2'\}}.$$

In other words, if the pole $\mathfrak{p}_\epsilon(z_1)$ lies outside $C_{r_2'}$, we choose z_2-contour to be a circle with radius $r_2 < r_2'$. If the pole $\mathfrak{p}_\epsilon(z_1)$ lies inside $C_{r_2'}$, we choose z_2-contour to be circle with radius $r_2'' > r_2'$. It is clear we always have for $t - s$ large enough that

$$|\mathfrak{p}_\epsilon(z_1) - z_2| \geq \frac{\beta}{\sqrt{t-s+1}}, \qquad \forall z_2 \in C_{r_2^*(z_1)}. \tag{6.7.30}$$

Referring to the expression of $\mathfrak{F}_\epsilon(z_1, z_2)$ (6.7.26), we find that

$$\mathsf{Res}_{z_2=\mathfrak{p}_\epsilon(z_1)}\mathfrak{F}_\epsilon(z_1, z_2) = \frac{q\nu - \nu + (\nu - q)q^{-\rho}\mathfrak{p}_\epsilon(z_1) + (1-q\nu)q^{-\rho}z_1 + (q-1)q^{-2\rho}z_1\mathfrak{p}_\epsilon(z_1)}{(q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho}}.$$

We set

$$\begin{aligned}
\mathfrak{H}_\epsilon(z_1) &= \mathfrak{D}_\epsilon(z_1)\mathfrak{D}_\epsilon(\mathfrak{p}_\epsilon(z_1)),\\
\mathfrak{J}_\epsilon(z_1) &= \mathsf{Res}_{z_2=\mathfrak{p}_\epsilon(z_1)}\mathfrak{F}_\epsilon(z_1, z_2) z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_2-y_1}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}},\\
&= \frac{q\nu - \nu + (\nu - q)q^{-\rho}\mathfrak{p}_\epsilon(z_1) + (1-q\nu)q^{-\rho}z_1 + (q-1)q^{-2\rho}z_1\mathfrak{p}_\epsilon(z_1)}{(q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho}} z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_2-y_1}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}.
\end{aligned} \tag{6.7.31}$$

From preceding discussion, we decompose $\mathbf{V}_\epsilon^{\text{in}} = \mathbf{V}_\epsilon^{\text{blk}} + \mathbf{V}_\epsilon^{\text{res}}$, where

$$\begin{aligned}
\mathbf{V}_\epsilon^{\text{blk}}\big((x_1, x_2), (y_1, y_2), t, s\big) &= \oint_{\Gamma(t-s,\epsilon)}\oint_{C_{r_2(z_1)}} \mathfrak{F}_\epsilon(z_1, z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor} z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i},\\
\mathbf{V}_\epsilon^{\text{res}}\big((x_1, x_2), (y_1, y_2), t, s\big) &= \oint_{\Gamma(t-s,\epsilon)} \mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}\mathfrak{J}_\epsilon(z_1)\mathfrak{H}_\epsilon(z_1)^{\lfloor\frac{t-s}{J}\rfloor}\frac{dz_1}{2\pi\mathbf{i}z_1\mathfrak{p}_\epsilon(z_1)}.
\end{aligned} \tag{6.7.32}$$

Note that we integrate under the indicator $1_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}$, which arises in the case that deforming the z_2-contour from C_r to $C_{r_2^*(z_1)}$ crosses the pole $\mathfrak{p}_\epsilon(z_1)$.

We want to perform the steepest descent argument for $\mathbf{V}_\epsilon^{\text{blk}}$ and $\mathbf{V}_\epsilon^{\text{res}}$, similar to what we have done in Section 6.7.1. More precisely, as $t - s \to \infty$ and $\epsilon \downarrow 0$, $\Gamma(t-s, \epsilon)$ converges to some fixed contour Γ_*.[13] We set

$$\mathfrak{p}_*(z) := \lim_{\epsilon \downarrow 0} \mathfrak{p}_\epsilon(z) = \frac{(I+1)z - 1}{z + (I-1)}. \tag{6.7.33}$$

Recall from (6.7.20) that

$$\mathfrak{D}_*(z) = z^{\frac{J}{I}} \frac{(Jb - (J-1))z - ((I+J)b - (I+J-1))}{z - (Ib - (I-1))}.$$

and set

$$\mathfrak{H}_*(z) = \mathfrak{D}_*(z)\mathfrak{D}_*(\mathfrak{p}_*(z)).$$

Note that

$$|\mathfrak{D}_*(z)| = \lim_{\epsilon \downarrow 0} |\mathfrak{D}_\epsilon(z)|, \qquad |\mathfrak{H}_*(z)| = \lim_{\epsilon \downarrow 0} |\mathfrak{H}_\epsilon(z)|.$$

We require the contour Γ_* satisfying the steepest descent condition.

$$(i)\left|\mathfrak{D}_*(z)\right| < 1, \quad z \in \Gamma_* \backslash \{1\}; \qquad (ii)\left|\mathfrak{H}_*(z)\right| < 1, \quad z \in \Gamma_* \backslash \{1\}. \tag{6.7.34}$$

As we see from (SD.C1) that if we take $\Gamma_* = C1$, (i) holds. However, (ii) does not hold. In truth, Figure 6.4 indicates the region where $|\mathfrak{D}_*(z)| \leq 1$ and $|\mathfrak{H}_*(z)| \leq 1$ for $I = 2$ and $b = 0.8$. We see that $C1$ lies fully inside $|\mathfrak{D}_*(z)| \leq 1$, but partially outside $|\mathfrak{H}_*(z)| \leq 1$.

Set $\mathcal{M} = \{\left|z - \frac{1}{I+1}\right| = \frac{I}{I+1}\}$, the following lemma says that $\mathcal{M}$ the satisfies steepest descent condition (6.7.34).

Lemma 6.7.5. *We have*

$$|\mathfrak{D}_*(z)| < 1, z \in \mathcal{M}\backslash\{1\}, \qquad |\mathfrak{H}_*(z)| < 1, z \in \mathcal{M}\backslash\{1\}. \tag{SD$\mathcal{M}$}$$

[13]We define the distance of two contours to be $\operatorname{dist}(\Gamma_1, \Gamma_2) = \sup_{x \in \Gamma_1, y \in \Gamma_2} \left(d(x, \Gamma_2) \vee d(y, \Gamma_1)\right)$. We say a sequence of contours Γ_n converges to Γ if $\lim_{n \to \infty} \operatorname{dist}(\Gamma_n, \Gamma) = 0$.

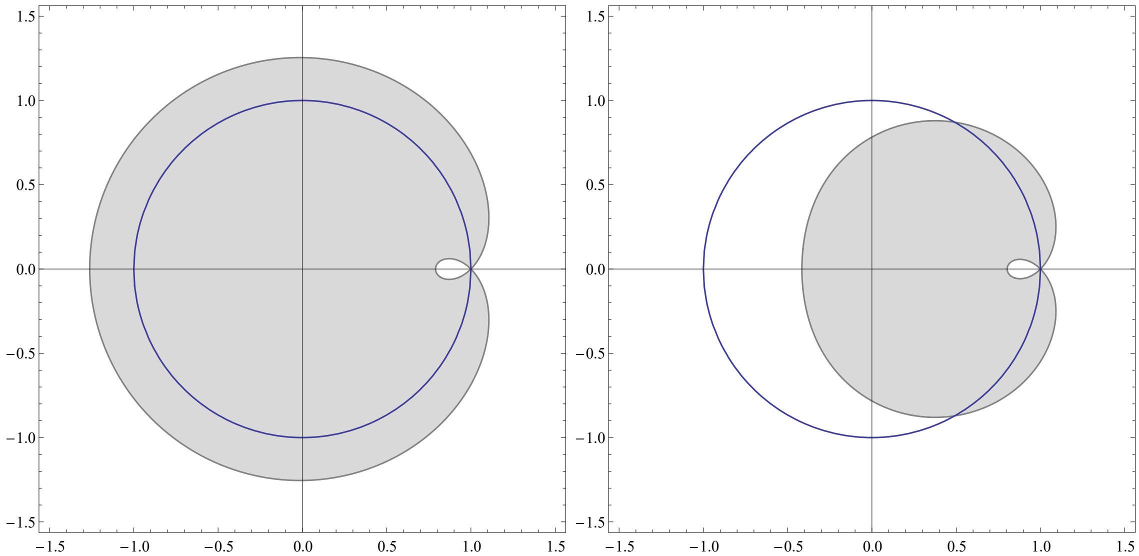


Figure 6.4: We choose $b = 0.8$ and $I = 2$. The figures on the left and right show respectively the region where $\left|\mathfrak{D}_*(z)\right| \le 1$ and $\left|\mathfrak{H}_*(z)\right| \le 1$, which is filled with gray color. The unit circle (with blue color) is drawn for comparison.

Proof. Parametrize $\mathcal{M}$ by $z(\theta) = \frac{1}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta}, \theta \in (-\pi, \pi]$, we compute

$$\begin{aligned}
|\mathfrak{D}_*(z(\theta))|^2 &\le |z(\theta)|^{\frac{2J}{I}} \left| \frac{(Jb-(J-1))z(\theta) - ((I+J)b-(I+J-1))}{z(\theta)-(Ib-(I-1))} \right|^2 \\
&\le \left| \frac{(Jb-(J-1))z(\theta) - ((I+J)b-(I+J-1))}{z(\theta)-(Ib-(I-1))} \right|^2 \\
&= \left| \frac{(Jb-(J-1))(\frac{1}{I+1} + \frac{I}{I+1}e^{i\theta}) - ((I+J)b-(I+J-1))}{\frac{I}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta} - (Ib-(I-1))} \right|^2 \\
&= 1 - \frac{2I^2J(1-b)((I+J+1)b-(I+J-1))(1-\cos\theta)}{\left|\frac{1}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta} - (Ib-(I-1))\right|^2 (1+I)^2} < 1, \qquad \theta \in (-\pi, \pi]\backslash\{0\}.
\end{aligned}$$

where in the first line we used the fact $|z(\theta)| \le 1$ and in the last line we used $\frac{I+J-2}{I+J-1} < b < 1$, note that when $I \ge 2$ and $J \ge 1$, we have

$$b \ge \frac{I+J-2}{I+J-1} > \frac{I+J-1}{I+J+1},$$

which concludes the last inequality.

For $\mathfrak{H}_*(z)$, note that

$$
\begin{aligned}
&\mathfrak{H}_*(z)\\
&= z^{\frac{J}{I}} \frac{(bJ-(J-1))z-((I+J)b-(I+J-1))}{z-(Ib-(I-1))} \mathfrak{p}_*(z)^{\frac{J}{I}} \frac{(bJ-(J-1))\mathfrak{p}_*(z)-((I+J)b-(I+J-1))}{\mathfrak{p}_*(z)-(Ib-(I-1))}\\
&= \left(z\mathfrak{p}_*(z)\right)^{\frac{J}{I}} \frac{(bJ-(J-1))z-((I+J)b-(I+J-1))}{z-(Ib-(I-1))} \cdot \frac{(bJ-(J-1))\mathfrak{p}_*(z)-((I+J)b-(I+J-1))}{\mathfrak{p}_*(z)-(Ib-(I-1))}
\end{aligned}
$$

A crucial observation is that $\left|z-\frac{1}{I+1}\right| = \frac{I}{I+1}$ implies

$$
\left|z\mathfrak{p}_*(z)\right| = \left|z\frac{(I+1)z-1}{z+(I-1)}\right| = \left|\frac{Iz}{z+(I-1)}\right| = 1.
$$

which can be verified by inserting $z(\theta) = \frac{1}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta}$. Consequently, we see that

$$
\begin{aligned}
\left|\mathfrak{H}_*(z(\theta))\right|^2 &= \left|\frac{bz(\theta)-(I+1)b-1}{z(\theta)-(Ib-(I-1))} \cdot \frac{b\mathfrak{p}_*(z(\theta))-((I+1)b-I)}{\mathfrak{p}_*(z(\theta))-(Ib-(I-1))}\right|^2\\
&= \left|\frac{I+J-(I+J+1)b+(Jb-(J-1))e^{\mathbf{i}\theta}}{I-(I+1)b+e^{\mathbf{i}\theta}} \cdot \frac{(I+J)b-(I+J-1)+((1-J)b+J-2)e^{\mathbf{i}\theta}}{Ib-(I-1)+(b-2)e^{\mathbf{i}\theta}}\right|^2\\
&= 1 + \frac{-4(b-1)J(2-J-I+b(J+I))(\cos\theta-1)(a_J-b_J\cos\theta)}{\left|(b-2)e^{\mathbf{i}\theta}+(1+(b-1)I)\right|^2\left|e^{\mathbf{i}\theta}-(b+(b-1)I)\right|^2} \qquad (6.7.35)
\end{aligned}
$$

where

$$
\begin{aligned}
a_J &= (J^2+JI)(1-b)^2+2+(2b-2)J+(b^2-1)I+(b-1)^2I^2\\
b_J &= (J^2+JI)(1-b)^2+(2b-2)J+(1+2b-b^2)+(-3+4b-b^2)I
\end{aligned}
$$

We claim that $|b_J| < a_J$, which implies $a_J - b_J\cos\theta > 0$. This claim is justified by showing

$$
\begin{aligned}
a_J + b_J &= (2J^2+2JI+I^2)(1-b)^2+(4b-4)(I+J)+3+2b-b^2\\
&= (J^2-1)(1-b)^2+((J+I)(b-1)+2)^2 > 0,\\
a_J - b_J &= (b-1)^2I^2+2(b-1)^2I+(b-1)^2 = (b-1)^2(I+1)^2 > 0.
\end{aligned}
$$

Therefore, by $\frac{I+J-2}{I+J-1} < b < 1$ and (6.7.35)

$$|\mathfrak{H}_*(z(\theta))| < 1, \qquad \theta \in (-\pi, \pi]\backslash\{0\},$$

which concludes our proof. □

We need to consider the following modification of $\mathcal{M}$

$$\mathcal{M}(u) := \partial\big(\{z : |z - \frac{1}{I+1}| = \frac{I}{I+1} + u\} \cap \{|z| \leq 1\}\big),$$

where u is some positive real number.

Lemma 6.7.6. *There exists $\delta > 0$ such that for all $0 < u < \delta$, one has*

$$\begin{aligned} |\mathfrak{D}_*(z)| < 1, \qquad z \in \mathcal{M}(u)\backslash\{1\}, \\ |\mathfrak{H}_*(z)| < 1, \qquad z \in \mathcal{M}(u)\backslash\{1\}. \end{aligned} \tag{SD$\mathcal{M}(u)$}$$

Proof. The proof of this lemma uses similar techniques which appear in [CGST20, Lemma 6.4]. By straightforward computation, one finds that

$$\begin{aligned} &\mathfrak{D}_*(1) = 1; \qquad \mathfrak{D}_*'(1) = 0; \qquad \mathfrak{D}_*''(1) = JV_*. \\ &\mathfrak{H}_*(1) = 1; \qquad \mathfrak{H}_*'(1) = 0; \qquad \mathfrak{H}_*''(1) = 2JV_*. \end{aligned}$$

Here, V_* is given by (6.1.12). We taylor expand $\mathfrak{D}_*(z)$ and $\mathfrak{H}_*(z)$ around $z = 1$ and get

$$\begin{aligned} \mathfrak{D}_*(z) &= 1 + \frac{1}{2}JV_*(z-1)^2 + O(|z-1|^3), \\ \mathfrak{H}_*(z) &= 1 + JV_*(z-1)^2 + O(|z-1|^3). \end{aligned}$$

Notice that in the vertical direction where $z - 1 \in \mathbf{i}\mathbb{R}$, $\frac{1}{2}(z-1)^2$ is negative. This implies that

$$|\mathfrak{D}_*(z)| < 1 \quad z \in \mathcal{A}_p\backslash\{1\}; \qquad |\mathfrak{H}_*(z)| < 1 \quad z \in \mathcal{A}_p\backslash\{1\}. \tag{6.7.36}$$

where $\mathcal{A}_p$ is a hourglass region centered at one, $\mathcal{A}_p = \{z : z = 1 + ve^{\mathbf{i}\phi}, |\phi - \frac{\pi}{2}| < \phi_0, |v| < v_0\}$ with $v_0, \phi_0 > 0$ fixed. For $z \in \mathcal{M}(u)\backslash\mathcal{A}_p$, due to $\lim_{u\downarrow 0} \mathrm{dist}\,(\mathcal{M}(u)\backslash\mathcal{A}_p, \mathcal{M}\backslash\mathcal{A}_p) = 0$ and Lemma 6.7.5, we find that there exists a small δ, such that for $0 < u < \delta$

$$\sup_{z\in\mathcal{M}(u)\backslash\mathcal{A}_p} |\mathfrak{D}_*(z)| < 1, \qquad \sup_{z\in\mathcal{M}(u)\backslash\mathcal{A}_p} |\mathfrak{H}_*(z)| < 1.$$

Combining this with (6.7.36) concludes the proof of Lemma 6.7.6. □

We fix a constant $0 < u_* < \delta \wedge \frac{1}{4I}$, and set $\mathcal{M}' := \mathcal{M}(u_*)$. From our discussion above, $\mathcal{M}'$ is admissible and satisfies (SD$\mathcal{M}(u)$).

To prove Proposition 6.7.3, we need to choose our contour such that it controls both $\mathbf{V}_\epsilon^{\mathrm{blk}}$ and $\mathbf{V}_\epsilon^{\mathrm{res}}$. The choice will depend on the sign of $x_2 - y_1$ and $x_1 - y_2$. We need to discuss separately for each of the following cases

(i): $(+-)$ case: $x_2 - y_1 \geq 0$ and $x_1 - y_2 \leq 0$,
(ii): $(--)$ case: $x_2 - y_1 \leq 0$ and $x_1 - y_2 \leq 0$,
(iii): $(++)$ case: $x_2 - y_1 \geq 0$ and $x_1 - y_2 \geq 0$.

Note that we don't need to consider the case where $x_2 - y_1 < 0$ and $x_1 - y_2 < 0$, since it contradicts our condition $x_1 \leq x_2$ and $y_1 \leq y_2$.

6.7.3 Estimate of $\mathbf{V}_\epsilon^{\mathrm{in}}$, the $(+-)$ case

In this case we shrink the z_1-contour from C_r to

$$\mathcal{M}(t-s, -\beta) := \{z_1 : \big|z_1 - \frac{1}{I+1}\big| = \frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\}.$$

It is clear that for $t - s$ large enough, $\mathcal{M}(t-s, -\beta)$ is admissible. Consequently, we have

$$\mathbf{V}_\epsilon^{\mathrm{in}}\big((x_1, x_2), (y_1, y_2), t, s\big) = \mathbf{V}_\epsilon^{\mathrm{blk}}\big((x_1, x_2), (y_1, y_2), t, s\big) + \mathbf{V}_\epsilon^{\mathrm{res}}\big((x_1, x_2), (y_1, y_2), t, s\big),$$

where

$$\mathbf{V}_\epsilon^{\mathrm{blk}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_{C_{r_2^*(z_1)}} \oint_{\mathcal{M}(t,-\beta)} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}, \tag{6.7.37}$$

$$\mathbf{V}_\epsilon^{\mathrm{res}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_{\mathcal{M}(t,-\beta)} \mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}} \mathfrak{I}_\epsilon(z_1) \mathfrak{H}_\epsilon(z_1)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_1,t,s) \mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s) \frac{dz_1}{2\pi \mathbf{i} z_1 \mathfrak{p}_\epsilon(z_1)}. \tag{6.7.38}$$

Parametrizing $z_1(\theta_1) = \frac{1}{I+1} + \big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\big)e^{\mathbf{i}\theta_1}$, we need the following lemma.

Lemma 6.7.7. *There exists positive $C(\beta,T), C$ such that*

$$|\mathfrak{D}_\epsilon(z(\theta))|^{t-s} \leq C(\beta,T)e^{-C(t-s+1)\theta^2}; |\mathfrak{H}_\epsilon(z(\theta))|^{t-s} \leq C(\beta,T)e^{-C(t-s+1)\theta^2}$$

with $z(\theta) = \frac{1}{I+1} + \big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\big)e^{\mathbf{i}\theta}$ for $\epsilon > 0$ small enough and $t - s \leq \epsilon^{-2}T$ large enough.

Proof. Similar to the proof of Lemma 6.7.4, it suffices to show there exists positive constants $C(\beta,T), C$ such that

$$\mathrm{Re}\,\log \mathfrak{D}_\epsilon(z(\theta)) \leq \frac{C(\beta,T)}{t-s+1} - C\theta^2; \qquad \mathrm{Re}\,\log \mathfrak{H}_\epsilon(z(\theta)) \leq \frac{C(\beta,T)}{t-s+1} - C\theta^2. \tag{6.7.39}$$

We prove the lemma for ($\theta = 0$), (θ small) and (θ large) respectively

- ($\theta = 0$) : $\mathrm{Re}\,\mathfrak{D}_\epsilon(z(0)), \mathrm{Re}\,\mathfrak{H}_\epsilon(z(0)) \leq \frac{C(\beta,T)}{t-s+1}$.
- (θ small): There exists $\zeta > 0$ and constants $C(\beta,T)$ and $C > 0$ such that (6.7.39) holds for $|\theta| \leq \zeta$.
- (θ large): There exists $\delta > 0$ such that $\big|\mathfrak{D}_\epsilon(z(\theta))\big|, \big|\mathfrak{H}_\epsilon(z(\theta))\big| < 1 - \delta$ for $|\theta| > \zeta$.

We consider the first two bullet points ($\theta = 0$) and (θ small). The analysis of ($\theta = 0$) and (θ small) case for $\mathfrak{D}_\epsilon$ is similar to Lemma 6.7.4, we do not repeat here. For $\mathfrak{H}_\epsilon(z) = \mathfrak{D}_\epsilon(z)\mathfrak{D}_\epsilon(\mathfrak{p}_\epsilon(z))$, by straightforward calculation,

$$\begin{aligned}
&\mathfrak{H}_\epsilon(1) = \mathfrak{D}_\epsilon(\mathfrak{p}_\epsilon(1)),\\
&\mathfrak{H}'_\epsilon(1) = \mathfrak{D}'_\epsilon(\mathfrak{p}_\epsilon(1))\mathfrak{p}'_\epsilon(1),\\
&\lim_{\epsilon\downarrow 0}\mathfrak{H}''_\epsilon(1) = 2JV_*.
\end{aligned} \tag{6.7.40}$$

For the first equation above, we taylor expand $\mathfrak{D}_\epsilon(z)$ at $z = 1$ and according to (6.7.44),

$$\mathfrak{H}_\epsilon(1) = 1 + \frac{1}{2}\mathfrak{D}''_\epsilon(1)(\mathfrak{p}_\epsilon(1)-1)^2 + O\big((\mathfrak{p}_\epsilon(1)-1)^3\big) = 1 + \frac{JV_*(\rho I - \rho^2)^2}{2I^2}\epsilon^2 + O(\epsilon^{\frac{5}{2}}). \tag{6.7.41}$$

For $\mathfrak{H}'_\epsilon(1) = \mathfrak{D}'_\epsilon(\mathfrak{p}_\epsilon(1))\mathfrak{p}'_\epsilon(1)$, taylor expanding $\mathfrak{D}'_\epsilon(z)$ around $z = 1$, according to (6.7.44),

$$\mathfrak{D}'_\epsilon(\mathfrak{p}_\epsilon(1)) = \mathfrak{D}'_\epsilon(1) + \mathfrak{D}''_\epsilon(1)(\mathfrak{p}_\epsilon(1)-1) + O(\mathfrak{p}_\epsilon(1)-1)^2 = \frac{JV_*(\rho I - \rho^2)}{2I}\epsilon + O(\epsilon^{\frac{3}{2}}),$$

Combining this with $\mathfrak{p}'_\epsilon(1) = 1 + O(\epsilon^{\frac{1}{2}})$ yields

$$\mathfrak{H}'_\epsilon(1) = \frac{JV_*(\rho I - \rho^2)}{2I}\epsilon + O(\epsilon^{\frac{3}{2}}). \tag{6.7.42}$$

Using (6.7.41), (6.7.42) and (6.7.40), we get

$$(\log\mathfrak{H}_\epsilon)(1) = \frac{JV_*(\rho I - \rho^2)^2}{2I^2}\epsilon^2 + O(\epsilon^{\frac{5}{2}}), \quad (\log\mathfrak{H}_\epsilon)'(1) = \frac{JV_*(\rho I - \rho^2)}{2I}\epsilon + O(\epsilon^{\frac{3}{2}}), \quad \lim_{\epsilon\downarrow 0},(\log\mathfrak{H}_\epsilon)''(1) = 2JV_*. \tag{6.7.43}$$

Moreover, straightforward calculation gives $|(\log\mathfrak{H}_\epsilon)'''(z)| \le C$ for $z \in O$ (which is a small neighborhood of 1). Thereby, by Taylor expansion we find that

$$\log\mathfrak{H}_\epsilon(z(0)) = \log\mathfrak{H}_\epsilon(1) + (\log\mathfrak{H}_\epsilon)'(1)(z(0)-1) + (\log\mathfrak{H}_\epsilon)''(1)(z(0)-1)^2 + O((z(0)-1)^3).$$

Using (6.7.43), $z(0) = 1 - \frac{\beta}{\sqrt{t-s+1}}$ and $\epsilon^2(t-s) \le T$, we see that there exists $C(\beta, T)$ such that for

$t-s$ large and ϵ small,

$$\log \mathfrak{H}_\epsilon(z(0)) \le \frac{C(\beta,T)}{t-s+1},$$

which gives the first bullet point.

For (θ small), we readily calculate

$$\partial_\theta(\log \mathfrak{H}_\epsilon(z(\theta)))\big|_{\theta=0} \in \mathbf{i}\mathbb{R},$$
$$\lim_{\epsilon\downarrow 0, t\to\infty} \partial_\theta^2(\log \mathfrak{H}_\epsilon(z(\theta)))\big|_{\theta=0} = -\frac{2I^2JV_*}{(I+1)^2},$$
$$\left|\partial_\theta^3(\log \mathfrak{H}_\epsilon(z(\theta)))\right| \le C, \qquad \text{for } |\theta| \le \zeta.$$

Thus, via Taylor expansion, we find that for $|\theta| \le \zeta$,

$$\mathrm{Re}\ \log \mathfrak{H}_\epsilon(z(\theta)) \le \mathrm{Re}\ \log \mathfrak{H}_\epsilon(z(0)) - \frac{I^2JV_*}{2(I+1)^2}\theta^2 \le \frac{C(\beta,T)}{t-s+1} - \frac{I^2JV_*}{2(I+1)^2}\theta^2,$$

which conclude the second bulletin point.

For (θ large), recall $z(\theta) = \frac{1}{I+1} + \left(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\right)e^{\mathbf{i}\theta}$, we notice that

$$\lim_{\epsilon\downarrow 0, t-s\to\infty} \left|\mathfrak{D}_\epsilon(z(\theta))\right| = \left|\mathfrak{D}_*(\frac{1}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta})\right|, \quad \text{uniformly for } \theta \in (-\pi,\pi].$$
$$\lim_{\epsilon\downarrow 0, t-s\to\infty} \left|\mathfrak{H}_\epsilon(z(\theta))\right| = \left|\mathfrak{H}_*(\frac{1}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta})\right|, \quad \text{uniformly for } \theta \in (-\pi,\pi].$$

Thanks to Lemma 6.7.5, there exists $\delta > 0$ such that for $t-s$ large enough and $\epsilon > 0$ small enough,

$$\left|\mathfrak{D}_\epsilon(z(\theta))\right|, \left|\mathfrak{H}_\epsilon(z(\theta))\right| < 1-\delta \text{ for } |\theta| > \zeta,$$

which completes our proof. □

For $\mathbf{V}_\epsilon^{\mathrm{res}}$ (6.7.38), we show that the indicator $\mathbf{1}_{\{\mathfrak{p}_\epsilon(z) > r_2'\}}$ prohibits θ to be too small.

Lemma 6.7.8. *We can choose k_2 large enough such that if $\left|\mathfrak{p}_\epsilon(z(\theta))\right| > r_2'$ with $z(\theta) = \frac{1}{I+1} + \big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\big)e^{\mathbf{i}\theta}$, then $|\theta| \geq (t-s+1)^{-\frac{1}{4}}$.*

Proof. Note that $r_2' = \mathsf{u}(t-s, 2k_2\beta) \geq 1 + \frac{2k_2\beta}{\sqrt{t-s+1}}$, it suffices to show that

$$\left|\mathfrak{p}_\epsilon(z(\theta))\right| > 1 + \frac{2k_2\beta}{\sqrt{t-s+1}} \text{ implies } |\theta| > C(t-s+1)^{-\frac{1}{4}}.$$

Referring to (6.7.28), we taylor expand $\mathfrak{p}_\epsilon(1)$ around $\epsilon = 0$

$$\mathfrak{p}_\epsilon(1) = \frac{e^{-I\sqrt{\epsilon}}(1-e^{\sqrt{\epsilon}}) + (e^{\sqrt{\epsilon}} - e^{-I\sqrt{\epsilon}})e^{-\rho\sqrt{\epsilon}}}{(1-e^{(1-I)\sqrt{\epsilon}})e^{-\rho\sqrt{\epsilon}} - (1-e^{\sqrt{\epsilon}})e^{-2\rho\sqrt{\epsilon}}} = 1 + \frac{\rho I - \rho^2}{I}\epsilon + O(\epsilon^{\frac{3}{2}}). \tag{6.7.44}$$

We highlight that there is no $\sqrt{\epsilon}$ term in the expansion, which is important for our proof.

We taylor expand $\mathfrak{p}_\epsilon(z)$ at $z = 1$. Using (6.7.44), $z(0) = 1 - \frac{\beta}{\sqrt{t-s+1}}$ and $\lim_{\epsilon\downarrow 0}\mathfrak{p}_\epsilon'(1) = 1$, we find that for $t-s$ large enough and ϵ small enough,

$$\mathfrak{p}_\epsilon(z(0)) = \mathfrak{p}_\epsilon(1) + \mathfrak{p}_\epsilon'(1)(z(0)-1) + O\big(z(0)-1\big)^2 \leq 1 + \frac{2(\rho I - \rho^2)}{I}\epsilon \leq 1 + \frac{C}{\sqrt{t-s+1}}. \tag{6.7.45}$$

In the last inequality, we used the condition $t-s \in [0, \epsilon^{-2}T]$. In addition, it is straightforward to see that $\frac{d}{d\theta}|\mathfrak{p}_\epsilon(z(\theta))|\big|_{\theta=0} = 0$ and there exists $\zeta, C' > 0$ such that $\left|\frac{d^2}{d\theta^2}|\mathfrak{p}_\epsilon(z(\theta))|\right| \leq C'$ for $|\theta| \leq \zeta$. Consequently, via Taylor expansion, for $|\theta| \leq \zeta$,

$$\left|p_\epsilon(z(\theta))\right| \leq \left|p_\epsilon(z(0))\right| + \frac{C'\theta^2}{2} \leq 1 + \frac{C}{\sqrt{t-s+1}} + \frac{C'\theta^2}{2}.$$

Consequently, we have that when $|\theta| \leq \zeta$,

$$\left|\mathfrak{p}_\epsilon(z(\theta))\right| > 1 + \frac{2k_2\beta}{\sqrt{t-s+1}} \text{ implies } 1 + \frac{C}{\sqrt{t-s+1}} + \frac{C'\theta^2}{2} \geq 1 + \frac{2k_2\beta}{\sqrt{t-s+1}}$$

By choosing k_2 large enough, we see that $|\theta| > (t-s+1)^{-1/4}$. ◻

We are ready to prove Theorem 6.7.3 for $(+-)$ case. As $\mathbf{V}_\epsilon^{\mathrm{in}} = \mathbf{V}_\epsilon^{\mathrm{blk}} + \mathbf{V}_\epsilon^{\mathrm{res}}$, it is enough to bound respectively $\mathbf{V}_\epsilon^{\mathrm{blk}}$ and $\mathbf{V}_\epsilon^{\mathrm{res}}$. We begin with $\mathbf{V}_\epsilon^{\mathrm{blk}}$ (6.7.37). The proof consists a sequence of bounds on terms appearing in the integrand (6.7.37). We parametrize by $z_1(\theta_1) = \frac{1}{I+1} + \big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\big)e^{\mathbf{i}\theta_1}$ and $z_2(\theta_2) = r^*(z_1)e^{\mathbf{i}\theta}$.

($\mathbf{V}_\epsilon^{\mathrm{blk}}, z_1^{x_2-y_1} z_2^{x_1-y_2}$): **Show that** $|z_1^{x_2-y_1} z_2^{x_1-y_2}| \leq Ce^{-\frac{\beta}{\sqrt{t-s+1}}(|x_1-y_2|+|x_2-y_1|)}$.
Observe that $|z_1(\theta_1)| = \big|\frac{1}{I+1} + \big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\big)e^{\mathbf{i}\theta_1}\big|$ reaches its maximum at $\theta_1 = 0$, hence

$$|z_1(\theta_1)| \leq |z_1(0)| = 1 - \frac{\beta}{\sqrt{t-s+1}} \leq e^{-\frac{\beta}{\sqrt{t-s+1}}},$$

which gives $|z_1|^{x_2-y_1} \leq e^{-\frac{\beta}{\sqrt{t-s+1}}|x_2-y_1|}$. By $|z_2| \geq \mathsf{u}(t-s,\beta)$, we deduce $|z_2|^{x_1-y_2} \leq e^{-\frac{\beta}{\sqrt{t-s+1}}|x_1-y_2|}$.

($\mathbf{V}_\epsilon^{\mathrm{blk}}, \frac{1}{z_i}$): **Show that** $|\frac{1}{z_i}| \leq C$.
Clearly, $\frac{1}{|z_i|}$ is bounded for $z_1 \in \mathcal{M}(t,-\beta)$ and $z_2 \in C_{r^*(z_1)}$.

($\mathbf{V}_\epsilon^{\mathrm{blk}}, \mathfrak{F}_\epsilon(z_1,z_2)$): **Show that** $\big|\mathfrak{F}_\epsilon(z_1,z_2)\big| \leq C + C\sqrt{t-s+1}(|\theta_1| + |\theta_2|)$.
To justify this claim, write

$$\begin{aligned}\mathfrak{F}_\epsilon(z_1,z_2) &= \frac{q\nu - \nu + (\nu-q)q^{-\rho}z_2 + (1-q\nu)q^{-\rho}z_1 + (q-1)q^{-2\rho}z_1z_2}{((q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho})(z_2 - \mathfrak{p}_\epsilon(z_1))} \\ &= 1 + \frac{q^{-\rho}(1+q)(\nu-1)}{(q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho}} \cdot (z_2 - z_1) \cdot \frac{1}{z_2 - \mathfrak{p}_\epsilon(z_1)}. \qquad (6.7.46)\end{aligned}$$

Let us bound each factor on the RHS of (6.7.46). Referring to (6.7.30), we know that $\frac{1}{|z_2-\mathfrak{p}_\epsilon(z_1)|} \leq C\sqrt{t-s+1}$. Furthermore, we note that

$$z_2 - z_1 = e^{\mathbf{i}r_2^*(z_1)\theta_2} - \Big(\frac{1}{I+1} + \big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\big)e^{\mathbf{i}\theta_1}\Big) = e^{\mathbf{i}r_2^*(z_1)\theta_2} - 1 - \Big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\Big)(e^{\mathbf{i}\theta_1} - 1) + \frac{\beta}{\sqrt{t-s+1}},$$

which implies $|z_2 - z_1| \leq C\big(\frac{1}{\sqrt{t-s+1}} + |\theta_1| + |\theta_2|\big)$.

In addition, we observe that

$$\lim_{\epsilon\downarrow 0} \frac{q^{-\rho}(1+q)(\nu-1)}{(q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho}} = -\frac{2I}{z_1 + I - 1}.$$

Thus, $|\frac{q^{-\rho}(1+q)(v-1)}{(q-1)q^{-2\rho}z_1+(1-qv)q^{-\rho}}|$ is uniformly bound over $\mathcal{M}(t-s,-\beta)$. Incorporating the bound for each factor on the RHS of (6.7.46) gives the desired bound.

($\mathbf{V}_\epsilon^{\mathrm{blk}}$, $\mathfrak{R}_\epsilon(z_i,t,s)$): **Show that** $|\mathfrak{R}_\epsilon(z_i,t,s)| \leq C$.

This is proved using the same reasoning for (6.7.17).

($\mathbf{V}_\epsilon^{\mathrm{blk}}$, $\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}$): **Show that** $|\mathfrak{D}_\epsilon(z_i(\theta_i))|^{\lfloor\frac{t-s}{J}\rfloor} \leq C(\beta,T)e^{-C(t-s+1)\theta_i^2}$.

The result $\mathfrak{D}_\epsilon(z_1(\theta_1))|^{\lfloor\frac{t-s}{J}\rfloor} \leq C(\beta,T)e^{-C(t-s+1)\theta_1^2}$ directly follows from Lemma 6.7.7. For $|\mathfrak{D}_\epsilon(z_2(\theta_2))|^{\lfloor\frac{t-s}{J}\rfloor}$, note that either $z_2(\theta_2) = \mathsf{u}(t,k_2\beta)e^{\mathbf{i}\theta_2}$ or $\mathsf{u}(t,3k_2\beta)e^{\mathbf{i}\theta_2}$ (depending on the choice of z_1). Lemma 6.7.4 implies $|\mathfrak{D}_\epsilon(z_2(\theta_2))|^{\lfloor\frac{t-s}{J}\rfloor} \leq C(\beta,T)e^{-C(t-s+1)\theta_2^2}$.

Via change of variable $z_1 = z_1(\theta_1)$ and $z_2 = z_2(\theta_2)$ and incorporating the preceding bounds, we arrive at

$$\begin{aligned}&\left|\mathbf{V}_\epsilon^{\mathrm{blk}}\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \\ &\leq C(\beta,T)e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(1+\sqrt{t-s+1}(|\theta_1|+|\theta_2|))e^{-C(t-s+1)(\theta_1^2+\theta_2^2)}d\theta_1 d\theta_2.\end{aligned}$$

Applying change of variable $\theta_i \to \frac{1}{\sqrt{t-s+1}}\theta_i$, we conclude

$$|\mathbf{V}_\epsilon^{\mathrm{blk}}\big((x_1,x_2),(y_1,y_2),t,s\big)| \leq \frac{C(\beta,T)}{t-s+1}e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}. \tag{6.7.47}$$

We turn to study $\mathbf{V}_\epsilon^{\mathrm{res}}$ in (6.7.38). The proof consists of bounds on terms involved in the integral (6.7.38). In the following we parametrize $z_1(\theta_1) = \frac{1}{I+1} + \big(\frac{I}{I+1} - \frac{\beta}{\sqrt{t-s+1}}\big)e^{\mathbf{i}\theta_1}$.

($\mathbf{V}_\epsilon^{\mathrm{res}}$, $\frac{1}{z_1\mathfrak{p}_\epsilon(z_1)}$) **Show that** $\frac{1}{|z_1\mathfrak{p}_\epsilon(z_1)|} \leq C$.

By $\lim_{\epsilon\downarrow 0}\mathfrak{p}_\epsilon(z_1) = \frac{(I+1)z_1-1}{z_1+(I-1)}$, we deduce that $\frac{1}{|z_1\mathfrak{p}_\epsilon(z_1)|} \leq C$ for $z_1 \in \mathcal{M}(t-s,-\beta)$.

($\mathbf{V}_\epsilon^{\mathrm{res}}$, $\mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s)$): **Show that** $|\mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s)| \leq C$.

By ($\mathbf{V}_\epsilon^{\mathrm{blk}}$, $\mathfrak{R}_\epsilon(z_i,t,s)$), we see that $|\mathfrak{R}_\epsilon(z_1,t,s)| \leq C$ for $z_1 \in \mathcal{M}(t-s,-\beta)$. We are left to show for $t-s$ large and ϵ small,

$$|\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1), t, s)| \leq C, \qquad z_1 \in \mathcal{M}(t-s, -\beta). \tag{6.7.48}$$

Recall from (6.7.14) that when $\epsilon > 0$ is small enough, all the singularity of $\mathfrak{R}_\epsilon(z, t, s)$ belongs to the interval $[0, \Theta]$ for some $\Theta < 1$. As $\lim_{\epsilon\downarrow 0} \mathfrak{p}_\epsilon(z) = \mathfrak{p}_*(z)$, it suffices to show that

$$|\mathfrak{p}_*(z_1)| \geq 1, \qquad z_1 \in \mathcal{M}.$$

To justify this, we parametrize by $z_1(\theta) = \frac{1}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta} \in \mathcal{M}$,

$$|\mathfrak{p}_*(z_1)|^2 = \frac{(I+1)^2}{I^2+1+2I\cos\theta} \geq 1.$$

Hence, we conclude (6.7.48).

$(\mathbf{V}_\epsilon^{\mathrm{res}}, \mathfrak{J}_\epsilon(z_1))$: **Show that** $|\mathfrak{J}_\epsilon(z_1)| \leq Ce^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}$.

Referring to (6.7.31),

$$\mathfrak{J}_\epsilon(z_1) = \frac{q\nu - \nu + (\nu-q)q^{-\rho}\mathfrak{p}_\epsilon(z_1) + (1-q\nu)q^{-\rho}z_1 + (q-1)q^{-2\rho}z_1\mathfrak{p}_\epsilon(z_1)}{(q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho}} z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_2-y_1}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}.$$

Let us first bound $z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}$. We know from the discussion in $(\mathbf{V}_\epsilon^{\mathrm{blk}}, z_1^{x_2-y_1}z_2^{x_1-y_2})$ that $|z_1| \leq e^{-\frac{\beta}{\sqrt{t-s+1}}}$. It is straightforward that $\left|\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}\right| \leq e^{-\frac{\beta}{\sqrt{t-s+1}}|x_1-y_2|}$, which implies

$$|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}| \leq e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}. \tag{6.7.49}$$

In addition, one can compute

$$\lim_{\epsilon\downarrow 0}\frac{q\nu - \nu + (\nu-q)q^{-\rho}\mathfrak{p}_\epsilon(z_1) + (1-q\nu)q^{-\rho}z_1 + (q-1)q^{-2\rho}z_1\mathfrak{p}_\epsilon(z_1)}{(q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho}} = \frac{1-(1+I)\mathfrak{p}_*(z) + (I-1)z + z\mathfrak{p}_*(z)}{z+I-1},$$

recall $\mathfrak{p}_*(z_1) = \frac{(I+1)z_1-1}{z_1+(I-1)}$. This implies that

$$\Big|\frac{q\nu - \nu + (\nu - q)q^{-\rho}\mathfrak{p}_\epsilon(z_1) + (1-q\nu)q^{-\rho}z_1 + (q-1)q^{-2\rho}z_1\mathfrak{p}_\epsilon(z_1)}{(q-1)q^{-2\rho}z_1 + (1-q\nu)q^{-\rho}}\Big| \leq C, \quad z_1 \in \mathcal{M}(t, -\beta). \tag{6.7.50}$$

Combining (6.7.49) and (6.7.50) yields

$$|\mathfrak{J}_\epsilon(z_1)| \leq Ce^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}.$$

$(\mathbf{V}_\epsilon^{\mathrm{res}}, \mathfrak{H}_\epsilon(z_1(\theta_1))^{\lfloor \frac{t-s}{J} \rfloor})$: **Show that** $|\mathfrak{H}_\epsilon(z_1(\theta_1))|^{\lfloor \frac{t-s}{J} \rfloor} \leq C(\beta, T)e^{-C(t-s+1)\theta_1^2}$.

This directly follows from Lemma 6.7.7.

Consequently, we find that

$$\begin{aligned}
|\mathbf{V}_\epsilon^{\mathrm{res}}\big((x_1,x_2),(y_1,y_2),t,s\big)| &\leq C\oint_{\mathcal{M}(t-s,-\beta)} \mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1(\theta_1))|>r_2'\}}|\mathfrak{J}_\epsilon(z_1(\theta_1))||\mathfrak{H}_\epsilon(z_1(\theta_1))|^{\lfloor \frac{t-s}{J} \rfloor}\frac{d\theta_1}{|\mathfrak{p}_\epsilon(z_1(\theta_1))|},\\
&\leq C(\beta,T)e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}\int_{-\pi}^{\pi}\mathbf{1}_{\{\mathfrak{p}_\epsilon(z_1(\theta_1))>r_2'\}}e^{-C(t-s+1)\theta_1^2}d\theta_1,\\
&\leq C(\beta,T)e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}\int_{|\theta_1|>(t-s+1)^{-\frac{1}{4}}} e^{-C(t-s+1)\theta_1^2}d\theta_1,
\end{aligned}$$

where the last inequality is due to Lemma 6.7.8. Via change of variable $\theta_1 \to \frac{\theta_1}{\sqrt{t-s+1}}$, we get

$$\int_{|\theta_1|>(t-s+1)^{-\frac{1}{4}}} e^{-C(t-s+1)\theta_1^2}d\theta_1 \leq \int_{|\theta_1|>(t-s+1)^{\frac{1}{2}}} e^{-C\theta_1^2}d\theta_1 \leq \frac{e^{-C(t-s+1)}}{\sqrt{t-s+1}} \leq \frac{C}{t-s+1}.$$

For the second inequality above, we used the fact $\int_b^\infty e^{-x^2}dx \leq \frac{C}{b}e^{-b^2}$. Thereby,

$$|\mathbf{V}_\epsilon^{\mathrm{res}}\big((x_1,x_2),(y_1,y_2),t,s\big)| \leq \frac{C(\beta,T)}{t-s+1}e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}.$$

Combining this with the upper bound over $\mathbf{V}_\epsilon^{\mathrm{blk}}$ (6.7.47) concludes Theorem 6.7.3 part (a).

For the gradient, note that applying ∇_{x_i} or ∇_{y_i} to (6.7.37) and (6.7.38) will gives an additional

$z_i^{\pm} - 1$ in the integrand of $\mathbf{V}_\epsilon^{\text{blk}}$ and $\mathbf{V}_\epsilon^{\text{res}}$, we bound $|z_i(\theta_i) - 1| \leq C(\frac{1}{\sqrt{t-s+1}} + |\theta_i|)$ and perform the change of variable $\theta_i \to \frac{1}{\sqrt{t-s+1}}\theta_i$ produces an extra factor of $\frac{1}{\sqrt{t-s+1}}$. Similarly, applying ∇_{x_1,x_2} will produce an additional factor $(z_1(\theta_1) - 1)(z_2(\theta_2) - 1)$. We bound

$$|z_1(\theta_1) - 1| \cdot |z_2(\theta_2) - 1| \leq C\big(\frac{1}{\sqrt{t-s+1}} + |\theta_1|\big) \cdot \big(\frac{1}{\sqrt{t-s+1}} + |\theta_2|\big),$$

performing change of variable $\theta_i \to \frac{1}{\sqrt{t-s+1}}\theta_i$ produces an extra factor of $\frac{1}{t-s+1}$. This completes the proof of Theorem 6.7.3 (b), (c).

6.7.4 Estimate of $\mathbf{V}_\epsilon^{\text{in}}$, the $(--)$ case

We turn to prove Theorem 6.7.1 when $x_2 - y_1 \leq 0$ and $x_1 - y_2 \leq 0$. This case is more involved than the previous one. One stumbling block is that we prefer to deform the z_1-contour to be $C_{\mathsf{u}(t,\beta)}$ to extract the spatial exponential decay. On the other hand, as depicted in Figure 6.4, the unit circle does not satisfy the steepest descent condition for $\mathfrak{H}_\epsilon(z)$. We resolve this issue by first shrinking the z_1-contour to $\mathcal{M}'(t-s,\beta)$, then for $\mathbf{V}_\epsilon^{\text{blk}}$, we re-deform the z_1-contour from $\mathcal{M}(t-s,\beta)$ to $C_{\mathsf{u}(t,\beta)}$.

We define

$$\mathcal{M}'(t,\beta) = \partial\left\{\{|z - \frac{1}{I+1}| \leq \frac{I}{I+1} + u_*\} \cap \{|z| \leq \mathsf{u}(t,\beta)\}\right\},$$

recall u_* is some fix constant which belongs to $(0, \delta \wedge \frac{1}{4I})$. Since $\mathcal{M}'(t,\beta) \to \mathcal{M}'$ as $t \to \infty$, it is clear that for $t-s$ large enough, $\mathcal{M}'(t-s,\beta)$ is admissible. Note that the parametrization of $\mathcal{M}'(t-s,\beta)$ is given by the right part of Figure 6.5.

We decompose $\mathbf{V}_\epsilon^{\text{in}} = \mathbf{V}_\epsilon^{\text{blk}} + \mathbf{V}_\epsilon^{\text{res}}$,

$$\mathbf{V}_\epsilon^{\text{blk}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_{\mathcal{M}'(t-s,\beta)} \oint_{C_{r_2^*(z_1)}} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi\mathbf{i}z_i},$$
$$\mathbf{V}_\epsilon^{\text{res}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_{\mathcal{M}'(t-s,\beta)} \mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}} \mathfrak{I}_\epsilon(z_1) \mathfrak{H}_\epsilon(z_1)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_1,t,s) \mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s) \frac{dz_1}{2\pi\mathbf{i}z_1\mathfrak{p}_\epsilon(z_1)}. \tag{6.7.51}$$

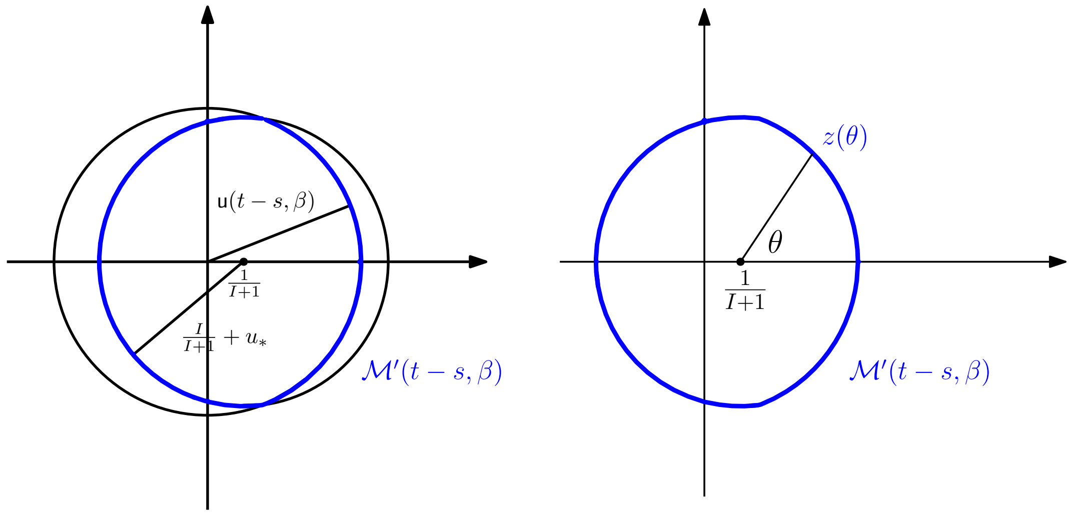


Figure 6.5: The contour $\mathcal{M}'(t-s,\beta)$ and its parametrization

Let us study $\mathbf{V}_\epsilon^{\mathrm{blk}}$ in the first place. As we mention at the beginning, when $x_2 - y_1 \le 0$, z_1 does not favor the contour $\mathcal{M}'(t-s,\beta)$ to extract spatial decay. We prove in the following that we can re-deform the z_1-contour from $\mathcal{M}'(t-s,\beta)$ to $C_{\mathsf{u}(t-s,\beta)}$.

Lemma 6.7.9. *For $t-s$ large enough and ϵ small enough,*

$$\oint_{\mathcal{M}'(t-s,\beta)} \oint_{C_{r_2^*(z_1)}} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}$$
$$= \oint_{C_{\mathsf{u}(t-s,\beta)}} \oint_{C_{r_2^*(z_1)}} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}.$$

Proof. The contour $\mathcal{M}'(t-s,\beta)$ and $C_{\mathsf{u}(t-s,\beta)}$ share a common part $\Lambda(t-s) := \mathcal{M}'(t-s,\beta) \cap C_{\mathsf{u}(t-s,\beta)}$. We denote by $\Lambda_1(t-s) := \mathcal{M}'(t-s,\beta) \backslash \Lambda(t-s)$ and $\Lambda_2(t-s) := C_{\mathsf{u}(t-s,\beta)} \backslash \Lambda(t-s)$. Decompose the contour $\mathcal{M}'(t-s,\beta) = \Lambda(t-s) \cup \Lambda_1(t-s)$, $C_{\mathsf{u}(t-s,\beta)} = \Lambda(t-s) \cup \Lambda_2(t-s)$, it suffices to prove

$$\oint_{\Lambda_1(t-s)} \oint_{C_{r_2^*(z_1)}} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i}$$
$$= \oint_{\Lambda_2(t-s)} \oint_{C_{r_2^*(z_1)}} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi \mathbf{i} z_i} \qquad (6.7.52)$$

To prove the above equation, we first claim that for ϵ small enough and $t-s \le \epsilon^{-2}T$ large enough,

$$r_2^*(z_1) = \mathsf{u}(t-s, k_2\beta), \quad \forall\, z_1 \in \Lambda_1(t-s) \cup \Lambda_2(t-s) \tag{6.7.53}$$

That is to say, the z_2-contour is always $C_{\mathsf{u}(t,k_2\beta)}$, which does not depend on the choice of z_1.

To justify this claim, we need to prove for ϵ small enough and $t-s$ large enough

$$|\mathfrak{p}_\epsilon(z_1)| > \mathsf{u}(t-s, 2k_2\beta).$$

We denote by $\Lambda^* = \mathcal{M}' \cap C_1$, $\Lambda_1^* = \mathcal{M}' \backslash \Lambda^*$ and $\Lambda_2^* = C_1 \backslash \Lambda^*$. Note that as $t-s \to \infty$ and $\epsilon \downarrow 0$,

$$\Lambda_1(t-s,\beta) \to \Lambda_1^*, \quad \Lambda_2(t-s,\beta) \to \Lambda_2^*, \quad \mathfrak{p}_\epsilon(z_1) \to \mathfrak{p}_*(z_1), \quad \mathsf{u}(t-s, 2k_2\beta) \to 1.$$

Therefore, it suffices to consider the limit case and show that there exists $\delta > 0$ s.t.

$$|\mathfrak{p}_*(z_1)| = \left| \frac{(I+1)z_1 - 1}{z_1 + (I-1)} \right| > 1 + \delta, \qquad z_1 \in \Lambda_1^* \cup \Lambda_2^*.$$

If $z_1 \in \Lambda_1^*$, we parametrize $z_1(\theta) = \frac{1}{I+1} + \frac{I}{I+1} e^{\mathbf{i}\theta}$, where $|\theta| \ge \zeta$ for some positive constant ζ. We readily compute

$$|\mathfrak{p}_*(z_1(\theta))|^2 = \frac{(I+1)^2}{I^2 + 1 + 2I\cos\theta} \ge \frac{(I+1)^2}{I^2 + 1 + 2I\cos\zeta} > 1.$$

If $z_1 \in \Lambda_2^*$, we parametrize $z_1(\theta) = e^{\mathbf{i}\theta}$ where $|\theta| \ge \zeta'$ for some positive constant ζ'.

$$|\mathfrak{p}_*(z_1)|^2 = \frac{(I+1)^2 + 1 - 2(I+1)\cos\theta}{(I-1)^2 + 1 + 2(I-1)\cos\theta} \ge \frac{(I+1)^2 + 1 - 2(I+1)\cos\zeta'}{(I-1)^2 + 1 + 2(I-1)\cos\zeta'} > 1,$$

where the first inequality above is due to the fact that $\frac{(I+1)^2+1-2(I+1)\cos\theta}{(I-1)^2+1+2(I-1)\cos\theta}$ increases as $|\theta| \in [0,\pi]$ increases.

Having shown (6.7.53), by Fubini's theorem, the desired identity (6.7.52) turns into

$$\oint_{C_{\mathsf{u}(t-s,k_2\beta)}}\oint_{\Lambda_1(t-s)}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}$$
$$=\oint_{C_{\mathsf{u}(t-s,k_2\beta)}}\oint_{\Lambda_2(t-s)}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}$$

In order justify the identity above, it is sufficient to show that for all $z_2\in C_{\mathsf{u}(t-s,k_2\beta)}$,

$$\oint_{\Lambda_1(t-s)}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}=\oint_{\Lambda_2(t-s)}\mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i}$$

which is equivalent to

$$\oint_{\partial\mathcal{G}(t-s)}\mathfrak{F}_\epsilon(z_1,z_2)\mathfrak{D}_\epsilon(z_1)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_1,t,s)z_1^{x_2-y_1}\frac{dz_1}{2\pi\mathbf{i}z_1}=0, \tag{6.7.54}$$

where $\partial\mathcal{G}(t-s)$ is the boundary of the crescent $\mathcal{G}(t-s)=\{|z|\le\mathsf{u}(t-s,\beta)\}\backslash\{|z-\frac{1}{I+1}|=\frac{I}{I+1}+u_*\}$, which is depicted in Figure 6.6 (note that $\partial\mathcal{G}(t-s)=\Lambda_1(t-s)\cup\Lambda_2(t-s)$).

We set out proving (6.7.54). Since $\partial\mathcal{G}(t-s)$ is a closed curve, according to Cauchy's theorem, we only need to prove that no pole of the integrand (6.7.54) lies inside of $\mathcal{G}(t-s)$. As we mentioned before, for ϵ small enough, the pole either equals $\mathfrak{s}_\epsilon(z_2)$ or belongs to $[0,\Theta]$. It is straightforward that $[0,\Theta]\cap\mathcal{G}(t-s)=\emptyset$. Hence, we only need to show that $\mathfrak{s}_\epsilon(z_2)\notin\mathcal{G}(t-s)$ for all $z_2\in C_{\mathsf{u}(t-s,k_2\beta)}$.

We claim that for $t-s$ large enough and ϵ small enough,

$$\inf_{z_2\in C_{\mathsf{u}(t-s,k_2\beta)}}\operatorname{Re}\mathfrak{s}_\epsilon(z_2)>\sup_{z_1\in\mathcal{G}(t-s)}\operatorname{Re}z_1.$$

Note that as $t-s\to\infty$ and $\epsilon\downarrow 0$,

$$C_{\mathsf{u}(t-s,k_2\beta)}\to C_1,\quad \mathcal{G}(t-s)\to\mathcal{G},\quad \mathfrak{s}_\epsilon(z)\to\mathfrak{s}_*(z)$$

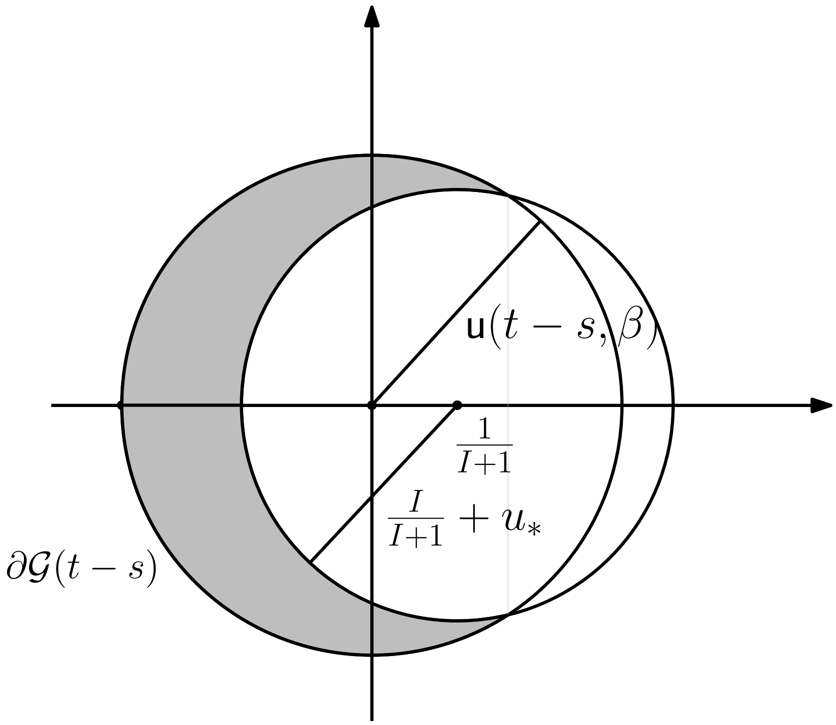


Figure 6.6: The crescent $\mathcal{G}(t-s)$ and its boundary $\partial\mathcal{G}(t-s)$.

where $\mathcal{G} := \{|z| \le 1\}\backslash\{|z-\frac{1}{I+1}| = \frac{I}{I+1}+u_*\}$ and recall $\mathfrak{s}_*(z) = \frac{(I-1)z+1}{I+1-z}$. Therefore, it suffices to show that

$$\inf_{z_2\in\mathcal{C}_1} \operatorname{Re}\mathfrak{s}_*(z_2) > \sup_{z_1\in\mathcal{G}} \operatorname{Re} z_1.$$

To justify the inequality above, we first observe that $\sup_{z_1\in\mathcal{G}} \operatorname{Re} z_1 < 1$. In addition, by setting $z_2 = e^{\mathbf{i}\theta}$, we see that

$$\operatorname{Re}\mathfrak{s}_*(e^{\mathbf{i}\theta}) = \operatorname{Re}\frac{(I-1)e^{\mathbf{i}\theta}+1}{I+1-e^{\mathbf{i}\theta}} = \frac{2+(I^2-2)\cos\theta}{(I+1)^2+1-2(I+1)\cos\theta} \ge 1.$$

Consequently, we proved $\mathfrak{s}_\epsilon(z_2) \notin \mathcal{G}(t-s)$, which completes the proof for Lemma 6.7.9. □

In summary, we can write $\mathbf{V}_\epsilon^{\mathrm{in}} = \mathbf{V}_\epsilon^{\mathrm{blk}} + \mathbf{V}_\epsilon^{\mathrm{res}}$, where

$$\mathbf{V}_\epsilon^{\mathrm{blk}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_{C_{\mathsf{u}(t-s,\beta)}}\oint_{C_{r_2^*(z_1)}} \mathfrak{F}_\epsilon(z_1,z_2)\prod_{i=1}^{2}\mathfrak{D}_\epsilon(z_i)^{\lfloor\frac{t-s}{J}\rfloor}\mathfrak{R}_\epsilon(z_i,t,s)z_i^{x_{3-i}-y_i}\frac{dz_i}{2\pi\mathbf{i}z_i} \tag{6.7.55}$$

and $\mathbf{V}_\epsilon^{\text{res}}$ is given by (6.7.51).

Lemma 6.7.10. *For the parametrization $z(\theta)$ given in Figure 6.5, we have for $t-s \le \epsilon^{-2}T$ large enough and $\epsilon > 0$ small enough*

$$|\mathfrak{D}_\epsilon(z(\theta))|^{t-s} \le C(\beta,T)e^{-C(t-s+1)\theta^2}, \qquad |\mathfrak{D}_\epsilon(z(\theta))|^{t-s} \le C(\beta,T)e^{-C(t-s+1)\theta^2}, \qquad z(\theta) \in \mathcal{M}'(t-s,\beta).$$

Proof. Similar to Lemma 6.7.7, it suffices to show that there exists $C(\beta,T), C > 0$ s.t.

$$\text{Re } \log \mathfrak{D}_\epsilon(z(\theta)) \le \frac{C(\beta,T)}{t-s+1} - C\theta^2; \qquad \text{Re } \log \mathfrak{H}_\epsilon(z(\theta)) \le \frac{C(\beta,T)}{t-s+1} - C\theta^2.$$

We split out proof for ($\theta = 0$), for (θ small) and for (θ large).

- ($\theta = 0$) : $\text{Re}\, \mathfrak{D}_\epsilon(z(0)), \text{Re}\, \mathfrak{H}_\epsilon(z(0)) \le \frac{C(\beta,T)}{t-s+1}$.
- (θ small): There exists $\zeta > 0$ and constants $C(\beta,T)$ and $C > 0$ such that (6.7.39) holds for $|\theta| \le \zeta$.
- (θ large): We can find $\delta > 0$ such that $\left|\mathfrak{D}_\epsilon(z(\theta))\right|, \left|\mathfrak{H}_\epsilon(z(\theta))\right| < 1-\delta$ for $|\theta| > \zeta$.

Recall that $\mathcal{M}'(t-s,\beta)$ is the same as $C_{\mathsf{u}(t-s,\beta)}$ in a neighborhood of 1, hence $z(\theta) \in C_{\mathsf{u}(t-s,\beta)}$ when θ is small. This being the case, the proof for ($\theta = 0$) and (θ small) is the same as in Lemma 6.7.7. For (θ large), since $\mathcal{M}'(t-s,\beta) \to \mathcal{M}'$ when $t-s \to \infty$ and $\mathcal{M}'$ satisfies the steepest descent condition, we find that for $t-s$ large and ϵ small,

$$|\mathfrak{D}_\epsilon(z(\theta))| < 1-\delta, \qquad |\mathfrak{H}_\epsilon(z(\theta))| < 1-\delta, \qquad \text{for } |\theta| \ge \zeta.$$

This completes our proof. □

We begin to estimate $\mathbf{V}_\epsilon^{\text{blk}}$ in (6.7.55). In what follows we check a sequence of bounds on terms involved in the integral (6.7.55), we parametrize $z_1 = \mathsf{u}(t-s,\beta)e^{\mathbf{i}\theta_1}$ and $z_2 = r_2^*(z_1)e^{\mathbf{i}\theta_2}$.

($\mathbf{V}_\epsilon^{\text{blk}}, z_1^{x_2-y_1} z_2^{x_1-y_2}$): **Show that** $|z_1^{x_2-y_1} z_2^{x_1-y_2}| \le e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}$.

Since $z_1 \in C_{\mathsf{u}(t-s,\beta)}$ and $z_2 \in C_{r_2^*(z_1)}$, we have $|z_i| \geq \mathsf{u}(t-s,\beta)$. Along with the condition $x_{3-i} - y_i \leq 0$ for $i = 1,2$, we obtain $|z_1|^{x_2-y_1}|z_2|^{x_1-y_2} \leq e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}$.

($\mathbf{V}_\epsilon^{\text{blk}}, \mathfrak{F}_\epsilon(z_1,z_2)$): **Show that** $\left|\mathfrak{F}_\epsilon(z_1,z_2)\right| \leq C + C\sqrt{t-s+1}(|\theta_1| + |\theta_2|)$.

The argument for this part is the same as in the (+−) case.

($\mathbf{V}_\epsilon^{\text{blk}}, \mathfrak{R}_\epsilon(z_i,t,s)$): **Show that** $|\mathfrak{R}_\epsilon(z_i,t,s)| \leq C$.

The argument is the same as (+−) case ($\mathbf{V}_\epsilon^{\text{blk}}, \mathfrak{R}_\epsilon(z_i,t,s)$).

($\mathbf{V}_\epsilon^{\text{blk}}, \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor}$): **Show that** $|\mathfrak{D}_\epsilon(z_i(\theta_i))|^{\lfloor \frac{t-s}{J} \rfloor} \leq C(\beta,T)\exp(-C(t-s+1)\theta_i^2)$.

This is the content of Lemma 6.7.4.

As a consequence, we perform the same procedure as in the (+−) case and get

$$
\begin{aligned}
&|\mathbf{V}_\epsilon^{\text{blk}}\big((x_1,x_2),(y_1,y_2),t,s\big)| \\
&\leq C(\beta,T)e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(1+\sqrt{t-s+1}(|\theta_1|+|\theta_2|))e^{-C(t-s+1)(\theta_1^2+\theta_2^2)}d\theta_1 d\theta_2 \\
&\leq \frac{C(\beta,T)}{t-s+1}e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}.
\end{aligned}
\tag{6.7.56}
$$

We turn our attention to study $\mathbf{V}_\epsilon^{\text{res}}$, the proof similarly consists of bounds on terms involved in the integral (6.7.51). In the following we parametrize $z_1 = z_1(\theta) \in \mathcal{M}'(t-s,\beta)$ as depicted in Figure 6.5.

($\mathbf{V}_\epsilon^{\text{res}}, \frac{1}{z_1\mathfrak{p}_\epsilon(z_1)}$): **Show that** $|\frac{1}{z_1\mathfrak{p}_\epsilon(z_1)}| \leq C$.

This is by the same argument as in the (+−) case.

($\mathbf{V}_\epsilon^{\text{res}}, \mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s)$): **Show that** $|\mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s)| \leq C$.

The argument for this part is the same as ($\mathbf{V}_\epsilon^{\text{res}}, \mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s)$) in the (+−) case.

($\mathbf{V}_\epsilon^{\text{res}}, \mathfrak{H}_\epsilon(z_1)^{\lfloor \frac{t-s}{J} \rfloor}$): **Show that** $|\mathfrak{H}_\epsilon(z_1)|^{\lfloor \frac{t-s}{J} \rfloor} \leq C(\beta,T)e^{-C(t-s+1)\theta^2}$.

This is the content of Lemma 6.7.10.

($\mathbf{V}_\epsilon^{\text{res}}, \mathfrak{J}_\epsilon(z_1)$): **Show that** $|\mathfrak{J}_\epsilon(z_1)| \leq Ce^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}$.

Similar to the discussion in $(\mathbf{V}_\epsilon^{\text{res}}, \mathfrak{J}_\epsilon(z_1))$ for the $(+-)$ case, it is sufficient to show

$$|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}| \le e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_1-y_2|+|x_2-y_1|)}.$$

Since for $z_1 \in \mathcal{M}(t-s,\beta)$, $|z_1|$ could be much less than 1, we can not bound z_1 and $\mathfrak{p}_\epsilon(z_1)$ separately. Instead, we write

$$|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}| = |z_1\mathfrak{p}_\epsilon(z_1)|^{x_2-y_1}|\mathfrak{p}_\epsilon(z_1)|^{x_1-x_2+y_1-y_2}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}} \tag{6.7.57}$$

Note that $x_1-x_2+y_1-y_2 \le 0$ (since $x_1 \le y_1$ and $x_2 \le y_2$), hence

$$|\mathfrak{p}_\epsilon(z_1)|^{x_1-x_2+y_1-y_2}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}} \le \mathsf{u}(t-s,\beta)^{x_2-x_1+y_2-y_1}.$$

We claim that

$$|z_1\mathfrak{p}_\epsilon(z_1)| > \mathsf{u}(t-s,\beta), \quad z_1 \in \mathcal{M}'(t-s,\beta). \tag{6.7.58}$$

Once this is proved, by (6.7.57)

$$|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}\mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}}| \le \mathsf{u}(t-s,\beta)^{x_2-y_1}\mathsf{u}(t-s,\beta)^{x_1-x_2+y_1-y_2} \le e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_1-y_2|+|x_2-y_1|)}.$$

Let us justify (6.7.58). We decompose $\mathcal{M}'(t-s,\beta) = \Lambda(t-s) \cup \Lambda_1(t-s)$, where $\Lambda(t-s) = \mathcal{M}'(t-s,\beta) \cap C_{\mathsf{u}(t-s,\beta)}$ and $\Lambda_1(t-s) = \mathcal{M}'(t-s,\beta)\backslash\Lambda(t-s)$. If $z_1 \in \Lambda(t-s) \subseteq C_{\mathsf{u}(t-s,\beta)}$, we reparametrize by $z_1(\theta_1) = \mathsf{u}(t-s,\beta)e^{\mathbf{i}\theta_1}$. It suffices to show that

$$|\mathfrak{p}_\epsilon(\mathsf{u}(t-s,\beta)e^{\mathbf{i}\theta_1})| \ge 1.$$

By straightforward computation, one sees that $|\mathfrak{p}_\epsilon(\mathsf{u}(t-s,\beta)e^{\mathbf{i}\theta_1})|$ reaches its minimum at $\theta_1 = 0$.

Hence we only need to prove that

$$\mathfrak{p}_\epsilon(\mathsf{u}(t-s,\beta)) \geq 1.$$

By (6.7.44), $\mathfrak{p}_\epsilon(1) = 1 + \frac{\rho I - \rho^2}{I}\epsilon + O(\epsilon^{\frac{3}{2}})$. In addition, direct computation yields $\lim_{\epsilon\downarrow 0} \mathfrak{p}'_\epsilon(1) = 1$ and $|\mathfrak{p}''_\epsilon(z)|$ uniformly bounded in a small neighborhood of 1. Consequently, we taylor expand $\mathfrak{p}_\epsilon(z)$ at 1,

$$\mathfrak{p}_\epsilon(\mathsf{u}(t-s,\beta)) = \mathfrak{p}_\epsilon(1) + \mathfrak{p}'_\epsilon(1)(\mathsf{u}(t-s,\beta)-1) + O((\mathsf{u}(t-s,\beta)-1)^2) \geq 1.$$

for $t-s$ large and ϵ small.

If $z_1 \in \Lambda_1(t-s)$, which means that $|z_1 - \frac{1}{I+1}| = \frac{I}{I+1} + u_*$. We see that

$$\lim_{\epsilon\downarrow 0} |z_1 \mathfrak{p}_\epsilon(z_1)| = |z_1 \mathfrak{p}_*(z_1)| = |(I+1)z_1 - 1| \cdot \Big|\frac{z_1}{z_1+I-1}\Big| = (I+(I+1)u_*)\cdot\Big|\frac{z_1}{z_1+I-1}\Big| \quad (6.7.59)$$

We claim that for $z_1 \in \Lambda_1(t-s)$, $\big|\frac{z_1}{z_1+I-1}\big| > \frac{1}{I}$. This could verify by inserting $z_1 = \frac{1}{I+1} + (\frac{I}{I+1}+u_*)e^{\mathbf{i}\theta}$ into (6.7.59). A geometric way to prove this inequality is that one has $|\frac{z}{z+I-1}| = \frac{1}{I}$ for all z satisfying $|z - \frac{1}{I+1}| = \frac{I}{I+1}$. If ones increase the radius of circle $|z - \frac{1}{I+1}| = \frac{I}{I+1}\}$ (by u_*), the value of $\big|\frac{z}{z+I-1}\big|$ will also increase. Thereby,

$$\lim_{\epsilon\downarrow 0} |z_1 \mathfrak{p}_\epsilon(z_1)| \geq \frac{I+(I+1)u_*}{I} > 1.$$

This implies when $z_1 \in \Lambda(t-s)$, $|z_1\mathfrak{p}_\epsilon(z_1)| > 1$ for $t-s$ large and ϵ small, which completes the proof of (6.7.58).

Similar to the proof of Lemma 6.7.8 in the (+−) case, we find that $\{|\mathfrak{p}_\epsilon(z_1(\theta))| > \mathsf{u}(t-s, 2k_2\beta)\} \subseteq \{|\theta| > (t-s+1)^{-\frac{1}{4}}\}$, hence

$$|\mathbf{V}^{\mathrm{res}}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)| \leq C(\beta,T)e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)} \int_{-\pi}^{\pi} \mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1(\theta))|\geq r'_2\}} e^{-C(t-s+1)\theta^2} d\theta$$

$$\le C(\beta,T)e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}\int_{|\theta|>(t-s+1)^{-\frac{1}{4}}} e^{-C(t-s+1)\theta^2}d\theta \le \frac{C(\beta,T)}{t-s+1}e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)} \tag{6.7.60}$$

Combining the bounds (6.7.56) and (6.7.60) implies Theorem 6.7.3 (a).

To estimate the gradient, the procedure is similar to in (+−) case, note that applying ∇_{x_i} or ∇_{y_i} to (6.7.55) and (6.7.51) gives an additional $z_i^{\pm}-1$ factor, applying ∇_{x_1,x_2} produces an additional factor $(z_1-1)(z_2-1)$. By $|z_i(\theta_i)-1| \le C(\frac{1}{\sqrt{t-s+1}}+|\theta_i|)$, we conclude Theorem 6.7.3 (b), (c).

6.7.5 Estimate of $\mathbf{V}_\epsilon^{\text{in}}$, the (++) case

In this section, we fix $k_2 = 1$ in (6.7.29). Note that $x_1 - y_2 \ge 0$, the difficulty for this case is to choose a suitable z_1-contour $\Gamma(t-s,\epsilon)$ so as to extract the spatial decay from $z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}$ in the integrand $\mathbf{V}_\epsilon^{\text{res}}$ (6.7.32). Let us write

$$|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}| = |z_1\mathfrak{p}_\epsilon(z_1)|^{x_1-y_2}|z_1|^{x_2-x_1+y_2-y_1}$$

We control respectively $|z_1\mathfrak{p}_\epsilon(z_1)|$ and $|z_1|$. We deform the z_1-contour to

$$\mathcal{M}''(t-s,\epsilon,-k_1\beta) = \{z_1 : |z_1\mathfrak{p}_\epsilon(z_1)| = \mathsf{u}(t-s,-k_1\beta)\},$$

where k_1 is a positive constant that we will specify later. Note that when $I \ge 2$, this contour can only be implicitly defined (when $I = 1$ it is a circle). The following lemma provides a few properties of the contour.

Lemma 6.7.11. *For $t-s$ large enough and ϵ small enough, given $\theta \in (-\pi,\pi]$, there exists a unique positive $r_{\epsilon,t-s}(\theta)$ such that*

$$|z_1\mathfrak{p}_\epsilon(z_1)| = \mathsf{u}(t-s,-k_1\beta), \quad z_1(\theta) = \frac{1}{I+1} + r_{\epsilon,t-s}(\theta)e^{\mathbf{i}\theta}. \tag{6.7.61}$$

$r_{\epsilon,t-s}(\theta)$ is infinitely differentiable with $r'_{\epsilon,t-s}(0) = 0$. Moreover, one has uniformly for $\theta \in (-\pi, \pi]$,

$$\lim_{\epsilon\downarrow 0, t-s\to\infty} r_{\epsilon,t-s}(\theta) = \frac{I}{I+1},$$
$$\lim_{\epsilon\downarrow 0, t-s\to\infty} r^{(n)}_{\epsilon,t-s}(\theta) = 0, \quad \forall n \in \mathbb{Z}_{\geq 1}.$$

where $f^{(n)}(\theta)$ represents the n-th derivative of $f(\theta)$.

Proof. Let $w = t - s$, as $w \to \infty$ and $\epsilon \downarrow 0$, the equation $|z_1 \mathfrak{p}_\epsilon(z_1)| = \mathsf{u}(w, -\beta)$ converges to

$$|z_1 \mathfrak{p}_*(z_1)| = \left|\frac{z_1((I+1)z_1 - 1)}{z_1 + (I-1)}\right| = 1. \tag{6.7.62}$$

(note $\mathfrak{p}_\epsilon(z_1) \to \mathfrak{p}_*(z)$ and $\mathsf{u}(w, \beta) \to 1$). Setting $z_1 = \frac{1}{I+1} + re^{\mathbf{i}\theta}$ in (6.7.62) yields

$$(I+1)^4 r^4 + 2(I+1)^3 r^3 \cos\theta - 2I^2(I+1)r\cos\theta - I^4 = 0. \tag{6.7.63}$$

Factorizing the LHS of (6.7.63) yields

$$\big((I+1)^2 r^2 - I^2\big)\big((I+1)^2 r^2 + I^2 + 2(I+1)r\cos\theta\big) = 0.$$

Thus, (6.7.63) permits four root at

$$r = \pm\frac{I}{I+1}, \frac{-1 \pm \mathbf{i}\sqrt{\cos\theta^2 - I^2}}{I+1}. \tag{6.7.64}$$

We only care about positive root, thus the contour (6.7.62) can be parametrized by $z_1(\theta) = \frac{1}{I+1} + \frac{I}{I+1}e^{\mathbf{i}\theta}$.

Similarly, inserting $z_1 = \frac{1}{I+1} + re^{\mathbf{i}\theta}$ in (6.7.61) yields

$$a_0(\epsilon, w)r^4 + 2a_1(\epsilon, w)r^3\cos\theta + a_2(\epsilon, w)r^2 + a_3(\epsilon, w)r\cos\theta + a_4(\epsilon, w) = 0$$

where $\{a_i(\epsilon, w)\}_{i=0}^4$ are a sequence of deterministic constants converging to the coefficient in (6.7.63)

$$\lim_{\epsilon\downarrow 0, w\to\infty} \big(a_0(\epsilon, w), a_1(\epsilon, w), a_2(\epsilon, w), a_3(\epsilon, w), a_4(\epsilon, w)\big) = \big((I+1)^4, 2(I+1)^3, 0, -2I^2(I+1), -I^4\big). \tag{6.7.65}$$

Denote by

$$P(\theta, r) = (I+1)^4 r^4 + 2(I+1)^3 r^3 \cos\theta - 2I^2(I+1) r\cos\theta - I^4$$
$$P_{\epsilon,w}(\theta, r) = a_0(\epsilon, w) r^4 + 2a_1(\epsilon, w) r^3 \cos\theta + a_2(\epsilon, w) r^2 + a_3(\epsilon, w) r\cos\theta + a_4(\epsilon, w).$$

By (6.7.65), when ϵ is small and w is large, $P_{\epsilon,w}(\theta, 0) < 0$ and $P_{\epsilon,w}(\theta, +\infty) = +\infty$. By continuity, for each $\theta \in (-\pi, \pi]$, $P_{\epsilon,w}(\theta, r) = 0$ admits a positive root. Since $P_{\epsilon,w}(\theta, r)$ is a perturbation of $P(\theta, r)$, as $\epsilon \downarrow 0$ and $w \to \infty$, the roots of $P_{\epsilon,w}(\theta, r)$ converge to those in (6.7.64), which implies the the positive root of $P_{\epsilon,w}(\theta)$ is unique for ϵ small and t large. We denote this unique positive root by $r_{\epsilon,w}(\theta)$. It is also clear that for $\theta \in (-\pi, \pi]$

$$\lim_{\epsilon\downarrow 0, w\to\infty} r_{\epsilon,w}(\theta) = \frac{I}{I+1} \text{ uniformly.} \tag{6.7.66}$$

Moreover, for all $\theta \in [-\pi, \pi]$, $r = \frac{I}{I+1}$ is a simple root of $P(\theta, r) = 0$. Hence, $\frac{\partial}{\partial r} P(\theta, r)\big|_{r=\frac{I}{I+1}} \neq 0$, using implicit function theorem shows that for ϵ small and w large, $r_{\epsilon,t-s}(\theta)$ is smooth over $(-\pi, \pi]$. Furthermore,

$$r'_{\epsilon,w}(0) = -\frac{\frac{\partial}{\partial\theta} P_{\epsilon,w}(\theta, r_{\epsilon,w}(0))\big|_{\theta=0}}{\frac{\partial}{\partial r} P_{\epsilon,w}(0, r)\big|_{r=r_{\epsilon,w}(0)}} = -\frac{\big(-2a_1(\epsilon, w) r_{\epsilon,w}(0)^3 \sin\theta + 2I^2(I+1) r_{\epsilon,w}(0)\sin\theta\big)\big|_{\theta=0}}{\frac{\partial}{\partial r} P_{\epsilon,w}(0, r)\big|_{r=r_{\epsilon,w}(0)}} = 0.$$

In addition, by (6.7.66) and implicit function theorem, uniformly over $\theta \in (-\pi, \pi]$

$$\lim_{\epsilon\downarrow 0, w\to\infty} r^{(n)}_{\epsilon,w}(\theta) = \Big(\frac{I}{I+1}\Big)^{(n)} = 0,$$

this completes our proof. □

We adopt the parametrization $z_1(\theta_1) = \frac{1}{I+1} + r_{\epsilon,t-s}(\theta_1)e^{\mathbf{i}\theta_1} \in \mathcal{M}''(t-s,\epsilon,-k_1\beta)$. From the preceding lemma, as $t-s \to \infty$ and $\epsilon \downarrow 0$, $\mathcal{M}''(t-s,\epsilon,-k_1\beta) \to \mathcal{M}$, thus the contour $\mathcal{M}''(t-s,\epsilon,-k_1\beta)$ is admissible for ϵ small and $t-s$ large. As before, we decompose $\mathbf{V}_\epsilon^{\text{in}} = \mathbf{V}_\epsilon^{\text{blk}} + \mathbf{V}_\epsilon^{\text{res}}$, where

$$\mathbf{V}_\epsilon^{\text{blk}}\big((x_1,x_2),(y_1,y_2),t,s\big) = \oint_{\mathcal{M}''(t-s,\epsilon,-k_1\beta)} \oint_{C_{r_2^*(z_1)}} \mathfrak{F}_\epsilon(z_1,z_2) \prod_{i=1}^{2} \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_i,t,s) z_i^{x_{3-i}-y_i} \frac{dz_i}{2\pi\mathbf{i}z_i}, \tag{6.7.67}$$

$$\begin{aligned}&\mathbf{V}_\epsilon^{\text{res}}\big((x_1,x_2),(y_1,y_2),t,s\big)\\ &= \oint_{\mathcal{M}''(t-s,\epsilon,-k_1\beta)} \mathbf{1}_{\{|\mathfrak{p}_\epsilon(z_1)|>r_2'\}} \mathfrak{J}_\epsilon(z_1)\mathfrak{H}_\epsilon(z_1)^{\lfloor \frac{t-s}{J} \rfloor} \mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s) \frac{dz_1}{2\pi\mathbf{i}z_1\mathfrak{p}_\epsilon(z_1)}.\end{aligned} \tag{6.7.68}$$

Lemma 6.7.12. *There exists $K > 0$ (which depends on k_1) such that for $t-s \le \epsilon^{-2}T$ large enough, $\epsilon > 0$ small enough, we have*

$$\begin{aligned} z_1(0) &\ge 1 - \frac{K\beta}{\sqrt{t-s+1}}, \\ |z_1(\theta)| &\le 1 - \frac{k_1\beta}{5\sqrt{t-s+1}}. \end{aligned}$$

Proof. Consider an alternate parametrization $\widetilde{z}_1(\theta) = \widetilde{r}_{\epsilon,t-s}(\theta)e^{\mathbf{i}\theta} \in \mathcal{M}''(t-s,\epsilon,-k_1\beta)$, where the existence and uniqueness of $\widetilde{r}_{\epsilon,t-s}(\theta)$ are confirmed by Lemma 6.7.11. It suffices to show for $t-s \le \epsilon^{-2}T$ large enough and $\epsilon > 0$ small enough,

$$\widetilde{r}_{\epsilon,t-s}(0) \ge 1 - \frac{K\beta}{\sqrt{t-s+1}}; \qquad |\widetilde{r}_{\epsilon,t-s}(\theta)| \le 1 - \frac{k_1\beta}{5\sqrt{t-s+1}}, \quad \forall \theta \in (-\pi,\pi]. \tag{6.7.69}$$

We prove (6.7.69) in two steps.

- First, $\frac{k_1\beta}{5\sqrt{t-s+1}} \le 1 - \widetilde{r}_{\epsilon,t-s}(0) \le \frac{K\beta}{\sqrt{t-s+1}}$.

- Second, $|\widetilde{r}_{\epsilon,t-s}(\theta)| \le \widetilde{r}_{\epsilon,t-s}(0)$ for $\theta \in (-\pi, \pi]$.

We verify the first bullet point. Note that uniformly in an neighborhood of 1,

$$\lim_{\epsilon \downarrow 0} \mathfrak{p}_\epsilon(z) = \mathfrak{p}_*(z), \qquad \lim_{\epsilon \downarrow 0} \mathfrak{p}'_\epsilon(z) = \mathfrak{p}'_*(z).$$

Referring to (6.7.33), $\frac{d}{dz} z\mathfrak{p}_*(z)\big|_{z=1} = 2$. Thus, there exists $\delta > 0$ such that for ϵ small enough and $z \in (1-\delta, 1+\delta)$,

$$|(z\mathfrak{p}_\epsilon(z))' - 2| < \frac{1}{2}. \tag{6.7.70}$$

We taylor expand $z\mathfrak{p}_\epsilon(z)$ around $z = 1$,

$$\mathsf{u}(t-s, -k_1\beta) = \widetilde{r}_{\epsilon,t-s}(0)\mathfrak{p}_\epsilon(\widetilde{r}_{\epsilon,t-s}(0)) = \mathfrak{p}_\epsilon(1) + \frac{d}{dz}(z\mathfrak{p}_\epsilon(z))\Big|_{z=x} \cdot (\widetilde{r}_{\epsilon,t-s}(0) - 1), \quad x \in (1-\delta, 1+\delta). \tag{6.7.71}$$

Referring to (6.7.44), we see $\mathfrak{p}_\epsilon(1) \ge 1$ for ϵ small enough, which implies

$$1 \ge \mathsf{u}(t-s, -k_1\beta) \ge 1 + \frac{d}{dz}(z\mathfrak{p}_\epsilon(z))\big|_{z=x} \cdot (\widetilde{r}_{\epsilon,t-s}(0) - 1).$$

Hence, $\widetilde{r}_{\epsilon,t-s}(0) \le 1$. We have by (6.7.70) and (6.7.71)

$$\begin{aligned} \mathsf{u}(t-s, -k_1\beta) &\ge \mathfrak{p}_\epsilon(1) + \frac{5}{2}(\widetilde{r}_{\epsilon,t-s}(0) - 1) \\ \mathsf{u}(t-s, -k_1\beta) &\le \mathfrak{p}_\epsilon(1) + \frac{3}{2}(\widetilde{r}_{\epsilon,t-s}(0) - 1) \end{aligned}$$

The first inequality yields

$$1 - \widetilde{r}_{\epsilon,t-s}(0) \ge \frac{2}{5}\big(\mathfrak{p}_\epsilon(1) - \mathsf{u}(t-s, -k_1\beta)\big) \ge \frac{2}{5}\big(1 - \mathsf{u}(t-s, -k_1\beta)\big) \ge \frac{k_1\beta}{5\sqrt{t-s+1}}.$$

which gives the lower bound. The second inequality indicates that (by (6.7.44))

$$1 - \widetilde{r}_{\epsilon,t-s}(0) \le \frac{2}{3}\big(\mathfrak{p}_\epsilon(1) - \mathsf{u}(t-s, -k_1\beta)\big) \le \frac{2}{3}\big(1 - \mathsf{u}(t-s, -k_1\beta)\big) + \frac{\rho I - \rho^2}{I}\epsilon.$$

Owing to $\epsilon \leq \sqrt{\frac{T}{t-s}}$, we see that $1 - \widetilde{r}_{\epsilon,t-s}(0) \leq \frac{K\beta}{\sqrt{t-s+1}}$ for constant K large enough, which concludes the first bullet point.

We move on proving the second bullet point. We set $F_\theta(r) = |r\mathfrak{p}_\epsilon(re^{\mathbf{i}\theta})|$. When ϵ small and $t-s$ large, we readily compute (note that $\widetilde{r}_{\epsilon,t-s}(0)$ is nearly $\frac{I}{I+1}$ and $\mathfrak{p}_\epsilon$ approximates $\mathfrak{p}_*$)

$$|F_\theta(\widetilde{r}_{\epsilon,t-s}(0))|^2 = \widetilde{r}_{\epsilon,t-s}(0)^2|\mathfrak{p}_\epsilon(\widetilde{r}_{\epsilon,t-s}(0)e^{\mathbf{i}\theta})|^2 = \frac{c_1^2 + c_2^2 - 2c_1c_2\cos\theta}{d_1^2 + d_2^2 + 2d_1d_2\cos\theta}, \qquad c_1, c_2, d_1, d_2 > 0,$$

which implies $|F_\theta(r_{\epsilon,t-s}(0))|$ reaches its minimum at $\theta = 0$. In other words, $F_\theta(r_{\epsilon,t-s}(0)) \geq F_0(r_{\epsilon,t-s}(0)) = \mathsf{u}(t-s, -k_1\beta)$. In addition, $F_\theta(0) = 0$. By intermediate value theorem, for each fixed $\theta \in (-\pi, \pi]$, the equation $F_\theta(r) = \mathsf{u}(t-s, -k_1\beta)$ admits a root $r \in (0, \widetilde{r}_{\epsilon,t-s}(0)]$. By uniqueness, this root equals $\widetilde{r}_{\epsilon,t-s}(\theta)$, thereby $\widetilde{r}_{\epsilon,t-s}(\theta) \leq \widetilde{r}_{\epsilon,t-s}(0)$ for all $\theta \in (-\pi, \pi]$. □

Lemma 6.7.13. *For k_1 large enough, $t - s \leq \epsilon^{-2}T$ large enough and $\epsilon > 0$ small enough, the condition $|\mathfrak{p}_\epsilon(z(\theta))| > r_2'$ with $z(\theta) = \frac{1}{I+1} + r_{\epsilon,t-s}(\theta)e^{\mathbf{i}\theta} \in \mathcal{M}''(t-s, \epsilon, \beta)$ implies $|\theta| \geq (t-s+1)^{-\frac{1}{4}}$.*

Proof. The proof is similar to Lemma 6.7.8. Since $k_2 = 1$, we have $r_2' = \mathsf{u}(t-s, -2\beta)$. Hence, $r_2' \geq 1 - \frac{4\beta}{\sqrt{t-s+1}}$. It suffices to show that

$$|\mathfrak{p}_\epsilon(z(\theta))| \geq 1 - \frac{4\beta}{\sqrt{t-s+1}} \Rightarrow |\theta| \geq (t-s+1)^{-\frac{1}{4}}.$$

Referring to (6.7.45), we see that

$$\mathfrak{p}_\epsilon(z(0)) = \mathfrak{p}_\epsilon(1) + \mathfrak{p}_\epsilon'(1)(z(0) - 1) + O\big(z(0) - 1\big)^2$$

By (6.7.44), we see $\mathfrak{p}_\epsilon(1) \leq 1 + \frac{C}{\sqrt{t-s+1}}$ for some positive constant C, together with the fact

$$z(0) - 1 \leq \frac{-k_1\beta}{5\sqrt{t-s+1}}, \qquad \lim_{\epsilon\downarrow 0} \mathfrak{p}_\epsilon'(1) = 1,$$

we obtain

$$\mathfrak{p}_\epsilon(z(0)) \leq 1 + \frac{C}{\sqrt{t-s+1}} - \frac{k_1\beta}{10\sqrt{t-s+1}}.$$

In addition, by Lemma 6.7.11, $r'_{\epsilon,t-s}(0) = 0$. Using this, it is straightforward to compute $\frac{d}{d\theta}|\mathfrak{p}_\epsilon(z(\theta))|\big|_{\theta=0} = 0$ and there exists $\zeta, C' > 0$ such that $\left|\frac{d^2}{d\theta^2}|\mathfrak{p}_\epsilon(z(\theta))|\right| \leq C'$ for $|\theta| < \zeta$. Consequently, one has by taylor expansion

$$|\mathfrak{p}_\epsilon(z(\theta))| \leq \mathfrak{p}_\epsilon(z(0)) + C'\theta^2 \leq 1 + \frac{10C - k_1\beta}{10\sqrt{t-s+1}} + C'\theta^2$$

Thereby, we can pick k_1 large enough s.t. $|\mathfrak{p}_\epsilon(z(\theta))| \geq 1 - \frac{4\beta}{\sqrt{t-s+1}}$ implies $|\theta| \geq (t-s+1)^{-\frac{1}{4}}$. □

Lemma 6.7.14. *For $t-s$ large and ϵ small, there exists positive constants $C(\beta,T), C$ such that*

$$|\mathfrak{D}_\epsilon(z(\theta))|^{t-s} \leq C(\beta,T)e^{-C(t-s+1)\theta^2}, \quad |\mathfrak{H}_\epsilon(z(\theta))|^{t-s} \leq C(\beta,T)e^{-C(t-s+1)\theta^2} \quad \textit{with } z(\theta) = \frac{1}{I+1} + r_{\epsilon,t-s}(\theta)e^{\mathbf{i}\theta}.$$

Proof. Similar to Lemma 6.7.7, it suffices to show that there exists $C(\beta,T), C > 0$ s.t.

$$\mathrm{Re}\ \log \mathfrak{D}_\epsilon(z(\theta)) \leq \frac{C(\beta,T)}{t-s+1} - C\theta^2; \qquad \mathrm{Re}\ \log \mathfrak{H}_\epsilon(z(\theta)) \leq \frac{C(\beta,T)}{t-s+1} - C\theta^2.$$

We split out proof for ($\theta = 0$), for (θ small) and for (θ large).

- ($\theta = 0$) : $\mathrm{Re}\,\mathfrak{D}_\epsilon(z(0)), \mathrm{Re}\,\mathfrak{H}_\epsilon(z(0)) \leq \frac{C(\beta,T)}{t-s+1}$.
- (θ small): There exists $\zeta > 0$ and constants $C(\beta,T)$ and $C > 0$ such that (6.7.39) holds for $|\theta| \leq \zeta$.
- (θ large): There exists $\delta > 0$ such that $\left|\mathfrak{D}_\epsilon(z(\theta))\right|, \left|\mathfrak{H}_\epsilon(z(\theta))\right| < 1-\delta$ for $|\theta| > \zeta$.

Owing to Lemma 6.7.12, $\frac{K}{\sqrt{t-s+1}} \leq 1 - z(0) \leq \frac{k_1}{5\sqrt{t-s+1}}$, hence the argument for ($\theta = 0$) is similar to Lemma 6.7.10.

For (θ small), using Lemma 6.7.11, one has

$$r'_{\epsilon,t-s}(0)=0,\qquad \lim_{\epsilon\downarrow 0,t-s\to\infty} r''_{\epsilon,t-s}(\theta)=0,\qquad \lim_{\epsilon\downarrow 0,t-s\to\infty} r'''_{\epsilon,t-s}(\theta)=0.$$

Using this, after a tedious but straightforward calculation (recall $z(\theta)=\frac{1}{I+1}+\frac{I}{I+1}r_{\epsilon,t-s}(\theta)$),

$$\begin{aligned}
&\partial_\theta(\log\mathfrak{D}_\epsilon(z(\theta)))\big|_{\theta=0}\in\mathbf{i}\mathbb{R}, && \partial_\theta(\log\mathfrak{H}_\epsilon(z(\theta)))\big|_{\theta=0}\in\mathbf{i}\mathbb{R}\\
&\lim_{\epsilon\downarrow 0,t-s\to\infty}\partial_\theta^2(\log\mathfrak{D}_\epsilon(z(\theta)))\big|_{\theta=0}=-\frac{I^2JV_*}{(I+1)^2}, && \lim_{\epsilon\downarrow 0,t-s\to\infty}\partial_\theta^2(\log\mathfrak{H}_\epsilon(z(\theta)))\big|_{\theta=0}=-\frac{2I^2JV_*}{(I+1)^2}\\
&\left|\partial_\theta^3(\log\mathfrak{D}_\epsilon(z(\theta)))\right|\leq C, && \left|\partial_\theta^3(\log\mathfrak{H}_\epsilon(z(\theta)))\right|\leq C.
\end{aligned}$$

The last line holds for all $|\theta|<\zeta$ where $\zeta>0$ is a constant. Hereafter, the argument is same as in Lemma 6.7.7, we do not repeat it here.

For (θ large), since

$$\lim_{\epsilon\downarrow 0,t-s\to\infty} r_{\epsilon,t-s}(\theta)=\frac{I}{I+1},\qquad \text{uniformly for } \theta\in(-\pi,\pi].$$

We have

$$\begin{aligned}
\lim_{\epsilon\downarrow 0,t-s\to\infty}\mathfrak{D}_\epsilon(z(\theta))&=\mathfrak{D}_*\Big(\frac{1}{I+1}+\frac{I}{I+1}e^{\mathbf{i}\theta}\Big),\qquad \text{uniformly over } \theta\in(-\pi,\pi],\\
\lim_{\epsilon\downarrow 0,t-s\to\infty}\mathfrak{H}_\epsilon(z(\theta))&=\mathfrak{H}_*\Big(\frac{1}{I+1}+\frac{I}{I+1}e^{\mathbf{i}\theta}\Big),\qquad \text{uniformly over } \theta\in(-\pi,\pi].
\end{aligned}$$

By the steepest descent condition (SD$\mathcal{M}$), we conclude (θ large). □

Now we are ready to bound $\mathbf{V}^{\text{blk}}_\epsilon$ and $\mathbf{V}^{\text{res}}_\epsilon$. We begin with $\mathbf{V}^{\text{blk}}_\epsilon$ given by (6.7.67). The proof consists of bounding each terms involved in the integrand (6.7.67). We parametrize $z_1(\theta_1)=r_{\epsilon,t-s}(\theta_1)e^{\mathbf{i}\theta_1}$, $z_2(\theta_2)=r_2^*(z_1)e^{\mathbf{i}\theta_2}$.

($\mathbf{V}^{\text{blk}}_\epsilon, z_1^{x_2-y_1}z_2^{x_1-y_2}$): **Show that** $|z_1^{x_2-y_1}z_2^{x_1-y_2}|\leq e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_1-y_2|+|x_2-y_1|)}$.
By Lemma 6.7.12, we see that $|z_1|\leq e^{-\frac{\beta}{\sqrt{t-s+1}}}$, since $r_2^*(z_1)$ equals $\mathsf{u}(t-s,-\beta)$ or $\mathsf{u}(t-s,-3\beta)$, we

find that $|z_2| \leq e^{-\frac{\beta}{\sqrt{t-s+1}}}$, which implies $|z_1^{x_2-y_1} z_2^{x_1-y_2}| \leq e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}$.

$(\mathbf{V}_\epsilon^{\mathrm{blk}}, \mathfrak{F}_\epsilon(z_1, z_2))$: **Show that** $\left|\mathfrak{F}_\epsilon(z_1, z_2)\right| \leq C + C\sqrt{t-s+1}(|\theta_1| + |\theta_2|)$.
By the argument in $(\mathbf{V}_\epsilon^{\mathrm{blk}}, \mathfrak{F}_\epsilon(z_1, z_2))$ in $(+-)$ case. It suffices to show that $|z_2 - z_1| \leq C(\frac{1}{\sqrt{t-s+1}} + |\theta_1| + |\theta_2|)$. Note that

$$|z_2(\theta_2) - z_1(\theta_1)| \leq |z_1(\theta_1) - 1| + |z_2(\theta_2) - 1| \leq |r_{\epsilon,t-s}(\theta_1)e^{\mathbf{i}\theta_1} - 1| + |r^*(z_1)e^{\mathbf{i}\theta_2} - 1|. \quad (6.7.72)$$

By Lemma 6.7.11 and Lemma 6.7.12, we know that $|r_{\epsilon,t-s}(0)-1| \leq \frac{C}{\sqrt{t-s+1}}$ and $\lim_{\epsilon\downarrow 0, t-s\to\infty} r'_{\epsilon,t-s}(\theta) = 0$ uniformly for $\theta \in (-\pi, \pi]$, we see that

$$|r_{\epsilon,t-s}(\theta_1)e^{\mathbf{i}\theta_1} - 1| \leq |r_{\epsilon,t-s}(\theta_1) - r_{\epsilon,t-s}(0)| + |r_{\epsilon,t-s}(0) - 1| + |e^{-\mathbf{i}\theta_1} - 1| \leq C(\frac{1}{\sqrt{t-s+1}} + |\theta_1|) \quad (6.7.73)$$

Since $r^*(z_1) = \mathsf{u}(t-s, \beta)$ or $r^*(z_1) = \mathsf{u}(t-s, 3\beta)$, we have

$$|r^*(z_1)e^{\mathbf{i}\theta_2} - 1| \leq C(\frac{1}{\sqrt{t-s+1}} + |\theta_2|) \quad (6.7.74)$$

Incorporating the bound (6.7.73) and (6.7.74) into the RHS of (6.7.72), we conclude $|z_2(\theta_2) - z_1(\theta_1)| \leq C(\frac{1}{\sqrt{t-s+1}} + |\theta_1| + |\theta_2|)$.

$(\mathbf{V}_\epsilon^{\mathrm{blk}}, \mathfrak{R}_\epsilon(z_i, t, s))$: **Show that** $|\mathfrak{R}_\epsilon(z_i, t, s)| \leq C$.
This is the same as $(+-)$ case $(\mathbf{V}_\epsilon^{\mathrm{blk}}, \mathfrak{R}_\epsilon(z_i, t, s))$.

$(\mathbf{V}_\epsilon^{\mathrm{blk}}, \mathfrak{D}_\epsilon(z_i)^{\lfloor \frac{t-s}{J} \rfloor})$: **Show that** $|\mathfrak{D}_\epsilon(z_i)|^{\lfloor \frac{t-s}{J} \rfloor} \leq C(\beta, T)\exp(-C(t-s+1)\theta_i^2)$.
This is the content of Lemma 6.7.14.

Consequently, we perform the same procedure as in the $(+-)$ case and get

$$\begin{aligned}|\mathbf{V}_\epsilon^{\mathrm{blk}}| &\leq C(\beta, T)\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(1 + \sqrt{t-s+1}(|\theta_1| + |\theta_2|))e^{-C(t-s+1)(\theta_1^2+\theta_2^2)}d\theta_1 d\theta_2 \\ &\leq \frac{C(\beta, T)}{t-s+1}e^{-\frac{\beta}{\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}.\end{aligned}$$

Let us move on bounding $\mathbf{V}_\epsilon^{\mathrm{res}}$ with integral expression (6.7.68). We parametrize by $z_1(\theta) = r_{\epsilon,t-s}(\theta)e^{\mathbf{i}\theta} \in \mathcal{M}''(t-s,\epsilon,-k_1\beta)$.

$(\mathbf{V}_\epsilon^{\mathrm{res}}, \frac{1}{z_1\mathfrak{p}_\epsilon(z_1)})$: **Show that** $|\frac{1}{z_1\mathfrak{p}_\epsilon(z_1)}| \le C$.
This is by the same argument as in the (+−) case.

$(\mathbf{V}_\epsilon^{\mathrm{res}}, \mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s))$: **Show that** $|\mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s)| \le C$.
The argument for this is the same as $(\mathbf{V}_\epsilon^{\mathrm{res}}, \mathfrak{R}_\epsilon(z_1,t,s)\mathfrak{R}_\epsilon(\mathfrak{p}_\epsilon(z_1),t,s))$ in the (+−) case.

$(\mathbf{V}_\epsilon^{\mathrm{res}}, \mathfrak{H}_\epsilon(z_1)^{\lfloor\frac{t-s}{J}\rfloor})$:**Show that** $|\mathfrak{H}_\epsilon(z_1)|^{\lfloor\frac{t-s}{J}\rfloor} \le C(\beta,T)e^{-C(t-s+1)\theta^2}$.
This is the content of Lemma 6.7.14.

$(\mathbf{V}_\epsilon^{\mathrm{res}}, \mathfrak{J}_\epsilon(z_1))$: **Show that** $|\mathfrak{J}_\epsilon(z_1)| \le Ce^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|}$.
By the discussion in $(\mathbf{V}_\epsilon^{\mathrm{res}}, \mathfrak{J}_\epsilon(z_1))$, It is sufficient to show that $|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}| \le e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_1-y_2|+|x_2-y_1|)}$.
We write

$$|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}| = |z_1\mathfrak{p}_\epsilon(z_1)|^{x_1-y_2}|z_1|^{x_2-x_1+y_2-y_1}$$

Since $z_1 \in \mathcal{M}''(t-s,\epsilon,-k_1\beta)$, $|z_1\mathfrak{p}_\epsilon(z_1)| = \mathsf{u}(t-s,-k_1\beta) \le e^{-\frac{\beta}{\sqrt{t-s+1}}}$. In addition, referring to Lemma 6.7.12, one has $|z_1| \le e^{-\frac{\beta}{\sqrt{t-s+1}}}$. Consequently,

$$|z_1^{x_2-y_1}\mathfrak{p}_\epsilon(z_1)^{x_1-y_2}| \le e^{-\frac{\beta}{\sqrt{t-s+1}}(x_1-y_2)}e^{-\frac{\beta}{\sqrt{t-s+1}}(x_2-x_1+y_2-y_1)} = e^{-\frac{\beta}{\sqrt{t-s+1}}(x_2-y_1)} \le e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}.$$

Thereby, using the same manner as (+−) case,

$$\begin{aligned}|\mathbf{V}_\epsilon^{\mathrm{res}}| &\le C(\beta,T)e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}\int_{-\pi}^{\pi}1_{\{|\mathfrak{p}_\epsilon(z_1(\theta))|\ge r_2'\}}e^{-C(t-s+1)\theta^2}d\theta,\\ &\le C(\beta,T)e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}\int_{|\theta|>(t-s+1)^{-\frac{1}{4}}}e^{-\frac{1}{C}(t-s+1)\theta^2}d\theta \le \frac{C(\beta,T)}{t-s+1}e^{-\frac{\beta}{2\sqrt{t-s+1}}(|x_2-y_1|+|x_1-y_2|)}.\end{aligned}$$

We conclude Theorem 6.7.3 (a).

To estimate the gradient, the procedure is similar to in (+−) case, note that applying ∇_{x_i} or ∇_{y_i} will give an additional factor $z_i^{\pm}-1$, while applying ∇_{x_1,x_2} will produce an additional factor $(z_1 -$

$1)(z_2-1)$. By $|z_i(\theta_i)-1|\leq C(\frac{1}{\sqrt{t-s+1}}+|\theta_i|)$, we conclude Theorem 6.7.3 (b), (c).

6.8 Proof of Proposition 6.6.8 via self-averaging

In this section, we apply the two Markov dualities in Corollary 6.3.9 and the estimate of $\mathbf{V}_\epsilon$ in Theorem 6.7.1 to conclude Proposition 6.6.8. The first step is to expand the term $\Theta_1(t,x)$ and $\Theta_2(t,x)$.

6.8.1 Expanding $\Theta_1(t,x)$ and $\Theta_2(t,x)$

We use $\mathscr{B}_\epsilon(t,x_1,\dots,x_n)$ to denote a generic uniformly bounded (random) process, which may differ from line to line. Define

$$u_\epsilon(t,i) := \sum_{j=i}^{\infty} \mathsf{p}_\epsilon(t+1,t,j-\mu_\epsilon(t))$$

Referring to (6.5.10) for the expression of $\Theta_1(t,x)$

$$\begin{aligned}
\epsilon^{-\frac{1}{2}}\Theta_1(t,x) &= \epsilon^{-\frac{1}{2}} q_\epsilon \lambda_\epsilon(t) Z(t,x) - \sum_{i=1}^{\infty} \epsilon^{-\frac{1}{2}} \mathsf{p}_\epsilon(t+1,t,i-\mu_\epsilon(t)) Z(t,x-i), \\
&= \epsilon^{-\frac{1}{2}}(q_\epsilon\lambda_\epsilon(t)-1)Z(t,x) + \sum_{i=1}^{\infty} \epsilon^{-\frac{1}{2}} \mathsf{p}_\epsilon(t+1,t,i-\mu_\epsilon(t)) \big(Z(t,x)-Z(t,x-i)\big), \\
&= \epsilon^{-\frac{1}{2}}(q_\epsilon\lambda_\epsilon(t)-1)Z(t,x) + \sum_{i=1}^{\infty} u_\epsilon(t,i)\big(\epsilon^{-\frac{1}{2}}\nabla Z(t,x-i)\big).
\end{aligned}$$

Here, we used the relation $Z(t,x)-Z(t,x-i)=\sum_{j=1}^{i}\nabla Z(t,x-j)$ and then changed the order of summation in the last equality.

Likewise, by the expression (6.5.11) of $\Theta_2(t,x)$

$$\epsilon^{-\frac{1}{2}}\Theta_2(t,x) = \epsilon^{-\frac{1}{2}}(1-\lambda_\epsilon(t))Z(t,x) - \sum_{i=1}^{\infty} u_\epsilon(t,i)(\epsilon^{-\frac{1}{2}}\nabla Z(t,x-i)).$$

Using Lemma 6.5.4, one has $\epsilon^{-\frac{1}{2}}(q_\epsilon\lambda_\epsilon(t)-1) = 1-\frac{\rho}{I}+O(\epsilon^{\frac{1}{2}})$ and $\epsilon^{-\frac{1}{2}}(1-\lambda_\epsilon(t)) = \frac{\rho}{I}+O(\epsilon^{\frac{1}{2}})$. Consequently,

$$\epsilon^{-\frac{1}{2}}\Theta_1(t,x) = \left(1-\frac{\rho}{I}\right)Z(t,x) + \sum_{i=1}^{\infty} u_\epsilon(t,i)(\epsilon^{-\frac{1}{2}}\nabla Z(t,x-i)) + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x), \tag{6.8.1}$$

$$\epsilon^{-\frac{1}{2}}\Theta_2(t,x) = \frac{\rho}{I}Z(t,x) - \sum_{i=1}^{\infty} u_\epsilon(t,i)(\epsilon^{-\frac{1}{2}}\nabla Z(t,x-i)) + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x). \tag{6.8.2}$$

For $x_1 \le x_2 \in \Xi(t)$ and $x \in \Xi(t)$, we denote by

$$Z_\nabla(t,x_1,x_2) := \epsilon^{-\frac{1}{2}}\nabla Z(t,x_1)Z(t,x_2),$$

$$Z_{\nabla,\nabla}(t,x_1,x_2) := \epsilon^{-1}\nabla Z(t,x_1)\nabla Z(t,x_2),$$

$$\mathcal{Y}_\nabla(t,x) := \sum_{i\in\mathbb{Z}_{\ge 1}} u_\epsilon(t,i)Z_\nabla(t,x-i,x), \tag{6.8.3}$$

$$\mathcal{Y}_{\nabla,\nabla}(t,x) := \sum_{i>j\in\mathbb{Z}_{\ge 1}} u_\epsilon(t,i)u_\epsilon(t,j)Z_{\nabla,\nabla}(t,x-i,x-j), \tag{6.8.4}$$

$$\widetilde{\mathcal{Y}}(t,x) := \sum_{i=1}^{\infty} u_\epsilon(t,i)^2\Big(Z_{\nabla,\nabla}(t,x-i,x-i) - \frac{\rho(I-\rho)}{I}Z(t,x-i)^2\Big). \tag{6.8.5}$$

Lemma 6.8.1. *Recall from* (6.6.22) *that*

$$\tau(t) = \frac{\rho(I-\rho)}{I^2}\cdot\frac{b(I+2\mathrm{mod}_J(t)+1)-(I+2\mathrm{mod}_J(t)-1)}{b(I+2\mathrm{mod}_J(t))-(I+2\mathrm{mod}_J(t)-2)},$$

we have

$$\begin{aligned}&\epsilon^{-1}\Theta_1(t,x)\Theta_2(t,x) - \tau(t)Z(t,x)^2\\ &= \left(\frac{2\rho}{I}-1\right)\mathcal{Y}_\nabla(t,x) + 2\mathcal{Y}_{\nabla,\nabla}(t,x) + \widetilde{\mathcal{Y}}(t,x) + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2.\end{aligned}$$

Proof. We name the three terms on the RHS of (6.8.1) (from left to right) as $A_{1,Z}, A_{1,\nabla}, A_{1,\mathrm{err}}$ respectively and those on the RHS of (6.8.2) as $A_{2,Z}$, $A_{2,\nabla}$, $A_{2,\mathrm{err}}$. Multiplying (6.8.1) by (6.8.2) gives

$$\epsilon^{-1}\Theta_1(t,x)\Theta_2(t,x) = \big(A_{1,Z} + A_{1,\nabla} + A_{1,\mathrm{err}}\big)\cdot\big(A_{2,Z} + A_{2,\nabla} + A_{2,\mathrm{err}}\big).$$

Expanding this product, it is straightforward that

$$A_{1,Z}A_{2,Z} = \frac{\rho}{I}(1-\frac{\rho}{I})Z(t,x)^2, \qquad A_{1,\nabla}A_{2,Z} + A_{2,\nabla}A_{1,Z} = \left(\frac{2\rho}{I}-1\right)\mathcal{Y}_{\nabla}(t,x),$$
$$A_{1,\nabla}A_{2,\nabla} = -\mathcal{Y}_{\nabla,\nabla}(t,x) - \sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z_{\nabla,\nabla}(t,x-k,x-k).$$

The sum of the rest of terms equals

$$A_{1,Z}A_{2,\mathrm{err}} + A_{1,\nabla}A_{2,\mathrm{err}} + A_{1,\mathrm{err}}A_{2,Z} + A_{1,\mathrm{err}}A_{2,\nabla} + A_{1,\mathrm{err}}A_{2,\mathrm{err}},$$
$$= \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)(\epsilon^{-\frac{1}{2}}\Theta_1(t,x) + \epsilon^{-\frac{1}{2}}\Theta_2(t,x)) - \epsilon\mathcal{B}_\epsilon(t,x)Z(t,x)^2 = \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2.$$

Therefore, we find that

$$\epsilon^{-1}\Theta_1(t,x)\Theta_2(t,x) = \frac{\rho}{I}(1-\frac{\rho}{I})Z(t,x)^2 + \mathcal{Y}_{\nabla}(t,x) - \mathcal{Y}_{\nabla,\nabla}(t,x) - \sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z_{\nabla,\nabla}(t,x-k,x-k)$$
$$+ \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2.$$

Thus,

$$\epsilon^{-1}\Theta_1(t,x)\Theta_2(t,x) - \frac{\rho}{I}(1-\frac{\rho}{I})Z(t,x)^2 = \mathcal{Y}_{\nabla}(t,x) - \mathcal{Y}_{\nabla,\nabla}(t,x) - \sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z_{\nabla,\nabla}(t,x-k,x-k)$$
$$+ \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2.$$

Adding $\frac{\rho(I-\rho)}{I}\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z(t,x-k)^2$ to both sides yields

$$\epsilon^{-1}\Theta_1(t,x)\Theta_2(t,x) - \frac{\rho}{I}(1-\frac{\rho}{I})Z(t,x)^2 + \frac{\rho(I-\rho)}{I}\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z(t,x-k)^2$$
$$= \mathcal{Y}_{\nabla}(t,x) - \mathcal{Y}_{\nabla,\nabla}(t,x) - \sum_{k=1}^{\infty} u_\epsilon(t,k)^2\left(Z_{\nabla,\nabla}(t,x-k,x-k) - \frac{\rho(I-\rho)}{I}Z(t,x-k)^2\right) + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2$$

$$= \mathscr{Y}_{\nabla}(t,x) - \mathscr{Y}_{\nabla,\nabla}(t,x) - \widetilde{\mathscr{Y}}(t,x) + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2. \tag{6.8.6}$$

We claim that

$$\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z(t,x-k)^2 = \frac{1-b}{I(b(I+2\mathrm{mod}_J(t)) - (I+2\mathrm{mod}_J(t)-2))} Z(t,x)^2 + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2. \tag{6.8.7}$$

If (6.8.7) holds, note that

$$\tau(t) = \frac{\rho}{I}(1-\frac{\rho}{I}) - \frac{\rho(I-\rho)}{I}\frac{1-b}{I(b(I+2\mathrm{mod}_J(t)) - (I+2\mathrm{mod}_J(t)-2))}.$$

Replacing the term $\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z(t,x-k)^2$ in the LHS of (6.8.6) by the RHS of (6.8.7), we prove Lemma 6.8.1.

To justify (6.8.7), we write

$$\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z(t,x-k)^2 = \sum_{k=1}^{\infty} u_\epsilon(t,k)^2 \big(Z(t,x-k)^2 - Z(t,x)^2\big) + \sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z(t,x)^2. \tag{6.8.8}$$

Let us analyze the first and second term on the RHS of (6.8.8) respectively. For the second term, we compute

$$u_\epsilon(t,k) = \sum_{j=k}^{\infty} \mathsf{p}_\epsilon(t+1,t,j) = \frac{\alpha(t)(1-q)}{1+\alpha(t)}\left(\frac{\nu+\alpha(t)}{1+\alpha(t)}\right)^{k-1}. \tag{6.8.9}$$

Here, we used $\mathfrak{p}_\epsilon(t+1,t,j) = \mathbb{P}(R(t)=j)$, the expression of which is given in (6.5.1). Using the preceding equation, we find that

$$\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 = \frac{\big(1-\frac{1+q\alpha(t)}{1+\alpha(t)}\big)^2}{1-\big(\frac{\nu+\alpha(t)}{1+\alpha(t)}\big)^2}.$$

Due to Lemma 6.5.4,

$$\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 = \frac{1-b}{I\big((I+2\mathrm{mod}_J(t))b - (I+2\mathrm{mod}_J(t)-2)\big)} + O(\epsilon^{\frac{1}{2}}).$$

Thereby, for the second term on the RHS of (6.8.8),

$$\sum_{k=1}^{\infty} u_\epsilon(t,k)^2 Z(t,x)^2 = \frac{1-b}{I(b(I+2\text{mod}_J(t)) - (I+2\text{mod}_J(t)-2))} Z(t,x)^2 + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2 \tag{6.8.10}$$

For the first term on the RHS of (6.8.8), noticing $Z(t,x-k) = e^{-\sqrt{\epsilon}\sum_{i=1}^{k}(\widetilde{\eta}_{x-i+1}(t)-\rho)} Z(t,x)$ (recall $\widetilde{\eta}_x(t) = \eta_x(x+\hat{\mu}(t))$), hence

$$Z(t,x-k)^2 - Z(t,x)^2 = Z(t,x)^2 \Big(e^{-2\sqrt{\epsilon}\left((\widetilde{\eta}_x(t)-\rho)+\cdots+(\widetilde{\eta}_{x-k+1}(t)-\rho)\right)} - 1 \Big)$$

Since $|\widetilde{\eta}_x(t) - \rho| \le I$,

$$\Big| \sum_{i=1}^{k} (\widetilde{\eta}_{x-i+1}(t) - \rho) \Big| \le kI.$$

Note that for any $K > 0$, there exists a constant C such that

$$|e^x - 1| \le C|x|, \text{ for } |x| \le K.$$

Thus, if $k \le \epsilon^{-\frac{1}{2}}$, one has

$$\Big| e^{-2\sqrt{\epsilon}\sum_{i=1}^{k}(\widetilde{\eta}_{x-i+1}(t)-\rho)} - 1 \Big| \le C\sqrt{\epsilon} kI.$$

If $k > \epsilon^{-\frac{1}{2}}$, one simply has

$$\Big| e^{-2\sqrt{\epsilon}\sum_{i=1}^{k}(\widetilde{\eta}_{x-i+1}(t)-\rho)} - 1 \Big| \le e^{2kI\sqrt{\epsilon}}.$$

Therefore,

$$\Big| e^{-2\sqrt{\epsilon}\sum_{i=1}^{k}(\widetilde{\eta}_{x-i+1}(t)-\rho)} - 1 \Big| \le C\big(\sqrt{\epsilon} kI \mathbf{1}_{\{k \le \epsilon^{-\frac{1}{2}}\}} + e^{2kI\sqrt{\epsilon}} \mathbf{1}_{\{k > \epsilon^{-\frac{1}{2}}\}} \big). \tag{6.8.11}$$

Referring to (6.8.9) for the expression of $u_\epsilon(t,k)$, using (6.7.4) we see that there exists $0 < \delta < 1$

s.t. for ϵ small enough and for all t, k

$$u_\epsilon(t,k) \leq \delta^{k-1}. \tag{6.8.12}$$

Combining this with (6.8.11) gives

$$\begin{aligned}\sum_{k=1}^{\infty} u_\epsilon(t,k)^2\big(Z(t,x-k)^2 - Z(t,x)^2\big) &= Z(t,x)^2\bigg(\sum_{k=1}^{\infty} u_\epsilon(t,k)^2\big(e^{-2\sqrt{\epsilon}\sum_{i=1}^{k}(\widetilde{\eta}_{x-i+1}(t)-\rho)} - 1\big)\bigg),\\ &\leq CZ(t,x)^2\bigg(\sum_{k=1}^{\lfloor \epsilon^{-\frac{1}{2}}\rfloor} \sqrt{\epsilon}k\delta^k + \sum_{k=\lfloor \epsilon^{-\frac{1}{2}}\rfloor+1}^{\infty} e^{2kI\sqrt{\epsilon}}\delta^k\bigg),\\ &= \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2.\end{aligned}$$

Combining this with (6.8.10), we prove the desired claim (6.8.7). □

By Lemma 6.8.1, we reduce the proof of Proposition 6.6.8 to the following lemmas.

Lemma 6.8.2. *For any given $T > 0$, there exists positive constants C and u such that for all $t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$, $x^\star \in \mathbb{Z}$*

$$\bigg\|\epsilon^2 \sum_{s=0}^{t} \mathcal{Y}_\nabla(s, x^\star(s))\bigg\|_2 \leq C\epsilon^{\frac{1}{4}}e^{2u\epsilon|x^\star|}, \tag{6.8.13}$$

$$\bigg\|\epsilon^2 \sum_{s=0}^{t} \mathcal{Y}_{\nabla,\nabla}(s, x^\star(s))\bigg\|_2 \leq C\epsilon^{\frac{1}{4}}e^{2u\epsilon|x^\star|}, \tag{6.8.14}$$

where we used the shorthand notation $x^\star(s) := x^\star - \hat{\mu}(s) + \lfloor\hat{\mu}(s)\rfloor$.

Lemma 6.8.3. *Fix $T > 0$, there exists positive constants C and u such that for all $t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$ and $x^\star \in \mathbb{Z}$,*

$$\bigg\|\epsilon^2 \sum_{s=0}^{t} \widetilde{\mathcal{Y}}(s, x^\star(s))\bigg\|_2 \leq C\epsilon^{\frac{1}{4}}e^{2u\epsilon|x^\star|}$$

We will prove Lemma 6.8.2 and Lemma 6.8.3 in the next two sections. Let us first conclude Proposition 6.6.8 based on them.

Proof of Proposition 6.6.8. Referring to Lemma 6.8.1, we have

$$\epsilon^2 \sum_{s=0}^{t} \Big(\epsilon^{-1}\Theta_1\Theta_2 - \tau(s)Z^2\Big)(s, x^\star(s)) = \epsilon^2 \sum_{s=0}^{t} \mathcal{Y}_\nabla(s, x^\star(s)) + \epsilon^2 \sum_{s=0}^{t} \mathcal{Y}_{\nabla,\nabla}(s, x^\star(s)) + \epsilon^2 \sum_{s=0}^{t} \widetilde{\mathcal{Y}}(s, x^\star(s)) + \epsilon^2 \sum_{s=0}^{t} \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(s,x) Z(s, x^\star(s))^2.$$

By Lemma 6.8.2 and Lemma 6.8.3, together with the bound $\|Z(s, x^\star(s))\|_2 \le Ce^{u\epsilon|x^\star|}$ (which follows from Proposition 6.6.1), one has

$$\begin{aligned}
&\Big\|\epsilon^2 \sum_{s=0}^{t} \Big(\epsilon^{-1}\Theta_1\Theta_2 - \tau(s)Z^2\Big)(s, x^\star(s))\Big\|_2 \\
&\quad \le \Big\|\epsilon^2 \sum_{s=0}^{t} \mathcal{Y}_\nabla(s, x^\star(s))\Big\|_2 + \Big\|\epsilon^2 \sum_{s=0}^{t} \mathcal{Y}_{\nabla,\nabla}(s, x^\star(s))\Big\|_2 + \Big\|\epsilon^2 \sum_{s=0}^{t} \widetilde{\mathcal{Y}}(s, x^\star(s))\Big\|_2 \\
&\quad + \epsilon^2 \sum_{s=0}^{t} \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(s,x) \|Z(s, x^\star(s))\|_2^2 \\
&\le C\big(\epsilon^{\frac{1}{4}} e^{2u\epsilon|x^\star|} + \epsilon^{\frac{5}{2}} t e^{2u\epsilon|x^\star|}\big).
\end{aligned}$$

Using $t \le \epsilon^{-2}T$, we obtain

$$\Big\|\epsilon^2 \sum_{s=0}^{t} \Big(\epsilon^{-1}\Theta_1\Theta_2 - \tau(s)Z^2\Big)(s, x^\star(s))\Big\|_2 \le C\epsilon^{\frac{1}{4}} e^{2u\epsilon|x^\star|}$$

This completes the proof of Proposition 6.6.8. □

6.8.2 Proof of Lemma 6.8.2

Recall the notation $\widetilde{\eta}_x(t) = \eta_{x+\hat{\mu}(t)}(t)$, we see that by Taylor expansion

$$\nabla Z(t,x) = Z(t,x)\big(e^{-\sqrt{\epsilon}(\widetilde{\eta}_{x+1}(t)-\rho)} - 1\big) = \sqrt{\epsilon} Z(t,x)(\rho - \widetilde{\eta}_{x+1}(t)) + \epsilon \mathcal{B}_\epsilon(t,x) Z(t,x).$$

Hence,

$$\epsilon^{-\frac{1}{2}}\nabla Z(t,x) = (\rho - \widetilde{\eta}_{x+1}(t))Z(t,x) + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x), \tag{6.8.15}$$

$$Z(t,x+1) = Z(t,x) + \nabla Z(t,x) = Z(t,x) + \epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x). \tag{6.8.16}$$

We will use these elementary relations frequently in the sequel.

The following lemma is crucial for the proof of Lemma 6.8.2.

Lemma 6.8.4. *Given $T > 0$ and $n \in \mathbb{Z}_{\geq 1}$, there exists constant C and u such that for all $s \leq t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$ such that for $x_1 \leq x_2 \in \Xi(t)$,*

$$\Big\|\mathbb{E}\big[Z_\nabla(t,x_1,x_2)\big|\mathcal{F}(s)\big]\Big\|_n \leq \frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{t-s+1}}e^{u\epsilon(|x_1|+|x_2|)}. \tag{6.8.17}$$

For $x_1 < x_2 \in \Xi(t)$,

$$\Big\|\mathbb{E}\big[Z_{\nabla,\nabla}(t,x_1,x_2)\big|\mathcal{F}(s)\big]\Big\|_n \leq \frac{C\epsilon^{-1}}{t-s+1}e^{u\epsilon(|x_1|+|x_2|)}. \tag{6.8.18}$$

Proof. Let us first justify (6.8.17). Recall the two point duality (6.5.21),

$$\mathbb{E}\big[Z(t,x_1)Z(t,x_2)\big|\mathcal{F}(s)\big] = \sum_{y_1 \leq y_2 \in \Xi(s)^2} \mathbf{V}\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2).$$

As $Z_\nabla(t,x_1,x_2) = \epsilon^{-\frac{1}{2}}\nabla Z(t,x_1)Z(t,x_2)$, it is straightforward that by this duality, if $x_1 < x_2$,

$$\mathbb{E}\big[Z_\nabla(t,x_1,x_2)\big|\mathcal{F}(s)\big] = \epsilon^{-\frac{1}{2}} \sum_{y_1 \leq y_2 \in \Xi(s)} \nabla_{x_1}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2). \tag{6.8.19}$$

If $x_1 = x_2$,

$$\mathbb{E}\big[Z_\nabla(t,x_1,x_2)\big|\mathcal{F}(s)\big] = \epsilon^{-\frac{1}{2}} \sum_{y_1 \leq y_2 \in \Xi(s)} \nabla_{x_2}\mathbf{V}_\epsilon\big((x_1,x_1),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2).$$

We assume $x_1 < x_2$ without loss of generosity, the proof of (6.8.17) for $x_1 = x_2$ will be similar (one only needs to replicate the estimate of $\nabla_{x_1}\mathbf{V}_\epsilon$ to $\nabla_{x_2}\mathbf{V}_\epsilon$). By the estimate of $\nabla_{x_1}\mathbf{V}_\epsilon$ provided in Theorem 6.7.1 (b), we see that

$$\left|\nabla_{x_1}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \leq \frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}}.$$

This, together with the moment bound of $Z(t,x)$ in (6.6.1) yields

$$\left\|\sum_{y_1\leq y_2}\nabla_{x_1}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2)\right\|_n \leq \sum_{y_1\leq y_2\in\Xi(s)}\frac{C(\beta,T)}{(t-s+1)^{\frac{3}{2}}} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}} e^{u\epsilon|y_1|}e^{u\epsilon|y_2|}$$

Due to Lemma 6.6.3, we see that we can choose β large enough so that

$$\begin{aligned}\sum_{y_1,y_2\in\Xi(s)} e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}} e^{u\epsilon(|y_1|+|y_2|)} &\leq \Big(\sum_{y_1\in\Xi(s)} e^{-\frac{\beta|x_1-y_1|}{\sqrt{t-s+1}+C(\beta)}} e^{u\epsilon(|y_1|)}\Big)\Big(\sum_{y_2\in\Xi(s)} e^{-\frac{\beta|x_2-y_2|}{\sqrt{t-s+1}+C(\beta)}} e^{u\epsilon(|y_2|)}\Big),\\ &\leq C(t-s+1)e^{u\epsilon(|x_1|+|x_2|)}.\end{aligned}$$

Thus,

$$\left\|\sum_{y_1\leq y_2}\nabla_{x_1}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2)\right\|_n \leq \frac{C(\beta,T)}{\sqrt{t-s+1}} e^{u\epsilon(|x_1|+|x_2|)}.$$

Referring to (6.8.19), we conclude (6.8.17).

We turn our attention to prove (6.8.18). With the aid of (6.5.21), one has for $x_1 < x_2 \in \Xi(t)$,

$$\begin{aligned}\mathbb{E}\big[Z_{\nabla,\nabla}(t,x_1,x_2)\big|\mathcal{F}(s)\big] &= \epsilon^{-1}\mathbb{E}\big[\nabla Z(t,x_1)\nabla Z(t,x_2)\big|\mathcal{F}(s)\big],\\ &= \epsilon^{-1}\sum_{y_1\leq y_2\in\Xi(s)}\nabla_{x_1,x_2}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2).\end{aligned} \tag{6.8.20}$$

Note that (6.8.20) does not hold when $x_1 = x_2$ (see Remark 6.8.5 below). Theorem 6.7.1 (c)

implies

$$\left|\nabla_{x_1,x_2}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)\right| \le \frac{C(\beta,T)}{(t-s+1)^2}e^{-\frac{\beta(|x_1-y_1|+|x_2-y_2|)}{\sqrt{t-s+1}+C(\beta)}}.$$

By same argument used in proving (6.8.17), one has

$$\left\|\mathbb{E}\big[Z_{\nabla,\nabla}(t,x_1,x_2)\big|\mathcal{F}(s)\big]\right\|_n \le \frac{C\epsilon^{-1}}{t-s+1}e^{u\epsilon(|x_1|+|x_2|)}.$$

This concludes the proof of the lemma. □

With the help of the preceding lemma, we proceed to prove Lemma 6.8.2.

Proof of Lemma 6.8.2. Referring to (6.8.3), (6.8.4) that

$$\sum_{s=0}^{t}\mathcal{Y}_\nabla(s,x^\star(s)) = \left(\frac{2\rho}{I}-1\right)\sum_{i\in\mathbb{Z}_{\ge1}}\sum_{s=0}^{t}u_\epsilon(s,i)Z_\nabla(s,x^\star(s)-i,x^\star(s)),$$
$$\sum_{s=0}^{t}\mathcal{Y}_{\nabla,\nabla}(s,x^\star(s)) = \sum_{i>j\in\mathbb{Z}_{\ge1}}\sum_{s=0}^{t}u_\epsilon(s,i)u_\epsilon(s,j)Z_{\nabla,\nabla}(s,x-i,x-j).$$

By triangle inequality, one has

$$\left\|\epsilon^2\sum_{s=0}^{t}\mathcal{Y}_\nabla(s,x^\star(s))\right\|_2 \le \left(\frac{2\rho}{I}-1\right)\sum_{i\in\mathbb{Z}_{\ge1}}\left\|\epsilon^2\sum_{s=0}^{t}u_\epsilon(s,i)Z_\nabla(s,x^\star(s)-i,x^\star(s))\right\|_2$$
$$\left\|\epsilon^2\sum_{s=0}^{t}\mathcal{Y}_{\nabla,\nabla}(s,x^\star(s))\right\|_2 \le \sum_{i>j\in\mathbb{Z}_{\ge1}}\left\|\epsilon^2\sum_{s=0}^{t}u_\epsilon(s,i)u_\epsilon(s,j)Z_{\nabla,\nabla}(s,x^\star(s)-i,x^\star(s)-j)\right\|_2.$$

To prove Lemma 6.8.2, it is sufficient to show that there exists constant C,u such that for all $t\in[0,\epsilon^{-2}T]\cap\mathbb{Z}$, $x^\star\in\mathbb{Z}$ and some constant $0<\delta<1$,

$$\left\|\epsilon^2\sum_{s=0}^{t}u_\epsilon(s,i)Z_\nabla(s,x^\star(s)-i,x^\star(s))\right\|_2 \le C\epsilon^{\frac14}e^{u\epsilon(2|x^\star|+i)}\delta^i, \qquad \forall i\in\mathbb{Z}_{\ge0}, \tag{6.8.21}$$

$$\left\|\epsilon^2\sum_{s=0}^{t}u_\epsilon(s,i)u_\epsilon(s,j)Z_{\nabla,\nabla}(s,x^\star(s)-i,x^\star(s)-j)\right\|_2 \le C\epsilon^{\frac14}e^{u\epsilon(2|x^\star|+i+j)}\delta^{i+j}, \qquad \forall i>j\in\mathbb{Z}_{\ge1}. \tag{6.8.22}$$

Note that here, we include $i = 0$ in (6.8.21), which is not needed to prove Lemma 6.8.2. We are going to use this in the proof of Lemma 6.8.3.

We begin with proving (6.8.21), by writing

$$
\begin{aligned}
&\Big\| \sum_{s=0}^{t} u_\epsilon(s,i) Z_\nabla(s, x^\star(s) - i, x^\star(s)) \Big\|_2^2 \\
&= 2 \sum_{0 \le s_1 < s_2 \le t} \mathbb{E}\Big[u_\epsilon(s_1,i) u_\epsilon(s_2,i) Z_\nabla(s_1, x^\star(s_1) - i, x^\star(s_1)) Z_\nabla(s_2, x^\star(s_2) - i, x^\star(s_2)) \Big] \\
&\quad + \sum_{s=0}^{t} \mathbb{E}\big[u_\epsilon(s,i)^2 Z_\nabla(s, x^\star(s) - i, x^\star(s))^2 \big] \\
&= 2 \sum_{0 \le s_1 < s_2 \le t} u_\epsilon(s_1,i) u_\epsilon(s_2,i) \mathbb{E}\Big[Z_\nabla(s_1, x^\star(s_1) - i, x^\star(s_1)) \mathbb{E}\big[Z_\nabla(s_2, x^\star(s_2) - i, x^\star(s_2)) \big| \mathcal{F}(s_1) \big] \Big] \\
&\quad + \sum_{s=0}^{t} u_\epsilon(s,i)^2 \mathbb{E}\big[Z_\nabla(s, x^\star(s) - i, x^\star(s))^2 \big]
\end{aligned}
$$

Using (6.8.12) to bound $u_\epsilon(s,i)$, one has

$$
\begin{aligned}
\Big\| \sum_{s=0}^{t} u_\epsilon(s,i) Z_\nabla(s, x^\star(s) - i, x^\star(s)) \Big\|_2^2 \le C\delta^{2i} \sum_{0 \le s_1 < s_2 \le t} \Big| \mathbb{E}\Big[Z_\nabla(s_1, x^\star(s_1) - i, x^\star(s_1)) \mathbb{E}\big[Z_\nabla(s_2, x^\star(s_2) - i, x^\star(s_2)) \big| \mathcal{F}
\\
+ C\delta^{2i} \sum_{s=0}^{t} \mathbb{E}\big[Z_\nabla(s, x^\star(s) - i, x^\star(s))^2 \big] \qquad (6.8.23)
\end{aligned}
$$

Let us analyze the two terms on the RHS of (6.8.23) respectively. For the first term, via Cauchy-Schwarz inequality $|\mathbb{E}\big[XY\big]| \le \|X\|_2 \|Y\|_2$, one has

$$
\begin{aligned}
&\Big| \mathbb{E}\Big[Z_\nabla(s_1, x^\star(s_1) - i, x^\star(s_1)) \mathbb{E}\big[Z_\nabla(s_2, x^\star(s_2) - i, x^\star(s_2)) \big| \mathcal{F}(s_1) \big] \Big] \Big| \\
&\le \| Z_\nabla(s_1, x^\star(s_1) - i, x^\star(s_1)) \|_2 \| \mathbb{E}\big[Z_\nabla(s_2, x^\star(s_2) - i, x^\star(s_2)) \big| \mathcal{F}(s_1) \big] \|_2
\end{aligned}
$$

By the moment bound in Proposition 6.6.1, we have $\|Z_\nabla(s, x_1, x_2)\|_2 \le Ce^{u\epsilon(|x_1|+|x_2|)}$. Combining this with (6.8.17),

$$\begin{aligned}
&\left|\mathbb{E}\Big[Z_\nabla(s_1, x^\star(s_1)-i, x^\star(s_1))\mathbb{E}\big[Z_\nabla(s_2, x^\star(s_2)-i, x^\star(s_2))\big|\mathcal{F}(s_1)\big]\Big]\right| \\
&\le Ce^{u\epsilon(|x^\star(s_1)-i|+|x^\star(s_1)|)}\frac{\epsilon^{-\frac{1}{2}}}{\sqrt{s_2-s_1+1}}e^{u\epsilon(|x^\star(s_2)-i|+|x^\star(s_2)|)} \\
&\le \frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{s_2-s_1+1}}e^{2u\epsilon(|x^\star|+|x^\star-i|)}.
\end{aligned}$$

Consequently, the first term in (6.8.23) is upper bounded by

$$\begin{aligned}
&\left|\sum_{0\le s_1<s_2\le t}\mathbb{E}\Big[Z_\nabla(s_1, x^\star(s_1)-i, x^\star(s_1))\mathbb{E}\big[Z_\nabla(s_2, x^\star(s_2)-i, x^\star(s_2))\big|\mathcal{F}(s_1)\big]\Big]\right| \\
&\le \sum_{0\le s_1<s_2\le t}\frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{s_2-s_1+1}}e^{2u\epsilon(|x^\star|+|x^\star-i|)} \le C\epsilon^{-\frac{1}{2}}t^{\frac{3}{2}}e^{2u\epsilon(2|x^\star|+i)} \le C\epsilon^{-\frac{7}{2}}e^{2u\epsilon(2|x^\star|+i)}. \qquad (6.8.24)
\end{aligned}$$

where in the second inequality above we used the integral approximation

$$\sum_{0\le s_1<s_2\le t}\frac{1}{\sqrt{s_2-s_1+1}} \le C\int_{0\le s_1\le s_2\le t}\frac{ds_1ds_2}{\sqrt{s_2-s_1}} = Ct^{\frac{3}{2}}$$

and in the last inequality we used $t \le \epsilon^{-2}T$.

Using again $\|Z_\nabla(s,x_1,x_2)\|_2 \le Ce^{u\epsilon(|x_1|+|x_2|)}$, the second term in (6.8.23) is readily upper bounded by

$$\left|\sum_{s=0}^{t}\mathbb{E}\big[Z_\nabla(s, x^\star(s)-i, x^\star(s))^2\big]\right| \le C\sum_{s=0}^{t}e^{2u\epsilon(|x^\star|+|x^\star-i|)} \le C\epsilon^{-2}e^{2u\epsilon(2|x^\star|+i)}. \qquad (6.8.25)$$

Incorporating the bounds (6.8.24) and (6.8.25) into the RHS of (6.8.23), we get (6.8.21).

We proceed to justify (6.8.22), the method is similar to the proof of (6.8.21). Write

$$\left\|\sum_{s=0}^{t}u_\epsilon(s,i)u_\epsilon(s,j)Z_{\nabla,\nabla}(s, x^\star(s)-i, x^\star(s)-j)\right\|_2^2$$

$$
\begin{aligned}
&= 2 \sum_{0 \le s_1 < s_2 \le t} u_\epsilon(s_1, i) u_\epsilon(s_1, j) u_\epsilon(s_2, i) u_\epsilon(s_2, j) \times \\
&\quad \mathbb{E}\Big[Z_{\nabla,\nabla}(s_1, x^\star(s_1) - i, x^\star(s_1) - j) \mathbb{E}\big[Z_{\nabla,\nabla}(s_2, x^\star(s_2) - i, x^\star(s_2) - j) \big| \mathcal{F}(s_1) \big] \Big] \\
&\quad + \sum_{s=0}^{t} u_\epsilon(s, i)^2 u_\epsilon(s, j)^2 \mathbb{E}\big[Z_{\nabla,\nabla}(s, x^\star(s) - i, x^\star(s) - j)^2 \big].
\end{aligned}
$$

Using again (6.8.12), one has

$$
\begin{aligned}
&\Big\| \sum_{s=0}^{t} u_\epsilon(s, i) u_\epsilon(s, j) Z_{\nabla,\nabla}(s, x^\star(s) - i, x^\star(s) - j) \Big\|_2^2 \\
&\le C\delta^{2(i+j)} \sum_{0 \le s_1 < s_2 \le t} \Big| \mathbb{E}\Big[Z_{\nabla,\nabla}(s_1, x^\star(s_1) - i, x^\star(s_1) - j) \mathbb{E}\big[Z_{\nabla,\nabla}(s_2, x^\star(s_2) - i, x^\star(s_2) - j) \big| \mathcal{F}(s_1) \big] \Big] \Big| \\
&\quad + C\delta^{2(i+j)} \sum_{s=0}^{t} \mathbb{E}\big[Z_{\nabla,\nabla}(s, x^\star(s) - i, x^\star(s) - j)^2 \big]. \qquad (6.8.26)
\end{aligned}
$$

Let us analyze the two terms on the RHS of (6.8.26) respectively. For the first term, by Cauchy Schwarz,

$$
\begin{aligned}
&\Big| \mathbb{E}\Big[Z_{\nabla,\nabla}(s_1, x^\star(s_1) - i, x^\star(s_1) - j) \mathbb{E}\big[Z_{\nabla,\nabla}(s_2, x^\star(s_2) - i, x^\star(s_2) - j) \big| \mathcal{F}(s_1) \big] \Big] \Big| \\
&\le \| Z_{\nabla,\nabla}(s_1, x^\star(s_1) - i, x^\star(s_1) - j) \|_2 \| \mathbb{E}\big[Z_{\nabla,\nabla}(s_2, x^\star(s_2) - i, x^\star(s_2) - j) \big| \mathcal{F}(s_1) \big] \|_2
\end{aligned}
$$

Using the bound $\|Z_\nabla(s, x_1, x_2)\|_2 \le Ce^{u\epsilon(|x_1|+|x_2|)}$ and (6.8.18), we have

$$
\begin{aligned}
&\Big| \mathbb{E}\Big[Z_{\nabla,\nabla}(s_1, x^\star(s_1) - i, x^\star(s_1) - j) \mathbb{E}\big[Z_{\nabla,\nabla}(s_2, x^\star(s_2) - i, x^\star(s_2) - j) \big| \mathcal{F}(s_1) \big] \Big] \Big| \\
&\le e^{u\epsilon(|x^\star - i| + |x^\star - j|)} \frac{C\epsilon^{-1}}{s_2 - s_1 + 1} e^{u\epsilon(|x^\star - i| + |x^\star - j|)} = \frac{C\epsilon^{-1}}{s_2 - s_1 + 1} e^{2u\epsilon(|x^\star - i| + |x^\star - j|)}.
\end{aligned}
$$

Therefore,

$$
\sum_{0 \le s_1 < s_2 \le t} \Big| \mathbb{E}\Big[Z_{\nabla,\nabla}(s_1, x^\star(s_1) - i, x^\star(s_1) - j) \mathbb{E}\big[Z_{\nabla,\nabla}(s_2, x^\star(s_2) - i, x^\star(s_2) - j) \big| \mathcal{F}(s_1) \big] \Big] \Big|
$$

$$\leq \sum_{0\leq s_1<s_2\leq t} \frac{C\epsilon^{-1}}{s_2-s_1+1} e^{2u\epsilon(|x^\star-i|+|x^\star-j|)} \tag{6.8.27}$$

$$\leq C\epsilon^{-1}(t+1)\log(t+1)e^{2u\epsilon(|x^\star-i|+|x^\star-j|)} \leq C\epsilon^{-\frac{7}{2}}e^{2u\epsilon(2|x^\star|+i+j)}.$$

In the second inequality above, we used the integral approximation

$$\sum_{0\leq s_1<s_2\leq t} \frac{1}{s_2-s_1+1} \leq C\int_{0\leq s_1\leq s_2\leq t} \frac{1}{s_2-s_1+1} ds_1 ds_2 \leq C(t+1)\log(t+1).$$

For the second term in (6.8.26), it is clear that

$$\sum_{s=0}^{t} \mathbb{E}\big[Z_{\nabla,\nabla}(s,x^\star(s)-i,x^\star(s)-j)^2\big] \leq Cte^{2u\epsilon(2|x^\star|+i+j)} \leq C\epsilon^{-2}e^{2u\epsilon(2|x^\star|+i+j)}. \tag{6.8.28}$$

Incorporating the bounds (6.8.27) and (6.8.28) into the RHS of (6.8.26), we prove the desired (6.8.22). □

Remark 6.8.5. In the argument above, we showed $Z_{\nabla,\nabla}(t,x_1,x_2) = (\epsilon^{-\frac{1}{2}}\nabla Z(t,x_1))(\epsilon^{-\frac{1}{2}}\nabla Z(t,x_2))$ vanishes after averaging over a long time interval when $x_1 \neq x_2$. The readers might wonder whether the same holds for $x_1 = x_2$? The answer is negative. In the case $x_1 \neq x_2$, we used two particle duality (6.5.21) to move the gradient from Z to $\mathbf{V}_\epsilon$

$$\mathbb{E}\big[Z_{\nabla,\nabla}(t,x_1,x_2)\big|\mathcal{F}(s)\big] = \epsilon^{-1}\sum_{y_1\leq y_2\in\Xi(s)} \nabla_{x_1,x_2}\mathbf{V}_\epsilon\big((x_1,x_2),(y_1,y_2),t,s\big)Z(s,y_1)Z(s,y_2).$$

However, if $x_1 = x_2$, the same two particle duality gives instead

$$\mathbb{E}\big[Z_{\nabla,\nabla}(t,x_1,x_2)\big|\mathcal{F}(s)\big]$$
$$= \epsilon^{-1}\sum_{y_1\leq y_2\in\Xi(s)} \Big(\mathbf{V}_\epsilon\big((x_1+1,x_1+1),(y_1,y_2),t,s\big) - 2\mathbf{V}_\epsilon\big((x_1,x_1+1),(y_1,y_2),t,s\big)+1\Big)Z(s,y_1)Z(s,y_2).$$

The same argument fails because we do not have an effective estimate of

$$\mathbf{V}_\epsilon\big((x_1+1,x_1+1),(y_1,y_2),t,s\big)-2\mathbf{V}_\epsilon\big((x_1,x_1+1),(y_1,y_2),t,s\big)+1.$$

In fact, when $x_1=x_2$, $Z_{\nabla,\nabla}(t,x_1,x_2)$ does not vanish after averaging. One quick way to see this is to use

$$\begin{aligned}Z_{\nabla,\nabla}(t,x_1,x_1) &= \big(\epsilon^{-\frac{1}{2}}\nabla Z(t,x_1)\big)^2\\ &= (\widetilde{\eta}_{x_1+1}(t)-\rho)^2 Z(t,x_1)^2+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon Z(t,x_1)^2\\ &\geq \min\big(1-\{\rho\},\{\rho\}\big)^2 Z(t,x_1)^2+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon Z(t,x_1)^2\end{aligned}$$

where $\{\rho\}$ represents the fractional part of ρ. This implies that $Z_{\nabla,\nabla}(t,x,x)$ is lower bounded by a constant times $Z(t,x)^2$, which does not vanish after averaging.

6.8.3 Proof of Lemma 6.8.3

The aim of this section is to justify Lemma 6.8.3, which indicates that $Z_{\nabla,\nabla}(t,x,x)-\frac{\rho(I-\rho)}{I}Z(t,x)^2$ vanishes after averaging over a long time interval. This was proved for the stochastic six vertex model [CGST20] (which corresponds to $I=1, J=1$). Note that when $I=1$, for all t,x one has $\widetilde{\eta}_x(t)\in\{0,1\}$, which yields $\widetilde{\eta}_x(t)^2=\widetilde{\eta}_x(t)$. [CGST20] utilizes this crucial observation to show that

$$\begin{aligned}Z_{\nabla,\nabla}(t,x,x) &= (\widetilde{\eta}_{x+1}(t)-\rho)^2 Z(t,x)^2+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2,\\ &= \rho^2 Z(t,x)^2+(1-2\rho)\widetilde{\eta}_{x+1}(t)Z(t,x)^2,\\ &= \rho(1-\rho)Z(t,x)^2+(2\rho-1)Z_\nabla(t,x,x)+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(t,x)Z(t,x)^2,\end{aligned}$$

where in the last line above, we used (6.8.15). We have seen in the previous section that $Z_\nabla(t,x,x)$ vanishes after averaging, which implies that $Z_{\nabla,\nabla}(t,x,x)-\rho(1-\rho)Z(t,x)^2$ will also vanish.

When $I \geq 2$, $\widetilde{\eta}_x(t)$ can takes more than two values so the $\widetilde{\eta}_x(t)^2 = \widetilde{\eta}_x(t)$ relation no longer holds. Notice that in the proof of Lemma 6.8.2, we have only leveraged the first duality (6.5.21) in the Lemma 6.5.2. To conclude Lemma 6.8.3, we will combine both of the dualities (6.5.21) and (6.5.22).

Before moving to the proof, we first offer a heuristic argument to explain why the $\lambda = \frac{\rho(I-\rho)}{I}$ is the value which makes $Z_{\nabla,\nabla}(t,x,x) - \lambda Z(t,x)^2$ vanish after averaging.

Heuristic argument for Lemma 6.8.3. Note that

$$Z_{\nabla,\nabla}(t,x,x) = (\widetilde{\eta}_{x+1}(t) - \rho)^2 Z(t,x)^2 + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2.$$

In Theorem C.0.3, we find that the stationary distribution of the (bi-infinite) SHS6V model is given by $\bigotimes \pi_\rho$, where π_ρ is defined in (C.0.1). It is straightforward to verify that $\bigotimes \pi_\rho$ is near stationary with density ρ (Definition 6.5.5). Start the SHS6V model from $\vec{\eta}(0) \sim \bigotimes \pi_\rho$, by stationarity $\eta_x(t) \sim \pi_\rho$ for all $t \in \mathbb{Z}_{\geq 0}$ and $x \in \mathbb{Z}$. Heuristically, we can approximate $(\widetilde{\eta}_{x+1}(t) - \rho)^2 Z(t,x)^2$ by $\mathbb{E}_{\pi_\rho}\big[(\widetilde{\eta}_{x+1}(t) - \rho)^2\big] Z(t,x)^2$. Note that

$$\mathbb{E}_{\pi_\rho}\big[(\widetilde{\eta}_{x+1}(t) - \rho)^2\big] Z(t,x)^2 = \mathrm{Var}\big[\pi_\rho\big] Z(t,x)^2$$

where $\mathrm{Var}\big[\pi_\rho\big]$ represents the variance of the probability distribution π_ρ. Referring to Lemma C.0.2, we have

$$\mathrm{Var}\big[\pi_\rho\big] = \rho - \sum_{i=1}^{I} \frac{\chi^2}{(q^i - \chi)^2}.$$

where χ is the unique negative real number satisfying $\sum_{i=1}^{I} \frac{\chi}{\chi - q^i} = \rho$. It is straightforward that under weak asymmetric scaling (6.5.30), one has $\lim_{\epsilon \downarrow 0} \chi_\epsilon = \frac{\rho}{\rho - I}$. Therefore,

$$\lim_{\epsilon \downarrow 0} \mathrm{Var}\big[\pi_\rho\big] = \frac{\rho(I-\rho)}{I},$$

which explains $\lambda = \frac{\rho(I-\rho)}{I}$. □

We proceed to prove Lemma 6.8.3 rigorously. The first step is to express $Z_{\nabla,\nabla}(t,x,x) - \frac{\rho(I-\rho)}{I} Z(t,x)^2$ in terms of the two duality functionals in Lemma 6.5.2,

$$
\begin{aligned}
&Z_{\nabla,\nabla}(t,x,x) - \frac{\rho(I-\rho)}{I} Z(t,x)^2 \\
&= \left((\widetilde{\eta}_{x+1}(t) - \rho)^2 - \frac{\rho(I-\rho)}{I} \right) Z(t,x)^2 + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2 \\
&= \left((I - \widetilde{\eta}_{x+1}(t))(I - 1 - \widetilde{\eta}_{x+1}(t)) - \frac{(I-1)(I-\rho)^2}{I} \right) Z(t,x)^2 - (2\rho + 1 - 2I) Z_\nabla(t,x,x) + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2 \\
&= \left((I - \widetilde{\eta}_{x+1}(t))(I - 1 - \widetilde{\eta}_{x+1}(t)) - \frac{(I-1)(I-\rho)^2}{I} \right) Z(t,x+1)^2 - (2\rho + 1 - 2I) Z_\nabla(t,x,x) \\
&\quad + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2 \qquad (6.8.29)
\end{aligned}
$$

In the last equality, we replaced $Z(t,x)$ by $Z(t,x+1)$, according to (6.8.16), this procedure produces an error term which can be absorbed in the $\epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2$.

Recall that $[n]_{q^{\frac{1}{2}}} = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$. Under weak asymmetric scaling, $q = e^{\sqrt{\epsilon}}$, one has

$$
[n]_{q^{\frac{1}{2}}} = n + O(\epsilon^{\frac{1}{2}}), \qquad q^{\eta_x(t)} = 1 + O(\epsilon^{\frac{1}{2}}). \qquad (6.8.30)
$$

These approximations imply that

$$
\begin{aligned}
&(I - \widetilde{\eta}_{x+1}(t))(I - 1 - \widetilde{\eta}_{x+1}(t)) Z(t,x+1)^2 \\
&= [I - \widetilde{\eta}_{x+1}(t)]_{q^{\frac{1}{2}}} [I - 1 - \widetilde{\eta}_{x+1}(t)]_{q^{\frac{1}{2}}} Z(t,x+1)^2 q^{\widetilde{\eta}_{x+1}(t)} + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2, \\
&= D(t, x+1, x+1) + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2. \qquad (6.8.31)
\end{aligned}
$$

where we recall the expression of the functional D from (6.5.19). Inserting (6.8.31) into the RHS of (6.8.29)

$$
Z_{\nabla,\nabla}(t,x,x) - \frac{\rho(I-\rho)}{I} Z(t,x)^2
$$

$$= D(t,x+1,x+1) - \frac{(I-1)(I-\rho)^2}{I} Z(t,x+1)^2 - (2\rho+1-2I) Z_\nabla(t,x,x) + \epsilon^{\frac{1}{2}} \mathcal{B}_\epsilon(t,x) Z(t,x)^2 \tag{6.8.32}$$

Recall that our goal is to show

$$\left\| \epsilon^2 \sum_{s=0}^{t} \widetilde{\mathcal{Y}}(s, x^\star(s)) \right\|_2 \leq C\epsilon^{\frac{1}{4}} e^{2u\epsilon|x^\star|}$$

Referring to the expression of $\widetilde{\mathcal{Y}}(s, x^\star(s))$ in (6.8.5), we need to prove that there exists some $0 < \delta < 1$ such that for all $i \in \mathbb{Z}_{\geq 1}$,

$$\left\| \epsilon^2 \sum_{s=0}^{t} u_\epsilon(s,i) \Big(Z_{\nabla,\nabla}(s, x^\star(s)-i, x^\star(s)-i) - \frac{\rho(I-\rho)}{I} Z(s, x^\star(s)-i)^2 \Big) \right\|_2 \leq C\epsilon^{\frac{1}{4}} e^{2u\epsilon|x^\star|} \delta^i.$$

Using (6.8.32), it suffices to show that for all $i \in \mathbb{Z}_{\geq 1}$,

$$\left\| \sum_{s=0}^{t} u_\epsilon(s,i) \Big(D(s, x^\star(s)+1-i, x^\star(s)+1-i) - \frac{(I-1)(I-\rho)^2}{I} Z(s, x^\star(s)+1-i)^2 \Big) \right\|_2 \leq C\epsilon^{\frac{1}{4}} e^{2u\epsilon|x^\star|} \delta^i. \tag{6.8.33}$$

and

$$\left\| \sum_{s=0}^{t} u_\epsilon(s,i) Z_\nabla(s, x^\star(s), x^\star(s)) \right\|_2 \leq C\epsilon^{\frac{1}{4}} e^{2u\epsilon|x^\star|} \delta^i. \tag{6.8.34}$$

Note that (6.8.34) is proved by taking $i = 0$ in (6.8.21). Therefore, we only need to prove (6.8.33). Similar to the proof in Lemma 6.8.2, to conclude (6.8.33), it suffices to prove the following lemma for upper bounding the conditional expectation. We do not repeat the rest of the proof here.

Lemma 6.8.6. *For $T > 0$ and $n \in \mathbb{Z}_{\geq 1}$, there exists constant C and u such that for all $x \in \Xi(t)$ and $s \leq t \in [0, \epsilon^{-2}T] \cap \mathbb{Z}$,*

$$\left\| \mathbb{E}\left[D(t,x,x) - \frac{(I-1)(I-\rho)^2}{I} Z(t,x)^2 \middle| \mathcal{F}(s) \right] \right\|_n \leq \frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{t-s+1}} e^{2u\epsilon|x|}. \tag{6.8.35}$$

Proof. Combining both of the dualities (6.5.21) and (6.5.22), one has

$$\mathbb{E}\Big[D(t,x,x) - \frac{(I-1)(I-\rho)^2}{I} Z(t,x)^2 \Big| \mathcal{F}(s)\Big] = \sum_{y_1 \le y_2 \in \Xi(s)} \mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\Big(D(s,y_1,y_2) - \frac{(I-1)(I-\rho)^2}{I} Z(t,y_1)Z(t,y_2)\Big)$$

We split the summation above according to the range of the value of $|y_1 - y_2|$,

$$\begin{aligned}
&\mathbb{E}\Big[D(t,x,x) - \frac{(I-1)(I-\rho)^2}{I} Z(t,x)^2 \Big| \mathcal{F}(s)\Big] \\
&= \sum_{\substack{y_1 < y_2 \in \Xi(s) \\ |y_1 - y_2| \ge 3}} \mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\Big(D(s,y_1,y_2) - \frac{(I-1)(I-\rho)^2}{I} Z(s,y_1)Z(s,y_2)\Big) \\
&\quad + \sum_{\substack{y_1 \le y_2 \in \Xi(s) \\ |y_1 - y_2| \le 2}} \mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\Big(D(s,y_1,y_2) - \frac{(I-1)(I-\rho)^2}{I} Z(s,y_1)Z(s,y_2)\Big).
\end{aligned} \tag{6.8.36}$$

We name the terms on the RHS of (6.8.36) by $\mathbf{E_1}$ and $\mathbf{E_2}$ respectively and we bound them separately. It follows from Proposition 6.6.1 that

$$\Big\| D(s,y_1,y_2) - \frac{(I-1)(I-\rho)^2}{I} Z(s,y_1)Z(s,y_2) \Big\|_n \le C e^{u\epsilon(|y_1| + |y_2|)}$$

Invoking Theorem 6.7.1 (a) and Lemma 6.6.3, we find that

$$\|\mathbf{E_2}\|_n \le \sum_{\substack{y_1 \le y_2 \in \Xi(s) \\ |y_1 - y_2| \le 2}} \frac{C(\beta,T)}{t-s+1} e^{\frac{-\beta(|y_1 - x| + |y_2 - x|)}{\sqrt{t-s+1} + C(\beta)}} e^{u\epsilon(|y_1| + |y_2|)} \le \frac{C}{\sqrt{t-s+1}} e^{2u\epsilon|x|}. \tag{6.8.37}$$

We proceed to bound $\mathbf{E_1}$, recall that when $y_1 < y_2$,

$$D(s,y_1,y_2) = \frac{[I-1]_{q^{\frac{1}{2}}}}{[I]_{q^{\frac{1}{2}}}} Z(s,y_1)Z(s,y_2)[I - \widetilde{\eta}_{y_1}(s)]_{q^{\frac{1}{2}}} [I - \widetilde{\eta}_{y_2}(s)]_{q^{\frac{1}{2}}} q^{\frac{1}{2}\widetilde{\eta}_{y_1}(s)} q^{\frac{1}{2}\widetilde{\eta}_{y_2}(s)},$$

which could be rewritten as (using (6.8.30))

$$D(s,y_1,y_2)=\frac{I-1}{I}(I-\widetilde{\eta}_{y_1}(s))(I-\widetilde{\eta}_{y_2}(s))Z(s,y_1)Z(s,y_2)+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(s,y_1,y_2)Z(s,y_1)Z(s,y_2).$$

Consequently, we write

$$\begin{aligned}
\mathbf{E_1}&=\frac{I-1}{I}\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|\geq 3}}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\Big((I-\widetilde{\eta}_{y_1}(s))(I-\widetilde{\eta}_{y_2}(s))-(I-\rho)^2\Big)Z(s,y_1)Z(s,y_2)\\
&\quad+\epsilon^{\frac{1}{2}}\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|\geq 3}}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\mathcal{B}_\epsilon(s,y_1,y_2)Z(s,y_1)Z(s,y_2)\\
&=\frac{I-1}{I}\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|\geq 3}}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\Big((\rho-\widetilde{\eta}_{y_1}(s))(I-\widetilde{\eta}_{y_2}(s))+(I-\rho)(\rho-\widetilde{\eta}_{y_2}(s))\Big)Z(s,y_1)Z(s,y_2)\\
&\quad+\epsilon^{\frac{1}{2}}\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|\geq 3}}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\mathcal{B}_\epsilon(s,y_1,y_2)Z(s,y_1)Z(s,y_2) \qquad (6.8.38)
\end{aligned}$$

It Is straightforward by (6.8.15) and (6.8.16) that

$$\begin{aligned}
&(\rho-\widetilde{\eta}_{y_1}(s))Z(s,y_1)\\
&=(\rho-\widetilde{\eta}_{y_1}(s))Z(s,y_1-1)+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(s,y_1)Z(s,y_1)=\epsilon^{-\frac{1}{2}}\nabla Z(s,y_1-1)+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(s,y_1)Z(s,y_1),\\
&(\rho-\widetilde{\eta}_{y_2}(s))Z(s,y_2)\\
&=(\rho-\widetilde{\eta}_{y_2}(s))Z(s,y_2-1)+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(s,y_2)Z(s,y_2)=\epsilon^{-\frac{1}{2}}\nabla Z(s,y_2-1)+\epsilon^{\frac{1}{2}}\mathcal{B}_\epsilon(s,y_2)Z(s,y_2).
\end{aligned}$$

Inserting these into the RHS of (6.8.38),

$$\begin{aligned}
\mathbf{E_1}=&\frac{I-1}{I}\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|>2}}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)(I-\widetilde{\eta}_{y_2}(s))(\epsilon^{-\frac{1}{2}}\nabla Z(s,y_1))Z(s,y_2)\\
&+\frac{I-1}{I}\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|>2}}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)(I-\rho)(\epsilon^{-\frac{1}{2}}\nabla Z(s,y_2))Z(s,y_1)
\end{aligned}$$

$$+\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|>2}} \epsilon^{\frac{1}{2}}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)\mathscr{B}_\epsilon(s,y_1,y_2)Z(s,y_1)Z(s,y_2).$$

Let us name respectively the three terms on the RHS above to be $\mathbf{J_1}$, $\mathbf{J_2}$, $\mathbf{J_3}$. Recall the summation by part formula (with notation $\nabla f(x) = f(x+1) - f(x)$)

$$\sum_{x<y} \nabla f(x)\cdot g(x) = f(y)g(y-1) - \sum_{x<y} f(x)\cdot \nabla g(x-1), \tag{6.8.39}$$

$$\sum_{x>y} \nabla f(x)\cdot g(x) = -f(y+1)g(y+1) - \sum_{x>y} f(x+1)\nabla g(x).$$

Note that

$$\mathbf{J_1} = \frac{I-1}{I}\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|>2}} \mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)(I-\widetilde{\eta}_{y_2}(s))(\epsilon^{-\frac{1}{2}}\nabla Z(s,y_1))Z(s,y_2),$$

by (6.8.39), we move the gradient from $\nabla Z(s,y_1)$ to $\mathbf{V}_\epsilon$,

$$\mathbf{J_1} = \frac{I-1}{I}\bigg[\sum_{y_2\in\Xi(s)} \epsilon^{-\frac{1}{2}}\mathbf{V}_\epsilon\big((x,x),(y_2-3,y_2),t,s\big)(I-\widetilde{\eta}_{y_2}(s))Z(s,y_2-3)Z(s,y_2)$$

$$-\sum_{\substack{y_1<y_2\in\Xi(s)\\|y_1-y_2|>2}} \epsilon^{-\frac{1}{2}}\nabla_{y_1}\mathbf{V}_\epsilon\big((x,x),(y_1,y_2),t,s\big)(I-\widetilde{\eta}_{y_2}(s))Z(s,y_1)Z(s,y_2)\bigg].$$

Using Theorem 6.7.1 part (a) and part (b) to control $\mathbf{V}_\epsilon$ and $\nabla\mathbf{V}_\epsilon$ respectively, we see that for $n\in\mathbb{Z}_{\geq 1}$,

$$\|\mathbf{J_1}\|_n \leq C(\beta,T)\bigg(\sum_{y_2\in\Xi(s)} \frac{\epsilon^{-\frac{1}{2}}}{t-s+1}e^{\frac{-\beta(|y_2-x|+|y_2-3-x|)}{\sqrt{t-s+1}+C(\beta)}}e^{u\epsilon(|y_2-3|+|y_2|)}$$

$$+\sum_{y_1\leq y_2\in\Xi(s)} \frac{\epsilon^{-\frac{1}{2}}}{(t-s+1)^{\frac{3}{2}}}e^{-\frac{\beta(|y_1-x_1|+|y_2-x_2|)}{\sqrt{t-s+1}+C(\beta)}}e^{u\epsilon(|y_1|+|y_2|)}\bigg).$$

Applying Lemma 6.6.3 yields $\|\mathbf{J_1}\|_n \leq \frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{t-s+1}}e^{2u\epsilon|x|}$. Likewise, we obtain $\|\mathbf{J_2}\|_n \leq \frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{t-s+1}}e^{2u\epsilon|x|}$.

For $\mathbf{J_3}$, applying Theorem 6.7.1 part (a) and Lemma 6.6.3 implies that

$$\|\mathbf{J_3}\|_n \leq \sum_{y_1 \leq y_2} \frac{C(\beta, T)\epsilon^{\frac{1}{2}}}{t-s+1} e^{-\frac{\beta(|x-y_1|+|x-y_2|)}{\sqrt{t-s+1}+C(\beta)}} e^{u\epsilon(|y_1|+|y_2|)} \leq C\epsilon^{\frac{1}{2}} e^{2u\epsilon|x|} \leq \frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{t-s+1}} e^{2u\epsilon|x|}.$$

In the last inequality above, we used the fact $s \leq t \in [0, \epsilon^{-2}T]$, which implies $t - s \leq \epsilon^{-2}T$.

Combining the bounds for $\|\mathbf{J_1}\|_n$, $\|\mathbf{J_2}\|_n$, $\|\mathbf{J_3}\|_n$, we have

$$\|\mathbf{E_1}\|_n \leq \frac{C\epsilon^{-\frac{1}{2}}}{\sqrt{t-s+1}} e^{2u\epsilon|x|}. \tag{6.8.40}$$

Recall from (6.8.36) that

$$\mathbb{E}\left[D(t,x,x) - \frac{(I-1)(I-\rho)^2}{I} Z(t,x)^2 \middle| \mathcal{F}(s)\right] = \mathbf{E_1} + \mathbf{E_2},$$

combining the bounds for $\mathbf{E_1}$ and $\mathbf{E_2}$ in (6.8.40) and (6.8.37), we conclude the desired (6.8.35). □

Chapter 7: The stochastic telegraph equation limit of the stochastic higher spin six vertex model

Chapter Abstract: In this paper, we prove that the stochastic telegraph equation arises as a scaling limit of the stochastic higher spin six vertex (SHS6V) model with general spin $I/2$,$J/2$. This extends results of Borodin and Gorin which focused on the $I = J = 1$ six vertex case and demonstrates the universality of the stochastic telegraph equation in this context. We also provide a functional extension of the central limit theorem obtained in [Borodin and Gorin 2019, Theorem 6.1].

This chapter is published at [Lin20c].

7.1 Introduction

7.1.1 Telegraph equation and stochastic telegraph equation

The *telegraph equation* is a hyperbolic PDE given by

$$\begin{cases} u_{XY}(X,Y) + \beta_1 u_Y(X,Y) + \beta_2 u_X(X,Y) = f(X,Y), \\ u(X,0) = \chi(X), \quad u(0,Y) = \psi(Y), \end{cases} \tag{7.1.1}$$

where the functions $\chi, \psi \in C^1$ satisfy $\chi(0) = \psi(0)$. When f is a deterministic function, the equation (7.1.1) is a classical object, see [CH08, Chapter V]. The stochastic versions of the telegraph equation were intensively studied in the last 50 years, we refer the reader to [BG19, Section 1.1] for a brief review. The solution theory of the telegraph equation goes back to [CH08], we present it in the way of [BG19, Section 4]. In fact, (7.1.1) admits a unique solution which reads

$$u(X,Y) = \psi(0)\mathcal{R}(X,Y,0,0) + \int_0^Y \mathcal{R}(X,Y;0,y)\big(\psi'(y) + \beta_2\psi(y)\big)dy + \\ + \int_0^X \mathcal{R}(X,Y;x,0)\big(\chi'(x) + \beta_1\chi(x)\big)dx \int_0^X \int_0^Y \mathcal{R}(X,Y,x,y)f(x,y)dxdy. \quad (7.1.2)$$

Here, $\mathcal{R}(X,Y,x,y)$ is the *Riemann function* defined as

$$\mathcal{R}(X,Y;x,y) = \frac{1}{2\pi\mathbf{i}}\oint_{-\beta_1} \frac{\beta_2-\beta_1}{(z+\beta_1)(z+\beta_2)}\exp\left[(\beta_1-\beta_2)\Big(-(X-x)\frac{z}{z+\beta_2} + (Y-y)\frac{z}{z+\beta_1}\Big)\right]dz, \quad (7.1.3)$$

where the contour of the complex integration is a small circle in positive direction which only includes the pole at $-\beta_1$. When f is given by $f(X,Y) = \sqrt{\theta(X,Y)}\eta(X,Y)$, where η is the space-time white noise with dirac delta correlation function and θ is a deterministic integrable function. By formula (7.1.2), the solution to the *stochastic telegraph equation* is a Gaussian field with covariance function

$$\operatorname{Cov}\big(u(X_1,Y_1),u(X_2,Y_2)\big) = \int_0^{X_1\wedge Y_1}\int_0^{X_2\wedge Y_2} \mathcal{R}(X_1,Y_1,x,y)\mathcal{R}(X_2,Y_2,x,y)\theta(x,y)dxdy. \quad (7.1.4)$$

[BG19, Section 4] identifies the following discretization of the telegraph equation

$$\begin{cases} \Phi(X+1,Y+1) - b_1\Phi(X,Y+1) - b_2\Phi(X+1,Y) + (b_1+b_2-1)\Phi(X,Y) = g(X+1,Y+1), \\ \Phi(X,0) = \chi(X), \quad \Phi(0,Y) = \psi(Y), \end{cases} \quad (7.1.5)$$

where $\chi(0) = \psi(0)$. The unique solution to (7.1.5) is given by [BG19, Theorem 4.7]:

$$\Phi(X,Y) = \psi(0)\mathcal{R}^d(X,Y;0,0) + \sum_{y=1}^{Y}\mathcal{R}^d(X,Y;0,y)\big(\psi(y) - b_2\psi(y-1)\big) \\ + \sum_{x=1}^{X}\mathcal{R}^d(X,Y;x,0)\big(\chi(x) - b_1\chi(x-1)\big) + \sum_{x=1}^{X}\sum_{y=1}^{Y}\mathcal{R}^d(X,Y;x,y)g(x,y). \quad (7.1.6)$$

where the discrete Riemann function $\mathcal{R}^d$ equals (see [BG19, Eq. 45])

$$\mathcal{R}^d(X,Y;x,y) = \frac{1}{2\pi \mathrm{i}} \oint_{-\frac{1}{b_2(1-b_1)}} \frac{(b_2-b_1)dz}{(1+b_2(1-b_1)z)(1+b_1(1-b_2)z)} \times \left(\frac{1+b_1(1-b_1)z}{1+b_2(1-b_1)z}\right)^{X-x} \left(\frac{1+b_2(1-b_2)z}{1+b_1(1-b_2)z}\right)^{Y-y}. \tag{7.1.7}$$

Here, the contour is a small circle going in positive direction which only encircles the pole at $-\frac{1}{b_2(1-b_1)}$.

In the first version of the arxiv paper [BG18], Borodin and Gorin showed that under a special scaling regime where the weight of the corner type vertex goes to zero, the height function of the stochastic six vertex model converges to the telegraph equation. They also conjectured that the fluctuation field will converge to the stochastic telegraph equation with some heuristic arguments and proved this result under a special situation called *low density boundary regime*. The result for general boundary condition was later proved in [ST19] and [BG19] via two distinct approaches. This result comes as a surprise. Since from [GS92, BCG16] we know that the stochastic six vertex model belongs to the KPZ universality class. The one point fluctuation of the models in this universality is governed by Tracy Widom distribution [TW94]. However, the solution to the stochastic telegraph equation does not lie in this universality (since it is a Gaussian field). In addition,[CGST20] shows that under weakly asymmetric scaling (which is a different scaling compared with the one in [BG19]), the stochastic six vertex model converges to the KPZ equation [KPZ86, Cor12], which is a parabolic stochastic PDE while the stochastic telegraph equation is hyperbolic!

It is natural to ask if the stochastic telegraph equation also arises as a scaling limit of other probabilistic models. In this paper, we show that the stochastic higher spin six vertex (SHS6V) model, which is a higher spin generalization of the stochastic six vertex model, converges to the stochastic telegraph equation under certain scaling regime. This extends the universality of the stochastic telegraph equation. In addition, [Lin20a] showed that under a different scaling than the one considered in this paper, the SHS6V model converges to the KPZ equation. This tells us that the SHS6V model converges to two distinct types of stochastic PDE under various choice of scaling.

7.1.2 The SHS6V model

The SHS6V model is a four-parameter family of quantum integrable system first introduced in [CP16] and has been intensely studied in recent years, from the perspective of symmetric polynomial [Bor17, Bor18], exact solvability [BCPS15, CP16, BP18], Markov duality [CP16, Kua18, Lin19] and scaling limit [CT17, IMS20, Lin20a, DR20]. In particular, it is a higher spin generalization of stochastic six vertex model from spin parameter $I = J = 1$ to general $I, J \in \mathbb{Z}_{\geq 1}$. In this paper, we discover a scaling regime for the SHS6V model (which degenerates to the scaling in [BG19] when $I = J = 1$), under which we prove that: 1) the hydrodynamic limit of the SHS6V model is a telegraph equation; 2) the fluctuation field of the model converges to a stochastic telegraph equation. To explain our result with more detail, we start with a brief review of the SHS6V model.

Definition 7.1.1 ($J = 1$ $\mathbb{L}$-matrix)**.** We define the $J = 1$ $\mathbb{L}$-matrix to be a matrix with row and column indexed by $\mathbb{Z}_{\geq 0} \times \{0, 1\}$. The element of the $J = 1$ $\mathbb{L}$-matrix is specified by

$$L_\alpha^{(1)}(m, 0; m, 0) = \frac{1 + \alpha q^m}{1 + \alpha}, \quad L_\alpha^{(1)}(m, 0; m - 1, 1) = \frac{\alpha(1 - q^m)}{1 + \alpha},$$
$$L_\alpha^{(1)}(m, 1; m, 1) = \frac{\alpha + \nu q^m}{1 + \alpha}, \quad L_\alpha^{(1)}(m, 1; m + 1, 0) = \frac{1 - \nu q^m}{1 + \alpha}$$

and $L_\alpha^{(1)}(i_1, j_1; i_2, j_2) = 0$ for all other values of (i_1, j_1), $(i_2, j_2) \in \mathbb{Z}_{\geq 0} \times \{0, 1\}$. As a convention, throughout the paper, we set $\nu = q^{-I}$ for some fixed $I \in \mathbb{Z}_{\geq 1}$. Note that $L_\alpha^{(1)}(I, 1; I + 1, 0) = 0$, hence the $J = 1$ $\mathbb{L}$-matrix transfers the subspace $\{0, 1, \ldots, I\} \times \{0, 1\}$ to itself and we will restrict ourselves on this subspace.

We call α the *spectral parameter* and in the notation of $L_\alpha^{(1)}$, where the dependence on other parameters is not made explicit. It is clear from the definition that for fixed $i_1 \in \{0, 1, \ldots, I\}$ and

$j_1 \in \{0,1\}$,

$$\sum_{(i_2,j_2)\in\{0,1,\ldots,I\}\times\{0,1\}} L_\alpha^{(1)}(i_1,j_1;i_2,j_2) = 1.$$

Moreover, $L_\alpha^{(1)}$ is stochastic if we impose the following condition.

Lemma 7.1.2. *$L_\alpha^{(1)}$ is stochastic if one of the following holds:*

- $q \in (0,1)$ *and* $\alpha < -q^{-I}$,
- $q > 1$ *and* $-q^{-I} < \alpha < 0$.

Proof. This follows from [CP16, Proposition 2.3], which can also be verified directly. □

For an entry $L_\alpha^{(1)}(i_1,j_1;i_2,j_2)$, we interpret the four tuple (i_1,j_1,i_2,j_2) as a vertex configuration in the sense that a vertex is associated with i_1 input lines and j_1 input lines coming from bottom and left, i_2 output lines and j_2 output lines flowing to above and right, see Figure 7.1. The quantity $L_\alpha^{(1)}(i_1,i_2;j_1,j_2)$ gives the weight of the vertex configuration. Note that for a vertex associated with $L_\alpha^{(1)}$, we allow up to I number of vertical lines and up to one horizontal line. We say that the $\mathbb{L}$-matrix is conservative in lines as it assigns zero weight to the entry $L_\alpha^{(1)}(i_1,j_1;i_2,j_2)$ unless $i_1+j_1 = i_2+j_2$.

We want to relax the restriction that the multiplicities of the horizontal line are bounded by 1, and instead, consider multiplicities bounded by any fixed J. This motivates us to define the $L_\alpha^{(J)}$ matrix, the construction of it follows the so-called fusion procedure, which was invented in a representation-theoretic context [KRS81, KR87] to produce higher-dimensional solutions of the Yang–Baxter equation from lower-dimensional ones. The explicit expression of general J $\mathbb{L}$-matrix is derived separately in [Man14] and [CP16]:

$$L_\alpha^{(J)}(i_1,j_1;i_2,j_2) = \mathbf{1}_{\{i_1+j_1=i_2+j_2\}} q^{\frac{2j_1-j_1^2}{4} - \frac{2j_2-j_2^2}{4} + \frac{i_2^2+i_1^2}{4} + \frac{i_2(j_2-1)+i_1 j_1}{2}} \times \frac{\nu^{j_1-i_2}\alpha^{j_2-j_1+i_2}(-\alpha\nu^{-1};q)_{j_2-i_1}}{(q;q)_{i_2}(-\alpha;q)_{i_2+j_2}(q^{J+1-j_1};q)_{j_1-j_2}} {}_4\overline{\phi}_3\left(\begin{matrix} q^{-i_2}; q^{-i_1}, -\alpha q^J, -q\nu\alpha^{-1} \\ \nu, q^{1+j_2-i_1}, q^{J+1-i_2-j_2} \end{matrix}\middle| q, q\right). \tag{7.1.8}$$

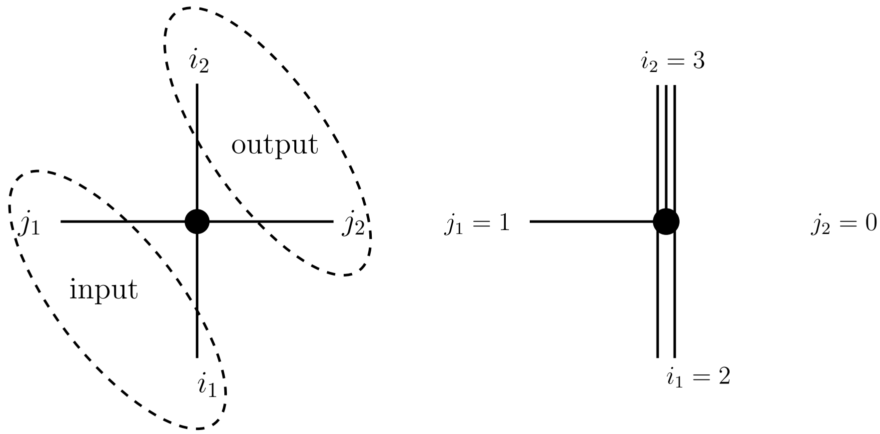


Figure 7.1: Left panel: The vertex configuration labeled by four tuple $(i_1, j_1; i_2, j_2)$ (from bottom and then in the clockwise order) has weight $L^{(1)}_\alpha(i_1, j_1; i_2, j_2)$, which absorbs $i_1 \in \{0, 1, \dots, I\}$ input lines from bottom, $j_1 \in \{0, 1\}$ input line from left, and produces $i_2 \in \{0, 1, \dots, I\}$ output lines to above, $j_2 \in \{0, 1\}$ output lines to right. Right panel: Visualization of the vertex configuration $(i_1, j_1; i_2, j_2) = (2, 1; 3, 0)$ in terms of lines.

Here, ${}_4\overline{\phi}_3$ is the regularized terminating basic hyper-geometric series defined by

$$ {}_{r+1}\overline{\phi}_r\left(\begin{matrix} q^{-n}, a_1, \dots, a_r \\ b, \dots, b_r \end{matrix}\middle| q, z\right) = \sum_{k=0}^{n} z^k \frac{(q^{-n}; q)_k}{(q; q)_k} \prod_{i=1}^{r} (a_i; q)_k (b_i q^k; q)_{n-k}. $$

It is a simple exercise to see when $J = 1$, the expression of $L^{(J)}_\alpha$ matches with $L^{(1)}_\alpha$ in Definition 7.1.1. We will show momentarily that $L^{(J)}_\alpha$ is stochastic (Corollary 7.1.4). This allows us to view the matrix element $L^{(J)}_\alpha(i_1, j_1; i_2, j_2)$ as a vertex configuration in the manner that we described in $J = 1$ case. Note that now we allow up to J lines in the horizontal direction.

Despite explicitness, the expression of the $\mathbb{L}$-matrix above is too complicated to manipulate. For instance, using (7.1.8) directly, it might be hard to demonstrate the stochasticity of $L^{(J)}_\alpha$. To this end, we recall a probabilistic derivation of $L^{(J)}_\alpha$ in [CP16] using the idea of fusion, which goes back to [KR87]. We start by introducing a few notations.

Define the stochastic matrix Ξ with rows and columns indexed by $\{0, 1\}^{\otimes J}$ and $\{0, 1, \dots, J\}$ such that

$$\Xi\big((h_1,\dots,h_J),h\big)=\begin{cases}1 & \text{if } h=\sum_{i=1}^J h_i\\ 0 & \text{else}\end{cases}$$

and the stochastic matrix Λ with row and column indexed by $\{0,1,\dots,J\}$ and $\{0,1\}^{\otimes J}$. The matrix element is given by

$$\Lambda\big(h,(h_1,\dots,h_J)\big)=\begin{cases}\frac{1}{Z_J(h)}\prod_{i:h_i=1} q^{i-1} & \text{if } h=\sum_{i=1}^J h_i\\ 0 & \text{else}\end{cases}$$

where $Z_J(h)=q^{h(h-1)/2}\frac{(q,q)_J}{(q,q)_h(q,q)_{J-h}}$ is the normalizing constant (it can be computed using q-binomial theorem).

We also define the matrix $L_\alpha^{\otimes_q J}$ with rows and columns indexed by $\{0,1,\dots,I\}\times\{0,1\}^{\otimes J}$ with matrix elements

$$L_\alpha^{\otimes_q J}(v,h_1,\dots,h_J;v',h'_1,\dots,h'_J)=\sum_{\substack{v_0,v_1,\dots,v_J\\ v_0=v,v_J=v'}}\prod_{i=1}^J L^{(1)}_{\alpha q^{i-1}}(v_{i-1},h_i;v_i,h'_i).$$

In terms of the right part of Figure 7.2, these matrix elements provide the transition probabilities from the lines coming into a column from bottom and left, to those leaving to the top and right.

The following lemma allows us to decompose the vertex with horizontal spin $J/2$ (i.e. the vertex associated with $L_\alpha^{(J)}$) in terms of a sequence of horizontal spin $1/2$ vertices, see Figure 7.2 for visualization.

Lemma 7.1.3. *The following identity holds*

$$L_\alpha^{(J)}(v,h;v',h')=\sum_{\substack{(h_1,\dots,h_J)\in\{0,1\}^J\\ (h'_1,\dots,h'_J)\in\{0,1\}^J}}\Lambda\big(h;(h_1,h_2,\dots h_J)\big)L_\alpha^{\otimes_q J}(v,h_1,\dots,h_J;v',h'_1,\dots,h'_J)\,\Xi\big((h'_1,\dots,h'_J);h'\big).$$

Proof. This was shown in [CP16, Theorem 3.15]. □

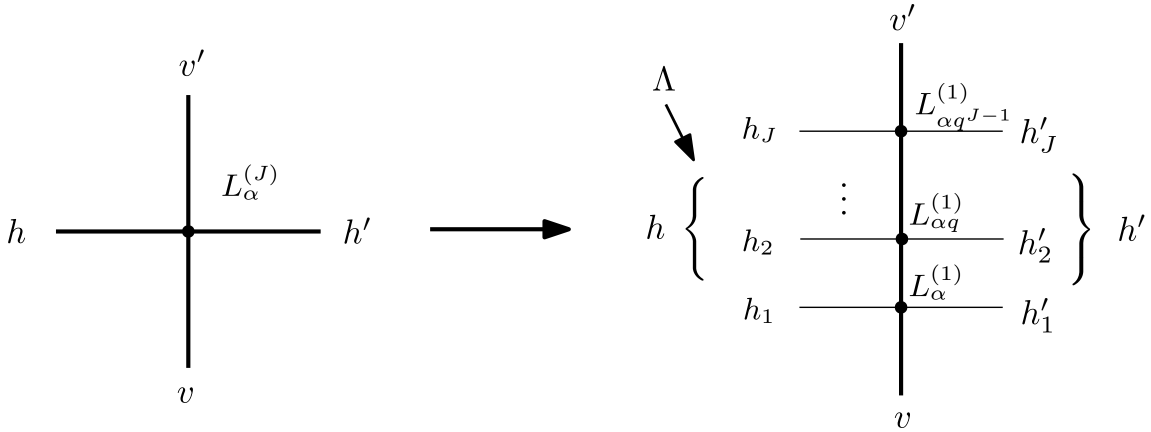


Figure 7.2: Pictorial representation of the identity in Lemma 7.1.3. Fixing h, v, h', v', the weight of vertex configuration on the left is given by $L_\alpha^{(J)}(v, h; v', h')$. It is equal to the weight of the column on the right, which is the summation of all $L_\alpha^{\otimes_q J}(v, h_1, \dots, h_J; v', h'_1, \dots, h'_J)$, under the condition $h_1 + \cdots + h_J = h$ and $h'_1 + \cdots + h'_J = h'$, each term in the summation is reweighted by multiplying $\Lambda\big(h; (h_1, \dots, h_J)\big)$.

Applying Lemma 7.1.3, we show that $L_\alpha^{(J)}$ is stochastic, under the following choice of parameters.

Corollary 7.1.4. *The matrix $L_\alpha^{(J)}$ is stochastic if either of the following condition holds*

- $q \in [0, 1)$ *and* $\alpha < -q^{-I-J+1}$,
- $q > 1$ *and* $-q^{-I-J+1} < \alpha < 0$.

Proof. Note that under the range imposed on q, α, referring to Lemma 7.1.2, the matrix $L^{(1)}_{\alpha q^i}$ is stochastic for each $i = 0, 1, \dots, J-1$. As the product of stochastic matrices is stochastic as well, the stochasticity of $L_\alpha^{(J)}$ follows directly from Lemma 7.1.3. □

We proceed to define the SHS6V model on the first quadrant $\mathbb{Z}^2_{\geq 0}$. For each vertex $(x, y) \in \mathbb{Z}^2_{\geq 0}$, we associate it with a four tuple $(v_{x,y}, h_{x,y}, v_{x,y+1}, h_{x+1,y}) \in \mathbb{Z}^4_{\geq 0}$ such that $v_{x,y}, h_{x,y}$ represent the number of lines entering into the vertex from bottom and left, $v_{x,y+1}, h_{x+1,y}$ denote the number of lines flowing from the vertex to above and right. Note that configurations chosen for two neighboring vertices need to be compatible in the sense that the lines keep flowing. For instance, $v_{x,y+1}$

also represents the number of vertical input lines flowing into $(x, y+1)$, $h_{x,y+1}$ equals the number of horizontal lines entering into $(x+1, y)$ (see the right part of Figure 7.3).

Definition 7.1.5. We define the SHS6V model to be a stochastic path ensemble on $\mathbb{Z}_{\geq 0}^2$. The boundary condition specified by $\{v_{x,0}\}_{x\in\mathbb{Z}_{\geq 0}}$ and $\{h_{0,y}\}_{y\in\mathbb{Z}_{\geq 0}}$ such that $v_{x,0} \in \{0, 1, \ldots, I\}$, $h_{0,y} \in \{0, 1, \ldots, J\}$. In other words, we have $h_{0,y}$ number of lines entering into the vertex $(0, y)$ from the left boundary and $v_{x,0}$ number of lines flowing into the vertex $(x, 0)$ from the bottom boundary. Sequentially taking (x, y) to be $(0, 0) \to (1, 0) \to (0, 1) \to (2, 0) \to (2, 1) \ldots$, for vertex at (x, y), given $v_{x,y}, h_{x,y}$ as the number of vertical and horizontal input lines, we randomly choose the number of vertical and horizontal output lines $(v_{x,y+1}, h_{x+1,y}) \in \{0, 1, \ldots, I\} \times \{0, 1, \ldots, J\}$ according to probability $L_\alpha^{(J)}(v_{x,y}, h_{x,y}; \cdot, \cdot)$. Proceeding with this sequential sampling, we get a collection of paths going to the up-right direction and we call this the SHS6V model.

We associate a *height function* $H : \mathbb{Z}_{\geq 0}^2 \to \mathbb{Z}$ to the path ensemble, where the paths play a role as the level lines of the height function (see Figure 7.3). Define for any $x, y \in \mathbb{Z}_{\geq 0}$,

$$H(x, y) = \sum_{j=1}^{y} h_{0,j-1} - \sum_{i=1}^{x} v_{i-1,y}.$$

Clearly, we have $H(0, 0) = 0$ and $H(x, y) - H(x-1, y) = -v_{x-1,y}$. Since the vertex is conservative, we also have

$$H(x, y) - H(x, y-1) = h_{x,y-1}.$$

Graphically, when we move across i number of vertical lines from left to right, the height function will decrease by i. When we move across j number of horizontal lines, the height function will increase by j. We further extend $H(x, y)$ to all $(x, y) \in \mathbb{R}_{\geq 0}^2$ by first linearly interpolating the height function first in the x-direction, then in the y-direction. It is obvious that the resulting function is Lipschitz and monotone.

For later use, we call $I/2, J/2$ the vertical and horizontal spin respectively. If a vertex is of horizontal spin $1/2$, we call it a $J = 1$ vertex, otherwise we call it a general J vertex.

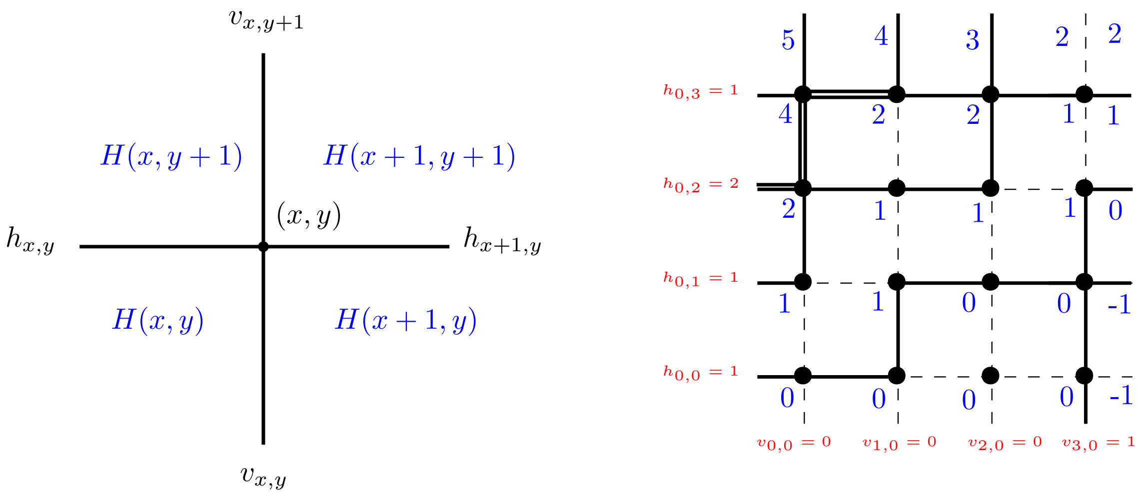


Figure 7.3: Left: Illustration of the height function around a vertex (x, y), note that $H(x, y + 1) = H(x, y) + h_{x,y}$, $H(x + 1, y) = H(x, y) - v_{x,y}$ and $H(x + 1, y + 1) = H(x, y) + h_{x,y} - v_{x,y+1} = H(x, y) - v_{x,y} + h_{x+1,y}$. Right: Sampled stochastic path ensemble on a quadrant. The red number indicates the number lines entering into the boundary, the blue number represents the height at each vertex.

7.1.3 Four point relation

[BG19] shows that the stochastic six vertex model height function converges to a telegraph equation and its fluctuation field converges to a stochastic telegraph equation. The key observation is the following *four point relation*, which says that if we define

$$\xi^{\mathrm{S6V}}(x + 1, y + 1) = q^{H(x+1,y+1)} - b_1 q^{H(x,y+1)} - b_2 q^{H(x+1,y)} + (b_1 + b_2 - 1)q^{H(x,y)},$$

Here b_1, b_2 are the weight of the six vertex model configuration (in our notation $b_1 = \frac{\alpha+\nu}{1+\alpha}$, $b_2 = \frac{1+\alpha q}{1+\alpha}$). Then the conditional expectation and variance of ξ read

$$\mathbb{E}\Big[\xi^{\mathrm{S6V}}(x + 1, y + 1)\Big|\mathcal{F}(x, y)\Big] = 0, \tag{7.1.9}$$

$$\mathbb{E}\Big[\xi^{\mathrm{S6V}}(x + 1, y + 1)^2\Big|\mathcal{F}(x, y)\Big] = \gamma_1\Delta_x\Delta_y + \gamma_2 q^{H(x,y)}\Delta_x + \gamma_3 q^{H(x,y)}\Delta_y, \tag{7.1.10}$$

where $\mathcal{F}(x, y)$ is a sigma algebra generated by $\{H(u, v) : u \le x \text{ or } v \le y\}$ and $\Delta_x := q^{H(x+1,y)} - q^{H(x,y)}$, $\Delta_y := q^{H(x,y+1)} - q^{H(x,y)}$. The parameters $\gamma_i, i = 1, 2, 3$ depend on b_1, b_2.

In our paper, we generalize the above relations to the SHS6V model. Define

$$\xi^{\text{S6SHV}}(x+1,y+1) = q^{H(x+1,y+1)} - \frac{\alpha+\nu}{1+\alpha} q^{H(x,y+1)} - \frac{1+\alpha q^J}{1+\alpha} q^{H(x+1,y)} + \frac{\nu+\alpha q^J}{1+\alpha} q^{H(x,y)},$$

We prove (respectively in Theorem 7.2.3 and Theorem 7.2.5) that

$$\mathbb{E}\Big[\xi^{\text{SHS6V}}(x+1,y+1)\Big|\mathcal{F}(x,y)\Big] = 0, \tag{7.1.11}$$

$$\mathbb{E}\Big[\xi^{\text{SHS6V}}(x+1,y+1)^2\Big|\mathcal{F}(x,y)\Big] = \gamma_1 \Delta_x \Delta_y + \gamma_2 q^{H(x,y)} \Delta_x + \gamma_3 q^{H(x,y)} \Delta_y + \mathbf{R}(x,y). \tag{7.1.12}$$

$\mathbf{R}(x,y)$ is an error term that is negligible under our scaling. From now on, we may also use ξ to denote ξ^{SHS6V}.

Why does such a generalization exist? In the context of the stochastic six vertex model, (7.1.9) is related to the self-duality discovered in [CP16, Proposition 2.20], though it is more of a local relation than the way duality is generally stated (it is unclear to us how to prove (7.1.9) from the duality directly). In fact, [CP16, Corollary 3.3] shows that the SHS6V model with general I, J enjoys the same self-duality, so it is natural to expect that (7.1.11), as a generalized version of (7.1.9) holds. For the quadratic variation, the situation is more subtle for the SHS6V model. We do not come up with a simple reason why (7.1.12) holds, though this may be understandable from our proof, which is briefly explained in the next paragraph. Here, we just emphasize that as shown in Remark 7.2.6, there exist no $\gamma_i, i = 1, 2, 3$ such that the identity without an error term holds for the SHS6V model. We also emphasize that it is only under our scaling (7.1.13) that $\mathbf{R}(x,y)$ is negligible.

Let us explain the ideas and techniques used in proving (7.1.11) and (7.1.12). In [BG19], the authors prove (7.1.9) and (7.1.10) via a direct computation, which corresponds to enumerating all possible six vertex configurations. In our case, the situation is more involved: when J is large, the expression of $L_\alpha^{(J)}$ is so complicated that it is hopeless to check these relations directly. Alternatively, we first verify them directly for $J = 1$, in which case the $\mathbb{L}$-matrix has a simple expression given by Definition 7.1.1. For general J, we use fusion, which allows us to decompose the general J vertex into a sequence of $J = 1$ vertices (see Figure 7.2). Repeatedly using the $J = 1$ version of (7.1.11) (where the spectral parameter α is replaced by αq^i in the expression of ξ), we get J identities. Summing up these identities in a clever way, we see a telescoping property and (7.1.11) follows. To prove (7.1.12), besides using fusion, we need to refer to the property of our scaling (7.1.13), which says that with a probability converging to 1, the lines entering into a vertex will keep flowing in the same direction (see Lemma 7.2.4).

In [CP16], the fusion was stated in a way that the spectral parameters progress geometrically by q from bottom to top when we decompose the general J vertex to a column of $J = 1$ vertices. It turns out that (Lemma 7.2.1) we can also reverse the direction and let the parameters progress geometrically by q from top to bottom (meanwhile we change the probability distribution assigned on the input lines from the left). We did not see this result elsewhere. Note that it is only after this reversal of the spectral parameters that we obtain the telescoping property mentioned in the previous paragraph.

7.1.4 Stochastic telegraph equation as a scaling limit of the SHS6V model

Having established the four point relation, we are ready to talk about our result. We show that under our scaling,

(i). (Hydrodynamic limit (or law of large numbers) – Theorem 7.1.6): The SHS6V model height function converges uniformly in probability to a telegraph equation.

(ii). (Functional central limit theorem – Theorem 7.1.7 (also see Corollary 7.1.9)): The fluctuation field of the height function around its hydrodynamic limit (viewed as a random continuous function) converges weakly to a stochastic telegraph equation.

Once we have proved the four point relation for the SHS6V model, the proof for the law of large numbers is akin to [BG19, Theorem 5.1]. For the functional central limit theorem, our proof breaks down into proving the finite dimensional weak convergence (Proposition 7.3.1) and tightness (Proposition 7.3.2). For finite dimensional convergence, the proof follows a similar idea as in [BG19, Theorem 6.1], subject to certain generalization. For the tightness, we rely on the Burkholder inequality and a careful control of joint moments of ξ at different locations (Lemma 7.3.3). We remark that the proof of the tightness may not fit to the regime of classical functional martingale CLT result (e.g. [Bro71, Section 6]), see Remark 7.3.4 for more discussion.

To present our results, let us first introduce our scaling. Fix $I, J \in \mathbb{Z}_{\geq 1}$ and positive β_1, β_2 such that $\beta_1 \neq \beta_2$, we scale the parameter q, α in the way that

$$q = e^{\frac{\beta_1 - \beta_2}{L}}, \qquad \frac{1 + \alpha q^J}{1 + \alpha} = e^{-\frac{J\beta_2}{L}}, \quad L \to \infty. \tag{7.1.13}$$

It is straightforward that as $L \to \infty$, α and q always satisfy one of the conditions given in Corollary 7.1.4, thus $L_\alpha^{(J)}$ is indeed stochastic.

Theorem 7.1.6. *Define* $\mathfrak{q} = e^{\beta_1 - \beta_2}$ *and fix* $A, B > 0$*, consider two monotone Lipschitz functions* χ *and* ψ*. Suppose that the boundary for the SHS6V model is chosen in the way that as* $L \to \infty$*,* $\frac{1}{L} H(Lx, 0) \to \chi(x)$ *and* $\frac{1}{L} H(0, Ly) \to \psi(y)$ *uniformly in probability for* $x \in [0, A]$ *and* $y \in [0, B]$*, then as* $L \to \infty$*,*

$$\frac{1}{L} \sup_{x \in [0,A] \times [0,B]} |H(Lx, Ly) - L\mathbf{h}(x, y)| \xrightarrow{p} 0,$$

where $\xrightarrow{p}$ *means the convergence in probability.* $\mathfrak{q}^{\mathbf{h}(x,y)}$ *is the unique solution to the telegraph equation*

$$\frac{\partial^2}{\partial x \partial y} \mathfrak{q}^{\mathbf{h}(x,y)} + J\beta_2 \frac{\partial}{\partial x} \mathfrak{q}^{\mathbf{h}(x,y)} + I\beta_1 \frac{\partial}{\partial y} \mathfrak{q}^{\mathbf{h}(x,y)} = 0, \tag{7.1.14}$$

with the boundary condition specified by $\mathfrak{q}^{\mathbf{h}(x,0)} = \mathfrak{q}^{\chi(x)}$ *and* $\mathfrak{q}^{\mathbf{h}(\mathbf{0},\mathbf{y})} = \mathfrak{q}^{\psi(y)}$.

We remark that there is a typo in [BG19, Eq. 69] about the boundary condition, $\mathfrak{q}^{\mathbf{h}(x,0)}, \mathfrak{q}^{\mathbf{h}(0,y)}$ should equal $\mathfrak{q}^{\chi(x)}$ and $\mathfrak{q}^{\psi(y)}$, instead of $\chi(x)$ and $\psi(y)$.

Having established the law of large number for the height function, we proceed to show the functional central limit theorem. As a convention, we endow the space $C(\mathbb{R}^2_{\geq 0})$ with the topology of uniform convergence over compact subsets and use " $\Rightarrow$ " to denote the weak convergence. Recall that we linearly extend $H(x,y)$ for non-integer x, y, so $H(x,y) \in C(\mathbb{R}^2_{\geq 0})$.

Theorem 7.1.7. *Assuming further that $\chi(x)$ and $\psi(y)$ are piecewise C^1-smooth, we have the weak convergence as $L \to \infty$,*

$$\sqrt{L}\Big(q^{H(Lx,Ly)} - \mathbb{E}\Big[q^{H(Lx,Ly)}\Big]\Big) \Rightarrow \varphi(x,y) \quad \text{in } C(\mathbb{R}^2_{\geq 0}),$$

where $\varphi(x,y)$ is a random continuous function which solves the stochastic telegraph equation

$$\varphi_{xy} + I\beta_1\varphi_y + J\beta_2\varphi_x = \eta \cdot \sqrt{(\beta_1+\beta_2)\mathfrak{q}^{\mathbf{h}}_x\mathfrak{q}^{\mathbf{h}}_y + J(\beta_2-\beta_1)\beta_2\mathfrak{q}^{\mathbf{h}}\mathfrak{q}^{\mathbf{h}}_x + I(\beta_1-\beta_2)\beta_1\mathfrak{q}^{\mathbf{h}}\mathfrak{q}^{\mathbf{h}}_y}, \quad (7.1.15)$$

Here, $\mathfrak{q}^{\mathbf{h}}_x := \partial_x(\mathfrak{q}^{\mathbf{h}(x,y)})$ and $\mathfrak{q}^{\mathbf{h}}_y := \partial_y(\mathfrak{q}^{\mathbf{h}(x,y)})$, the boundary of φ is given by zero.

Remark 7.1.8. By (7.1.4), it is clear that φ is a Gaussian field with covariance function

$$\begin{aligned}Cov\Big(\varphi(X_1,Y_1),\varphi(X_2,Y_2)\Big) = &\int_0^{X_1\wedge Y_1}\int_0^{X_2\wedge Y_2}\mathcal{R}_{IJ}(X_1,Y_1,x,y)\mathcal{R}_{IJ}(X_2,Y_2,x,y)\\ &\times\Big((\beta_1+\beta_2)\mathfrak{q}^{\mathbf{h}}_x\mathfrak{q}^{\mathbf{h}}_y + J(\beta_2-\beta_1)\beta_2\mathfrak{q}^{\mathbf{h}}\mathfrak{q}^{\mathbf{h}}_x + I(\beta_1-\beta_2)\beta_1\mathfrak{q}^{\mathbf{h}}\mathfrak{q}^{\mathbf{h}}_y\Big)dxdy,\end{aligned}$$

where $\mathcal{R}_{IJ}$ is the Riemann function in (7.1.3) with β_1 and β_2 replaced by $I\beta_1$ and $J\beta_2$ respectively, i.e.

$$\mathcal{R}_{IJ}(X,Y;x,y) = \frac{1}{2\pi\mathbf{i}}\oint_{-I\beta_1}\frac{J\beta_2 - I\beta_1}{(z+I\beta_1)(z+J\beta_2)}\exp\left[(I\beta_1 - J\beta_2)\Big(-(X-x)\frac{z}{z+J\beta_2} + (Y-y)\frac{z}{z+I\beta_1}\Big)\right]dz, \quad (7.1.16)$$

As a corollary of the previous results, we have the following.

Corollary 7.1.9. *As $L \to \infty$,*

$$\frac{H(Lx, Ly) - \mathbb{E}\big[H(Lx, Ly)\big]}{\sqrt{L}} \Rightarrow \phi(x, y) \quad in\ C(\mathbb{R}^2_{\geq 0}),$$

$\phi(x, y)$ is a Gaussian field given by $\phi(x, y) := \frac{\varphi(x,y)}{\mathfrak{q}^{\mathbf{h}(x,y)} \log \mathfrak{q}}$, which solves

$$\phi_{xy} + I\beta_1\phi_y + J\beta_2\phi_x + (\beta_1 - \beta_2)(\phi_y\mathbf{h}_x + \phi_x\mathbf{h}_y) = \eta \cdot \sqrt{(\beta_1 + \beta_2)\mathbf{h}_x\mathbf{h}_y - J\beta_2\mathbf{h}_x + I\beta_1\mathbf{h}_y}. \quad (7.1.17)$$

The rest of the paper is organized as follows. In Section 7.2, we first establish an identity (Lemma 7.2.1), which gives an alternative way to apply fusion. Then, we prove our four point relation (Theorem 7.2.3 and Theorem 7.2.5). We also discuss some properties of our scaling (Lemma 7.2.4). In Section 7.3, we first use the four point relation to prove the law of large numbers (Theorem 7.1.6) and the finite dimensional version of the CLT (Proposition 7.3.1). Then we establish the tightness (Proposition 7.3.2) and improve our CLT to the functional level (Theorem 7.1.7).

7.1.5 Acknowledgment

The author wants to thank Ivan Corwin for many valuable comments on the paper; Vadim Gorin for helpful comments and discussion; and Shalin Parekh for an inspiring discussion about the tightness result. The author was supported by Ivan Corwin through the NSF grants DMS-1811143, DMS-1664650 and also by the Minerva Foundation Summer Fellowship program.

7.2 Four point relation

In this section, we prove the four point relation (7.1.11) and (7.1.12) that mentioned in Section 7.1.3. To begin with, we present a lemma that allows us to reverse the spectral parameters upside down when we decompose the general J vertex into a column of $J = 1$ vertices, see Figure 7.4 for visualization. The key for our proof is an identity that allows us to switch a pair of vertices with different spectral parameters, see Figure 7.5. We do not find such identity in the literature. It seems to us that this identity does not follow directly from the Yang-Baxter equation.

Define the stochastic matrix $\widetilde{\Lambda}$,

$$\widetilde{\Lambda}(h,(h_1,\dots,h_J)) := \begin{cases} \dfrac{1}{Z_J(h)} \displaystyle\prod_{h_i=1} q^{J-i} & \text{if } h = \sum_{i=1}^J h_i \\ 0 & \text{else} \end{cases}$$

and

$$\widetilde{L}_\alpha^{\otimes_q J}(v,h_1,\dots,h_J;v',h'_1,\dots,h'_J) := \sum_{\substack{v_0,v_1,\dots,v_J \\ v_0=v,v_J=v'}} \prod_{i=1}^J L^{(1)}_{\alpha q^{J-i}}(v_{i-1},h_i;v_i,h'_i).$$

Note that comparing with the expression of Λ and $L_\alpha^{\otimes_q J}$, the term q^{i-1} is replaced by q^{J-i}, which corresponds to reversing the spectral parameters upside down.

Lemma 7.2.1. *For fixed h, v, h', v', the following identity holds,*

$$\begin{aligned} &\sum_{\substack{(h_1,\dots,h_J)\in\{0,1\}^J \\ (h'_1,\dots,h'_J)\in\{0,1\}^J}} \Lambda(h;h_1,h_2,\dots h_J) L_\alpha^{\otimes_q J}(v,h_1,\dots,h_J;v',h'_1,\dots,h'_J)\, \Xi(h'_1,\dots,h'_J;h') \\ &= \sum_{\substack{(h_1,\dots,h_J)\in\{0,1\}^J \\ (h'_1,\dots,h'_J)\in\{0,1\}^J}} \widetilde{\Lambda}(h;h_1,h_2,\dots h_J) \widetilde{L}_\alpha^{\otimes_q J}(v,h_1,\dots,h_J;v',h'_1,\dots,h'_J)\, \Xi(h'_1,\dots,h'_J;h'). \end{aligned} \tag{7.2.1}$$

Consequently, we have alternate expression for the general J vertex weight

$$L_\alpha^{(J)}(v,h;v',h') = \sum_{\substack{(h_1,\dots,h_J)\in\{0,1\}^J \\ (h'_1,\dots,h'_J)\in\{0,1\}^J}} \widetilde{\Lambda}(h;h_1,h_2,\dots h_J) \widetilde{L}_\alpha^{\otimes_q J}(v,h_1,\dots,h_J;v',h'_1,\dots,h'_J)\, \Xi(h'_1,\dots,h'_J;h'). \tag{7.2.2}$$

Proof. By Lemma 7.1.3, it is clear that (7.2.1) implies (7.2.2). It suffices to prove (7.2.1), which says, graphically

When $J = 1$, the proof is trivial. When $J = 2$, the identity (7.2.1) reduces to Figure 7.5. Since $h, h' \in \{0,1,2\}$, there are nine cases in total. One can verify each case directly and here, we only

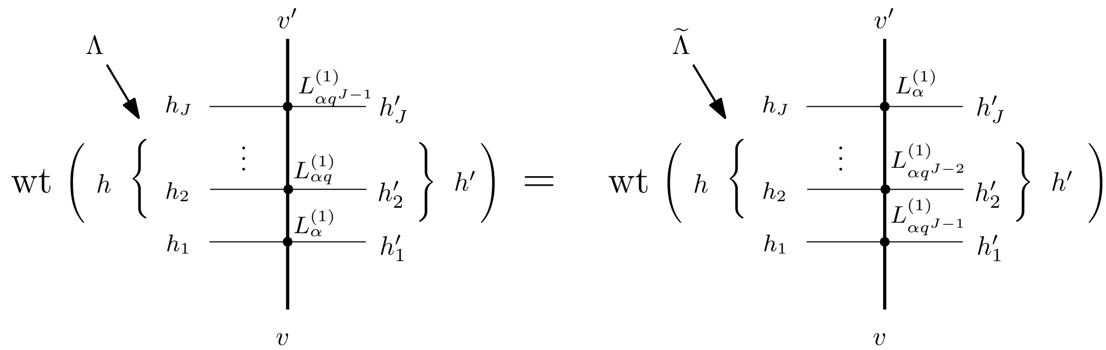


Figure 7.4: Pictorial representation of the identity (7.2.1). The weight (wt) of a diagram is given by a summation of products of $\mathbb{L}$-matrices over $h_1, \dots, h_J$, with condition $h_1 + \cdots + h_J = h$ and $h'_1 + \cdots + h'_J = h'$. Each product on the left (resp. right) hand side in the summation is reweighted by $\Lambda(h; h_1, \dots, h_J)$ (resp. $\widetilde{\Lambda}(h; h_1, \dots, h_J)$).

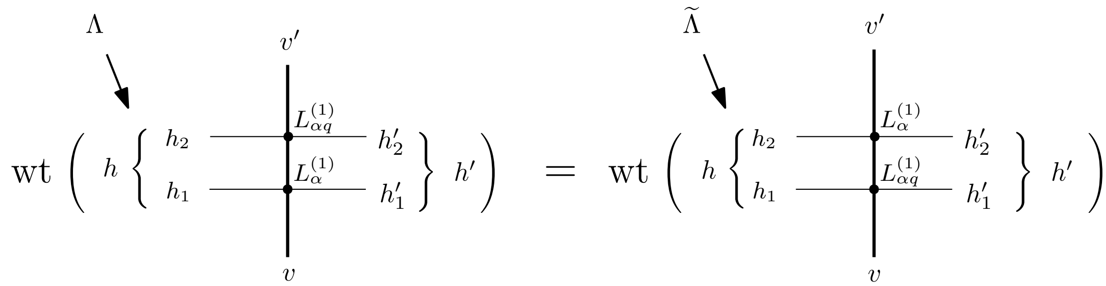


Figure 7.5: Identity (7.2.1) when $J = 2$.

show our verification for $h = 1$ and $h' = 1$, in which case the computation is more involved. The LHS in Figure 7.5 equals

$$\begin{aligned}
&\Lambda\big(1;(1,0)\big)\Big(L^{(1)}_{\alpha}(v,1;v,1)L^{(1)}_{\alpha q}(v,0;v,0) + L^{(1)}_{\alpha}(v,1;v+1,0)L^{(1)}_{\alpha q}(v+1,0;v,1)\Big)\\
&+\Lambda\big(1;(0,1)\big)\Big(L^{(1)}_{\alpha}(v,0;v-1,1)L^{(1)}_{\alpha q}(v-1,1;v,0) + L^{(1)}_{\alpha}(v,0;v,0)L^{(1)}_{\alpha q}(v,1;v,1)\Big)\\
&=\frac{1}{1+q}\Big(\frac{\alpha+vq^v}{1+\alpha}\frac{1+\alpha q^{v+1}}{1+\alpha q} + \frac{1-vq^v}{1+\alpha}\frac{\alpha q(1-q^{v+1})}{1+\alpha q}\Big) + \frac{q}{1+q}\Big(\frac{\alpha(1-q^v)}{1+\alpha}\frac{1-vq^{v-1}}{1+\alpha q} + \frac{1+\alpha q^v}{1+\alpha}\frac{\alpha q+vq^v}{1+\alpha q}\Big)
\end{aligned} \tag{7.2.3}$$

and the RHS equals

$$\begin{aligned}
&\widetilde{\Lambda}\big(1;(1,0)\big)\Big(L^{(1)}_{\alpha q}(v,1;v,1)L^{(1)}_{\alpha}(v,0;v,0) + L^{(1)}_{\alpha q}(v,1;v+1,0)L^{(1)}_{\alpha}(v+1,0;v,1)\Big)\\
&+\widetilde{\Lambda}\big(1;(0,1)\big)\Big(L^{(1)}_{\alpha q}(v,0;v-1,1)L^{(1)}_{\alpha}(v-1,1;v,0) + L^{(1)}_{\alpha q}(v,0;v,0)L^{(1)}_{\alpha}(v,1;v,1)\Big)
\end{aligned}$$

$$= \frac{q}{1+q}\Big(\frac{\alpha q+\nu q^v}{1+\alpha q}\frac{1+\alpha q^v}{1+\alpha} + \frac{1-\nu q^v}{1+\alpha q}\frac{\alpha(1-q^{v+1})}{1+\alpha}\Big) + \frac{1}{1+q}\Big(\frac{\alpha q(1-q^v)}{1+\alpha q}\frac{1-\nu q^{v-1}}{1+\alpha} + \frac{1+\alpha q^{v+1}}{1+\alpha q}\frac{\alpha+\nu q^v}{1+\alpha}\Big) \tag{7.2.4}$$

It is not hard to see directly that the RHS of (7.2.3) and (7.2.4) are both the sum of the following four terms (divided by a common denominator $(1+q)(1+\alpha)(1+\alpha q)$)

$$q(\alpha q+\nu q^v)(1+\alpha q^v), \qquad q\alpha(1-\nu q^v)(1-q^{v+1}), \quad \alpha q(1-q^v)(1-\nu q^{v-1}), \quad (1+\alpha q^{v+1})(\alpha+\nu q^v).$$

For the verification of other $h, h' \in \{0, 1, 2\}$, we omit the details of our computation.

For general J, we look at the column of vertices on the LHS of the equation illustrated in Figure 7.4. From bottom to top, we label the vertices from 1 to J. Sequentially for $i = 1, \dots, J-1$, we apply the $J = 2$ identity (that we just verified) for the vertex i and $i+1$ in that column. Then, the spectral parameters of the vertices (looking from bottom to top) change from $(\alpha, \alpha q, \dots, \alpha q^{J-1})$ to $(\alpha q, \alpha q^2, \dots, \alpha q^{J-1}, \alpha)$, note that the vertex with spectral parameter α moves from bottom to top. The Λ also changes accordingly. Then we apply the $J = 2$ identity for $i = 1, \dots, J-2$ to move the spectral parameter αq to the second top place. If we keep implementing this procedure, finally we get a column of vertices with spectral parameters $(\alpha q^{J-1}, \alpha q^{J-2}, \dots, \alpha)$. The left input lines are weighted by $\widetilde{\Lambda}$. □

Remark 7.2.2. It turns out that following the same argument, the identities (7.2.1), (7.2.2) also hold when we replace the stochastic matrix $\widetilde{\Lambda}$ with

$$\Lambda_\sigma\big(h, (h_1, \dots, h_J)\big) := \begin{cases} \dfrac{1}{Z_J(h)} \displaystyle\prod_{h_i=1} q^{\sigma(i)-1} & \textit{if } h = \sum_{i=1}^J h_i, \\ 0 & \textit{else,} \end{cases}$$

and replace $\widetilde{L}_{\alpha}^{\otimes_q J}(v, h_1, \ldots, h_J; v', h'_1, \ldots, h'_J)$ with

$$L_{\sigma,\alpha}^{\otimes_q J}(v, h_1, \ldots, h_J; v', h'_1, \ldots, h'_J) := \sum_{\substack{v_0, v_1, \ldots, v_J \\ v_0 = v, v_J = v'}} \prod_{i=1}^{J} L^{(1)}_{\alpha q^{\sigma(i)-1}}(v_{i-1}, h_i; v_i, h'_i).$$

where σ is an arbitrary permutation of $\{1, 2, \ldots, J\}$. We do not include this generalization in the lemma since we are not going to use it.

Theorem 7.2.3. *Consider the SHS6V model associated with the height function H, define for $x, y \in \mathbb{Z}_{\geq 0}$,*

$$\xi(x+1, y+1) = q^{H(x+1,y+1)} - \frac{\alpha + \nu}{1+\alpha} q^{H(x,y+1)} - \frac{1 + \alpha q^J}{1+\alpha} q^{H(x+1,y)} + \frac{\nu + \alpha q^J}{1+\alpha} q^{H(x,y)}, \qquad (7.2.5)$$

then we have,

$$\mathbb{E}\Big[\xi(x+1, y+1) \Big| \mathcal{F}(x, y)\Big] = 0, \qquad (7.2.6)$$

where $\mathcal{F}(x, y) = \sigma\big(H(i, j) : i \leq x \text{ or } j \leq y\big)$.

Proof. Since our model is homogeneous, i.e. every vertex is assigned with the same $\mathbb{L}$-matrix, we suppress the dependence on x, y in our notation and denote by

$$\xi := \xi(x+1, y+1), \qquad \mathsf{H} := H(x, y), \qquad h := H(x, y+1) - H(x, y), \qquad v := H(x, y) - H(x+1, y).$$

In addition, we let

$$\mathcal{F} := \sigma\big(H(x, y), H(x, y+1), H(x+1, y)\big) = \sigma\big(\mathsf{H}, h, v\big).$$

By the sequential update rule specified in Definition 7.1.5, $H(x+1, y+1)$ only depends on the information of H, h, v, so

$$\mathbb{E}\Big[\xi \Big| \mathcal{F}(x, y)\Big] = \mathbb{E}\Big[\xi \Big| \mathcal{F}\Big].$$

To prove (7.2.6), it suffices to show that

$$\mathbb{E}\Big[\xi\Big|\mathcal{F}\Big] = 0. \tag{7.2.7}$$

We prove this identity in two steps:

Step 1 ($J = 1$): We assume $J = 1$, in which case the vertex weight (7.1.8) reduces to the weights in Definition 7.1.1. Let us verify (7.2.7) directly,

$$\begin{aligned}
\mathbb{E}\Big[\xi\Big|\mathcal{F}\Big] &= \mathbb{E}\Big[q^{H(x+1,y+1)} - \frac{\alpha+\nu}{1+\alpha}q^{H(x,y+1)} - \frac{1+\alpha q}{1+\alpha}q^{H(x+1,y)} + \frac{\nu+\alpha q}{1+\alpha}q^{H(x,y)}\Big|\mathcal{F}\Big],\\
&= \mathbb{E}\Big[q^{H(x+1,y+1)}\Big|\mathcal{F}\Big] - \frac{\alpha+\nu}{1+\alpha}q^{H(x,y+1)} - \frac{1+\alpha q}{1+\alpha}q^{H(x+1,y)} + \frac{\nu+\alpha q}{1+\alpha}q^{H(x,y)},\\
&= \mathbb{E}\Big[q^{H(x+1,y+1)}\Big|\mathcal{F}\Big] - \frac{\alpha+\nu}{1+\alpha}q^{\mathsf{H}+h} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-\nu} + \frac{\nu+\alpha q}{1+\alpha}q^{\mathsf{H}}.
\end{aligned}$$

Since $J = 1$, h is either 0 or 1, we discuss them respectively.

If $h = 0$, i.e. $H(x, y+1) = \mathsf{H}$, by Definition 7.1.1,

$$\mathbb{P}\Big(H(x+1, y+1) = \mathsf{H} - \nu\Big) = \frac{1+\alpha q^{\nu}}{1+\alpha}; \quad \mathbb{P}\Big(H(x+1, y+1) = \mathsf{H} - \nu + 1\Big) = \frac{\alpha(1-q^{\nu})}{1+\alpha}. \tag{7.2.8}$$

Hence,

$$\mathbb{E}\Big[\xi\Big|\mathcal{F}\Big] = \frac{1+\alpha q^{\nu}}{1+\alpha}q^{\mathsf{H}-\nu} + \frac{\alpha(1-q^{\nu})}{1+\alpha}q^{\mathsf{H}-\nu+1} - \frac{\alpha+\nu}{1+\alpha}q^{\mathsf{H}} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-\nu} + \frac{\nu+\alpha q}{1+\alpha}q^{\mathsf{H}} = 0.$$

If $h = 1$, i.e. $H(x, y+1) = \mathsf{H} + 1$, we have

$$\mathbb{P}\Big(H(x+1, y+1) = \mathsf{H} - \nu\Big) = \frac{1-\nu q^{\nu}}{1+\alpha}; \quad \mathbb{P}\Big(H(x+1, y+1) = \mathsf{H} - \nu + 1\Big) = \frac{\alpha+\nu q^{\nu}}{1+\alpha}, \tag{7.2.9}$$

which yields

$$\mathbb{E}\Big[\xi\Big|\mathcal{F}\Big] = \frac{1-\nu q^{\nu}}{1+\alpha}q^{\mathsf{H}-\nu} + \frac{\alpha+\nu q^{\nu}}{1+\alpha}q^{\mathsf{H}-\nu+1} - \frac{\alpha+\nu}{1+\alpha}q^{\mathsf{H}+1} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-\nu} + \frac{\nu+\alpha q}{1+\alpha}q^{\mathsf{H}} = 0.$$

Step 2 (General J): Using fusion, we decompose the general J vertex with input (v, h) into a column of $J = 1$ vertices with input $(v, h_1, \ldots, h_J)$, where $(h_1, \ldots h_J)$ is weighted by $\Lambda(h; h_1, \ldots, h_J)$, see Figure 7.6. Define $H_i, H'_i, i = 0, 1, \ldots, J$ in the way that

$$H_0 = H(x, y), \qquad H'_0 = H(x+1, y), \tag{7.2.10}$$

$$H_i = H_0 + \sum_{j=1}^{i} h_j, \qquad H'_i = H'_0 + \sum_{j=1}^{i} h'_j. \tag{7.2.11}$$

Since $h = h_1 + \cdots + h_J$, $H_J = H(x, y+1)$. Furthermore, $H'_J = H(x+1, y+1)$ in law. It suffices to

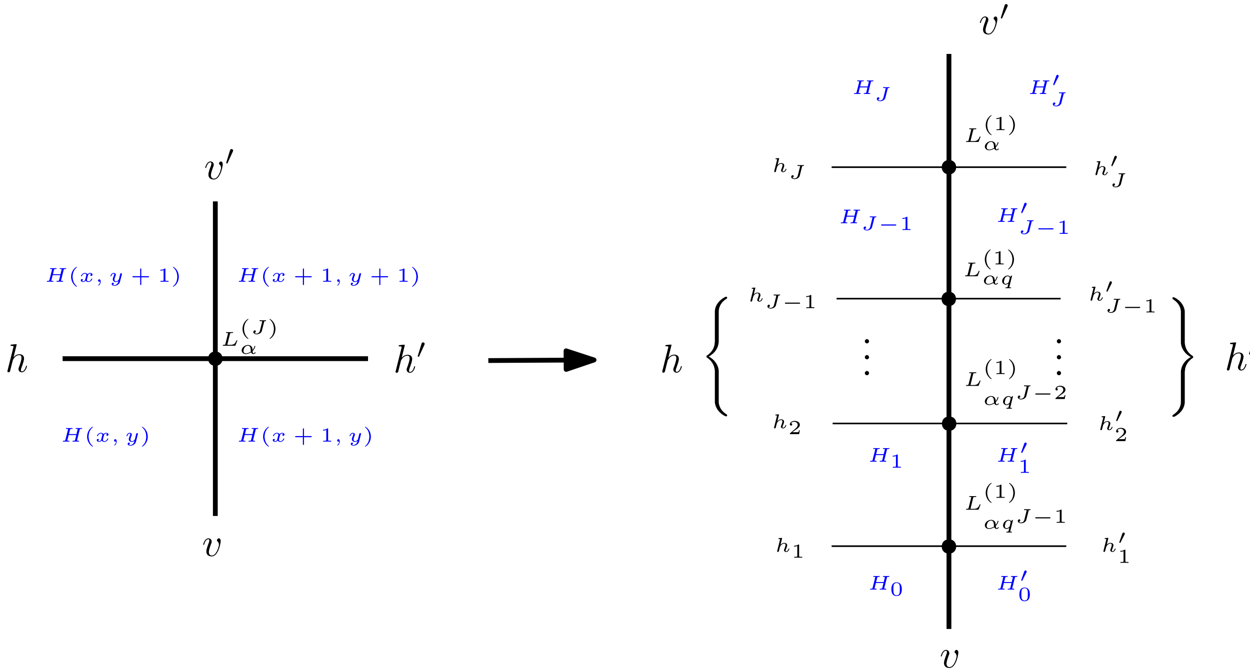


Figure 7.6: Given $H(x, y) = H_0, H(x+1, y) = H'_0, H(x, y+1) = H_J$. By fusion (the spectral parameters have been reversed upside down thanks to Lemma 7.2.1), we have the distributional identity $H(x+1, y+1) = H'_J$. The advantage of utilizing fusion is that we can apply $J = 1$ version of (7.2.6) to each vertex in the column, where the heights around the i-th vertex are $H_{i-1}, H'_{i-1}, H_i, H'_i$. The horizontal input $(h_1, \ldots, h_J)$ is weighted by $\widetilde{\Lambda}(h; h_1, \ldots, h_J)$.

prove

$$\mathbb{E}\Big[q^{H'_J} - \frac{\alpha + \nu}{1+\alpha} q^{H_J} - \frac{1+\alpha q^J}{1+\alpha} q^{H'_0} + \frac{\nu + \alpha q^J}{1+\alpha} q^{H_0} \Big| \mathcal{F}\Big] = 0$$

This is equivalent to

$$\mathbb{E}\Big[q^{H'_J} - \frac{\alpha+\nu}{1+\alpha} q^{H_J} \Big| \mathcal{F}\Big] = \mathbb{E}\Big[\frac{1+\alpha q^J}{1+\alpha} q^{H'_0} - \frac{\nu+\alpha q^J}{1+\alpha} q^{H_0} \Big| \mathcal{F}\Big]. \tag{7.2.12}$$

We define the sigma algebra $\mathcal{F}_i = \sigma(H_i, H'_i, H_{i+1})$ for $i = 0, 1, \ldots, J-1$. Since all the vertices are of horizontal spin $1/2$ now, using the $J = 1$ version of (7.2.6) (proved in **Step 1**) for the i-th vertex (with the spectral parameter αq^{J-i}) looking from the bottom, we have

$$\begin{aligned}
&\mathbb{E}\Big[q^{H'_i} - \frac{\nu+\alpha q^{J-i}}{1+\alpha q^{J-i}} q^{H_i} - \frac{1+\alpha q^{J+1-i}}{1+\alpha q^{J-i}} q^{H'_{i-1}} + \frac{\nu+\alpha q^{J+1-i}}{1+\alpha q^{J-i}} q^{H_{i-1}} \Big| \mathcal{F}\Big] \\
&= \mathbb{E}\Bigg[\mathbb{E}\Big[q^{H'_i} - \frac{\nu+\alpha q^{J-i}}{1+\alpha q^{J-i}} q^{H_i} - \frac{1+\alpha q^{J+1-i}}{1+\alpha q^{J-i}} q^{H'_{i-1}} + \frac{\nu+\alpha q^{J+1-i}}{1+\alpha q^{J-i}} q^{H_{i-1}} \Big| \mathcal{F}_{i-1}\Big] \Bigg| \mathcal{F}\Bigg] = 0.
\end{aligned}$$

In other words,

$$\mathbb{E}\Big[q^{H'_i} - \frac{\nu+\alpha q^{J-i}}{1+\alpha q^{J-i}} q^{H_i} \Big| \mathcal{F}\Big] = \frac{1+\alpha q^{J+1-i}}{1+\alpha q^{J-i}} \mathbb{E}\Big[q^{H'_{i-1}} - \frac{\nu+\alpha q^{J+1-i}}{1+\alpha q^{J+1-i}} q^{H_{i-1}} \Big| \mathcal{F}\Big].$$

Iterating the above equation from $i = J$ to $i = 1$, one concludes the desired (7.2.12). □

To prove relation (7.1.12), we need the following fact which says that under our scaling (7.1.13), it is unlikely that a vertex will change the direction of lines entering into it. More specifically, if a vertex has i vertical input lines and j horizontal input lines, with probability going to 1, it produces i vertical and j horizontal output lines.

We use $O(a)$ to denote some quantity bounded by a constant times a, when the scaling parameter L is large.

Lemma 7.2.4. *For any fixed $i_1, i_2 \in \{0, 1, \ldots, I\}$ and $j_1, j_2 \in \{0, 1, \ldots, J\}$, as $L \to \infty$*

$$L_\alpha^{(J)}(i_1, j_1; i_2, j_2) = \mathbf{1}_{\{i_1=i_2, j_1=j_2\}} + O(L^{-1}).$$

Proof. Via Lemma 7.1.3, it suffices to show that for every $i \in \{0, 1, \ldots, J-1\}$

$$L^{(1)}_{\alpha q^i}(i_1, j_1; i_2, j_2) = \mathbf{1}_{\{i_1 = i_2, j_1 = j_2\}} + O(L^{-1}). \tag{7.2.13}$$

Indeed, by direct computation, under our scaling,

$$L^{(1)}_{\alpha q^i}(m, 0; m, 0) = \frac{1 + \alpha q^{m+i}}{1 + \alpha q^i} = 1 - \frac{\beta_2 m}{L} + O(L^{-2}), \qquad L^{(1)}_{\alpha q^i}(m, 0; m-1, 1) = \frac{\beta_2 m}{L} + O(L^{-2}),$$
$$L^{(1)}_{\alpha q^i}(m, 1; m, 1) = \frac{\alpha q^i + \nu q^m}{1 + \alpha q^i} = 1 + \frac{\beta_1 (m - I)}{L} + O(L^{-2}), \quad L^{(1)}_{\alpha q^i}(m, 1; m+1, 0) = \frac{\beta_1 (I - m)}{L} + O(L^{-2}),$$

which implies (7.2.13). □

Theorem 7.2.5. *Define*

$$\Delta_x := q^{H(x+1,y)} - q^{H(x,y)}, \qquad \Delta_y := q^{H(x,y+1)} - q^{H(x,y)}.$$

Fix $A, B > 0$, under scaling (7.1.13), for any $x \in [0, LA] \cap \mathbb{Z}$ and $y \in [0, LB] \cap \mathbb{Z}$ and $L > 1$,

$$\begin{aligned} &\mathbb{E}\Big[\xi(x+1, y+1)^2 \Big| \mathcal{F}(x, y)\Big] \\ &= L^{-1}(\beta_1 + \beta_2)\Delta_x \Delta_y + J L^{-2}(\beta_2 - \beta_1)\beta_2 q^{H(x,y)} \Delta_x + I L^{-2}(\beta_1 - \beta_2)\beta_1 q^{H(x,y)} \Delta_y + \mathbf{R}(x, y), \end{aligned}$$

where $\mathbf{R}(x, y)$ is a random field with the uniform upper bound

$$|\mathbf{R}(x, y)| \le C L^{-4}, \tag{7.2.14}$$

for all $x \in [0, LA] \cap \mathbb{Z}$ and $y \in [0, LB] \cap \mathbb{Z}$, C is some constant that only depends on A, B.

Proof. We only need to show that the random field $\mathbf{R}(x, y)$ defined via

$$\mathbf{R}(x, y) = \mathbb{E}\Big[\xi(x+1, y+1)^2 \Big| \mathcal{F}(x, y)\Big] - L^{-1}(\beta_1 + \beta_2)\Delta_x \Delta_y - J L^{-2}(\beta_2 - \beta_1)\beta_2 q^{H(x,y)} \Delta_x - I L^{-2}(\beta_1 - \beta_2)\beta_1 q^{H(x,y)} \Delta_y$$

satisfies (7.2.14). Using same notation as in the proof of Theorem 7.2.3,

$$\xi := \xi(x+1, y+1), \qquad \mathcal{F} := \sigma\big(\mathsf{H}, h, v\big),$$

and

$$\mathsf{H} := H(x,y), \qquad h := H(x, y+1) - H(x,y), \qquad v := H(x,y) - H(x+1,y).$$

It is clear that $\mathbb{E}\big[\xi(x+1,y+1)^2|\mathcal{F}(x,y)\big] = \mathbb{E}\big[\xi^2|\mathcal{F}\big]$. Our proof is divided into two steps.

Step 1 ($J = 1$): When $J = 1$, $h \in \{0, 1\}$. We discuss the $h = 0$ and $h = 1$ case separately.

If $h = 0$,

$$\mathbb{E}\Big[\xi^2\Big|\mathcal{F}\Big] = \mathbb{E}\Big[\Big(q^{H(x+1,y+1)} - \frac{\alpha+v}{1+\alpha}q^{\mathsf{H}} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-v} + \frac{v+\alpha q}{1+\alpha}q^{\mathsf{H}}\Big)^2\Big|\mathcal{F}\Big]$$

Referring to (7.2.8), we have (recall $v = q^{-I}$)

$$\begin{aligned}\mathbb{E}\Big[\xi^2\Big|\mathcal{F}\Big] &= \frac{1+\alpha q^v}{1+\alpha}\Big(q^{\mathsf{H}-v} - \frac{\alpha+v}{1+\alpha}q^{\mathsf{H}} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-v} + \frac{v+\alpha q}{1+\alpha}q^{\mathsf{H}}\Big)^2 \\ &\quad + \frac{\alpha(1-q^v)}{1+\alpha}\Big(q^{\mathsf{H}-v+1} - \frac{\alpha+v}{1+\alpha}q^{\mathsf{H}} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-v} + \frac{v+\alpha q}{1+\alpha}q^{\mathsf{H}}\Big)^2 \\ &= \frac{\alpha(q-1)^2q^{-2v}(1-q^v)(1+\alpha q^v)}{(1+\alpha)^2}q^{2\mathsf{H}}, \end{aligned} \tag{7.2.15}$$

The second equality in the above display follows from a straightforward calculation.

Let $\mathsf{b} = \frac{1+\alpha q}{1+\alpha}$ and rewrite (7.2.15) as

$$\mathbb{E}\Big[\xi^2\Big|\mathcal{F}\Big] = (1-\underline{\ })q^{-2v}(q^v-1)\big(-\mathsf{b}+q+(\underline{-}1)q^v\big)q^{2\mathsf{H}} \tag{7.2.16}$$

Referring to scaling (7.1.13), we see that $q^{\mathsf{H}} = e^{\frac{\beta_1-\beta_2}{L}\mathsf{H}}$ is bounded, since for $x \in [0, LA]$ and $y \in [0, LB]$ $|\mathsf{H}| = |H(x,y)| \le L(A+B)$. In addition,

$$q = 1 + \frac{\beta_1-\beta_2}{L} + \mathcal{O}(L^{-2}), \qquad \mathsf{b} = 1 - \frac{\beta_2}{L} + \mathcal{O}(L^{-2}) \tag{7.2.17}$$

Using the expansion of q and b in (7.2.17), we have

$$\mathbb{E}\left[\xi^2\middle|\mathcal{F}\right] = q^{2\mathsf{H}}\beta_2(\beta_2-\beta_1)^2\nu L^{-3} + O(L^{-4}). \tag{7.2.18}$$

When $h=0$, $\Delta_y = q^{H(x,y+1)} - q^{H(x,y)} = 0$. Under scaling (7.1.13),

$$\Delta_x = q^{H(x+1,y)} - q^{H(x,y)} = q^{\mathsf{H}}(q^{-\nu}-1) = q^{\mathsf{H}}\frac{\nu(\beta_2-\beta_1)}{L} + O(L^{-2}).$$

Thereby,

$$\begin{aligned}
&L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y \\
&= L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x, \\
&= q^{2\mathsf{H}}\beta_2(\beta_2-\beta_1)^2\nu L^{-3} + O(L^{-4}).
\end{aligned} \tag{7.2.19}$$

It follows from (7.2.18) and (7.2.19) (note that $J=1$)

$$\mathbf{R}(x,y) = \mathbb{E}\left[\xi^2\middle|\mathcal{F}\right] - \left(L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y\right) = O(L^{-4}).$$

If $h=1$, $H(x+1,y+1)$ is distributed as (7.2.9), then (recall $\nu = q^{-I}$)

$$\begin{aligned}
\mathbb{E}\left[\xi^2\middle|\mathcal{F}\right] &= \frac{1-\nu q^{v}}{1+\alpha}\left(q^{\mathsf{H}-v} - \frac{\alpha+\nu}{1+\alpha}q^{\mathsf{H}+1} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-v} + \frac{\nu+\alpha q}{1+\alpha}q^{\mathsf{H}}\right)^2, \\
&+ \frac{\alpha+\nu q^{v}}{1+\alpha}\left(q^{\mathsf{H}+1-v} - \frac{\alpha+\nu}{1+\alpha}q^{\mathsf{H}+1} - \frac{1+\alpha q}{1+\alpha}q^{\mathsf{H}-v} + \frac{\nu+\alpha q}{1+\alpha}q^{\mathsf{H}}\right)^2, \\
&= \frac{(q-1)^2 q^{-2(I+v)}(q^I-q^v)(\alpha q^I+q^v)}{(1+\alpha)^2}q^{2\mathsf{H}}
\end{aligned}$$

Rewrite the RHS above as (recall $\mathsf{b} = \frac{1+\alpha q}{1+\alpha}$)

$$\mathbb{E}\left[\xi^2\middle|\mathcal{F}\right] = (q-\_)q^{-2(I+v)}(q^I-q^v)\left((-1+\_)q^I + q^v(q-\_)\right)q^{2\mathsf{H}}$$

Using the expansion in (7.2.17), we deduce

$$\mathbb{E}\Big[\xi^2\Big|\mathcal{F}\Big] = q^{2\mathsf{H}}(I-\nu)(\beta_2-\beta_1)^2\beta_1 L^{-3} + O(L^{-4}). \tag{7.2.20}$$

When $h = 1$,

$$\Delta_x = q^{\mathsf{H}}(\beta_2-\beta_1)\nu L^{-1} + O(L^{-2}) \qquad \Delta_y = q^{\mathsf{H}}(\beta_1-\beta_2)L^{-1} + O(L^{-2}),$$

which yields

$$\begin{aligned} &L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y \\ &= q^{2\mathsf{H}}(I-\nu)(\beta_2-\beta_1)^2\beta_1 L^{-3} + O(L^{-4}). \end{aligned} \tag{7.2.21}$$

Combining (7.2.20) and (7.2.21) yields

$$\mathbf{R}(x,y) = \mathbb{E}\Big[\xi^2\Big|\mathcal{F}\Big] - \Big(L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y\Big) = O(L^{-4}).$$

This concludes (7.2.14).

Step 2 (general J): Similar as what we did in Theorem 7.2.3, we apply fusion (see Figure 7.6). Recall H_i, H_i' from (7.2.10) and (7.2.11) and define

$$\begin{aligned} \xi_i &:= q^{H_i'} - \frac{\nu+\alpha q^{J-i}}{1+\alpha q^{J-i}} q^{H_i} - \frac{1+\alpha q^{J-i+1}}{1+\alpha q^{J-i}} q^{H_{i-1}'} + \frac{\nu+\alpha q^{J+1-i}}{1+\alpha q^{J-i}} q^{H_{i-1}}, \\ &= q^{H_i'} - \frac{\nu+\alpha q^{J-i}}{1+\alpha q^{J-i}} q^{H_i} - \frac{1+\alpha q^{J-i+1}}{1+\alpha q^{J-i}}\Big(q^{H_{i-1}'} - \frac{\nu+\alpha q^{J+1-i}}{1+\alpha q^{J+1-i}} q^{H_{i-1}}\Big). \end{aligned}$$

By straightforward calculation,

$$\sum_{i=1}^{J} \frac{1+\alpha q^{J-i}}{1+\alpha}\xi_i = \sum_{i=1}^{J}\Bigg(\frac{1+\alpha q^{J-i}}{1+\alpha}\Big(q^{H_i'} - \frac{\nu+\alpha q^{J-i}}{1+\alpha q^{J-i}} q^{H_i}\Big) - \frac{1+\alpha q^{J-i+1}}{1+\alpha}\Big(q^{H_{i-1}'} - \frac{\nu+\alpha q^{J+1-i}}{1+\alpha q^{J+1-i}} q^{H_{i-1}}\Big)\Bigg)$$

$$= q^{H'_J} - \frac{\nu+\alpha}{1+\alpha} q^{H_J} - \frac{1+\alpha q^J}{1+\alpha}\left(q^{H'_0} - \frac{\nu+\alpha q^J}{1+\alpha q^J} q^{H_0}\right) = \xi, \tag{7.2.22}$$

where the second equality follows from the telescoping property of the summation.

By Theorem 7.2.3, ξ_i are martingale increments, so $\mathbb{E}\left[\xi_i\xi_j|\mathcal{F}\right] = 0$ for $i \neq j$. It follows from (7.2.22) that

$$\mathbb{E}\Big[\xi^2\big|\mathcal{F}\Big] = \mathbb{E}\Big[\sum_{i=1}^{J}\Big(\frac{1+\alpha q^{J-i}}{1+\alpha}\xi_i\Big)^2\big|\mathcal{F}\Big] = \sum_{i=1}^{J}\Big(\frac{1+\alpha q^{J-i}}{1+\alpha}\Big)^2\mathbb{E}\Big[\xi_i^2\big|\mathcal{F}\Big]. \tag{7.2.23}$$

Using the $J = 1$ version of (7.2.16) proved in **Step 1** for the i-th vertex counting from bottom (here, though the spectral parameter changes from α to αq^i, it does not matter under our scaling)

$$\mathbb{E}\Big[\xi_i^2\big|\mathcal{F}_{i-1}\Big] = L^{-1}(\beta_1+\beta_2)\Delta_x^i\Delta_y^i + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x^i + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y^i + \mathbf{R}_i(x,y) \tag{7.2.24}$$

where $\Delta_x^i = q^{H'_{i-1}} - q^{H_{i-1}}$ and $\Delta_y^i = q^{H_i} - q^{H_{i-1}}$ and, also recall that $\mathcal{F}_i = \sigma\big(H_i, H_{i+1}, H'_i\big)$. By **Step 1**, there exists constant C only depending on A, B such that

$$\sup_{\substack{i\in\{1,\dots,J\}\\(x,y)\in[0,LA]\times[0,LB]}} |\mathbf{R}_i(x,y)| \le CL^{-4}. \tag{7.2.25}$$

By conditioning, $\mathbb{E}\left[\xi_i^2|\mathcal{F}\right] = \mathbb{E}\Big[\mathbb{E}\left[\xi_i^2|\mathcal{F}_{i-1}\right]\big|\mathcal{F}\Big]$ (note that here we are not using the tower property but instead the sequential update rule). Using (7.2.23) and (7.2.24), we get

$$\begin{aligned}\mathbb{E}\Big[\xi^2\big|\mathcal{F}\Big] = \sum_{i=1}^{J}\Big(\frac{1+\alpha q^{J-i}}{1+\alpha}\Big)^2\mathbb{E}\Big[&L^{-1}(\beta_1+\beta_2)\Delta_x^i\Delta_y^i + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x^i \\ &+ IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y^i + \mathbf{R}_i(x,y)\big|\mathcal{F}\Big].\end{aligned}$$

Note that under our scaling, $\lim_{L\to\infty} \frac{1+\alpha q^{J-i}}{1+\alpha} = 1$, along with (7.2.25),

$$\mathbb{E}\Big[\xi^2\Big|\mathcal{F}\Big] = \sum_{i=1}^{J} \mathbb{E}\Big[L^{-1}(\beta_1+\beta_2)\Delta_x^i\Delta_y^i + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x^i + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y^i\Big|\mathcal{F}\Big] + O(L^{-4}). \tag{7.2.26}$$

It is clear that

$$\sum_{i=1}^{J} \Delta_y^i = \sum_{i=1}^{J}\left(q^{H_i} - q^{H_{i-1}}\right) = q^{H_J} - q^{H_0} = \Delta_y.$$

Furthermore, by Lemma 7.2.4,

$$\mathbb{P}\Big(\exists\, i \text{ such that } \Delta_x^i \neq \Delta_x\Big|\mathcal{F}\Big) = 1 - O(L^{-1}).$$

Hence, we can simplify (7.2.26) and get

$$\begin{aligned}
\mathbb{E}\Big[\xi^2\Big|\mathcal{F}\Big] &= \sum_{i=1}^{J} \mathbb{E}\Big[L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y^i + L^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y^i\Big|\mathcal{F}\Big] + O(L^{-4}),\\
&= \mathbb{E}\Big[L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y + JL^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y\Big|\mathcal{F}\Big] + O(L^{-4}),\\
&= L^{-1}(\beta_1+\beta_2)\Delta_x\Delta_y + JL^{-2}(\beta_2-\beta_1)\beta_2 q^{\mathsf{H}}\Delta_x + IL^{-2}(\beta_1-\beta_2)\beta_1 q^{\mathsf{H}}\Delta_y + O(L^{-4}).
\end{aligned}$$

The last line is because Δ_x and Δ_y and H are measurable with respect to $\mathcal{F}$. □

Remark 7.2.6. The identity (7.1.10) which holds for stochastic six vertex model no long works for the SHS6V model. For example, consider $I = 2$ and $J = 1$. For an arbitrary vertex (x, y), if there exists three parameters $\gamma_1, \gamma_2, \gamma_3$ such that (7.1.10) is true. When $h = 0$, referring to (7.2.15), we have

$$\mathbb{E}\Big[\xi(x+1, y+1)^2\Big|\mathcal{F}(x, y)\Big] = \frac{\alpha(q-1)^2 q^{-2v}(1-q^v)(1+\alpha q^v)}{(1+\alpha)^2} q^{2\mathsf{H}}$$

Since $\Delta_y = 0$, the right hand side of (7.1.10) reduces to

$$\gamma_1 \Delta_x \Delta_y + \gamma_2 \Delta_x q^{\mathsf{H}} + \gamma_3 \Delta_y q^{\mathsf{H}} = \gamma_2 \Delta_x q^{\mathsf{H}} = \gamma_2 (q^{-v} - 1) q^{2\mathsf{H}}$$

So for all $v \in \{0, 1, \ldots, I\}$,

$$\frac{\alpha(q-1)^2 q^{-v}(q^{-v}-1)(1+\alpha q^v)}{(1+\alpha)^2} q^{2\mathsf{H}} = \gamma_2 (q^{-v} - 1) q^{2\mathsf{H}}$$

Canceling the factor $(q^{-v} - 1)q^{2\mathsf{H}}$ on both sides, we get

$$\frac{\alpha(q-1)^2 q^{-v}(1+\alpha q^v)}{(1+\alpha)^2} = \gamma_2$$

Since γ_2 does not depend on v, so the previous equation could not hold for $v = 1, 2$ simultaneously.

The following corollary is a direct consequence of Theorem 7.2.5.

Corollary 7.2.7. *Fix $A, B > 0$, there exists constant C s.t. for every $x \in [0, LA] \cap \mathbb{Z}$, $y \in [0, LB] \cap \mathbb{Z}$ and $L > 1$*

$$\mathbb{E}\Big[\xi(x+1, y+1)^2 \Big| \mathcal{F}(x, y)\Big] \le CL^{-3}.$$

Proof. It is clear that there exists C such that for any $x \in [0, LA] \cap \mathbb{Z}$ and $y \in [0, LB] \cap \mathbb{Z}$,

$$|\Delta_x| = \left| q^{H(x+1,y)} - q^{H(x,y)} \right| = q^{H(x,y)} \left| e^{\frac{(\beta_1 - \beta_2)h}{L}} - 1 \right| \le CL^{-1}.$$

Similarly, $|\Delta_y| \le CL^{-1}$. Referring to Theorem 7.2.5 (note that $q^{H(x,y)}$ is bounded), the corollary follows. □

7.3 Proof of the main results

Having established the four point relation, we move on proving Theorem 7.1.6 and Theorem 7.1.7. Corollary 7.1.9 follows from a straightforward argument once we proved Theorem 7.1.7. For the ensuing discussion, we will usually write C for constants, we might not generally specify when irrelevant terms are being absorbed into the constants. We might also write for example $C(n)$ when we want to specify which parameter the constant depends on.

Proof of Theorem 7.1.6. Given Theorem 7.2.3, our proof is akin to [BG19, Theorem 5.1]. We provide the detail for the sake of completeness. Recall $\mathfrak{q} = q^{\frac{1}{L}}$, to prove $\frac{1}{L}H(Lx, Ly) \to \mathbf{h}(x, y)$ uniformly in probability for $x \in [0, A]$ and $y \in [0, B]$, it suffices to show that $q^{H(Lx,Ly)} \to \mathfrak{q}^{\mathbf{h}(x,y)}$ uniformly in probability. To this end, we write

$$q^{H(Lx,Ly)} = \mathbb{E}\Big[q^{H(Lx,Ly)}\Big] + q^{H(Lx,Ly)} - \mathbb{E}\Big[q^{H(Lx,Ly)}\Big].$$

It suffices to show that as $L \to \infty$,

(i). $\mathbb{E}[q^{H(Lx,Ly)}] \to \mathfrak{q}^{\mathbf{h}(x,y)}$ uniformly for $(x, y) \in [0, A] \times [0, B]$,

(ii). $q^{H(Lx,Ly)} - \mathbb{E}[q^{H(Lx,Ly)}] \to 0$ uniformly in probability for $(x, y) \in [0, A] \times [0, B]$.

We first demonstrate (i). By Theorem 7.2.3,

$$\mathbb{E}\Big[q^{H(x+1,y+1)}\Big] - b_1\mathbb{E}\Big[q^{H(x,y+1)}\Big] - b_2\mathbb{E}\Big[q^{H(x+1,y)}\Big] + (b_1 + b_2 - 1)\mathbb{E}\Big[q^{H(x,y)}\Big] = 0,$$

where $b_1 = \frac{\alpha+\nu}{1+\alpha}$, $b_2 = \frac{1+\alpha q^J}{1+\alpha}$. Summing this equation over $x = 0, 1, \dots, LX - 1$ and $y = 0, 1, \dots LY - 1$ yields

$$\begin{aligned}
&-(1-b_1)\sum_{x=1}^{LX-1}\mathbb{E}\Big[q^{H(x,0)}\Big] - (1-b_2)\sum_{y=1}^{LY-1}\mathbb{E}\Big[q^{H(0,y)}\Big] + (1-b_1)\sum_{x=1}^{LX-1}\mathbb{E}\Big[q^{H(x,LY)}\Big] \\
&+(1-b_2)\sum_{y=1}^{LY-1}\mathbb{E}\Big[q^{H(LX,y)}\Big] + (b_1+b_2-1)\mathbb{E}\Big[q^{H(0,0)}\Big] - b_2\mathbb{E}\Big[q^{H(LX,0)}\Big] - b_1\mathbb{E}\Big[q^{H(0,LY)}\Big] + \mathbb{E}\Big[q^{H(LX,LY)}\Big] = 0
\end{aligned} \tag{7.3.1}$$

Since H is Lipschitz, the sequence of deterministic functions $\mathbb{E}[q^{H(L\cdot,L\cdot)}] = \mathbb{E}[\mathfrak{q}^{\frac{1}{L}H(L\cdot,L\cdot)}] \in C([0, A] \times [0, B])$ is uniformly bounded and equi-continuous. By Arzela-Ascoli Theorem, it has a limit point $\mathfrak{q}^{\widetilde{h}}$.

Under scaling (7.1.13), when $L \to \infty$,

$$b_1 = 1 - \beta_1 I L^{-1} + O(L^{-2}), \qquad b_2 = 1 - \beta_2 J L^{-1} + O(L^{-2}). \tag{7.3.2}$$

Combining this with (7.3.1) and taking the $L \to \infty$ limit, $\mathfrak{q}^{\widetilde{\mathbf{h}}}$ satisfies the integral equation

$$\begin{aligned}&- I\beta_1 \int_0^X \mathfrak{q}^{\widetilde{\mathbf{h}}(x,0)} dx - J\beta_2 \int_0^Y \mathfrak{q}^{\widetilde{\mathbf{h}}(0,y)} + I\beta_1 \int_0^X \mathfrak{q}^{H(x,Y)} dx + J\beta_2 \int_0^Y \mathfrak{q}^{H(X,y)} dy \\ &+ \mathfrak{q}^{\widetilde{\mathbf{h}}(0,0)} - \mathfrak{q}^{\widetilde{\mathbf{h}}(X,0)} - \mathfrak{q}^{\widetilde{\mathbf{h}}(0,Y)} + \mathfrak{q}^{\widetilde{\mathbf{h}}(X,Y)} = 0\end{aligned}$$

In other words, any limit point $\mathfrak{q}^{\widetilde{\mathbf{h}}}$ of $\mathbb{E}\left[\mathfrak{q}^{\frac{1}{L}H(Lx,Ly)}\right]$ as $L \to \infty$ satisfies the telegraph equation

$$\frac{\partial^2}{\partial x \partial y}\mathfrak{q}^{\widetilde{\mathbf{h}}(x,y)} + I\beta_1 \frac{\partial}{\partial y}\mathfrak{q}^{\widetilde{\mathbf{h}}(x,y)} + J\beta_2 \frac{\partial}{\partial x}\mathfrak{q}^{\widetilde{\mathbf{h}}(x,y)} = 0.$$

By our assumption on the boundary, we also know that $\mathfrak{q}^{\widetilde{\mathbf{h}}(x,0)} = \mathfrak{q}^{\chi(x)}$ and $\mathfrak{q}^{\widetilde{\mathbf{h}}(0,y)} = \mathfrak{q}^{\psi(y)}$. This implies that $\widetilde{\mathbf{h}} = \mathbf{h}$, which concludes (i).

To verify (ii), we denote by $U(x,y) = q^{H(x,y)} - \mathbb{E}\left[q^{H(x,y)}\right]$. Using Theorem 7.2.3, $q^{H(x,y)}$ and $\mathbb{E}\left[q^{H(x,y)}\right]$ satisfy the discrete telegraph equation (7.1.5) with g given by ξ and 0 respectively, hence by linearity,

$$U(x+1,y+1) - b_1 U(x,y+1) - b_2 U(x+1,y) + (b_1+b_2-1)U(x,y) = \xi(x+1,y+1).$$

Summing over $x = 0, 1, \ldots, LX-1$ and $y = 0, 1, \ldots LY-1$, along with the fact $U(x,0) = U(0,y) = 0$ yields

$$U(LX, LY) + (1-b_1)\sum_{x=1}^{LX-1} U(x, LY) + (1-b_2)\sum_{y=1}^{LY-1} U(LX, y) = \sum_{x=1}^{LX}\sum_{y=1}^{LY}\xi(x,y). \tag{7.3.3}$$

Since $\xi(x, y)$ is a martingale increment, using Corollary 7.2.7

$$\mathbb{E}\Big[\Big(\sum_{x=1}^{LA}\sum_{y=1}^{LB}\xi(x, y)\Big)^2\Big] = \sum_{x=1}^{LA}\sum_{y=1}^{LB}\mathbb{E}\Big[\xi(x, y)^2\Big] \le CABL^{-1}.$$

Applying Doob's L^p maximal inequality, it is clear that

$$\sup_{(X,Y)\in[0,A]\times[0,B]}\Big|\sum_{x=1}^{LX}\sum_{y=1}^{LY}\xi(x, y)\Big| \xrightarrow{p} 0. \tag{7.3.4}$$

Observing that $U(L\cdot, L\cdot)$ are uniformly bounded and uniformly Lipschitz on $[0, A]\times[0, B]$. Therefore, their law are tight, any subsequential limit $\widetilde{U}$ has continuous trajectories must solve the $L = \infty$ version of (7.3.3), which reads (the right hand side is zero by (7.3.4))

$$\widetilde{U}(X,Y) + I\beta_1\int_0^X \widetilde{U}(x,Y)dx + J\beta_2\int_0^Y \widetilde{U}(X, y)dy = 0.$$

According to [BG19, Prop 4.1], the only solution to the above equation is given by $\widetilde{U} = 0$, which implies (ii). □

We move on proving the functional CLT for the SHS6V model. The proof of Theorem 1.7 is composed of showing the finite dimensional weak convergence and demonstrating the tightness, which is formulated into the following two propositions.

Denote by

$$U_L(x, y) := \sqrt{L}\Big(q^{H(Lx,Ly)} - \mathbb{E}\Big[q^{H(Lx,Ly)}\Big]\Big) = \sqrt{L}U(Lx, Ly).$$

Proposition 7.3.1 (finite dimensional convergence)**.** *With the same setup as in Theorem 1.7, we have the weak convergence in finite dimension as $L \to \infty$,*

$$U_L(x, y) \Rightarrow \varphi(x, y).$$

Recall that we linearly interpolate $H(x, y)$ for non-integer x, y, thus H is a function in $C(\mathbb{R}^2_{\ge 0})$, so is $U_L(x,y)$.

Proposition 7.3.2 (tightness)**.** *For each fixed $A, B > 0$ and $n \in \mathbb{N}$, there is a constant C (only depends on n, A, B) such that for all $L > 1$ and $(X_1, Y_1), (X_2, Y_2) \in [0, LA] \times [0, LB]$,*

$$\mathbb{E}\left[\Big(U_L(X_1,Y_1) - U_L(X_2,Y_2)\Big)^{2n}\right] \leq C\big(|X_1 - X_2| + |Y_1 - Y_2|\big)^n. \tag{7.3.5}$$

Consequently, the sequence of random function $U_L(\cdot,\cdot) \in C(\mathbb{R}^2_{\geq 0})$ is tight.

Proof of Theorem 7.1.7. The proof is a direct combination of Proposition 7.3.1 and Proposition 7.3.2. □

We first prove the finite dimensional weak convergence.

Proof of Proposition 7.3.1. Recall that in the proof of Theorem 7.1.6, we set $U(x,y) = q^{H(x,y)} - \mathbb{E}[q^{H(x,y)}]$. As shown earlier, we have

$$U(x+1,y+1) - b_1 U(x,y+1) - b_2 U(x+1,y) + (b_1 + b_2 - 1)U(x,y) = \xi(x+1,y+1)$$

Furthermore, since $H(x,0)$ and $H(0,y)$ are deterministic, we have $U(x,0) = U(y,0) = 0$. By (7.1.6), one has

$$U(X,Y) = \sum_{x=1}^{X}\sum_{y=1}^{Y} \mathcal{R}^d(X,Y;x,y)\xi(x,y). \tag{7.3.6}$$

Here $\mathcal{R}^d$ is defined through (7.1.7) with $b_1 = \frac{\alpha+\nu}{1+\alpha}$, $b_2 = \frac{1+\alpha q^J}{1+\alpha}$.

We need to show that $U_L(\cdot,\cdot) = \sqrt{L}U(L\cdot, L\cdot)$ converges weakly to $\varphi(\cdot,\cdot)$ (given by (7.1.15)) in finite dimension. As in the proof of [BG19, Theorem 6.1], we use the martingale central limit theorem [HH14, Section 3] for the martingale (note that $U_L(X,Y) = M_L(L^2XY)$)

$$\left(M_L(t) := \sum_{i=1}^{t} \sqrt{L}\mathcal{R}^d(LX, LY, x(i), y(i))\xi(x(i), y(i)), 1 \leq t \leq L^2XY\right) \tag{7.3.7}$$

where we linearly order points in $[1, LX] \times [1, LY]$ by sequentially tracing the diagonals $x + y =$ const

$$(x(1), y(1)) := (1,1), \quad (x(2), y(2)) := (2,1), \quad (x(3), y(3)) := (1,2), \quad (x(4), y(4)) := (3,1) \dots \tag{7.3.8}$$

Note that we will only deal with the one point convergence $M_L(L^2XY) \Rightarrow \varphi(X,Y)$ for simplicity, the finite dimensional convergence can be proved by invoking multi-dimensional version of martingale CLT (see [ST19, Theorem 3.1]) for a multi-dimensional version of the martingale in (7.3.7).

The key for the proof is to study the conditional variance of $M_L(t)$ at $t = L^2XY$. We show that as $L \to \infty$, it converges to the variance of φ (7.1.15) in probability. In other words, we need to prove

$$\begin{aligned}
&L \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 \mathbb{E}\Big[\xi(x+1,y+1)^2 \Big| \mathcal{F}(x,y)\Big] \\
&\xrightarrow{p} \int_0^X \int_0^Y \mathcal{R}_{IJ}(X,Y,x,y)^2 \Big((\beta_1+\beta_2)\mathfrak{q}_x^{\mathbf{h}}\mathfrak{q}_y^{\mathbf{h}} + J(\beta_2-\beta_1)\beta_2 \mathfrak{q}^{\mathbf{h}}\mathfrak{q}_x^{\mathbf{h}} + I(\beta_1-\beta_2)\beta_1\mathfrak{q}^{\mathbf{h}}\mathfrak{q}_y^{\mathbf{h}}\Big) dxdy.
\end{aligned} \tag{7.3.9}$$

where the RHS above is the variance of $\varphi(X,Y)$, see Remark 7.1.8.

To prove this convergence, we first use Theorem 7.2.5,

$$\begin{aligned}
&L \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 \mathbb{E}\Big[\xi(x+1,y+1)^2 \Big| \mathcal{F}(x,y)\Big] \\
&= \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 \times \\
&\Big((\beta_1+\beta_2)\Delta_x\Delta_y + JL^{-1}(\beta_2-\beta_1)\beta_2 q^{H(x,y)}\Delta_x + IL^{-1}(\beta_1-\beta_2)\beta_1 q^{H(x,y)}\Delta_y\Big) \\
&\quad + L \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 \mathbf{R}(x,y).
\end{aligned}$$

By (7.2.14), $\sup_{x\in[0,LA], y\in[0,LB]} |\mathbf{R}(x,y)| \le CL^{-4}$, together with the fact $\mathcal{R}^d$ is uniformly bounded in $[0, LA] \times [0, LB]$, we have almost surely,

$$L \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 \mathbf{R}(x, y) \to 0$$

uniformly in $(x, y) \in [0, LA] \times [0, LB]$. As a result, to demonstrate (7.3.9), it suffices to prove that as $L \to \infty$

$$L^{-1} \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 q^{H(x,y)} \Delta_x \xrightarrow{p} \int_0^X \int_0^Y \mathcal{R}_{IJ}(X, Y, x, y)^2 \mathfrak{q}_x^{\mathbf{h}} \mathfrak{q}^{\mathbf{h}} dx dy \tag{7.3.10}$$

$$L^{-1} \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 q^{H(x,y)} \Delta_y \xrightarrow{p} \int_0^X \int_0^Y \mathcal{R}_{IJ}(X, Y, x, y)^2 \mathfrak{q}_y^{\mathbf{h}} \mathfrak{q}^{\mathbf{h}} dx dy \tag{7.3.11}$$

$$\sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 \Delta_x \Delta_y \xrightarrow{p} \int_0^X \int_0^Y \mathcal{R}_{IJ}(X, Y, x, y)^2 \mathfrak{q}_x^{\mathbf{h}} \mathfrak{q}_y^{\mathbf{h}} dx dy \tag{7.3.12}$$

To demonstrate these approximations, as in the proof of [BG19, Theorem 6.1], we split the the interval $[0, LX] \times [0, LY]$ into squares such as $[LX_0, L(X_0 + \theta)] \times [LY_0, L(Y_0 + \theta)]$ (where θ is small) and apply the discrete to continuous approximation in each square.

We first demonstrate (7.3.10), for $x \in [LX_0, L(X_0 + \theta)]$ and $y \in [LY_0, L(Y_0 + \theta)]$, it is not hard to see that $\mathcal{R}^d(LX, LY, Lx, Ly) \to \mathcal{R}_{IJ}(X, Y, x, y)$ uniformly for $0 \le x \le X \le A$ and $0 \le y \le Y \le B$ (see [ST19, Eq 2.9] for $I = J = 1$ case). Thus,

$$\mathcal{R}^d(LX, LY, x, y) = \mathcal{R}_{IJ}(X, Y, X_0, Y_0) + O(\theta) + o(1), \qquad q^{H(LX,LY)} = \mathfrak{q}^{\mathbf{h}(X_0,Y_0)} + O(\theta) + o(1). \tag{7.3.13}$$

where $o(1)$ represents the term converging to zero as $L \to \infty$. Using these expansions, we have

$$
\begin{aligned}
&L^{-1} \sum_{\substack{x \in [LX_0, L(X_0+\theta)] \\ y \in [LY_0, L(Y_0+\theta)]}} \mathcal{R}^d(LX, LY, x+1, y+1)^2 q^{H(x,y)} \Delta_x \\
&= L^{-1} \mathcal{R}_{IJ}(X, Y, X_0, Y_0)^2 \mathfrak{q}^{\mathbf{h}(X_0, Y_0)} \times \Big(\sum_{y \in [LY_0, L(Y_0+\theta)]} \big(q^{H(L(X_0+\theta), y)} - q^{H(LX_0, y)} \big) \Big) + O(\theta^3) + \theta^2 o(1)
\end{aligned}
\tag{7.3.14}
$$

Using law of large number proved in Theorem 7.1.6, uniformly for $y' \in [Y_0, Y_0 + \theta]$

$$
q^{H(L(X_0+\theta), Ly')} - q^{H(LX_0, Ly')} = \mathfrak{q}^{\mathbf{h}(X_0+\theta, y')} - \mathfrak{q}^{\mathbf{h}(X_0, y')} + o(1).
$$

Consequently, it follows from (7.3.14)

$$
\begin{aligned}
&L^{-1} \sum_{\substack{x \in [LX_0, L(X_0+\theta)] \\ y \in [LY_0, L(Y_0+\theta)]}} \mathcal{R}^d(LX, LY, x+1, y+1)^2 q^{H(x,y)} \Delta_x \\
&= L^{-1} \mathcal{R}_{IJ}(X, Y, X_0, Y_0)^2 \mathfrak{q}^{\mathbf{h}(X_0, Y_0)} \int_{Y_0}^{Y_0+\theta} \Big(\mathfrak{q}^{\mathbf{h}(X_0+\theta, y)} - \mathfrak{q}^{\mathbf{h}(X_0, y)} \Big) dy + \theta o(1) + O(\theta^3) + \theta^2 o(1), \\
&= L^{-1} \mathcal{R}_{IJ}(X, Y, X_0, Y_0)^2 \mathfrak{q}^{\mathbf{h}(X_0, Y_0)} \mathfrak{q}_x^{\mathbf{h}(X_0, Y_0)} \theta^2 + \theta o(1) + O(\theta^3) + \theta^2 o(1).
\end{aligned}
\tag{7.3.15}
$$

Note that in the last line, we used the property that the solution $\mathfrak{q}^{\mathbf{h}}$ to the (7.1.14) is piecewise C^1 (since we assume additionally the boundary χ and ψ are smooth). By (7.3.15),

$$
\begin{aligned}
&L^{-1} \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY, x+1, y+1)^2 q^{H(x,y)} \Delta_x \\
&= \sum_{0 \le i \le X/\theta} \sum_{0 \le j \le Y/\theta} \mathcal{R}_{IJ}(X, Y, \theta i, \theta j)^2 \mathfrak{q}^{\mathbf{h}(\theta i, \theta j)} \mathfrak{q}_x^{\mathbf{h}(\theta i, \theta j)} \theta^2 + (1 + \theta^{-1}) o(1) + O(\theta).
\end{aligned}
\tag{7.3.16}
$$

By first letting $L \to \infty$ then $\theta \to 0$, we conclude the desired (7.3.10). The approximation for (7.3.11) is similar, we omit the detail.

Things become more involved for (7.3.12), note that

$$\begin{aligned}
&\sum_{x=LX_0}^{L(X_0+\theta)}\sum_{y=LY_0}^{L(Y_0+\theta)} \mathcal{R}^d(LX,LY,x+1,y+1)^2\Delta_x\Delta_y \\
&= \mathcal{R}^d(LX,LY,LX_0,Y_0)\left(\sum_{x=LX_0}^{L(X_0+\theta)}\sum_{y=LY_0}^{L(Y_0+\theta)}\Delta_x\Delta_y\right) + O(\theta^3), \\
&= L^{-2}(\log\mathfrak{q})^2\mathcal{R}_{IJ}(X,Y,X_0,Y_0)\mathfrak{q}^{2\mathbf{h}(X_0,Y_0)}\left(\sum_{x=LX_0}^{L(X_0+\theta)}\sum_{y=LY_0}^{L(Y_0+\theta)}\nabla_x H(x,y)\nabla_y H(x,y)\right) + o(1)O(\theta^2) + O(\theta^3),
\end{aligned} \tag{7.3.17}$$

where we denote by $\nabla_x H(x,y) := H(x+1,y) - H(x,y), \nabla_y H(x,y) := H(x,y+1) - H(x,y)$. In the last equality, we used the approximation in (7.3.13) and

$$\begin{aligned}
\Delta_x &= q^{H(x+1,y)} - q^{H(x,y)} = L^{-1}\nabla_x H(x,y)\mathfrak{q}^{\mathbf{h}(X_0,Y_0)}\log\mathfrak{q} + L^{-1}o(1), \\
\Delta_y &= q^{H(x,y+1)} - q^{H(x,y)} = L^{-1}\nabla_y H(x,y)\mathfrak{q}^{\mathbf{h}(X_0,Y_0)}\log\mathfrak{q} + L^{-1}o(1).
\end{aligned}$$

Note that $-\nabla_x H(x,y), \nabla_y H(x,y)$ indicate the number of lines entering into the vertex (x,y) from bottom and left.

For a vertex associated with four tuple $(i_1,j_1;i_2,j_2)$, we say this vertex is *unusual* if $i_1 \neq i_2$ or $j_1 \neq i_2$. Let $\square$ denote the square $[LX_0, LX_0+L\theta]\times[LY_0, LY_0+L\theta]$ and suppose that there are respectively n and m lines entering inside $\square$ from bottom and left. Let C be the number of unusual vertices in the square. If $C=0$, it is clear that

$$\sum_{x=LX_0}^{L(X_0+\theta)}\sum_{y=LY_0}^{L(Y_0+\theta)}\nabla_x H(x,y)\nabla_y H(x,y) = -nm.$$

Each unusual vertex might change the LHS summation at most by $2IJ\theta L$. As an analogue of [BG19, Eq. 93],

$$\left|\sum_{x=LX_0}^{L(X_0+\theta)}\sum_{y=LY_0}^{L(Y_0+\theta)}\nabla_x H(x,y)\nabla_y H(x,y) + nm\right| \leq IJ\theta L\cdot C.$$

It follows from Lemma 7.2.4 that the probability that a vertex is unusual is upper bounded by CL^{-1}, where C is a constant. Thus,

$$\Big| \sum_{x=LX_0}^{L(X_0+\theta)} \sum_{y=LY_0}^{L(Y_0+\theta)} \nabla_x H(x,y) \nabla_y H(x,y) + nm \Big| \leq \text{const} \cdot \theta^3 L^2, \tag{7.3.18}$$

with high probability as $L \to \infty$. Noting that

$$H\big(L(X_0+\theta), LY_0\big) - H(LX_0, LY_0) = -n, \qquad H\big(LX_0, L(Y_0+Y)\big) - H(LX_0, LY_0) = m.$$

Combining (7.3.17) and (7.3.18) (together with Theorem 7.2.3) yields

$$\sum_{x=LX_0}^{L(X_0+\theta)} \sum_{y=LY_0}^{L(Y_0+\theta)} \mathcal{R}^d(LX, LY, x+1, y+1)^2 \Delta_x \Delta_y$$
$$= L^{-2} (\log \mathfrak{q})^2 \mathcal{R}_{IJ}(X, Y, X_0, Y_0) \mathfrak{q}^{2\mathbf{h}(X_0,Y_0)} \big(H(L(X_0+\theta), LY_0) - H(LX_0, LY_0)\big)\big(H(LX_0, L(Y_0+\theta)\big) - H(LX_0, LY_0$$
$$\quad + o(1) \mathcal{O}(\theta^2) + \mathcal{O}(\theta^3)$$
$$= \mathcal{R}_{IJ}(X, Y, X_0, Y_0)(\log \mathfrak{q})^2 \mathfrak{q}^{2\mathbf{h}(X_0,Y_0)} \big(\mathbf{h}(X_0+\theta, Y_0) - \mathbf{h}(X_0, Y_0)\big)\big(\mathbf{h}(X_0, Y_0+\theta) - \mathbf{h}(X_0, Y_0)\big) + o(1)\mathcal{O}(\theta^2) + \mathcal{O}(\theta^3$$

Using similar approximation as in (7.3.16), by first letting $L \to \infty$ then $\theta \to 0$, we demonstrate (7.3.12). Having proved (7.3.10)-(7.3.12), we simply obtain the desired (7.3.9).

We conclude the theorem using martingale CLT [HH14, Section 3]. Recall that

$$M_L(t) = \sqrt{L} \sum_{i=1}^{t} \mathcal{R}^d\big(LX, LY, x(i), y(i)\big) \xi\big(x(i), y(i)\big), \qquad t \in [1, L^2 XY],$$

We want to show $U_L(X,Y) = M_L(L^2XY) \to \varphi(X,Y)$ in law as $L \to \infty$. By Theorem 7.2.3, $M_L(t)$ is a martingale with respect to the its own filtration. The proof of Theorem 7.1.7 reduces to verify the following conditions for martingale CLT:

(i). The conditional covariance of $M_L(t)$ at $t = L^2XY$, which equals

$$L \sum_{x=0}^{LX-1} \sum_{y=0}^{LY-1} \mathcal{R}^d(LX, LY; x+1, y+1)^2 \mathbb{E}\Big[\xi(x+1, y+1)^2 \Big| \mathcal{F}(x, y)\Big],$$

has the same $L \to \infty$ behavior as its unconditional variance, in the sense that their ratio tends to 1 in probability.

(ii). The Lindeberg's condition, i.e. $\lim_{L\to\infty} \sum_{i=1}^{L^2XY} \mathbb{E}\Big[(M_L(i)-M_L(i-1))^2 \mathbf{1}_{\{(M_L(i)-M_L(i-1))^2>\epsilon\}}\Big] = 0$.

Using Corollary 7.2.7, it is clear that the conditional variance on the LHS of (7.3.9) is uniformly bounded. By the convergence in (7.3.9) together with dominated convergence theorem, we know that both the conditional and unconditional variance of $M_L(t)$ at $t = L^2XY$ converge to the RHS of (7.3.9) (which equals to variance of $\varphi(X, Y)$ given in Remark 7.1.8), this concludes (i).

The Lindeberg's condition (ii) follows directly from how ξ is defined: By straightforward computation, there exists constant C such that $|\xi(x+1, y+1)| \leq CL^{-1}$ for all $x \in [0, LA]$ and $y \in [0, LB]$. In addition, $\mathcal{R}^d(LX, LY, x, y)$ is uniformly bounded. So when L is large enough,

$$\Big\{\big(M_L(i) - M_L(i-1)\big)^2 > \epsilon\Big\} = \Big\{L\mathcal{R}^d(LX, LY, x(i), y(i))^2 \xi(x(i), y(i))^2 > \epsilon\Big\} = \emptyset,$$

which implies that for every $i \in [1, L^2XY]$,

$$\mathbb{E}\Big[(M_L(i) - M_L(i-1))^2 \mathbf{1}_{\{(M_L(i)-M_L(i-1))^2>\epsilon\}}\Big] = 0.$$

Having verified (i) and (ii), we conclude our proof using the martingale central limit theorem. □

We move on proving Proposition 7.3.2. Before presenting our proof, we require the following result.

Lemma 7.3.3. *Fixed $A, B \geq 0$ and $n, \ell_1, \dots \ell_n \in \mathbb{N}$, there exists constant C (only depends on A, B, n) such that for all $L > 1$ and arbitrary distinct points $(x_i, y_i) \in [1, LA] \times [1, LB]$, $i = 1, \dots n$,*

$$\mathbb{E}\Big[\prod_{i=1}^{n} |\xi(x_i, y_i)|^{\ell_i}\Big] \leq CL^{-\sum_{i=1}^{n}(\ell_i+1)}.$$

Proof. It suffices to prove that for $(x, y) \in [0, LA - 1] \times [0, LB - 1]$,

$$\mathbb{E}\Big[|\xi(x+1, y+1)|^{\ell}\Big|\mathcal{F}(x, y)\Big] \leq CL^{-\ell-1}. \tag{7.3.19}$$

We first finish the proof of the lemma by assuming (7.3.19). Consider the ordering (7.3.8) of integer points in $[1, LA] \times [1, LB]$, without loss of generality, we assume $(x_i, y_i) = (x(s_i), y(s_i))$ so that $s_1 < \cdots < s_n$. Recall that $\mathcal{F}(x, y) = \sigma\big(H(i, j) : i \leq x \text{ or } j \leq y\big)$, so $\xi(x_i, y_i) \in \mathcal{F}(x_n - 1, y_n - 1)$ for $i = 1, \dots, n-1$. By (7.3.19) and conditioning,

$$\begin{aligned}\mathbb{E}\Big[\prod_{i=1}^{n} |\xi(x_i, y_i)|^{\ell_i}\Big] &= \mathbb{E}\Big[\prod_{i=1}^{n-1} |\xi(x_i, y_i)|^{\ell_i}\Big]\mathbb{E}\Big[|\xi(x_n, y_n)|^{\ell_n}\Big|\mathcal{F}(x_n - 1, y_n - 1)\Big] \\ &\leq CL^{-\ell_n-1}\mathbb{E}\Big[\prod_{i=1}^{n-1} |\xi(x_i, y_i)|^{\ell_i}\Big].\end{aligned}$$

Iterating the above inequality, we conclude the lemma.

We move on showing (7.3.19). Denote v, v' to be the vertical input and output for the vertex (x, y) and h to be the horizontal input, i.e.

$$v := H(x, y) - H(x+1, y), \quad v' := H(x, y+1) - H(x+1, y+1), \quad h := H(x, y+1) - H(x, y).$$

It is evident that we can rewrite $\xi(x+1, y+1)$ in (7.2.5) as

$$\xi(x+1, y+1) = q^{H(x,y)}\big(q^{h-v'} - b_1 q^h - b_2 q^{-v} + b_1 + b_2 - 1\big), \tag{7.3.20}$$

recall $b_1 = \frac{\alpha+\nu}{1+\alpha}$ and $b_2 = \frac{1+\alpha q^J}{1+\alpha}$. Since $q = \mathfrak{q}^{\frac{1}{L}}$ where $\mathfrak{q}$ is fixed, so for $(x, y) \in [0, LA] \times [0, LB]$, there exists C such that $\frac{1}{C} \le q^{H(x,y)} \le C$. In addition, by (7.3.2),

$$q^{h-v'} - b_1 q^h - b_2 q^{-v} + b_1 + b_2 - 1 = q^{h-v'} - q^h - q^{-v} + 1 + (1-b_1)(q^h - 1) + (1-b_2)(q^{-v} - 1)$$
$$= \ln \mathfrak{q}(v - v')L^{-1} + O(L^{-2})$$

Referring to (7.3.20), we conclude that for fixed A and B there exists a constant C such that for arbitrary $L > 1$, $(x, y) \in [0, LA] \times [0, LB]$,

$$\begin{aligned} |\xi(x+1, y+1)| \le CL^{-2} \qquad & \text{if } (h, v) = (h', v') \\ |\xi(x+1, y+1)| \le CL^{-1} \qquad & \text{if } (h, v) \ne (h', v') \end{aligned} \tag{7.3.21}$$

Note that

$$\begin{aligned} \mathbb{E}\Big[|\xi(x+1,y+1)|^{\ell}\Big|\mathcal{F}(x,y)\Big] &= \mathbb{E}\Big[|\xi(x+1,y+1)|^{\ell}\Big|\sigma(H(x,y),h,v)\Big] \\ &= \sum_{(h',v'):h'+v'=h+v} L_{\alpha}^{(J)}(h,v;h',v')|\xi(x+1,y+1)|^{\ell} \end{aligned} \tag{7.3.22}$$

Using Lemma 7.2.4 and (7.3.21), we know that for each term in the summation: Either $(h', v') \ne (h, v)$, which implies $L_{\alpha}^{(J)}(h, v; h', v') \le CL^{-1}$ and $|\xi(x+1, y+1)| \le CL^{-1}$; Either $(h, v) = (h', v')$, which yields $|\xi(x + 1, y + 1)| \le CL^{-2}$. Hence, the absolute value for each term in the summation (7.3.22) is upper bounded by $CL^{-\ell-1}$. As the summation is finite, we conclude (7.3.19). □

Proof of Proposition 7.3.2. Using the Kolmogorov-Chentsov criterion, the tightness of $U_L(\cdot,\cdot)$ follows directly from (7.3.5). To prove (7.3.5), it suffices to show that there exists constant C such that for $X \in [0, LA]$ and $0 \le Y_1 \le Y_2 \le LB$,

$$\mathbb{E}\Big[\Big(U_L(X,Y_1) - U_L(X,Y_2)\Big)^{2n}\Big] \le C|Y_1 - Y_2|^n, \tag{7.3.23}$$

Since we linearly interpolate $H(X, Y)$ for non-integer X, Y and $U_L(X, Y)$ is expressed in terms of

$H(LX, LY)$, we can assume $Y_2 - Y_1 \geq L^{-1}$. Referring to (7.3.6), we know that

$$U_L(X,Y) = \sqrt{L} \sum_{x=1}^{LX} \sum_{y=1}^{LY} \mathcal{R}^d(LX, LY, x, y)\xi(x, y), \tag{7.3.24}$$

which implies

$$\begin{aligned} U_L(X,Y_2) - U_L(X,Y_1) = &\sum_{x=1}^{LX} \sum_{y=1}^{LY_1} \sqrt{L}\big(\mathcal{R}^d(LX, LY_1, x, y) - \mathcal{R}^d(LX, LY_2, x, y)\big)\xi(x, y) \\ &+ \sum_{x=1}^{LX} \sum_{y=LY_1+1}^{LY_2} \sqrt{L}\mathcal{R}^d(LX, LY_2, x, y)\xi(x, y) \end{aligned}$$

Taking the n-th power of both sides in the above display and apply the inequality $(a + b)^{2n} \leq 2^{2n-1}(a^{2n} + b^{2n})$ to the RHS, we have

$$\begin{aligned} \mathbb{E}\Big[\big(U_L(X,Y_2) - U_L(X,Y_1)\big)^{2n}\Big] \leq &\, 2^{2n-1}\mathbb{E}\Big[\Big(\sum_{x=1}^{LX} \sum_{y=1}^{LY_1} \sqrt{L}\big(\mathcal{R}^d(LX, LY_1, x, y) - \mathcal{R}^d(LX, LY_2, x, y)\big)\xi(x, y)\Big)^{2n}\Big] \\ &+ 2^{2n-1}\mathbb{E}\Big[\Big(\sum_{x=1}^{LX} \sum_{y=LY_1+1}^{LY_2} \sqrt{L}\mathcal{R}^d(LX, LY_2, x, y)\xi(x, y)\Big)^{2n}\Big] \end{aligned} \tag{7.3.25}$$

Denote the first and second term above (without the constant multiplier) by $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively. We proceed to upper bound $\mathbf{M}_1$ and $\mathbf{M}_2$ respectively.

For $\mathbf{M}_1$, since $\xi(x, y)$ is a martingale increment, by Burkholder–Davis–Gundy inequality, we have

$$\mathbf{M}_1 \leq C(n)L^n\mathbb{E}\Big[\Big(\sum_{x=1}^{LX} \sum_{y=1}^{LY_1} \big(\mathcal{R}^d(LX, LY_1, x, y) - \mathcal{R}^d(LX, LY_2, x, y)\big)^2\xi(x, y)^2\Big)^n\Big],$$

where the constant $C(n)$ only depends on n. Under scaling (7.3.2), there exists a constant C such that for $L > 1$, $X \in [0, LA]$ and $Y_1, Y_2 \in [0, LB]$ (one can see this from the expression of $\mathcal{R}^d$ in (7.1.7)),

$$\Big|\mathcal{R}^d(LX, LY_1, x, y) - \mathcal{R}^d(LX, LY_2, x, y)\Big| \leq C|Y_1 - Y_2|,$$

this implies

$$\mathbf{M}_1 \le C(n)|Y_1 - Y_2|^{2n} \cdot L^n \mathbb{E}\Big[\Big(\sum_{x=1}^{LX}\sum_{y=1}^{LY_1} \xi(x,y)^2\Big)^n\Big]. \tag{7.3.26}$$

We claim that for all $L > 1$, the term $L^n \mathbb{E}\Big[\big(\sum_{x=1}^{LX}\sum_{y=1}^{LY_1} \xi(x,y)^2\big)^n\Big]$ is uniformly upper bounded for $(x,y) \in [0, LA] \times [0, LB]$. To see this, we expand the n-th power of the double summation in the expectation above. It is not hard to see that there exists a constant C such that

$$L^n \mathbb{E}\Big[\Big(\sum_{x=1}^{LX}\sum_{y=1}^{LY_1} \xi(x,y)^2\Big)^n\Big] \le C L^n \sum_{\lambda \vdash n} \sum_{\substack{(x_i,y_i)\in[1,LX]\times[1,LY_1] \\ i=1,\dots,\ell(\lambda),(x_i,y_i) \text{ are distinct}}} \mathbb{E}\Big[\prod_{i=1}^{\ell(\lambda)} \xi(x_i,y_i)^{2\lambda_i}\Big]$$

Here, the summation above is taken over the partition λ of n, that is to say, $\lambda = (\lambda_1 \ge \cdots \ge \lambda_s) \in \mathbb{Z}_{\ge 1}^s$ with $\sum_{i=1}^s \lambda_i = n$, $\ell(\lambda) = s$ is the length of the partition λ. We want to upper bound the right hand side in the above display. By Lemma 7.3.3, we know that the $\mathbb{E}\Big[\prod_{i=1}^{\ell(\lambda)} \xi(x_i,y_i)^{2\lambda_i}\Big]$ can be upper bounded by a constant times $L^{-2n-\ell(\lambda)}$. In addition, it is clear that $\#\big\{(x_i,y_i) \in [1,LX]\times[1,LY_1], i = 1,\dots,\ell(\lambda), (x_i,y_i) \text{ are distinct}\big\} \le (L^2 X Y_1)^{\ell(\lambda)}$ ($\#A$ denotes the number of elements in the set A). Consequently

$$L^n \mathbb{E}\Big[\Big(\sum_{x=1}^{LX}\sum_{y=1}^{LY_1} \xi(x,y)^2\Big)^n\Big] \le C L^n \sum_{\lambda \vdash n} (L^2 X Y_1)^{\ell(\lambda)} L^{-2n-\ell(\lambda)} \le C.$$

Inserting the above upper bound into (7.3.26) implies

$$\mathbf{M}_1 \le C(n)|Y_2 - Y_1|^{2n}. \tag{7.3.27}$$

We proceed to upper bound $\mathbf{M}_2$. Again, using Burkholder-Davis-Gundy inequality, one obtains

$$\mathbf{M}_2 \le C(n) L^n \mathbb{E}\Big[\Big(\sum_{x=1}^{LX}\sum_{y=LY_1+1}^{LY_2} \mathcal{R}^d(LX, LY_2, x, y)^2 \xi(x,y)^2\Big)^n\Big].$$

Expanding the n-th power for the double summation and upper bounding the square of $\mathcal{R}^d$ by a constant,

$$\mathbf{M}_2 \le C(n)L^n \sum_{\lambda \vdash n} \sum_{\substack{(x_i,y_i)\in[1,LX]\times(LY_1,LY_2] \\ i=1,\dots,\ell(\lambda),(x_i,y_i) \text{ are distinct}}} \mathbb{E}\Big[\prod_{i=1}^{\ell(\lambda)} \xi(x_i,y_i)^{2\lambda_i}\Big].$$

Using Lemma 7.3.3 and by similar argument for upper bounding $\mathbf{M}_1$, we have

$$\mathbf{M}_2 \le C(n)L^n \sum_{\lambda \vdash n} (L^2 X(Y_2-Y_1))^{\ell(\lambda)} L^{-2n-\ell(\lambda)} \le C(n)L^{\ell(\lambda)-n}(Y_2-Y_1)^{\ell(\lambda)} \le C(n)|Y_2-Y_1|^n \tag{7.3.28}$$

The last inequality in the above display is due to our assumption $Y_2 - Y_1 \ge L^{-1}$.

Referring to (7.3.25), we have

$$\mathbb{E}\Big[\big(U_L(X,Y_2) - U_L(X,Y_1)\big)^{2n}\Big] \le 2^{2n-1}(\mathbf{M}_1 + \mathbf{M}_2).$$

Combining (7.3.27) with (7.3.28), we conclude (7.3.23). □

Remark 7.3.4. It is worth remarking that the classical theory for martingale functional CLT, e.g. [Bro71, Section 6], might not be helpful for proving our tightness. In order to get the tightness, the classical theory requires $U_L(X,Y)$ to be a martingale in (X,Y) in order to control (using martingale inequalities) the modulus

$$\sup_{|X_1-X_2|+|Y_1-Y_2|\le\delta} |U_L(X_1,Y_1) - U_L(X_2,Y_2)|,$$

for small $\delta > 0$, and then apply the Arzela-Ascoli. See [Bil13, Theorem 7.3]. In our case, though $\xi(x,y)$ is a martingale increment, $U_L(X,Y)$ fails to be a martingale due to dependence of $\mathcal{R}^d$ on X, Y in (7.3.24).

Proof of Corollary 7.1.9. It suffices to prove the weak convergence for arbitrary interval $[0,A] \times$

$[0, B]$. Note that $U(x, y) = q^{H(x,y)} - \mathbb{E}\big[q^{H(x,y)}\big]$, then

$$H(Lx, Ly) = L \log_{\mathfrak{q}} \Big(q^{H(Lx,Ly)}\Big) = L \log_{\mathfrak{q}} \mathbb{E}\Big[q^{H(Lx,Ly)}\Big] + L \log_{\mathfrak{q}} \left(1 + \frac{U(Lx, Ly)}{\mathbb{E}\big[q^{H(Lx,Ly)}\big]}\right). \quad (7.3.29)$$

Since $H(x, y)$ is Lipschitz and $q = \mathfrak{q}^{\frac{1}{L}}$ (where $\mathfrak{q}$ is fixed), there exists constant C such that for $(x, y) \in [0, LA] \times [0, LB]$,

$$\frac{1}{C} \leq q^{H(Lx,Ly)} \leq C, \qquad \frac{1}{C} \leq \mathbb{E}\Big[q^{H(Lx,Ly)}\Big] \leq C.$$

For the second term on the right hand side of (7.3.29), we taylor expand the function $\log_{\mathfrak{q}}(1 + x)$ around $x = 0$,

$$H(Lx, Ly) = L \log_{\mathfrak{q}} \mathbb{E}\Big[q^{H(Lx,Ly)}\Big] + \frac{LU(Lx, Ly)}{\log \mathfrak{q} \cdot \mathbb{E}\big[q^{H(Lx,Ly)}\big]} + Lr_L(x, y),$$

where $|r_L(x, y)| \leq CU(Lx, Ly)^2 / \big(\mathbb{E}[q^{H(Lx,Ly)}]\big)^2 \leq CU(Lx, Ly)^2$. Consequently, since $\mathbb{E}\big[U(Lx, Ly)\big] = 0$,

$$\frac{H(Lx, Ly) - \mathbb{E}\Big[H(Lx, Ly)\Big]}{\sqrt{L}} = \frac{\sqrt{L}U(Lx, Ly)}{\mathbb{E}\Big[q^{H(Lx,Ly)}\Big] \log \mathfrak{q}} + \sqrt{L}\Big(r_L(x, y) - \mathbb{E}\big[r_L(x, y)\big]\Big).$$

By Proposition 7.3.2, we know that $U_L(\cdot, \cdot) = \sqrt{L}U(L\cdot, L\cdot)$ is tight. Thus, for any fixed $A, B > 0$, as $L \to \infty$,

$$\sup_{x \in [0,A]\times[0,B]} L^{\frac{1}{2}} U(Lx, Ly)^2 \to 0 \quad \text{in probability.}$$

Since $|r_L(x, y)| \leq CU(Lx, Ly)^2$,

$$\sup_{(x,y) \in [0,A]\times[0,B]} \sqrt{L}\Big|r_L(x, y) - \mathbb{E}\big[r_L(x, y)\big]\Big| \to 0 \quad \text{in probability.}$$

Therefore, we have the weak convergence in $C([0, A] \times [0, B])$,

$$\lim_{L\to\infty} \frac{H(Lx, Ly) - \mathbb{E}\Big[H(Lx, Ly)\Big]}{\sqrt{L}} = \lim_{L\to\infty} \frac{\sqrt{L}U(Lx, Ly)}{\mathbb{E}\big[q^{H(Lx,Ly)}\big] \log \mathfrak{q}} = \frac{\varphi(x, y)}{\mathfrak{q}^{\mathbf{h}(x,y)} \log \mathfrak{q}}.$$

To get the second equality above, we apply Theorem 7.1.6 and Theorem 7.1.7 to the denominator and numerator respectively. By straightforward computation, $\phi(x, y) := \frac{\varphi(x,y)}{\mathfrak{q}^{\mathbf{h}(x,y)} \log \mathfrak{q}}$ solves (7.1.17), which concludes the corollary. □

Bibliography

[AAR00] George E Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71. Cambridge university press, 2000.

[AB16] Amol Aggarwal and Alexei Borodin. Phase transitions in the ASEP and stochastic six-vertex model. *arXiv:1607.08684, to appear in Annals of Probability*, 2016.

[ACQ11] Gideon Amir, Ivan Corwin, and Jeremy Quastel. Probability distribution of the free energy of the continuum directed random polymer in 1+ 1 dimensions. *Communications on pure and applied mathematics*, 64(4):466–537, 2011.

[Agg17] Amol Aggarwal. Convergence of the stochastic six-vertex model to the ASEP. *Mathematical Physics, Analysis and Geometry*, 20(2):3, 2017.

[Agg18a] Amol Aggarwal. Current fluctuations of the stationary ASEP and six-vertex model. *Duke Mathematical Journal*, 167(2):269–384, 2018.

[Agg18b] Amol Aggarwal. Dynamical stochastic higher spin vertex models. *Selecta Mathematica*, 24(3):2659–2735, 2018.

[AGZ10] Greg W Anderson, Alice Guionnet, and Ofer Zeitouni. *An introduction to random matrices*, volume 118. Cambridge university press, 2010.

[AS48] Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55. US Government printing office, 1948.

[Bax16] Rodney J Baxter. *Exactly solved models in statistical mechanics*. Elsevier, 2016.

[BBC16] Alexei Borodin, Alexey Bufetov, and Ivan Corwin. Directed random polymers via nested contour integrals. *Annals of Physics*, 368:191–247, 2016.

[BBC20] Guillaume Barraquand, Alexei Borodin, and Ivan Corwin. Half-space Macdonald processes. *Forum of Mathematics, Pi*, 8, 2020.

[BBCS18] Jinho Baik, Guillaume Barraquand, Ivan Corwin, and Toufic Suidan. Pfaffian Schur processes and last passage percolation in a half-quadrant. *The Annals of Probability*, 46(6):3015–3089, 2018.

[BBCW18] Guillaume Barraquand, Alexei Borodin, Ivan Corwin, and Michael Wheeler. Stochastic six-vertex model in a half-quadrant and half-line open asymmetric simple exclusion process. *Duke Mathematical Journal*, 167(13):2457–2529, 2018.

[BC95] Lorenzo Bertini and Nicoletta Cancrini. The stochastic heat equation: Feynman-Kac formula and intermittence. *Journal of statistical Physics*, 78(5-6):1377–1401, 1995.

[BC14a] A. Borodin and I. Corwin. Macdonald processes. *Probability Theory and Related Fields*, 158(1-2):225–400, 2014.

[BC14b] A. Borodin and I. Corwin. Moments and Lyapunov exponents for the parabolic Anderson model. *Ann. Appl. Probab.*, 24(3):1172–1198, 2014.

[BC16] R. M. Balan and D. Conus. Intermittency for the wave and heat equations with fractional noise in time. *Ann. Probab.*, 44(2):1488–1534, 2016.

[BCG16] Alexei Borodin, Ivan Corwin, and Vadim Gorin. Stochastic six-vertex model. *Duke Mathematical Journal*, 165(3):563–624, 2016.

[BCPS15] Alexei Borodin, Ivan Corwin, Leonid Petrov, and Tomohiro Sasamoto. Spectral theory for interacting particle systems solvable by coordinate Bethe ansatz. *Communications in Mathematical Physics*, 339(3):1167–1245, 2015.

[BCPS19] Alexei Borodin, Ivan Corwin, Leonid Petrov, and Tomohiro Sasamoto. Correction to: Spectral theory for interacting particle systems solvable by coordinate bethe ansatz. *Communications in Mathematical Physics*, 370(3):1069–1072, 2019.

[BCS14] Alexei Borodin, Ivan Corwin, and Tomohiro Sasamoto. From duality to determinants for q-TASEP and ASEP. *The Annals of Probability*, 42(6):2314–2382, 2014.

[BDM08] Amarjit Budhiraja, Paul Dupuis, and Vasileios Maroulas. Large deviations for infinite dimensional stochastic dynamical systems. *Ann Probab*, pages 1390–1420, 2008.

[BDSG+15] Lorenzo Bertini, Alberto De Sole, Davide Gabrielli, Giovanni Jona-Lasinio, and Claudio Landim. Macroscopic fluctuation theory. *Rev Modern Phys*, 87(2):593, 2015.

[BFO20] Dan Betea, Patrik L Ferrari, and Alessandra Occelli. Stationary half-space last passage percolation. *Communications in Mathematical Physics*, pages 1–47, 2020.

[BG97] L. Bertini and G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. *Comm. Math. Phys.*, 183(3):571–607, 1997.

[BG16] Alexei Borodin and Vadim Gorin. Moments match between the KPZ equation and the Airy point process. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 12:102, 2016.

[BG18] Alexei Borodin and Vadim Gorin. A stochastic telegraph equation from the six-vertex model. *arXiv preprint arXiv:1803.09137*, 2018.

[BG19] Alexei Borodin and Vadim Gorin. A stochastic telegraph equation from the six-vertex model. *The Annals of Probability*, 47(6):4137–4194, 2019.

[BGS17] Riddhipratim Basu, Shirshendu Ganguly, and Allan Sly. Upper tail large deviations in first passage percolation. *arXiv preprint arXiv:1712.01255*, 2017.

[BGS19] Riddhipratim Basu, Shirshendu Ganguly, and Allan Sly. Delocalization of polymers in lower tail large deviation. *Commun Math Phys*, 370(3):781–806, 2019.

[Bil13] Patrick Billingsley. *Convergence of probability measures*. John Wiley & Sons, 2013.

[BKD20] Guillaume Barraquand, Alexandre Krajenbrink, and Pierre Le Doussal. Half-space stationary kardar-parisi-zhang equation. *arXiv preprint arXiv:2003.03809*, 2020.

[BL19] Jinho Baik and Zhipeng Liu. Multipoint distribution of periodic TASEP. *Journal of the American Mathematical Society*, 2019.

[Bor17] Alexei Borodin. On a family of symmetric rational functions. *Advances in Mathematics*, 306:973–1018, 2017.

[Bor18] Alexei Borodin. Stochastic higher spin six vertex model and Macdonald measures. *Journal of Mathematical Physics*, 59(2):023301, 2018.

[BP18] Alexei Borodin and Leonid Petrov. Higher spin six vertex model and symmetric rational functions. *Selecta Mathematica*, 24(2):751–874, 2018.

[BR01] Jinho Baik and Eric M Rains. The asymptotics of monotone subsequences of involutions. *Duke Mathematical Journal*, 109(2):205–281, 2001.

[Bro71] Bruce M Brown. Martingale central limit theorems. *The Annals of Mathematical Statistics*, 42(1):59–66, 1971.

[BS15a] Vladimir Belitsky and Gunter M Schütz. Quantum algebra symmetry of the asep with second-class particles. *Journal of statistical physics*, 161(4):821–842, 2015.

[BS15b] Vladimir Belitsky and Gunter M Schütz. Self-duality for the two-component asymmetric simple exclusion process. *Journal of mathematical physics*, 56(8):083302, 2015.

[CC19] Mattia Cafasso and Tom Claeys. A Riemann-Hilbert approach to the lower tail of the KPZ equation. *arXiv preprint arXiv:1910.02493*, 2019.

[CD15] L. Chen and R. C. Dalang. Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions. *Ann. Probab.*, 43(6):3006–3051, 11 2015.

[CD19] S. Cerrai and A. Debussche. Large deviations for the two-dimensional stochastic Navier-Stokes equation with vanishing noise correlation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 55(1):211–236, 2019.

[CG20a] Ivan Corwin and Promit Ghosal. KPZ equation tails for general initial data. *Electronic Journal of Probability*, 25, 2020.

[CG20b] Ivan Corwin and Promit Ghosal. Lower tail of the KPZ equation. *Duke Mathematical Journal*, 169(7):1329–1395, 2020.

[CGH19] I. Corwin, P. Ghosal, and A. Hammond. KPZ equation correlations in time. *arXiv preprint arXiv:1907.09317*, 2019.

[CGK+18] Ivan Corwin, Promit Ghosal, Alexandre Krajenbrink, Pierre Le Doussal, and Li-Cheng Tsai. Coulomb-gas electrostatics controls large fluctuations of the Kardar-Parisi-Zhang equation. *Physical review letters*, 121(6):060201, 2018.

[CGRS16] Gioia Carinci, Cristian Giardinà, Frank Redig, and Tomohiro Sasamoto. A generalized asymmetric exclusion process with $U_q(sl2)$ stochastic duality. *Probability Theory and Related Fields*, 166(3-4):887–933, 2016.

[CGST20] Ivan Corwin, Promit Ghosal, Hao Shen, and Li-Cheng Tsai. Stochastic PDE limit of the six vertex model. *Communications in Mathematical Physics*, pages 1–94, 2020.

[CH08] Richard Courant and David Hilbert. *Methods of Mathematical Physics: Partial Differential Equations*. John Wiley & Sons, 2008.

[CH16] I. Corwin and A. Hammond. KPZ line ensemble. *Probability Theory and Related Fields*, 166(1-2):67–185, 2016.

[Che15] Xia Chen. Precise intermittency for the parabolic Anderson equation with an (1+1)-dimensional time–space white noise. *Annales de l'IHP Probabilités et statistiques*, 51(4):1486–1499, 2015.

[Che17] L. Chen. Nonlinear stochastic time-fractional diffusion equations on $\mathbb{R}$: moments, Hölder regularity and intermittency. *Trans. Amer. Math. Soc.*, 369(12):8497–8535, 2017.

[CHKN18] L. Chen, Y. Hu, K. Kalbasi, and D. Nualart. Intermittency for the stochastic heat equation driven by a rough time fractional Gaussian noise. *Probab. Theory Related Fields*, 171(1-2):431–457, 2018.

[CHN16] Le Chen, Yaozhong Hu, and David Nualart. Regularity and strict positivity of densities for the nonlinear stochastic heat equation. *arXiv:1611.03909*, 2016.

[CHN19] L. Chen, Y. Hu, and D. Nualart. Nonlinear stochastic time-fractional slow and fast diffusion equations on $\mathbb{R}^d$. *Stochastic Process. Appl.*, 129(12):5073–5112, 2019.

[CJK13] Daniel Conus, Mathew Joseph, and Davar Khoshnevisan. On the chaotic character of the stochastic heat equation, before the onset of intermittency. *The Annals of Probability*, 41(3B):2225–2260, 2013.

[CJKS13] Daniel Conus, Mathew Joseph, Davar Khoshnevisan, and Shang-Yuan Shiu. On the chaotic character of the stochastic heat equation, II. *Probability Theory and Related Fields*, 156(3-4):483–533, 2013.

[CM94] R. A. Carmona and S. A. Molchanov. Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.*, 108(518):viii+125, 1994.

[CM97] Fabien Chenal and Annie Millet. Uniform large deviations for parabolic SPDEs and applications. *Stochastic Process Appl*, 72(2):161–186, 1997.

[Com17] F. Comets. *Directed polymers in random environments*, volume 2175 of *Lecture Notes in Mathematics*. Springer, Cham, 2017. Lecture notes from the 46th Probability Summer School held in Saint-Flour, 2016.

[Cor12] Ivan Corwin. The Kardar–Parisi–Zhang equation and universality class. *Random matrices: Theory and applications*, 1(01):1130001, 2012.

[Cor18] Ivan Corwin. Exactly solving the KPZ equation. *arXiv:1804.05721*, 2018.

[CP] Ivan Corwin and Leonid Petrov. Personal communication.

[CP16] Ivan Corwin and Leonid Petrov. Stochastic higher spin vertex models on the line. *Communications in Mathematical Physics*, 343(2):651–700, 2016.

[CP19] Ivan Corwin and Leonid Petrov. Correction to: Stochastic higher spin vertex models on the line. *Communications in Mathematical Physics*, 371(1):353–355, 2019.

[CS18] Ivan Corwin and Hao Shen. Open ASEP in the weakly asymmetric regime. *Communications on Pure and Applied Mathematics*, 71(10):2065–2128, 2018.

[CS19] Ivan Corwin and Hao Shen. Some recent progress in singular stochastic PDEs. *Bull Amer Math Soc*, 57:409–454, 2019.

[CST18] Ivan Corwin, Hao Shen, and Li-Cheng Tsai. ASEP(q, j) converges to the KPZ equation. *Annales de l'Institut Henri Poincaré, Probabilités et Statistiques*, 54(2):995–1012, 2018.

[CT17] Ivan Corwin and Li-Cheng Tsai. KPZ equation limit of higher-spin exclusion processes. *The Annals of Probability*, 45(3):1771–1798, 2017.

[CW17] Ajay Chandra and Hendrik Weber. Stochastic PDEs, regularity structures, and interacting particle systems. *Annales de la faculté des sciences de Toulouse Mathématiques*, 26(4):847–909, 2017.

[DL98] Bernard Derrida and Joel L Lebowitz. Exact large deviation function in the asymmetric exclusion process. *Physical review letters*, 80(2):209, 1998.

[DNKDT19] Jacopo De Nardis, Alexandre Krajenbrink, Pierre Le Doussal, and Thimothée Thiery. Delta-Bose gas on a half-line and the KPZ equation: boundary bound states and unbinding transitions. *arXiv preprint arXiv:1911.06133*, 2019.

[DOV18] Duncan Dauvergne, Janosch Ortmann, and Bálint Virág. The directed landscape. *arXiv preprint arXiv:1812.00309*, 2018.

[DR20] Evgeni Dimitrov and Mark Rychnovsky. Gue corners process in boundary-weighed six-vertex models. *arXiv preprint arXiv:2005.06836*, 2020.

[DS01] Jean-Dominique Deuschel and Daniel W Stroock. *Large deviations*, volume 342. American Mathematical Soc., 2001.

[DT16] Amir Dembo and Li-Cheng Tsai. Weakly asymmetric non-simple exclusion process and the Kardar–Parisi–Zhang equation. *Communications in Mathematical Physics*, 341(1):219–261, 2016.

[DT19] S. Das and L.-C. Tsai. Fractional moments of the stochastic heat equation. *arXiv preprint arXiv:1910.09271*, 2019.

[DZ94] Amir Dembo and Ofer Zeitouni. Large deviations techniques and applications. *Applications of Mathematics (New York)*, 38, 1994.

[EK04] Vlad Elgart and Alex Kamenev. Rare event statistics in reaction-diffusion systems. *Phys Rev E*, 70(4):041106, 2004.

[Eva98] Lawrence C Evans. Partial differential equations. *Graduate studies in mathematics*, 19(2), 1998.

[Fer04] Patrik L Ferrari. Polynuclear growth on a flat substrate and edge scaling of GOE eigenvalues. *Communications in mathematical physics*, 252(1-3):77–109, 2004.

[FGV01] G Falkovich, K Gawedzki, and Massimo Vergassola. Particles and fields in fluid turbulence. *Rev Modern Phys*, 73(4):913, 2001.

[FK09] M. Foondun and D. Khoshnevisan. Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.*, 14:no. 21, 548–568, 2009.

[FKLM96] G Falkovich, I Kolokolov, V Lebedev, and A Migdal. Instantons and intermittency. *Physical Review E*, 54(5):4896, 1996.

[Flo14] Gregorio R Moreno Flores. On the (strict) positivity of solutions of the stochastic heat equation. *Ann Probab*, 42(4):1635–1643, 2014.

[Fog98] Hans C Fogedby. Soliton approach to the noisy burgers equation: Steepest descent method. *Phys Rev E*, 57(5):4943, 1998.

[FS10] Patrik L Ferrari and Herbert Spohn. Random growth models. *arXiv:1003.0881*, 2010.

[GGS15] Tobias Grafke, Rainer Grauer, and Tobias Schäfer. The instanton method and its numerical implementation in fluid mechanics. *J Phys A: Math Theor*, 48(33):333001, 2015.

[GH19] Máté Gerencsér and Martin Hairer. Singular SPDEs in domains with boundaries. *Probability Theory and Related Fields*, 173(3-4):697–758, 2019.

[Gho17] Promit Ghosal. Hall-Littlewood-PushTASEP and its KPZ limit. *arXiv preprint arXiv:1701.07308*, 2017.

[Gho18] Promit Ghosal. Moments of the SHE under delta initial measure. *arXiv preprint arXiv:1808.04353*, 2018.

[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[GJ14] Patrícia Gonçalves and Milton Jara. Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Archive for Rational Mechanics and Analysis*, 212(2):597–644, 2014.

[GKM07] Jürgen Gärtner, Wolfgang König, and Stanislav Molchanov. Geometric characterization of intermittency in the parabolic Anderson model. *The Annals of Probability*, 35(2):439–499, 2007.

[GL20] Promit Ghosal and Yier Lin. Lyapunov exponents of the SHE for general initial data. *arXiv preprint arXiv:2007.06505*, 2020.

[GLD12] Thomas Gueudré and Pierre Le Doussal. Directed polymer near a hard wall and KPZ equation in the half-space. *EPL (Europhysics Letters)*, 100(2):26006, 2012.

[GM90] J. Gärtner and S. A. Molchanov. Parabolic problems for the Anderson model. I. Intermittency and related topics. *Comm. Math. Phys.*, 132(3):613–655, 1990.

[GP17] M. Gubinelli and N. Perkowski. KPZ reloaded. *Comm. Math. Phys.*, 349(1):165–269, 2017.

[GP18] Massimiliano Gubinelli and Nicolas Perkowski. Energy solutions of KPZ are unique. *Journal of the American Mathematical Society*, 31(2):427–471, 2018.

[GPS20] Patricia Gonçalves, Nicolas Perkowski, and Marielle Simon. Derivation of the stochastic burgers equation with dirichlet boundary conditions from the wasep. *Annales Henri Lebesgue*, 3:87–167, 2020.

[GS92] Leh-Hun Gwa and Herbert Spohn. Six-vertex model, roughened surfaces, and an asymmetric spin Hamiltonian. *Physical review letters*, 68(6):725, 1992.

[Hai13] M. Hairer. Solving the KPZ equation. *Ann. of Math. (2)*, 178(2):559–664, 2013.

[Hai14] Martin Hairer. A theory of regularity structures. *Inventiones mathematicae*, 198(2):269–504, 2014.

[HH14] Peter Hall and Christopher C Heyde. *Martingale limit theory and its application*. Academic press, 2014.

[HHF85] D. A. Huse, C. L. Henley, and D. S. Fisher. Huse, henley, and fisher respond. *Phys. Rev. Lett.*, 55:2924–2924, Dec 1985.

[HHNT15] Y. Hu, J. Huang, D. Nualart, and S. Tindel. Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency. *Electron. J. Probab.*, 20:no. 55, 50, 2015.

[HL66] BI Halperin and Melvin Lax. Impurity-band tails in the high-density limit. i. minimum counting methods. *Phys Rev*, 148(2):722, 1966.

[HL18] Yaozhong Hu and Khoa Lê. Asymptotics of the density of parabolic Anderson random fields. *arXiv:1801.03386*, 2018.

[HLDM+18a] A. K. Hartmann, P. Le Doussal, S. N. Majumdar, A. Rosso, and G. Schehr. High-precision simulation of the height distribution for the KPZ equation. *Europhys. Lett.*, 121(6):67004, mar 2018.

[HLDM+18b] Alexander K Hartmann, Pierre Le Doussal, Satya N Majumdar, Alberto Rosso, and Gregory Schehr. High-precision simulation of the height distribution for the KPZ equation. *EPL (Europhysics Letters)*, 121(6):67004, 2018.

[HMS19] Alexander K Hartmann, Baruch Meerson, and Pavel Sasorov. Optimal paths of nonequilibrium stochastic fields: The kardar-parisi-zhang interface as a test case. *Physical Review Research*, 1(3):032043, 2019.

[HW15a] M. Hairer and H. Weber. Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions. *Ann. Fac. Sci. Toulouse Math. (6)*, 24(1):55–92, 2015.

[HW15b] Martin Hairer and Hendrik Weber. Large deviations for white-noise driven, nonlinear stochastic PDEs in two and three dimensions. *Annales de la Faculté des sciences de Toulouse: Mathématiques*, 24(1):55–92, 2015.

[IMS20] Takashi Imamura, Matteo Mucciconi, and Tomohiro Sasamoto. Stationary stochastic Higher Spin Six Vertex Model and q-Whittaker measure. *Probability Theory and Related Fields*, pages 1–120, 2020.

[JKM16] M. Janas, A. Kamenev, and B. Meerson. Dynamical phase transition in large-deviation statistics of the Kardar-Parisi-Zhang equation. *Phys. Rev. E*, 94:032133, Sep 2016.

[Kar85] Mehran Kardar. Depinning by quenched randomness. *Physical review letters*, 55(21):2235, 1985.

[Kar87] Mehran Kardar. Replica bethe ansatz studies of two-dimensional interfaces with quenched random impurities. *Nuclear Physics B*, 290:582–602, 1987.

[Kho14] D. Khoshnevisan. *Analysis of stochastic partial differential equations*, volume 119 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2014.

[Kim19] Yujin H Kim. The lower tail of the half-space KPZ equation. *arXiv preprint arXiv:1905.07703*, 2019.

[KK07] IV Kolokolov and SE Korshunov. Optimal fluctuation approach to a directed polymer in a random medium. *Physical Review B*, 75(14):140201, 2007.

[KK09] IV Kolokolov and SE Korshunov. Explicit solution of the optimal fluctuation problem for an elastic string in a random medium. *Physical Review E*, 80(3):031107, 2009.

[KKX17] Davar Khoshnevisan, Kunwoo Kim, and Yimin Xiao. Intermittency and multifractality: A case study via parabolic stochastic pdes. *The Annals of Probability*, 45(6A):3697–3751, 2017.

[KLD17] Alexandre Krajenbrink and Pierre Le Doussal. Exact short-time height distribution in the one-dimensional Kardar–Parisi–Zhang equation with Brownian initial condition. *Phys Rev E*, 96(2):020102, 2017.

[KLD18a] Alexandre Krajenbrink and Pierre Le Doussal. Large fluctuations of the KPZ equation in a half-space. *SciPost Phys*, 5:032, 2018.

[KLD18b] Alexandre Krajenbrink and Pierre Le Doussal. Simple derivation of the $(\lambda H)^{5/2}$ tail for the 1D KPZ equation. *Journal of Statistical Mechanics: Theory and Experiment*, 2018(6):063210, 2018.

[KLD19] A. Krajenbrink and P. Le Doussal. Linear statistics and pushed coulomb gas at the edge of β -random matrices: Four paths to large deviations. *Europhys. Lett.*, 125(2):20009, feb 2019.

[KLD20] Alexandre Krajenbrink and Pierre Le Doussal. Replica Bethe Ansatz solution to the Kardar-Parisi-Zhang equation on the half-line. *SciPost Phys.*, 8, 2020.

[KLDP18] Alexandre Krajenbrink, Pierre Le Doussal, and Sylvain Prolhac. Systematic time expansion for the Kardar–Parisi–Zhang equation, linear statistics of the GUE at the edge and trapped fermions. *Nucl Phys B*, 936:239–305, 2018.

[KMHH92] Joachim Krug, Paul Meakin, and Timothy Halpin-Healy. Amplitude universality for driven interfaces and directed polymers in random media. *Physical Review A*, 45(2):638, 1992.

[KMS16] Alex Kamenev, Baruch Meerson, and Pavel V Sasorov. Short-time height distribution in the one-dimensional kardar-parisi-zhang equation: Starting from a parabola. *Physical Review E*, 94(3):032108, 2016.

[KPZ86] Mehran Kardar, Giorgio Parisi, and Yi-Cheng Zhang. Dynamic scaling of growing interfaces. *Physical Review Letters*, 56(9):889, 1986.

[KR87] Anatol N Kirillov and N Yu Reshetikhin. Exact solution of the integrable XXZ Heisenberg model with arbitrary spin. I. The ground state and the excitation spectrum. *Journal of Physics A: Mathematical and General*, 20(6):1565, 1987.

[Kra19] Alexandre Krajenbrink. *Beyond the typical fluctuations: a journey to the large deviations in the Kardar-Parisi-Zhang growth model*. PhD thesis, PSL Research University, 2019.

[KRS81] PP Kulish, N Yu Reshetikhin, and EK Sklyanin. Yang-baxter equation and representation theory: I. *Letters in Mathematical Physics*, 5(5):393–403, 1981.

[Kua] Jeffrey Kuan. Personal communication.

[Kua16] Jeffrey Kuan. Stochastic duality of ASEP with two particle types via symmetry of quantum groups of rank two. *Journal of Physics A: Mathematical and Theoretical*, 49(11):115002, 2016.

[Kua17] Jeffrey Kuan. A multi-species asep and-tazrp with stochastic duality. *International Mathematics Research Notices*, 2018(17):5378–5416, 2017.

[Kua18] Jeffrey Kuan. An Algebraic Construction of Duality Functions for the Stochastic $U_q(A_n^1)$ Vertex Model and Its Degenerations. *Communications in Mathematical Physics*, 359(1):121–187, 2018.

[Lab17] Cyril Labbé. Weakly asymmetric bridges and the KPZ equation. *Communications in Mathematical Physics*, 353(3):1261–1298, 2017.

[LDMRS16] Pierre Le Doussal, Satya N. Majumdar, Alberto Rosso, and Grégory Schehr. Exact short-time height distribution in the one-dimensional kardar-parisi-zhang equation and edge fermions at high temperature. *Phys. Rev. Lett.*, 117:070403, Aug 2016.

[LDMS16] Pierre Le Doussal, Satya N Majumdar, and Grégory Schehr. Large deviations for the height in 1D Kardar-Parisi-Zhang growth at late times. *EPL (Europhysics Letters)*, 113(6):60004, 2016.

[Le 19] P. Le Doussal. Large deviations for the KPZ equation from the KP equation. *arXiv e-prints*, October 2019.

[Led96] Michel Ledoux. Isoperimetry and gaussian analysis. In *Lectures on probability theory and statistics*, pages 165–294. Springer, 1996.

[Lie74] Elliott H Lieb. Residual entropy of square ice. *Matematika*, 18(4):64–84, 1974.

[Lif68] IM Lifshitz. Theory of fluctuating levels in disordered systems. *Sov Phys JETP*, 26(462):012110–9, 1968.

[Lig12] Thomas M Liggett. *Interacting particle systems*, volume 276. Springer Science & Business Media, 2012.

[Lig13] Thomas M Liggett. *Stochastic interacting systems: contact, voter and exclusion processes*, volume 324. Springer Science & Business Media, 2013.

[Lin19] Yier Lin. Markov Duality for Stochastic Six Vertex Model. *Electronic Communications in Probability*, 24(67):1–17, 2019.

[Lin20a] Yier Lin. KPZ equation limit of stochastic higher spin six vertex model. *Mathematical Physics, Analysis and Geometry*, 23(1):1, 2020.

[Lin20b] Yier Lin. Lyapunov exponents of the half-line SHE. *arXiv preprint arXiv:2007.10212*, 2020.

[Lin20c] Yier Lin. The stochastic telegraph equation limit of the stochastic higher spin six vertex model. *Electronic Journal of Probability*, 25, 2020.

[LT21] Yier Lin and Li-Cheng Tsai. Short time large deviations of the KPZ equation. *Communications in Mathematical Physics*, pages 1–35, 2021.

[Man14] Vladimir V Mangazeev. On the Yang–Baxter equation for the six-vertex model. *Nuclear Physics B*, 882:70–96, 2014.

[MKV16] Baruch Meerson, Eytan Katzav, and Arkady Vilenkin. Large deviations of surface height in the Kardar-Parisi-Zhang equation. *Physical review letters*, 116(7):070601, 2016.

[MN08] Carl Mueller and David Nualart. Regularity of the density for the stochastic heat equation. *Electron J Probab*, 13:2248–2258, 2008.

[Mol96] S. Molchanov. Reaction-diffusion equations in the random media: localization and intermittency. In *Nonlinear stochastic PDEs (Minneapolis, MN, 1994)*, volume 77 of *IMA Vol. Math. Appl.*, pages 81–109. Springer, New York, 1996.

[MQR16] Konstantin Matetski, Jeremy Quastel, and Daniel Remenik. The KPZ fixed point. *arXiv preprint arXiv:1701.00018*, 2016.

[MS11] Baruch Meerson and Pavel V Sasorov. Negative velocity fluctuations of pulled reaction fronts. *Phys Rev E*, 84(3):030101, 2011.

[MS17] Baruch Meerson and Johannes Schmidt. Height distribution tails in the Kardar–Parisi–Zhang equation with Brownian initial conditions. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(10):103207, 2017.

[Mue91] Carl Mueller. On the support of solutions to the heat equation with noise. *Stochastics: An International Journal of Probability and Stochastic Processes*, 37(4):225–245, 1991.

[MV18] Baruch Meerson and Arkady Vilenkin. Large fluctuations of a Kardar-Parisi-Zhang interface on a half line. *Physical Review E*, 98(3):032145, 2018.

[Nua06] David Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[Olv97] Frank Olver. *Asymptotics and special functions*. CRC Press, 1997.

[OP17] Daniel Orr and Leonid Petrov. Stochastic higher spin six vertex model and q-TASEPs. *Advances in Mathematics*, 317:473–525, 2017.

[Par19a] Shalin Parekh. Positive random walks and an identity for half-space SPDEs. *arXiv preprint arXiv:1901.09449*, 2019.

[Par19b] Shalin Parekh. The KPZ limit of ASEP with boundary. *Communications in Mathematical Physics*, 365(2):569–649, 2019.

[Pau35] Linus Pauling. The structure and entropy of ice and of other crystals with some randomness of atomic arrangement. *Journal of the American Chemical Society*, 57(12):2680–2684, 1935.

[QS15] Jeremy Quastel and Herbert Spohn. The one-dimensional KPZ equation and its universality class. *J Stat Phys*, 160(4):965–984, 2015.

[Qua11] Jeremy Quastel. Introduction to KPZ. *Current developments in mathematics*, 2011(1), 2011.

[Rai00] Eric M Rains. Correlation functions for symmetrized increasing subsequences. *arXiv preprint math/0006097*, 2000.

[Sch97] Gunter M Schütz. Duality relations for asymmetric exclusion processes. *Journal of statistical physics*, 86(5-6):1265–1287, 1997.

[SI04] T Sasamoto and T Imamura. Fluctuations of the one-dimensional polynuclear growth model in half-space. *Journal of statistical physics*, 115(3-4):749–803, 2004.

[SMP17] Pavel Sasorov, Baruch Meerson, and Sylvain Prolhac. Large deviations of surface height in the 1+ 1-dimensional Kardar–Parisi–Zhang equation: exact long-time results for $\lambda H < 0$. *Journal of Statistical Mechanics: Theory and Experiment*, 2017(6):063203, 2017.

[Spo12] Herbert Spohn. KPZ scaling theory and the semidiscrete directed polymer model. *Random Matrix Theory, Interacting Particle Systems and Integrable Systems*, 65, 2012.

[ST19] Hao Shen and Li-Cheng Tsai. Stochastic telegraph equation limit for the stochastic six vertex model. *Proceedings of the American Mathematical Society*, 147(6):2685–2705, 2019.

[Tsa18] Li-Cheng Tsai. Exact lower tail large deviations of the KPZ equation. *arXiv preprint arXiv:1809.03410*, 2018.

[TW94] Craig A Tracy and Harold Widom. Level-spacing distributions and the Airy kernel. *Communications in Mathematical Physics*, 159(1):151–174, 1994.

[TW96] Craig A Tracy and Harold Widom. On orthogonal and symplectic matrix ensembles. *Communications in Mathematical Physics*, 177(3):727–754, 1996.

[TW08] Craig A Tracy and Harold Widom. Integral formulas for the asymmetric simple exclusion process. *Communications in Mathematical Physics*, 279(3):815–844, 2008.

[TW09] Craig A Tracy and Harold Widom. The distributions of random matrix theory and their applications. In *New trends in mathematical physics*, pages 753–765. Springer, 2009.

[Wal86] John B Walsh. An introduction to stochastic partial differential equations. In *École d'Été de Probabilités de Saint Flour XIV-1984*, pages 265–439. Springer, 1986.

[Wu18] Xuan Wu. Intermediate disorder regime for half-space directed polymers. *arXiv preprint arXiv:1804.09815*, 2018.

[ZL66] J Zittartz and JS Langer. Theory of bound states in a random potential. *Phys Rev*, 148(2):741, 1966.

Appendix A: Basic facts of Airy function

In this section, we review some basic properties of the Airy function. As a notation convention, we say $f(x) \sim g(x)$ as $x \to a$ (where a can be $\pm\infty$) if $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$.

Lemma A.0.1. *We have the following asymptotics for Airy function*

$$\mathrm{Ai}(x) \sim \begin{cases} \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}} & x \to +\infty, \\ \frac{1}{\sqrt{\pi}|x|^{\frac{1}{4}}} \cos\left(\frac{\pi}{4} - \frac{2|x|^{\frac{3}{2}}}{3}\right) & x \to -\infty. \end{cases} \qquad \mathrm{Ai}'(x) \sim \begin{cases} -\frac{x^{\frac{1}{4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}} & x \to +\infty, \\ -\frac{|x|^{\frac{1}{4}}}{\sqrt{\pi}} \cos\left(\frac{\pi}{4} + \frac{2|x|^{\frac{3}{2}}}{3}\right) & x \to -\infty. \end{cases}$$

Proof. See Eq 10.4.59-10.4.62 of [AS48]. □

Lemma A.0.2. *We have* $\int_{-\infty}^{\infty} \mathrm{Ai}(x)dx = 1$ *and* $\int_{-\infty}^{0} \mathrm{Ai}(x)dx = 1/3$.

Proof. See page 431 of [Olv97]. □

Lemma A.0.3. *There exists constant C such that*

$$\begin{aligned} \frac{1}{C(x+1)}e^{-\frac{4}{3}x^{\frac{3}{2}}} \leq \int_0^\infty \mathrm{Ai}(x+\lambda)^2 d\lambda \leq \frac{C}{x+1}e^{-\frac{4}{3}x^{\frac{3}{2}}} \quad & \forall x \geq 0 \\ \frac{1}{C}(\sqrt{|x|}+1) \leq \int_0^\infty \mathrm{Ai}(x+\lambda)^2 d\lambda \leq C(\sqrt{|x|}+1), \qquad & \forall x \leq 0 \end{aligned}$$

Proof. This is Eq 2.8 and Eq 2.9 of [DT19]. □

Appendix B: Estimate of the Pfaffian Kernel entries

In this section, we provide various bounds for the entries in the GOE Pfaffian kernel.

Lemma B.0.1. *There exists a constant $C > 0$ such that*

(i) $\frac{\exp(-\frac{2}{3}x^{\frac{3}{2}})}{C(1+x)^{\frac{1}{4}}} \leq K_{12}(x,x) \leq \frac{C\exp(-\frac{2}{3}x^{\frac{3}{2}})}{(1+x)^{\frac{1}{4}}} \qquad \forall x \geq 0,$

(ii) $0 \leq K_{12}(x,x) \leq C\sqrt{1-x} \qquad \forall\ x \leq 0.$

Proof. We first prove (i). By setting $x = y$ in (4.2.2), we get

$$K_{12}(x,x) = \int_0^\infty \mathrm{Ai}(x+\lambda)^2 d\lambda + \frac{1}{2}\mathrm{Ai}(x)\int_{-\infty}^x \mathrm{Ai}(\lambda)d\lambda. \tag{B.0.1}$$

For the second term in the above display, by Lemma A.0.1 and Lemma A.0.2, we have as $x \to +\infty$

$$\mathrm{Ai}(x)\int_{-\infty}^x \mathrm{Ai}(\lambda)d\lambda \sim \frac{e^{-\frac{2}{3}x^{\frac{3}{2}}}}{2\sqrt{\pi}x^{\frac{1}{4}}}$$

Combining this with the first inequality of Lemma A.0.3, which controls the first term on the right hand side, the upper bound in (i) naturally follows. To prove the lower bound of (i), due to the above displayed asymptotic and the non-negativity of $\int_0^\infty \mathrm{Ai}(x+\lambda)^2 d\lambda$, there exists constant M and C such that for $x > M$,

$$K_{12}(x,x) \geq C^{-1}x^{-\frac{1}{4}}\exp(-\frac{2}{3}x^{\frac{3}{2}}).$$

To conclude the lower bound in (i), it suffices to show that the minimum of $K_{12}(x,x)$ is positive over $[0,M]$ ($K_{12}(x,x)$ is continuous, so admits a minimum). Due to Eq. (B.0.1) and Lemma A.0.2, we can rewrite $K_{12}(x,x) = \int_0^\infty \mathrm{Ai}(x+\lambda)^2 d\lambda + \frac{1}{3}\mathrm{Ai}(x) + \frac{1}{2}\mathrm{Ai}(x)\int_0^x \mathrm{Ai}(\lambda)d\lambda$. Since $\mathrm{Ai}(x)$ is positive for $x \geq 0$, this implies $K_{12}(x,x) > 0$ for all $x > 0$, which completes the proof of the lower bound.

We move on proving (ii). The lower bound follows directly since $K_{12}(x,x)$ is the first order correlation function of a Pfaffian point process, thus is negative. For the upper bound, by the asymptotic of $\mathrm{Ai}(x)$ at $-\infty$, there exists constant C such that for all $x \leq 0$,

$$\left|\mathrm{Ai}(x)\int_{-\infty}^{x}\mathrm{Ai}(\lambda)d\lambda\right| \leq C(1+|x|)^{-\frac{1}{4}}.$$

The result then follows from the second inequality of Lemma A.0.3 and (B.0.1). □

Recall that we defined $F_{\alpha,\beta}(x) = C\big(e^{-\alpha x^{\frac{3}{2}}}\mathbf{1}_{\{x\geq 0\}} + (1-x)^{\beta}\mathbf{1}_{\{x<0\}}\big)$.

Lemma B.0.2. *There exists a constant C, such that for all $x, y \in \mathbb{R}$, we have the following upper bounds for the Pfaffian kernel entries:*

(a) $|K_{11}(x,y)| \leq C\big(F_{\frac{2}{3},\frac{5}{4}}(x) \wedge F_{\frac{2}{3},\frac{3}{4}}(x)F_{\frac{2}{3},\frac{3}{4}}(y)\big)$

(b) $|K_{12}(x,y)| \leq C\big(F_{\frac{2}{3},\frac{3}{4}}(x) \wedge F_{0,\frac{3}{4}}(y)\big)$

(c) $|K_{22}(x,y)| \leq CF_{0,\frac{3}{4}}(x)$

Proof. For (a), it suffices to show that $|K_{11}(x,y)| \leq CF_{\frac{2}{3},\frac{5}{4}}(x)$ and $|K_{11}(x,y)| \leq CF_{\frac{2}{3},\frac{3}{4}}(x)F_{\frac{2}{3},\frac{3}{4}}(y)$. Recall the expression of $K_{11}(x,y)$ from (4.2.1). Using integration by parts for the right hand side of (4.2.1), we get $K_{11}(x,y) = \mathrm{Ai}(x)\mathrm{Ai}(y) - 2\int_0^\infty \mathrm{Ai}(y+\lambda)\mathrm{Ai}'(x+\lambda)d\lambda$. This implies that $|K_{11}(x,y)| \leq |\mathrm{Ai}(x)\mathrm{Ai}(y)| + 2\int_0^\infty |\mathrm{Ai}(y+\lambda)\mathrm{Ai}'(x+\lambda)|d\lambda$. Since $|\mathrm{Ai}(x)|$ is a bounded function, there exists constant C such that

$$|K_{11}(x,y)| \leq C|\mathrm{Ai}(x)| + C\int_0^\infty |\mathrm{Ai}'(x+\lambda)|d\lambda = C + C\int_x^\infty |\mathrm{Ai}'(\lambda)|d\lambda.$$

To obtain the upper bound for $|\mathrm{Ai}(x)|$ and $\int_x^\infty |\mathrm{Ai}(\lambda)|d\lambda$, it suffices to look at their behavior as $x \to \pm\infty$. The asymptotic $\mathrm{Ai}'(x)$ at $\pm\infty$ is specified in Lemma A.0.1. Therefore,

$$\int_x^\infty |\mathrm{Ai}'(\lambda)|d\lambda \leq Ce^{-\frac{2}{3}x^{\frac{3}{2}}},\ \text{if } x \geq 0; \qquad \int_x^\infty |\mathrm{Ai}'(\lambda)|d\lambda \leq C(1-x)^{\frac{5}{4}} \text{ if } x \leq 0.$$

This implies that $|K_{11}(x,y)| \leq CF_{\frac{2}{3},\frac{5}{4}}(x)$. In addition, since

$$K_{11}(x,y) = \int_0^\infty \mathrm{Ai}(x+\lambda)\mathrm{Ai}'(y+\lambda)d\lambda - \int_0^\infty \mathrm{Ai}'(x+\lambda)\mathrm{Ai}(y+\lambda)d\lambda = A_1 - A_2$$

By Cauchy Schwartz inequality,

$$A_1^2 \leq \int_0^\infty \mathrm{Ai}(x+\lambda)^2 d\lambda \int_0^\infty \mathrm{Ai}'(y+\lambda)^2 d\lambda = \int_x^\infty \mathrm{Ai}(\lambda)^2 d\lambda \int_y^\infty \mathrm{Ai}'(\lambda)^2 d\lambda.$$

By Lemma A.0.1, $\mathrm{Ai}(x)^2$ decays asymptotically as $\exp(-\frac{4}{3}x^{\frac{3}{2}})$ as $x \to +\infty$ and is asymptotically upper bounded by $|x|^{-\frac{1}{2}}$ as $x \to -\infty$. This implies that $\int_x^\infty \mathrm{Ai}(\lambda)^2 d\lambda \leq CF_{\frac{4}{3},\frac{1}{2}}(x)$. Similarly, $\mathrm{Ai}'(y)^2$ decays asymptotically as $\exp(-\frac{4}{3}y^{\frac{3}{2}})$ and is asymptotically upper bounded by $|y|^{\frac{1}{2}}$, we get $\int_y^\infty \mathrm{Ai}'(\lambda)^2 d\lambda \leq CF_{\frac{4}{3},\frac{3}{2}}(y)$. As a result,

$$|A_1| \leq \Big(\int_x^\infty \mathrm{Ai}(\lambda)^2 d\lambda\Big)^{\frac{1}{2}}\Big(\int_y^\infty \mathrm{Ai}'(\lambda)^2 d\lambda\Big)^{\frac{1}{2}} \leq CF_{\frac{2}{3},\frac{1}{4}}(x)F_{\frac{2}{3},\frac{3}{4}}(y) \leq CF_{\frac{2}{3},\frac{3}{4}}(x)F_{\frac{2}{3},\frac{3}{4}}(y).$$

For the second inequality above, we use the property that $\sqrt{F_{\alpha,\beta}} = F_{\alpha/2,\beta/2}$ and for the third inequality, $F_{\alpha,\beta}(x)$ is increasing in β. Interchanging the role of x and y, we also have $|A_2| \leq CF_{\frac{2}{3},\frac{3}{4}}(x)F_{\frac{2}{3},\frac{3}{4}}(y)$. Therefore, the same upper bound holds for $|K_{11}(x,y)|$ and we conclude the proof of (a).

We move on showing (b). We will prove $|K_{12}(x,y)| \leq CF_{\frac{2}{3},\frac{3}{4}}(x)$ and $|K_{12}(x,y)| \leq CF_{0,\frac{3}{4}}(y)$ respectively. Recall $K_{12}(x,y)$ from (4.2.2). Note that both $|\mathrm{Ai}(y+\lambda)|$ and $|\int_{-\infty}^y \mathrm{Ai}(\lambda)d\lambda|$ are bounded function of y (see Lemma A.0.2), by using triangle inequality,

$$|K_{12}(x,y)| \leq \frac{1}{2}\int_0^\infty |\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)|d\lambda + \frac{1}{2}|\mathrm{Ai}(x)| \cdot \Big|\int_{-\infty}^y \mathrm{Ai}(\lambda)d\lambda\Big| \leq C\int_x^\infty |\mathrm{Ai}(\lambda)|d\lambda + C|\mathrm{Ai}(x)|.$$

By the asymptotic of $\mathrm{Ai}(x)$ at $\pm\infty$, (use the similar approach as in part (a)), we see that $|K_{12}(x,y)| \leq CF_{\frac{2}{3},\frac{3}{4}}(x)$. We proceed to obtain a different upper bound for K_{12}. Referring to the right hand side of the first inequality in the above display and upper bounding $|\mathrm{Ai}(x+\lambda)|$ and $|\frac{1}{2}\mathrm{Ai}(x)\int_{-\infty}^y \mathrm{Ai}(\lambda)d\lambda|$

by a constant, we find that

$$|K_{12}(x,y)| \leq C \int_0^\infty |\mathrm{Ai}(y+\lambda)| d\lambda + C \leq C F_{0,\frac{3}{4}}(y).$$

This concludes our proof of (b).

Finally, let us demonstrate (c). Recall from (4.2.3) that

$$\begin{aligned} K_{22}(x,y) = &\frac{1}{4}\int_0^\infty \mathrm{Ai}(x+\lambda)\Big(\int_\lambda^\infty \mathrm{Ai}(y+\mu)d\mu\Big)d\lambda - \frac{1}{4}\int_0^\infty \mathrm{Ai}(y+\lambda)\Big(\int_\lambda^\infty \mathrm{Ai}(x+\mu)d\mu\Big)d\lambda \\ &- \frac{1}{4}\int_0^\infty \mathrm{Ai}(x+\lambda)d\lambda + \frac{1}{4}\int_0^\infty \mathrm{Ai}(y+\lambda)d\lambda - \frac{\mathrm{sgn}(x-y)}{4} \end{aligned} \tag{B.0.2}$$

and recall that sgn is the sign function. By Fubini's theorem,

$$\begin{aligned} \int_0^\infty \mathrm{Ai}(y+\lambda)\Big(\int_\lambda^\infty \mathrm{Ai}(x+\mu)d\mu\Big)d\lambda = &\Big(\int_0^\infty \mathrm{Ai}(x+\lambda)d\lambda\Big)\Big(\int_0^\infty \mathrm{Ai}(y+\lambda)d\lambda\Big) \\ &- \int_0^\infty \mathrm{Ai}(x+\lambda)\Big(\int_\lambda^\infty \mathrm{Ai}(y+\mu)d\mu\Big)d\lambda \end{aligned}$$

Replacing the term $\int_0^\infty \mathrm{Ai}(y+\lambda)\big(\int_\lambda^\infty \mathrm{Ai}(x+\mu)d\mu\big)d\lambda$ in (B.0.2) with the right hand side in the above display,

$$\begin{aligned} K_{22}(x,y) = &\frac{1}{2}\int_0^\infty \mathrm{Ai}(x+\lambda)\Big(\int_\lambda^\infty \mathrm{Ai}(y+\mu)d\mu\Big)d\lambda - \frac{1}{4}\Big(\int_0^\infty \mathrm{Ai}(x+\lambda)d\lambda\Big)\Big(\int_0^\infty \mathrm{Ai}(y+\lambda)d\lambda\Big) \\ &- \frac{1}{4}\int_0^\infty \mathrm{Ai}(x+\lambda)d\lambda + \frac{1}{4}\int_0^\infty \mathrm{Ai}(y+\lambda)d\lambda - \frac{\mathrm{sgn}(x-y)}{4}. \end{aligned}$$

We know that $\big|\int_0^\infty \mathrm{Ai}(x+\lambda)d\lambda\big|, \big|\int_0^\infty \mathrm{Ai}(y+\lambda)d\lambda\big|$ can upper bounded by a constant. Applying triangle inequality to the above display,

$$|K_{22}(x,y)| \leq C \int_0^\infty |\mathrm{Ai}(x+\lambda)| d\lambda + C.$$

Using the asymptotic of $\mathrm{Ai}(x)$ at $\pm\infty$ in Lemma A.0.1, we find that $|K_{22}(x,y)| \leq C F_{0,\frac{3}{4}}(x)$, thus conclude (c). □

Appendix C: Stationary distribution of the SHS6V model

In this section, we provide a one parameter family of stationary distribution for the unfused SHS6V model. It is worth to remark that in the recent work of [IMS20], a translation-invariant Gibbs measure was obtained (using the idea from [Agg18a]) for the space-time inhomogeneous SHS6V model on the full lattice, see Proposition 4.5 of [IMS20]. However, It is not obvious that the dynamic of SHS6V model under this Gibbs measure coincides with the one of the bi-infinite SHS6V model specified in Lemma 6.2.1. This being the case, we choose to proceed without relying on the result from [IMS20].

We start with a well-known combinatoric lemma.

Lemma C.0.1 (q-binomial formula)**.** *Set $\nu = q^{-I}$ as usual, the following identity holds for all $q \in \mathbb{C}$,*

$$\sum_{n=0}^{I} \frac{(\nu; q)_n}{(q; q)_n} z^n = \frac{(\nu z; q)_\infty}{(z; q)_\infty}.$$

Proof. According to q-binomial theorem [AAR00],

$$\sum_{n=0}^{\infty} \frac{(\nu; q)_n}{(q; q)_n} z^n = \frac{(\nu z; q)_\infty}{(z; q)_\infty}.$$

When $\nu = q^{-I}$, $(\nu, q)_n = 0$ for $n > I$. Therefore,

$$\sum_{n=0}^{I} \frac{(\nu; q)_n}{(q; q)_n} z^n = \sum_{n=0}^{\infty} \frac{(\nu; q)_n}{(q; q)_n} z^n = \frac{(\nu z; q)_\infty}{(z; q)_\infty}.$$

□

Lemma C.0.2. *Fix $q > 1$, $\nu = q^{-I}$ and $\rho \in (0, I)$, define a probability measure π_ρ on $\{0, 1, \ldots, I\}$:*

$$\pi_\rho(i) = \frac{(\chi, q)_\infty}{(\chi\nu, q)_\infty} \frac{(\nu, q)_i}{(q, q)_i} \chi^i, \quad i \in \{0, 1, \ldots, I\}, \tag{C.0.1}$$

where χ is the unique negative real number satisfying

$$\sum_{i=1}^{I} \frac{\chi}{\chi - q^i} = \rho. \tag{C.0.2}$$

Furthermore, we have

$$\mathbb{E}\big[\pi_\rho\big] = \rho, \qquad Var\big[\pi_\rho\big] = \rho - \sum_{i=1}^{I} \frac{\chi^2}{(q^i - \chi)^2}.$$

Proof. We first show that π_ρ is indeed a probability measure. It is clear that $\pi_\rho(i) \geq 0$ for all $i \in \{0, 1, \ldots, I\}$. By Lemma C.0.1,

$$\sum_{i=0}^{I} \pi_\rho(i) = \frac{(\chi, q)_\infty}{(\chi\nu, q)_\infty} \sum_{i=0}^{I} \frac{(\nu, q)_i}{(q, q)_i} \chi^i = \frac{(\chi, q)_\infty}{(\chi\nu, q)_\infty} \frac{(\nu\chi, q)_\infty}{(\chi, q)_\infty} = 1.$$

Next, we compute the expectation and the variance of π_ρ. Using again Lemma C.0.1, the moment generating function is given by

$$\Lambda(z) = \frac{(\chi, q)_\infty}{(\chi\nu, q)_\infty} \sum_{i=0}^{I} \frac{(\nu, q)_i}{(q, q)_i} \chi^i z^i = \frac{(\chi, q)_\infty}{(\chi\nu, q)_\infty} \frac{(\nu\chi z, q)_\infty}{(\chi z, q)_\infty} = \frac{(\chi, q)_\infty}{(\chi\nu, q)_\infty} \prod_{i=1}^{I} (1 - \nu q^{i-1} \chi z). \tag{C.0.3}$$

It is clear that

$$\begin{aligned} \mathbb{E}\big[\pi_\rho\big] &= \Lambda'(1), \\ \mathrm{Var}\big[\pi_\rho\big] &= \Lambda''(1) + \Lambda'(1) - \Lambda'(1)^2. \end{aligned}$$

Via (C.0.3), one has

$$\Lambda'(z) = \frac{(\chi,q)_\infty}{(\chi\nu,q)_\infty}\Big(\prod_{i=1}^{I}(1-\nu q^{i-1}\chi z)\Big)\Big(\sum_{i=1}^{I}\frac{-\nu q^{i-1}\chi}{1-\nu q^{i-1}\chi z}\Big),$$
$$\Lambda''(z) = \frac{(\chi,q)_\infty}{(\chi\nu,q)_\infty}\Big(\prod_{i=1}^{I}(1-\nu q^{i-1}\chi z)\Big)\Big[\Big(\sum_{i=1}^{I}\frac{-\nu q^{i-1}\chi}{1-\nu q^{i-1}\chi z}\Big)^2 - \sum_{i=1}^{I}\frac{(\nu q^{i-1}\chi)^2}{(1-\nu q^{i-1}\chi z)^2}\Big]$$

Note that

$$\frac{(\chi,q)_\infty}{(\chi\nu,q)_\infty}\prod_{i=1}^{I}(1-\nu q^{i-1}\chi) = 1,$$

combining this with (C.0.2) yields

$$\Lambda'(1) = \rho, \qquad \Lambda''(1) = \rho^2 - \sum_{i=1}^{I}\frac{\chi^2}{(q^i-\chi)^2},$$

which concludes the lemma. □

Theorem C.0.3. *For $\rho \in (0, I)$, the product measure $\bigotimes \pi_\rho$ is stationary for the unfused SHS6V model $\vec{\eta}(t)$ (Definition 6.2.3).*

Proof. It suffices to show that if $\vec{\eta}(t) \sim \bigotimes \pi_\rho$, then $\vec{\eta}(t+1) \sim \bigotimes \pi_\rho$.

Recall that $K(t,y) = N(t,y) - N(t+1,y)$ records the number of particles (either zero or one) that move across location y at time t. We first show that $K(t,y) \sim \text{Ber}\,(\frac{\alpha(t)\chi}{\alpha(t)\chi+1})$ (recall that $\alpha(t) = \alpha q^{\text{mod}_J(t)}$). To this end, referring to (6.2.4),

$$K(t,y) = \sum_{y'=-\infty}^{y}\prod_{z=y'+1}^{y}\Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t))\Big)B(t,y',\eta_{y'}(t)), \tag{C.0.4}$$

Recalling from (6.2.1), $B(t,y,\eta) \sim \text{Ber}\,\big(\frac{\alpha(t)(1-q^\eta)}{1+\alpha(t)}\big)$, $B'(t,y,\eta) \sim \text{Ber}\,\big(\frac{\alpha(t)+\nu q^\eta}{1+\alpha(t)}\big)$. Since the random variables B, B' are all independent,

$$\mathbb{E}\Big[\prod_{z=y'+1}^{y}\Big(B'(t,z,\eta_z(t)) - B(t,z,\eta_z(t)\Big)B(t,y',\eta_{y'}(t))\Big|\mathcal{F}(t)\Big] = \frac{\alpha(t)(1-q^{\eta_{y'}(t)})}{1+\alpha(t)}\prod_{z=y'+1}^{y}\frac{(\alpha(t)+\nu)q^{\eta_z(t)}}{1+\alpha(t)}$$

Therefore, by tower property

$$\begin{aligned}\mathbb{E}\big[K(t,y)\big] &= \sum_{y'=-\infty}^{y} \mathbb{E}\Big[\prod_{z=y'+1}^{y} \frac{\alpha(t)(1-q^{\eta_{y'}(t)})}{1+\alpha(t)} \prod_{z=y'+1}^{y} \frac{(\alpha(t)+\nu)q^{\eta_z(t)}}{1+\alpha(t)}\Big],\\ &= \sum_{y'=-\infty}^{y} \frac{\alpha(t)}{1+\alpha(t)}\left(\frac{\alpha(t)+\nu}{1+\alpha(t)}\right)^{y-y'} \big(\mathbb{E}\big[q^{\eta_y(t)}\big]\big)^{y-y'}(1-\mathbb{E}\big[q^{\eta_y(t)}\big]). \qquad \text{(C.0.5)}\end{aligned}$$

As $\eta_y(t) \sim \pi_\rho$, we obtain using Lemma C.0.1

$$\mathbb{E}\big[q^{\eta_y(t)}\big] = \frac{(\chi,q)_\infty}{(\chi\nu,q)_\infty}\sum_{i=0}^{\infty}\frac{(\nu,q)_i}{(q,q)_i}(\chi q)^i = \frac{(\chi\nu q;q)_\infty}{(\chi q;q)_\infty}\frac{(\chi;q)_\infty}{(\chi\nu;q)_\infty} = \frac{1-\chi}{1-\chi\nu}.$$

Inserting the value of $\mathbb{E}\big[q^{\eta_y(t)}\big]$ into (C.0.5) yields that

$$\mathbb{E}\big[K(t,y)\big] = \sum_{y'=-\infty}^{y}\frac{\alpha(t)}{1+\alpha(t)}\left(\frac{(\alpha(t)+\nu)(1-\chi)}{(1+\alpha(t))(1-\chi\nu)}\right)^{y-y'}\left(1-\frac{1-\chi}{1-\chi\nu}\right) = \frac{\alpha(t)\chi}{\alpha(t)\chi+1}.$$

Since $K(t,y)\in\{0,1\}$, we conclude that

$$K(t,y) \sim \mathrm{Ber}\,(\frac{\alpha(t)\chi}{\alpha(t)\chi+1}). \qquad \text{(C.0.6)}$$

The next step is to show that the marginal of $\vec{\eta}(t+1)$ is distributed as π_ρ for each coordinate. Referring to (C.0.4), it is straightforward that the following recursion holds

$$K(t,y) = B(t,y,\eta_y(t)) + \Big(B'(t,y,\eta_y(t)) - B(t,y,\eta_y(t))\Big)K(t,y-1) \qquad \text{(C.0.7)}$$

Therefore,

$$\begin{aligned}\eta_y(t)-\eta_y(t+1) &= N(t,y)-N(t,y-1)+N(t+1,y-1)-N(t+1,y),\\ &= K(t,y)-K(t,y-1),\\ &= K(t,y-1)\Big(B'(t,y,\eta_y(t))-B(t,y,\eta_y(t))-1\Big)+B(t,y,\eta_y(t)).\end{aligned}$$

For the second equality above, we used $K(t, y) = N(t, y) - N(t+1, y)$. Therefore,

$$\eta_y(t+1) = \begin{cases} \eta_y(t) - B(t, y, \eta_y(t)), & K(t, y-1) = 0, \\ \eta_y(t) + 1 - B'(t, y, \eta_y(t)), & K(t, y-1) = 1. \end{cases} \tag{C.0.8}$$

Due to (C.0.4), we see that $K(t, y-1) \in \sigma\Big(B(t, z, \eta), B'(t, z, \eta), \eta_z(t) : z \le y-1, \eta \in \{0, 1, \ldots, I\}\Big)$. Note that we have assumed $\vec{\eta}(t) \sim \bigotimes \pi_\rho$, which implies the independence between $\eta_y(t)$ and $\eta_z(t)$ for $z \neq y$. Therefore, $\eta_y(t)$ and $K(t, y-1)$ are independent. Using (C.0.8) we get

$$\begin{aligned} \mathbb{P}\big(\eta_y(t+1) = i\big) &= \mathbb{P}\big(K(t, y-1) = 0\big)\mathbb{P}\big(\eta_y(t) - B(t, y, \eta_y(t)) = i\big) \\ &\quad + \mathbb{P}\big(K(t, y-1) = 1\big)\mathbb{P}\big(\eta_y(t) - B'(t, y, \eta_y(t)) = i - 1\big) \end{aligned}$$

By $K(t, y-1) \sim \mathrm{Ber}\,(\frac{\alpha(t)\chi}{\alpha(t)\chi+1})$ and $\eta_y(t) \sim \pi_\rho$, one readily has

$$\begin{aligned} &\mathbb{P}\big(\eta_y(t+1) = i\big) \\ &= \frac{1}{1+\alpha(t)\chi}\left[\pi_\rho(i)\frac{1+\alpha(t)q^i}{1+\alpha(t)} + \pi_\rho(i+1)\frac{\alpha(t)(1-q^{i+1})}{1+\alpha(t)}\right] + \frac{\alpha(t)\chi}{1+\alpha(t)\chi}\left[\pi_\rho(i)\frac{\alpha(t)+\nu q^i}{1+\alpha(t)} + \pi_\rho(i-1)\frac{1-\nu q^{i-1}}{1+\alpha(t)}\right] \\ &= \pi_\rho(i). \end{aligned}$$

To conclude Theorem C.0.3, it suffices to show the independence among $\eta_y(t+1)$ for different value of y. It is enough to show that

$$\eta_y(t+1) \text{ is independent with } \{\eta_{y+1}(t+1), \eta_{y+2}(t+1), \ldots\} \text{ for all } y \in \mathbb{Z}. \tag{C.0.9}$$

We need the following lemma.

Lemma C.0.4. *For all $y \in \mathbb{Z}$, $\eta_y(t+1)$ is independent with $K(t, y)$.*

Let us first see how this lemma leads to (C.0.9). We have via (C.0.4),

$$K(t,y) \in \sigma\Big(B(t,z,\eta), B'(t,z,\eta), \eta_z(t) : z \leq y, \eta \in \{0,1,\ldots,I\}\Big).$$

Combining this with (C.0.8),

$$\eta_y(t+1) \in \sigma\Big(B(t,z,\eta), B'(t,z,\eta), \eta_z(t) : z \leq y, \eta \in \{0,1,\ldots,I\}\Big).$$

Since $\eta_i(t)$ are all independent for different i, one has

$$\Big(B(t,z,\eta), B'(t,z,\eta), \eta_z(t) : z \leq y, \eta \in \{0,1,\ldots,I\}\Big) \text{ is independent with } \big(\eta_{y+1}(t), \eta_{y+2}(t), \ldots\big).$$

We achieve

$$\big(K(t,y), \eta_y(t+1)\big) \text{ is independent with } \big(\eta_{y+1}(t), \eta_{y+2}(t), \ldots\big).$$

Using Lemma C.0.4, we conclude

$$\eta_y(t+1) \text{ is independent with } \big(K(t,y), \eta_{y+1}(t), \eta_{y+2}(t), \ldots\big).$$

Therefore,

$$\eta_y(t+1) \text{ is independent with } \sigma\Big(K(t,y), \eta_z(t), B(t,z,\eta), B'(t,z,\eta) : z \geq y+1, \eta \in \{0,1,\ldots,I\}\Big). \tag{C.0.10}$$

On the other hand, by (C.0.7) and (C.0.8), we conclude for all $y \in \mathbb{Z}$

$$\big(\eta_{y+1}(t+1), \eta_{y+2}(t+1), \ldots\big) \in \sigma\Big(K(t,y), B(t,z,\eta), B'(t,z,\eta), \eta_z(t) : z \geq y+1, \eta \in \{0,1,\ldots,I\}\Big). \tag{C.0.11}$$

Combining (C.0.10) and (C.0.11), we find that for all $y \in \mathbb{Z}$

$$\eta_y(t+1) \text{ is independent with } \big(\eta_{y+1}(t+1), \eta_{y+2}(t+1), \ldots\big),$$

which concludes (C.0.9). □

Proof of Lemma C.0.4. As $K(t,y) \in \{0,1\}$, it suffices to show that for all $j \in \{0,1,\dots,I\}$, one has

$$\mathbb{P}\big(\eta_y(t+1) = j, K(t,y) = 1\big) = \mathbb{P}\big(\eta_y(t+1) = j\big)\mathbb{P}\big(K(t,y) = 1\big)$$

Due to (C.0.7),

$$K(t,y) = \begin{cases} B(t,y,\eta_y(t)), & K(t,y-1) = 0, \\ B'(t,y,\eta_y(t)), & K(t,y-1) = 1. \end{cases}$$

Together with (C.0.8), we obtain that if $K(t,y-1) = 0$,

$$\big(\eta_y(t+1), K(t,y)\big) = (j,1) \text{ is equivalent to } \big(\eta_y(t), B(t,y,\eta_y(t))\big) = (j+1,1).$$

If $K(t,y-1) = 1$,

$$\big(\eta_y(t+1), K(t,y)\big) = (j,1) \text{ is equivalent to } \big(\eta_y(t), B(t,y,\eta_y(t))\big) = (j,1).$$

The discussion above yields (using the independence between $\eta_y(t)$ and $K(t,y-1)$)

$$\begin{aligned}
&\mathbb{P}\big(\eta_y(t+1) = j, K(t,y) = 1\big), \\
&= \mathbb{P}\big(K(t,y-1) = 0\big)\mathbb{P}\big(\eta_y(t) = j+1, B(t,y,\eta_y(t)) = 1\big) + \mathbb{P}\big(K(t,y-1) = 1\big)\mathbb{P}\big(\eta_y(t) = j, B'(t,y,\eta_y(t)) = 1\big), \\
&= \frac{1}{1+\alpha(t)\chi}\frac{\alpha(t)(1-q^{j+1})}{1+\alpha(t)}\pi_\rho(j+1) + \frac{\alpha(t)\chi}{1+\alpha(t)\chi}\frac{\alpha(t)+\nu q^j}{1+\alpha(t)}\pi_\rho(j), \\
&= \frac{\alpha(t)\chi\pi_\rho(j)}{\alpha(t)\chi+1} = \mathbb{P}\big(\eta_{y+1}(t+1) = j\big)\mathbb{P}\big(K(t,y) = 1\big),
\end{aligned}$$

which concludes Lemma C.0.4. □

Remark C.0.5. Since $\vec{g}(t) = \vec{\eta}(Jt)$, it is clear that for all $\rho \in (0,I)$, $\bigotimes \pi_\rho$ is also stationary for the fused SHS6V model $\vec{g}(t)$.

Appendix D: KPZ scaling theory

The KPZ scaling theory has been developed in a landmark contribution by [KMHH92]. The scaling theory is a physics approach which makes prediction for the non-universal coefficients of the KPZ equation. In this appendix, we show how the coefficients of the KPZ equation (6.1.11) arise from the microscopic observables of the fused SHS6V model using the KPZ scaling theory.

Recall that Theorem 6.1.6 reads

$$\sqrt{\epsilon}\big(N^{\mathsf{f}}_\epsilon(\epsilon^{-2}t, \epsilon^{-1}x + \epsilon^{-2}\mu_\epsilon t) - \rho(\epsilon^{-1}x + \epsilon^{-2}\mu_\epsilon t) - t\log\lambda_\epsilon\big) \Rightarrow \mathcal{H}(t,x) \text{ in } C([0,\infty), C(\mathbb{R})) \text{ as } \epsilon \downarrow 0.$$

Here, $N^{\mathsf{f}}_\epsilon(t,x)$ is the fused height function and $\mathcal{H}(t,x)$ solves the KPZ equation

$$\mathcal{H}(t,x) = \frac{\alpha_1}{2}\partial_x^2\mathcal{H}(t,x) - \frac{\alpha_2}{2}\big(\partial_x\mathcal{H}(t,x)\big)^2 + \sqrt{\alpha_3}\xi(t,x),$$

where

$$\begin{aligned}\alpha_1 &= \alpha_2 = JV_* = \frac{J\big((I+J)b - (I+J-2)\big)}{I^2(1-b)},\\ \alpha_3 &= JD_* = \frac{\rho(I-\rho)}{I}\cdot\frac{J\big((I+J)b - (I+J-2)\big)}{I^2(1-b)}.\end{aligned}$$

The first step in the KPZ scaling theory is to derive the stationary distribution of the fused SHS6V model, which is exactly what we did in Appendix C (see Remark C.0.5). Under stationary distribution $\bigotimes \pi_\rho$, we proceed to define two natural quantity of the models

- The *average steady state current* $j(\rho)$ is defined as

$$j(\rho) = \epsilon^{-\frac{1}{2}}\big(\big\langle N^{\mathsf{f}}(t,x) - N^{\mathsf{f}}(t,x+1)\big\rangle_\rho - \rho\mu\big), \tag{D.0.1}$$

where $\langle \cdot \rangle_\rho$ means that we are taking the expectation under stationary distribution $\bigotimes \pi_\rho$ and μ is given in (6.1.9). Note that under stationary distribution, the average steady state current $j(\rho)$ depends neither on space or time. Let us explain the meaning of (D.0.1). Note that $N^{\mathsf{f}}(t,x) - N^{\mathsf{f}}(t+1,x)$ records the number of particles in the fused SHS6V model that move across location x at time t, we subtract $\rho\mu$ here because we are in a frame of reference that moves to right with speed $\rho\mu$.

- *The integrated covariance* is defined as

$$A(\rho) := \lim_{r\to\infty} \frac{1}{2r} \Big\langle N^{\mathsf{f}}(t,x+r) - N^{\mathsf{f}}(t,x-r) - \big\langle N^{\mathsf{f}}(t,x+r) - N^{\mathsf{f}}(t,x-r) \big\rangle_\rho \Big\rangle_\rho .$$

The KPZ scaling theory (equation (12) and (15) of [KMHH92]) predicts that

$$(i)\ \alpha_2 = -\lim_{\epsilon \downarrow 0} j_\epsilon''(\rho), \qquad (ii)\ \frac{\alpha_3}{\alpha_1} = \lim_{\epsilon\downarrow 0} A_\epsilon(\rho),$$

where $A_\epsilon(\rho)$ and $j_\epsilon(\rho)$ depend on ϵ under weakly asymmetry scaling (6.5.30).

Let us first verify (ii), note that under stationary distribution, $N^{\mathsf{f}}_\epsilon(t,x+r) - N^{\mathsf{f}}_\epsilon(t,x-r)$ is the sum of $2r$ i.i.d. random variable with distribution π_ρ, which implies $A_\epsilon(\rho) = \mathrm{Var}\big[\pi_\rho\big]$. By Lemma C.0.2, we know that

$$\mathrm{Var}\big[\pi_\rho\big] = \rho - \sum_{i=1}^{I} \frac{\chi^2}{(q^i - \chi)^2},$$

where χ is the unique negative solution of

$$\sum_{i=1}^{I} \frac{\chi}{\chi - q^i} = \rho. \tag{D.0.2}$$

Under weakly asymmetric scaling, one has $q = e^{\sqrt{\epsilon}}$, which yields $\lim_{\epsilon\downarrow 0} \chi_\epsilon = \frac{\rho}{\rho - I}$. Therefore,

$$\lim_{\epsilon\downarrow 0} A_\epsilon(\rho) = \lim_{\epsilon \downarrow 0} \mathrm{Var}\big[\pi_\rho\big] = \frac{\rho(I-\rho)}{I}.$$

This matches with the value of $\frac{\alpha_3}{\alpha_1}$.

We proceed to verify (i). First, note that by $N^{\mathfrak{f}}(t,x) = N(Jt,x)$,

$$N^{\mathfrak{f}}(t,x) - N^{\mathfrak{f}}(t+1,x) = N(Jt,x) - N((J+1)t,x) = \sum_{s=Jt}^{(J+1)t-1} K(s,x),$$

where $K(s,x) = N(s,x) - N(s+1,x)$. We have shown in (C.0.6) that $K(s,x) \sim \text{Ber}\,(\frac{\alpha(s)\chi}{1+\alpha(s)\chi})$, where $\alpha(s) = \alpha q^{\text{mod}_J(s)}$. Therefore,

$$\mathbb{E}\big[N^{\mathfrak{f}}(t,x) - N^{\mathfrak{f}}(t+1,x)\big] = \mathbb{E}\Bigg[\sum_{s=Jt}^{(J+1)t-1} K(s,x)\Bigg] = \sum_{k=0}^{J-1} \frac{\alpha q^k \chi}{1+\alpha q^k \chi},$$

which yields

$$j(\rho) = \epsilon^{-\frac{1}{2}}\Bigg(\sum_{k=0}^{J-1} \frac{\alpha q^k \chi}{1+\alpha q^k \chi} - \rho\mu\Bigg).$$

We proceed to taylor expand $j_\epsilon(\rho)$ around $\epsilon = 0$. Note that χ is implicitly defined through (D.0.2), we expand χ_ϵ around $\epsilon = 0$

$$\chi_\epsilon = \frac{\rho}{\rho - I} + \frac{(I+1)\rho}{2(\rho - I)}\sqrt{\epsilon} + O(\epsilon).$$

Note that α depends on ϵ through $\alpha_\epsilon = \frac{1-b}{b-e^{\sqrt{\epsilon}}}$. Via straightforward calculation, one has

$$\frac{\alpha q^k \chi}{1+\alpha q^k \chi} = \frac{\alpha_\epsilon e^{k\sqrt{\epsilon}}\chi_\epsilon}{1+\alpha_\epsilon e^{k\sqrt{\epsilon}}\chi_\epsilon} = \frac{\rho}{I} + \frac{(I\rho - \rho^2)((2k+I+1)b + 1 - I - 2k)}{2(b-1)I^2}\sqrt{\epsilon} + O(\epsilon),$$

which implies

$$\sum_{k=0}^{J-1} \frac{\alpha q^k \chi}{1+\alpha q^k \chi} = \frac{J\rho}{I} + \frac{J(I\rho - \rho^2)\big((I+J)b - (I+J-2)\big)}{2(b-1)I^2}\sqrt{\epsilon} + O(\epsilon).$$

Referring to the expression of μ in (6.1.9), one has the asymptotic expansion

$$\mu_\epsilon = \frac{J}{I} + \frac{J(I-2\rho)(2+(b-1)(I+J))}{2(b-1)I^2}\sqrt{\epsilon} + O(\epsilon).$$

Consequently,

$$\begin{aligned} j_\epsilon(\rho) &= \epsilon^{-\frac{1}{2}}\left(\sum_{k=0}^{J-1}\frac{\alpha q^k \chi}{1+\alpha q^k \chi} - \rho\mu\right) \\ &= \frac{\rho^2 J(b(I+J)-(I+J-2)}{2(b-1)I^2} + O(\epsilon^{\frac{1}{2}}). \end{aligned}$$

We have

$$\lim_{\epsilon\downarrow 0} -j_\epsilon''(\rho) = \frac{J(b(I+J)-(I+J-2))}{(1-b)I^2},$$

which coincides with the value of α_2.