The logic of systems of granular partitions

Thomas Bittner[1234], Barry Smith[14], and Maureen Donnelly[13]
[1]Department of Philosophy, [2]Department of Geography,
[3]New York State Center of Excellence in Bioinformatics and Life Sciences
[4]National Center of Geographic Information and Analysis (NCGIA)
State University of New York at Buffalo

Abstract

The theory of granular partitions is designed to capture in a formal framework important aspects of the selective character of common-sense views of reality. It comprehends not merely the ways in which we can view reality by conceiving its objects as gathered together not merely into sets, but also into wholes of various kinds, partitioned into parts at various levels of granularity. We here represent granular partitions as triples consisting of a rooted tree structure as first component, a domain satisfying the axioms of Extensional Mereology as second component, and a mapping (called 'projection') of the first into the second as a third component. We define ordering relations among granular partitions the resulting structures are called partition frames. We then introduce an axiomatic theory which sentences are interpreted in partition frames.

1 Introduction

Human beings have a variety of ways of dividing up, classifying, mapping, sorting and listing the objects in reality. The theory of granular partitions presented in [BS03, SB02] seeks to provide a general and unified basis for understanding such phenomena in formal terms. Its aim is to contribute to an understanding of the granular and selective character of human common sense. Related work in this area includes [Hob85, BWJ98, Ste, Ste00, Don01, Bit02].

The theory of granular partitions has two parts. The first is a theory of classification (Theory A), which describes the tree structures of familiar classificatory systems. The second is a theory of reference or intentionality (Theory B). It provides an account of how those tree-structures relate to objects in reality.

Consider, for example, the Figure 1. On the left side we have a simple tree representation of the (incomplete) subdivision of the category *food* into subcategories *fruit* and *vegetables*. Theory A governs how to build nested *cell structures* in such a way that they correspond to the mentioned category trees. In the middle of Figure 1 such a cell structure is represented as a Venn diagram. Theory B governs the way these cell-structures project onto reality indicated by the arrows connecting the middle and the right parts of the Figure.

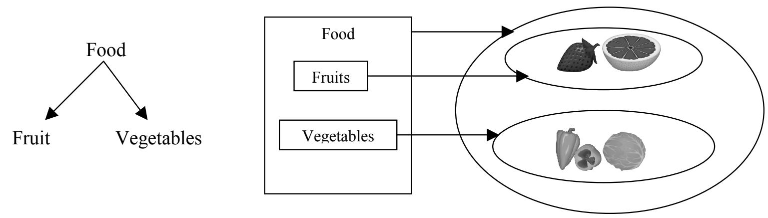

Figure 1: Relationships between cells and objects

Bittner and Smith use the notion of projection to characterize the relation between the cells in a partition and objects in reality. Briefly, we can think of cells as being projected onto objects in something like the way in which floodlights are projected upon objects on the stage in a theater. Projection is involved also when proper names are used to refer to the objects they denote or when acts of perception are directed towards objects in the immediate environment of the perceiving subject. (Projection is thus close to what philosophers call 'intentionality' [Ser83].) In 1 the cell labeled 'Vegetables' projects onto the class of all vegetables in reality.

Granular partitions are not only at work in the realm of classes of things such as food, vegetables, etc., but also in the realm of objects. Consider Figure 2. On the left side we have the tree representation of certain aspects of the mereological structure of the human being Fred. In the middle we have a corresponding cell structure and at the right hand side we have the target domain – your friend Fred. We assume the obvious 'Fred's Head' $\mapsto$ *Fred's head*, 'Fred's limbs' $\mapsto$ *Fred's left arm + Fred's right arm + Fred's left leg + Fred's right leg* ... projection.

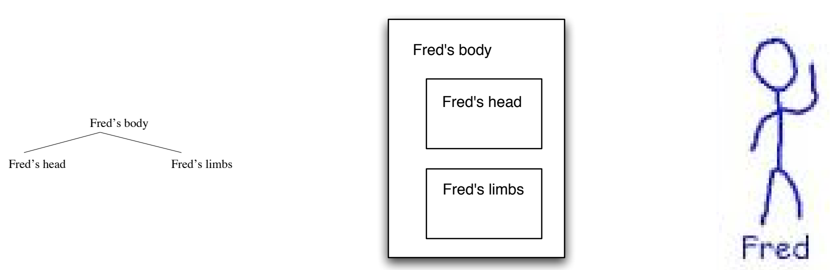

Figure 2: Relationships between cells and objects (2)

All granular partitions are both selective and granular. Selectivity of projection means that a partition does not project onto all objects. Consider Figure 2. Granularity of projection means more specifically that a partition projects onto a whole without projecting onto all of its parts. The depicted partition of Fred is granular since there is a cell projecting onto Fred's head but there are no cells projecting onto parts of Fred's head such as his nose, his ears, etc., and similarly for all other cells which do not have subcells.

In order to see what selectivity means, consider the cell structure in the middle of Figure 2. Here we have only the subcells ‘Head’ and ‘Limbs’. There is no cell ‘Torso’ in this cell structure. This may be because this cell tree is a part of a partition which deals only with parts of Fred that ‘stick out of the torso’. In this case, the partition selectively projects only on parts which are relevant given the purpose for which the partition was created.

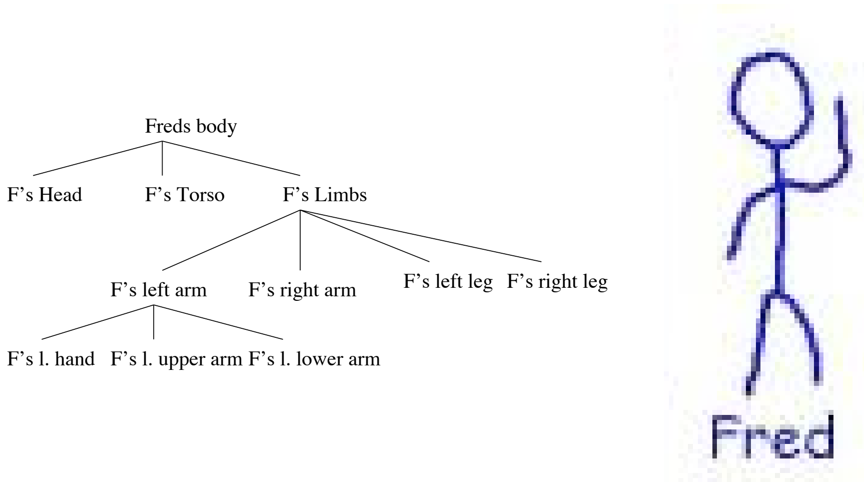

Figure 3: Relationships between cells and objects (3)

In their paper [BS03], Bittner and Smith focus on single granular partitions and their projective relation to reality. In the present paper, we will talk about the relations between granular partitions, and we will define structures on sets of granular partitions. Consider Figures 2 and 3. The granular partitions in both figures project onto Fred, but the partition in Figure 3 includes more detail than the partition in Figure 2. In this paper, we will define a *refinement* relation on partitions, according to which the partition in Figure 3 is a refinement of the partition in Figure 2.

To better understand these kinds of relations among granular partitions, we will introduce a class of structures called *labeled typed granular partitions* and define an ordering on these structures. We will show that these structures form *frame structures* in the sense of [HC04], which will then provide the formal semantics for our partition logic $\mathcal{L}$. This logic is a predicate modal logic of type S4. We show that reasoning in $\mathcal{L}$ is sound with respect to our partition theoretic semantics and we claim that reasoning within $\mathcal{L}$ has many properties of commonsense reasoning due to its underlying partition-theoretic semantics.

2 Individual objects, cell trees, and types of objects

We begin by presenting the two mereological systems that are needed for the definition of typed granular partitions.

The primitive relation of mereology is the part-of relation. This binary relation is reflexive, antisymmetric, and transitive, i.e., it is a partial ordering relation. As pointed out by authors such as [WCH87, GP95, AFG96], there are different kinds of parthood relations, which can be further classified by additional axioms. In this paper two kinds of parthood relations are of relevance:

1. The parthood relation characterized by the axiomatic system of extensional mereology (EM) [Sim87, CV99]. We will use the symbol $\leq$ for this relation. We call the entities among which this parthood relation holds *objects*. (That is, objects are the members of the domain of EM.) 'Object' here is used in a very wide sense, to include also scattered mereological sums. We will use the letters x, x_1, x_2, y, y_1, y_2, etc. as variables for objects.

2. The parthood relation characterized by what we call rooted tree mereology (RTM). We will use the symbol $\sqsubseteq$ for this relation. We call the entities among which this parthood relation holds *cells*. (That is, cells are the members of the domain of RTM.) We will use the letters z, z_1, z_2, etc. as variables for cells.

To specify the axioms for EM and RTM, we need to introduce an additional mereological relation. We say that x_1 and x_2 *overlap* if and only if there is some x that is a part of both x_1 and x_2. We will use the same symbol O for overlap in both EM and RTM the kinds of variables (variables for objects or variables for cells) will make clear which relation is meant. The formal definitions of the overlap relation in EM and RTM can be stated as follows.

$$\text{DO-EM} \quad x_1 O x_2 \equiv (\exists x)(x \leq x_1 \wedge x \leq x_2)$$
$$\text{O-RTM} \quad z_1 O z_2 \equiv (\exists z)(z \sqsubseteq z_1 \wedge z \sqsubseteq z_2)$$

In EM, there is one additional axioms besides those requiring $\leq$ to be a partial ordering (i.e., reflexive, antisymmetric, and transitive) [Sim87]: the axiom of extensionality, which tells us that if every object that overlaps x also overlaps y, then x is a part of y:

$$\text{AE-GM} \quad \forall x(xOx_1 \rightarrow xOx_2) \rightarrow x_1 \leq x_2$$

Note that it follows from AE-EM and the anti-symmetry of $\leq$ that O is extensional in EM.

$$\text{TE-EM} \quad \forall x(xOx_1 \leftrightarrow xOx_2) \rightarrow x_1 = x_2$$

Structures which satisfy the axioms of rooted tree mereology (RTM) form rooted trees similar to the one depicted in the left part of Figure 3. The rooted tree structure is ensured by the axioms below, which are added to the axioms requiring $\sqsubseteq$ to be a partial ordering.

We use the following definition in the axioms.

$$\text{DI-RTM} \quad z_1 \overline{\sqsubseteq} z_2 \equiv z_1 \sqsubseteq z_2 \wedge \forall z(z_1 \sqsubseteq z \sqsubseteq z_2 \rightarrow z = z_1 \vee z = z_2)$$

When $z_1 \overline{\sqsubseteq} z_2$, we say that z_1 is an *immediate subcell* of z_2.

We now give the following axioms for the partial ordering $\sqsubseteq$:

$$\text{ARoot-RTM} \quad (\exists z)(\forall z_1) z_1 \sqsubseteq z$$

ARoot-RTM requires that each model, Z, of RTM have a maximal cell. It follows from the anti-symmetry of $\sqsubseteq$ that this maximal cell is unique. We will let *root(Z)* stand for the unique maximal cell of the cell tree Z.

AChain-RTM	each cell $z \in Z$ there is a finite chain $z \overline{\sqsubseteq} z_1 \overline{\sqsubseteq} \dots z_n \overline{\sqsubseteq}$*root*$(Z)$ of immediate subcells connecting z to *root*(Z);
AO-RTM	$z_1 O z_2 \rightarrow z_1 \sqsubseteq z_2 \vee z_2 \sqsubseteq z_1$

AO-RTM restricts overlap to cells that stand in the subcell relation. Thus, there are no instances of proper overlap in RTM models. Notice that it follows from AO-RTM and the anti-symmetry of $\sqsubseteq$ that the graph induced by $\sqsubseteq$ contains no circles, i.e. is a tree. AO-RTM is also called the no-partial-overlap principle.

Finally, to do justice to the fact that cells and partitions are cognitive artifacts [Smi04], we add the following axiom.

AFin-RTM There are only finitely many cells in any model of RTM.

EM is designed to capture mereological reality: if x is part of y, then the EM representation of the part-whole structure of y must do justice to this fact. RTM, in contrast, is designed to capture the selectivity of cognition: RTM is a mereology, in which not all parts need be represented; in particular, RTM is devised in such a way that we can do justice to the granularity of cognition: when we see paint on a wall, we do not see the molecules by which this paint is constituted. Thus models of RTM need not satisfy the axiom of extensionality. The axiom of extensionality will fail in trees that include a cell, x, which has exactly one immediate proper subcell, y. In this case, x and y will be distinct even though they overlap exactly the same cells. We allow these kinds of models because we want our cell trees to be able to represent the selectivity of human cognition. For example, in a partition representing the parts of a particular yacht, called 'Maude', the cell representing the whole boat may have only one proper subcell representing, Maude's engine, because in a particular context we may only be interested in Maude's engine parts. And it is unlikely that there will ever be a partitition projecting onto Maude which includes cells projecting onto the separate molecules in these engine parts.

We use the variables $e, e_1, e_2, \dots$ to range over types (or classes) – (the type *human being*, the type *national state*, the type mountain, and so forth). The relation of instantiation holds between objects and their types (in that order). For example New York City is an instance of the type *city*, I am an instance of the type *human being*. We write *Inst* xe to signify that the object x instantiates the type c.

The relation *Inst* is irreflexive and asymmetric. Since in our ontology types and objects are represented as disjoint sorts of variables we do not need to add explicit irreflexivity and asymmetry axioms for *Inst*. We require every object is member of some type (AI1); every type has some object as its member (AI2); if x is instance of e_1

if and only if x is an instance of e_2 then e_1 and e_2 are identical (AI3).

$$
\begin{array}{ll}
AI1 & (\exists e)(\textit{Inst}\ xe) \\
AI2 & (\exists x)(\textit{Inst}\ xe) \\
AI3 & (x)(\textit{Inst}\ xe_1 \leftrightarrow \textit{Inst}\ xe_2) \rightarrow e_1 = e_2
\end{array}
$$

We define the sub-type relation in terms of instantiation: e_1 is a sub-type of e_2 if and only if the instances of e_1 are also instances of e_2. For example, the type (class) federal state is a sub-type of the type socio-economic unit. Therefore every instance of federal state (e.g., New York State) is also an instance of socio-economic unit.

$$D_{\subseteq} \quad e_1 \subseteq e_2 \equiv (x)(\textit{Inst}\ xe_1 \rightarrow \textit{Inst}\ xe_2)$$

We can prove that $\subseteq$ is reflexive, antisymmetric, and transitive. We call the theory formed byAI1-3 Minimal Type Theory (MTT).

3 Typed and labeled granular partitions

In this section, we first define a mathematical framework for the theory of granular partitions following the strategy outlined in [BS03]. We then extend this framework in two directions: Firstly, we require that cells of granular partitions are always *labeled*. The label of a cell is the name of the object onto which the cell projects. Secondly, we require that cells of granular partitions have always *an associated type*. If a given cell z projects on an object of a given type e, then e is the associated type of the projecting cell z.

3.1 Granular partitions

We introduce the notation $\mathcal{EM}$ and $\mathcal{RTM}$ to denote the classes of structures satisfying EM and RTM. We now define *granular partitions* x as triples of the form

$$(Z, \Delta, \rho)$$

where $Z \in \mathcal{RTM}$ is called the *cell tree* of the partition, $\Delta \in \mathcal{EM}$ is called the *target domain* of the partition, and the *projection-mapping* of signature $\rho : \mathcal{Z} \rightarrow \Delta$ has the following properties:

(i) ρ is a one-one mapping, i.e., if $\rho(z_1) = \rho(z_2)$ then $z_1 = z_2$;

(ii) ρ is *order-preserving* in the sense that if $z_1 \sqsubseteq z_2$ then $\rho(z_1) \leq \rho(z_2)$. This ensures that the tree structure in Z does not distort the mereological structure in Δ;

(iii) ρ is not an empty mapping: $(\exists z)(\exists x)(\rho(z) = x)$. It follows that every granular partition has at least one cell in its cell tree and at least one object in its target domain ;

(iv) ρ is a total mapping. This equivalent to requiring that granular partitions do not contain empty cells in the sense of [BS03].

In general the ρ will be not an onto mapping due to the selective and granular character of granular partitions.

3.2 Labeling

Consider the tree structures in Figure 2 and the way the corresponding cell trees project onto the object *Fred*. The *labels* on the nodes of the tree and the cells are an important aspect of the representations of Fred's parts. We will interpret the labels of granular partitions as *names* of the entities the labeled cell projects on. Notice that the label 'Fred's left leg' is not understood as a definite description [Rus19], i.e., the unique instance of the type (class, kind) *left human leg* that is part of Fred at a given time. This leg keeps its name during its existence. If Fred donates his left leg and the leg becomes a part of Bill then the name the name of Bill's new leg is still 'Fred's left leg'.

Let Λ be the set of names in language λ and let (Z, Δ, ρ) be a granular partition. A *labeled granular partition* is then a quintuple of the form

$$(Z, \Delta, \rho, \Lambda, \phi),$$

which is such that the labeling function $\phi : \Lambda \to Z$ is a one-one and onto mapping, i.e. each cell in the tree Z has a unique label. It follows that if $\lambda_i \in \Lambda$ is a label for a cell, then there is an entity in $x \in \Delta$ such that λ_i is the name of x. Names are finite strings of some alphabet λ. Since a cell tree has finitely many cells, it is always possible to assign finite strings of λ to the cells of a given partition. The labeling mappings ϕ will in general be partial, since finite partitions do not exhaust all strings of the underlying alphabet.

Consider the left part of Figure 4. The corresponding labeled granular partition $(Z, \Delta, \rho, \alpha, \phi)$ has projection and labeling mappings ρ and ϕ which are such that the following holds:

$$\rho = \{(\phi(\text{`Montana'}), \textit{Montana}), (\phi(\text{`Idaho'}), \textit{Idaho}), (\phi(\text{`Wyoming'}), \textit{Wyoming}), \ldots\}. \quad (1)$$

Here $\phi(\text{`Montana'})$ stands for "the cell labeled 'Montana"' and *Montana* refers to the targeted portions of reality (in this case, the portion of the surface of Earth that is occupied by the Federal State Montana).

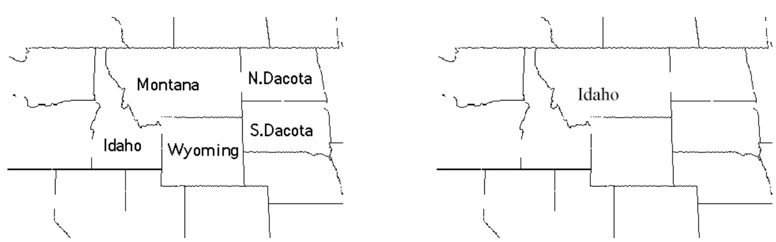

Figure 4: (left) A labeled granular partition (some labels are omitted) ; (right) a miss-labeled granular partition.

Consider the right part of Figure 4. Here we have a 'mislabeling' of the form $\rho(\phi(\text{'Idaho'})) = \textit{Montana}$, which means that the cell labeled 'Idaho' projects onto the piece of land which is usually referred to as Montana. Intuitively, this means that the labeling of this partition is in a certain way incompatible with the way the vast majority of other partitions which target the same domain are labeled. In particular, it is incompatible with the way the federal government of the United States labels their maps (which are special kinds of partitions [BS01]).

3.3 Typed granular partitions

Let (Z, Δ, ρ) be a granular partition and let Ω be a set of types which together with their instances – objects in $\mathcal{EM}$ – satisfy the axioms of our Minimal Type Theory (MTT). A *typing* for partition (Z, Δ, ρ) is a mapping ψ of signature $\psi : Z \rightarrow \Omega$ assigning cells in Z to members of the set Ω. If $\psi(z) = c$ then we say that the cell z is of type c. A *typed granular partition* then is a quintuple of the form

$$(Z, \Delta, \rho, \Omega, \psi)$$

such that the typing function ψ has the following properties:

1. ψ is a total function, i.e., each cell in the tree Z has exactly one type but there can be multiple cells in Z that have the same type,
2. if cell z is of type e and z projects onto x then x is an instance of e, i.e., if $\psi(z) = e$ then *Inst* $\rho(z)e$.

Consider the left part of Figure 3. The corresponding typed granular partition $(Z, \Delta, \rho, \Omega, \psi)$ has projection and typing mappings ρ and ψ which are such that the following holds:

$$\begin{aligned}
&\psi = \{(\phi(\text{'Fred's body'}), \textit{human body}), (\phi(\text{'Fred's head'}), \textit{human heads}),\\
&(\phi(\text{'Fred's left leg'}), \textit{left human leg}), \ldots\}.\\
&\text{and}\\
&\textit{Inst} = \{(\rho(\phi(\text{'Fred's body'})), \textit{human body}), (\rho(\phi(\text{'Fred's head'})), \textit{human heads}),\\
&(\rho(\phi(\text{'Fred's left leg'})), \textit{left human leg}), \ldots\}.
\end{aligned} \tag{2}$$

A *labeled and typed granular partition* then is a seven-tuple of the form

$$(Z, \Delta, \rho, \Lambda, \phi, \Omega, \psi)$$

such that (Z, Δ, ρ) is a granular partition, $(Z, \Delta, \rho, \Lambda, \phi)$ is a labeled granular partition, and $(Z, \Delta, \rho, \Omega, \psi)$ is a typed granular partition.

4 Refinement relations between granular partitions

So far we have discussed single granular partitions and their projective relation to reality. In this section we will discuss relations between granular partitions, and we will define structures on sets of granular partitions. As discussed above the granular partitions in Figures 2 and 3 project onto Fred, but the partition in Figure 3 includes more detail than the partition in Figure 2. This will now be captured formally in our discussion of *refinement* relation between labeled typed granular partitions.

4.1 Refinement as ordering

Let Π be a set of labeled typed granular partitions. Let $\Gamma_1 = (Z_1, \Delta_1, \rho_1, \Lambda_1, \phi_1, \Omega_1, \psi_1)$ and $\Gamma_2 = (Z_2, \Delta_2, \rho_2, \Lambda_2, \phi_2, \Omega_2, \psi_2)$ be labeled, typed granular partitions in Π. And let Γ_1 and Γ_2 be the labeled, typed granular partitions in Figures 2 and 3. One can see that Γ_1 and Γ_2 stand in a kind of refinement relation to each other. We will use the symbol $\preceq$ to refer to this relation and write $\Gamma_1 \preceq \Gamma_2$ to express the fact that the granular partition Γ_1 is a refined by the granular partition Γ_2.

We give a formal account of the relation $\preceq$ as follows. For labeled typed granular partitions $\Gamma_1, \Gamma_2 \in \Pi$ we say that $\Gamma_1 \preceq \Gamma_2$ if and only if there exists a mapping $f : Z_1 \rightarrow Z_2$ with the following properties:

(i) f is one-one and total,

(ii) f is *order-preserving*, i.e., if $z_i \sqsubseteq z_j$ then $f(z_i) \sqsubseteq f(z_j)$,

(iii) f is *target-preserving*, i.e., $\rho_1(z) = \rho_2(f(z))$,

(iv) f is *label-preserving*, i.e., $\phi_2(\lambda_i) = f(\phi_1(\lambda_i))$, and

(v) f is *type-preserving*, i.e., $\psi_1(c) = \psi_2(f(c))$.

The existence of the mapping f with its particular properties (i-v) ensures that if partition Γ_1 is a refinement of partition Γ_2, then we can map cells in Z_1 to cells in Z_2 in such a way that: (a) if two cells in $z_i, z_j \in Z_1$ are subcells of each other then so are their counterparts in $f(z_1), f(z_2) \in Z_2$; (b) the target $\rho_1(z)$ of the cell $z \in Z_1$ is identical to the target $\rho_2(f(z))$ of its counterpart $f(z) \in Z_2$; (c) the cells $z \in Z_1$ and $f(z) \in Z_2$ have the same labels; and (d) the cells $z \in Z_1$ and $f(z) \in Z_2$ have the same type. In other words we require that if partition Γ_1 is a refinement of partition Γ_2 then there exists an order-, label-, type-, and target-preserving mapping f such that the diagrams in Figure 5 commute.

Let $\Gamma, \Gamma_1, \Gamma_2$ and Γ_3 be a labeled, typed granular partitions. We can show that the relation $\preceq$ is reflexive (ref) and transitive (tr):

(ref) We have $\Gamma \preceq \Gamma$ since the identity map of a cell tree onto itself, defined by $z = id(z)$ is always order-, label-, type-, and target-preserving.

(tr) For transitivity we have to show that if $f_1 : Z_1 \rightarrow Z_2$ and $f_2 : Z_2 \rightarrow Z_3$ are order-, label-, type-, and target-preserving then so is their composition $f_1 \circ f_2 : Z_1 \rightarrow Z_3$. That this is the case can be seen in the diagrams in Figure 6.

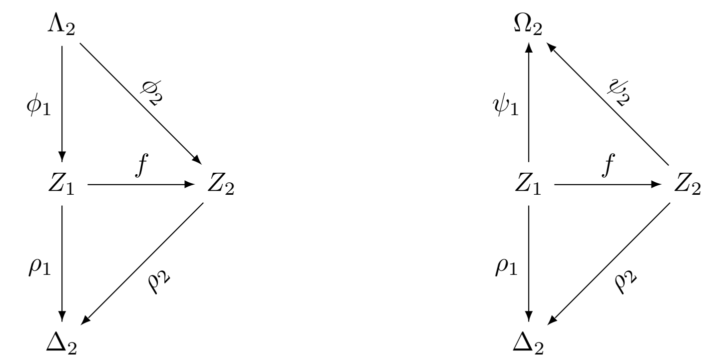

Figure 5: Commutative diagram illustration for the definition of $\preceq$.

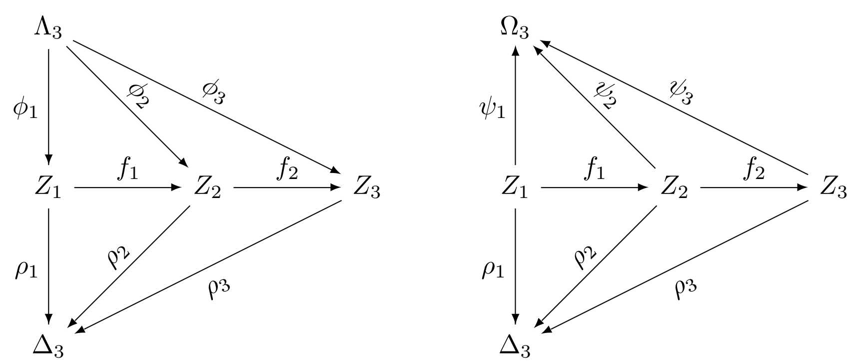

Figure 6: Commutative diagram illustration fothe proof of transitivity of $\preceq$.

4.2 Refinement vs. extension

Consider the left part of Figure 7. We have a partition Γ_x with Δ_x being the collection of Fred's body parts, $\Lambda_x = \{$'Fred's body', 'Fred's right arm'$, \ldots\}$, $\Omega_x = \{$*human body*, *right human arm*, *upper human body*$, \ldots\}$, cells labeled 'Fred's body' and 'Fred's right arm' with $\phi($'Fred's right arm'$) \sqsubseteq \phi($'Fred's body'$)$ and with the cell labeled 'Fred's right arm' projecting onto your friend Freds' right arm, i.e., $\rho_x(\phi_x($'Fred's right arm'$)) =$ *Freds' right arm*, and the cell labeled 'Fred's body' projecting onto Fred's whole body, i.e., $\rho_x(\phi_x($'Fred's' body'$)) =$ *Freds' body*. (In Figure 7 we use the stretched bracket $<$ to indicate that the cell labeled 'Fred's body' targets Fred's whole body.) The cell labeled 'Fred's right arm' is of type *right human arm* and the cell labeled 'Fred's body' is of type *human body*.

We also have a partition Γ_y with $\Lambda_y = \Lambda_x$, $\Omega_y = \Omega_x$, $\Delta_x = \Delta_y$, and $\phi($'Fred's right arm'$) \sqsubseteq \phi($'Fred's upper body'$) \sqsubseteq \phi($'Fred's body'$)$, with 'Fred's right arm' and 'Freds body' being of the same type and projecting as above, and with 'Fred's upper body' with the obvious type and projection. (In the figure we use the small bracket $<$ to indicate that the cell labeled 'Fred's upper body' targets Fred's upper body.) It is easy to see that the induced mapping $f_1 : Z_x \rightarrow Z_y$ is order-, target-, type-, and label-preserving. Thus $\Gamma_x \preceq \Gamma_y$.

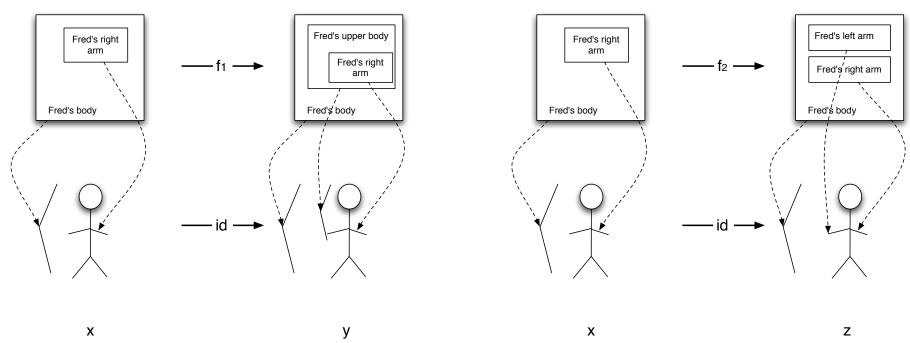

Figure 7: Examples of partitions between the relation $\preceq$ holds (1).

The situation in the right part of Figure 7 is similar. We have Γ_x as before. However we have a refinement Γ_z in with a third cell labeled 'Fred's right arm' which is not a supercell of 'Fred's left arm' and in which projects onto Freds' left arm. Again, the induced mapping $f_2 : Z_x \rightarrow Z_z$ is order-, target-, type-, and label-preserving. Thus $\Gamma_x \preceq \Gamma_z$.

In the left part of Figure 8, we have a refinement of Γ_x by Γ_u similar to the refinement in the left part of Figure 7. The refinement partition Γ_y and Γ_u recognize the same parts of Fred: Fred as a whole, Freds upper body, and Freds right arm. They differ however in the following respect: The partition Γ_y recognizes the fact that Freds right arm is a part of Freds upper body. This aspect of mereological ordering is traced over in the partition Γ_u.

Note that not only is Γ_y is a refinement of Γ_x. It is also a refinement of Γ_u. To see

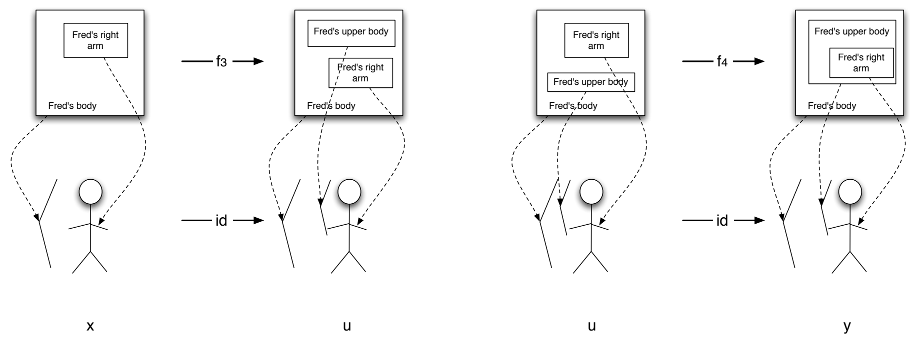

Figure 8: Examples of partitions between the relation $\preceq$ holds (2).

this consider the mapping $f_4 : Z_u \to Z_y$ mapping cells in Z_u to cells with matching labels in Z_y. Clearly, f_4 is order-, target-, type-, and label-preserving, hence $\Gamma_u \preceq \Gamma_y$. On the other hand the partitions Γ_y and Γ_z are not comparable with respect to $\preceq$ since no commutative diagram like the one in the left of Figure 5 can be constructed for the two partitions.

The refinement relations in Figures 7 and 8 are examples of what we call *proper refinement*. In proper refinement the object targeted by the root cell – the cell 'Fred's body' in Figures 7 and 8– remains the same. A proper refinement can target additional objects as long as these objects are parts of objects targeted by the original partition (e.g., $\Gamma_x \preceq \Gamma_y$ in Figure 7). Or, a proper refinement may target the same set of objects but include more information about mereological relations between objects (e.g., $\Gamma_u \preceq \Gamma_y$ in Figure 8).

As an example of another way of how a granular partition can be refined consider a granular partition Γ_{US} which recognizes the Federal States of the US and let Γ_{US-EU} represent a granular partition which recognizes the Federal States of the US as well as the states of the European Community together with a root cell labeled 'The United States and the States of the EU'. It is easy to see that we have $\Gamma_{US} \preceq \Gamma_{US-EU}$.

This is an example of what we will call *extensions*. When one partition is an extension of another, then the target of the original root cell is always a proper part of the extension's root cell.

Assume $\Gamma_1 \preceq \Gamma_2$ and consider the corresponding commutative diagrams in Figure 5. As sketched above, we can further analyze the two different uses of refinement by considering the projection of those cells in Z_2 which are not targeted by the mapping f. Intuitively, in the case of proper refinement those cells project onto objects in Δ_2 which are parts of objects in the image of ρ_1. In the case of extension, those cells project onto objects in Δ_2 which are not parts of objects in the image of ρ_1. Formally we define now define the binary relations *RP* (x is-properly-refined-by y) and EP (x

is-properly-extended-by y) which both are subrelations of $\preceq$ as follows:

$$RP(\Gamma_1, \Gamma_2) \equiv \quad \Gamma_1 \preceq \Gamma_2 \text{ and } \forall z_2 \in Z_2(\exists z_1 \in Z_1(\rho_2(z_2) \leq \rho_1(z_1))),$$
$$EP(\Gamma_1, \Gamma_2) \equiv \quad \Gamma_1 \preceq \Gamma_2 \text{ and } \forall z_2 \in Z_2 \neg(\exists z_1 \in Z_1(\rho_2(z_2) \leq \rho_1(z_1))).$$

Obviously, there are also 'mixed' cases where $\Gamma_1 \preceq \Gamma_2$ but neither $RP(\Gamma_1, \Gamma_2)$ nor $EP(\Gamma_1, \Gamma_2)$.

4.3 Counterparts

Consider the granular partitions Γ_x, Γ_y, Γ_z, and Γ_u in Figures 7 and 8. Each of these partitions has a cell labeled 'Fred's body' of type *human body* which projects onto Fred's body. (Similarly, each of these partitions has a cell labeled 'Fred's right arm' of type *right human arm* which projects onto Fred's right arm.) Notice that that the cell labeled 'Fred's body' in partition Γ_x is distinct from the cell labeled 'Fred's body' in partition Γ_y (which in turn is distinct from the cell labeled 'Fred's body' in partitions Γ_z and Γ_u). We call cells like the cells labeled 'Fred's body' in Γ_x, Γ_y, Γ_z, and Γ_u *counterparts*.

Let $\Gamma_1 = (Z_1, \Delta_1, \rho_1, \Lambda_1, \phi_1, \Omega_1, \psi_1)$ and $\Gamma_2 = (Z_2, \Delta_2, \rho_2, \Lambda_2, \phi_2, \Omega_2, \psi_2)$ be labeled, typed granular partitions in $\mathcal{P}$. The cells $z_1 \in Z_1$ and $z_2 \in Z_2$ are counterparts, $z_1 \; C \; z_2$, if and only if there is a target-, label-, and type-preserving one-one mapping $f : Z_1 \to Z_2$ such that $f(z_1) = z_2$. Counterparthood is reflexive, symmetric, and transitive, i.e., an equivalence relation:

- ref C is reflexive since the identity mapping of the cell structure of a partition onto itself is a target-, label-, and type-preserving one-one mapping;
- sym C is symmetric since the inverse mapping (f^{-1}) of every target-, label-, and type-preserving one-one mapping (f) between the cell structures of two partitions is a (possibly partial) target-, label-, and type-preserving one-one mapping. Hence if $f(z_1) = z_2$ then $f^{-1}(z_2) = z_1$.
- trans C is transitive since the composition of two target-, label-, and type-preserving one-one mappings is a target-, label-, and type-preserving one-one mapping.

It immediately follows that if partition Γ_1 is a refinement of partition Γ_2 then every cell in Γ_1 has a counterpart in Γ_2. It also follows that counterparts of partitions which stand in the refinement relation have identical labels. These points can be verified easily in Figures 7 and 8.

5 Partition logic

So far we discussed granular partitions from an 'God's eye perspective'. That is, we discussed the projective relationship of granular partitions to reality and explored (ordering) relationships among granular partitions. In this section we develop a formal language, $\mathcal{L}$, of how we 'see' mereological structure *through* granular partitions. In $\mathcal{L}$ we can to formally express the following statements relative to the collection of typed, labeled granular partitions $\Pi = \{\Gamma_x, \Gamma_y, \Gamma_z, \Gamma_u\}$ as depicted in Figures 7 and 8:

A Partition Γ_x recognizes that there exists an object named 'Fred's body' of type human body.
B Partition Γ_x recognizes that the object named 'Fred's body' has an object named 'Fred's right arm' as a part.
C Partition Γ_x and all its refinements in Π recognize that the object named 'Fred's body' has an object named 'Fred's right arm' as part.
D Some refinement of Γ_x in Π recognizes that the object named 'Fred's body' has an object named 'Fred's left arm' as a part.
E All partitions in Π recognize that human bodies have right human arms as parts.
F Partition Γ_x does not recognize an object named 'Fred's left hand'.
G An object named 'Fred's left hand' is absent in partition Γ_x.

We use a first order modal logic with identity to formally express statements like A-G. In this language, $\mathcal{L}$, we have the additional binary predicate '*SC*' which is interpreted as the subcell relation ($\sqsubseteq$) and the unary existence predicate '*E*'. We also introduce a binary predicate *IO* which holds between cell z and type t if and only if t is type assigned to z (to be defined more precisely below).

In $\mathcal{L}$ we use the letter z with indexes to designate variables $z, z_1, z_2, \ldots$. We use the letter c with indexes to designate object-constants $c, c_1, c_2, \ldots$. We use the letter t with indexes to designate type-constants $t, t_1, t_2, \ldots$. Atomic formulas of $\mathcal{L}$ are of the form '*E* z', '*E* c', '*SC* $z_1 z_2$', '*SC* $c_1 z_2$', '*SC* $z_1 c_2$', '*SC* $c_1 c_2$', '$z_1 = z_2$' $\ldots$, '*IO* zt', '*IO* ct'. α and β are complex formulas which are defined recursively as follows: If $\alpha, \beta \in \mathcal{L}$ then so are $\sim \alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $\alpha \leftrightarrow \beta$, $\Box\alpha$, $\Diamond\alpha$, $(\exists z)(\alpha)$ and $(x)(\alpha)$.

5.1 Semantics

Let Π be a set of labeled, typed granular granular partitions. Let $\mathcal{Z}$ be the set of all cells of granular partitions in Π, i.e, $\mathcal{Z} = \bigcup\{Z \mid (Z, \Delta, \rho, \Lambda, \phi, \Omega, \psi) \in \Pi\}$. Let $\mathbf{\Lambda}$ be the set of all labels of cells of granular partitions in Π, i.e, $\mathbf{\Lambda} = \bigcup\{\Lambda \mid (Z, \Delta, \rho, \Lambda, \phi, \Omega, \psi) \in \Pi\}$. Let $\mathcal{T}$ be the set of all types of cells of granular partitions in Π, i.e, $\mathcal{T} = \bigcup\{\Omega \mid (Z, \Delta, \rho, \Lambda, \phi, \Omega, \psi) \in \Pi\}$. Let $\preceq$ be the refinement ordering among members of Π, and let C be the counterpart relation between cells in $\mathcal{Z}$. A *partition frame* then is a sixtuple

$$(\Pi, \mathcal{Z}, \mathbf{\Lambda}, \mathcal{T}, \preceq, C).$$

In the semantics of $\mathcal{L}$ the granular partitions in Π are treated as 'worlds' and $\preceq$ is treated as an accessibility relation between worlds in the sense of the standard possible world semantics of modal logic [HC04]. Hence, the granular partition Γ_2 is accessible from the granular partition Γ_1 if and only if Γ_2 is a refinement of Γ_1. Notice that, since distinct granular partitions do not have cells in common, every cell $z \in \mathcal{Z}$ exists in exactly one world. Cells $z_1, z_2 \in \mathcal{Z}$ are counterparts in David Lewis' sense if and only if $z_1\ C\ z_2$ [Lew86]. Since the accessibility relation $\preceq$ is reflexive and transitive and the counterpart relation C is an equivalence relation, the modal logic underlying $\mathcal{L}$ will be of type S4.

The object-constants $c_1, \ldots, c_n$ of $\mathcal{L}$ are the members of $\mathbf{\Lambda}$, i.e., $\{c_1, \ldots, c_n\} = \mathbf{\Lambda}$. The interpretation function for object-constants is a binary function $I_o : \mathbf{\Lambda} \times \Pi \rightarrow \mathcal{Z}$

such that $I_o(c, \Gamma) = \phi_\Gamma(c)$, where ϕ_Γ is the labeling function of partition Γ. The type-constants $t_1, \ldots, t_m$ of $\mathcal{L}$ are interpreted as the members of $\mathcal{T}$, i.e., I_t is a total one-one onto mapping such that $I_t(t) \in \mathcal{T}$.

The variables in $\mathcal{L}$ range over the members of $\mathcal{Z}$. μ and ν are functions which assign members of $\mathcal{Z}$ to the variables $z, z_1, z_2, \ldots$ Let $\Gamma \in \Pi$ be a labeled, typed granular partition with cell tree Z in the partition frame $\mathcal{F} = (\Pi, \mathcal{Z}, \mathbf{\Lambda}, \mathcal{T}, \preceq, C)$, and let $\sqsubseteq_\Gamma$ be the subcell relation between cells in Γ and let ψ_Γ be the function assigning types to cells in Γ. Atomic formulas in $\mathcal{L}$ are interpreted in $\mathcal{F}$ as follows:

$$\begin{array}{l}
\mu \models^{\mathcal{F}}_{\Gamma} [E\ z] \text{ iff } \mu(z) \in Z_\Gamma \\
\mu \models^{\mathcal{F}}_{\Gamma} [E\ c] \text{ iff } I_o(c,\Gamma) \in Z_\Gamma \\
\mu \models^{\mathcal{F}}_{\Gamma} [IO\ zt] \text{ iff } \psi_\Gamma(\mu(z)) = I_t(t) \\
\mu \models^{\mathcal{F}}_{\Gamma} [IO\ ct] \text{ iff } \psi_\Gamma(I_o(c,\Gamma)) = I_t(t) \\
\mu \models^{\mathcal{F}}_{\Gamma} [SC\ z_1 z_2] \text{ iff } \mu(z_1) \sqsubseteq_\Gamma \mu(z_2) \\
\mu \models^{\mathcal{F}}_{\Gamma} [SC\ c_1 z_2] \text{ iff } \phi(I_o(c_1,\Gamma)) \sqsubseteq_\Gamma \mu(z_2) \\
\mu \models^{\mathcal{F}}_{\Gamma} [SC\ zc] \text{ iff } \mu(z) \sqsubseteq_\Gamma \phi(I_o(c,\Gamma)) \\
\mu \models^{\mathcal{F}}_{\Gamma} [SC\ cz] \text{ iff } \phi(I_o(c,\Gamma)) \sqsubseteq_\Gamma \mu(z)
\end{array} \tag{3}$$

Following [HC04] complex formulas of $\mathcal{L}$ then are interpreted as follows:

$$\begin{array}{l}
\mu \models^{\mathcal{F}}_{\Gamma} [\sim \alpha] \text{ iff } \mu \not\models^{\mathcal{F}}_{\Gamma} [\alpha] \\
\mu \models^{\mathcal{F}}_{\Gamma} [\alpha \wedge \beta] \text{ iff } \mu \models^{\mathcal{F}}_{\Gamma} [\alpha] \text{ and } \mu \models^{\mathcal{F}}_{\Gamma} [\beta] \\
\mu \models^{\mathcal{F}}_{\Gamma} [\alpha \vee \beta] \text{ iff } \mu \models^{\mathcal{F}}_{\Gamma} [\alpha] \text{ or } \mu \models^{\mathcal{F}}_{\Gamma} [\beta] \\
\mu \models^{\mathcal{F}}_{\Gamma} [\alpha \rightarrow \beta] \text{ iff if } \mu \models^{\mathcal{F}}_{\Gamma} [\alpha] \text{ then } \mu \models^{\mathcal{F}}_{\Gamma} [\beta] \\
\mu \models^{\mathcal{F}}_{\Gamma} [\alpha \leftrightarrow \beta] \text{ iff } \mu \models^{\mathcal{F}}_{\Gamma} [\alpha] \rightarrow \beta \text{ and } \mu \models^{\mathcal{F}}_{\Gamma} [\beta \rightarrow \alpha] \\
\mu \models^{\mathcal{F}}_{\Gamma} [(z_i)\alpha] \text{ iff } \nu \models^{\mathcal{F}}_{\Gamma} [\alpha] \text{ for every } \nu \text{ with } \nu(z_j) = \mu(z_j) \text{ if } j \neq i \text{ and } \nu(z_i) \in Z_\Gamma \\
\mu \models^{\mathcal{F}}_{\Gamma} [(\exists z_i)\alpha] \text{ iff } \nu \models^{\mathcal{F}}_{\Gamma} [\alpha] \text{ for some } \nu \text{ with } \nu(z_j) = \mu(z_j) \text{ if } j \neq i \text{ and } \nu(z_i) \in Z_\Gamma \\
\mu \models^{\mathcal{F}}_{\Gamma} [\Box\alpha] \text{ iff for all } \Gamma_1 \in \Pi: \text{ if } \Gamma \preceq \Gamma_1 \text{ then there is some } \nu \text{ such that } \nu \models^{\mathcal{F}}_{\Gamma_1} [\alpha] \text{ and} \\
\qquad \mu(z)\ C\ \nu(z) \text{ and } \nu(z) \in Z_{\Gamma_1} \text{ for all variables } z \text{ in } \alpha \\
\mu \models^{\mathcal{F}}_{\Gamma} [\Diamond\alpha] \text{ iff for some } \Gamma_1 \in \Pi: \Gamma \preceq \Gamma_1 \text{ and there is some } \nu \text{ such that } \nu \models^{\mathcal{F}}_{\Gamma_1} [\alpha] \text{ and} \\
\qquad \mu(z)\ C\ \nu(z) \text{ and } \nu(z) \in Z_{\Gamma_1} \text{ for all variables } z \text{ in } \alpha
\end{array} \tag{4}$$

Notice that we employ an 'actualist' semantics in the sense that the evaluation of the truth of quantified formulas α in partition Γ is performed with respect to the cells of Γ, i.e., $\models^{\mathcal{L}}_{\Gamma} [(z)\alpha]$ and $\models^{\mathcal{L}}_{\Gamma} [(\exists z)\alpha]$ are evaluated with respect to the cells in Z_Γ.

The well-formed formula α is true in partition Γ of frame $\mathcal{F}$ on the interpretation I_o and I_t (i.e., the partition Γ recognizes that α), $\models^{\mathcal{F}}_{\Gamma} [\alpha]$, if and only if there is an assignment μ of the variables of $\mathcal{L}$ with members of $\mathcal{Z}$ such that $\mu \models^{\mathcal{F}}_{\Gamma} [\alpha]$ holds. α is valid in frame $\mathcal{F}$ (α is recognized by all partitions of frame $\mathcal{F}$), $\models^{\mathcal{F}} [\alpha]$, if and only if $\mu \models^{\mathcal{F}}_{\Gamma} [\alpha]$ holds for every $\Gamma \in \Pi$ and every assignment μ of variables of $\mathcal{L}$ to members of $\mathcal{Z}$, under the interpretations I_o and I_t. α is valid, $\models \alpha$, if and only if α is valid in every partition frame $\mathcal{F}$ under interpretations $I_o^{\mathcal{F}}$ and $I_t^{\mathcal{F}}$.

Consider the following example. Let $\Pi_e = \{\Gamma_x, \Gamma_y, \Gamma_z, \Gamma_u\}$ be as depicted in

Figures 7 and 8. We then have

$\mathbf{\Lambda}_e = \{$‘Fred’s body’, ‘Fred’s right arm’, ‘Fred’s left arm’, ‘Fred’s upper body’$\}$
$\mathcal{Z}_e = \{\phi_x($‘Fred’s body’$), \phi_y($‘Fred’s body’$), \phi_z($‘Fred’s body’$), \phi_u($‘Fred’s body’$),$
$\phi_x($‘Fred’s right arm’$), \phi_y($‘Fred’s right arm’$), \ldots\}$
$\mathcal{T}_e = \{$*human body*, *left human arm*, *right human arm*, *upper human body*$\}$

The counterpart relation holds between cells with identical labels and the refinement relation $\preceq$ holds as discussed above. The corresponding partition frame is

$$\mathcal{F}_e = (\Pi_e, \mathcal{Z}_e, \mathbf{\Lambda}_e, \mathcal{T}_e, \preceq_e, C_e).$$

Now assume that the object-constants ‘Fred’s body’, ‘Fred’s right arm’, ... in $\mathcal{L}$ are interpreted as the corresponding names in $\mathbf{\Lambda}_e$ and that type-constants *HumanBody*, *RightHumanHand*, ... are interpreted as the corresponding types in $\mathcal{T}_e$. Given the semantics of $\mathcal{L}$ the sentences A-G can be now be expressed formally as follows:

A $\models^{\mathcal{F}_e}_{\Gamma_x}$ [*E* ‘Fred’s body’ $\wedge$ *IO* ‘Fred’s body’ *HumanBody*]
B $\models^{\mathcal{F}_e}_{\Gamma_x}$ [*SC* ‘Fred’s right arm’ ‘Fred’s body’]
C $\models^{\mathcal{F}_e}_{\Gamma_x}$ [$\Box$(*SC* ‘Fred’s right arm’ ‘Fred’s body’)]
D $\models^{\mathcal{F}_e}_{\Gamma_x}$ [$\Diamond$(*SC* ‘Fred’s left arm’ ‘Fred’s body’)]
E $\models^{\mathcal{F}_e}$ $[\Box[(z_1)(E\, z_1 \wedge IO\, z_1 HumanBody \rightarrow (\exists z_2)(IO\, z_2 RightHumanArm \wedge SC\, z_2 z_1))]]$
F $\models^{\mathcal{F}_e}_{\Gamma_x}$ [$\sim$ *E* ‘Fred’s left arm’]
G $\models^{\mathcal{F}_e}_{\Gamma_x}$ [$\sim$ *E* ‘Fred’s left arm’ $\wedge$ $\Diamond$*E* ‘Fred’s left arm’]

Notice, that (A-G except E) are assertions about what is recognized by partition Γ_x (or it’s refinements). E is an assertion of what is true in (recognized by) all partitions in the frame $\mathcal{F}_e$. Notice also, that (E) is not true in every frame since there are surely partitions in frames other than $\mathcal{F}_e$ which do recognize human beings which do not have a right arm or which trace over (do not recognize) a particular right arm. Thus, the partition frame $\mathcal{F}_e$ can be considered as a *context* and the evaluation of statements of $\mathcal{L}$ with respect to $\mathcal{F}_e$ as the evaluation of the truth of the statements (A-G) in the context $\mathcal{F}_e$. Consider sentence (F). $\models^{\mathcal{F}_e}_{\Gamma_x}$ [$\sim$ *E* ‘Fred’s left arm’] does NOT mean that Fred’s left arm does not exist. It merely means that Γ_x fails to recognize that an object with the name ‘Fred’s left arm’ exists. (G) shows that in $\mathcal{L}$ the sentence ‘An object named ‘Fred’s left hand’ is absent in partition Γ_x’ means that Γ_x does not recognize the existence of an object named ‘Fred’s left hand’ but that some refinement of Γ_x does recognize the existence of this object.

5.2 Valid principles

We now prove that the following principles that are characteristic for a modal S4 predicate logic are valid in any partition frame under the semantics given above:

K-S	$\models [\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)]$
T-S	$\models [\Box\alpha \rightarrow \alpha]$
4-S	$\models [\Box\alpha \rightarrow \Box\Box\alpha]$
D$\Diamond$-S	$\models [\Diamond\alpha \leftrightarrow \sim\Box\sim\alpha]$
RN-S	if $\models [\alpha]$ then $\models [\Box\alpha]$
BFC-S	if $\models [\Box(z)\alpha \rightarrow (z)\Box\alpha]$

K-S Assume $\mu \models^{\mathcal{L}}_{\Gamma} [\Box(\alpha \rightarrow \beta)]$. Thus, if $\Gamma \preceq \Gamma_1$ then there is some ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\alpha \rightarrow \beta]$ and $\mu(z)\ C\ \nu(z)$. Assume $\Gamma \preceq \Gamma_1$. Then if there is some ν such that $\mu(z)\ C\ \nu(z)$ and $\nu \models^{\mathcal{F}}_{\Gamma_1} [\alpha]$ then there is some ν such that $\mu(z)\ C\ \nu(z)$ and $\nu \models^{\mathcal{F}}_{\Gamma_1} [\beta]$. Thus if $\mu \models^{\mathcal{L}}_{\Gamma} [\Box\alpha]$ then $\mu \models^{\mathcal{L}}_{\Gamma} [\Box\beta]$. Hence $\mu \models^{\mathcal{L}}_{\Gamma} [\Box\alpha \rightarrow \Box\beta]$. Thus $\models [\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)]$.

T-S Assume $\mu \models^{\mathcal{L}}_{\Gamma} [\Box\alpha]$. Thus, if $\Gamma \preceq \Gamma_1$ then there is some ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\alpha]$ and $\mu(z)\ C\ \nu(z)$. Moreover, if $\Gamma \preceq \Gamma_1$ then $\nu(z) = f(\mu(z))$. By reflexivity of $\preceq$ we have $\Gamma \preceq \Gamma$. Thus $\nu \models^{\mathcal{F}}_{\Gamma} [\alpha]$ and $\nu(z) = \mathrm{id}(\mu(z))$. Hence $\mu \models^{\mathcal{F}}_{\Gamma} [\alpha]$. Thus $\mu \models^{\mathcal{F}}_{\Gamma} [\Box\alpha \rightarrow \alpha]$.

4-S Assume $\mu \models^{\mathcal{L}}_{\Gamma_1} [\Box\alpha]$. Thus if $\Gamma \preceq \Gamma'$ then there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma'} [\alpha]$ and $\mu(z)\ C\ \nu(z)$ for all z in α and all Γ'. Assume $\Gamma_1 \preceq \Gamma_2$ and $\Gamma_2 \preceq \Gamma_3$. Since $\preceq$ is transitive we have $\Gamma_1 \preceq \Gamma_3$. Thus there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma_3} [\alpha]$ and $\mu(z)\ C\ \nu(z)$ and $\nu(z) = f_{23}(f_{12}(\mu(z)))$ for all z in α, where $f_{12} : Z_1 \rightarrow Z_2$ and $f_{23} : Z_2 \rightarrow Z_3$ are label, target, type preserving total one-one mappings. Hence there is some ι such that $\iota \models^{\mathcal{F}}_{\Gamma_2} [\Box\alpha]$ and $\iota(z)\ C\ \nu(z)$ and $\iota(z) = f_{12}(\mu(z))$ for all z in α. Thus $\mu \models^{\mathcal{F}}_{\Gamma_1} [\Box\Box\alpha]$ and $\mu(z)\ C\ \iota(z)$ for all z in α. Hence $\mu \models^{\mathcal{L}}_{\Gamma_1} [\Box\alpha \rightarrow \Box\Box\alpha]$.

D$\Diamond$-S Assume $\mu \models^{\mathcal{F}}_{\Gamma} [\neg\Box\neg\alpha]$. Iff $\mu \not\models^{\mathcal{F}}_{\Gamma} [\Box\neg\alpha]$. Iff not for all Γ_1: if $\Gamma \preceq \Gamma_1$ then there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\neg\alpha]$. Iff not for all Γ_1: if $\Gamma \preceq \Gamma_1$ then there is a ν such that $\nu \not\models^{\mathcal{F}}_{\Gamma_1} [\alpha]$. Iff there is a Γ_1 such that $\Gamma \preceq \Gamma_1$ and there is a ν such that $\nu \not\models^{\mathcal{F}}_{\Gamma_1} [\alpha]$. Iff $\mu \models^{\mathcal{F}}_{\Gamma} [\Diamond\alpha]$.

RN-S Assume $\models [\alpha]$. Then for all $\mathcal{F}$ and all Γ of $\mathcal{F}$ and assignments μ: $\mu \models^{\mathcal{L}}_{\Gamma} [\alpha]$. Assume $\Gamma \preceq \Gamma_1$. Then there is some ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\alpha]$ and $\nu(x) = f(\mu(z))$ for all z in α where $f : Z \rightarrow Z_1$ is target, label, and type-preserving total one-one mapping. Thus if $\Gamma \preceq \Gamma_1$ then there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\alpha]$ and $\mu(z)\ C\ \nu(z)$. Hence $\mu \models^{\mathcal{L}}_{\Gamma} [\Box\alpha]$. Thus $\models [\Box\alpha]$.

BFC-S Assume $\mu \models^{\mathcal{L}}_{\Gamma} [\Box(z_i)\alpha]$. Thus, if $\Gamma \preceq \Gamma_1$ then there is some ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [(z_i)\alpha]$ and $\mu(z_i)\ C\ \nu(z_i)$. Assume $\Gamma \preceq \Gamma_1$. Thus there is some ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [(z_i)\alpha]$ and $\mu(z_i)\ C\ \nu(z_i)$. Since $\nu \models^{\mathcal{F}}_{\Gamma_1} [(z_i)\alpha]$ we have $\iota \models^{\mathcal{F}}_{\Gamma_1} [\alpha]$ for every ι with $\iota(z_i) \in Z_{\Gamma_1}$ and $\iota(z_j) = \nu(z_j)$ if $j \neq i$. Let $\iota(z_i) = \nu(z_i)$. Since C is reflexive, symmetric, and transitive we have $\nu(z_i)\ C\ \iota(z_i)$ and $\mu(z_i)\ C\ \iota(z_i)$. Thus, if $\Gamma \preceq \Gamma_1$ then there is some ι such that $\iota \models^{\mathcal{F}}_{\Gamma_1} \alpha$ and $\mu(z)\ C\ \iota(z)$. Hence $\mu \models^{\mathcal{L}}_{\Gamma} [\Box\alpha]$.

Let δ be such that $\delta(z_i) \in Z_{\Gamma}$ and $\delta(z_j) = \mu(z_j)$ if $j \neq i$. Then $\delta \models^{\mathcal{L}}_{\Gamma} [\Box\alpha]$ since there is a ι such that $\iota \models^{\mathcal{F}}_{\Gamma_1} [\alpha]$ with $\iota(z_i) \in Z_{\Gamma_1}$ and $\iota(z_j) = f(\delta(z_j))$ for all $z_j \in Z_{\Gamma}$ where $f : Z_{\Gamma} \rightarrow \Gamma_1$ is a target, label, type, total one-one mapping which exists due to $\Gamma \preceq \Gamma_1$. Thus $\mu \models^{\mathcal{L}}_{\Gamma} [(z_i)\Box\alpha]$.

We can also prove that if the formula α that does not contain negation, universal quantification, and implication is recognized by a given partition Γ of frame $\mathcal{F}$, then α is recognized by all refinements of Γ of $\mathcal{F}$:

Theorem 1 *If $\alpha \neq \neg\beta$ and $\alpha \neq (z)\beta$ and $\alpha \neq [\beta \rightarrow \gamma]$ then if $\mu \models^{\mathcal{F}}_{\Gamma} [\alpha]$ then $\mu \models^{\mathcal{F}}_{\Gamma} [\Box\alpha]$.*

Proof by induction over the complexity of α. Assume $\alpha = E\ c$ and let $\mu \models^{\mathcal{F}}_{\Gamma} [E\ c]$. Then $I_o(c, \Gamma) \in Z_\Gamma$. Assume $\Gamma \preceq \Gamma_1$ then there exists a total target, label, type, one-one mapping $f : Z_\Gamma \rightarrow Z_{\Gamma_1}$ such that $f(I_o(c, \Gamma)) = I_o(c, \Gamma_1)$. Thus $I_o(c, \Gamma_1)) \in Z_{\Gamma_1}$ and $I_o(c, \Gamma)\ C\ I_o(c, \Gamma_1)$. Hence if $\Gamma \preceq \Gamma_1$ then there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [E\ c]$ and $\mu(z)\ C\ \nu(z)$. Thus $\mu \models^{\mathcal{F}}_{\Gamma} [\Box E\ c]$.

Assume $\alpha = E\ z$ and let $\mu \models^{\mathcal{F}}_{\Gamma} [E\ z]$. Then $\mu(c) \in Z_\Gamma$. Assume $\Gamma \preceq \Gamma_1$ then there exists a total target, label, type, one-one mapping $f : Z_\Gamma \rightarrow Z_{\Gamma_1}$ such that $f(\mu(z)) \in Z_{\Gamma_1}$. Thus there is a ν such that $\nu(z) \in Z_{\Gamma_1}$ and $\mu(z)\ C\ \nu(z)$. Thus $\nu \models^{\mathcal{F}}_{\Gamma_1} [E\ z]$ and $\mu(z)\ C\ \nu(z)$. Hence $\mu \models^{\mathcal{F}}_{\Gamma} [\Box E\ z]$.

Assume $\alpha = IO\ ct$ and let $\mu \models^{\mathcal{F}}_{\Gamma} [IO\ ct]$. Then $\psi_\Gamma(I_o(c, \Gamma)) = I_t(t)$. Assume $\Gamma \preceq \Gamma_1$ then there exists a total target, label, type, one-one mapping $f : Z_\Gamma \rightarrow Z_{\Gamma_1}$ such that $f(I_o(c, \Gamma)) = I_o(c, \Gamma_1)$ and $\psi_\Gamma(I_o(c, \Gamma_1)) = \psi_{\Gamma_1}(f(I_o(c, \Gamma)))$. Thus $I_t(t) = \psi_{\Gamma_1}(f(I_o(c, \Gamma)))$. Thus $I_t(t) = \psi_{\Gamma_1}(I_o(c, \Gamma_1))$. Hence if $\Gamma \preceq \Gamma_1$ then there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [IO\ ct]$ and $\mu(c)\ C\ \nu(c)$. Thus $\mu \models^{\mathcal{F}}_{\Gamma} [\Box IO\ ct]$.

Assume $\alpha = IO\ zt$ and let $\mu \models^{\mathcal{F}}_{\Gamma} [IO\ zt]$. Then $\psi_\Gamma(\mu(z)) = I_t(t)$. Assume $\Gamma \preceq \Gamma_1$ then there exists a total target, label, type, one-one mapping $f : Z_\Gamma \rightarrow Z_{\Gamma_1}$ such that $f(\mu(z)) \in Z_{\Gamma_1}$ and $\psi_\Gamma(\mu(z)) = \psi_{\Gamma_1}(f(I_o(\mu(z))))$. Thus $I_t(t) = \psi_{\Gamma_1}(f(\mu(z)))$. Hence if $\Gamma \preceq \Gamma_1$ then there is a ν such that $I_t(t) = \psi_{\Gamma_1}(\nu(z))$ and $\mu(z)\ C\ \nu(z)$. Thus $\nu \models^{\mathcal{F}}_{\Gamma_1} [IO\ zt]$ and $\mu(c)\ C\ \nu(c)$. Thus $\mu \models^{\mathcal{F}}_{\Gamma} [\Box IO\ zt]$.

The treatment of $\alpha = Eq\ z_1 z_2$, $\alpha = Eq\ cz$, $\alpha = Eq\ zc$, $\alpha = Eq\ cc$, $\alpha = SC\ z_1 z_2$, $\alpha = SC\ cz$, $\alpha = SC\ zc$, and $\alpha = SC\ cc$ is similar and omitted here.

Now assume that if $\mu \models^{\mathcal{F}}_{\Gamma} [\beta]$ then $\mu \models^{\mathcal{F}}_{\Gamma} [\Box\beta]$ for all β, μ and Γ (IA).

Let $\alpha = \Box\beta$ and assume $\mu \models^{\mathcal{F}}_{\Gamma} [\Box\beta]$. Then if $\Gamma \preceq \Gamma_1$ then there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma} [\beta]$ and $\mu(z)\ C\ \nu(z)$ for all variables z of β. Suppose $\Gamma \preceq \Gamma_1$. Thus, by IA, $\nu \models^{\mathcal{F}}_{\Gamma_1} [\Box\beta]$. Thus, if $\Gamma \preceq \Gamma_1$ then there is a ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\Box\beta]$ and $\mu(z)\ C\ \nu(z)$ for all variables z of β. Hence $\mu \models^{\mathcal{F}}_{\Gamma} [\Box\Box\beta]$.

Let $\alpha = \Diamond\beta$ and assume $\mu \models^{\mathcal{F}}_{\Gamma} [\Diamond\beta]$. Then for some $\Gamma_1 \in \Pi$ such that $\Gamma \preceq \Gamma_1$ there is some ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\beta]$ and $\mu(z)\ C\ \nu(z)$ for all variables z in β. Thus, by IA, $\nu \models^{\mathcal{F}}_{\Gamma_1} [\Box\beta]$. Thus if $\Gamma_1 \preceq \Gamma_2$ then there is a ι such that $\iota \models^{\mathcal{F}}_{\Gamma_2} [\beta]$ and $\nu(z)\ C\ \iota(z)$ for all variables z in β. By the reflexivity of $\preceq$ we have $\Gamma_1 \preceq \Gamma_1$ and there is a ι such that $\iota \models^{\mathcal{F}}_{\Gamma_1} [\beta]$ and $\nu(z)\ C\ \iota(z)$ for all variables z in β. Thus $\nu \models^{\mathcal{F}}_{\Gamma_1} [\Diamond\beta]$. Assume $\Gamma \preceq \Gamma_1$. Then there is some ν such that $\nu \models^{\mathcal{F}}_{\Gamma_1} [\Diamond\beta]$ and and $\mu(z)\ C\ \nu(z)$ for all variables z in β. Thus $\mu \models^{\mathcal{F}}_{\Gamma} [\Box\Diamond\beta]$.

Let $\alpha = (\exists z)\beta$ and assume $\mu \models^{\mathcal{F}}_{\Gamma} [(\exists z)\beta]$. Then $\nu \models^{\mathcal{F}}_{\Gamma} \beta$ for some ν with $\nu(z_j) = \mu(z_j)$ if $j \neq i$ and $\nu(z_i) \in \mathcal{Z}$. Thus $\nu \models^{\mathcal{F}}_{\Gamma} [\Box\beta]$ by (IA). Hence if $\Gamma \preceq \Gamma_1$ then there is a ι such that $\iota \models^{\mathcal{F}}_{\Gamma_1} [\beta]$ and $\nu(z)\ C\ \iota(z)$ for all z in β. Assume $\Gamma \preceq \Gamma_1$. Then there is a ι such that $\iota \models^{\mathcal{F}}_{\Gamma_1} [\beta]$ and $\nu(z)\ C\ \iota(z)$ for all z in β. Thus $\iota \models^{\mathcal{F}}_{\Gamma_1} \beta$ for some ι with $\iota(z_j) = \iota(z_j)$ if $j \neq i$ and $\iota(z_i) \in \mathcal{Z}$. Hence $\iota \models^{\mathcal{F}}_{\Gamma_1} [\Diamond\beta]$. Thus $\mu \models^{\mathcal{F}}_{\Gamma} [\Box\Diamond\beta]$. ∎

From theorem 1 it follows that the following formulas are valid in partition frames:

S1 $\models [(z_1)(z_2)(SC\ z_1z_2 \rightarrow \Box SC\ z_1z_2)]$
S2 $\models [(z_1)(z_2)(z_1 = z_2 \rightarrow \Box(z_1 = z_2))]$
S3 $\models [(z)(IO\ zt \rightarrow \Box IO\ zt)]$

That is, if partition Γ recognizes that cell $\mu(z_1)$ is a subcell of $\mu(z_2)$ then all refinements of Γ recognize that their counterparts of $\mu(z_1)$ are subcells of their counterparts of $\mu(z_2)$ (S1). Similarly, for $=$ (S2) and *IO* (S3).

That the following mereological principles are valid in any partition frame follows immediately from the properties of the subcell relation $\sqsubseteq$ as specified in Section 2 and the validity of RN-S:

M1-S $\models [\Box(z)(SC\ zz)]$
M2-S $\models [\Box(z_1)(z_2)(SC\ z_1z_2 \wedge SC\ z_2z_1 \rightarrow z_2 = z_1)]$
M3-S $\models [\Box(z_1)(z_2)(z_3)(SC\ z_1z_2 \wedge SC\ z_2z_3 \rightarrow SC\ z_1z_3)]$
M4-S $\models [\Box(z_1)(z_2)[(\exists z_3)(SC\ z_3z_1 \wedge SC\ z_3z_2) \rightarrow (SC\ z_1z_2 \vee SC\ z_2z_1)]]$

Consider M1-S, assignment μ and partition Γ. We have $\mu \models^{\mathcal{F}}_{\Gamma} [(z)(SC\ z)]$ since whatever member of Z_Γ the function μ assigns to the variable z ($\mu(z) \in Z_\Gamma$ by Equation 4) it holds that $\mu(z) \sqsubseteq_\Gamma \mu(z)$ due to the reflexivity of $\sqsubseteq_\Gamma$. Since if $\Gamma \preceq \Gamma_1$ there is an order-preserving total one-one mapping $f_1 : Z_\Gamma \rightarrow Z_{\Gamma_1}$ such that $f_1(\mu(z)) \sqsubseteq_{\Gamma_1} f_1(\mu(z))$. Hence $\mu \models^{\mathcal{F}}_{\Gamma} [\Box(z)(SC\ z)]$ for any μ, Γ, and $\mathcal{F}$.

5.3 The theory

Let $\mathcal{L}$ the language of our partition logic with the semantics given above. We add to $\mathcal{L}$ axioms sufficient for a first order logic with identity. We define the possibility operator $\Diamond$ as usual ($D_\Diamond$), add the S4-axioms T, 4, and K, and include the additional rule of inference (RN).

$D_\Diamond$	$\Diamond\alpha \equiv \neg\Box\neg\alpha$	RN	$\dfrac{\alpha}{\Box\alpha}$
T	$\Box\alpha \rightarrow \alpha$		
4	$\Box\alpha \rightarrow \Box\Box\alpha$	BFC	$\Box(x)\alpha \rightarrow (x)\Box\alpha$
K	$\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$	Id	$(z_1)(z_2)(z_1 = z_2 \rightarrow \Box z_1 = z_2$

We can prove that if all refinements Γ_1 of Γ recognize that for all $z \in Z_{\Gamma_1}$ it holds that α then for all $z \in \Gamma$ it holds that all refinements of Γ recognize that α, i.e., we can prove the converse of the so-called Bacon formula (BFC). We can also prove that if z_1 is identical to z_2 in some partition Γ then the counterpart of z_1 is identical to the counterpart of z_2 in all refinements of Γ (Id).

We then include axioms of reflexivity, asymmetry, and transitivity for *SC* as well as an axiom for the no-partial overlap principle:

$A1$ $(z)(SC\ zz)$
$A2$ $(z_1)(z_2)(SC\ z_1z_2 \wedge SC\ z_2z_1 \rightarrow z_1 = z_2$
$A3$ $(z_1)(z_2)(z_3)(SC\ z_1z_2 \wedge SC\ z_2z_3 \rightarrow SC\ z_1z_3$
$A4$ $(\exists z)(SC\ zz_1 \wedge SC\ zz_2) \rightarrow (SC\ z_1z_2 \vee SC\ z_2z_1)$

Using RN we can immediately derive:

$$
\begin{array}{ll}
T1 & \Box(z)SC\ zz \\
T2 & \Box(z_1)(z_2)(SC\ z_1z_2 \wedge SC\ z_2z_1 \rightarrow z_1 = z_2) \\
T3 & \Box(z_1)(z_2)(z_3)(SC\ z_1z_2 \wedge SC\ z_2z_3 \rightarrow SC\ z_1z_3) \\
T4 & \Box[(\exists z)(SC\ zz_1 \wedge SC\ zz_2) \rightarrow (SC\ z_1z_2 \vee SC\ z_2z_1)]
\end{array}
$$

Corresponding to (S1) and (S3) we require: if z_1 is a subcell of z_2 in some partition Γ then the counterpart of z_1 is a subcell of the counterpart of z_2 in all refinements of Γ (A5); if z of type t in some partition Γ then the counterpart of z is of type t in all refinements of Γ (A6).

$$
\begin{array}{ll}
\text{A5} & (z_1)(z_2)(SC\ z_1z_2 \rightarrow \Box SC\ z_1z_2) \\
\text{A6} & (z)(t)(IO\ zt \rightarrow \Box IO\ zt)
\end{array}
$$

It is easy to see that the axioms T, K, and 4 as well as definition $D_\Diamond$ and theorems BFC and Id are valid in partition frames in virtue of T-S, K-S, 4-S and $D_\Diamond$-S, BFC-S, and S-2 respectively. TM1-4 are valid in partition frames in virtue of M1-S - M4-S. Axioms A5 and A6 are valid in virtue of S1 and S3. Since RN preserves validity as we proved in RN-S it follows that reasoning in $\mathcal{L}$ is sound with respect to the semantics given above.

Within $\mathcal{L}$ we can, for example, define: z exists if and only if z is a subcell of itself (D_E); z is absent in a given partition Γ if and only if Γ does not recognize that z exists but some refinement of Γ recognizes that z exists (D_A); z_1 is an essential subcell of z_2 if and only if in all refinements of the partition which recognizes z_2, the counterpart of z_1 is a subcell of z_2 (D_{EP}).

$$
\begin{array}{ll}
D_E & E\ z \equiv SC\ zz \\
D_A & A\ z \equiv \Diamond E\ z \wedge \neg Ez \\
D_{EP} & EP\ z_1z_2 \equiv E\ z_2 \rightarrow \Box(z_1 \sqsubseteq z_2)
\end{array}
$$

These notions are interesting and useful, particularly due to their partition-theoretic interpretation. For example, our formal definitions of existence (or presence) and absence on their partition-theoretic interpretation are very close to the notions of presence and absence medical doctors use to reason about the outcome of medical tests [**?**]. Notice also that although the syntactic structure of D_{EP} is quite similar to the standard definition of essential parthood, its interpretation is quite different from the standard interpretation of what is meant by 'essential part'. To further explore the expressive and the reasoning power of $\mathcal{L}$, however, is beyond the scope of this paper.

6 Conclusions

In this paper we continued our work on the formalization of granular partitions which we started in [BS03]. We represented granular partitions as triples consisting of a rooted tree structure as first component, a domain satisfying the axioms of Extensional Mereology as second component, and a (projection) mapping of the first component (the cell tree) into the second component (the target domain) as a third component. We then assigned labels and types to the cells of the cell tree such that if the cell z projects onto the object x in the target domain, then the label assigned to z is the name of x and the type assigned to z is the type of x. The resulting structures are called labeled typed granular partitions.

We defined an ordering (refinement) relation, $\preceq$ among labeled typed granular partitions and a counterpart relation, C, which holds between cells in distinct granular partitions that project onto the same object in reality. We proved that $\preceq$ is reflexive and transitive and that C is reflexive, symmetric and transitive. Partition frames are structures formed by a set of labeled typed granular partitions, refinement relations between the partition in this set, and counterpart relations among cells of the granular partitions.

We then introduced the formal language $\mathcal{L}$ in which we can express sentences like: 'Partition Γ_x recognizes that there exists an object named 'Fred's body' of type human body', 'All partitions in context Π recognize that human bodies have right human arms as parts', 'Partition Γ_x does not recognize an object named 'Fred's left hand' ', 'An object named 'Fred's left hand' is absent in partition Γ_x'.

We gave a formal semantics of $\mathcal{L}$ with respect to the interpretation in partition frames, provided a formal system for $\mathcal{L}$ that facilitates formal reasoning, and showed that reasoning within this system is sound with respect to the given semantics.

Important properties of the formal system are:

- $\mathcal{L}$ is not interpreted directly in reality but in cell structures of granular partitions that have a projective relationship to reality;
- $\mathcal{L}$ is a modal predicate logic of type S4;
- granular partitions are treated as worlds in the sense of the standard multiple world semantics of modal logic and the refinement relation between between granular partitions is treated as an accessibility relation between worlds;
- the underlying semantics is an 'actualist' semantics in the sense that quantification is restricted to the cells of the granular partition with respect to which a quantified formula is evaluated;
- since distinct granular partitions do not share cells we establish counterpart relations between cells in distinct granular partitions that project onto the same object in reality;
- the negation in $\mathcal{L}$ is rather weak: $\sim \alpha$ means that a given partition does not recognize that α is the case;
- in $\mathcal{L}$ we can express facts about the absence of certain objects and relations between them.

To further explore the expressive and reasoning power of $\mathcal{L}$ and to firmly show its usefulness in practical applications such as bio-medicine or geographic information science is subject of ongoing research.

References

[AFG96] Alessandro Artale, Enrico Franconi, and Nicola Guarino. Open problems for part-whole relations. In *International Workshop on Description Logics*, Boston MA, 1996.

[Bit02] T. Bittner. Reasoning about qualitative spatial relations at multiple levels of granularity. In F. van Harmelen, editor, *ECAI 2002. Proceedings of the 15th European Conference on Artificial Intelligence*, pages 317–321. IOS Press, Amsterdam, 2002.

[BS01] T. Bittner and B. Smith. A taxonomy of granular partitions. In D.R. Montello, editor, *Spatial Information Theory, COSIT '01*, volume 2205 of *Lecture Notes in Computer Science*, pages 28–43. Berlin/New York: Springer, 2001.

[BS03] T. Bittner and B. Smith. A theory of granular partitions. In M. Duckham, M. F. Goodchild, and M. F. Worboys, editors, *Foundations of Geographic Information Science*, pages 117–151. London: Taylor & Francis, 2003.

[BWJ98] C. Bettini, X. Wang, and S. Jajodia. A general framework for time granularity and its application to temporal reasoning. *Annals of Mathematics and Artificial Intelligence*, 22:29–58, 1998.

[CV99] R. Casati and A. C. Varzi. *Parts and Places*. Cambridge, MA: MIT Press., 1999.

[Don01] M. Donnelly. Introducing granularity-dependent qualitative distance and diameter measures in common-sense reasoning contexts. In C. Welty and B. Smith, editors, *Formal Ontology in Information Systems*, pages 321–332. ACM Press, 2001.

[GP95] P. Gerstl and S. Pribbenow. Midwinters, end games, and body parts: a classification of part-whole relations. *Int. J. Human-Computer Studies*, 43:865–889, 1995.

[HC04] G.E. Hughes and M.J. Cresswell. *A new Introduction to Modal Logic*. Routledge, 2004.

[Hob85] J. Hobbs. Granularity. In *Proceedings of the IJCAI 85*, pages 432–435, 1985.

[Lew86] D. Lewis. *The Plurality of Worlds*. New York: Basil Blackwell, 1986.

[Rus19] B. Russell. *Introduction to Mathematical Philosophy*, chapter Descriptions, pages 167–180. Georg Allen and Unwin Ltd., 1919.

[SB02] B. Smith and B. Brogaard. Quantum mereotopology. *Annals of Mathematics and Artificial Intelligence*, 35(1–2), 2002.

[Ser83] John R. Serle. *Intentionality: An Essay in the Philosophy of Mind*. Cambridge University Press, 1983.

[Sim87] P. Simons. *Parts, A Study in Ontology*. Clarendon Press, Oxford, 1987.

[Smi04] B. Smith. Carving up reality. In M. Gorman and J. Sanford, editors, *Categories: Historical and Systematic Essays*, pages 225–237. Washington: Catholic University of America Press, 2004.

[Ste] J. G. Stell. Granulation for graphs. In C. Freksa and D. M. Mark, editors, *Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science*, Lecture Notes in Computer Science, 1661, pages 417–432. Berlin/New York: Springer.

[Ste00] J.G. Stell. The representation of discrete multi-resolution spatial knowledge. In A.G. Cohn, F. Giunchiglia, and B. Selman, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the Seventh International Conference (KR2000)*, pages 38–49. Morgan-Kaufmann, 2000.

[WCH87] M.E. Winston, R. Chaffin, and D. Herrmann. A taxonomy of Part-Whole relations. *Cognitive Science*, 11:417–444, 1987.