Volume 00, Number 0, Pages 000–000
S 0025-5718(XX)0000-0

STRONG CONVERGENCE AND ASYMPTOTIC STABILITY OF EXPLICIT NUMERICAL SCHEMES FOR NONLINEAR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this article we introduce a number of explicit schemes, which are amenable to Khasminski's technique and are particularly suitable for highly nonlinear stochastic differential equations (SDEs). We show that without additional restrictions to those which guarantee the exact solutions possess their boundedness in expectation with respect to certain Lyapunov-type functions, the numerical solutions converge strongly to the exact solutions in finite-time. Moreover, based on the convergence theorem of nonnegative semimartingales, positive results about the ability of the explicit numerical scheme proposed to reproduce the well-known LaSalle-type theorem of SDEs are proved here, from which we deduce the asymptotic stability of numerical solutions. Some examples are discussed to demonstrate the validity of the new numerical schemes and computer simulations are performed to support the theoretical results.

1. Introduction

In 1949, Itô established the well known stochastic calculus. Since then, the theory of stochastic differential equations (SDEs) has been developed quickly. Particularly, the Lyapunov method has been used to deal with the dynamical behaviors of SDEs in both finite and infinite intervals by many authors, and here we only mention Arnold [2], Friedman [6], Khasminskii [19], Kushner [21], Mao [29,30], and Yin and Zhu [42]. These studies also showed that the integrability and the stability of solutions to SDEs can be obtained via stochastic Lyapunov analysis. However, these properties are not necessarily inherited by standard numerical approximations. The main goal of this article is to develop the approximation techniques of nonlinear SDEs that are flexible enough for the stochastic Lyapunov method. Despite of the lack of a discrete version of Itô's formula, asymptotic and qualitative properties in discrete version are obtained by our explicit schemes. In particular, this article is to construct numerical solutions and to prove that they converge to the true solution of the underlying SDEs. In addition to obtaining the V-integrability

2010 *Mathematics Subject Classification.* Primary 65C30, 60H35, 65H10.

Key words and phrases. Stochastic differential equation, local Lipschitz condition, explicit scheme, Lyapunov functions, LaSalle's theorem, strong convergence.

The research of the first author was supported in part by the National Natural Science Foundation of Chain (11971096), the Natural Science Foundation of Jilin Province (No. YDZJ202101ZYTS154), the Fundamental Research Funds for the Central Universities.

The research of the second author was supported in part by the Royal Society (WM160014, Royal Society Wolfson Research Merit Award), the Royal Society and the Newton Fund (NA160317, Royal Society-Newton Advanced Fellowship), the EPSRC (EP/K503174/1).

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(see, [39, Definition 2.3]) and convergence rate, we consider the explicit numerical approximations to reproduce the well-known LaSalle-type theorem of SDEs, from which we deduce the asymptotic stability of numerical solutions.

Explicit Euler-Maruyama (EM) scheme is very popular for approximating the solutions of SDEs with the globally Lipschitz continuous coefficients (see, e.g., [9,20,29]). However, the coefficients of many important SDE models are not only locally Lipschitzian but also superlinear (see, e.g., [14,16] and the references therein). If the global Lipschitz condition does not hold for either of the coefficients of an SDE with bounded moments, Hutzenthaler, Jentzen and Kloeden [14] showed that the explicit EM scheme may have unbounded moments, and consequently the classical EM scheme may fail to converge strongly. Implicit methods were developed to approximate the solutions of these SDEs. Higham et al. [9] showed that the backward EM schemes converge if the diffusion coefficients are globally Lipschitz while the drift coefficients satisfy a one-sided Lipschitz condition. Therefore, the methods with implicit structure are often employed as the alternatives, more details on the implicit methods can be found in [20, 29, 37, 38]. Nevertheless, additional computational efforts are required for its implementation since the solution of an algebraic equation has to be found before each iteration.

Due to the advantages of explicit schemes (e.g., simple structure and cheap computational cost), several modified EM methods have been developed for nonlinear SDEs including the tamed EM method [15, 16, 35, 36], the tamed Milstein method [40], the stopped EM method [25] and the truncated EM method [7, 22, 23, 33, 34]. These modified EM methods have shown their advantages to approximate the solutions of nonlinear SDEs in any finite time interval. In almost all of the above mentioned papers, they all imposed the popular *global monotonicity assumption* that there is a pair of constants $q \geq 2$ and L such that for all $x, y \in \mathbb{R}^d$,

$$\langle x-y, f(x)-f(y)\rangle + \frac{q-1}{2}|g(x)-g(y)|^2 \leq L|x-y|^2.$$

Of course, it should also be mentioned that the convergence rates of numerical approximations were investigated for several scalar SDEs without the global monotonicity assumption, e.g., the square-root diffusion processes (see, e.g., [1, 4, 8]). More recently, Hutzenthaler and Jentzen [18] introduced a new approach, which is based on the general perturbation theory and exponential integrability properties of the exact and numerical solutions, to obtain the strong convergence rate of the stopped-tamed EM scheme for multidimensional SDEs with nonglobally monotonic coefficients.

More general V-integrability results of some specific modified (tamed) EM schemes were established (see, e.g., [16–18, 39] and the references therein). For examples, Hutzenthaler and Jentzen [16] showed some criteria for the moment boundedness under priori estimates (see, [16, Proposition 2.7]) and then used them to prove the strong convergence of the proposed numerical method. Szpruch and Zhang [39] investigated the V-integrability of explicit numerical approximations for SDEs with additional restrictions on coefficients, and provided the $(1/2)$-order rate of convergence in the strong mean square sense for the projected scheme. It is observed that to construct the appropriate scheme to inherit the integrability is challenging. Motivated by this, we construct the explicit EM schemes for nonlinear SDEs and establish their V-integrability by requiring only that the drift and diffusion coefficients satisfy a structure condition (2.4) for the V-integrability of the exact solution. We also demonstrate the convergence of the algorithm under weak conditions. Then under slightly stronger conditions, we prove the strong convergence rate for the explicit schemes, with respect to a larger class of Lyapunov functions.

On the other hand, long-time behaviors of SDEs are also one of the hot topics in the study of stochastic processes, systems theory, control and optimization (see, e.g., the monographs [19, 29, 42] and the references therein). So far, the dynamical properties of SDEs are investigated deeply including stochastic stability (see, e.g., [19,29,39]), ergodicity (see, e.g., [3,19,27,28]) and so on. Here we focus on the asymptotic stability. Although the finite-time convergence of numerical methods is one of the fundamental issues, how to preserve the asymptotic stability of the underlying SDEs is significant and challenging. Recently, considerable effort has been made in this direction (mainly for implicit schemes) in [10–12,32,41], where the diffusion coefficients of SDEs are always required to be globally Lipschitz continuous. Thus, in order to close the gap, a few modified EM methods have been developed to approximate the asymptotic stability for nonlinear SDEs. For instance, Guo et al. [7] showed that the partially truncated EM method can preserve the mean square exponential stability of the underlying SDEs. Liu and Mao [26] made use of the EM method with random variable stepsize to reproduce the almost sure stability of the underlying SDEs. Szpruch and Zhang [39] established the asymptotic stability properties for the tamed EM scheme and the projected scheme, making use of some Lyapunov functions. For the further development of numerical schemes for SDEs, we refer readers to [17, 23, 39], for example, and the references therein. To the best of our knowledge, much research on numerical stability relies on simple Lyapunov functions such as $|\cdot|^2$, with the exception of [23,39]. Here our aim is to handle more general cases by new schemes using more general Lyapunov functions.

In this paper, borrowing the truncation idea from [23, 33] and using novel approximation technique, we construct a new explicit scheme that preserve the V-integrability of SDEs with respect to a larger class of Lyapunov functions, and derive strong convergence result in a finite time interval. We go further to improve the scheme according to the structure condition of the LaSalle-type theorem so that it is easily implementable for approximating the underlying stability of the SDEs, admitting a large class of Lyapunov functions. The schemes proposed in this paper are very much different from those of [7, 26, 33, 39]. More precisely, the numerical solutions at the grid points are modified before each iteration according to the growth rate of the drift and diffusion coefficients such that the numerical solutions preserve the underlying nice properties of the exact solutions of SDEs. Our main contributions are as follows:

- We construct an explicit scheme with respect to a larger class of Lyapunov functions for the SDEs which do not satisfy the global monotonicity assumption. The numerical solutions preserve the V-integrability of the exact solution almost perfectly. The finite-time strong convergence results are established.
- We reconstruct a more carefully modified explicit scheme to reproduce the LaSalle-type results in stochastic version making use of a large class of auxiliary Lyapunov functions. Especially, the explicit scheme inherits the exponential stability of the exact solution very well.

- The range of the auxiliary Lyapunov functions that can be used to design the explicit schemes for asymptotic stability is much wider than the existing results, for examples, in [7, 23, 26, 39].

The rest of the paper is organized as follows. Section 2 begins with notations and preliminaries on the properties of the exact solution. Section 3 constructs an explicit scheme, and yields the strong convergence and integrability in a finite time interval, with respect to a larger class of Lyapunov functions. Section 4 provides the rate of convergence. Section 5 reconstructs a more carefully modified scheme to preserve the LaSalle-type results in stochastic version. Section 6 presents a couple of examples and simulations to illustrate our results. Section 7 gives some further remarks to conclude the paper.

2. Notations and preliminaries

Throughout this paper, we use the following notations. Let d, m and n denote finite positive integers, $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^d := \mathbb{R}^{d\times 1}$ and the trace norm in $\mathbb{R}^{d\times m}$, $\langle\cdot,\cdot\rangle$ stand for the dot product (usual Euclidean scalar product) on $\mathbb{R}^d$. For any $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$, and $a \wedge b := \min\{a, b\}$. If $[[0, \beta]]$ is a set, its indicator function is denoted by $I_{[[0,\beta]]}$, namely $I_{[[0,\beta]]}(x) = 1$ if $x \in [[0, \beta]]$ and 0 otherwise. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\mathbb{E}$ denotes the expectation corresponding to $\mathbb{P}$. Let $B(t) = \big(B^{(1)}(t), \dots, B^{(m)}(t)\big)^T$ be an m-dimensional Brownian motion defined on this probability space. Suppose $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration defined on this probability space satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets) such that $B(t)$ is $\mathcal{F}_t$ adapted. Let $\mathbb{R}_+ := (0, \infty)$, $\bar{\mathbb{R}}_+ := [0, \infty)$ and $\mathbf{0}$ denote a null matrix whose dimension may change in different appearances. Also let C_u and C denote two generic positive real constants respectively, whose value may change in different appearances, where C_u is dependent on u. In this paper, we consider the d-dimensional stochastic differential equation (SDE)

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}B(t) \tag{2.1}$$

with an initial value $X(0) = x_0 \in \mathbb{R}^d$, where the drift and diffusion terms

$$f : \mathbb{R}^d \to \mathbb{R}^d, \qquad g : \mathbb{R}^d \to \mathbb{R}^{d\times m},$$

are local Lipschitz continuous, this is, for any $N > 0$ there exists a positive constant C_N such that, for any $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq N$,

$$|f(x) - f(y)| \vee |g(x) - g(y)| \leq C_N|x - y|.$$

Let $\mathcal{C}^p(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ denote the family of all nonnegative functions V on $\mathbb{R}^d$ which are continuously pth differentiable in x. Let $\mathcal{C}^p_\infty(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ denote the family of all functions $V \in \mathcal{C}^p(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ with the property $\lim_{|x|\to\infty} V(x) = \infty$. For convenience, we cite the following notations introduced by [5, p.617]. A vector of the form $\alpha = (\alpha_1, \dots, \alpha_d)$, where each component α_i is nonnegative integer, is called a *multiindex* of order $|\alpha| := \alpha_1 + \cdots + \alpha_d$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$ and any vector $x \in \mathbb{R}^d$, we set as usual $\alpha! := \alpha_1! \cdots \alpha_d!$, $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. If $\beta = (\beta_1, \dots, \beta_d)$ is also a multiindex, then $\alpha \geq \beta$ means each component $\alpha_i \geq \beta_i$ for any $1 \leq i \leq d$, multiindex $\alpha - \beta := (\alpha_1 - \beta_1, \dots, \alpha_d - \beta_d)$, and multiindex $\alpha + \beta := (\alpha_1 + \beta_1, \dots, \alpha_d + \beta_d)$. For each $V \in \mathcal{C}^p(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ and a multiindex α with

$|\alpha| \le p$, define

$$D^\alpha V(x) := \frac{\partial^{|\alpha|} V(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}.$$

For any nonnegative integer $n \le p$, define $D^{(n)}V(x) := \big\{ D^\alpha V(x) \big| |\alpha| = n \big\}$, the set of all partial derivatives with nth order. Assigning some ordering to the various partial derivatives, we can also regard $D^{(n)}V(x)$ as a point in $\mathbb{R}^{d^n}$ and define $|D^{(n)}V(x)| = \Big(\sum_{|\alpha|=n} |D^\alpha V(x)|^2 \Big)^{\frac{1}{2}}$. Especially, if $n = 1$, we regard the elements of $D^{(1)}V(x)$ as being arranged in a vector

$$D^{(1)}V(x) = \left(\frac{\partial V(x)}{\partial x_1}, \dots, \frac{\partial V(x)}{\partial x_d} \right), \quad \text{and} \quad |D^{(1)}V(x)| = \left[\sum_{i=1}^d \Big(\frac{\partial V(x)}{\partial x_i} \Big)^2 \right]^{\frac{1}{2}};$$

If $n = 2$, we regard the elements of $D^{(2)}V(x)$ as being arranged in a matrix

$$D^{(2)}V(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j} \right)_{d\times d}, \quad \text{and} \quad |D^{(2)}V(x)| = \left[\sum_{i,j=1}^d \Big(\frac{\partial^2 V(x)}{\partial x_i \partial x_j} \Big)^2 \right]^{\frac{1}{2}}.$$

For each $V \in \mathcal{C}^2(\mathbb{R}^d; \bar{\mathbb{R}}_+)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^d$ to $\mathbb{R}$ by

$$\mathcal{L}V(x) = \langle D^{(1)}V(x), f(x) \rangle + \frac{1}{2} \mathrm{tr}\Big[g^T(x) D^{(2)}V(x) g(x) \Big]. \tag{2.2}$$

As [16, 39], for a pair of integers $p \in [2, +\infty)$ and $1/\delta_p \in [p, +\infty)$, define

$$\begin{aligned} \mathcal{V}^p_{\delta_p} := \Big\{ V \in \mathcal{C}^p_\infty(\mathbb{R}^d; \bar{\mathbb{R}}_+) \Big| \exists\ c > 0 \text{ s.t.} \\ |D^{(n)}V(\cdot)| \le c\big(1 + V(\cdot)\big)^{1 - n\delta_p},\ n = 1, 2, \dots, p \Big\}. \end{aligned} \tag{2.3}$$

Note that many frequently-used functions belong to the set $\mathcal{V}^p_{\delta_p}$ (see [16] for more details).

Now we prepare the regularity and V-integrability of the exact solution.

Theorem 2.1. *Assume there is a function $V \in \mathcal{C}^2_\infty(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ and a pair of positive constants ρ and λ such that*

$$\begin{aligned} \mathcal{L}(1 + V(x))^\rho &= \frac{\rho}{2}\big(1 + V(x)\big)^{\rho - 2} \Big[2\big(1 + V(x)\big)\mathcal{L}V(x) + (\rho - 1)|D^{(1)}V(x) g(x)|^2 \Big] \\ &\le \lambda\big[1 + V^\rho(x)\big], \qquad \forall x \in \mathbb{R}^d. \end{aligned} \tag{2.4}$$

Then the SDE (2.1) with any initial value $x_0 \in \mathbb{R}^d$ has a unique regular solution $X(t)$ satisfying

$$\sup_{0 \le t \le T} \mathbb{E} V^\rho(X(t)) \le C_T, \qquad \forall T > 0,$$

where throughout this paper C_T denotes a positive constant dependent on T but independent of Δ (which will be used later).

Remark 2.2. In fact this is the Khasminskii test as if we set $\bar{V}(x) = (1 + V(x))^\rho$. In other words, the conditions of Theorem 2.1 is an alternative to Khasminskii's condition that there exists a positive constant $\bar{\lambda}$ such that $\mathcal{L}\bar{V}(x) \le \bar{\lambda}(1 + \bar{V}(x))$, see [19, Theorem 3.5, p.75].

3. V-integrability and strong convergence

In this section, we aim to construct an explicit scheme and show that its numerical solution converges strongly to the exact solution of SDE (2.1). If it can efficiently prevent the diffusion term from producing extra-ordinary large values, the numerical method will keep the properties of the exact solution by using the Taylor expansion. Thus we define the explicit scheme by the appropriate truncation map.

Let $V \in \mathcal{V}^4_{\delta_4}$ for some integer $1/\delta_4 \in [4, +\infty)$. To define appropriate numerical solutions, we choose a strictly increasing continuous function $\varphi : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ such that $\varphi(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\le u}\left(\frac{|f(x)|}{\big(1+V(x)\big)^{\delta_4}} \vee \frac{|g(x)|^2}{\big(1+V(x)\big)^{2\delta_4}}\right) \le \varphi(u), \qquad \forall\, u \ge 1. \tag{3.1}$$

Denote by φ^{-1} the inverse function of φ. Obviously $\varphi^{-1} : [\varphi(1), \infty) \to \bar{\mathbb{R}}_+$ is a strictly increasing continuous function. It is worth mentioning that there are many functions φ which satisfy (3.1). The user can choose any of them. It is used to define the truncation mapping π_Δ below but our convergence rate is independent of the choice of φ. In practice, it is wise to choose φ so that its inverse function φ^{-1} can be obtained explicitly (see examples in Section 6). We choose a constant $K \ge \varphi(|x_0| \vee 1)$, where K is independent of the iteration order k and the time stepsize $\triangle$. For the given $\triangle \in (0, 1]$, define a truncation mapping $\pi_\triangle : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\pi_\triangle(x) = \Big(|x| \wedge \varphi^{-1}\big(K\triangle^{-\theta}\big)\Big)\frac{x}{|x|},$$

where $\theta \in (0, 1/2]$, and we use the convention $\frac{x}{|x|} = \mathbf{0}$ when $x = \mathbf{0} \in \mathbb{R}^d$. Obviously,

$$|f(\pi_\triangle(x))| \le \varphi(\varphi^{-1}(K\triangle^{-\theta}))\big(1+V(\pi_\triangle(x))\big)^{\delta_4} = K\triangle^{-\theta}\big(1+V(\pi_\triangle(x))\big)^{\delta_4},$$
$$|g(\pi_\triangle(x))|^2 \le \varphi(\varphi^{-1}(K\triangle^{-\theta}))\big(1+V(\pi_\triangle(x))\big)^{2\delta_4} = K\triangle^{-\theta}\big(1+V(\pi_\triangle(x))\big)^{2\delta_4}$$

for any $x \in \mathbb{R}^d$.

Next we propose our numerical method to approximate the exact solution of the SDE (2.1). For any given stepsize $\triangle \in (0, 1]$, define

$$\begin{cases} Y_0 = x_0, \\ \tilde{Y}_{k+1} = Y_k + f(Y_k)\triangle + g(Y_k)\triangle B_k, \\ Y_{k+1} = \pi_\triangle(\tilde{Y}_{k+1}), \end{cases} \tag{3.2}$$

for any integer $k \ge 0$, where $t_k = k\triangle$ and $\triangle B_k = (\triangle B_k^{(1)}, \dots, \triangle B_k^{(m)})^T = B(t_{k+1}) - B(t_k)$. The explicit method (3.2) is called the *V-truncated EM scheme* which modifies the values of nodes before each iteration avoiding the extra-ordinary large deviations. One further observes that

$$|f(Y_k)| \le K\triangle^{-\theta}\big(1+V(Y_k)\big)^{\delta_4}, \qquad |g(Y_k)|^2 \le K\triangle^{-\theta}\big(1+V(Y_k)\big)^{2\delta_4}. \tag{3.3}$$

To obtain the continuous-time approximations, define $\tilde{Y}(t)$ and $Y(t)$ by

$$\tilde{Y}(t) := \tilde{Y}_k, \qquad Y(t) := Y_k, \quad \forall t \in [t_k, t_{k+1}).$$

We write $\mathbb{E}_k[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_{t_k}]$ for simplicity. The following lemmas will play their important roles in the proof of the V-integrability of the numerical solutions.

Lemma 3.1. *If $V \in \mathcal{V}^4_{\delta_4}$ for some integer $1/\delta_4 \in [4, +\infty)$, then the V-truncated EM scheme (3.2) has the property that*

$$\sum_{|\alpha|=1}^{3} \frac{D^\alpha V(Y_k)}{\alpha!}\big(\tilde{Y}_{k+1} - Y_k\big)^\alpha \leq \mathcal{L}V(Y_k)\triangle + \mathcal{R}^\triangle V(Y_k) + \sum_{i=1}^{3} \mathcal{S}_i^\triangle V(Y_k), \tag{3.4}$$

where $\mathcal{S}_1^\triangle V(\cdot), \mathcal{S}_2^\triangle V(\cdot)$ and $\mathcal{S}_3^\triangle V(\cdot)$ are defined by (A.1), (A.2), (A.6), *respectively,*

$$\mathcal{R}^\triangle V(Y_k) := C \sum_{i=2}^{3} \sum_{j=0}^{i-2} |f(Y_k)|^{i-2j} |g(Y_k)|^{2j} |D^{(i)} V(Y_k)| \triangle^{i-j}.$$

We also have $\mathbb{E}_k\big[\mathcal{S}_i^\triangle V(Y_k)\big] = 0$ for $i = 1, 2, 3$.

Lemma 3.2. *If $V \in \mathcal{V}^4_{\delta_4}$ for some integer $1/\delta_4 \in [4, +\infty)$, we then have*

$$\big|\mathcal{L}V(Y_k)\big| \leq 2cK\big(1 + V(Y_k)\big)\triangle^{-\theta}, \qquad \mathcal{R}^\triangle V(Y_k) \leq C\big(1 + V(Y_k)\big)\triangle^{2(1-\theta)}, \tag{3.5}$$

where $\mathcal{L}V$ is defined by (2.2). *Moreover, we have*

$$\mathbb{E}_k\big[|\mathcal{S}_1^\triangle V(Y_k)|^2\big] = |D^{(1)}V(Y_k)g(Y_k)|^2\triangle, \quad \mathbb{E}_k\big[|\mathcal{S}_i^\triangle V(Y_k)|^2\big] \leq C\big(1 + V(Y_k)\big)^2\triangle^{1-\theta}$$

for $i = 1, 2, 3$ and

$$\mathbb{E}_k\big[\mathcal{S}_1^\triangle V(Y_k)\mathcal{S}_j^\triangle V(Y_k)\big] \geq -C\big(1 + V(Y_k)\big)^2\triangle^{2(1-\theta)} \quad \text{for} \quad j = 2, 3.$$

The proofs of both lemmas above can be found in Appendix A. Let us begin to establish the criterion on the V-integrability of the scheme (3.2). It makes use of Lemma 3.1 and Lemma 3.2 above.

Theorem 3.3. *Let the conditions of Theorem 2.1 hold. Assume moreover that the function $V \in \mathcal{V}^4_{\delta_4}$ for some integer $1/\delta_4 \in [4, +\infty)$ and has the property*

$$V(\epsilon x) \leq V(x) \qquad \forall\, x \in \mathbb{R}^d, \quad 0 < \epsilon \leq 1.$$

Then the truncation scheme defined by (3.2) *has the property*

$$\sup_{\triangle \in (0,1]} \sup_{0 \leq k\triangle \leq T} \mathbb{E}V^\rho(Y_k) \leq C_T, \quad \forall T > 0. \tag{3.6}$$

Proof. Due to $V \in \mathcal{V}^4_{\delta_4}$, using the Taylor formula with integral remainder term,

$$V(\tilde{Y}_{k+1}) = V(Y_k) + \sum_{|\alpha|=1}^{3} \frac{D^\alpha V(Y_k)}{\alpha!}\big(\tilde{Y}_{k+1} - Y_k\big)^\alpha + J(\tilde{Y}_{k+1}, Y_k), \tag{3.7}$$

where

$$J(\tilde{Y}_{k+1}, Y_k) := 4 \sum_{|\alpha|=4} \frac{\big(\tilde{Y}_{k+1} - Y_k\big)^\alpha}{\alpha!} \int_0^1 (1-t)^3 D^\alpha V\big(Y_k + t(\tilde{Y}_{k+1} - Y_k)\big)\mathrm{d}t.$$

One observes that

$$\begin{aligned}\big|J(\tilde{Y}_{k+1}, Y_k)\big| \leq& 4 \sum_{|\alpha|=4} \frac{\big|\big(\tilde{Y}_{k+1} - Y_k\big)^\alpha\big|}{\alpha!} \int_0^1 (1-t)^3 \big|D^{(4)} V\big(Y_k + t(\tilde{Y}_{k+1} - Y_k)\big)\big|\mathrm{d}t \\ \leq& \frac{c}{3!}\bigg(\sum_{i=1}^{d} \Big|f_i(Y_k)\triangle + \sum_{j=1}^{m} g_{ij}(Y_k)\triangle B_k^{(j)}\Big|\bigg)^4\end{aligned}$$

$$\times \int_0^1 (1-t)^3 \Big[1+V\big(Y_k+t\big(\tilde{Y}_{k+1}-Y_k\big)\big)\Big]^{1-4\delta_4} \mathrm{d}t.$$

Note that for any $U \in \mathcal{V}^4_{\delta_4}$ we know $|D^{(1)}(1+U(x))| \le c(1+U(x))^{1-\delta_4}$. By the result of [16, Lemma 2.12, p.22] we have

$$1+U(x+y) \le c^{\frac{1}{\delta_4}} 2^{\frac{1}{\delta_4}-1}\Big(\big|1+U(x)\big| + |y|^{\frac{1}{\delta_4}}\Big), \qquad \forall x, y \in \mathbb{R}^d,$$

which leads to

$$\begin{aligned}\Big[1+V\big(Y_k+t\big(\tilde{Y}_{k+1}-Y_k\big)\big)\Big]^{1-4\delta_4} \le & \Big[c^{\frac{1}{\delta_4}} 2^{\frac{1}{\delta_4}-1}\Big(1+V(Y_k)+t^{\frac{1}{\delta_4}}|\tilde{Y}_{k+1}-Y_k|^{\frac{1}{\delta_4}}\Big)\Big]^{1-4\delta_4} \\ \le & C\Big[\big(1+V(Y_k)\big)^{1-4\delta_4} + |\tilde{Y}_{k+1}-Y_k|^{\frac{1}{\delta_4}-4}\Big]\end{aligned}$$

for any $V \in \mathcal{V}^4_{\delta_4}$ with $1/\delta_4 \in [4, +\infty)$. Therefore, we derive from (3.3) that

$$\begin{aligned}\big|J(\tilde{Y}_{k+1}, Y_k)\big| \le & C\Big[\Big(|f(Y_k)|^4 \triangle^4 + |g(Y_k)|^4|\triangle B_k|^4\Big)\big(1+V(Y_k)\big)^{1-4\delta_4} \\ & + \Big(|f(Y_k)|^{\frac{1}{\delta_4}} \triangle^{\frac{1}{\delta_4}} + |g(Y_k)|^{\frac{1}{\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big)\Big] \\ \le & C\Bigg\{\Big[\big(1+V(Y_k)\big)^{4\delta_4}\triangle^{4(1-\theta)} \\ & + \big(1+V(Y_k)\big)^{4\delta_4}\triangle^{-2\theta}|\triangle B_k|^4\Big]\big(1+V(Y_k)\big)^{1-4\delta_4} \\ & + \big(1+V(Y_k)\big)\triangle^{\frac{1-\theta}{\delta_4}} + \big(1+V(Y_k)\big)\triangle^{\frac{-\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Bigg\} \\ \le & C\big(1+V(Y_k)\big)\triangle^{4(1-\theta)} + \mathcal{J}^{\triangle}V(Y_k),\end{aligned} \tag{3.8}$$

where

$$\mathcal{J}^{\triangle}V(Y_k) = C\big(1+V(Y_k)\big)\big(\triangle^{-2\theta}|\triangle B_k|^4 + \triangle^{\frac{-\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\big). \tag{3.9}$$

For any $\rho > 0$, substituting (3.4) and (3.8) into (3.7), then using the second inequality of (3.5), we yield that

$$\big(1+V(\tilde{Y}_{k+1})\big)^{\rho} \le \big(1+V(Y_k)\big)^{\rho}\big(1+\xi_k\big)^{\rho}, \tag{3.10}$$

where

$$\xi_k = \frac{\mathcal{L}V(Y_k)\triangle + C\big(1+V(Y_k)\big)\triangle^{2(1-\theta)} + \sum_{i=1}^3 \mathcal{S}_i^{\triangle}V(Y_k) + \mathcal{J}^{\triangle}V(Y_k)}{1+V(Y_k)},$$

and we can see that $\xi_k > -1$. By the virtue of [23, Inequality (3.12)], without loss of generality we prove (3.6) only for $0 < \rho \le 1$. It follows from (3.10) that

$$\begin{aligned}\mathbb{E}_k\Big[\big(1+V(\tilde{Y}_{k+1})\big)^{\rho}\Big] \le \big(1+V(Y_k)\big)^{\rho}\Bigg(&1+\rho\mathbb{E}_k\big[\xi_k\big] + \frac{\rho(\rho-1)}{2}\mathbb{E}_k\big[\xi_k^2\big] \\ & + \frac{\rho(\rho-1)(\rho-2)}{6}\mathbb{E}_k\big[\xi_k^3\big]\Bigg).\end{aligned} \tag{3.11}$$

In order to independently estimate each of the expectations on the right-hand side of inequality (3.11), we divide it into three steps.

Step 1. We estimate $\mathbb{E}_k\big[\xi_k\big]$. Due to (3.9) and (A.5), we deduce that

$$\begin{aligned}\mathbb{E}_k\big[|\mathcal{J}^{\triangle}V(Y_k)|\big] \le & C\big(1+V(Y_k)\big)\big(\triangle^{2(1-\theta)} + \triangle^{\frac{1-\theta}{2\delta_4}}\big) \\ \le & C\big(1+V(Y_k)\big)\triangle^{2(1-\theta)}.\end{aligned} \tag{3.12}$$

This together with Lemma 3.1 implies

$$\begin{aligned}\mathbb{E}_k\big[\xi_k\big] \le&\big(1+V(Y_k)\big)^{-1}\Big[\mathcal{L}V(Y_k)\triangle+C\big(1+V(Y_k)\big)\triangle^{2(1-\theta)}\Big]\\=&\big(1+V(Y_k)\big)^{-1}\mathcal{L}V(Y_k)\triangle+C\triangle^{2(1-\theta)}.\end{aligned}\tag{3.13}$$

Step 2. We estimate $\mathbb{E}_k\big[\xi_k^2\big]$. Similar to (3.12), combining (A.1) and (3.9) imply

$$\begin{aligned}&\mathbb{E}_k\Big[\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{J}^{\triangle}V(Y_k)\Big]\\=&C\big(1+V(Y_k)\big)\mathbb{E}_k\Big[\langle D^{(1)}V(Y_k),g(Y_k)\triangle B_k\rangle\Big(\triangle^{-2\theta}|\triangle B_k|^4+\triangle^{\frac{-\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big)\Big]\\\ge&-C\triangle^{\frac{-\theta}{2\delta_4}}\big(1+V(Y_k)\big)^{2-\delta_4}|g(Y_k)|\mathbb{E}_k\Big[|\triangle B_k|^{1+\frac{1}{\delta_4}}\Big]\ge-C\triangle^{2(1-\theta)}\big(1+V(Y_k)\big)^2\end{aligned}$$

for $1/\delta_4\in[4,+\infty)$. This together with Lemma 3.2 implies

$$\begin{aligned}\mathbb{E}_k\big[\xi_k^2\big]=&\big(1+V(Y_k)\big)^{-2}\mathbb{E}_k\Big[\Big(\mathcal{L}V(Y_k)\triangle+C\big(1+V(Y_k)\big)\triangle^{2(1-\theta)}\\&+\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Y_k)+\mathcal{J}^{\triangle}V(Y_k)\Big)^2\Big]\\\ge&\big(1+V(Y_k)\big)^{-2}\mathbb{E}_k\Big\{|\mathcal{S}_1^{\triangle}V(Y_k)|^2+2\mathcal{S}_1^{\triangle}V(Y_k)\Big[\mathcal{L}V(Y_k)\triangle\\&+C\big(1+V(Y_k)\big)\triangle^{2(1-\theta)}+\sum_{i=2}^{3}\mathcal{S}_i^{\triangle}V(Y_k)+\mathcal{J}^{\triangle}V(Y_k)\Big]\Big\}\\\ge&\big(1+V(Y_k)\big)^{-2}\Big[|D^{(1)}V(Y_k)g(Y_k)|^2\triangle+2\mathbb{E}_k\Big(\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{S}_2^{\triangle}V(Y_k)\Big)\\&+2\mathbb{E}_k\Big(\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{S}_3^{\triangle}V(Y_k)\Big)+2\mathbb{E}_k\Big(\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{J}^{\triangle}V(Y_k)\Big)\Big]\\\ge&\big(1+V(Y_k)\big)^{-2}|D^{(1)}V(Y_k)g(Y_k)|^2\triangle-C\triangle^{2(1-\theta)}.\end{aligned}\tag{3.14}$$

Step 3. We estimate $\mathbb{E}_k\big[\xi_k^3\big]$. Similar to (3.12), we can also prove that

$$\begin{aligned}\mathbb{E}_k\big[|\mathcal{J}^{\triangle}V(Y_k)|^2\big]\le&C\big(1+V(Y_k)\big)^2\mathbb{E}_k\Big[\triangle^{-4\theta}|\triangle B_k|^8+\triangle^{\frac{-\theta}{\delta_4}}|\triangle B_k|^{\frac{2}{\delta_4}}\Big]\\\le&C\big(1+V(Y_k)\big)^2\triangle^{4(1-\theta)},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_k\big[|\mathcal{J}^{\triangle}V(Y_k)|^3\big]\le&C\big(1+V(Y_k)\big)^3\mathbb{E}_k\Big[\triangle^{-6\theta}|\triangle B_k|^{12}+\triangle^{\frac{-3\theta}{2\delta_4}}|\triangle B_k|^{\frac{3}{\delta_4}}\Big]\\\le&C\big(1+V(Y_k)\big)^3\triangle^{6(1-\theta)}.\end{aligned}$$

This together with (3.12) as well as Lemma 3.2 implies

$$\begin{aligned}\mathbb{E}_k\big[\xi_k^3\big]\le&\big(1+V(Y_k)\big)^{-3}\mathbb{E}_k\Big[\Big(C\big(1+V(Y_k)\big)\triangle^{1-\theta}+\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Y_k)+\mathcal{J}^{\triangle}V(Y_k)\Big)^3\Big]\\\le&C\big(1+V(Y_k)\big)^{-3}\mathbb{E}_k\Big\{\big(1+V(Y_k)\big)^3\triangle^{3(1-\theta)}+\Big(\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Y_k)+\mathcal{J}^{\triangle}V(Y_k)\Big)^3\end{aligned}$$

$$
\begin{aligned}
&\qquad + \big(1+V(Y_k)\big)^2 \triangle^{2(1-\theta)} \bigg(\sum_{i=1}^{3} \mathcal{S}_i^{\triangle} V(Y_k) + \mathcal{J}^{\triangle} V(Y_k) \bigg) \\
&\qquad + \big(1+V(Y_k)\big) \triangle^{1-\theta} \bigg(\sum_{i=1}^{3} \mathcal{S}_i^{\triangle} V(Y_k) + \mathcal{J}^{\triangle} V(Y_k) \bigg)^2 \bigg\} \\
\le & C\big(1+V(Y_k)\big)^{-3} \bigg\{ \big(1+V(Y_k)\big)^3 \triangle^{3(1-\theta)} + \mathbb{E}_k\Big[\big|\mathcal{J}^{\triangle} V(Y_k)\big|^3 \Big] \\
& + \sum_{i=1}^{3} \mathbb{E}_k \Big[\big|\mathcal{S}_i^{\triangle} V(Y_k)\big|^2 \big|\mathcal{J}^{\triangle} V(Y_k)\big| \Big] + \big(1+V(Y_k)\big)^2 \triangle^{2(1-\theta)} \mathbb{E}_k \Big[|\mathcal{J}^{\triangle} V(Y_k)| \Big] \\
&\quad + \big(1+V(Y_k)\big) \triangle^{1-\theta} \bigg(\sum_{i=1}^{3} \mathbb{E}_k \Big[\big|\mathcal{S}_i^{\triangle} V(Y_k)\big|^2 \Big] + \mathbb{E}_k \Big[\big|\mathcal{J}^{\triangle} V(Y_k)\big|^2 \Big] \bigg) \bigg\} \\
\le & C\big(1+V(Y_k)\big)^{-3} \bigg\{ \big(1+V(Y_k)\big)^3 \triangle^{3(1-\theta)} + \big(1+V(Y_k)\big)^3 \triangle^{2(1-\theta)} \\
&\qquad + \sum_{i=1}^{3} \mathbb{E}_k \Big[\big|\mathcal{S}_i^{\triangle} V(Y_k)\big|^2 \big|\mathcal{J}^{\triangle} V(Y_k)\big| \Big] \bigg\}.
\end{aligned}
\tag{3.15}
$$

On the other hand, by (A.5), (A.7) and (3.9), one observes

$$
\begin{aligned}
&\mathbb{E}_k \Big[\big|\mathcal{S}_1^{\triangle} V(Y_k)\big|^2 |\mathcal{J}^{\triangle} V(Y_k)| \Big] \\
\le & C\big(1+V(Y_k)\big) \mathbb{E}_k \bigg[\Big(\big(1+V(Y_k)\big)^2 \triangle^{1-\theta} \\
&\qquad + \mathcal{H}_{1,1}^{\triangle} V(Y_k) \Big) \Big(\triangle^{-2\theta} |\triangle B_k|^4 + \triangle^{\frac{-\theta}{2\delta_4}} |\triangle B_k|^{\frac{1}{\delta_4}} \Big) \bigg] \\
\le & C\big(1+V(Y_k)\big)^3 \mathbb{E}_k \Big[\big(\triangle^{1-\theta} + \triangle^{-\theta} |\triangle B_k|^2 \big) \big(\triangle^{-2\theta} |\triangle B_k|^4 + \triangle^{\frac{-\theta}{2\delta_4}} |\triangle B_k|^{\frac{1}{\delta_4}} \big) \Big] \\
\le & C\big(1+V(Y_k)\big)^3 \triangle^{3(1-\theta)}.
\end{aligned}
$$

Similarly, by (A.5), (A.8), (A.9) and (3.9), we can also prove that

$$
\begin{aligned}
&\mathbb{E}_k \Big[\big|\mathcal{S}_2^{\triangle} V(Y_k)\big|^2 |\mathcal{J}^{\triangle} V(Y_k)| \Big] \\
\le & C\big(1+V(Y_k)\big)^3 \mathbb{E}_k \bigg[\Big(\triangle^{-2\theta} |\triangle B_k|^4 + \triangle^{2(1-\theta)} \\
&\qquad + |\triangle B_k|^2 \triangle^{2-3\theta} \Big) \Big(\triangle^{-2\theta} |\triangle B_k|^4 + \triangle^{\frac{-\theta}{2\delta_4}} |\triangle B_k|^{\frac{1}{\delta_4}} \Big) \bigg] \\
\le & C\big(1+V(Y_k)\big)^3 \triangle^{4(1-\theta)},
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}_k \Big[\big|\mathcal{S}_3^{\triangle} V(Y_k)\big|^2 |\mathcal{J}^{\triangle} V(Y_k)| \Big] \\
\le & C\big(1+V(Y_k)\big)^3 \mathbb{E}_k \bigg[\Big(|\triangle B_k|^2 \triangle^{4-5\theta)} + \triangle^{2-4\theta} |\triangle B_k|^4 + \triangle^{4(1-\theta)} \\
&\qquad + \triangle^{-3\theta} |\triangle B_k|^6 \Big) \Big(\triangle^{-2\theta} |\triangle B_k|^4 + \triangle^{\frac{-\theta}{2\delta_4}} |\triangle B_k|^{\frac{1}{\delta_4}} \Big) \bigg] \\
\le & C\big(1+V(Y_k)\big)^3 \triangle^{5(1-\theta)}.
\end{aligned}
$$

Thus the above inequalities and (3.15) imply

$$
\mathbb{E}_k \big[\xi_k^3 \big] \le C \triangle^{2(1-\theta)} \Big(1 + \triangle^{1-\theta} + \triangle^{2(1-\theta)} + \triangle^{3(1-\theta)} \Big) \le C \triangle^{2(1-\theta)}. \tag{3.16}
$$

Using (3.13), (3.14) and (3.16) as well as (2.4) we now establish inequality (3.6). To this end, combining (3.11)-(3.16), we know that for any integer $k \geq 0$,

$$\begin{aligned}\mathbb{E}_k\Big[\big(1+V(\tilde{Y}_{k+1})\big)^{\rho}\Big] \leq &\big(1+V(Y_k)\big)^{\rho}\Big[1+C\triangle^{2(1-\theta)}\\ &+\frac{\rho\triangle}{2}\frac{2\big(1+V(Y_k)\big)\mathcal{L}V(Y_k)-(1-\rho)|D^{(1)}V(Y_k)g(Y_k)|^2}{\big(1+V(Y_k)\big)^2}\Big]\\ \leq &\big(1+V(Y_k)\big)^{\rho}\Big(1+C\triangle^{2(1-\theta)}\Big)+\mathcal{L}\big(1+V(Y_k)\big)^{\rho}.\end{aligned}$$

Making use of the above inequality as well as (2.4) yields

$$\begin{aligned}\mathbb{E}_k\Big[\big(1+V(\tilde{Y}_{k+1})\big)^{\rho}\Big] \leq &\big(1+V(Y_k)\big)^{\rho}\big(1+C\triangle\big)+\lambda\big(1+V^{\rho}(Y_k)\big)\triangle\\ \leq &\big(1+V(Y_k)\big)^{\rho}\big(1+C\triangle\big)+\lambda\triangle\end{aligned}$$

for any integer $k \geq 0$. Furthermore,

$$V(\pi_{\triangle}(x)) \leq V(x) \qquad \forall\, x \in \mathbb{R}^d,$$

which implies that

$$\mathbb{E}_k\Big[\big(1+V(Y_{k+1})\big)^{\rho}\Big] \leq \mathbb{E}_k\Big[\big(1+V(\tilde{Y}_{k+1})\big)^{\rho}\Big] \leq \big(1+V(Y_k)\big)^{\rho}\big(1+C\triangle\big)+\lambda\triangle.$$

Repeating this procedure we obtain

$$\mathbb{E}\big[\big(1+V(Y_k)\big)^{\rho}\big|\mathcal{F}_0\big] \leq (1+C\triangle)^k\big(1+V(x_0)\big)^{\rho}+\lambda\triangle\sum_{i=0}^{k-1}(1+C\triangle)^i.$$

Taking expectations on both sides yields

$$\mathbb{E}\big[V^{\rho}(Y_k)\big] \leq C(1+C\triangle)^k \leq C\exp\left(Ck\triangle\right) \leq C\exp\left(CT\right)$$

for any integer k satisfying $0 \leq k\triangle \leq T$. Therefore the desired result follows. □

Remark 3.4. If $V \in \mathcal{V}_{\delta_p}^{p} \cap \{D^{(p+1)}V(\cdot) \equiv 0\}$ for $p = 2$ or 3 and some integer $1/\delta_p \in [p, +\infty)$, then $V \in \mathcal{V}_{\delta_4}^{4}$ with $1/\delta_4 = 4 \vee 1/\delta_p$.

Remark 3.5. If $\tilde{V} \in \mathcal{V}_{\delta_4}^{4}$ in Theorem 3.3 is replaced by $\tilde{V} \in \mathcal{V}_{\delta_3}^{3}$ for some integer $1/\delta_3 \in [3, +\infty)$, we can then choose a strictly increasing continuous function $\varphi : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ such that $\varphi(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\leq u}\left(\frac{|f(x)|}{\big(1+\tilde{V}(x)\big)^{\delta_3}} \vee \frac{|g(x)|^2}{\big(1+\tilde{V}(x)\big)^{2\delta_3}}\right) \leq \varphi(u), \qquad \forall\, u \geq 1.$$

Then for any given $0 < \theta \leq 1/3$ the corresponding V-truncated EM scheme (3.2) still has V-integrability (3.6). It turns out that the smoothness of $V(x)$ affects the construction of the scheme (3.2).

Remark 3.6. V-integrability of numerical schemes has already been well studied in [16, 39]. However the results are based on some priori inequality estimates (see, [16, Proposition 2.7] and [39, Theorem 2.5]). Here we show that without additional restricted conditions except those which guarantee the V-integrability of exact solutions, the V-truncated EM scheme (3.2) has V-integrability just like the form of (3.6).

Lemma 3.7. *Under the conditions of Theorem 3.3, for any* $\triangle,\ \triangle_1 \in (0,1]$, *define*

$$\eta_{\triangle}^{\triangle_1} =: \inf\big\{t \geq 0 : |\tilde{Y}(t)| \geq \varphi^{-1}\big(K\triangle_1^{-\theta}\big)\big\}. \tag{3.17}$$

Then for any $T > 0$ *and any* $\triangle \in (0, \triangle_1] \subseteq (0,1]$,

$$\mathbb{E}V^{\rho}\big(\tilde{Y}(T \wedge \eta_{\triangle}^{\triangle_1})\big) \leq C_T,$$

where C_T *is a positive constant independent of* Δ.

Proof. For any $\triangle \in (0, \triangle_1] \subseteq (0,1]$, define $\beta_{\triangle}^{\triangle_1} =: \inf\Big\{k \geq 0 : |\tilde{Y}_k| \geq \varphi^{-1}\big(K\triangle_1^{-\theta}\big)\Big\}$, we have $\eta_{\triangle}^{\triangle_1} = \triangle\beta_{\triangle}^{\triangle_1}$. We write $\eta = \eta_{\triangle}^{\triangle_1}$ and $\beta = \beta_{\triangle}^{\triangle_1}$ shortly. Obviously, both η and β are $\mathcal{F}_{t_k}$ stopping times. For $\omega \in \{\beta \geq k+1\}$, we have $Y_k = \tilde{Y}_{k\wedge\beta}$ due to $\varphi^{-1}\big(K\triangle^{-\theta}\big) \geq \varphi^{-1}\big(K\triangle_1^{-\theta}\big)$ for any $\triangle \in (0, \triangle_1]$, and

$$\tilde{Y}_{(k+1)\wedge\beta} = \tilde{Y}_{k+1}.$$

On the other hand, for $\omega \in \{\beta < k+1\}$, we have $\beta \leq k$ and hence

$$\tilde{Y}_{(k+1)\wedge\beta} = \tilde{Y}_{\beta} = \tilde{Y}_{k\wedge\beta}.$$

Therefore, we derive from (3.2) that for any integer $k \geq 0$,

$$\tilde{Y}_{(k+1)\wedge\beta} = \tilde{Y}_{k\wedge\beta} + \Big[f(\tilde{Y}_{k\wedge\beta})\triangle + g(\tilde{Y}_{k\wedge\beta})\triangle B_{k\wedge\beta}\Big]\, I_{[[0,\beta]]}(k+1),$$

where $[[0,\beta]] := \{(t,\omega) \in \mathbb{R}_+ \times \Omega : 0 \leq t \leq \beta(\omega)\}$ is called a stochastic interval. Since $V \in \mathcal{V}_{\delta_4}^4$, using the Taylor formula with integral remainder term we get

(3.18)

$$V(\tilde{Y}_{(k+1)\wedge\beta}) = V(\tilde{Y}_{k\wedge\beta}) + \sum_{|\alpha|=1}^{3} \frac{D^{\alpha}V(\tilde{Y}_{k\wedge\beta})}{\alpha!}\big(\tilde{Y}_{(k+1)\wedge\beta} - \tilde{Y}_{k\wedge\beta}\big)^{\alpha} + J(\tilde{Y}_{(k+1)\wedge\beta}, \tilde{Y}_{k\wedge\beta}).$$

Note that

$$\triangle B_{k\wedge\beta} I_{[[0,\beta]]}(k+1) = \big(B(t_{(k+1)\wedge\beta}) - B(t_{k\wedge\beta})\big) I_{[[0,\beta]]}(k+1).$$

Since $B(t)$ is a continuous local martingale, by the virtue of the Doob martingale stopping time theorem (see e.g., [29, pp. 11-12]), we know that

$$\mathbb{E}_{k\wedge\beta}\Big[\big(\triangle B_{k\wedge\beta}^{(l)}\big)^{2r+1} I_{[[0,\beta]]}(k+1)\Big] = \mathbb{E}\Big[\big(\triangle B_{k\wedge\beta}^{(l)}\big)^{2r+1} I_{[[0,\beta]]}(k+1)\big|\mathcal{F}_{t_{k\wedge\beta}}\Big] = 0,$$

and

$$\mathbb{E}_{k\wedge\beta}\Big[\big(\triangle B_{k\wedge\beta}^{(l)}\big)^{2r} I_{[[0,\beta]]}(k+1)\Big] = (2r-1)!!\triangle^{r}\mathbb{E}_{k\wedge\beta}\big[I_{[[0,\beta]]}(k+1)\big],$$

where $\mathbb{E}_{k\wedge\beta}[\cdot] := \mathbb{E}\big[\,\cdot\,|\mathcal{F}_{t_{k\wedge\beta}}\big]$ and $\triangle B_{k\wedge\beta} = \big(\triangle B_{k\wedge\beta}^{(1)}, \ldots, \triangle B_{k\wedge\beta}^{(m)}\big)^{T}$. Using the techniques in the proof of Lemma 3.1, we deduce that

$$\begin{aligned}
&\sum_{|\alpha|=1}^{3} \frac{D^{\alpha}V(\tilde{Y}_{k\wedge\beta})}{\alpha!}\big(\tilde{Y}_{(k+1)\wedge\beta} - \tilde{Y}_{k\wedge\beta}\big)^{\alpha} \\
&\leq \Big[\mathcal{L}V(\tilde{Y}_{k\wedge\beta})\triangle + \mathcal{R}^{\triangle}V(\tilde{Y}_{k\wedge\beta}) + \sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(\tilde{Y}_{k\wedge\beta})\Big] I_{[[0,\beta]]}(k+1).
\end{aligned}$$

This together with (3.8) and (3.18) implies

$$\big(1 + V(\tilde{Y}_{(k+1)\wedge\beta})\big)^{\rho} \leq \Big[1 + V(\tilde{Y}_{k\wedge\beta}) + \Big(\mathcal{L}V(\tilde{Y}_{k\wedge\beta})\triangle + \mathcal{R}^{\triangle}V(\tilde{Y}_{k\wedge\beta})$$

$$+\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(\tilde{Y}_{k\wedge\beta})+J(\tilde{Y}_{(k+1)\wedge\beta},\tilde{Y}_{k\wedge\beta})\Big)I_{[[0,\beta]]}(k+1)\Big]^{\rho}$$
$$\le\big(1+V(\tilde{Y}_{k\wedge\beta})\big)^{\rho}\big(1+\bar{\xi}_{k\wedge\beta}I_{[[0,\beta]]}(k+1)\big)^{\rho},$$

where

$$\bar{\xi}_{k\wedge\beta}=\frac{\mathcal{L}V(\tilde{Y}_{k\wedge\beta})\triangle+C\big(1+V(\tilde{Y}_{k\wedge\beta})\big)\triangle^{2(1-\theta)}+\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(\tilde{Y}_{k\wedge\beta})+\mathcal{J}^{\triangle}V(\tilde{Y}_{k\wedge\beta})}{1+V(\tilde{Y}_{k\wedge\beta})}.$$

Using the techniques in the proof of Theorem 3.3, we show that

$$\begin{aligned}
&\mathbb{E}_{k\wedge\beta}\Big[\big(1+V(\tilde{Y}_{(k+1)\wedge\beta})\big)^{\rho}\Big]\\
\le&\big(1+V(\tilde{Y}_{k\wedge\beta})\big)^{\rho}\Bigg\{1+\mathbb{E}_{k\wedge\beta}\Bigg[\frac{\rho\triangle}{2}\big(1+V(\tilde{Y}_{k\wedge\beta})\big)^{-2}\Big(2\big(1+V(\tilde{Y}_{k\wedge\beta})\big)\mathcal{L}V(\tilde{Y}_{k\wedge\beta})\\
&\qquad-(1-\rho)|D^{(1)}V(\tilde{Y}_{k\wedge\beta})g(\tilde{Y}_{k\wedge\beta})|^2\Big)I_{[[0,\beta]]}(k+1)\Bigg]+C\triangle^{2(1-\theta)}\Bigg\}\\
\le&\big(1+V(\tilde{Y}_{k\wedge\beta})\big)^{\rho}\big(1+C\triangle\big)+\lambda\triangle.
\end{aligned}$$

Repeating this procedure we obtain

$$\mathbb{E}\big[\big(1+V(\tilde{Y}_{k\wedge\beta})\big)^{\rho}|\mathcal{F}_0\big]\le\big(1+V(x_0)\big)^{\rho}(1+C\triangle)^k+\lambda\triangle\sum_{i=0}^{k-1}(1+C\triangle)^i$$

for any integer k satisfying $0\le k\triangle\le T$. Taking expectations on both sides yields

$$\mathbb{E}\big[V^{\rho}(\tilde{Y}_{k\wedge\beta})\big]\le C(1+C\triangle)^k\le C\exp\left(Ck\triangle\right)\le C\exp\left(CT\right).$$

Therefore, the desired assertion follows from

$$\mathbb{E}V^{\rho}\big(\tilde{Y}(T\wedge\eta)\big)=\mathbb{E}V^{\rho}\big(\tilde{Y}_{\lfloor\frac{T}{\triangle}\rfloor\wedge\beta}\big)\le C_T,$$

where $\lfloor\frac{T}{\triangle}\rfloor$ represents the integer part of $T/\triangle$. The proof is complete. □

Let $\mathcal{K}$ denote the family of all continuous increasing functions $\kappa:\bar{\mathbb{R}}_+\to\bar{\mathbb{R}}_+$ such that $\kappa(0)=0$ while $\kappa(u)>0$ for $u>0$. Denote by κ^{-1} the inverse function of $\kappa\in\mathcal{K}$. Let $\mathcal{K}_{\vee}$ denote the family of all convex functions $\kappa\in\mathcal{K}$, and $\mathcal{K}_{\wedge}$ denote the family of all concave functions $\kappa\in\mathcal{K}$.

Theorem 3.8. *Let the conditions of Theorem 3.3 hold. If there moreover exists a function $\kappa\in\mathcal{K}_{\vee}$ and a constant $\bar{p}>0$ such that*

$$\kappa\big(|x|^{\bar{p}}\big)\le V^{\rho}(x)\quad\forall x\in\mathbb{R}^d,$$

then for any $q\in(0,\bar{p})$,

$$\lim_{\triangle\to 0}\mathbb{E}|X(T)-Y(T)|^q=0,\quad\forall\ T>0.\tag{3.19}$$

Proof. Let $\eta_{\triangle}^{\triangle_1}$ be defined as before. For any $N>0$, $T>0$, define

(3.20)

$$\tau_N=:\inf\{t\ge 0:|X(t)|\ge N\},\quad\theta_{\triangle}^{N,\triangle_1}:=\tau_N\wedge\eta_{\triangle}^{\triangle_1},\quad e_{\triangle}(T):=X(T)-Y(T).$$

For any $l>0$, using the Young inequality we obtain that

$$\mathbb{E}|e_{\triangle}(T)|^q=\mathbb{E}\left(|e_{\triangle}(T)|^qI_{\{\theta_{\triangle}^{N,\triangle_1}\ge T\}}\right)+\mathbb{E}\left(|e_{\triangle}(T)|^qI_{\{\theta_{\triangle}^{N,\triangle_1}\le T\}}\right)$$

$$\leq \mathbb{E}\left(|e_\triangle(T)|^q I_{\{\theta_\triangle^{N,\triangle_1} \geq T\}}\right) + \frac{ql}{\bar{p}}\mathbb{E}\left(|e_\triangle(T)|^{\bar{p}}\right) + \frac{\bar{p}-q}{\bar{p}l^{q/(\bar{p}-q)}}\mathbb{P}\{\theta_\triangle^{N,\triangle_1} \leq T\}. \tag{3.21}$$

It follows from the virtues of Theorem 2.1 and Theorem 3.3 that

$$\begin{aligned}\mathbb{E}|e_\triangle(T)|^{\bar{p}} \leq& 2^{\bar{p}}\big(\mathbb{E}|X(T)|^{\bar{p}} + \mathbb{E}|Y(T)|^{\bar{p}}\big)\\ \leq& 2^{\bar{p}}\Big(\kappa^{-1}\big(\mathbb{E}V^\rho(X(T))\big) + \kappa^{-1}\big(\mathbb{E}V^\rho(Y(T))\big)\Big) \leq C_T,\end{aligned}$$

where κ^{-1} is the inverse function of κ. Now let $\varepsilon > 0$ be arbitrary. Choose $l > 0$ small sufficiently such that $C_T ql/\bar{p} \leq \varepsilon/3$, then we have

$$\frac{ql}{\bar{p}}\mathbb{E}\left(|e_\triangle(T)|^{\bar{p}}\right) \leq \frac{\varepsilon}{3}. \tag{3.22}$$

Then due to the monotonicity of φ^{-1} we can choose $\triangle_1 \in (0,1]$ small sufficiently such that

$$\frac{C_T(\bar{p}-q)}{\kappa\big(|\varphi^{-1}\big(K\triangle_1^{-\theta}\big)|^{\bar{p}}\big)\bar{p}l^{q/(\bar{p}-q)}} \leq \frac{\varepsilon}{6}.$$

Let $N = \varphi^{-1}\big(K\triangle_1^{-\theta}\big) > 0$. Then by the virtue of Theorem 2.1 and Lemma 3.7, we have

$$\mathbb{P}\{\tau_N \leq T\} \leq \frac{C_T}{\kappa\big(N^{\bar{p}}\big)}, \qquad \mathbb{P}\{\eta_\triangle^{\triangle_1} \leq T\} \leq \frac{C_T}{\kappa\big(|\varphi^{-1}\big(K\triangle_1^{-\theta}\big)|^{\bar{p}}\big)}.$$

This implies that

$$\frac{\bar{p}-q}{\bar{p}l^{q/(\bar{p}-q)}}\mathbb{P}\{\theta_\triangle^{N,\triangle_1} \leq T\} \leq \frac{\bar{p}-q}{\bar{p}l^{q/(\bar{p}-q)}}\Big(\mathbb{P}\{\tau_N \leq T\} + \mathbb{P}\{\eta_\triangle^{\triangle_1} \leq T\}\Big) \leq \frac{\varepsilon}{3}. \tag{3.23}$$

Combining (3.21), (3.22) and (3.23), we know that for the chosen N, $\triangle_1$ and all $\triangle \in (0, \triangle_1]$,

$$\mathbb{E}|e_\triangle(T)|^q \leq \mathbb{E}\left(|e_\triangle(T)|^q I_{\{\theta_\triangle^{N,\triangle_1} \geq T\}}\right) + \frac{2\varepsilon}{3}.$$

If we can show that

$$\lim_{\triangle \to 0}\mathbb{E}\left(|e_\triangle(T)|^q I_{\{\theta_\triangle^{N,\triangle_1} \geq T\}}\right) = 0, \tag{3.24}$$

the required assertion follows. For this purpose we define the truncated functions

$$f_N(x) = f\Big(\big(|x| \wedge N\big)\frac{x}{|x|}\Big) \text{ and } g_N(x) = g\Big(\big(|x| \wedge N\big)\frac{x}{|x|}\Big)$$

for any $x \in \mathbb{R}^d$. Consider the truncated SDE

$$dz(t) = f_N(z(t))dt + g_N(z(t))dB(t) \tag{3.25}$$

with the initial value $z(0) = x_0$. For the chosen N, we know that $f_N(\cdot)$ and $g_N(\cdot)$ are globally Lipschitz continuous with the Lipschitz constant C_N. Therefore, SDE (3.25) has a unique regular solution $z(t)$ satisfying

$$X(t \wedge \tau_N) = z(t \wedge \tau_N) \quad \text{a.s.} \qquad \forall\, t \geq 0. \tag{3.26}$$

On the other hand, for each $\triangle \in (0, \triangle_1]$, we apply the EM method to (3.25) and denote by $u(t)$ the piecewise constant EM solution (see [9, 20]) which has the property

$$\mathbb{E}\Big(\sup_{0\leq t\leq T}|z(t) - u(t)|^q\Big) \leq C_N\triangle^{q/2}, \qquad \forall\, T \geq 0. \tag{3.27}$$

Due to $\varphi^{-1}\big(K\triangle^{-\theta}\big) \geq \varphi^{-1}\big(K\triangle_1^{-\theta}\big)$ for any $\triangle \in (0, \triangle_1]$, we have $Y(t \wedge \theta_\triangle^{N,\triangle_1}) = u(t \wedge \theta_\triangle^{N,\triangle_1})$ a.s. This together with (3.26) implies

$$\begin{aligned}\mathbb{E}\left(|e_\triangle(T)|^q I_{\{\theta_\triangle^{N,\triangle_1} \geq T\}}\right) =& \mathbb{E}\left(|e_\triangle(T \wedge \theta_\triangle^{N,\triangle_1})|^q I_{\{\theta_\triangle^{N,\triangle_1} \geq T\}}\right) \\ \leq& \mathbb{E}\Big(\sup_{0 \leq t \leq T \wedge \theta_\triangle^{N,\triangle_1}} |z(t) - u(t)|^q\Big).\end{aligned}$$

This together with (3.27) implies (3.24) as desired. The proof is complete. $\square$

4. Convergence rate

In this section, our aim is to establish a rate of convergence result under the conditions of Theorem 3.8 and additional conditions on f and g. The convergence rate to be established is optimal as it is similar to the standard results of the explicit EM scheme for SDEs with globally Lipschitz coefficients. The work of Higham et al. [9] gave the optimal rate in pth moment for the implicit EM scheme for $p \geq 2$ with the global Lipschitz g and a one-sided Lipschitz f together with the polynomial growth condition. Using the similar conditions to those in [9], the rate for the tamed Euler was obtained by Hutzenthaler et al. [15]. Using the special Lyapunov function $|x|^p$ with $p \geq 2$, Sabanis [36] developed the tamed EM scheme, and obtained the optimal convergence rate.

For convenience, for a pair of integers $p \in [2, +\infty)$ and $1/\delta_p \in [4, +\infty)$, define

$$\begin{aligned}\bar{\mathcal{V}}_{\delta_p}^p := \Big\{V \in \mathcal{C}_\infty^p(\mathbb{R}^d; \bar{\mathbb{R}}_+)\Big| V(\mathbf{0}) = 0,\ \exists\ c > 0 \text{ s.t.}& \\ |D^{(n)}V(\cdot)| \leq cV^{1-n\delta_p}(\cdot),\ n = 1, 2, \ldots, p\Big\}.&\end{aligned} \tag{4.1}$$

To estimate the (strong) convergence rate, we need somewhat stronger conditions compared with the convergence alone, which are stated as follows.

Assumption 1. There exist functions $\kappa^2 \in \mathcal{K}_\wedge$, $V \in \mathcal{V}_{\delta_4}^4$ and $U \in \bar{\mathcal{V}}_{\delta_2}^2$ for a pair of integers $1/\delta_4 \in [4, +\infty)$ and $1/\delta_2 \in [2, +\infty)$, as well as positive constants a, q, τ and c_1 such that $\Delta^\tau \leq \kappa(\Delta^{\frac{q}{2}})$ for any $\Delta \in (0, \Delta^*]$,

$$U(x) \leq \kappa(|x|^q) \leq V^a(x), \qquad U(x+y) \leq c_1(U(x) + U(y)) \qquad \forall\ x, y \in \mathbb{K}, \tag{4.2}$$

for any compact subset $\mathbb{K} \subset \mathbb{R}^d$. Moreover, for some $\iota > 0$, there exist positive constants $\bar{K}$ and r such that $r > 2(\delta_4/a - \delta_2)$,

$$2U_x(x-y)\big(f(x) - f(y)\big) + (1+\iota)|U_{xx}(x-y)||g(x) - g(y)|^2 \leq \bar{K}U(x-y),$$

and

$$\begin{aligned}|f(x) - f(y)| &\leq \bar{K}\big(1 + U^r(x) + U^r(y)\big)U^{\delta_2}(x-y), \\ |g(x) - g(y)|^2 &\leq \bar{K}\big(1 + U^r(x) + U^r(y)\big)U^{2\delta_2}(x-y),\end{aligned}$$

for any $x, y \in \mathbb{R}^d$.

Remark 4.1. One observes that if Assumption 1 holds, by Young's inequality

$$\begin{aligned}|g(x)| \leq& |g(x) - g(\mathbf{0})| + |g(\mathbf{0})| \\ \leq& \sqrt{\bar{K}}\big(U^{2\delta_2}(x) + U^{r+2\delta_2}(x)\big)^{\frac{1}{2}} + |g(\mathbf{0})| \leq C\big(1 + U^{\frac{r}{2}+\delta_2}(x)\big),\end{aligned} \tag{4.3}$$

Remark 4.2. Let $\ell := r+2\delta_2-2\delta_4/a$. Under Assumption 1, $\ell > 0$ and we may define $\varphi(u) = C\big(1+\kappa^{\ell}(u^q)\big)$ for any $u > 0$ in (3.1), then $\varphi^{-1}(u) = \big[\kappa^{-1}\big((u/C-1)^{\frac{1}{\ell}}\big)\big]^{\frac{1}{q}}$ for all $u > C$.

Making use of scheme (3.2), we define an auxiliary approximation process by

$$\bar{Y}(t) = Y_k + f(Y_k)(t-t_k) + g(Y_k)(B(t)-B(t_k)), \qquad \forall t \in [t_k, t_{k+1}). \tag{4.4}$$

Note that $\bar{Y}(t_k) = Y(t_k) = Y_k$, that is $\bar{Y}(t)$ and $Y(t)$ coincide with the discrete solution at the grid points.

Lemma 4.3. *Let Assumption 1 and the conditions of Theorem 3.3 hold. Then for any $T > 0$ and $q_0 \in (0, 2\rho/a(r+2\delta_2)]$, the process defined by (4.4) satisfies*

$$\mathbb{E}|\bar{Y}(t) - Y(t)|^{q_0} \le C_T \triangle^{\frac{q_0}{2}}, \quad \forall t \in [0,T].$$

Proof. For any $t \in [0,T]$, there is an integer $k \ge 0$ such that $t \in [t_k, t_{k+1})$. Then,

$$\begin{aligned}\mathbb{E}\big(|\bar{Y}(t) - Y(t)|^{q_0}\big) =& \mathbb{E}\big(|\bar{Y}(t) - Y(t_k)|^{q_0}\big)\\ \le& 2^{q_0}\mathbb{E}\big(|f(Y_k)|^{q_0}\big)\triangle^{q_0} + 2^{q_0}\mathbb{E}\big(|g(Y_k)|^{q_0}|B(t)-B(t_k)|^{q_0}\big)\\ \le& C\left(\mathbb{E}|f(Y_k)|^{q_0}\triangle^{q_0} + \mathbb{E}|g(Y_k)|^{q_0}\triangle^{\frac{q_0}{2}}\right).\end{aligned}$$

Due to (3.3), (4.2) and (4.3) as well as Theorem 3.3,

$$\begin{aligned}\mathbb{E}\big(|\bar{Y}(t) - Y(t)|^{q_0}\big) &\le C\mathbb{E}\big(1+V(Y_k)\big)^{q_0\delta_4}\triangle^{q_0(1-\theta)} + C\mathbb{E}\big(1+U^{\frac{r}{2}+\delta_2}(Y_k)\big)^{q_0}\triangle^{\frac{q_0}{2}}\\ &\le C\triangle^{\frac{q_0}{2}}\left[\mathbb{E}\big(1+V(Y_k)\big)^{a(\frac{r}{2}+\delta_2)q_0} + \mathbb{E}\big(1+U^{\frac{r}{2}+\delta_2}(Y_k)\big)^{q_0}\right]\\ &\le C\triangle^{\frac{q_0}{2}}\left[1 + \big(\mathbb{E}V^{\rho}(Y_k)\big)^{\frac{aq_0(r+2\delta_2)}{2\rho}} + \big(\mathbb{E}V^{\rho}(Y_k)\big)^{\frac{aq_0(r+2\delta_2)}{2\rho}}\right]\\ &\le C_T\triangle^{\frac{q_0}{2}}.\end{aligned}$$

The required assertion follows. □

Using techniques in the proofs of Theorem 3.3 and Lemma 3.7, we obtain the following lemmas.

Lemma 4.4. *Under the conditions of Theorem 3.3, the process defined by (4.4) has the property that*

$$\sup_{\triangle\in(0,1]}\sup_{0\le t\le T}\mathbb{E}V^{\rho}(\bar{Y}(t)) \le C_T, \quad \forall\ T > 0.$$

Lemma 4.5. *Under the conditions of Theorem 3.3, for any $\triangle \in (0,1]$, define*

$$\bar{\eta}_{\triangle} := \inf\big\{t \ge 0 : |\bar{Y}(t)| \ge \varphi^{-1}\big(K\triangle^{-\theta}\big)\big\}. \tag{4.5}$$

Then for any $T > 0$,

$$\mathbb{E}V^{\rho}(\bar{Y}(T\wedge\bar{\eta}_{\triangle})) \le C_T.$$

Theorem 4.6. *Let Assumption 1 hold. Assume that the conditions of Theorem 3.3 hold for $\rho > a$ and $\rho \ge (2ar/\delta_2)\vee aq(r/2+\delta_2)$. If $\tau \in (0, \theta(\rho-a)/a\ell]$, then the process defined by (4.4) has the property that*

$$\mathbb{E}U(\bar{Y}(T) - X(T)) \le C_T\kappa(C_T\triangle^{\frac{q}{2}}), \quad \forall\ T > 0.$$

Proof. Define $\bar\theta_\triangle = \tau_{\varphi^{-1}(K\triangle^{-\theta})}\wedge\eta_\triangle^\triangle\wedge\bar\eta_\triangle$, $\Omega_1 := \{\omega:\ \bar\theta_\triangle > T\}$, $\bar e(t) = \bar Y(t) - X(t)$, for any $t\in[0,T]$, where τ_N, $\eta_\triangle^\triangle$ and $\bar\eta_\triangle$ are defined by (3.20), (3.17) and (4.5), respectively. Using the Young inequality, we have

$$
\begin{aligned}
\mathbb{E}U(\bar e(T)) =&\mathbb{E}\left[U(\bar e(T))I_{\Omega_1}\right] + \mathbb{E}\left[U(\bar e(T))I_{\Omega_1^c}\right]\\
\le&\mathbb{E}\left[U(\bar e(T))I_{\Omega_1}\right] + \frac{a\kappa(\triangle^{\frac q2})}{\rho}\mathbb{E}\left[U^{\rho/a}(\bar e(T))\right] + \frac{\rho-a}{\rho\kappa^{\frac{a}{\rho-a}}(\triangle^{\frac q2})}\mathbb{P}(\Omega_1^c).
\end{aligned}\tag{4.6}
$$

It follows from the results of Theorem 2.1 and Lemma 4.4 that

$$
\frac{a\kappa(\triangle^{\frac q2})}{\rho}\mathbb{E}U^{\rho/a}\big(\bar e_\triangle(T)\big) \le C\kappa(\triangle^{\frac q2})\Big[\mathbb{E}V^\rho(X(T)) + \mathbb{E}V^\rho(\bar Y(T))\Big] \le C_T\kappa(\triangle^{\frac q2}).\tag{4.7}
$$

Moreover, by the virtue of Theorem 2.1, Lemmas 3.7 and 4.5 we yield

$$
\begin{aligned}
&\frac{\rho-a}{\rho\big[\kappa(\triangle^{\frac q2})\big]^{\frac{a}{\rho-a}}}\mathbb{P}(\Omega_1^c)\\
\le&\frac{\rho-a}{\rho\big[\kappa(\triangle^{\frac q2})\big]^{\frac{a}{\rho-a}}}\Big(\mathbb{P}\{\tau_{\varphi^{-1}(K\triangle^{-\theta})}\le T\} + \mathbb{P}\{\eta_\triangle^\triangle\le T\} + \mathbb{P}\{\bar\eta_\triangle\le T\}\Big)\\
\le&\frac{\rho-a}{\rho\big[\kappa(\triangle^{\frac q2})\big]^{\frac{a}{\rho-a}}}\frac{3C_T}{\big[\kappa(|\varphi^{-1}(K\triangle^{-\theta})|^q)\big]^{\frac{\rho}{a}}} \le C_T\big[\kappa(\triangle^{\frac q2})\big]^{\frac{a}{a-\rho}}\Big(K\triangle^{-\theta}/C - 1\Big)^{-\frac{\rho}{a\ell}}\\
\le&C_T\big[\kappa(\triangle^{\frac q2})\big]^{\frac{a}{a-\rho}+\frac{\theta\rho}{a\ell\tau}} \le C_T\kappa(\triangle^{\frac q2}).
\end{aligned}\tag{4.8}
$$

On the other hand, note that for any $t\in(0, T\wedge\bar\theta_\triangle]$,

$$
\bar e(t) = \int_0^t\Big(f(X(s)) - f(Y(s))\Big)\mathrm{d}s + \int_0^t\Big(g(X(s)) - g(Y(s))\Big)\mathrm{d}B(s).
$$

Using Itô's formula we obtain

$$
\begin{aligned}
&U\big(\bar e(T\wedge\bar\theta_\triangle)\big)\\
=&\frac12\int_0^{T\wedge\bar\theta_\triangle}\Big[2U_x\big(\bar e(s)\big)\Big(f(X(s)) - f(Y(s))\Big)\\
&\qquad + \mathrm{tr}\Big[\Big(g(X(s)) - g(Y(s))\Big)^T U_{xx}\big(\bar e(s)\big)\Big(g(X(s)) - g(Y(s))\Big)\Big]\mathrm{d}s + M(T\wedge\bar\theta_\triangle)\\
\le&\frac12\int_0^{T\wedge\bar\theta_\triangle}\Big[2U_x\big(\bar e(s)\big)\Big(f(X(s)) - f(Y(s))\Big)\\
&\qquad + |U_{xx}\big(\bar e(s)\big)||g(X(s)) - g(Y(s))|^2\Big]\mathrm{d}s + M(T\wedge\bar\theta_\triangle),
\end{aligned}
$$

where $M(t) = \int_0^t U_x\big(\bar e(s)\big)\big(g(X(s)) - g(Y(s))\big)\mathrm{d}B(s)$ is a local martingale with initial value 0. This implies

$$
\begin{aligned}
\mathbb{E}U\big(\bar e(T\wedge\bar\theta_\triangle)\big) \le&\frac12\mathbb{E}\int_0^{T\wedge\bar\theta_\triangle}\Big[2U_x\big(\bar e(s)\big)\Big(f(X(s)) - f(Y(s))\Big)\\
&\qquad + |U_{xx}\big(\bar e(s)\big)||g(X(s)) - g(Y(s))|^2\Big]\mathrm{d}s.
\end{aligned}\tag{4.9}
$$

Then an application of Young's inequality together with Assumption 1 leads to

$$
\begin{aligned}
&2U_x\big(\bar e(s)\big)\Big(f(X(s)) - f(Y(s))\Big) + |U_{xx}\big(\bar e(s)\big)||g(X(s)) - g(Y(s))|^2\\
=&2U_x\big(\bar e(s)\big)\Big(f(X(s)) - f(\bar Y(s)) + f(\bar Y(s)) - f(Y(s))\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + |U_{xx}\big(\bar e(s)\big)||g(X(s)) - g(\bar Y(s)) + g(\bar Y(s)) - g(Y(s))|^2\\
&\le 2U_x\big(\bar e(s)\big)\Big(f(X(s)) - f(\bar Y(s))\Big) + 2U_x\big(\bar e(s)\big)\Big(f(\bar Y(s)) - f(Y(s))\Big)\\
&\quad + (1+\iota)|U_{xx}\big(\bar e(s)\big)||g(X(s)) - g(\bar Y(s))|^2 + \big(1+\frac{1}{\iota}\big)|U_{xx}\big(\bar e(s)\big)||g(\bar Y(s)) - g(Y(s))|^2\\
&\le C\Big[U\big(\bar e(s)\big) + U^{1-\delta_2}\big(\bar e(s)\big)\big|f(\bar Y(s)) - f(Y(s))\big| + U^{1-2\delta_2}\big(\bar e(s)\big)|g(\bar Y(s)) - g(Y(s))|^2\Big]\\
&\le C\Big[U\big(\bar e(s)\big) + \big|f(\bar Y(s)) - f(Y(s))\big|^{\frac{1}{\delta_2}} + |g(\bar Y(s)) - g(Y(s))|^{\frac{1}{\delta_2}}\Big]\\
&\le C\Big[U\big(\bar e(s)\big) + \big(1 + U^r(Y(s)) + U^r(\bar Y(s))\big)^{\frac{1}{\delta_2}}U\big(\bar Y(s) - Y(s)\big)\\
&\qquad + \big(1 + U^r(Y(s)) + U^r(\bar Y(s))\big)^{\frac{1}{2\delta_2}}U\big(\bar Y(s) - Y(s)\big)\Big]\\
&\le C\Big[U\big(\bar e(s)\big) + \big(1 + U^{\frac{r}{\delta_2}}(Y(s)) + U^{\frac{r}{\delta_2}}(\bar Y(s))\big)U\big(\bar Y(s) - Y(s)\big)\Big].
\end{aligned}
$$

Inserting the above inequality into (4.9) and applying Hölder's equality and Jensen's equality, and then Lemmas 4.3 and 4.4, we have

$$
\begin{aligned}
&\mathbb{E}\Big[U\big(\bar e(T\wedge\bar\theta_\triangle)\big)\Big]\\
&\le C\int_0^T \mathbb{E}\Big[U\big(\bar e(s\wedge\bar\theta_\triangle)\big)\Big]\mathrm{d}s\\
&\qquad + C\int_0^T \bigg[\mathbb{E}\Big(1 + U^{\frac{r}{\delta_2}}(Y(s)) + U^{\frac{r}{\delta_2}}(\bar Y(s))\Big)^2\bigg]^{\frac12}\kappa\big(\mathbb{E}|Y(s) - \bar Y(s)|^q\big)\mathrm{d}s\\
&\le C\int_0^T \mathbb{E}\Big[U\big(\bar e(s\wedge\bar\theta_\triangle)\big)\Big]\mathrm{d}s + C\int_0^T \bigg[\Big(1 + \big(\mathbb{E}V^\rho(Y(s))\big)^{\frac{2ar}{\rho\delta_2}}\\
&\qquad + \big(\mathbb{E}V^\rho(\bar Y(s))\big)^{\frac{2ar}{\rho\delta_2}}\Big)^{\frac12}\times\kappa\Big(\big(\mathbb{E}|Y(s) - \bar Y(s)|^{\frac{2\rho}{a(r+2\delta_2)}}\big)^{\frac{a(r+2\delta_2)q}{2\rho}}\Big)\bigg]\mathrm{d}s\\
&\le C\int_0^T \mathbb{E}\Big[U\big(\bar e(s\wedge\bar\theta_\triangle)\big)\Big]\mathrm{d}s + C_T\kappa(C_T\triangle^{\frac{q}{2}}).
\end{aligned}
$$

Applying the Gronwall inequality, we yield that

$$
\mathbb{E}\Big[U(\bar e(T))I_{\Omega_1}\Big] \le \mathbb{E}\Big[U\big(\bar e(T\wedge\bar\theta_\triangle)\big)\Big] \le C_T\kappa(C_T\triangle^{\frac{q}{2}}). \tag{4.10}
$$

Inserting (4.7), (4.8) and (4.10) into (4.6) yields the desired assertion. □

Next we provide two remarks to demonstrate (4.2) of Assumption 1.

Remark 4.7. For any $x\in\mathbb{R}$, let $\kappa(|x|^2) = \log(1+|x|^2)$, for any $\Delta\in(0,0.6]$ we have $\Delta^\tau \le \kappa(\Delta^{\frac{q}{2}})$ with $\tau\ge 1.5$, $q=2$. Let $V(x) = U(x) = \log(1+|x|^2)$, we have

$$
|U_x(x)| = \frac{2|x|}{1+x^2} \le 3\sqrt{\log(1+x^2)}, \qquad |U_{xx}(x)| = \Big|\frac{2}{1+x^2} - \frac{4x^2}{(1+x^2)^2}\Big| \le 3.
$$

Thus the above inequalities imply $a=1$, $\delta_2 = 1/2$, $U\in\bar{\mathcal{V}}^2_{1/2}$ and $V\in\mathcal{V}^4_{\delta_4}$ with $\delta_4\in(0,1/4]$. For any $x,y\in\mathbb{R}$, we have

$$
U(x+y) = \log\big(1+|x+y|^2\big) \le 4\Big(\log\big(1+|x|^2\big) + \log\big(1+|y|^2\big)\Big) \le 4\big(U(x)+U(y)\big),
$$

then (4.2) holds for $c_1 = 4$. Clear, for any $r>0$, we have $r > 2(\delta_4/a - \delta_2)$.

Remark 4.8. For any $x, y \in \mathbb{R}^d$, let $\kappa(|x|^q) = |x|^{\bar{q}}$, for any $\Delta \in (0,1]$ we have $\Delta^\tau \leq \kappa(\Delta^{\frac{q}{2}})$ with $\tau \geq \bar{q}/2$, $q = 2\bar{q}$. Let $V(x) = |x|^2$ and $U(x) = |x|^{\bar{q}}$ with $\bar{q} \geq 2$, we have

$$U(x+y) \leq 2^{\bar{q}-1}\big(U(x) + U(y)\big), \quad |U_x(x)| = \left|\bar{q}|x|^{\bar{q}-2}x^T\right| = \bar{q}|x|^{\bar{q}-1} \leq \bar{q}^2\big(|x|^{\bar{q}}\big)^{1-\frac{1}{\bar{q}}},$$

and

$$|U_{xx}(x)| = \left|\bar{q}(\bar{q}-2)|x|^{\bar{q}-4}xx^T + \bar{q}|x|^{\bar{q}-2}\mathbb{I}_d\right| \leq \bar{q}(\bar{q}-1)|x|^{\bar{q}-2} \leq \bar{q}^2\big(|x|^{\bar{q}}\big)^{1-\frac{2}{\bar{q}}}.$$

Thus the above inequalities imply $a = \bar{q}/2$, $\delta_2 = 1/\bar{q}$, $U \in \bar{\mathcal{V}}^2_{1/\bar{q}}$ and $V \in \mathcal{V}^4_{\delta_4}$ with $\delta_4 \in (0, 1/\bar{q}]$, then (4.2) holds for $c_1 = 2^{\bar{q}-1}$. Clearly, for any $r > 0$, we have $r > 2(\delta_4/a - \delta_2)$. Choosing $\rho = p/2, \delta_4 = 1/\bar{q}$ and $p > \bar{q}$, we can get the optimal rate of convergence $1/2$ as in the literature [15, 23, 36].

Remark 4.9. In order to highlight our contributions to the study of convergence rate, we make comparisons between Theorem 4.6 and some results in [15, 23, 34, 36] as follows.

- Condition (3.31) of Theorem 3.8 in [34] is not imposed in our Theorem 4.6. Moreover, the 1/2-order rate of convergence under mild conditions can be obtained by Theorem 4.6 while this rate can only be arbitrarily close to 1/2 by Theorem 3.8 in [34].
- Although the conditions of Theorem 4.6 look slightly more complicated than those in [15, 23, 36] due to the nature of the use of general Lyapunov-type functions, the optimal convergence rate 1/2 can be obtained by taking special functions for $\kappa(\cdot)$, $V(\cdot)$ and $U(\cdot)$, which is emphasised in Remark 4.8.

5. Asymptotic stability

Since the stability is one of the major concerns in many applications, the easily implementable scheme preserving the underlying stability is desired eagerly. In this section we cite the stability criterion of the exact solution for SDEs (see, e.g. [29,30]) and go a further step to construct the new explicit scheme while keeping this long-time property well. Moreover, for the stability purpose, we assume furthermore in this section that SDE (2.1) with drift and diffusion satisfying $f(x^*) \equiv \mathbf{0} \in \mathbb{R}^d$, $g(x^*) \equiv \mathbf{0} \in \mathbb{R}^{d\times m}$ for some $x^* \in \mathbb{R}^d$, which implies $X(t) \equiv x^*$ is a *trivial solution* of SDE (2.1) with the initial value $x_0 = x^*$. Without loss of generality, we assume in this section that $x^* = \mathbf{0}$, namely, $f(\mathbf{0}) \equiv \mathbf{0}$ and $g(\mathbf{0}) \equiv \mathbf{0}$.

5.1. Stability of the exact solution.

To begin this subsection, we cite a stochastic version of the LaSalle theorem on almost sure stability, please see details in [30].

Lemma 5.1 ([30]). *Assume that for some $\rho > 0$, there are functions $V \in \mathcal{C}^2_\infty(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ and $w \in \mathcal{C}(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ such that*

$$\mathcal{L}V^\rho(x) \leq -w(x), \qquad \forall x \in \mathbb{R}^d. \tag{5.1}$$

Then, for every $x_0 \in \mathbb{R}^d$, $\mathrm{Ker}(w) := \{x \in \mathbb{R}^d | w(x) = 0\} \neq \emptyset$, and the solution $X(t)$ of (2.1) has the property

$$\limsup_{t\to\infty} V^\rho(X(t)) < \infty, \qquad \lim_{t\to\infty} d(X(t), \mathrm{Ker}(w)) = 0 \qquad \text{a.s.}$$

Moreover, if $w(x) = 0$ iff $x = \mathbf{0} \in \mathbb{R}^d$, then $\lim_{t\to\infty} X(t) = \mathbf{0}$ a.s.

Based on the above lemma, we further estimate the exponential convergence rate.

Corollary 5.2. *Assume that the conditions of Lemma 5.1 hold and, moreover, $w(x) \geq \mu V^\rho(x)$ for all $x \in \mathbb{R}^d$, where μ is a positive constant. Then the solution $X(t)$ of SDE (2.1) has the properties that*

$$\limsup_{t\to\infty} \frac{\log\big(\mathbb{E}V^\rho(X(t))\big)}{t} \leq -\mu, \tag{5.2}$$

and

$$\limsup_{t\to\infty} \frac{\log\big(V(X(t))\big)}{t} \leq -\frac{\mu}{\rho} \qquad a.s. \tag{5.3}$$

Proof. For any given $\rho > 0$, using Itô's formula (see e.g., [29, Theorem 6.4, p.36]) and $w(x) \geq \mu V^\rho(x)$ we derive that $\mathbb{E}\big[\mathrm{e}^{\mu t}V^\rho(X(t))\big] \leq V^\rho(x_0)$. This implies the desired assertion (5.2). Moreover, by the nonnegative semimartingale convergence theorem (see, e.g., [29, Theorem 3.9, p.14]), we obtain $\limsup_{t\to\infty} \mathrm{e}^{\mu t}V^\rho(X(t)) < \infty$ *a.s.* We can easily carry out the proof of this corollary and hence omit it to avoid repetition. □

5.2. **Stability of numerical solutions.** The conditions of Lemma 5.1 provide us an opportunity to construct a more precise scheme approximating the underlying stability. Recall that for an integer $1/\delta_4 \in [4, +\infty)$, $\bar{\mathcal{V}}^4_{\delta_4}$ has been defined by (4.1). If a Lyapunov function $V \in \bar{\mathcal{V}}^4_{\delta_4}$ satisfying (5.1) is found, the almost surely asymptotic property of the solution process follows from Lemma 5.1. In order to approximate this property in this subsection, assume

$$\sup_{0<|x|\leq u} \frac{|f(x)|^2 \vee |g(x)|^2}{\Lambda_\rho(x)V^{2\delta_4}(x)} < +\infty \tag{5.4}$$

for any $u > 0$, where $\Lambda_\rho(x) := 1 \wedge \big(w(x)/V^\rho(x)\big)$. Under the above assumption we estimate the growth rate of f and g. Choose a strictly increasing continuous function $\bar{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\bar{\varphi}(u) \to \infty$ as $u \to \infty$ such that

$$\sup_{0<|x|\leq u} \left[\frac{|f(x)|}{\Lambda_\rho^{1/2}(x)V^{\delta_4}(x)} \vee \frac{|g(x)|^2}{\Lambda_\rho(x)V^{2\delta_4}(x)}\right] \leq \bar{\varphi}(u), \qquad \forall\ u \geq 1. \tag{5.5}$$

Due to (5.4), $\bar{\varphi}$ can be found easily and denote by $\bar{\varphi}^{-1}$ the inverse function of $\bar{\varphi}$, obviously $\bar{\varphi}^{-1} : [\bar{\varphi}(1), \infty) \to \mathbb{R}_+$ is a strictly increasing continuous function. Then choose a constant $K \geq \bar{\varphi}(|x_0| \vee 1)$, where K is a positive constant independent of $\triangle$. For the given $\triangle \in (0, 1]$, define a truncation mapping $\bar{\pi}_\triangle : \mathbb{R}^d \to \mathbb{R}^d$ by $\bar{\pi}_\triangle(x) = \big(|x| \wedge \bar{\varphi}^{-1}(K\triangle^{-\bar{\theta}})\big)\frac{x}{|x|}$, where $\bar{\theta} \in (0, 1/2)$, and we use the convention $\frac{x}{|x|} = \mathbf{0}$ when $x = \mathbf{0} \in \mathbb{R}^d$. Next, for any given stepsize $\triangle \in (0, 1]$, define the V-truncated EM scheme by

$$\begin{cases} Z_0 = x_0, \\ \tilde{Z}_{k+1} = Z_k + f(Z_k)\triangle + g(Z_k)\triangle B_k, \\ Z_{k+1} = \bar{\pi}_\triangle(\tilde{Z}_{k+1}), \end{cases} \tag{5.6}$$

for any integer $k \geq 0$, where $t_k = k\triangle$ and $\triangle B_k = B(t_{k+1}) - B(t_k)$. To obtain the continuous-time approximation, we define $Z(t)$ by $Z(t) := Z_k$ for any $t \in [t_k, t_{k+1})$.

Then the drift and diffusion terms have the property

$$|f(Z_k)|^2 \le K^2 \triangle^{-2\bar{\theta}} \Lambda_\rho(Z_k) V^{2\delta_4}(Z_k), \quad |g(Z_k)|^2 \le K \triangle^{-\bar{\theta}} \Lambda_\rho(Z_k) V^{2\delta_4}(Z_k), \tag{5.7}$$

which implies (3.3) holds. Thus, the conclusion of Theorem 3.8 still holds for the new scheme (5.6) under the conditions of Lemma 5.1. Next we prepare the discrete version of the nonnegative semimartingale convergence theorem (see, e.g. [24, Theorem 7, p.139] or [39, Lemma 3.7]).

Lemma 5.3. *Consider a non-negative stochastic process $\{V_k\}$ with representation*

$$V_k = V_0 + A_k - U_k + \mathcal{M}_k,$$

where $\{A_k\}$ and $\{U_k\}$ are almost surely non-decreasing, predictable processes with $A_0 = U_0 = 0$, and $\mathcal{M}_k$ is a local martingale adapted to $\mathcal{F}_{t_k}$ with $\mathcal{M}_0 = 0$. Then

$$\left\{\lim_{k\to\infty} A_k < \infty\right\} \subset \left\{\lim_{k\to\infty} U_k < \infty\right\} \cap \left\{\lim_{k\to\infty} V_k < \infty \text{ exists}\right\} \quad a.s.$$

The following lemmas will play an important role in the proof of the asymptotic stability of the numerical solutions.

Lemma 5.4. *If $V \in \bar{\mathcal{V}}^4_{\delta_4}$ for some integer $1/\delta_4 \in [4, +\infty)$, we have*

$$\begin{aligned} V(\tilde{Z}_{k+1}) \le & V(Z_k) + \mathcal{L}V(Z_k)\triangle + C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar{\theta})} \\ & + \sum_{i=1}^{3} \mathcal{S}_i^\triangle V(Z_k) + \tilde{\mathcal{J}}^\triangle V(Z_k), \end{aligned}$$

and $\mathbb{E}_k\big[\mathcal{S}_i^\triangle V(Z_k)\big] = 0$,

$$|\tilde{\mathcal{J}}^\triangle V(Z_k)| \le C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar{\theta})} + \mathcal{A}_1^\triangle V(Z_k), \qquad \mathbb{E}_k\big[\mathcal{A}_1^\triangle V(Z_k)\big] = 0, \tag{5.8}$$

where $\tilde{\mathcal{J}}^\triangle V(\cdot)$ and $\mathcal{A}_1^\triangle V(\cdot)$ are defined by (A.13) *and* (A.15)*. Moreover, we also have*

$$|\mathcal{S}_1^\triangle V(Z_k)|^2 = |D^{(1)}V(Z_k)g(Z_k)|^2\triangle + \mathcal{H}_{1,1}^\triangle V(Z_k), \qquad \mathbb{E}_k\big[\mathcal{H}_{1,1}^\triangle V(Z_k)\big] = 0,$$

and for $i = 1, 2, 3$,

$$|\mathcal{S}_i^\triangle V(Z_k)|^2 \le C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{1-\bar{\theta}} + \mathcal{H}_{i,i}^\triangle V(Z_k), \qquad \mathbb{E}_k\big[\mathcal{H}_{i,i}^\triangle V(Z_k)\big] = 0,$$

where $\mathcal{H}_{i,i}^\triangle V(\cdot)$ is defined by (A.7), (A.8) *and* (A.9)*, respectively, and we also have*

$$\mathcal{S}_1^\triangle V(Z_k)\mathcal{S}_j^\triangle V(Z_k) \ge -C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{2(1-\bar{\theta})} + \mathcal{H}_{1,j}^\triangle V(Z_k),$$

and $\mathbb{E}_k\big[\mathcal{H}_{1,j}^\triangle V(Z_k)\big] = 0$ for $j = 2, 3$, where $\mathcal{H}_{1,j}^\triangle V(\cdot)$ are defined by (A.10) *and* (A.11)*, respectively.*

Lemma 5.5. *If $V \in \bar{\mathcal{V}}^4_{\delta_4}$ for some integer $1/\delta_4 \in [4, +\infty)$, we have*

$$|\tilde{\mathcal{J}}^\triangle V(Z_k)|^2 \le C\Lambda_\rho(Z_k)V^2(Y_k)\triangle^{4(1-\bar{\theta})} + \mathcal{A}_2^\triangle V(Z_k), \qquad \mathbb{E}_k\big[\mathcal{A}_2^\triangle V(Z_k)\big] = 0,$$

and

$$|\tilde{\mathcal{J}}^\triangle V(Z_k)|^3 \le C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{6(1-\bar{\theta})} + \mathcal{A}_3^\triangle V(Z_k), \qquad \mathbb{E}_k\big[\mathcal{A}_3^\triangle V(Z_k)\big] = 0,$$

where $\mathcal{A}_2^\triangle V(\cdot)$ and $\mathcal{A}_3^\triangle V(\cdot)$ are defined by (A.17) *and* (A.19)*. Moreover, we also have*

$$\big|\mathcal{S}_1^\triangle V(Z_k)\big|^2 |\tilde{\mathcal{J}}^\triangle V(Z_k)| \le C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{3(1-\bar{\theta})} + \mathcal{A}_4^\triangle V(Z_k),$$

$$\big|\mathcal{S}_2^{\triangle}V(Z_k)\big|^2|\tilde{\mathcal{J}}^{\triangle}V(Z_k)| \le C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{4(1-\bar{\theta})} + \mathcal{A}_5^{\triangle}V(Z_k),$$

and $\mathbb{E}_k\big[\mathcal{A}_4^{\triangle}V(Z_k)\big]=0$, $\mathbb{E}_k\big[\mathcal{A}_5^{\triangle}V(Z_k)\big]=0$,

$$\big|\mathcal{S}_3^{\triangle}V(Z_k)\big|^2|\tilde{\mathcal{J}}^{\triangle}V(Z_k)| \le C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{5(1-\bar{\theta})} + \mathcal{A}_6^{\triangle}V(Z_k),$$

and $\mathbb{E}_k\big[\mathcal{A}_6^{\triangle}V(Z_k)\big]=0$, *where* $\mathcal{A}_4^{\triangle}V(\cdot)$, $\mathcal{A}_5^{\triangle}V(\cdot)$ *and* $\mathcal{A}_6^{\triangle}V(\cdot)$ *are defined by* (A.20), (A.21) *and* (A.22), *respectively.*

The proofs of both above lemmas can be found in Appendix A. Now we analyze the asymptotic properties of the numerical solutions of the V-truncated EM scheme (5.6). It makes use of Lemma 5.4 and Lemma 5.5 above.

Theorem 5.6. *Let the conditions of Lemma 5.1 hold. Assume also that the function* $V\in\bar{\mathcal{V}}^4_{\delta_4}$ *for some integer* $1/\delta_4\in[4,+\infty)$ *with the property*

$$V(\epsilon x)\le V(x) \qquad \forall\, x\in\mathbb{R}^d,\ 0<\epsilon\le 1.$$

Then there is a constant $\triangle^{**}\in(0,1]$ *such that for any* $\triangle\in(0,\triangle^{**}]$ *the corresponding* V*-truncated EM scheme* (5.6) *has the property that*

$$\limsup_{k\to\infty} V^\rho(Z_k)<\infty, \qquad \lim_{k\to\infty} d(Z_k,\mathrm{Ker}(w))=0 \qquad \text{a.s.} \tag{5.9}$$

Moreover, if $w(x)=0$ *iff* $x=\mathbf{0}\in\mathbb{R}^d$, *then for any* $\triangle\in(0,\triangle^{**}]$,

$$\lim_{k\to\infty} Z_k=\mathbf{0} \quad \text{a.s.}$$

Proof. For any $\rho>0$, we deduce by virtue of Lemma 5.4 that

$$V^\rho(\tilde{Z}_{k+1})\le V^\rho(Z_k)\big(1+\zeta_k\big)^\rho, \tag{5.10}$$

where

$$\zeta_k=V^{-1}(Z_k)\bigg[\mathcal{L}V(Z_k)\triangle + C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar{\theta})}+\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Z_k)+|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|\bigg],$$

if $V(Z_k)\ne 0$, otherwise it is equal to -1. Clear, $\zeta_k\ge -1$. Now we only prove the case that $0<\rho\le 1$ and the proofs for other cases are similar. By the virtue of [23, Inequality (3.12)], for any $0<\rho\le 1$, we derive from (5.10) that

$$V^\rho(\tilde{Z}_{k+1})\le V^\rho(Z_k)I_{\{V(Z_k)\ne 0\}}(Z_k)\bigg(1+\rho\zeta_k+\frac{\rho(\rho-1)}{2}\zeta_k^2+\frac{\rho(\rho-1)(\rho-2)}{6}\zeta_k^3\bigg). \tag{5.11}$$

It is easy to see that

$$\begin{aligned}
&I_{\{V(Z_k)\ne 0\}}(Z_k)\zeta_k\\
=&I_{\{V(Z_k)\ne 0\}}(Z_k)V^{-1}(Z_k)\Big[\mathcal{L}V(Z_k)\triangle + C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar{\theta})}\\
&\qquad\qquad\qquad\qquad +\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Z_k)+|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|\Big]\\
\le&I_{\{V(Z_k)\ne 0\}}(Z_k)V^{-1}(Z_k)\mathcal{L}V(Z_k)\triangle + C\Lambda_\rho(Z_k)\triangle^{2(1-\bar{\theta})}+\mathcal{M}_1^{\triangle}V(Z_k),
\end{aligned} \tag{5.12}$$

where

$$\mathcal{M}_1^{\triangle}V(Z_k)=I_{\{V(Z_k)\ne 0\}}(Z_k)V^{-1}(Z_k)\Big[\sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Z_k)+\mathcal{A}_1^{\triangle}V(Z_k)\Big],$$

and by Lemma 5.4 we know $\mathbb{E}_k\big[\mathcal{M}_1^{\triangle}V(Z_k)\big]=0$. We can now analyze the last two terms of the summation in (5.11). First, combining (A.1) and (A.14) we obtain

$$\begin{aligned}
&\mathcal{S}_1^{\triangle}V(Z_k)|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|\\
=&C\mathcal{S}_1^{\triangle}V(Z_k)\Lambda_\rho(Z_k)V(Z_k)|\triangle B_k|^4\triangle^{-2\bar\theta}\\
&+C\Lambda_\rho(Z_k)V(Z_k)\Big(\langle D^{(1)}V(Z_k),g(Z_k)\triangle B_k\rangle|\triangle B_k|^{\frac{1}{\delta_4}}\triangle^{\frac{-\bar\theta}{2\delta_4}}\Big)\\
\geq&C\mathcal{S}_1^{\triangle}V(Z_k)\Lambda_\rho(Z_k)V(Z_k)|\triangle B_k|^4\triangle^{-2\bar\theta}-C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{\frac{(\delta_4+1)(1-\bar\theta)}{2\delta_4}}\\
&-C\triangle^{\frac{-\bar\theta}{2\delta_4}}\Lambda_\rho(Z_k)V(Z_k)|D^{(1)}V(Z_k)||g(Z_k)|\Big(|\triangle B_k|^{1+\frac{1}{\delta_4}}-K_{1+\frac{1}{\delta_4}}\triangle^{\frac{\delta_4+1}{2\delta_4}}\Big)
\end{aligned}$$

for $1/\delta_4\in[4,+\infty)$. This together with Lemma 5.4 implies

(5.13)

$$\begin{aligned}
&I_{\{V(Z_k)\neq0\}}(Z_k)\zeta_k^2\\
\geq&I_{\{V(Z_k)\neq0\}}(Z_k)V^{-2}(Z_k)\Big[2\mathcal{S}_1^{\triangle}V(Z_k)\Big(\mathcal{L}V(Z_k)\triangle+C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar\theta)}\Big)\\
&\qquad+2\mathcal{S}_1^{\triangle}V(Z_k)\Big(\mathcal{S}_2^{\triangle}V(Z_k)+\mathcal{S}_3^{\triangle}V(Z_k)+|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|\Big)+|\mathcal{S}_1^{\triangle}V(Z_k)|^2\Big]\\
\geq&I_{\{V(Z_k)\neq0\}}(Z_k)V^{-2}(Z_k)\Big[2\mathcal{S}_1^{\triangle}V(Z_k)\Big(\mathcal{L}V(Z_k)\triangle+C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar\theta)}\Big)\\
&\qquad+2\mathcal{S}_1^{\triangle}V(Z_k)|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|-C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{2(1-\bar\theta)}\\
&\qquad+2\sum_{i=2}^{3}\mathcal{H}_{1,i}^{\triangle}V(Z_k)+|D^{(1)}V(Z_k)g(Z_k)|^2\triangle+\mathcal{H}_{1,1}^{\triangle}V(Z_k)\Big]\\
\geq&I_{\{V(Z_k)\neq0\}}(Z_k)V^{-2}(Z_k)|D^{(1)}V(Z_k)g(Z_k)|^2\triangle-C\Lambda_\rho(Z_k)\triangle^{2(1-\bar\theta)}+\mathcal{M}_2^{\triangle}V(Z_k),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{M}_2^{\triangle}V(Z_k)=&I_{\{V(Z_k)\neq0\}}(Z_k)V^{-2}(Z_k)\Big[\mathcal{H}_{1,1}^{\triangle}V(Z_k)+2\sum_{i=2}^{3}\mathcal{H}_{1i}^{\triangle}V(Z_k)\\
&+2\mathcal{S}_1^{\triangle}V(Z_k)\Big(\mathcal{L}V(Z_k)\triangle+C\Lambda_\rho(Z_k)V(Z_k)\big(\triangle^{2(1-\bar\theta)}+\triangle^{-2\bar\theta}|\triangle B_k|^4\big)\Big)\\
&-C\triangle^{\frac{-\bar\theta}{2\delta_4}}\Lambda_\rho(Z_k)V(Z_k)|D^{(1)}V(Z_k)||g(Z_k)|\Big(|\triangle B_k|^{1+\frac{1}{\delta_4}}-K_{1+\frac{1}{\delta_4}}\triangle^{\frac{\delta_4+1}{2\delta_4}}\Big)\Big],
\end{aligned}$$

and from Lemma 5.4 and (A.5) we have $\mathbb{E}_k\big[\mathcal{M}_2^{\triangle}V(Z_k)\big]=0$. By (4.1) and (5.7), we deduce that

$$\begin{aligned}
\big|\mathcal{L}V(Z_k)\big|\triangle=&\Big|\langle D^{(1)}V(Z_k),f(Z_k)\rangle+\frac{1}{2}\mathrm{tr}\Big[g^T(Z_k)D^{(2)}V(Z_k)g(Z_k)\Big]\Big|\triangle\\
\leq&|D^{(1)}V(Z_k)||f(Z_k)|\triangle+|D^{(2)}V(Z_k)||g(Z_k)|^2\triangle\\
\leq&cK\triangle^{1-\bar\theta}\Big(\Lambda_\rho^{\frac{1}{2}}(Z_k)V(Z_k)+\Lambda_\rho(Z_k)V(Z_k)\Big)\\
\leq&C\Lambda_\rho^{\frac{1}{2}}(Z_k)V(Z_k)\triangle^{1-\bar\theta}.
\end{aligned}$$

This together with Lemmas 5.4 and 5.5 implies

(5.14)

$$
\begin{aligned}
&I_{\{V(Z_k)\neq 0\}}(Z_k)\zeta_k^3\\
\le& I_{\{V(Z_k)\neq 0\}}(Z_k)V^{-3}(Z_k)\Big[C\Lambda_\rho^{\frac12}(Z_k)V(Z_k)\triangle^{1-\bar\theta}+\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)+|\tilde{\mathcal{J}}^\triangle V(Z_k)|\Big]^3\\
\le& I_{\{V(Z_k)\neq 0\}}(Z_k)V^{-3}(Z_k)\Big\{C\Lambda_\rho^{\frac32}(Z_k)V^3(Z_k)\triangle^{3(1-\bar\theta)}+\Big[\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)+|\tilde{\mathcal{J}}^\triangle V(Z_k)|\Big]^3\\
&\qquad+C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{2(1-\bar\theta)}\Big[\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)+|\tilde{\mathcal{J}}^\triangle V(Z_k)|\Big]\\
&\qquad+C\Lambda_\rho^{\frac12}(Z_k)V(Z_k)\triangle^{1-\bar\theta}\Big[\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)+|\tilde{\mathcal{J}}^\triangle V(Z_k)|\Big]^2\Big\}\\
\le& I_{\{V(Z_k)\neq 0\}}(Z_k)V^{-3}(Z_k)\Big\{C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{3(1-\bar\theta)}+\Big(\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)\Big)^3\\
&\qquad+C\big|\tilde{\mathcal{J}}^\triangle V(Z_k)\big|^3+C\sum_{i=1}^3\big|\mathcal{S}_i^\triangle V(Z_k)\big|^2|\tilde{\mathcal{J}}^\triangle V(Z_k)|\\
&\qquad+C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{2(1-\bar\theta)}\Big(\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)+|\tilde{\mathcal{J}}^\triangle V(Z_k)|\Big)\\
&\qquad+C\Lambda_\rho^{\frac12}(Z_k)V(Z_k)\triangle^{1-\bar\theta}\Big(\sum_{i=1}^3\big|\mathcal{S}_i^\triangle V(Z_k)\big|^2+\big|\tilde{\mathcal{J}}^\triangle V(Z_k)\big|^2\Big)\Big\}\\
\le& CI_{\{V(Z_k)\neq 0\}}(Z_k)V^{-3}(Z_k)\Lambda_\rho(Z_k)\Big[V^3(Z_k)\triangle^{3(1-\bar\theta)}+V^3(Z_k)\triangle^{6(1-\bar\theta)}\\
&\qquad+V^3(Z_k)\triangle^{3(1-\bar\theta)}+V^3(Z_k)\triangle^{4(1-\bar\theta)}+V^3(Z_k)\triangle^{5(1-\bar\theta)}+V^3(Z_k)\triangle^{4(1-\bar\theta)}\\
&\qquad+V^3(Z_k)\triangle^{2(1-\bar\theta)}+V^3(Z_k)\triangle^{5(1-\bar\theta)}\Big]+\mathcal{M}_3^\triangle V(Z_k)\\
\le& C\Lambda_\rho(Z_k)\triangle^{2(1-\bar\theta)}+\mathcal{M}_3^\triangle V(Z_k),
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{M}_3^\triangle V(Z_k)\\
=&I_{\{V(Z_k)\neq 0\}}(Z_k)V^{-3}(Z_k)\Big\{C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{2(1-\bar\theta)}\Big(\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)+\mathcal{A}_1^\triangle V(Z_k)\Big)\\
&+C\Lambda_\rho^{\frac12}(Z_k)V(Z_k)\triangle^{1-\bar\theta}\Big(\sum_{i=1}^3\mathcal{H}_{i,i}^\triangle V(Z_k)+\mathcal{A}_2^\triangle V(Z_k)\Big)+\Big(\sum_{i=1}^3\mathcal{S}_i^\triangle V(Z_k)\Big)^3\\
&\qquad+C\sum_{i=4}^6\mathcal{A}_i^\triangle V(Z_k)+C\mathcal{A}_3^\triangle V(Z_k)\Big\},
\end{aligned}
$$

and from Lemmas 5.4 and 5.5 it is easy to see that $\mathbb{E}_k\big[\mathcal{M}_3^{\triangle}V(Z_k)\big]=0$. Thus, for any integer $k\geq 0$, substituting (5.12)-(5.14) into (5.11), we deduce from (5.1) that

$$\begin{aligned}
V^{\rho}(\tilde{Z}_{k+1}) \leq& I_{\{V(Z_k)\neq 0\}}(Z_k)V^{\rho}(Z_k)\bigg\{1+C\Lambda_{\rho}(Z_k)\triangle^{2(1-\bar{\theta})}\\
&+\frac{\rho\triangle}{2}V^{-2}(Z_k)\Big[2V(Z_k)\mathcal{L}V(Z_k)-(1-\rho)|D^{(1)}V(Z_k)g(Z_k)|^2\Big]\bigg\}+\mathcal{N}_k^{\triangle}\\
=&I_{\{V(Z_k)\neq 0\}}(Z_k)\bigg\{V^{\rho}(Z_k)+C\Lambda_{\rho}(Z_k)V^{\rho}(Z_k)\triangle^{2(1-\bar{\theta})}\\
&+\frac{\rho\triangle}{2}V^{\rho-2}(Z_k)\Big[2V(Z_k)\mathcal{L}V(Z_k)-(1-\rho)|D^{(1)}V(Z_k)g(Z_k)|^2\Big]\bigg\}+\mathcal{N}_k^{\triangle}\\
\leq& I_{\{V(Z_k)\neq 0\}}(Z_k)\Big(V^{\rho}(Z_k)+Cw(Z_k)\triangle^{2(1-\bar{\theta})}-w(Z_k)\triangle\Big)+\mathcal{N}_k^{\triangle}\\
\leq& V^{\rho}(Z_k)-\big(1-C\triangle^{1-2\bar{\theta}}\big)w(Z_k)\triangle+\mathcal{N}_k^{\triangle},
\end{aligned}\tag{5.15}$$

where

$$\mathcal{N}_k^{\triangle}:=V^{\rho}(Z_k)\big[\mathcal{M}_1^{\triangle}V(Z_k)+\mathcal{M}_2^{\triangle}V(Z_k)+\mathcal{M}_3^{\triangle}V(Z_k)\big],$$

and we can see that $\mathbb{E}_k\big[\mathcal{N}_k^{\triangle}\big]=0$. Thus the required inequality (5.15) for the case $\rho>1$ can be proved similarly. Furthermore,

$$V(\bar{\pi}_{\triangle}(x))\leq V(x)\qquad \forall\ x\in\mathbb{R}^d,$$

which implies that

$$V^{\rho}(Z_{k+1})\leq V^{\rho}(\tilde{Z}_{k+1})\leq V^{\rho}(Z_k)-\big(1-C\triangle^{1-2\bar{\theta}}\big)w(Z_k)\triangle+\mathcal{N}_k^{\triangle}.$$

For any given $\gamma\in(0,1)$, we choose $\triangle^{**}\in(0,1]$ small sufficiently such that

$$\triangle^{**}\leq\big[(1-\gamma)/C\big]^{1/(1-2\bar{\theta})},$$

which implies that for all integer $k\geq 1$ and any $\triangle\in(0,\triangle^{**}]$,

$$V^{\rho}(Z_{k+1})\leq V^{\rho}(Z_k)-\gamma w(Z_k)\triangle+\mathcal{N}_k^{\triangle},$$

Then taking sum on k, we have

$$V^{\rho}(Z_k)\leq V^{\rho}(Z_0)-\gamma\sum_{i=0}^{k-1}w(Z_i)\triangle+\mathcal{M}_k,$$

where $\mathcal{M}_k=\sum_{i=0}^{k-1}\mathcal{N}_i^{\triangle}$. It is easy to show that

$$\mathbb{E}\left[\mathcal{M}_k|\,\mathcal{F}_{k-1}\right]=\mathcal{M}_{k-1}+\mathbb{E}\Big[\mathcal{N}_{k-1}^{\triangle}|\mathcal{F}_{k-1}\Big]=\mathcal{M}_{k-1}.$$

This implies immediately that $\mathcal{M}_k$ is a martingale with $\mathcal{M}_0=0$. Set

$$V_k:=V_0-\gamma\sum_{i=0}^{k-1}w(Z_i)\triangle+\mathcal{M}_k$$

with $V_0=V^{\rho}(Z_0)$, and $\sum_{i=0}^{k-1}w(Z_i)\triangle$ is increasing in k. It is easy to see that $V^{\rho}(Z_k)\leq V_k$ a.s. Thus, by Lemma 5.3, we infer that

$$\lim_{k\to\infty}V^{\rho}(Z_k)<\infty\quad\text{a.s.}\qquad\text{and}\qquad\sum_{i=0}^{\infty}w(Z_i)\triangle<\infty\ \text{ a.s.}$$

which implies that

$$\lim_{k\to\infty} w(Z_k) = 0 \quad \text{a.s.} \qquad \text{and} \qquad \sup_{0\le k<\infty} V^\rho(Z_k) < +\infty \quad \text{a.s.} \tag{5.16}$$

Define $\phi : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ by $\phi(r) = \inf_{|x|\ge r} V^\rho(x)$ for $r \ge 0$. Clear, $\phi(|Z_k|) \le V^\rho(Z_k)$. So

$$\sup_{0\le k<\infty} \phi(|Z_k|) \le \sup_{0\le k<\infty} V^\rho(Z_k) < +\infty \quad \text{a.s.}$$

On the other hand, due to $V \in \mathcal{C}^4_\infty(\mathbb{R}^d; \bar{\mathbb{R}}_+)$ we know $\lim_{r\to\infty} \phi(r) = \infty$, which implies

$$\sup_{0\le k<\infty} |Z_k| < +\infty \quad \text{a.s.} \tag{5.17}$$

Moreover, when $w(x) = 0$ iff $x = \mathbf{0}$ one concludes that

$$\lim_{k\to\infty} Z_k = \mathbf{0} \quad \text{a.s.}$$

Observe, from (5.16) and (5.17) that there is an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$, such that

$$\lim_{k\to\infty} w(Z_k(\omega)) = 0, \qquad \sup_{0\le k<\infty} |Z_k(\omega)| < +\infty \qquad \forall \omega \in \Omega_0. \tag{5.18}$$

We shall now show that

$$\lim_{k\to\infty} d\left(Z_k(\omega), \mathrm{Ker}(w)\right) = 0 \qquad \forall \omega \in \Omega_0. \tag{5.19}$$

If this is false, then there is some $\bar{\omega} \in \Omega_0$ such that

$$\limsup_{k\to\infty} d\left(Z_k(\bar{\omega}), \mathrm{Ker}(w)\right) > 0,$$

whence there is a subsequence $\{Z_{k_n}(\bar{\omega})\}_{n\ge1}$ of $\{Z_k(\bar{\omega})\}_{k\ge0}$ such that

$$d\left(Z_{k_n}(\bar{\omega}), \mathrm{Ker}(w)\right) \ge \epsilon, \quad n \ge 1,$$

for some $\epsilon > 0$. Since $\{Z_{k_n}(\bar{\omega})\}_{n\ge0}$ is bounded, we can find a subsequence $\{Z_{k_{n_j}}(\bar{\omega})\}_{j\ge0}$ of $\{Z_{k_n}(\bar{\omega})\}_{n\ge0}$ and

$$\lim_{j\to\infty} Z_{k_{n_j}}(\bar{\omega}) = \hat{z}, \qquad \hat{z} \in \mathbb{R}^d.$$

Clearly, $\hat{z} \notin \mathrm{Ker}(w)$ so $w(\hat{z}) > 0$. However, by (5.18),

$$w(\hat{z}) = \lim_{j\to\infty} w\big(Z_{k_{n_j}}(\bar{\omega})\big) = 0,$$

which contradicts $w(\hat{z}) > 0$. Hence (5.19) must hold and the required assertion (5.9) follows since $\mathbb{P}(\Omega_0) = 1$. Therefore, the proof is complete. $\square$

Remark 5.7. If $V \in \bar{\mathcal{V}}^4_{\delta_4}$ in Theorem 5.6 is replaced by $V \in \bar{\mathcal{V}}^p_{\delta_p} \cap \{D^{(p+1)}V(\cdot) \equiv 0\}$ for $p = 2$ or 3 and some integer $1/\delta_p \in [p, +\infty)$, then the conclusion of Theorem 5.6 still holds. This implies $1/\delta_4$ may be less than 4, since Taylor's formula with integral remainder term $J(\tilde{Z}_{k+1}, Z_k) \equiv 0$ in the estimation of (5.10).

Remark 5.8. If $\tilde{V} \in \bar{\mathcal{V}}^4_{\delta_4}$ in Theorem 5.6 is replaced by $\tilde{V} \in \bar{\mathcal{V}}^3_{\delta_3}$ for some integer $1/\delta_3 \in [3, +\infty)$, we choose a strictly increasing continuous function $\bar{\varphi} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\bar{\varphi}(u) \to \infty$ as $u \to \infty$ such that

$$\sup_{0<|x|\le u} \left[\frac{|f(x)|}{\Lambda_\rho^{1/2}(x)\tilde{V}^{\delta_3}(x)} \vee \frac{|g(x)|^2}{\Lambda_\rho(x)\tilde{V}^{2\delta_3}(x)} \right] \le \bar{\varphi}(u), \qquad \forall\ u > 0.$$

Then for any given $0 < \bar{\theta} < 1/3$ the corresponding V-truncated EM scheme (5.6) may reproduce the LaSalle-type theorem. It turns out that the smoothness of $V(x)$ affects the construction of the scheme (5.6).

Next, we shall discuss the exponential stability of V-truncated EM scheme (5.6). Here, the key idea is to show not only that the quantity $V(Z_k)$ decays with time, but also that the decay is exponential.

Corollary 5.9. *Under the conditions of Theorem 5.6, and $w(x) \geq \mu V^{\rho}(x)$ for all $x \in \mathbb{R}^d$, where μ is a positive constant. Then for any $\varrho \in (0, \mu)$ there is a constant $\triangle^{**} \in (0, 1]$ such that for any $\triangle \in (0, \triangle^{**}]$ the V-truncated EM scheme* (5.6) *has the property that*

$$\limsup_{k\to\infty} \frac{\log\big(\mathbb{E}V^{\rho}(Z_k)\big)}{k\triangle} \leq -\big(\mu - \varrho\big) < 0, \tag{5.20}$$

and

$$\limsup_{k\to\infty} \frac{\log\big(V(Z_k)\big)}{k\triangle} \leq -\frac{\mu - \varrho}{\rho} \qquad a.s. \tag{5.21}$$

Proof. If $w(x) \geq \mu V^{\rho}(x)$ for some $\mu > 0$ and $\rho > 0$, then instead of (5.15) one runs the same calculation to get

$$\mathbb{E}_k\big[V^{\rho}(\tilde{Z}_{k+1})\big] \leq V^{\rho}(Z_k) + CV^{\rho}(Z_k)\triangle^{2(1-\bar{\theta})} - \mu V^{\rho}(Z_k)\triangle.$$

For any $\varrho \in (0, \mu)$, choose $\triangle^{**} \in (0, 1]$ sufficiently small such that $C\big(\triangle^{**}\big)^{1-2\bar{\theta}} \leq \varrho\triangle^{**}$, $\triangle^{**} < 1/(\mu - \varrho)$. Taking expectations on both sides, then we have

$$\mathbb{E}\big[V^{\rho}(Z_{k+1})\big] \leq \mathbb{E}\big[V^{\rho}(\tilde{Z}_{k+1})\big] \leq \Big(1 - (\mu - \varrho)\triangle\Big)\mathbb{E}\big[V^{\rho}(Z_k)\big]$$

for any $\triangle \in (0, \triangle^{**}]$. Repeating this procedure we obtain

$$\mathbb{E}\big[V^{\rho}(Z_k)\big] \leq \Big(1 - (\mu - \varrho)\triangle\Big)^k V^{\rho}(x_0) \leq \exp\Big(-(\mu - \varrho)k\triangle\Big)V^{\rho}(x_0), \quad \forall k \geq 1,$$

which implies the required assertion (5.20) holds. Moreover, the other required assertion (5.21) follows from the Chebyshev inequality and the Borel-Cantelli lemma (see, e.g., [29, p.7]) directly, please refer to [13, p.600]. □

Remark 5.10. In the special case where $V^{\rho}(x) = |x|^{2\rho}$ with $\rho > 0$ and $w(x) \geq \mu V^{\rho}(x)$ for all $x \in \mathbb{R}^d$. We choose $\delta_4 = 1/2$, then $V \in \bar{\mathcal{V}}^2_{\delta_4} \cap \{D^{(3)}V(\cdot) \equiv 0\}$ and equation (5.5) become much simpler. That is to say, we only need to choose a strictly increasing continuous function $\bar{\varphi} : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ such that $\bar{\varphi}(u) \to \infty$ as $u \to \infty$ and

$$\sup_{0<|x|\leq u} \left\{\frac{|f(x)|}{|x|} \vee \frac{|g(x)|^2}{|x|^2}\right\} \leq \bar{\varphi}(u) \qquad \forall\, u \geq 1.$$

5.3. **Further results of numerical solutions.** In practice we always wish that the numerical solutions will preserve the stability of the exact solution perfectly for some given Lyapunov functions. Although most practically relevant Lyapunov functions can be found in the subset $\bar{\mathcal{V}}^p_{\delta_p}$ defined in (4.1), we may treat them as a special case. In this subsection, we will consider the following class of Lyapunov functions $\hat{\mathcal{V}}^p_{\delta_p}$. It is sufficiently large so is rich enough for one to choose suitable

Lyapunov functions for many of important SDEs (see [14,39] for more details). To be precise, define

$$\hat{\mathcal{V}}^p_{\delta_p} := \mathcal{V}^p_{\delta_p} \cap \big\{\mathrm{Ker}(V) = \{\mathbf{0}\}\big\}.$$

It is easy to see that $\hat{\mathcal{V}}^p_{\delta_p}$ includes almost all Lyapunov functions presented in [14,39]. In this subsection, we assume there is a function $V \in \hat{\mathcal{V}}^4_{\delta_4}$ such that

$$\sup_{0<|x|\le u} \frac{(1+V(x))^{1-2\delta_4}\big(|f(x)|^2 \vee |g(x)|^2\big)}{\Lambda_\rho(x)V(x)} < +\infty \tag{5.22}$$

for any $u > 0$. To define the truncation mapping for a super-linear diffusion and drift terms, we choose function $V \in \hat{\mathcal{V}}^4_{\delta_4}$ and a strictly increasing continuous function $\hat{\varphi} : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ such that $\hat{\varphi}(u) \to \infty$ as $u \to \infty$ and

$$\sup_{0<|x|\le u} \left\{ \frac{(1+V(x))^{\frac{1}{2}-\delta_4}|f(x)|}{\big(\Lambda_\rho(x)V(x)\big)^{1/2}} \vee \frac{(1+V(x))^{1-2\delta_4}|g(x)|^2}{\Lambda_\rho(x)V(x)} \right\} \le \hat{\varphi}(u)$$

for any $u \ge 1$. Owing to f and g satisfy (5.22), the function $\hat{\varphi}$ can be well defined as well as its inverse function $\hat{\varphi}^{-1} : [\hat{\varphi}(1), \infty) \to \bar{\mathbb{R}}_+$. Then choose a constant $K \ge \hat{\varphi}(|x_0| \vee 1)$, where K is a positive constant independent of the iteration order k and the time stepsize $\triangle$. For the given $\triangle \in (0,1]$, define a truncation mapping $\hat{\pi}_\triangle : \mathbb{R}^d \to \mathbb{R}^d$ by $\hat{\pi}_\triangle(x) = \Big(|x| \wedge \hat{\varphi}^{-1}\big(K\triangle^{-\hat{\theta}}\big)\Big)\frac{x}{|x|}$, where $\hat{\theta} \in (0,1/2)$, and we use the convention $\frac{x}{|x|} = \mathbf{0}$ when $x = \mathbf{0} \in \mathbb{R}^d$ is a zero vector. Next, for any given stepsize $\triangle \in (0,1]$, define the V-truncated EM scheme by

$$\begin{cases} Z_0 = x_0, \\ \tilde{Z}_{k+1} = Z_k + f(Z_k)\triangle + g(Z_k)\triangle B_k, \\ Z_{k+1} = \hat{\pi}_\triangle(\tilde{Z}_{k+1}), \end{cases} \tag{5.23}$$

for any integer $k \ge 0$, where $t_k = k\triangle$ and $\triangle B_k = B(t_{k+1}) - B(t_k)$. To obtain the continuous-time approximation, we define $Z(t)$ by $Z(t) := Z_k$ for any $t \in [t_k, t_{k+1})$. Then the drift and diffusion terms have the property

$$|f(Z_k)|^2 \le \frac{\big(K\triangle^{-\hat{\theta}}\big)^2\Lambda_\rho(Z_k)V(Z_k)}{(1+V(Z_k))^{1-2\delta_4}}, \qquad |g(Z_k)|^2 \le \frac{K\triangle^{-\hat{\theta}}\Lambda_\rho(Z_k)V(Z_k)}{(1+V(Z_k))^{1-2\delta_4}}. \tag{5.24}$$

which implies (3.3) holds. As the special case $V \in \bar{\mathcal{V}}^4_{\delta_4}$, all of the conclusions in Section 5.2 holds for the scheme (5.23) under the conditions of Lemma 5.1 and we don't mention them to avoid duplication.

Theorem 5.11. *Assume the conditions of Theorem 5.6 hold for $\rho = 1$ and $V \in \hat{\mathcal{V}}^4_{\delta_4}$. Then there is a constant $\triangle^{**} \in (0,1]$ such that for any $\triangle \in (0, \triangle^{**}]$ the V-truncated EM scheme* (5.6) *has the property that*

$$\limsup_{k\to\infty} V(Z_k) < \infty, \qquad \lim_{k\to\infty} d(Z_k, \mathrm{Ker}(w)) = 0 \quad \text{a.s.}$$

Moreover, if $w(x) = 0$ iff $x = \mathbf{0} \in \mathbb{R}^d$, then for any $\triangle \in (0, \triangle^{**}]$,

$$\lim_{k\to\infty} Z_k = \mathbf{0}, \quad \text{a.s.}$$

Proof. For $\rho = 1$, making use of the techniques in the proof of Lemma 5.4 as well as (5.24), (5.1) yields

$$\begin{aligned} V(\tilde{Z}_{k+1}) \le & V(Z_k) + \mathcal{L}V(Z_k)\triangle + C\Lambda_1(Z_k)V(Z_k)\triangle^{2(1-\hat{\theta})} + \sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Z_k) + \mathcal{A}_1^{\triangle}V(Z_k) \\ \le & V(Z_k) - \big(1 - C\triangle^{1-2\hat{\theta}}\big)w(Z_k)\triangle + \sum_{i=1}^{3}\mathcal{S}_i^{\triangle}V(Z_k) + \mathcal{A}_1^{\triangle}V(Z_k), \end{aligned}$$

where $\mathbb{E}_k\big[\mathcal{S}_i^{\triangle}V(Z_k)\big] = 0$ and $\mathbb{E}_k\big[\mathcal{A}_1^{\triangle}V(Z_k)\big] = 0$. Using the same method as employed in the proof of Theorem 5.6, we can easily carry out the proof of this theorem and hence is omitted to avoid repetition. □

Based on the above theorem, we further obtain the following corollary.

Corollary 5.12. *Assume that the conditions of Theorem 5.11 hold and, moreover, $w(x) \ge \mu V(x)$ for all $x \in \mathbb{R}^d$, where μ is a positive constant. Then for any $\varrho \in (0, \mu)$ there is a constant $\triangle^{**} \in (0, 1]$ such that for any $\triangle \in (0, \triangle^{**}]$ the V-truncated EM scheme* (5.6) *has the property that*

$$\limsup_{k\to\infty} \frac{\log\big(\mathbb{E}V(Z_k)\big)}{k\triangle} \le -\big(\mu - \varrho\big) < 0,$$

and

$$\limsup_{k\to\infty} \frac{\log\big(V(Z_k)\big)}{k\triangle} \le -\big(\mu - \varrho\big) \quad a.s.$$

6. Examples and simulations

Example 6.1. Consider the stochastic Ginzburg-Laudau equation (see [14, 20])

$$\mathrm{d}X(t) = \big(-0.5X(t) - X^3(t)\big)\mathrm{d}t + X(t)\mathrm{d}B(t), \tag{6.1}$$

with the initial value $x_0 = 19$, where $B(t)$ is a scalar Brownian motion. Obviously, the drift and diffusion coefficients are locally Lipschitz continuous. Let $V(x) = |x|^2$, then $V \in \bar{\mathcal{V}}^2_{1/2}$ with $0 < \rho < 1$. One observes that

$$\begin{aligned} \mathcal{L}V^{\rho}(x) =& \frac{\rho}{2}V^{\rho-2}(x)\Big[2V(x)\mathcal{L}V(x) + (\rho-1)|D^{(1)}V(x)g(x)|^2\Big] \\ =& \frac{\rho}{2}V^{\rho-2}(x)\big(-4|x|^6 + 4(\rho-1)|x|^4\big) \le -2\rho(1-\rho)V^{\rho}(x). \end{aligned}$$

By virtue of Theorem 2.1 and Corollary 5.2, equation (6.1) with any initial value $x_0 > 0$ has a unique regular solution $X(t)$, which is exponentially stable for $0 < \rho < 1$. It can be verified that condition (5.4) holds. Note that for all $u > 0$,

$$\sup_{0<|x|\le u}\left\{\frac{|f(x)|}{|x|} \vee \frac{|g(x)|^2}{|x|^2}\right\} \le \sup_{0<|x|\le u}\left\{\frac{|x| + |x|^3}{|x|} \vee \frac{|x|^2}{|x|^2}\right\} \le u^2 + 1.$$

Take $\bar{\varphi}(u) = u^2 + 1$, $\forall u > 0$, then $\bar{\varphi}^{-1}(u) = \sqrt{u-1}$, $\forall u > 1$. Define $K = \bar{\varphi}(|x_0| \vee 1)$ and $\bar{\theta} = 1/4$. For a fixed $\triangle \in (0, 1]$, the V-truncated EM scheme for (6.1) is

$$\begin{cases} Z_0 = x_0, \\ \tilde{Z}_{k+1} = Z_k - \big(0.5Z_k + Z_k^3\big)\triangle + Z_k\triangle B_k, \\ Z_{k+1} = \Big(|\tilde{Z}_{k+1}| \wedge \sqrt{(|x_0|^2 \vee 1 + 1)\triangle^{-1/4} - 1}\Big)\frac{\tilde{Z}_{k+1}}{|\tilde{Z}_{k+1}|}. \end{cases} \tag{6.2}$$

By virtue of Remark 4.8, the numerical solution of this scheme approximates the exact solution

$$X(t) = \frac{x_0 \exp\big(B(t) - t\big)}{\sqrt{1 + 2x_0^2 \int_0^t \exp\big(2B(s) - 2s\big)\mathrm{d}s}}$$

given by [20, Equation (4.25), p.125] in the root mean square sense with error estimate $\triangle^{1/2}$. To test the efficiency of the V-truncated EM scheme, we carry out numerical experiments by implementing (6.2) using MATLAB. Fig. 1 plots the root mean square approximation error $\big(\mathbb{E}|X(1) - Z(1)|^2\big)^{1/2}$ between the exact solution of SDE (6.1) and the numerical solution by the V-truncated EM scheme (6.2) as the function of stepsize $\triangle \in \{2^{-12}, 2^{-13}, \dots, 2^{-19}, 2^{-20}\}$.

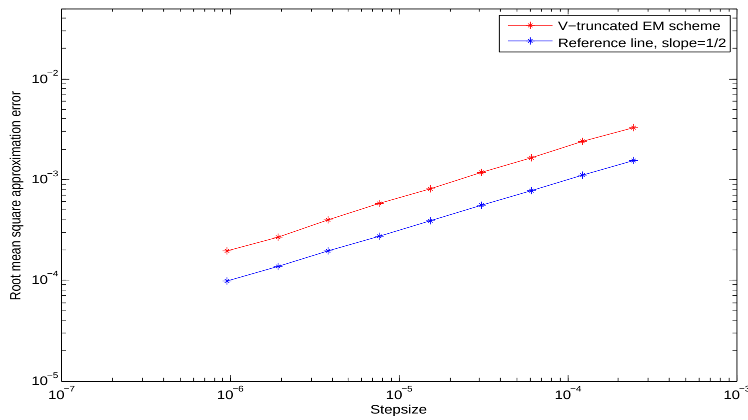

FIGURE 1. The root square approximation error for 1000 sample points between the exact solution X(1) of SDE (6.1) and the numerical solution $Z(1)$ of scheme (6.2) as the function of stepsize $\triangle \in \{2^{-12}, 2^{-13}, \dots, 2^{-19}, 2^{-20}\}$.

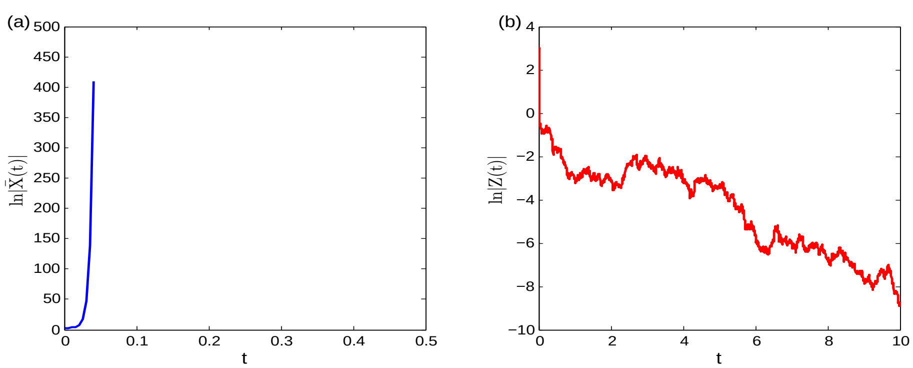

FIGURE 2. (a) Sample path of the classical EM numerical solution $\ln|\bar{X}(t)|$, (b) Sample path of the V-truncated EM numerical solution $\ln|Z(t)|$ with the same initial value $x_0 = 19$ for stepsize $\triangle = 0.005$ and $t \in [0, 10]$.

Moreover, as a fairly well-known result (see, e.g., [13]), the classical EM numerical solution for a nonlinear stable SDE is unstable with a positive probability. However, Corollary 5.9 reveals that the V-truncated EM solution $Z(t)$ preserves the underlying stability perfectly. Fig. 2 gives sample path of the classical EM solution $\ln|\bar{X}(t)|$ and of the V-truncated EM solution $\ln|Z(t)|$ with the same initial value $x_0 = 19$ for stepsize $\triangle = 0.005$ and $t \in [0, 10]$. Fig. 2(a) displays that the classical EM solution blows up quickly, so it cannot capture the stability behavior of SDE (6.1). Fig. 2(b) displays clearly that the V-truncated EM solution reproduces the almost sure stability of SDE (6.1).

Example 6.2. Consider the two-dimensional nonlinear SDE

$$\left\{ \begin{array}{l} \mathrm{d}X_1(t) = \big(-2X_1^3(t) - 2X_2^2(t)X_1(t)\big)\mathrm{d}t + 2\sqrt{2}\big(X_1^2(t) + X_2^2(t)\big)\mathrm{d}B^{(1)}(t), \\ \mathrm{d}X_2(t) = \big(-2X_1^2(t)X_2(t) - 2X_2^3(t)\big)\mathrm{d}t + 2\sqrt{2}\big(X_1^2(t) + X_2^2(t)\big)\mathrm{d}B^{(2)}(t), \end{array} \right. \tag{6.3}$$

with the initial value $x_0 = (1, \sqrt{3})^T$, where $B(t) = (B^{(1)}(t), B^{(2)}(t))^T$ is a two-dimensional Brownian motion. Obviously, its drift and diffusion coefficients

$$f(x) = \begin{bmatrix} -2x_1^3 - 2x_2^2x_1 \\ -2x_1^2x_2 - 2x_2^3 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 2\sqrt{2}\big(x_1^2 + x_2^2\big) & 0 \\ 0 & 2\sqrt{2}\big(x_1^2 + x_2^2\big) \end{bmatrix}.$$

are locally Lipschitz continuous for any $x \in \mathbb{R}^2$. Let $V(x) = |x|^2$, then $V \in \bar{\mathcal{V}}^2_{1/2}$ with $0 < \rho < 1/4$. One observes that

$$\mathcal{L}V^\rho(x) = \frac{\rho}{2}V^{\rho-2}(x)\big(24|x|^6 + 32(\rho-1)|x|^6\big) = 4\rho\big(4\rho-1\big)|x|^{2\rho+2} =: -w(x).$$

We choose $\rho = 1/8$, then $w(x) = 1/4|x|^{2\rho+2}$ and $w(x) = 0$ iff $x = \mathbf{0} \in \mathbb{R}^2$. One notices that the exact solution $X(t)$ of SDE (6.3) tends to $\mathbf{0}$ almost surely but the exponentially stability in moment is uncertain, see Lemma 5.1. To the best of our knowledge the numerical methods in the literatures such as [7, 23, 26, 39] cannot treat this case. However, the V-truncated EM scheme (5.6) performs perfectly for this case, see Fig. 4.

To carry out the simulations by the scheme (5.6), we divide it into seven steps.

Step 1. Examine the hypothesis. Since that

$$\sup_{0<|x|\le u} \frac{|f(x)|^2 \vee |g(x)|^2}{\Lambda_\rho(x)V^{2\delta_4}(x)} = \sup_{0<|x|\le u} \frac{4|x|^4 \vee 16|x|^2}{\big(1/4|x|^2\big) \wedge 1} < +\infty,$$

which implies condition (5.4) is satisfied.

Step 2. Choose $\bar{\varphi}(\cdot)$ and $\bar{\theta}$. For any $x \in \mathbb{R}^2$, compute

$$\begin{aligned} \sup_{0<|x|\le u} \left[\frac{|f(x)|}{\Lambda_\rho^{1/2}(x)V^{\delta_4}(x)} \vee \frac{|g(x)|^2}{\Lambda_\rho(x)V^{2\delta_4}(x)}\right] &= \sup_{0<|x|\le u} \left[\frac{4|x|^2}{|x| \wedge 2} \vee \frac{64|x|^2}{|x|^2 \wedge 4}\right] \\ &\le 16(u+2)^2 =: \bar{\varphi}(u) \qquad \forall\ u > 0. \end{aligned}$$

Then $\bar{\varphi}^{-1}(u) = 0.25\sqrt{u} - 2$, $\forall u > 64$. Define $K = \bar{\varphi}(|x_0| \vee 1)$ and $\bar{\theta} = 0.4$.

Step 3. Construct an explicit scheme. For a fixed $\triangle \in (0, 1]$, the V-truncated EM scheme for (6.3) is

$$\left\{ \begin{array}{l} Z_0 = x_0, \\ \tilde{Z}_{k+1} = Z_k + f(Z_k)\triangle + g(Z_k)\triangle B_k, \\ Z_{k+1} = \Big[|\tilde{Z}_{k+1}| \wedge \bar{\varphi}^{-1}\big(\bar{\varphi}(|x_0| \vee 1)\triangle^{-0.4}\big)\Big] \frac{\tilde{Z}_{k+1}}{|\tilde{Z}_{k+1}|}. \end{array} \right. \tag{6.4}$$

We can therefore conclude by Theorem 3.8 that the numerical solution $Z(t)$ of the V-truncated EM scheme (6.4) satisfies $\lim_{\triangle\to 0}\mathbb{E}|X(T)-Z(T)|^2=0$ for any $T\in[0,+\infty)$.

Step 4. MATLAB code. Next we specify the MATLAB code for simulating $Z(t)$:

```
% MATLAB code for simulating the V-truncated EM scheme
function Z=main
clc
clear all;
Z(:,1)=[1; sqrt(3)];T=1;dt=2^(-22);dB=sqrt(dt)*randn(2,T/dt);
v=varphi_inv(varphi(max(norm(Z(:,1)),1))*dt^(-0.4));
%Obviously, v>norm(Z(:,1));
for n=1:T/dt
    Z(:,n+1)=Z(:,n)-2*Z(:,n)*norm(Z(:,n))^2*dt+...
        [2*sqrt(2)   0;0  2*sqrt(2)]*norm(Z(:,n))^(2)*dB(:,n);
    if norm(Z(:,n+1))>v
        Z(:,n+1)=v*Z(:,n+1)/norm(Z(:,n+1));
    end
end
end
function y=varphi(u)
y=16*(u+2)^2;
end
function y=varphi_inv(u)
y=0.25*sqrt(u)-2;
end
```

Step 5. The root mean square approximation error $\big(\mathbb{E}|X(T)-Z(T)|^2\big)^{1/2}$. Due to no closed-form of the solution, using the V-truncated EM scheme (6.4), we regard the better approximation with $\triangle=2^{-22}$ as the exact solution $X(t)$ and compare it with the numerical solution $Z(t)$ with $\triangle=2^{-18},2^{-19},2^{-20},2^{-21}$. To compute the approximation error, we run M independent trajectories where $X^{(j)}(t)$ and $Z^{(j)}(t)$ represent the jth trajectories of the exact solution $X(t)$ and the numerical solution $Z(t)$ respectively. Thus

$$\Big(\mathbb{E}|X(1)-Z(1)|^2\Big)^{\frac{1}{2}}=\Big(\frac{1}{M}\sum_{j=1}^{M}|X^{(j)}(1)-Z^{(j)}(1)|^2\Big)^{\frac{1}{2}}.$$

Step 6. The log-log error plot with $M=1000$. The simulation procedure is carried out by steps 4 and 5. The red solid line depicts log-log error while the blue dashed is a reference line of slope 1/2 in Fig. 3. Fig. 3 depicts the approximation error $\big(\mathbb{E}|X(1)-Z(1)|^2\big)^{1/2}$ of the exact solution and the numerical solution of (6.4) as the function of stepsize $\triangle\in\{2^{-18},2^{-19},2^{-20},2^{-21}\}$.

Step 7. Simulate sample path by using MATLAB. Let $T=100$ and stepsize $\triangle=10^{-4}$. The numerical experiment is carried out by step 4. By virtue of Theorem 5.6, using the V-truncated EM scheme (6.4) we can preserves the underlying stability perfectly, see Fig. 4.

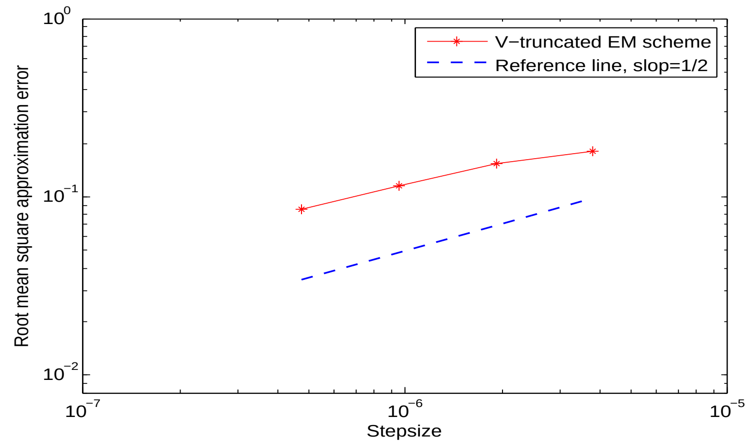

FIGURE 3. The root square approximation error of the exact solution and the numerical solution by the V-truncated EM scheme (6.4) as the function of stepsize $\triangle \in \{2^{-18}, 2^{-19}, 2^{-20}, 2^{-21}\}$.

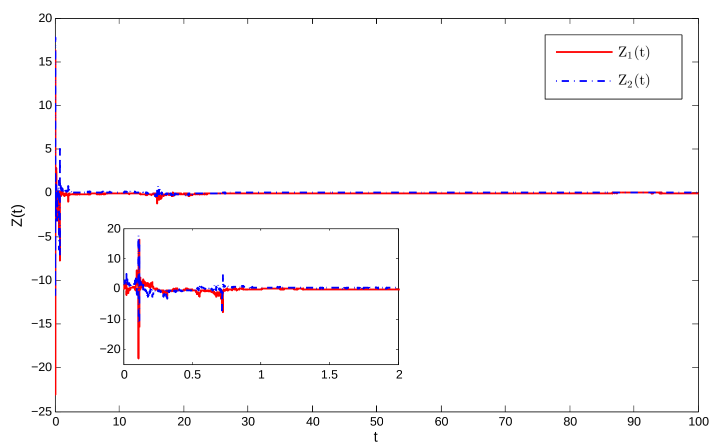

FIGURE 4. Sample path of the V-truncated EM numerical solution $Z(t)$ with the initial value $x_0 = (1, \sqrt{3})^T$ for stepsize $\triangle = 10^{-4}$ and $t \in [0, 100]$.

Example 6.3. Consider a stochastic Duffing-van der Pol oscillator (see [16, p.81])

$$\ddot{z}(t) + 3z(t) + 2\dot{z}(t) + 2\dot{z}(t)z^2(t) + z^3(t) = \sqrt{2}z(t)\mathrm{d}B^{(1)}(t) + \sqrt{2.5}\dot{z}(t)\mathrm{d}B^{(2)}(t) \tag{6.5}$$

for $t \in \mathbb{R}_+$, where $B(t) = [B^{(1)}(t), B^{(2)}(t)]^T$. Introducing a new variable $(x_1, x_2)^T = (z, \dot{z})^T$, we can write this Duffing-van der Pol oscillator as a two-dimensional SDE with drift and diffusion coefficients

$$f(x) = \begin{bmatrix} x_2 \\ -3x_1 - 2x_2 - 2x_2x_1^2 - x_1^3 \end{bmatrix}, \qquad g(x) = \begin{bmatrix} 0 & 0 \\ \sqrt{2}x_1 & \sqrt{2.5}x_2 \end{bmatrix}.$$

Obviously, its coefficients are locally Lipschitz continuous for any $x \in \mathbb{R}^2$. Therefore, Lemma 5.1 applies here with the Lyapunov-type function $V : \mathbb{R}^2 \to \mathbb{R}_+$ given by $V(x) = x_1^4 + x_2^2 + x_1x_2 + 4x_1^2$, which is from a broader class $\hat{\mathcal{V}}^4_{1/4}$ and

$$\mathcal{L}V(x) = -4x_1^2x_2^2 - x_1^2 - 0.5x_2^2 - x_1^4 \le -0.5|x|^2 =: -w(x).$$

Note that in this case the solution is not necessarily pth moment exponentially stable for $p \ge 2$, but Lemma 5.1 still holds. We can then conclude that the SDE (6.5) has the property that

$$\lim_{t\to\infty} [|z(t)| + |\dot{z}(t)|] = 0 \qquad \text{a.s.}$$

On the other hand, in order to represent the simulations of the scheme (5.6), we divide it into three steps.

Step 1. Examine the hypothesis. Since that $V(x) \ge w(x)$ for any $x \in \mathbb{R}^2$ and

$$\sup_{|x|\le N} \big(|f(x)|^2 \vee |g(x)|^2\big) \le C_N w(x), \qquad \forall N > 0,$$

which implies condition (5.22) is satisfied.

Step 2. Choose $\hat{\varphi}(\cdot)$ and $\hat{\theta}$. For any $x \in \mathbb{R}^2$, compute

$$\begin{aligned}&\sup_{0<|x|\le u} \left[\frac{\big(1+V(x)\big)^{1/4}|f(x)|}{\sqrt{0.5}|x|} \vee \frac{\big(1+V(x)\big)^{1/2}|g(x)|^2}{0.5|x|^2}\right]\\ \le& 5\big(8u^4+21\big)^{1/4}\big(36+16u^4\big)^{\frac{1}{2}} \le \big(36+16u^4\big)^{\frac{3}{4}} =: \hat{\varphi}(u), \qquad \forall\ u \ge 1,\end{aligned}$$

which implies $\hat{\varphi}^{-1}(u) = 0.5\big(u^{\frac{4}{3}} - 36\big)^{\frac{1}{4}}$ $\forall\ u \ge 20$. Then define $K = \hat{\varphi}(|x_0| \vee 1)$ and $\hat{\theta} = 0.4$.

Step 3. Construct an explicit scheme. For a fixed $\triangle \in (0, 1]$, the V-truncated EM scheme for (6.5) is

$$\left\{\begin{aligned} &Z_0 = x_0,\\ &\tilde{Z}_{k+1} = Z_k + f(Z_k)\triangle + g(Z_k)\triangle B_k,\\ &Z_{k+1} = \Big(|\tilde{Z}_{k+1}| \wedge \hat{\varphi}^{-1}\big(\hat{\varphi}(|x_0| \vee 1)\triangle^{-0.4}\big)\Big)\frac{\tilde{Z}_{k+1}}{|\tilde{Z}_{k+1}|}.\end{aligned}\right. \tag{6.6}$$

By virtue of Theorem 3.8, using this scheme we can approximate the exact solution in the mean square sense. Moreover, by Theorem 5.11, we can conclude that the V-truncated EM scheme (6.6) has the property that

$$\lim_{k\to\infty} \big[|Z_{1,k}| + |Z_{2,k}|\big] = 0 \qquad \text{a.s.}$$

7. Concluding remarks

This paper deals with numerical solutions of nonlinear SDEs. The schemes proposed are flexible in terms of Lyapunov-type functions. We have constructed two explicit numerical schemes for the SDEs whose drift and diffusion coefficients are not globally Lipschitz but grow faster than linearly. A novelty of this paper is to establish two explicit schemes to approximate the dynamical properties of SDEs in terms of Lyapunov-type functions. One of the schemes is concerned with the convergence in the finite time interval. We obtained the convergence and V-integrability of the numerical solutions under the local Lipschitz condition and the structure conditions required by the exact solutions. Moreover, the convergence rate (see Theorem 4.6) was also obtained under certain conditions and our result generalises the results in the related literature. In the case of infinite time interval, we used the other scheme to produce the LaSalle-type theorem of SDEs, from which we deduced the asymptotic stability of numerical solutions. Several examples were discussed to demonstrate the validity of our approaches and computer simulations were performed to support the theoretical results.

Appendix A.

In this appendix, we will provide the proofs of some results in Sections 3 and 5.

Proof of Lemma 3.1. First of all, note that Y_k is $\mathcal{F}_{t_k}$-measurable, we have

$$\begin{aligned}
&\sum_{|\alpha|=1} \frac{D^\alpha V(Y_k)}{\alpha!}\big(\tilde{Y}_{k+1}-Y_k\big)^\alpha = \sum_{|\alpha|=1} \frac{D^\alpha V(Y_k)}{\alpha!}\big(f(Y_k)\triangle+g(Y_k)\triangle B_k\big)^\alpha \\
&=\sum_{|\alpha|=1} \frac{D^\alpha V(Y_k)}{\alpha!}\big(f(Y_k)\triangle\big)^\alpha + \sum_{|\beta|=1} \frac{D^\beta V(Y_k)}{\beta!}\big(g(Y_k)\triangle B_k\big)^\beta \\
&=\langle D^{(1)}V(Y_k), f(Y_k)\rangle\triangle + \mathcal{S}_1^\triangle V(Y_k),
\end{aligned}$$

where

$$\mathcal{S}_1^\triangle V(Y_k) := \langle D^{(1)}V(Y_k), g(Y_k)\triangle B_k\rangle. \tag{A.1}$$

Moreover,

$$\begin{aligned}
&\sum_{|\alpha|=2} \frac{D^\alpha V(Y_k)}{\alpha!}\big(\tilde{Y}_{k+1}-Y_k\big)^\alpha \\
&=\sum_{|\alpha|=2} \frac{D^\alpha V(Y_k)}{\alpha!}\big(f(Y_k)\triangle\big)^\alpha + \sum_{\substack{|\alpha|=2,|\beta|=1\\ \alpha\geq\beta}} \frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\triangle\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta \\
&\quad+\sum_{|\beta|=2} \frac{D^\beta V(Y_k)}{\beta!}\big(g(Y_k)\triangle B_k\big)^\beta \\
&=\frac{1}{2}\langle f(Y_k), D^{(2)}V(Y_k)f(Y_k)\rangle\triangle^2 + \sum_{\substack{|\alpha|=2,|\beta|=1\\ \alpha\geq\beta}} \frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\triangle\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta \\
&\quad+\frac{1}{2}\mathrm{tr}\Big[\big(g(Y_k)\triangle B_k\big)^T D^{(2)}V(Y_k)\big(g(Y_k)\triangle B_k\big)\Big] \\
&=:\frac{1}{2}\langle f(Y_k), D^{(2)}V(Y_k)f(Y_k)\rangle\triangle^2 + \frac{1}{2}\mathrm{tr}\Big[g^T(Y_k)D^{(2)}V(Y_k)g(Y_k)\Big]\triangle + \mathcal{S}_2^\triangle V(Y_k) \\
&\leq \Big|D^{(2)}V(Y_k)\Big|\,|f(Y_k)|^2\triangle^2 + \frac{1}{2}\mathrm{tr}\Big[g^T(Y_k)D^{(2)}V(Y_k)g(Y_k)\Big]\triangle + \mathcal{S}_2^\triangle V(Y_k),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}_2^\triangle V(Y_k) :=& \sum_{\substack{|\alpha|=2,|\beta|=1\\ \alpha\geq\beta}} \frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\triangle\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta \\
&+\frac{1}{2}\mathrm{tr}\Big[D^{(2)}V(Y_k)g(Y_k)\big(\triangle B_k\triangle B_k^T-\mathbb{I}_m\triangle\big)g^T(Y_k)\Big],
\end{aligned} \tag{A.2}$$

and $\mathbb{I}_m$ denotes the $m\times m$ identity matrix. The fact that $\triangle B_k$ is independent of $\mathcal{F}_{t_k}$ implies that

$$\mathbb{E}_k\big[\triangle B_k\big]=\mathbb{E}\big[\triangle B_k|\mathcal{F}_{t_k}\big]=\mathbb{E}(\triangle B_k)=\mathbf{0}\in\mathbb{R}^m,\quad \mathbb{E}_k\big[\triangle B_k\triangle B_k^T\big]=\mathbb{I}_m\triangle. \tag{A.3}$$

Hence $\mathbb{E}_k\big[\mathcal{S}_i^{\triangle}V(Y_k)\big]=0$ for $i=1,2$. We can now analyse the rest of the expansion for $|\alpha|=3$, we have

$$\begin{aligned}
&\sum_{|\alpha|=3}\frac{D^\alpha V(Y_k)}{\alpha!}\big(\tilde{Y}_{k+1}-Y_k\big)^\alpha\\
=&\sum_{|\alpha|=3}\frac{D^\alpha V(Y_k)}{\alpha!}\big(f(Y_k)\big)^\alpha\triangle^3+\sum_{\substack{|\alpha|=3,|\beta|=1\\ \alpha\ge\beta}}\frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta\triangle^2\\
&+\sum_{\substack{|\alpha|=3,|\beta|=2\\ \alpha\ge\beta}}\frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta\triangle+\sum_{|\beta|=3}\frac{D^\beta V(Y_k)}{\beta!}\big(g(Y_k)\triangle B_k\big)^\beta\\
\le&\sum_{\substack{|\alpha|=3,|\beta|=1\\ \alpha\ge\beta}}\frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta\triangle^2+\sum_{|\beta|=3}\frac{D^\beta V(Y_k)}{\beta!}\big(g(Y_k)\triangle B_k\big)^\beta\\
&+C|D^{(3)}V(Y_k)|\Big(|f(Y_k)|^3\triangle^3+|g(Y_k)|^2|f(Y_k)||\triangle B_k|^2\triangle\Big).
\end{aligned}\tag{A.4}$$

Using the properties

$$\mathbb{E}_k\big[|\triangle B_k|^i\big]=K_i\triangle^{\frac{i}{2}},\qquad i=1,2,3,\dots \tag{A.5}$$

where K_i is a positive constant dependent on i, as well as (A.4) yields

$$\begin{aligned}
&\sum_{|\alpha|=3}\frac{D^\alpha V(Y_k)}{\alpha!}\big(\tilde{Y}_{k+1}-Y_k\big)^\alpha\\
\le&C|D^{(3)}V(Y_k)|\triangle^2\Big(|f(Y_k)|^3\triangle+|g(Y_k)|^2|f(Y_k)|\Big)+\mathcal{S}_3^{\triangle}V(Y_k),
\end{aligned}$$

and $\mathbb{E}_k\big[\mathcal{S}_3^{\triangle}V(Y_k)\big]=0$, where

$$\begin{aligned}
\mathcal{S}_3^{\triangle}V(Y_k):=&\sum_{\substack{|\alpha|=3,|\beta|=1\\ \alpha\ge\beta}}\frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta\triangle^2\\
&+C|D^{(3)}V(Y_k)||g(Y_k)|^2|f(Y_k)|\big(|\triangle B_k|^2-K_2\triangle\big)\triangle\\
&+\sum_{|\beta|=3}\frac{D^\beta V(Y_k)}{\beta!}\big(g(Y_k)\triangle B_k\big)^\beta.
\end{aligned}\tag{A.6}$$

The proof is therefore complete. □

Proof of Lemma 3.2. First of all, note that Y_k is $\mathcal{F}_{t_k}$-measurable. Then, using (2.2), (3.3) as well as the estimates of $|D^{(n)}V(\cdot)|$, we derive that

$$\begin{aligned}
\big|\mathcal{L}V(Y_k)\big|\le&|D^{(1)}V(Y_k)||f(Y_k)|+|D^{(2)}V(Y_k)||g(Y_k)|^2\\
\le&cK\triangle^{-\theta}\big[1+V(Y_k)\big]^{1-\delta_4}\big[1+V(Y_k)\big]^{\delta_4}\\
&+cK\triangle^{-\theta}\big[1+V(Y_k)\big]^{1-2\delta_4}\big[1+V(Y_k)\big]^{2\delta_4}\le 2cK\big(1+V(Y_k)\big)\triangle^{-\theta},
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{R}^{\triangle}V(Y_k) =&C\sum_{i=2}^{3}\sum_{j=0}^{i-2}|f(Y_k)|^{i-2j}|g(Y_k)|^{2j}|D^{(i)}V(Y_k)|\triangle^{i-j}\\
&\le C\big(1+V(Y_k)\big)\Big[\triangle^{2(1-\theta)}+\sum_{r=0}^{1}\triangle^{(3-r)(1-\theta)}\Big]\le C\big(1+V(Y_k)\big)\triangle^{2(1-\theta)}.
\end{aligned}
$$

Thus, the required assertion (3.5) is obtained. Moreover,

$$
\begin{aligned}
|\mathcal{S}_1^{\triangle}V(Y_k)|^2 =&\langle D^{(1)}V(Y_k),g(Y_k)\triangle B_k\rangle\langle D^{(1)}V(Y_k),g(Y_k)\triangle B_k\rangle\\
=&\mathrm{tr}\Big[\big(D^{(1)}V(Y_k)\big)^T\big(g(Y_k)\triangle B_k\big)\big(g(Y_k)\triangle B_k\big)^TD^{(1)}V(Y_k)\Big]\\
=&|D^{(1)}V(Y_k)g(Y_k)|^2\triangle+\mathcal{H}_{1,1}^{\triangle}V(Y_k),
\end{aligned}
$$

where

$$
\mathcal{H}_{1,1}^{\triangle}V(Y_k)=\mathrm{tr}\Big[\big(D^{(1)}V(Y_k)\big)^Tg(Y_k)\big(\triangle B_k\triangle B_k^T-\mathbb{I}_m\triangle\big)g^T(Y_k)D^{(1)}V(Y_k)\Big], \tag{A.7}
$$

Thus, we obtain that

$$
\begin{aligned}
&\mathbb{E}_k\big[\mathcal{H}_{1,1}^{\triangle}V(Y_k)\big]\\
=&\mathbb{E}_k\Big\{\mathrm{tr}\Big[\big(D^{(1)}V(Y_k)\big)^Tg(Y_k)\big(\triangle B_k\triangle B_k^T-\mathbb{I}_m\triangle\big)g^T(Y_k)D^{(1)}V(Y_k)\Big]\Big\}\\
=&\mathrm{tr}\Big\{\big(D^{(1)}V(Y_k)\big)^Tg(Y_k)\big[\mathbb{E}_k\big(\triangle B_k\triangle B_k^T\big)-\mathbb{I}_m\triangle\big]g^T(Y_k)D^{(1)}V(Y_k)\Big\}=0.
\end{aligned}
$$

Using (3.3) as well as the estimates of $|D^{(n)}V(\cdot)|$, we have

$$
\mathbb{E}_k\big[|\mathcal{S}_1^{\triangle}V(Y_k)|^2\big]\le|D^{(1)}V(Y_k)|^2|g(Y_k)|^2\triangle\le C\big(1+V(Y_k)\big)^2\triangle^{1-\theta}.
$$

One further observes that

$$
\begin{aligned}
|\mathcal{S}_2^{\triangle}V(Y_k)|^2 =&\Bigg(\sum_{\substack{|\alpha|=2,|\beta|=1\\ \alpha\ge\beta}}\frac{D^{\alpha}V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\triangle\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^{\beta}\\
&+\frac{1}{2}\mathrm{tr}\Big[D^{(2)}V(Y_k)g(Y_k)\big(\triangle B_k\triangle B_k^T-\mathbb{I}_m\triangle\big)g^T(Y_k)\Big]\Bigg)^2\\
\le&C|D^{(2)}V(Y_k)|^2\Big(|f(Y_k)|^2|g(Y_k)|^2|\triangle B_k|^2\triangle^2\\
&+|g(Y_k)|^4|\triangle B_k|^4+|g(Y_k)|^4\triangle^2\Big)\\
\le&C|D^{(2)}V(Y_k)|^2\Big(|f(Y_k)|^2|g(Y_k)|^2\triangle^3+|g(Y_k)|^4\triangle^2\Big)+\mathcal{H}_{2,2}^{\triangle}V(Y_k),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{2,2}^{\triangle}V(Y_k)=C|D^{(2)}V(Y_k)|^2\Big[&|f(Y_k)|^2|g(Y_k)|^2\triangle^2\big(|\triangle B_k|^2-K_2\triangle\big)\\
&+|g(Y_k)|^4\big(|\triangle B_k|^4-K_4\triangle^2\big)\Big].
\end{aligned}\tag{A.8}
$$

Using (3.3), (A.5) as well as the estimates of $|D^{(n)}V(\cdot)|$, we have

$$
\begin{aligned}
\mathbb{E}_k\big[|\mathcal{S}_2^{\triangle}V(Y_k)|^2\big]\le&C\big(1+V(Y_k)\big)^2\Big(\triangle^{2(1-\theta)}+\triangle^{3(1-\theta)}\Big)+\mathbb{E}_k\big[\mathcal{H}_{2,2}^{\triangle}V(Y_k)\big]\\
\le&C\big(1+V(Y_k)\big)^2\triangle^{2(1-\theta)}.
\end{aligned}
$$

Similarly, we can also prove that

$$
\begin{aligned}
&|\mathcal{S}_3^{\triangle} V(Y_k)|^2\\
=&\Big[\sum_{\substack{|\alpha|=3,|\beta|=1\\ \alpha\geq\beta}} \frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta\triangle^2 + C|D^{(3)}V(Y_k)||g(Y_k)|^2\\
&\qquad\times |f(Y_k)|\big(|\triangle B_k|^2 - K_2\triangle\big)\triangle + \sum_{|\beta|=3}\frac{D^\beta V(Y_k)}{\beta!}\big(g(Y_k)\triangle B_k\big)^\beta\Big]^2\\
\leq& C|D^{(3)}V(Y_k)|^2\Big(|f(Y_k)|^4|g(Y_k)|^2\triangle^5 + |g(Y_k)|^6\triangle^3\\
&\qquad + |f(Y_k)|^2|g(Y_k)|^4\triangle^4\Big) + \mathcal{H}_{3,3}^{\triangle}V(Y_k),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{3,3}^{\triangle}V(Y_k) =& C|D^{(3)}V(Y_k)|^2\Big[|g(Y_k)|^2|f(Y_k)|^4\big(|\triangle B_k|^2 - K_2\triangle\big)\triangle^4\\
&+ |g(Y_k)|^6\big(|\triangle B_k|^6 - K_6\triangle^3\big) + |f(Y_k)|^2|g(Y_k)|^4\big(|\triangle B_k|^4 - K_4\triangle^2\big)\triangle^2\Big].
\end{aligned}
\tag{A.9}
$$

Thus, we obtain that $\mathbb{E}_k\big[|\mathcal{S}_3^{\triangle}V(Y_k)|^2\big] \leq C\big(1+V(Y_k)\big)^2\triangle^{3(1-\theta)}$. Returning to (A.1), (A.2) and using (3.3) as well as the estimates of $|D^{(n)}V(\cdot)|$, we obtain

$$
\begin{aligned}
&\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{S}_2^{\triangle}V(Y_k)\\
=&\langle D^{(1)}V(Y_k), g(Y_k)\triangle B_k\rangle\Big\{\sum_{\substack{|\alpha|=2,|\beta|=1\\ \alpha\geq\beta}}\frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\triangle\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta\\
&\qquad + \frac{1}{2}\mathrm{tr}\Big[D^{(2)}V(Y_k)g(Y_k)\big(\triangle B_k\triangle B_k^T - \mathbb{I}_m\triangle\big)g^T(Y_k)\Big]\Big\}\\
\geq& -C|D^{(1)}V(Y_k)||D^{(2)}V(Y_k)||f(Y_k)||g(Y_k)|^2\triangle^2 + \mathcal{H}_{1,2}^{\triangle}V(Y_k)\\
\geq& -C\big(1+V(Y_k)\big)^2\triangle^{2(1-\theta)} + \mathcal{H}_{1,2}^{\triangle}V(Y_k),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{1,2}^{\triangle}V(Y_k) =& -C|D^{(1)}V(Y_k)||D^{(2)}V(Y_k)||f(Y_k)||g(Y_k)|^2\big(|\triangle B_k|^2 - K_2\triangle\big)\triangle\\
&+ \frac{1}{2}\mathrm{tr}\Big[D^{(2)}V(Y_k)g(Y_k)\big(\triangle B_k\triangle B_k^T - \mathbb{I}_m\triangle\big)g^T(Y_k)\Big]\mathcal{S}_1^{\triangle}V(Y_k).
\end{aligned}
\tag{A.10}
$$

Using (A.3) and (A.5), it is easy to see that

$$
\begin{aligned}
&\mathbb{E}_k\big[\mathcal{H}_{1,2}^{\triangle}V(Y_k)\big]\\
=& -C|D^{(1)}V(Y_k)||D^{(2)}V(Y_k)||f(Y_k)||g(Y_k)|^2\Big[\mathbb{E}_k\big(|\triangle B_k|^2\big) - K_2\triangle\Big]\triangle = 0.
\end{aligned}
$$

Thus, we obtain that

$$
\mathbb{E}_k\big[\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{S}_2^{\triangle}V(Y_k)\big] \geq -C\big(1+V(Y_k)\big)^2\triangle^{2(1-\theta)}.
$$

One further observes that

$$
\begin{aligned}
&\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{S}_3^{\triangle}V(Y_k)\\
=&\langle D^{(1)}V(Y_k), g(Y_k)\triangle B_k\rangle\Big[\sum_{\substack{|\alpha|=3,|\beta|=1\\ \alpha\geq\beta}}\frac{D^\alpha V(Y_k)}{\beta!(\alpha-\beta)!}\big(f(Y_k)\big)^{\alpha-\beta}\big(g(Y_k)\triangle B_k\big)^\beta\triangle^2
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + C|D^{(3)}V(Y_k)||g(Y_k)|^2|f(Y_k)|\triangle\big(|\triangle B_k|^2 - K_2\triangle\big)\\
&\qquad + \sum_{|\beta|=3}\frac{D^\beta V(Y_k)}{\beta!}\big(g(Y_k)\triangle B_k\big)^\beta\Big]\\
&\geq - C|D^{(1)}V(Y_k)||D^{(3)}V(Y_k)|\Big(|f(Y_k)|^2|g(Y_k)|^2\triangle^3 + |g(Y_k)|^4\triangle^2\Big) + \mathcal{H}_{1,3}^{\triangle}V(Y_k),
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{H}_{1,3}^{\triangle}V(Y_k) =& - C|D^{(1)}V(Y_k)||D^{(3)}V(Y_k)|\Big[|f(Y_k)|^2|g(Y_k)|^2\big(|\triangle B_k|^2 - K_2\triangle\big)\triangle^2\\
&\qquad + |g(Y_k)|^4\big(|\triangle B_k|^4 - K_4\triangle^2\big)\Big]\\
&+ C|D^{(3)}V(Y_k)||g(Y_k)|^2|f(Y_k)|\triangle\big(|\triangle B_k|^2 - K_2\triangle\big)\mathcal{S}_1^{\triangle}V(Y_k).
\end{aligned}
\tag{A.11}
$$

Using (3.3), (A.3) and (A.5) as well as the estimates of $|D^{(n)}V(\cdot)|$, it is easy to see that

$$
\mathbb{E}_k\big[\mathcal{S}_1^{\triangle}V(Y_k)\mathcal{S}_3^{\triangle}V(Y_k)\big] \geq - C\big(1+V(Y_k)\big)^2\triangle^{2(1-\theta)}.
$$

Therefore the desired result follows. □

Proof of Lemma 5.4. Due to $V \in \bar{\mathcal{V}}_{\delta_4}^4$, using the Taylor formula with integral remainder term we get

$$
V(\tilde{Z}_{k+1}) = V(Z_k) + \sum_{|\alpha|=1}^{3}\frac{D^\alpha V(Z_k)}{\alpha!}\big(\tilde{Z}_{k+1} - Z_k\big)^\alpha + J(\tilde{Z}_{k+1}, Z_k).
\tag{A.12}
$$

One observes that

$$
\begin{aligned}
\big|J(\tilde{Z}_{k+1}, Z_k)\big| \leq& 4\sum_{|\alpha|=4}\frac{\big|\big(\tilde{Z}_{k+1} - Z_k\big)^\alpha\big|}{\alpha!}\int_0^1(1-t)^3\big|D^{(4)}V\big(Z_k + t\big(\tilde{Z}_{k+1} - Z_k\big)\big)\big|\mathrm{d}t\\
\leq& \frac{c}{3!}\bigg(\sum_{i=1}^{d}\Big|f_i(Z_k)\triangle + \sum_{j=1}^{m}g_{ij}(Z_k)\triangle B_k^{(j)}\Big|\bigg)^4\\
&\qquad \times\int_0^1(1-t)^3V^{1-4\delta_4}\big(Z_k + t\big(\tilde{Z}_{k+1} - Z_k\big)\big)\mathrm{d}t.
\end{aligned}
$$

Note that for any $U \in \bar{\mathcal{V}}_{\delta_4}^4$ we know $|D^{(1)}U(x)| \leq cU^{1-\delta_4}(x)$. By the result of [16, Lemma 2.12, p.22] we have

$$
U(x+y) \leq c^{\frac{1}{\delta_4}}2^{\frac{1}{\delta_4}-1}\Big(\big|U(x)\big| + |y|^{\frac{1}{\delta_4}}\Big), \qquad \forall x, y \in \mathbb{R}^d,
$$

which leads to

$$
\begin{aligned}
\big[V\big(Z_k + t\big(\tilde{Z}_{k+1} - Z_k\big)\big)\big]^{1-4\delta_4} \leq& \Big[c^{\frac{1}{\delta_4}}2^{\frac{1}{\delta_4}-1}\Big(V(Z_k) + t^{\frac{1}{\delta_4}}|\tilde{Z}_{k+1} - Z_k|^{\frac{1}{\delta_4}}\Big)\Big]^{1-4\delta_4}\\
\leq& C\Big[V^{1-4\delta_4}(Z_k) + |\tilde{Z}_{k+1} - Z_k|^{\frac{1}{\delta_4}-4}\Big]
\end{aligned}
$$

for $1/\delta_4 \in [4, +\infty)$. Therefore, we derive from (5.7) that for any integer $k \geq 0$,

$$
\begin{aligned}
\big|J(\tilde{Z}_{k+1}, Z_k)\big| \leq& C\Big[\Big(|f(Z_k)|^4\triangle^4 + |g(Z_k)|^4|\triangle B_k|^4\Big)V^{1-4\delta_4}(Z_k)\\
&\qquad + \Big(|f(Z_k)|^{\frac{1}{\delta_4}}\triangle^{\frac{1}{\delta_4}} + |g(Z_k)|^{\frac{1}{\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big)\Big]\\
\leq& C\Lambda_\rho(Z_k)\Big\{\Big[\triangle^{4(1-\bar{\theta})} + \triangle^{-2\bar{\theta}}|\triangle B_k|^4\Big]V(Z_k)
\end{aligned}
$$

$$+ V(Z_k)\Big[\triangle^{\frac{1}{\delta_4}(1-\bar\theta)} + \triangle^{-\frac{\bar\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big]\Big\}$$

$$\leq C\Lambda_\rho(Z_k)V(Z_k)\triangle^{4(1-\bar\theta)} + \tilde{\mathcal{J}}^\triangle V(Z_k), \tag{A.13}$$

where

$$\tilde{\mathcal{J}}^\triangle V(Z_k) = C\Lambda_\rho(Z_k)V(Z_k)\Big[\triangle^{-2\bar\theta}|\triangle B_k|^4 + \triangle^{-\frac{\bar\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big]. \tag{A.14}$$

Using the similar techniques in the proofs of Lemmas 3.1 and 3.2, we can also prove that

$$\sum_{|\alpha|=1}^{3}\frac{D^\alpha V(Z_k)}{\alpha!}\big(\tilde Z_{k+1}- Z_k\big)^\alpha \leq \mathcal{L}V(Z_k)\triangle + C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar\theta)} + \sum_{i=1}^{3}\mathcal{S}_i^\triangle V(Z_k).$$

This together with (A.12) and (A.13) implies

$$V(\tilde Z_{k+1}) \leq V(Z_k) + \mathcal{L}V(Z_k)\triangle + C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar\theta)} + \sum_{i=1}^{3}\mathcal{S}_i^\triangle V(Z_k) + |\tilde{\mathcal{J}}^\triangle V(Z_k)|,$$

and by (A.14), one observes

$$\begin{aligned}|\tilde{\mathcal{J}}^\triangle V(Z_k)| =& C\Lambda_\rho(Z_k)V(Z_k)\Big[K_4\triangle^{2(1-\bar\theta)} + K_{\frac{1}{\delta_4}}\triangle^{\frac{1-\bar\theta}{2\delta_4}}\Big] + \mathcal{A}_1^\triangle V(Z_k)\\ \leq& C\Lambda_\rho(Z_k)V(Z_k)\triangle^{2(1-\bar\theta)} + \mathcal{A}_1^\triangle V(Z_k),\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_1^\triangle V(Z_k) = C\Lambda_\rho(Z_k)V(Z_k)\Big[&\triangle^{-2\bar\theta}\big(|\triangle B_k|^4 - K_4\triangle^2\big)\\ &+ \triangle^{-\frac{\bar\theta}{2\delta_4}}\Big(|\triangle B_k|^{\delta_4} - K_{\frac{1}{\delta_4}}\triangle^{\frac{1}{2\delta_4}}\Big)\Big].\end{aligned} \tag{A.15}$$

Using the property (A.5) we have $\mathbb{E}_k\big[\mathcal{A}_1^\triangle V(Z_k)\big] = 0$. Thus, the required assertion (5.8) is obtained. Using the similar techniques in the proof of Lemma 3.2, we can also prove that

$$|\mathcal{S}_1^\triangle V(Z_k)|^2 = |D^{(1)}V(Z_k)g(Z_k)|^2\triangle + \mathcal{H}_{1,1}^\triangle V(Z_k),$$

and for $i = 1, 2, 3$,

$$|\mathcal{S}_i^\triangle V(Z_k)|^2 \leq C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{1-\bar\theta} + \mathcal{H}_{i,i}^\triangle V(Z_k), \qquad \mathbb{E}_k\big[\mathcal{H}_{i,i}^\triangle V(Z_k)\big] = 0,$$

where $\mathcal{H}_{i,i}^\triangle V(\cdot)$ is defined by (A.7), (A.8) and (A.9), respectively,

$$\mathcal{S}_1^\triangle V(Z_k)\mathcal{S}_j^\triangle V(Z_k) \geq -C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{2(1-\bar\theta)} + \mathcal{H}_{1,j}^\triangle V(Z_k),$$

and $\mathbb{E}_k\big[\mathcal{H}_{1,j}^\triangle V(Z_k)\big] = 0$ for $j = 2, 3$, where $\mathcal{H}_{1,j}^\triangle V(\cdot)$ are defined by (A.10) and (A.11), respectively. □

Proof of Lemma 5.5. First, for an integer $1/\delta_4 \in [4, +\infty)$, by (A.14) and (A.5), we deduce that

$$\begin{aligned}|\tilde{\mathcal{J}}^\triangle V(Z_k)|^2 \leq& C\Lambda_\rho^2(Z_k)V^2(Z_k)\big(\triangle^{-4\bar\theta}|\triangle B_k|^8 + \triangle^{-\frac{\bar\theta}{\delta_4}}|\triangle B_k|^{\frac{2}{\delta_4}}\big)\\ \leq& C\Lambda_\rho(Z_k)V^2(Y_k)\triangle^{4(1-\bar\theta)} + \mathcal{A}_2^\triangle V(Z_k),\end{aligned} \tag{A.16}$$

and $\mathbb{E}_k\big[\mathcal{A}_2^\triangle V(Z_k)\big] = 0$, where

$$\mathcal{A}_2^\triangle V(Z_k) = C\Lambda_\rho^2(Z_k)V^2(Z_k)\Big[\triangle^{-4\bar\theta}\Big(|\triangle B_k|^8 - K_8\triangle^4\Big)$$

$$+\triangle^{-\frac{\bar\theta}{\delta_4}}\Big(|\triangle B_k|^{\frac{2}{\delta_4}}-K_{\frac{2}{\delta_4}}\triangle^{\frac{1}{\delta_4}}\Big)\Big]. \tag{A.17}$$

One further observes that

$$\begin{aligned}|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|^3 \le& C\Lambda_\rho^3(Z_k)V^3(Z_k)\big(\triangle^{-6\bar\theta}|\triangle B_k|^{12}+\triangle^{-\frac{3\bar\theta}{2\delta_4}}|\triangle B_k|^{\frac{3}{\delta_4}}\big)\\ \le& C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{6(1-\bar\theta)}+\mathcal{A}_3^{\triangle}V(Z_k),\end{aligned} \tag{A.18}$$

and $\mathbb{E}_k\big[\mathcal{A}_3^{\triangle}V(Z_k)\big]=0$, where

$$\begin{aligned}\mathcal{A}_3^{\triangle}V(Z_k) =& C\Lambda_\rho^3(Z_k)V^3(Z_k)\Big[\triangle^{-6\bar\theta}\Big(|\triangle B_k|^{12}-K_{12}\triangle^6\Big)\\ &+\triangle^{-\frac{3\bar\theta}{2\delta_4}}\Big(|\triangle B_k|^{\frac{3}{\delta_4}}-K_{\frac{3}{\delta_4}}\triangle^{\frac{3}{2\delta_4}}\Big)\Big].\end{aligned} \tag{A.19}$$

On the other hand, by (A.7), (5.7) and (A.14) as well as the estimates of $D^{(n)}V(\cdot)$, one observes

$$\begin{aligned}\big|\mathcal{S}_1^{\triangle}V(Z_k)\big|^2|\tilde{\mathcal{J}}^{\triangle}V(Z_k)| \le& C\Lambda_\rho(Z_k)V(Z_k)\Big(C\Lambda_\rho(Z_k)V^2(Z_k)\triangle^{1-\bar\theta}+\mathcal{H}_{1,1}^{\triangle}V(Z_k)\Big)\\ &\times\Big(\triangle^{-2\bar\theta}|\triangle B_k|^4+\triangle^{-\frac{\bar\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big)\\ \le& C\Lambda_\rho(Z_k)V^3(Z_k)\Big(\triangle^{1-\bar\theta}+\triangle^{-\bar\theta}|\triangle B_k|^2\Big)\\ &\times\Big(\triangle^{-2\bar\theta}|\triangle B_k|^4+\triangle^{-\frac{\bar\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big)\\ \le& C\Lambda_\rho(Z_k)V^3(Z_k)\Big(\triangle^{1-3\bar\theta}|\triangle B_k|^4+\triangle^{1-\frac{\bar\theta}{2\delta_4}-\bar\theta}|\triangle B_k|^{\frac{1}{\delta_4}}\\ &+\triangle^{-3\bar\theta}|\triangle B_k|^6+\triangle^{-\frac{\bar\theta}{2\delta_4}-\bar\theta}|\triangle B_k|^{2+\frac{1}{\delta_4}}\Big)\\ \le& C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{3(1-\bar\theta)}+\mathcal{A}_4^{\triangle}V(Z_k)\end{aligned}$$

for $1/\delta_4\in[4,+\infty)$, where

$$\begin{aligned}\mathcal{A}_4^{\triangle}V(Z_k) =& C\Lambda_\rho(Z_k)V^3(Z_k)\Big[\triangle^{-3\bar\theta}\Big(|\triangle B_k|^6-K_6\triangle^3+\triangle|\triangle B_k|^4-K_4\triangle^3\Big)\\ &+\triangle^{-\frac{\bar\theta}{2\delta_4}-\bar\theta}\Big(|\triangle B_k|^{2+\frac{1}{\delta_4}}-K_{2+\frac{1}{\delta_4}}\triangle^{1+\frac{1}{2\delta_4}}\Big)\\ &+\triangle^{1-\frac{\bar\theta}{2\delta_4}-\bar\theta}\Big(|\triangle B_k|^{\frac{1}{\delta_4}}-K_{\frac{1}{\delta_4}}\triangle^{\frac{1}{2\delta_4}}\Big)\Big].\end{aligned} \tag{A.20}$$

and it is easy to see that $\mathbb{E}_k\big[\mathcal{A}_4^{\triangle}V(Z_k)\big]=0$. Similarly, we can also prove that

$$\begin{aligned}&\big|\mathcal{S}_2^{\triangle}V(Z_k)\big|^2|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|\\ \le& C\Lambda_\rho(Z_k)V^3(Z_k)\Big(\triangle^{-2\bar\theta}|\triangle B_k|^4+\triangle^{2(1-\bar\theta)}\\ &+\triangle^{2-3\bar\theta}|\triangle B_k|^2\Big)\Big(\triangle^{-2\bar\theta}|\triangle B_k|^4+\triangle^{-\frac{\bar\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big)\\ \le& C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{4(1-\bar\theta)}+\mathcal{A}_5^{\triangle}V(Z_k),\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_5^{\triangle}V(Z_k) =& C\Lambda_\rho(Z_k)V^3(Z_k)\Big\{\triangle^{-4\bar\theta}\Big(|\triangle B_k|^8-K_8\triangle^4\Big)+\triangle^{2-5\bar\theta}\Big(|\triangle B_k|^6-K_6\triangle^3\Big)\\ &+\triangle^{-(\frac{1}{2\delta_4}+2)\bar\theta}\Big(|\triangle B_k|^{\frac{1}{\delta_4}+4}-K_{\frac{1}{\delta_4}+4}\triangle^{\frac{1}{2\delta_4}+2}\Big)\\ &+\triangle^{2(1-\bar\theta)}\Big[\triangle^{-2\bar\theta}\Big(|\triangle B_k|^4-K_4\triangle^2\Big)+\triangle^{\frac{-\bar\theta}{2\delta_4}}\Big(|\triangle B_k|^{\frac{1}{\delta_4}}-K_{\frac{1}{\delta_4}}\triangle^{\frac{1}{2\delta_4}}\Big)\Big]\end{aligned}$$

$$+ \triangle^{2-\bar\theta(\frac{1}{2\delta_4}+3)}\Big(|\triangle B_k|^{\frac{1}{\delta_4}+2} - K_{\frac{1}{\delta_4}+2}\triangle^{\frac{1}{2\delta_4}+1}\Big)\bigg\}, \tag{A.21}$$

and it is easy to see that $\mathbb{E}_k\big[\mathcal{A}_5^{\triangle}V(Z_k)\big]=0$. Moreover,

$$\begin{aligned}
&\big|\mathcal{S}_3^{\triangle}V(Z_k)\big|^2|\tilde{\mathcal{J}}^{\triangle}V(Z_k)|\\
\le& C\Lambda_\rho(Z_k)V^3(Z_k)\Big(|\triangle B_k|^2\triangle^{4-5\bar\theta}+\triangle^{2-4\bar\theta}|\triangle B_k|^4\\
&\qquad +\triangle^{4(1-\bar\theta)}+\triangle^{-3\bar\theta}|\triangle B_k|^6\Big)\Big(\triangle^{-2\bar\theta}|\triangle B_k|^4+\triangle^{\frac{-\bar\theta}{2\delta_4}}|\triangle B_k|^{\frac{1}{\delta_4}}\Big)\\
\le& C\Lambda_\rho(Z_k)V^3(Z_k)\triangle^{5(1-\bar\theta)}+\mathcal{A}_6^{\triangle}V(Z_k),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_6^{\triangle}V(Z_k)=&C\Lambda_\rho(Z_k)V^3(Z_k)\bigg\{\triangle^{-5\bar\theta}\Big(|\triangle B_k|^{10}-K_{10}\triangle^5\Big)\\
&+\triangle^{2-6\bar\theta}\Big(|\triangle B_k|^8-K_8\triangle^4\Big)+\triangle^{4-7\bar\theta}\Big(|\triangle B_k|^6-K_6\triangle^3\Big)\\
&+\triangle^{-\bar\theta(\frac{1}{2\delta_4}+3)}\Big(|\triangle B_k|^{\frac{1}{\delta_4}+6}-K_{\frac{1}{\delta_4}+6}\triangle^{\frac{1}{2\delta_4}+3}\Big)\\
&+\triangle^{4(1-\bar\theta)}\Big[\triangle^{-2\bar\theta}\Big(|\triangle B_k|^4-K_4\triangle^2\Big)+\triangle^{\frac{-\bar\theta}{2\delta_4}}\Big(|\triangle B_k|^{\frac{1}{\delta_4}}-K_{\frac{1}{\delta_4}}\triangle^{\frac{1}{2\delta_4}}\Big)\Big]\\
&+\triangle^{2-\bar\theta(\frac{1}{2\delta_4}+4)}\Big(|\triangle B_k|^{\delta_4+4}-K_{\frac{1}{\delta_4}+4}\triangle^{\frac{1}{2\delta_4}+2}\Big)
\end{aligned}$$

$$+\triangle^{4-\bar\theta(\frac{1}{2\delta_4}+5)}\Big(|\triangle B_k|^{\frac{1}{\delta_4}+2}-K_{\frac{1}{\delta_4}+2}\triangle^{\frac{1}{2\delta_4}+1}\Big)\bigg\}, \tag{A.22}$$

and it is easy to see that $\mathbb{E}_k\big[\mathcal{A}_6^{\triangle}V(Z_k)\big]=0$. □

Acknowledgements

We would like to thanks the editor and reviewers for their helpful comments and suggestions.

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