


Stabilization of hybrid systems by intermittent feedback controls based on discrete-time observations with a time delay

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Funding information

Fundamental Research Funds for the Central Universities, Grant/Award Number: 2232020D-37; National Natural Science Foundation of China, Grant/Award Number: 11401261; “333 High-level Project” of Jiangsu Province and the Qing Lan Project of Jiangsu Province; Fundamental Research Funds for the Central Universities and Graduate Student Innovation Fund of Donghua University, Grant/Award Number: CUSF-DH-D-2021054

Abstract

This paper mainly investigates stabilization of hybrid stochastic differential equations (SDEs) via periodically intermittent feedback controls based on discrete-time state observations with a time delay. First, by using the theory of M-matrix and intermittent control strategy, we establish sufficient conditions for the stability of hybrid SDEs. Then, we prove the intermittent stabilization for a given unstable nonlinear hybrid SDE by comparison theorem. Two numerical examples are discussed to support our results of theoretical analysis.

1 | INTRODUCTION

Stochastic systems have been applied to model practical problems in many fields such as science and technology, information engineering, social economy and so on. As an important type of stochastic systems, hybrid stochastic differential equations (SDEs; also known as SDEs with Markovian switching) can well describe the actual systems whose structures and parameters are suddenly changed. Therefore, hybrid SDEs have been studied by many researchers (see, e.g. [1–5]).

Stabilization is one of the hot topics in the research of hybrid stochastic systems (see, e.g. [6, 7]). That is, to design a feedback control in the drift part to make the given system become stable. Regular feedback controls are designed based on the continuous-time observations of current state $x(t)$ (see, e.g. [1, 3, 4, 8]). To reduce the high cost of continuous-time state observations, Mao [9] introduced the feedback controls based on discrete-time state observations to stabilize the given

hybrid stochastic system. Mao et al. [10] improved method to study the discrete-time state feedback control system, and stabilize a given hybrid stochastic systems in the sense of mean-square exponential stability. You et al. [11] discussed not only stability of controlled systems in the sense of mean-square exponential stability (as Mao does), but also H_∞ stability and asymptotic stability in mean square and other senses. Dong [12] discussed almost sure exponential stabilization by stochastic feedback control based on discrete-time observations. However in real life, there is a time lag between state observations and true value of the current system states. Chen et al. [13] studied stabilization of hybrid neutral stochastic differential delay equations by delay feedback control. Mao et al. [14] and Hu et al. [15] investigated stabilization of hybrid SDEs by delay feedback control. Li et al. [16] discussed the high non-linear hybrid stochastic delay differential equations by Lyapunov function. Qiu et al. [17] and Zhu et al. [18] took both discrete time and delay into account when designing the controller, they studied

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exponential stability problem of hybrid SDEs by feedback control based on discrete-time state observations with a time delay. Song et al. [19] studied stabilization based on discrete-time observations of state and mode.

In order to reduce the cost of continuous working time of the controller, intermittent control is an efficient strategy to stabilize the unstable systems (see, e.g. [20, 21]). Intermittent control divides time into two parts: working time and rest time. The controller runs at working time, closes at rest. In other words, the controlled system can be regarded as a transformation between the closed loop system and the open loop system. Obviously, it reduces the control cost in practical application; At the same time, it improves the control efficiency and operability. Zhang et al. [22] applied intermittent stochastic noise to stabilize the non-linear differential equation, and established a class of theory about intermittent stochastic disturbance stability. Ren et al. [23] showed the quasi-sure exponential stabilization of non-linear differential equations via intermittent G-Brownian motion. Liu et al. [24, 25] investigated the stochastic stabilization based on the intermittent control strategy with discrete-time feedback or time delay feedback. Mao et al. [26] studied the stabilization by intermittent control for hybrid stochastic differential delay equations. Yin et al. [27] discussed the almost sure exponential stabilization of non-linear differential equations by intermittent stochastic perturbation with jumps. Recently, intermittent control has been applied in many fields, such as complex network (see, e.g. [28–30]), multi-agent system (see, e.g. [31]), synchronization of memory neural network (see, e.g. [32]) and so on.

In order to obtain better control effect, more and more scholars have studied the stabilization problem of using hybrid strategy (also known as two or more control strategies at the same time, see, e.g. [29, 31]). Here, we take both discrete-time state observation with delay and intermittent control strategy into account when designing the controller, there are rare literatures on this topic.

The hybrid design strategy of controller is very novel and it has not been applied to stabilize an unstable non-linear hybrid stochastic system. We will design the feedback controller based on discrete-time state observations, time delay and the intermittent control strategy. Taking both the observation time lag and the observation frequency into account, we can make the decision on designing the controller. Under the different assumptions, we will prove the controlled hybrid system is exponential stable by some stochastic analysis techniques and dynamical property.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries. In Section 3, we investigate the intermittent stabilization for hybrid stochastic systems by feedback controls based on discrete-time state observations with time delay. While in Section 4 we give two examples to illustrate our theory.

2 | PRELIMINARIES

Here, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual condition. Let $B(t) =$

$(B_1(t), \dots, B_m(t))^T$ be an m -dimensional Brownian motion defined on the probability space. Let $r(t), t \geq 0$ denote a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & i \neq j, \\ 1 + \gamma_{ij}\Delta + o(\Delta), & i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Let τ_0 be a positive number. Denote by $C([-\tau_0, 0], R^n)$ the family of continuous functions $\xi : [-\tau_0, 0] \rightarrow R^n$ with the norm $\|\xi\| = \sup_{-\tau_0 \leq x \leq 0} |\xi(x)|$. For $p > 0$ and $t \geq 0$, $L^p_{\mathcal{F}_t}([-\tau_0, 0]; R^n)$ denote the family of all \mathcal{F}_t -measurable $C([-\tau_0, 0], R^n)$ -valued random variables $\xi = \{\xi(x) : -\tau_0 \leq x \leq 0\}$ such that $\mathbb{E}\|\xi\|^p < \infty$, where \mathbb{E} is the expectation with respect to the probability measure \mathbb{P} .

Consider an unstable hybrid SDE:

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t), \tag{2.1}$$

on $t \geq 0$ with initial value on $x(0) = x_0$ and $r(0) = r_0$, where $f : R^n \times S \times R_+ \rightarrow R^n$ and $g : R^n \times S \times R_+ \rightarrow R^{n \times m}$, for the unstable hybrid SDE (2.1), we aim to design the control function with some time delay $\tau_0 > 0$ and the gap of the discrete-time state observation $\tau > 0$ to make it stable. Moreover, we will combine stabilization technology with the intermittent control strategy, Thus, the controlled system is as follows:

$$\begin{aligned} dx(t) &= (f(x(t), r(t), t) + u(x(\delta_t), r(t), t)I(t))dt \\ &\quad + g(x(t), r(t), t)dB(t), \end{aligned} \tag{2.2}$$

where $\delta_t = [t/\tau]\tau - \tau_0$ and

$$I(t) = \sum_{k=0}^{\infty} I_{[t_k, t_k + \theta\Delta)}(t).$$

$t_k = k\Delta, \Delta > 0$ and $\theta \in [0, 1]$.

Noting we naturally impose the initial data

$$x(u) : -\tau_0 \leq u \leq 0 = \xi \in C([-\tau_0, 0]; R^n) \text{ and } r(0) = r_0 \in S. \tag{2.3}$$

Here, the coefficients are assumed to satisfy the following assumption.

Assumption 2.1. There exist three positive constants β_1, β_2 , and β_3 such that

$$|f(x, i, t) - f(y, i, t)| \leq \beta_1|x - y|,$$

$$|u(x, i, t) - u(y, i, t)| \leq \beta_2|x - y|,$$

$$|g(x, i, t) - g(y, i, t)| \leq \beta_3|x - y|,$$

for all $x, y \in R^n$ and $i \in S$, we assume that $f(0, i, t) = 0$, $u(0, i, t) = 0$, $g(0, i, t) = 0$ for all $t \geq 0$.

We see this assumption implies

$$\begin{aligned} |f(x, i, t)| &\leq \beta_1 |x|, & |u(x, i, t)| &\leq \beta_2 |x|, \\ |g(x, i, t)| &\leq \beta_3 |x|. \end{aligned} \quad (2.4)$$

3 | MAIN RESULTS

The stabilization problem by intermittent feedback controls based on discrete-time observations with a time delay could be transferred to the classic stabilization problem by intermittent feedback controls without discrete-time observations state and delay, the form of function is as follows (3.1). In [15], Hu et al. used this approach to build up the connection between the delay feedback control and the control function without delay.

$$\begin{aligned} dy(t) &= (f(y(t), r(t), t) + u(y(t), r(t), t)I(t))dt \\ &+ g(y(t), r(t), t)dB(t), \end{aligned} \quad (3.1)$$

where

$$I(t) = \sum_{k=0}^{\infty} I_{[t_k, t_k + \theta\Delta)}(t).$$

$t_k = k\Delta$, $\Delta > 0$, and $\theta \in [0, 1]$.

Assumption 3.1. Let $p > 0$, assume that there are nonnegative numbers b_i, λ_i and $\alpha_i, i \in S$, such that

$$\frac{x^T u(x, i, t)}{|x|^2} \leq -b_i,$$

and

$$\frac{1}{|x|^2} \left(x^T f(x, i, t) + \frac{1}{2} |g(x, i, t)|^2 \right) - \frac{2-p}{2|x|^4} |x^T g(x, i, t)|^2 \leq \alpha_i,$$

for all $(x, i, t) \in (R^n - \{0\}) \times S \times R_+$.

$$\alpha_i - b_i \leq -\lambda_i.$$

Assumption 3.2. There is a constant $p > 0$ such that the $N \times N$ matrix

$$\mathcal{A}(p) = \text{diag}(\rho_1(p), \dots, \rho_N(p)) - \Gamma \quad (3.2)$$

is a non-singular M-matrix, where

$$\rho_i(p) = \lambda_i p.$$

Define

$$(\varphi_1, \dots, \varphi_N)^T = \mathcal{A}^{-1}(p)(1, \dots, 1)^T. \quad (3.3)$$

Let

$$\gamma = \max_{i \in S} (\alpha_i + \lambda_i), \quad C_1 = \min_{i \in S} \varphi_i, \quad C_2 = \max_{i \in S} \varphi_i. \quad (3.4)$$

Remark 3.3. Under Assumption 2.1, $y(t; y_0, r_0, t_0)$ denotes the solution of the hybrid stochastic system (3.1), we can hence highlight a significant property given in Mao [[3], Lemma 5.1], which then leads to

$$\mathbb{P}\{y(t; y_0, r_0, t_0) \neq 0 \text{ on } t \geq t_0\} = 1.$$

For all $(x, i, t) \in (R^n - \{0\}) \times S \times R_+$, that is, if any initial solution of system (3.1) is a non-zero state, almost all the trajectories of system (3.1) will never converge to the origin. Thus, Lyapunov functions can be chosen in a variety of ways.

Lemma 3.4. Under Assumptions 3.1 and 3.2, when $1 - \frac{1}{p\gamma C_2} \vee 0 < \theta < 1$ ($p\gamma C_2 > 1$ is almost impossible), the solution of the controlled system has the property that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|y(t)|^p) < 0.$$

Therefore, the controlled SDE (3.1) is exponentially stable in L^p .

Proof. We first consider $y(t) \neq 0$ a.s for any $t \geq 0$ (see [3], Lemma 5.1 on page 164). Define a function $V : (R^n - \{0\}) \times S \times R_+$ by $V(y, i, t) = \varphi_i |y|^p$,

$$\begin{aligned} LV(y, i, t) &= \varphi_i p |y|^{p-1} \left[\frac{1}{|y|^2} (y^T (f(y, i, t) + u(y, i, t)I(t)) \right. \\ &+ \frac{1}{2} |g(y, i, t)|^2) - \frac{(2-p)}{2|y|^4} |y^T g(y, i, t)|^2 \left. \right] \\ &+ \sum_{j=1}^N \gamma_{ij} \varphi_j |y|^p, \end{aligned}$$

when $t \in [t_k, t_k + \theta\Delta)$, $I = 1$,

$$\begin{aligned} LV(y, i, t) &\leq -\varphi_i \rho_i(p) |y|^p + \sum_{j=1}^N \gamma_{ij} \varphi_j |y|^p \\ &\leq -|y|^p \left(\rho_i(p) \varphi_i - \sum_{j=1}^N \gamma_{ij} \varphi_j \right). \end{aligned}$$

However, by (3.3) and (3.2)

$$\rho_i(p) \varphi_i - \sum_{j=1}^N \gamma_{ij} \varphi_j = 1.$$

Hence

$$LV(y, i, t) \leq -|y|^p = -\varphi_i |y|^p \cdot \frac{1}{\varphi_i} < -\frac{1}{C_2} V(y, i), \quad (3.5)$$

when $t \in (t_k + \theta\Delta, t_{k+1})$, $I = 0$,

$$\begin{aligned} LV(y, i, t) &\leq \varphi_i p |y|^p \alpha_i + \sum_{j=1}^N \gamma_{ij} \varphi_j |y|^p \\ &\leq (\gamma - \lambda_i) \varphi_i p |y|^p + \sum_{j=1}^N \gamma_{ij} \varphi_j |y|^p \\ &\leq p\gamma |y|^p \varphi_i - p\lambda_i \varphi_i |y|^p + \sum_{j=1}^N \gamma_{ij} \varphi_j |y|^p \\ &\leq p\gamma |y|^p \varphi_i - |y|^p \left(\rho_i(p) \varphi_i - \sum_{j=1}^N \gamma_{ij} \varphi_j \right). \end{aligned}$$

Since

$$\rho_i(p) \varphi_i - \sum_{j=1}^N \gamma_{ij} \varphi_j = 1,$$

$$\begin{aligned} LV(y, i, t) &\leq \varphi_i |y|^p p\gamma - |y|^p = \varphi_i |y|^p \left(p\gamma - \frac{1}{\varphi_i} \right) \\ &\leq \left(p\gamma - \frac{1}{C_2} \right) V(y, i). \end{aligned} \quad (3.6)$$

From (3.5) and (3.6), we get

$$LV(y, i, t) \leq \left[-\frac{1}{C_2} I(t) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(t)) \right] V(y, i).$$

For each integer $d \geq 1$, define a stopping time $\rho_d = \inf\{t \geq t_0 : |y(t)| \geq d\}$. Clearly, $\rho_d \rightarrow \infty$ almost surely as $d \rightarrow \infty$. For $t \geq t_0$, $t_0 = 0$, the generalized Itô formula shows that

$$\begin{aligned} &\mathbb{E} \left[V(y(t \wedge \rho_d), r(t \wedge \rho_d)) e^{-\int_0^{t \wedge \rho_d} \left[-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right] ds} \right] \\ &= EV(y_0, r_0) + \mathbb{E} \int_0^{t \wedge \rho_d} e^{-\int_0^s \left[-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right] du} \\ &\quad \times \left(LV(y(s), r(s)) - \left[-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right] \right. \\ &\quad \left. \times V(y(s), r(s)) \right) ds. \end{aligned}$$

We have

$$\begin{aligned} &\mathbb{E} \left[V(y(t \wedge \rho_d), r(t \wedge \rho_d)) e^{-\int_0^{t \wedge \rho_d} \left[-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right] ds} \right] \\ &\leq EV(y_0, r_0), \end{aligned}$$

when $d \rightarrow \infty$, we get

$$\mathbb{E} \left[V(y(t), r(t)) e^{-\int_0^t \left[-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right] ds} \right] \leq EV(y_0, r_0).$$

This implies

$$C_1 \mathbb{E}|y(t)|^p \leq C_2 \mathbb{E}|y_0|^p e^{\int_0^t \left[-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right] ds}.$$

By $1 - \frac{1}{p\gamma C_2} \vee 0 < \theta < 1$. Let $t_k = k\Delta$, $k > N$, $-\frac{1}{C_2} \leq -\frac{1}{C_2} \theta \leq \frac{1}{C_2} \theta + \left(p\gamma - \frac{1}{C_2} \right) (1 - \theta)$, hence when $t \in [t_k, k\Delta + \theta\Delta)$, let

$$\begin{aligned} &\int_0^t \left(-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right) ds \\ &= \left(p\gamma - \frac{1}{C_2} \right) t - p\gamma (k\theta\Delta + t - k\Delta) \\ &= \left(p\gamma - \frac{1}{C_2} - p\gamma\theta \right) k\Delta - \frac{1}{C_2} (t - k\Delta) \\ &\leq \left(p\gamma - \frac{1}{C_2} - p\gamma\theta \right) t, \end{aligned}$$

when $t \in [k\Delta + \theta\Delta, t_{k+1})$,

$$\begin{aligned} &\int_0^t \left(-\frac{1}{C_2} I(s) + \left(p\gamma - \frac{1}{C_2} \right) (1 - I(s)) \right) ds \\ &= \left(p\gamma - \frac{1}{C_2} \right) t - p\gamma (k + 1)\theta\Delta \\ &\leq \left(p\gamma - \frac{1}{C_2} \right) t - p\gamma\theta t \\ &\leq \left(p\gamma - \frac{1}{C_2} - p\gamma\theta \right) t. \end{aligned}$$

$$E|y(t)|^p \leq \frac{C_2}{C_1} |y_0|^p \exp \left\{ \left(p\gamma - \frac{1}{C_2} - p\gamma\theta \right) t \right\}.$$

This implies

$$E|y(t)|^p \leq C_3 |y_0|^p e^{-\tilde{\gamma}t}.$$

We have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|y(t)|^p) \leq \left(p\gamma - \frac{1}{C_2} - p\gamma\theta \right) < 0.$$

□

Lemma 3.5. *When Assumptions 2.1, 3.1, 3.2 and Lemma 3.4 hold, then the solution of the controlled system has the property that*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |y(t)| < 0, \text{ a.s.}$$

Therefore, the controlled SDE is almost surely exponentially stable.

Proof. Applying Itô formula again on $|y(t)|^p$, gives

$$\begin{aligned} \mathbb{E}|y(t)|^p &\leq |y(0)|^p \\ &+ \mathbb{E} \int_0^t p|y(s)|^{p-2} \left[\frac{1}{|y(s)|^2} (y(s)^T (f(y(s), r(s)) + u(y(s), r(s))I(t))) \right. \\ &\left. + \frac{1}{2} |g(y(s), r(s))|^2 - \frac{(2-p)}{2|y(s)|^4} |y(s)^T g(y(s), r(s))|^2 \right] ds. \end{aligned}$$

By Assumption 3.1, however when $t \in [t_k, t_k + \theta\Delta)$, $I = 1$,

$$\mathbb{E}|y(t)|^p \leq |y(0)|^p - \lambda_i p \int_0^t |y(s)|^p ds,$$

when $t \in [t_k + \theta\Delta, t_{k+1})$, $I = 0$,

$$\mathbb{E}|y(t)|^p \leq |y(0)|^p + \alpha_i p \int_0^t |y(s)|^p ds.$$

By (3.4), we have

$$\sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^p \leq |y(0)|^p + \gamma p \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|y(u)|^p ds. \quad (3.7)$$

The well-known Gronwall inequality yields that

$$\sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^p \leq |y(0)|^p \exp\{\gamma p t\}. \quad (3.8)$$

By (3.7) and (3.8), we have

$$\int_0^\Delta \sup_{0 \leq u \leq s} \mathbb{E}|y(u)|^p ds = (\exp\{\gamma p \Delta\} - 1) \gamma^{-1} p^{-1} \mathbb{E}|y(0)|^p.$$

Hence, by Hölder inequality, we have

$$\begin{aligned} \int_0^\Delta \sup_{0 \leq u \leq s} \mathbb{E}|y(u)|^2 ds &\leq \left(\int_0^\Delta \sup_{0 \leq u \leq s} \mathbb{E}|y(u)|^p ds \right)^{2/p} \\ &= (\exp\{\gamma p \Delta\} - 1)^{2/p} \gamma^{-2/p} p^{-2/p} \mathbb{E}|y(0)|^2. \end{aligned} \quad (3.9)$$

Let k be any non-negative integer. We first prove the following equation for $p = 2$, we have

$$\begin{aligned} \mathbb{E} \left(\sup_{t_k \leq t \leq t_{k+1}} |y(t)|^2 \right) &\leq 3\mathbb{E}|y(t_k)|^2 + 3\mathbb{E} \int_{t_k}^{t_{k+1}} (|f(y(t), r(t), t) \\ &+ u(y(t), r(t))I(t)|^2) dt \\ &+ 3\mathbb{E} \left(\sup_{t_k \leq s \leq t_{k+1}} \int_{t_k}^s |g(y(s), r(s), s)|^2 dB(s) \right)^2. \end{aligned} \quad (3.10)$$

By Assumption 2.1, we can obtain that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq \Delta} |y(t)|^2 \right) &\leq 3\mathbb{E}|y(0)|^2 + 3\Delta \int_0^\Delta \mathbb{E}(|f(y(t), r(t), t)|^2 \\ &+ |u(y(t), r(t))I(t)|^2) dt \\ &+ 12 \int_0^\Delta \mathbb{E}|g(y(t), r(t), t)|^2 dt \leq 3\mathbb{E}|y(0)|^2 \\ &+ (3\Delta\beta_1^2 + 12\beta_3^2) \int_0^\Delta \mathbb{E}(|y(t)|^2) dt \\ &+ 3\Delta\beta_2^2 \int_0^\Delta \sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^2 dt \\ &\leq 3\mathbb{E}|y(0)|^2 + (3\Delta\beta_1^2 + 3\Delta\beta_2^2 + 12\beta_3^2) \\ &\times \int_0^\Delta \sup_{0 \leq s \leq t} \mathbb{E}|y(s)|^2 dt. \end{aligned}$$

By (3.9), we have

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq \Delta} |y(t)|^2 \right) &\leq 3\mathbb{E}|y(0)|^2 \\ &+ (3\Delta\beta_1^2 + 3\Delta\beta_2^2 + 12\beta_3^2) \\ &\times (\exp\{\gamma p \Delta\} - 1)^{2/p} \gamma^{-2/p} p^{-2/p} \mathbb{E}|y(0)|^2. \end{aligned}$$

Which means

$$\mathbb{E} \left(\sup_{0 \leq t \leq \Delta} |y(t)|^p \right) \leq C_4 \mathbb{E}|y(0)|^p.$$

Where

$$C_4 = 3 + (3\Delta\beta_1^2 + 3\Delta\beta_2^2 + 12\beta_3^2) (\exp\{\gamma p \Delta\} - 1)^{2/p} \gamma^{-2/p} p^{-2/p}.$$

Repeating the above procedure, we get

$$\mathbb{E} \left(\sup_{i\Delta \leq t \leq (i+1)\Delta} |y(t)|^p \right) \leq C_4 \mathbb{E}|y(i\Delta)|^p.$$

By Chebyshev inequality, we get

$$\mathbb{P}\left(\sup_{i\Delta \leq t \leq (i+1)\Delta} |y(t)|^p \geq e^{-0.5\bar{\gamma}i\Delta}\right) \leq C_4 e^{-0.5\bar{\gamma}i\Delta} \mathbb{E}|y(0)|^p.$$

By Borel–Cantelli Lemma show that for almost all $\omega \in \Omega$, there is a positive integer $i_0 = i_0(\omega)$ such that

$$\sup_{i\Delta \leq t \leq (i+1)\Delta} |y(t)|^p \leq e^{-0.5\bar{\gamma}i\Delta}, \quad \forall i > i_0, a.s.$$

So, for almost all $\omega \in \Omega$,

$$\frac{1}{t} \log |y(t)| \leq -\frac{0.5\bar{\gamma}i\Delta}{(i+1)\Delta}, \quad i\Delta \leq t \leq (i+1)\Delta, \quad a.s.$$

We have

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |y(t)| \leq -0.5\bar{\gamma} < 0, a.s.$$

□

To prove Theorem 3.8, we present some lemmas.

Lemma 3.6. *When Assumption 2.1 holds, for any $T > 0$ such that*

$$\sup_{0 \leq t \leq T + \tau_1} \mathbb{E}|x(t)|^p \leq H_1(p, \tau_1, T) \mathbb{E}\|x(0)\|^p, \quad (3.11)$$

$$\mathbb{E}\left(\sup_{0 \leq t \leq T + \tau_1} |x(t)|^p\right) \leq H_2(p, \tau_1, T) \mathbb{E}\|x(0)\|^p, \quad (3.12)$$

$$\sup_{0 \leq t \leq T} \mathbb{E}\left(\sup_{0 \leq u \leq \tau_1} |x(t+u) - x(t)|^p\right) \leq H_3(p, \tau_1, T) \mathbb{E}\|x(0)\|^p, \quad (3.13)$$

where

$$H_1(p, \tau_1, T) = \begin{cases} (1 + \tau_1 p \beta_2) e^{(T + \tau_1)p(\beta_1 + 0.5(p-1)\beta_3^2 + \beta_2 p)}, & p \geq 2, \\ (1 + 2\tau_1 \beta_2)^{p/2} e^{p(T + \tau_1)(\beta_1 + 0.5\beta_3^2 + 2\beta_2)}, & p \in (0, 2), \end{cases}$$

$$H_2(p, \tau_1, T) = \begin{cases} (3^{p-1} + (6(T + \tau_1))^{p-1} \beta_2^p) + [(6(T + \tau_1))^{p-1} \beta_1^p + (6(T + \tau_1))^{p-1} \beta_2^p \\ + 3^{p-1} \left(\frac{p^3}{2p-2}\right)^{\frac{p}{2}} (T + \tau_1)^{\frac{p-2}{2}} \beta_3^p] \times (1 + \tau_1 p \beta_2) \\ (e^{(T + \tau_1)p(\beta_1 + 0.5(p-1)\beta_3^2 + \beta_2 p)} - 1) (\beta_1 p + 0.5p(p-1)\beta_3^2 p + p^2 \beta_2)^{-1}, & p \geq 2, \\ \left\{ (3 + (6(T + \tau_1))\beta_2^2) + [(6(T + \tau_1))\beta_1^2 + (6(T + \tau_1))\beta_2^2 + 12(T + \tau_1)\beta_3^2] \right. \\ \left. \times (1 + 2\tau_1 \beta_2) (e^{2(T + \tau_1)(\beta_1 + 0.5\beta_3^2 + 2\beta_2)} - 1) (2\alpha + 4\beta_2)^{-1} \right\}^{p/2}, & p \in (0, 2), \end{cases}$$

$$H_3(p, \tau_1, T)$$

$$= \begin{cases} 3^{p-1} (1 + \tau_1 p \beta_2) e^{(T + \tau_1)p(\beta_1 + 0.5(p-1)\beta_3^2 + \beta_2 p)} \\ \times \left[\tau_1^p (\beta_1^p + \beta_2^p) + \left(\frac{p^3}{2(p-1)}\right)^{\frac{p}{2}} \tau_1^2 \beta_3^p \right], & p \geq 2, \\ \left\{ 3(1 + 2\tau_1 \beta_2) e^{(T + \tau_1)2(\beta_1 + 0.5\beta_3^2 + 2\beta_2)} \right. \\ \left. \times [\tau_1^2 (\beta_1^2 + \beta_2^2) + 4\tau_1 \beta_3^2] \right\}^{p/2}, & p \in (0, 2). \end{cases}$$

Proof. We first prove (3.11) for $p \geq 2$, and we only consider the deterministic data $\xi \in C([-\tau_1, 0], R^n)$, where $\tau_1 = \tau_0 + \tau$, by the Itô formula, we have

$$\begin{aligned} \mathbb{E}|x(t)|^p &\leq |x(0)|^p \\ &+ \mathbb{E} \int_0^t p|x(s)|^{p-2} [x(s)^T f(x, s) + 0.5(p-1)|g(x, s)|^2] ds \\ &+ \mathbb{E} \int_0^t p|x(s)|^{p-1} u(x(\delta_s)) I(s) ds. \end{aligned}$$

By Assumption 2.1, we have

$$\begin{aligned} \mathbb{E}|x(t)|^p &\leq |x(0)|^p \\ &+ (p\beta_1 + 0.5p(p-1)\beta_3^2) \mathbb{E} \int_0^t |x(s)|^p ds \\ &+ \mathbb{E} \int_0^t p\beta_2 |x(s)|^{p-1} |x(\delta_s)| ds. \end{aligned}$$

Since

$$|x(s)|^{p-1} |x(\delta_s)| \leq (p-1)|x(s)|^p + |x(\delta_s)|^p.$$

Note that

$$\begin{aligned} \int_0^t \mathbb{E}|x(\delta_s)|^p ds &\leq \int_0^t \sup_{-\tau_1 \leq u \leq s} \mathbb{E}|x(u)|^p ds \leq \tau_1 \mathbb{E}\|x(0)\|^p \\ &+ \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^p ds. \end{aligned} \quad (3.14)$$

Hence, we have

$$\begin{aligned} \sup_{-\tau_1 \leq u \leq t} \mathbb{E}|x(u)|^p &\leq (1 + \tau_1 p \beta_2) \mathbb{E}\|x(0)\|^p \\ &+ (p\beta_1 + 0.5p(p-1)\beta_3^2 + p^2\beta_2) \\ &\times \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^p ds. \end{aligned}$$

The well-known Gronwall inequality yields

$$\begin{aligned} \sup_{0 \leq u \leq T+\tau_1} \mathbb{E}|x(u)|^p \\ \leq (1 + \tau_1 p \beta_2) e^{(T+\tau_1)p(\beta_1+0.5(p-1)\beta_3^2+\beta_2p)} \mathbb{E}\|x(0)\|^p. \end{aligned} \quad (3.15)$$

For $p \in (0, 2)$, we have

$$\mathbb{E}(|x(t)|^2 | \mathcal{F}_0) \leq (1 + 2\tau_1 \beta_2) e^{(T+\tau_1)2(\beta_1+0.5\beta_3^2+2\beta_2)} \mathbb{E}\|x(0)\|^2.$$

Then

$$\begin{aligned} \mathbb{E}(|x(t)|^p | \mathcal{F}_0) &\leq (\mathbb{E}(|x(t)|^2 | \mathcal{F}_0))^{p/2} \\ &\leq (1 + \tau_1 2\beta_2)^{p/2} e^{(T+\tau_1)p(\beta_1+0.5\beta_3^2+2\beta_2)} \mathbb{E}\|x(0)\|^p. \end{aligned}$$

The proof of the assertion (3.11) is complete. Let us proceed to prove the second assertion.

Let us proceed to prove the second assertion, It is easy to show from (3.10) and (3.14) that

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq T+\tau_1} |x(s)|^p \right) \\ \leq 3^{p-1} \mathbb{E}|x(0)|^p \\ + (3(T+\tau_1))^{p-1} 2^{p-1} \int_0^{T+\tau_1} \mathbb{E}(|f(x(s), r(s), s)|^p \\ + |u(x(\delta_s), r(t))I(t)|^p) dt \\ + 3^{p-1} \left(\frac{p^3}{2p-2} \right)^{\frac{p}{2}} (T+\tau_1)^{\frac{p-2}{2}} \int_0^{T+\tau_1} \mathbb{E}|g(x(t), r(t), t)|^p dt \\ \leq 3^{p-1} \mathbb{E}|x(0)|^p + [(3(T+\tau_1))^{p-1} 2^{p-1} \beta_1^p \end{aligned}$$

$$\begin{aligned} + 3^{p-1} \left(\frac{p^3}{2p-2} \right)^{\frac{p}{2}} (T+\tau_1)^{\frac{p-2}{2}} \beta_3^p] \int_0^{T+\tau_1} \mathbb{E}(|x(t)|^p) dt \\ + (3(T+\tau_1))^{p-1} 2^{p-1} \beta_2^p \int_0^{T+\tau_1} \sup_{-\tau_1 \leq u \leq s} \mathbb{E}|x(u)|^p ds \\ \leq (3^{p-1} + (3(T+\tau_1))^{p-1} 2^{p-1} \beta_2^p) \mathbb{E}|x(0)|^p \\ + [(3(T+\tau_1))^{p-1} 2^{p-1} \beta_1^p + (3(T+\tau_1))^{p-1} 2^{p-1} \beta_2^p \\ + 3^{p-1} \left(\frac{p^3}{2p-2} \right)^{\frac{p}{2}} (T+\tau_1)^{\frac{p-2}{2}} \beta_3^p] \\ \times \int_0^{T+\tau_1} \sup_{0 \leq u \leq s} \mathbb{E}|x(u)|^p ds. \end{aligned}$$

By (3.15), we have

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq T+\tau_1} |x(s)|^p \right) \\ \leq \left(3^{p-1} + (6(T+\tau_1))^{p-1} \beta_2^p \right) \mathbb{E}\|x(0)\|^p \\ + \left[(6(T+\tau_1))^{p-1} \beta_1^p + (6(T+\tau_1))^{p-1} \beta_2^p \right. \\ \left. + 3^{p-1} \left(\frac{p^3}{2p-2} \right)^{\frac{p}{2}} (T+\tau_1)^{\frac{p-2}{2}} \beta_3^p \right] \\ \times (1 + \tau_1 p \beta_2) (e^{(T+\tau_1)p(\alpha+\beta_2p)} - 1) (\alpha p + p^2 \beta_2)^{-1} \mathbb{E}\|x(0)\|^p \\ \leq \{ (3^{p-1} + (6(T+\tau_1))^{p-1} \beta_2^p) \\ + \left[(6(T+\tau_1))^{p-1} \beta_1^p + (6(T+\tau_1))^{p-1} \beta_2^p \right. \\ \left. + 3^{p-1} \left(\frac{p^3}{2p-2} \right)^{\frac{p}{2}} (T+\tau_1)^{\frac{p-2}{2}} \beta_3^p \right] \\ \times (1 + \tau_1 p \beta_2) (e^{(T+\tau_1)p(\beta_1+0.5(p-1)\beta_3^2+\beta_2p)} - 1) \\ \left. (\beta_1 p + 0.5p(p-1)\beta_3^2 + p^2\beta_2)^{-1} \} \mathbb{E}\|x(0)\|^p. \end{aligned}$$

For $p \in (0, 2)$, we have

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq s \leq T+\tau_1} |x(s)|^2 | \mathcal{F}_0 \right) \\ \leq (3 + (6(T+\tau_1))\beta_2^2) + [(6(T+\tau_1))\beta_1^2 \\ + (6(T+\tau_1))\beta_2^2 + 12(T+\tau_1)\beta_3^2] \\ \times (1 + 2\tau_1 \beta_2) \left(e^{2(T+\tau_1)(\beta_1+0.5\beta_3^2+2\beta_2)} - 1 \right) \\ (2\beta_1 + \beta_3^2 + 4\beta_2)^{-1} \mathbb{E}\|x(0)\|^2. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E} \left(\sup_{0 \leq t \leq T + \tau_1} |x(t)|^p \middle| \mathcal{F}_0 \right) &\leq \left(\mathbb{E} \left(\sup_{0 \leq t \leq T + \tau_1} |x(t)|^2 \middle| \mathcal{F}_0 \right) \right)^{p/2} \\ &\leq \left\{ (3 + (6(T + \tau_1))\beta_2^2) + \left[(6(T + \tau_1))\beta_1^2 \right. \right. \\ &\quad \left. \left. + (6(T + \tau_1))\beta_2^2 + 12(T + \tau_1)\beta_3^2 \right] \right. \\ &\quad \times (1 + 2\tau_1\beta_2) \left. \left(e^{2(T + \tau_1)(\beta_1 + 0.5\beta_3^2 + 2\beta_2)} - 1 \right) \right. \\ &\quad \left. (2\beta_1 + \beta_3^2 + 4\beta_2)^{-1} \right\}^{p/2} \mathbb{E} \|x(0)\|^p. \end{aligned}$$

The proof is complete. \square

Similarly, we show the third assertion. Using Itô formula, Hölder inequality and Burkholder–Davis–Gundy give

$$\begin{aligned} &\mathbb{E} \left(\sup_{0 \leq u \leq \tau_1} |x(t + u) - x(t)|^p \right) \\ &\leq (3\tau_1)^{p-1} \beta_1^p \int_t^{t+\tau_1} \mathbb{E} |x(s)|^p ds + (3\tau_1)^{p-1} \beta_2^p \int_t^{t+\tau_1} \mathbb{E} |x(\delta_s)|^p ds \\ &\quad + 3^{p-1} \left(\frac{p^3}{2(p-1)} \right)^{\frac{p}{2}} \tau_1^{\frac{p-2}{2}} \beta_3^p \int_t^{t+\tau_1} \mathbb{E} |x(s)|^p ds \\ &\leq ((3\tau_1)^{p-1} \beta_1^p + 3^{p-1} \left(\frac{p^3}{2(p-1)} \right)^{\frac{p}{2}} \tau_1^{\frac{p-2}{2}} \beta_3^p) \int_t^{t+\tau_1} \mathbb{E} |x(s)|^p ds \\ &\quad + (3\tau_1)^{p-1} \beta_2^p \int_t^{t+\tau_1} \mathbb{E} |x(\delta_s)|^p ds. \end{aligned}$$

For any $t \leq s \leq t + \tau_1$, using inequality (3.15) gives that

$$\mathbb{E} |x(s)|^p \leq (1 + \tau_1 p \beta_2) e^{(t+\tau_1)p(\beta_1 + 0.5(p-1)\beta_3^2 + \beta_2 p)} \mathbb{E} \|x(0)\|^p,$$

$$\mathbb{E} |x(\delta_s)|^p \leq (1 + \tau_1 p \beta_2) e^{(t+\tau_1 - \tau_0)p(\beta_1 + 0.5(p-1)\beta_3^2 + \beta_2 p)} \mathbb{E} \|x(0)\|^p.$$

Therefore, we have

$$\begin{aligned} &\sup_{0 \leq t \leq T} \mathbb{E} \left(\sup_{0 \leq u \leq \tau_1} |x(t + u) - x(t)|^p \right) \\ &\leq 3^{p-1} (1 + \tau_1 p \beta_2) e^{(T + \tau_1)p(\beta_1 + 0.5(p-1)\beta_3^2 + \beta_2 p)} \\ &\quad \times [\tau_1^p (\beta_1^p + \beta_2^p) + \left(\frac{p^3}{2(p-1)} \right)^{\frac{p}{2}} \tau_1^{\frac{p-2}{2}} \beta_3^p] \mathbb{E} \|x(0)\|^p. \end{aligned}$$

For $p \in (0, 2)$, we have

$$\begin{aligned} &\sup_{0 \leq t \leq T} \mathbb{E} \left(\sup_{0 \leq u \leq \tau_1} |x(t + u) - x(t)|^2 \right) \\ &\leq 3(1 + 2\tau_1\beta_2) e^{(T + \tau_1)2(\beta_1 + 0.5\beta_3^2 + 2\beta_2)} \\ &\quad \times [\tau_1^2(\beta_1^2 + \beta_2^2) + 4\tau_1\beta_3^2] \mathbb{E} \|x(0)\|^2. \end{aligned}$$

Then

$$\begin{aligned} &\sup_{0 \leq t \leq T} \mathbb{E} \left(\sup_{0 \leq u \leq \tau_1} |x(t + u) - x(t)|^p \right) \\ &\leq \left\{ 3(1 + 2\tau_1\beta_2) e^{(T + \tau_1)2(\beta_1 + 0.5\beta_3^2 + 2\beta_2)} \right. \\ &\quad \left. \times [\tau_1^2(\beta_1^2 + \beta_2^2) + 4\tau_1\beta_3^2] \right\}^{p/2} \mathbb{E} \|x(0)\|^p. \end{aligned}$$

Lemma 3.7. *When Assumption 2.1 hold and $T > 0$, let $y(t; x_0, r_0, 0) = y(t)$, Then, for $t \in [0, T + \tau_1]$ such that*

$$\mathbb{E} |x(t) - y(t)|^p \leq H_4(p, \tau, T) \mathbb{E} \|x(0)\|^p,$$

where

$$H_4(p, \tau, T) = \begin{cases} 6^{p-1} \beta_2^p (T + \tau_1)^p H_3(p, \tau_1, T) \\ \quad \times e^{[3^{p-1} \beta_1^p + 3^{p-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} (T + \tau_1)^{-\frac{p}{2}} \beta_3^p + 6^{p-1} \beta_2^p] (T + \tau_1)^p}, & p \geq 2. \\ (6\beta_2^2 (T + \tau_1)^2 H_3(2, \tau_1, T) \\ \quad \times e^{[3\beta_1^2 + 3(T + \tau_1)^{-1} \beta_3^2 + 6\beta_2^2] (T + \tau_1)^2})^{\frac{p}{2}}, & p \in (0, 2). \end{cases}$$

$H_3(p, \tau, T)$ has been defined in Lemma 3.6.

Proof.

$$\begin{aligned} x(t) - y(t) &= \int_0^t (f(x(s)) - f(y(s))) ds \\ &\quad + \int_0^t (u(x(\delta_s)) - u(y(s))) I(s) ds \\ &\quad + \int_0^t (g(x(s)) - g(y(s))) dB(s). \end{aligned}$$

Thus

$$\begin{aligned} &\mathbb{E} |x(t) - y(t)|^p \\ &\leq (3t)^{p-1} \int_0^t \mathbb{E} |f(x(s)) - f(y(s))|^p ds + (3t)^{p-1} \\ &\quad \int_0^t \mathbb{E} |u(x(\delta_s)) - u(y(s))|^p ds \\ &\quad + 3^{p-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} t^{\frac{p-2}{2}} \int_0^t \mathbb{E} |g(x(s)) - g(y(s))|^p ds \\ &\leq (3t)^{p-1} \beta_1^p \int_0^t \mathbb{E} |x(s) - y(s)|^p ds \\ &\quad + (6t)^{p-1} \beta_2^p \left(\int_0^t \mathbb{E} |x(\delta_s) - x(s)|^p ds \right) \end{aligned}$$

$$\begin{aligned}
& + \int_0^t \mathbb{E}|x(s) - y(s)|^p ds \\
& + 3^{p-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} t^{\frac{p-2}{2}} \beta_3^p \int_0^t \mathbb{E}|x(s) - y(s)|^p ds \\
& \leq \left[(3t)^{p-1} \beta_1^p + 3^{p-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} t^{\frac{p-2}{2}} \beta_3^p \right. \\
& \quad \left. + (6t)^{p-1} \beta_2^p \right] \int_0^t \mathbb{E}|x(s) - y(s)|^p ds \\
& \quad + (6t)^{p-1} \beta_2^p \int_0^t \mathbb{E}|x(\delta_s) - x(s)|^p ds.
\end{aligned}$$

Let $\Phi(t) = \mathbb{E}|x(t) - y(t)|^p$, we have

$$\begin{aligned}
\Phi(t) & \leq \left[(3t)^{p-1} \beta_1^p + 3^{p-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} t^{\frac{p-2}{2}} \beta_3^p \right. \\
& \quad \left. + (6t)^{p-1} \beta_2^p \right] \int_0^t \Phi(s) ds \\
& \quad + (6t)^{p-1} \beta_2^p (T + \tau_1) H_3(p, \tau_1, T) \mathbb{E}\|x(0)\|^p.
\end{aligned}$$

By the Gronwall inequality, we have

$$\begin{aligned}
\mathbb{E}|x(t) - y(t)|^p & \leq (6t)^{p-1} \beta_2^p (T + \tau_1) H_3(p, \tau_1, T) \\
& \quad \times e^{[(3t)^{p-1} \beta_1^p + 3^{p-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} t^{\frac{p-2}{2}} \beta_3^p + (6t)^{p-1} \beta_2^p] (T + \tau_1)} \\
\mathbb{E}\|x(0)\|^p & \leq 6^{p-1} \beta_2^p (T + \tau_1)^p H_3(p, \tau_1, T) \\
& \quad \times e^{[3^{p-1} \beta_1^p + 3^{p-1} \left(\frac{p(p-1)}{2} \right)^{\frac{p}{2}} (T + \tau_1)^{-\frac{p}{2}} \beta_3^p + 6^{p-1} \beta_2^p] (T + \tau_1)^p \mathbb{E}\|x(0)\|^p}.
\end{aligned}$$

Let us consider the case when $p \in (0, 2)$. Similarly to how Lemma 3.6 was proved. For $t \in [0, T + \tau_1]$, we can show that

$$\begin{aligned}
\mathbb{E}|x(t) - y(t)|^p & \leq 6\beta_2^2 (T + \tau_1)^2 H_3(2, \tau_1, T) \\
& \quad e^{[3\beta_1^2 + 3(T + \tau_1)^{-1} \beta_3^2 + 6\beta_2^2] (T + \tau_1)^2 \mathbb{E}\|x(0)\|^p}.
\end{aligned}$$

The proof is therefore complete. \square

Theorem 3.8. *Let Assumptions 2.1, 3.1 and 3.2 hold. Choose a free parameter $\zeta \in (0, 1)$ and take $T = \frac{1}{\gamma} \log\left(\frac{2^{2\hat{p}} C_2}{C_1 \zeta}\right)$. Let $\tau^* > 0$ be the*

unique root to the following equation:

$$\begin{aligned}
& \zeta(1 + \tau_1 p_1 \beta_2)^{0.5\hat{p}_2} e^{\tau_1 p(\beta_1 + 0.5(p_1 - 1)\beta_3^2 + \beta_2 p_1)} \\
& \quad + 2^{\hat{p}} (2^{\hat{p}} H_4(p, \tau_1, T) + H_3(p, \tau_1, T)) = 1. \quad (3.16)
\end{aligned}$$

Where $p_1 = 2 \vee p$, $p_2 = 2 \wedge p$, $\hat{p} = 0 \vee (p - 1)$, and $p > 0$, then for each $\tau_1 \in (0, \tau^*)$, we can choose a period of the intermittent control Δ such that $\Delta = (T + 2\tau_1)/N\tau_1$ and $\theta \in (1 - \frac{1}{C_2 p \gamma}, 1)$ in order for the controlled system (2.2) to be exponentially stable in p th moment and in probability one.

Proof. We will simply write $H_3(p, \tau_1, T) = H_3$ and $H_4(p, \tau_1, T) = H_4$. Fix $\tau_1 \in (0, \tau^*)$ and the initial data (2.3). For simplicity, we write $x(t; \xi, r_0, 0) = x(t)$, $r(t; r_0, 0) = r(t)$ for $t \geq 0$. Likewise, we write $y(\tau_1 + T; \tau_1, x(\tau_1), r(\tau_1)) = y(\tau_1 + T)$. By Lemmas 3.4 and 3.6, let $p_1 = 2 \vee p$, $p_2 = 2 \wedge p$, $C_1 = \min_{i \in S} \varphi_i$, $C_2 = \max_{i \in S} \varphi_i$, we have

$$\begin{aligned}
& \mathbb{E}|y(\tau_1 + T)|^p \\
& \leq \frac{C_2}{C_1} \mathbb{E}|y(\tau_1)|^p e^{-\gamma T} \leq \frac{C_2}{C_1} (1 + \tau_1 p_1 \beta_2)^{\frac{p_2}{2}} \\
& \quad e^{\tau_1 p(\beta_1 + 0.5(p_1 - 1)\beta_3^2 + \beta_2 p_1)} \mathbb{E}\|x(0)\|^p e^{-\gamma T}. \quad (3.17)
\end{aligned}$$

Moreover, by the elementary inequality $(a + b)^p \leq 2^{\hat{p}}(a^p + b^p)$ for any $a, b \geq 0$ and $\hat{p} = 0 \vee (p - 1)$, we have $\mathbb{E}|x(\tau_1 + T)|^p \leq \mathbb{E}|y(\tau_1 + T)|^p + \mathbb{E}|x(\tau_1 + T) - y(\tau_1 + T)|^p$. By (3.17) and Lemma 3.7, we have

$$\begin{aligned}
\mathbb{E}|x(\tau_1 + T)|^p & \leq 2^{\hat{p}} \left(\frac{C_2}{C_1} (1 + \tau_1 p_1 \beta_2)^{\frac{p_2}{2}} \right. \\
& \quad \left. e^{\tau_1 p(\beta_1 + 0.5(p_1 - 1)\beta_3^2 + \beta_2 p_1)} e^{-\gamma T} + H_4 \right) \mathbb{E}\|x(0)\|^p. \quad (3.18)
\end{aligned}$$

On the other hand, by Lemma 3.6, we have

$$\begin{aligned}
\mathbb{E}\|x_{2\tau_1+T}\|^p & \leq 2^{\hat{p}} (\mathbb{E}|x(\tau_1 + T)|^p \\
& \quad + \mathbb{E}(\sup_{0 \leq u \leq \tau_1} |x(\tau_1 + T) - x(u + \tau_1 + T)|^p)) \\
& \leq 2^{\hat{p}} (\mathbb{E}|x(\tau_1 + T)|^p + H_3 \mathbb{E}\|x(0)\|^p). \quad (3.19)
\end{aligned}$$

Insert (3.18) into (3.19). Let $\zeta = 2^{2\hat{p}} \frac{C_2}{C_1} e^{-\gamma T}$, we obtain that

$$\begin{aligned}
\mathbb{E}\|x_{2\tau_1+T}\|^p & \leq [\zeta(1 + \tau_1 p_1 \beta_2)^{\frac{p_2}{2}} e^{\tau_1 p(\beta_1 + 0.5(p_1 - 1)\beta_3^2 + \beta_2 p_1)} \\
& \quad + 2^{\hat{p}} (2^{\hat{p}} H_4 + H_3)] \mathbb{E}\|x(0)\|^p. \quad (3.20)
\end{aligned}$$

But, as $\tau_1 < \tau^*$, we see from (3.16) that

$$\zeta(1 + \tau_1 p_1 \beta_2)^{\frac{p_2}{2}} e^{\tau_1 p(\beta_1 + 0.5(p_1 - 1)\beta_3^2 + \beta_2 p_1)} + 2^{\hat{p}} (2^{\hat{p}} H_4 + H_3) < 1.$$

We may therefore write

$$\zeta(1 + \tau_1 p_1 \beta_2)^{\frac{p_2}{2}} e^{\tau_1 p(\beta_1 + 0.5(p_1 - 1)\beta_3^2 + \beta_2 p_1)} + 2^{\hat{p}}(2^{\hat{p}} H_4 + H_3) = e^{-\lambda(2\tau_1 + T)}$$

for some $\lambda > 0$. We see from (3.20) that

$$\mathbb{E} \|x_{2\tau_1 + T}\|^p \leq e^{-\lambda(2\tau_1 + T)} \mathbb{E} \|x(0)\|^p. \tag{3.21}$$

Let us proceed to consider the solution $x(t)$ on $t \geq 2\tau_1 + T$. There exists a τ_1 and a positive constant N such that $T + 2\tau_1 = N\Delta\tau_1$. By the flow property, this can be regarded as the solution of Equation (2.2) with the initial data $x_{N\Delta\tau_1}$ and $r(N\Delta\tau_1)$ at $t = N\Delta\tau_1$. In the same way as above, we can show that

$$\mathbb{E} \|x_{2N\Delta\tau_1}\|^p \leq e^{-\lambda N\Delta\tau_1} \mathbb{E} \|x_{N\Delta\tau_1}\|^p.$$

By (3.21), this implies

$$\mathbb{E} \|x_{2N\Delta\tau_1}\|^p \leq e^{-2\lambda N\Delta\tau_1} \mathbb{E} \|x(0)\|^p.$$

Repeating this procedure, we have

$$\mathbb{E} \|x_{kN\Delta\tau_1}\|^p \leq e^{-k\lambda N\Delta\tau_1} \mathbb{E} \|x(0)\|^p.$$

For all $k = 1, 2, \dots$. Now, by Lemma 3.6, we have

$$\mathbb{E} \left(\sup_{kN\Delta\tau_1 \leq t \leq (k+1)N\Delta\tau_1} \mathbb{E} |x(t)|^p \right) \leq H_2(p, \tau_1, N\Delta\tau_1 - \tau_1) e^{-k\lambda N\Delta\tau_1} \mathbb{E} \|x(0)\|^p, \tag{3.22}$$

for all $k = 0, 1, 2, \dots$. Hence, for $t \in [kN\Delta\tau_1, (k + 1)N\Delta\tau_1]$,

$$\begin{aligned} & \frac{1}{t} \log(\mathbb{E} |x(t)|^p) \\ & \leq \frac{\log(H_2(p, \tau_1, N\Delta\tau_1 - \tau_1) \mathbb{E} \|x(0)\|^p) - k\lambda N\Delta\tau_1}{kN\Delta\tau_1}. \end{aligned}$$

This implies

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E} |x(t)|^p \leq -\lambda.$$

Using Markov inequality and (3.22), we get

$$\begin{aligned} & \mathbb{P} \left(\sup_{kN\Delta\tau_1 \leq t \leq (k+1)N\Delta\tau_1} |x(t)|^p \geq e^{-0.5k\lambda N\Delta\tau_1} \right) \\ & \leq H_2(p, \tau_1, N\Delta\tau_1 - \tau_1) e^{-0.5k\lambda N\Delta\tau_1} \mathbb{E} \|x(0)\|^p, \end{aligned}$$

for all $k \geq 0$. By the Borel–Cantelli Lemma, we can obtain that for almost all $\omega \in \Omega$, there exists an integer $k_0 = k_0(\omega)$ such that

$$\sup_{kN\Delta\tau_1 \leq t \leq (k+1)N\Delta\tau_1} |x(t)|^p < e^{-0.5k\lambda N\Delta\tau_1},$$

for any $k > k_0(\omega)$. This implies that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(|x(t, \omega)|) \leq -\frac{\lambda}{2p},$$

for almost all $\omega \in \Omega$. The proof is therefore complete. □

4 | SIMULATIONS

We present two numerical examples in this section to support our theoretical results.

Example 4.1. Consider a hybrid SDE

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t), \tag{4.1}$$

Let $R(t)$ be a Markov chain with the state space $S = \{1, 2\}$ and the generator is

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

initial value $x(0) = 2, r(0) = 1$, where

$$f(x, 1) = 0.1x, g(x, 1) = 0.2x, \tag{4.2}$$

$$f(x, 2) = 0.2x, g(x, 2) = 0.3x. \tag{4.3}$$

It is obvious that the hybrid SDE is unstable (see Figure 1). In our example, we will design a control function $u : R \times S \rightarrow R$ defined by

$$u(x, 1) = -0.4x, u(x, 2) = -0.3x.$$

It is straightforward to show that Assumption 2.1 is satisfied with $\beta_1 = 0.2, \beta_2 = 0.5, \beta_3 = 0.5$. Then, we choose $p = 0.99$, it is easy to see that $\lambda_1, \lambda_2, \alpha_1, \alpha_2$ in Assumption 3.1 are

$$\lambda_1 = 0.2, \lambda_2 = 0.1, \alpha_1 = 0.2, \alpha_2 = 0.3.$$

The matrix defined by (3.2) as

$$\mathcal{A} = \begin{pmatrix} 1.198 & -1 \\ -1 & 1.099 \end{pmatrix},$$

which is a nonsingular M-matrix. By (3.3), we have $\varphi_1 = 6.9425, \varphi_2 = 6.6298$, since $C_2 = 6.9425, C_1 = 6.6298, \gamma = 0.4$,

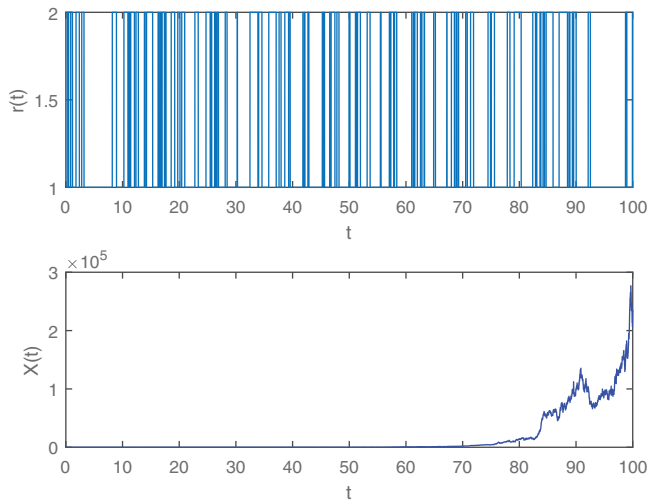


FIGURE 1 The computer simulation of the sample paths of the Markov chain and the solution of the unstable hybrid SDE (2.1) with the initial data $x(0) = 2$ and $r(0) = 1$ using the Euler–Maruyama method with step size 0.01

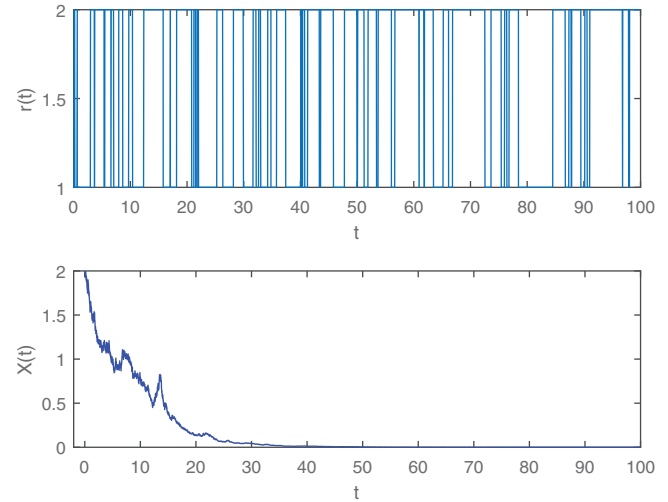


FIGURE 3 The computer simulation of the sample paths of the Markov chain and the solution of the controlled hybrid SDE (2.2) with the initial data $x(t) = 1 + \cos(t)$ for $t \in [10^{-5}, 0]$ and $r(0) = 1$ using the Euler–Maruyama method with step size 10^{-5}

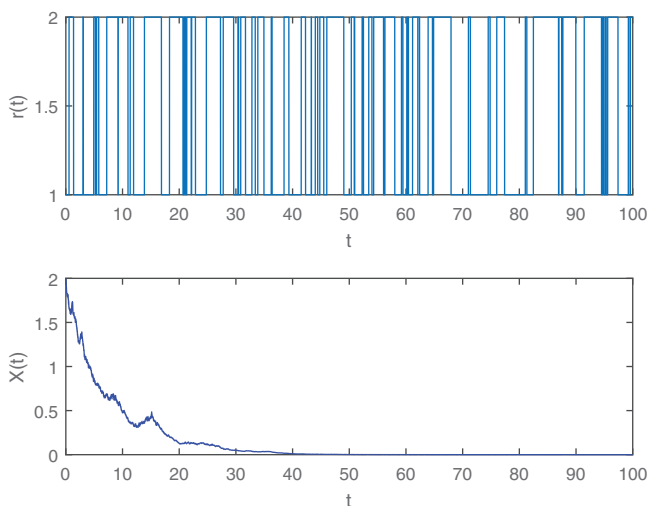


FIGURE 2 The computer simulation of the sample paths of the Markov chain and the solution of the controlled hybrid SDE (3.1) with the initial data $x(0) = 2$ and $r(0) = 1$ using the Euler–Maruyama method with step size 0.01

by Lemma 3.4, we can conclude that if $\theta \in (0.636, 1)$, then the controlled SDE (3.1) has exponential stability property. Figure 2 shows that the system is stable if the intermittent parameters $\theta = 0.9$. Besides, our aim is to use the discrete-time feedback control with delay time. For this purpose, we choose $\theta = 0.9$, and $\zeta = 0.9$, by Lemma 3.4, we get $\bar{\gamma} = 0.1044$, so we compute $T = 0.6298$, then Equation (3.16) becomes

$$0.9(1 + \tau_1)^{0.495} e^{1.3118\tau_1} + H_4(0.99, \tau_1, 0.6298) + H_3(0.99, \tau_1, 0.6298) = 1,$$

TABLE 1 Four cases of different θ

θ	$\bar{\gamma}$	T	τ^*
0.85	0.0846	0.7775	4.2125×10^{-5}
0.90	0.1044	0.6298	1.1031×10^{-4}
0.95	0.1240	0.5304	1.9177×10^{-4}
0.99	0.1435	0.4584	2.8470×10^{-4}

which has the unique positive root $\tau^* = 1.1031 \times 10^{-4}$ (which is about microseconds if the time unit is of year). By Theorem 3.8, we can conclude that the controlled system (2.2) is almost surely exponentially stable provided $\tau_1 < 1.1031 \times 10^{-4}$, we let $(\tau, \tau_0) = (10^{-4}, 10^{-5})$ and $\Delta = 10^{-5}$ (see Figure 3). The computer simulation supports this theoretical result clearly.

When $\theta = 0.85, 0.9, 0.95, 0.99$, the calculation shows that delay time changes along with the intermittent parameters θ . The larger the intermittent parameter value θ is, the longer time is taken to control. The larger the value τ^* is, the frequency of control is less. We can choose the different values of θ according to the actual situation (see Table 1).

Example 4.2. Consider the two-dimensional hybrid SDEs

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t). \quad (4.4)$$

Let $R(t)$ be a Markov chain with the state space $S = \{1, 2\}$ and the generator is

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix},$$

initial value $x_1(0) = 2, x_2(0) = 2, r(0) = 1$, where

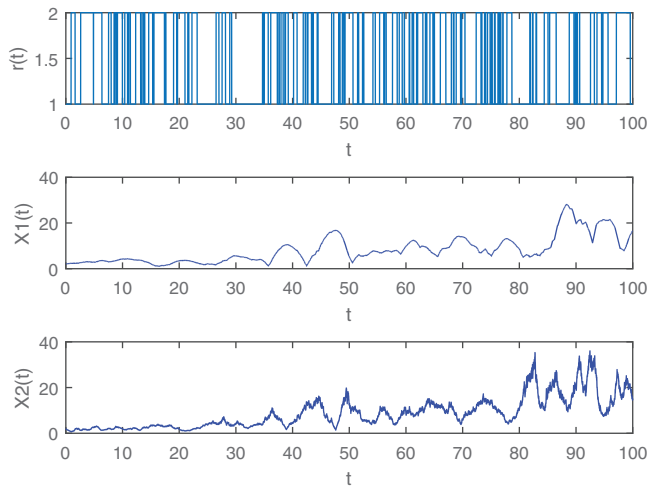


FIGURE 4 The computer simulation of the sample paths of the Markov chain and the solution of the unstable hybrid SDE (2.1) with the initial data $x_1(0) = 2, x_2(0) = 2$ and $r(0) = 1$ using the Euler–Maruyama method with step size 0.01

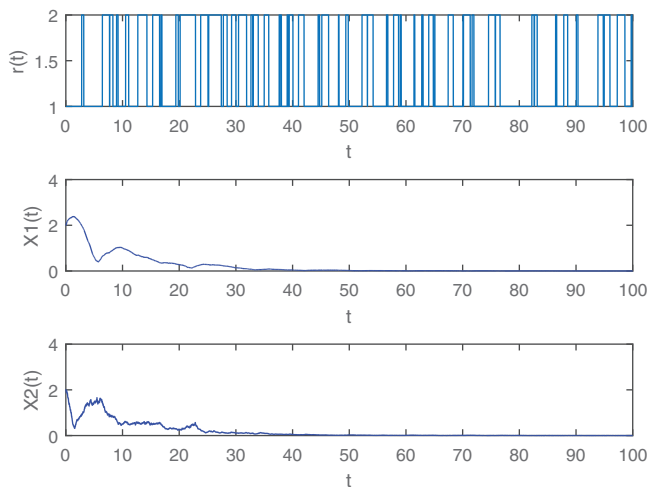


FIGURE 5 The computer simulation of the sample paths of the Markov chain and the solution of the controlled hybrid SDE (3.1) with the initial data $x_1(0) = 2, x_2(0) = 2$ and $r(0) = 1$ using the Euler–Maruyama method with step size 0.01

$$f(x, 1) = \begin{pmatrix} 0.5x_2 \\ -0.5x_1 + 0.01x_2 \end{pmatrix}, \quad g(x, 1) = \begin{pmatrix} 0 \\ -0.4x_2 \end{pmatrix}, \tag{4.5}$$

$$f(x, 2) = \begin{pmatrix} 0.02x_2 \\ -0.02x_1 + 0.1x_2 \end{pmatrix}, \quad g(x, 2) = \begin{pmatrix} 0 \\ -0.5x_2 \end{pmatrix}. \tag{4.6}$$

From Figure 4, we know the hybrid SDE is unstable, we will design a control function $u : R^2 \times S \rightarrow R^2$ defined by

TABLE 2 Four cases of different θ values

θ	$\bar{\gamma}$	T	τ^*
0.85	0.0626	0.7566	1.9265×10^{-4}
0.90	0.0725	0.6528	3.0171×10^{-4}
0.95	0.0823	0.5751	3.7453×10^{-4}
0.99	0.0903	0.5243	5.1235×10^{-4}

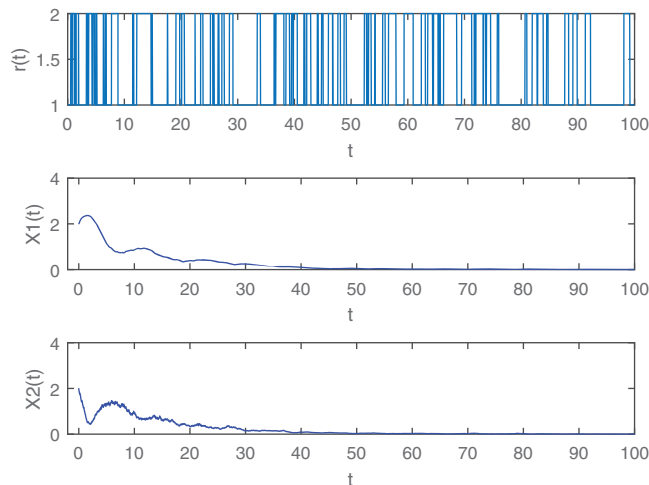


FIGURE 6 The computer simulation of the sample paths of the Markov chain and the solution of the controlled hybrid SDE (2.2) with the initial data $x(t) = 1 + \cos(t)$ for $t \in [10^{-5}, 0]$ and $r(0) = 1$ using the Euler–Maruyama method with step size 10^{-5}

$$u(x, 1) = \begin{pmatrix} -0.1x_1 \\ -0.1x_2 \end{pmatrix}, \quad u(x, 2) = \begin{pmatrix} -0.1x_1 \\ -0.2x_2 \end{pmatrix}.$$

By Assumption 2.1, we have $\beta_1 = 0.6, \beta_2 = 0.2, \beta_3 = 0.5$. Then, we choose $p = 0.99$ the same as example 4.1, it is easy to see that $\lambda_1, \lambda_2, \alpha_1, \alpha_2$ in Assumption 3.1 are

$$\lambda_1 = 0.09, \quad \lambda_2 = 0.1, \quad \alpha_1 = 0.01, \quad \alpha_2 = 0.1.$$

The matrix is defined by (3.2) as

$$A = \begin{pmatrix} 1.0891 & -1 \\ -2 & 2.099 \end{pmatrix}.$$

By (3.3), we have $\varphi_1 = 10.8394, \varphi_2 = 10.8003$, since $C_2 = 10.8394, C_1 = 10.8003, \gamma = 0.2$, by Lemma 3.4, we can conclude that if $\theta \in (0.534, 1)$, then the controlled SDE (3.1) has exponential stability property. Figure 5 shows that the system is stable if the intermittent parameters $\theta = 0.9$. To use the discrete-time feedback control with delay, we choose $\theta = 0.9$, and $\zeta = 0.9$, by Lemma 3.4, we get $\bar{\gamma} = 0.0725$, so we compute

$T = 0.6528$, then Equation (3.16) becomes

$$0.9(1 + 0.4\tau_1)^{0.495} e^{1.11375\tau_1} + H_4(0.99, \tau_1, 0.6528) \\ + H_3(0.99, \tau_1, 0.6528) = 1,$$

which has the unique positive root $\tau^* = 3.0171 \times 10^{-4}$ (which is about microseconds if the time unit is of year). By Theorem 3.8, we can conclude that the controlled system (2.2) is almost surely exponentially stable provided $\tau_1 < 3.0171 \times 10^{-4}$, we let $(\tau, \tau_0) = (10^{-4}, 10^{-5})$ and $\Delta = 10^{-5}$ (see Figure 6). The computer simulation supports this theoretical result clearly. We consider when $\theta = 0.85, 0.9, 0.95, 0.99$, the calculation shows that delay time changes along with the intermittent parameters θ (see Table 2).

5 | CONCLUSION

Here, we have discussed the stabilization of continuous-time hybrid SDEs by *intermittent feedback controls based on discrete-time state observations with a time delay*. The stabilities here mainly refer to exponential stability in p th moment and almost sure. We point out that the problems become harder when we take discrete-time state observation, time delay feedback and the intermittent control strategy into consideration at the same time. Finally, we obtain the upper bound of (τ, τ_0) and intermittent parameter θ . Two examples and computer simulations are illustrated to support our theory.

ACKNOWLEDGEMENTS

The authors would like to thank for the financial support by the Fundamental Research Funds for the Central Universities, 2232020D-37. The research of Wei Mao was supported by the National Natural Science Foundation of China(11401261), 333 High-level Project of Jiangsu Province and the Qing Lan Project of Jiangsu Province.

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How to cite this article: Jiang, Y., et al.: Stabilization of hybrid systems by intermittent feedback controls based on discrete-time observations with a time delay. *IET Control Theory Appl.* 15, 2039–2052 (2021).
<https://doi.org/10.1049/cth2.12160>