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Studies On the Recurrence Properties of Generalized Pentanacci Sequence

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Abstract

In this paper, we investigate the recurrence properties of the generalized Pentanacci sequence and present how the generalized Pentanacci sequence at negative indices can be expressed by the sequence itself at positive indices.

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11B37, 11B39, 11B83.

Keywords

Pentanacci numbers, Pentanacci sequence, negative indices, recurrence relations.

1. Introduction

We can propose an open problem as follows: Whether and how can the generalized Pentanacci sequence W_n at negative indices be expressed by the sequence itself at positive indices?

We present our main result as follows which completely solves the above problem for the generalized Pentanacci sequence W_n.

Theorem 1. *For $n \in \mathbb{Z}$, we have*

$W_{-n} = \frac{1}{24}v^{-n}(W_0H_n^4 - 4W_nH_n^3 + 3W_0H_{2n}^2 + 12H_n^2W_{2n} - 6W_0H_n^2H_{2n} - 6W_0H_{4n} - 8W_nH_{3n} - 12H_{2n}W_{2n} - 24H_nW_{3n} + 24W_{4n} + 8W_0H_nH_{3n} + 12W_nH_nH_{2n})$

$= v^{-n}(W_{4n} - H_nW_{3n} + \frac{1}{2}(H_n^2 - H_{2n})W_{2n} - \frac{1}{6}(H_n^3 + 2H_{3n} - 3H_{2n}H_n)W_n + \frac{1}{24}(H_n^4 + 3H_{2n}^2 - 6H_n^2H_{2n} - 6H_{4n} + 8H_{3n}H_n)W_0).$

Note that H_n can be written in terms of W_n using Remark 2 below.

The generalized (r, s, t, u, v) sequence (the generalized Pentanacci sequence or 5-step Fibonacci sequence)

$$\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, W_3, W_4; r, s, t, u, v)\}_{n\geq 0}$$

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is defined by the fifth-order recurrence relations

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4} + vW_{n-5}, \qquad W_0 = a, W_1 = b, W_2 = c, W_3 = d, W_4 = e \qquad (1)$$

where the initial values W_0, W_1, W_2, W_3, W_4 are arbitrary complex (or real) numbers and r, s, t, u, v are real numbers. Pentanacci sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [2,3,4,9] and references therein. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{u}{v}W_{-(n-1)} - \frac{t}{v}W_{-(n-2)} - \frac{s}{v}W_{-(n-3)} - \frac{r}{v}W_{-(n-4)} + \frac{1}{v}W_{-(n-5)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1) holds for all integer n.

In the following Table 1 we present some special cases of generalized Pentanacci sequence.

Table 1 A few special case of generalized Pentanacci sequences.

No	Sequences (Numbers)	Notation
1	Generalized Pentanacci	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 1)\}$
2	Generalized Fifth Order Pell	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 1, 1, 1, 1)\}$
3	Generalized Fifth Order Jacobsthal	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 1, 1, 1, 1, 2)\}$
4	Generalized 5-primes	$\{V_n\} = \{W_n(W_0, W_1, W_2, W_3, W_4; 2, 3, 5, 7, 11)\}$

In literature, for example, the following names and notations (see Table 2) are used for the special case of r, s, t, u, v and initial values.

Table 2. A few special case of generalized Pentanacci sequences.

Sequences (Numbers)	Notation	OEIS [5]	Ref
Pentanacci	$\{P_n\} = \{W_n(0, 1, 1, 2, 4; 1, 1, 1, 1, 1)\}$	A001591	[6]
Pentanacci-Lucas	$\{Q_n\} = \{W_n(5, 1, 3, 7, 15; 1, 1, 1, 1, 1)\}$	A074048	[6]
fifth order Pell	$\{P_n^{(5)}\} = \{W_n(0, 1, 2, 5, 13; 2, 1, 1, 1, 1)\}$	A141448	[7]
fifth order Pell-Lucas	$\{Q_n^{(5)}\} = \{W_n(5, 2, 6, 17, 46; 2, 1, 1, 1, 1)\}$		[7]
modified fifth-order Pell	$\{E_n^{(5)}\} = \{W_n(0, 1, 1, 3, 8; 2, 1, 1, 1, 1)\}$		[7]
fifth order Jacobsthal	$\{J_n^{(5)}\} = \{W_n(0, 1, 1, 1, 1; 1, 1, 1, 1, 2)\}$	A226310	[10,1]
fifth order Jacobsthal-Lucas	$\{j_n^{(5)}\} = \{W_n(2, 1, 5, 10, 20; 1, 1, 1, 1, 2)\}$	A226311	[10,1]
modified fifth order Jacobsthal	$\{K_n^{(5)}\} = \{W_n(3, 1, 3, 10, 20; 1, 1, 1, 1, 2)\}$		[10]
fifth-order Jacobsthal Perrin	$\{Q_n^{(5)}\} = \{W_n(3, 0, 2, 8, 16; 1, 1, 1, 1, 2)\}$		[10]
adjusted fifth-order Jacobsthal	$\{S_n^{(5)}\} = \{W_n(0, 1, 1, 2, 4; 1, 1, 1, 1, 2)\}$		[10]
modified fifth-order Jacobsthal-Lucas	$\{R_n^{(5)}\} = \{W_n(5, 1, 3, 7, 15; 1, 1, 1, 1, 2)\}$		[10]
5-primes	$\{G_n\} = \{W_n(0, 0, 0, 1, 2; 2, 3, 5, 7, 11)\}$		[8]
Lucas 5-primes	$\{H_n\} = \{W_n(5, 2, 10, 41, 150; 2, 3, 5, 7, 11)\}$		[8]
modified 5-primes	$\{E_n\} = \{W_n(0, 0, 0, 1, 1; 2, 3, 5, 7, 11)\}$		[8]

Here, OEIS stands for On-line Encyclopedia of Integer Sequences. For easy writing, from now on, we drop the superscripts from the sequences, for example we write J_n for $J_n^{(5)}$.

It is well known that the generalized (r, s, t, u, v) numbers (the generalized Pentanacci numbers) can be expressed, for all integers n, using Binet's formula

$$W_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n + A_5\lambda^n$$

where

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)}, \\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)}, \\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)}, \\ A_4 &= \frac{p_4}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)}, \\ A_5 &= \frac{p_5}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}. \end{aligned}$$

and

$$\begin{aligned} p_1 &= W_4-(\beta+\gamma+\delta+\lambda)W_3+(\beta\lambda+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta)W_2-(\beta\lambda\gamma+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta)W_1+(\beta\lambda\gamma\delta)W_0, \\ p_2 &= W_4-(\alpha+\gamma+\delta+\lambda)W_3+(\alpha\lambda+\alpha\gamma+\alpha\delta+\lambda\gamma+\lambda\delta+\gamma\delta)W_2-(\alpha\lambda\gamma+\alpha\lambda\delta+\alpha\gamma\delta+\lambda\gamma\delta)W_1+(\alpha\lambda\gamma\delta)W_0, \\ p_3 &= W_4-(\alpha+\beta+\delta+\lambda)W_3+(\alpha\beta+\alpha\lambda+\beta\lambda+\alpha\delta+\beta\delta+\lambda\delta)W_2-(\alpha\beta\lambda+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\delta)W_1+(\alpha\beta\lambda\delta)W_0, \\ p_4 &= W_4-(\alpha+\beta+\gamma+\lambda)W_3+(\alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\beta\gamma+\lambda\gamma)W_2-(\alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\beta\lambda\gamma)W_1+(\alpha\beta\lambda\gamma)W_0, \\ p_5 &= W_4-(\alpha+\beta+\gamma+\delta)W_3+(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)W_2-(\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta)W_1+(\alpha\beta\gamma\delta)W_0. \end{aligned}$$

and $\alpha,\beta,\gamma,\delta,\lambda$ are the roots of characteristic equation of W_n which are given by

$$x^5-rx^4-sx^3-tx^2-ux-v=0 \tag{2}$$

Note that we have the following identities

$$\left\{ \begin{array}{c} \alpha+\beta+\gamma+\delta+\lambda=r, \\ \alpha\beta+\alpha\lambda+\alpha\gamma+\beta\lambda+\alpha\delta+\beta\gamma+\lambda\gamma+\beta\delta+\lambda\delta+\gamma\delta=-s, \\ \alpha\beta\lambda+\alpha\beta\gamma+\alpha\lambda\gamma+\alpha\beta\delta+\alpha\lambda\delta+\beta\lambda\gamma+\alpha\gamma\delta+\beta\lambda\delta+\beta\gamma\delta+\lambda\gamma\delta=t, \\ \alpha\beta\lambda\gamma+\alpha\beta\lambda\delta+\alpha\beta\gamma\delta+\alpha\lambda\gamma\delta+\beta\lambda\gamma\delta=-u, \\ \alpha\beta\gamma\delta\lambda=v. \end{array} \right. \tag{3}$$

Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n. Now we define two special cases of the generalized (r,s,t,u,v) sequence $\{W_n\}$. (r,s,t,u,v) sequence $\{G_n\}_{n\geq 0}$ and Lucas (r,s,t,u,v) sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the fiifth-order recurrence relations

$$\begin{aligned} G_{n+5} &= rG_{n+4}+sG_{n+3}+tG_{n+2}+uG_{n+1}+vG_n, \\ G_0 &= 0, G_1=1, G_2=r, G_3=r^2+s, G_4=r^3+2sr+t, \\ H_{n+5} &= rH_{n+4}+sH_{n+3}+tH_{n+2}+uH_{n+1}+vH_n, \\ H_0 &= 5, H_1=r, H_2=2s+r^2, H_3=r^3+3sr+3t, H_4=r^4+4r^2s+4tr+2s^2+4u. \end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= -\frac{u}{v}G_{-(n-1)}-\frac{t}{v}G_{-(n-2)}-\frac{s}{v}G_{-(n-3)}-\frac{r}{v}G_{-(n-4)}+\frac{1}{v}G_{-(n-5)}, \\ H_{-n} &= -\frac{u}{v}H_{-(n-1)}-\frac{t}{v}H_{-(n-2)}-\frac{s}{v}H_{-(n-3)}-\frac{r}{v}H_{-(n-4)}+\frac{1}{v}H_{-(n-5)}, \end{aligned}$$

for $n=1,2,3,...$ respectively.

Some special cases of (r,s,t,u,v) sequence $\{G_n(0,1,r,r^2+s,r^3+2sr+t;r,s,t,u,v)\}$ and Lucas (r,s,t,u,v) sequence $\{H_n(4,r,2s+r^2,r^3+3sr+3t,r^4+4r^2s+4tr+2s^2+4u;r,s,t,u,v)\}$ are as follows:

1. $G_n(0,1,1,2,4;1,1,1,1,1) = P_n$, Pentanacci sequence,
2. $H_n(5,1,3,7,15;1,1,1,1,1) = Q_n$, Pentanacci-Lucas sequence,
3. $G_n(0,1,2,5,13;2,1,1,1,1) = P_n$, fifth-order Pell sequence,
4. $H_n(5,2,6,17,46;2,1,1,1,1) = Q_n$, fifth-order Pell-Lucas sequence,

For all integers n, (r,s,t,u,v) and Lucas (r,s,t,u,v) numbers can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\lambda)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\lambda)} + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)(\gamma-\lambda)} \\ &\quad + \frac{\delta^{n+3}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)(\delta-\lambda)} + \frac{\lambda^{n+3}}{(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma)(\lambda-\delta)}, \\ H_n &= \alpha^n + \beta^n + \gamma^n + \delta^n + \lambda^n, \end{aligned}$$

respectively.

2 The Proof of Theorem 1

To prove Theorem 1, we need following lemma.

Lemma 1. *For $n \in \mathbb{Z}$, denote*

$$S_n = \alpha^n\beta^n\lambda^n\gamma^n + \alpha^n\beta^n\lambda^n\delta^n + \alpha^n\beta^n\gamma^n\delta^n + \alpha^n\lambda^n\gamma^n\delta^n + \beta^n\lambda^n\gamma^n\delta^n$$

where $\alpha, \beta, \gamma, \delta$ and λ are as in defined in Formula (3). Then the followings hold:

(a) *For $n \in \mathbb{Z}$, we have $S_n = v^n H_{-n}$ and $S_{-n} = v^{-n} H_n$.*

(b) *S_n has the recurrence relation so that*

$$S_n = -uS_{n-1} - tvS_{n-2} - sv^2S_{n-3} - rv^3S_{n-4} + v^4S_{n-5}$$

with the initial conditions $S_0 = 5,\ S_1 = -u,\ S_2 = u^2 - 2tv, S_3 = -u^3 + 3tuv - 3sv^2, S_4 = 2t^2v^2 - 4tu^2v + u^4 + 4suv^2 - 4rv^3$. The sequence at negative indices is given by

$$S_{-n} = -\frac{-rv^3}{v^4}S_{-(n-1)} - \frac{-sv^2}{v^4}S_{-(n-2)} - \frac{-tv}{v^4}S_{-(n-3)} - \frac{-u}{v^4}S_{-(n-4)} + \frac{1}{v^4}S_{-(n-5)},\ \text{for } n = 1,2,3,....$$

(c) *S_n has the identity so that*

$$S_n = \frac{1}{24}(H_n^4 + 3H_{2n}^2 - 6H_n^2H_{2n} - 6H_{4n} + 8H_{3n}H_n).$$

Proof.

(a) From the definition of S_n and H_n, we obtain

$$\begin{aligned} H_{-n} &= \alpha^{-n} + \beta^{-n} + \gamma^{-n} + \delta^{-n} + \lambda^{-n} \\ &= \frac{\alpha^n\beta^n\lambda^n\gamma^n + \alpha^n\beta^n\lambda^n\delta^n + \alpha^n\beta^n\gamma^n\delta^n + \alpha^n\lambda^n\gamma^n\delta^n + \beta^n\lambda^n\gamma^n\delta^n}{\alpha^n\beta^n\lambda^n\gamma^n\delta^n} \\ &= \frac{S_n}{v^n} \end{aligned}$$

i.e., $S_n = v^n H_{-n}$ and so $S_{-n} = v^{-n} H_n$.

(b) With Formula (3) or using the formula $S_n = v^n H_{-n}$, we obtain initial values of S_n as

$$\begin{aligned}
S_0 &= v^0 H_0 = 5,\\
S_1 &= v^1 H_{-1} = v^1(-\frac{u}{v}) = -u,\\
S_2 &= v^2 H_{-2} = v^2(\frac{1}{v^2}(u^2 - 2tv)) = u^2 - 2tv,\\
S_3 &= v^3 H_{-3} = v^3(-\frac{1}{v^3}(u^3 - 3tuv + 3sv^2)) = -u^3 + 3tuv - 3sv^2,\\
S_4 &= v^4 H_{-4} = v^4(\frac{1}{v^4}()) = 2t^2v^2 - 4tu^2v + u^4 + 4suv^2 - 4rv^3,
\end{aligned}$$

For $n \geq 5$, we have

$$\begin{aligned}
(-u)S_{n-1} &= S_1 S_{n-1}\\
&= S_n\\
&\quad +\alpha\beta\gamma\delta\lambda(\alpha^{n-2}\beta^{n-2}\lambda^{n-2}\gamma^{n-2}(\alpha\beta\lambda + \alpha\beta\gamma + \alpha\lambda\gamma + \beta\lambda\gamma)\\
&\quad +\alpha^{n-2}\beta^{n-2}\lambda^{n-2}\delta^{n-2}(\alpha\beta\lambda + \alpha\beta\delta + \alpha\lambda\delta + \beta\lambda\delta)\\
&\quad +\alpha^{n-2}\beta^{n-2}\gamma^{n-2}\delta^{n-2}(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)\\
&\quad +\alpha^{n-2}\lambda^{n-2}\gamma^{n-2}\delta^{n-2}(\alpha\lambda\gamma + \alpha\lambda\delta + \alpha\gamma\delta + \lambda\gamma\delta)\\
&\quad +\beta^{n-2}\lambda^{n-2}\gamma^{n-2}\delta^{n-2}(\beta\lambda\gamma + \beta\lambda\delta + \beta\gamma\delta + \lambda\gamma\delta))\\
&= S_n + v(tS_{n-2} - v(-sS_{n-3} - v(rS_{n-4} - vS_{n-5})))
\end{aligned}$$

and so

$$S_n = -uS_{n-1} - tvS_{n-2} - sv^2 S_{n-3} - rv^3 S_{n-4} + v^4 S_{n-5}.$$

(c) From the definition of S_n and H_n, we get

$$\begin{aligned}
4S_n + \frac{1}{2}(H_n^2 - H_{2n})H_{2n} - H_{3n}H_n + H_{4n} &= (\alpha^n\beta^n\lambda^n + \alpha^n\beta^n\gamma^n + \alpha^n\lambda^n\gamma^n + \alpha^n\beta^n\delta^n\\
&\quad +\alpha^n\lambda^n\delta^n + \beta^n\lambda^n\gamma^n + \alpha^n\gamma^n\delta^n + \beta^n\lambda^n\delta^n\\
&\quad +\beta^n\gamma^n\delta^n + \lambda^n\gamma^n\delta^n)H_n\\
H_n^3 &= H_{3n} + 3H_{2n}H_n - 3H_{3n} + 6(\alpha^n\beta^n\lambda^n + \alpha^n\beta^n\gamma^n\\
&\quad +\alpha^n\lambda^n\gamma^n + \alpha^n\beta^n\delta^n + \alpha^n\lambda^n\delta^n + \beta^n\lambda^n\gamma^n\\
&\quad +\alpha^n\gamma^n\delta^n + \beta^n\lambda^n\delta^n + \beta^n\gamma^n\delta^n + \lambda^n\gamma^n\delta^n)
\end{aligned}$$

It now follows that

$$\begin{aligned}
H_n^4 &= H_n^3 H_n\\
&= 6H_n^2 H_{2n} - 8H_{3n}H_n - 3H_{2n}^2 + 24S_n + 6H_{4n}\\
&\Rightarrow\\
S_n &= \frac{1}{24}(H_n^4 + 3H_{2n}^2 - 6H_n^2 H_{2n} - 6H_{4n} + 8H_{3n}H_n). \quad \square
\end{aligned}$$

Now, we shall complete the proof of Theorem 1.
The Proof of Theorem 1:
Note that for $n \in \mathbb{Z}$, we have the following:

$$\begin{aligned}
(\alpha^n\beta^n + \alpha^n\lambda^n + \alpha^n\gamma^n + \beta^n\lambda^n + \alpha^n\delta^n + \beta^n\gamma^n + \lambda^n\gamma^n + \beta^n\delta^n + \lambda^n\delta^n + \gamma^n\delta^n) &= \frac{1}{2}(H_n^2 - H_{2n})\\
A_1 + A_2 + A_3 + A_4 + A_5 &= W_0\\
\delta^n\gamma^n\beta^n\alpha^n A_5 + \gamma^n\lambda^n\beta^n\alpha^n A_4 + \delta^n\lambda^n\beta^n\alpha^n A_3 + \delta^n\gamma^n\lambda^n\alpha^n A_2 + \delta^n\gamma^n\lambda^n\beta^n A_1 &= v^n W_{-n}
\end{aligned}$$

Now, for $n\in\mathbb{Z}$, we obtain

$$\begin{aligned}
& W_n\times(4S_n+\frac{1}{2}(H_n^2-H_{2n})H_{2n}-H_{3n}H_n+H_{4n}) \\
= & (A_1\alpha^n+A_2\beta^n+A_3\gamma^n+A_4\delta^n+A_5\lambda^n) \\
& (\alpha^n\beta^n\lambda^n+\alpha^n\beta^n\gamma^n+\alpha^n\lambda^n\gamma^n+\alpha^n\beta^n\delta^n \\
& +\alpha^n\lambda^n\delta^n+\beta^n\lambda^n\gamma^n+\alpha^n\gamma^n\delta^n+\beta^n\lambda^n\delta^n+\beta^n\gamma^n\delta^n+\lambda^n\gamma^n\delta^n)H_n \\
= & (\frac{1}{2}(H_n^2-H_{2n})W_{2n}-W_{3n}H_n+W_{4n}+S_nW_0-v^nW_{-n})H_n
\end{aligned}$$

By Lemma 1 (c) (using $S_n=\frac{1}{24}(H_n^4+3H_{2n}^2-6H_n^2H_{2n}-6H_{4n}+8H_{3n}H_n)$), it follows that
$W_{-n}=\frac{1}{24}v^{-n}(W_0H_n^4-4W_nH_n^3+3W_0H_{2n}^2+12H_n^2W_{2n}-6W_0H_n^2H_{2n}-6W_0H_{4n}-8W_nH_{3n}-12H_{2n}W_{2n}-24H_nW_{3n}+24W_{4n}+8W_0H_nH_{3n}+12W_nH_nH_{2n})$
$=v^{-n}(W_{4n}-H_nW_{3n}+\frac{1}{2}(H_n^2-H_{2n})W_{2n}-\frac{1}{6}(H_n^3+2H_{3n}-3H_{2n}H_n)W_n+\frac{1}{24}(H_n^4+3H_{2n}^2-6H_n^2H_{2n}-6H_{4n}+8H_{3n}H_n)W_0)$. □

Next, we present a remark which presents how H_n can be written in terms of W_n.

To express W_{-n} by the sequence itself at positive indices we need that H_n can be written in terms of W_n. For this, writing

$$H_n=a\times W_{n+4}+b\times W_{n+3}+c\times W_{n+2}+d\times W_{n+1}+e\times W_n$$

and solving the system of equations

$$\begin{aligned}
H_0 &= a\times W_4+b\times W_3+c\times W_2+d\times W_1+e\times W_0\\
H_1 &= a\times W_5+b\times W_4+c\times W_3+d\times W_2+e\times W_1\\
H_2 &= a\times W_6+b\times W_5+c\times W_4+d\times W_3+e\times W_2\\
H_3 &= a\times W_7+b\times W_6+c\times W_5+d\times W_4+e\times W_3\\
H_4 &= a\times W_8+b\times W_7+c\times W_6+d\times W_5+e\times W_4
\end{aligned}$$

or

$$\begin{pmatrix} a\\ b\\ c\\ d\\ e\end{pmatrix}=\begin{pmatrix} W_4 & W_3 & W_2 & W_1 & W_0\\ W_5 & W_4 & W_3 & W_2 & W_1\\ W_6 & W_5 & W_4 & W_3 & W_2\\ W_7 & W_6 & W_5 & W_4 & W_3\\ W_8 & W_7 & W_6 & W_5 & W_4\end{pmatrix}^{-1}\begin{pmatrix} H_0\\ H_1\\ H_2\\ H_3\\ H_4\end{pmatrix}$$

we find a,b,c,d,e so that H_n can be written in terms of W_n and we can replace this H_n in Theorem 1.

Using Theorem 1, we have the following corollary.

Corollary 1. *For $n\in\mathbb{Z}$, we have*
$H_{-n}=\frac{1}{24}v^{-n}(H_n^4+3H_{2n}^2-6H_n^2H_{2n}-6H_{4n}+8H_{3n}H_n)$.

Using Theorem 1 and Remark 2, we can give some formulas for the special cases of generalized Pentanacci sequence (generalized (r,s,t,u,v)-sequence) as follows.

We have the following corollary which gives the connection between the special cases of generalized Pentanacci sequence at the positive index and the negative index.

Corollary 2. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a) *Pentanacci sequence:*

$P_{-n}=\frac{1}{6}(6P_{4n}-3P_{2n}(8P_{2n+1}-P_{2n}+P_{2n+2}-P_{2n+4})+3P_{2n}(P_n-8P_{n+1}-P_{n+2}+P_{n+4})^2-2P_n(8P_{3n+1}-P_{3n}+P_{3n+2}-P_{3n+4})+P_n(P_n-8P_{n+1}-P_{n+2}+P_{n+4})^3+6P_{3n}(P_n-8P_{n+1}-P_{n+2}+P_{n+4})-3P_n(P_n-8P_{n+1}-P_{n+2}+P_{n+4})(8P_{2n+1}-P_{2n}+P_{2n+2}-P_{2n+4}))$.

(b) *Pentanacci-Lucas sequence:*

$Q_{-n}=\frac{1}{24}(Q_n^4+3Q_{2n}^2-6Q_n^2Q_{2n}-6Q_{4n}+8Q_{3n}Q_n).$

The following corollary illustrates the connection between the special cases of generalized fifth-order Pell sequence at the positive index and the negative index.

Corollary 3. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a) *fifth order Pell sequence:*

$P_{-n}=\frac{1}{6}(3P_{2n}(9P_{n+1}-5P_n+2P_{n+2}+P_{n+3}-P_{n+4})^2+6P_{4n}-2P_n(9P_{3n+1}-5P_{3n}+2P_{3n+2}+P_{3n+3}-P_{3n+4})-P_n(9P_{n+1}-5P_n+2P_{n+2}+P_{n+3}-P_{n+4})^3-6P_{3n}(9P_{n+1}-5P_n+2P_{n+2}+P_{n+3}-P_{n+4})-3P_{2n}(9P_{2n+1}-5P_{2n}+2P_{2n+2}+P_{2n+3}-P_{2n+4})+3P_n(9P_{n+1}-5P_n+2P_{n+2}+P_{n+3}-P_{n+4})(9P_{2n+1}-5P_{2n}+2P_{2n+2}+P_{2n+3}-P_{2n+4})).$

(b) *fifth order Pell-Lucas sequence:*

$Q_{-n}=\frac{1}{24}(Q_n^4+3Q_{2n}^2-6Q_n^2Q_{2n}-6Q_{4n}+8Q_{3n}Q_n).$

(c) *modified fifth-order Pell sequence:*

$E_{-n}=\frac{1}{6}(6E_{4n}-6E_{3n}(\frac{6}{5}E_n+\frac{37}{5}E_{n+1}-\frac{2}{5}E_{n+2}-\frac{6}{5}E_{n+3}+\frac{1}{5}E_{n+4})+3E_{2n}(\frac{6}{5}E_n+\frac{37}{5}E_{n+1}-\frac{2}{5}E_{n+2}-\frac{6}{5}E_{n+3}+\frac{1}{5}E_{n+4})^2-3E_{2n}(\frac{6}{5}E_{2n}+\frac{37}{5}E_{2n+1}-\frac{2}{5}E_{2n+2}-\frac{6}{5}E_{2n+3}+\frac{1}{5}E_{2n+4})-E_n(\frac{6}{5}E_n+\frac{37}{5}E_{n+1}-\frac{2}{5}E_{n+2}-\frac{6}{5}E_{n+3}+\frac{1}{5}E_{n+4})^3-2E_n(\frac{6}{5}E_{3n}+\frac{37}{5}E_{3n+1}-\frac{2}{5}E_{3n+2}-\frac{6}{5}E_{3n+3}+\frac{1}{5}E_{3n+4})+3E_n(\frac{6}{5}E_{2n}+\frac{37}{5}E_{2n+1}-\frac{2}{5}E_{2n+2}-\frac{6}{5}E_{2n+3}+\frac{1}{5}E_{2n+4})(\frac{6}{5}E_n+\frac{37}{5}E_{n+1}-\frac{2}{5}E_{n+2}-\frac{6}{5}E_{n+3}+\frac{1}{5}E_{n+4})).$

The following corollary illustrates the connection between the special cases of generalized generalized 5-primes sequence at the positive index and the negative index.

Corollary 4. *For $n\in\mathbb{Z}$, we have the following recurrence relations:*

(a) *5-primes sequence:*

$G_{-n}=\frac{1}{6\times 11^n}(G_n(\frac{61}{11}G_n+\frac{64}{11}G_{n+1}+\frac{67}{11}G_{n+2}-\frac{69}{11}G_{n+3}+\frac{7}{11}G_{n+4})^3+6G_{3n}(\frac{61}{11}G_n+\frac{64}{11}G_{n+1}+\frac{67}{11}G_{n+2}-\frac{69}{11}G_{n+3}+\frac{7}{11}G_{n+4})+6G_{4n}+3G_{2n}(\frac{61}{11}G_{2n}+\frac{64}{11}G_{2n+1}+\frac{67}{11}G_{2n+2}-\frac{69}{11}G_{2n+3}+\frac{7}{11}G_{2n+4})+3G_{2n}(\frac{61}{11}G_n+\frac{64}{11}G_{n+1}+\frac{67}{11}G_{n+2}-\frac{69}{11}G_{n+3}+\frac{7}{11}G_{n+4})^2+2G_n(\frac{61}{11}G_{3n}+\frac{64}{11}G_{3n+1}+\frac{67}{11}G_{3n+2}-\frac{69}{11}G_{3n+3}+\frac{7}{11}G_{3n+4})+3G_n(\frac{61}{11}G_{2n}+\frac{64}{11}G_{2n+1}+\frac{67}{11}G_{2n+2}-\frac{69}{11}G_{2n+3}+\frac{7}{11}G_{2n+4})(\frac{61}{11}G_n+\frac{64}{11}G_{n+1}+\frac{67}{11}G_{n+2}-\frac{69}{11}G_{n+3}+\frac{7}{11}G_{n+4})).$

(b) *Lucas 5-primes sequence:*

$H_{-n}=\frac{1}{24\times 11^n}(H_n^4+3H_{2n}^2-6H_n^2H_{2n}-6H_{4n}+8H_{3n}H_n).$

(c) *modified 5-primes sequence:*

$E_{-n}=\frac{1}{6\times 11^n}(6E_{3n}(\frac{130}{27}E_n+\frac{7}{3}E_{n+1}-\frac{35}{27}E_{n+2}-\frac{164}{27}E_{n+3}+\frac{29}{27}E_{n+4})+6E_{4n}+3E_{2n}(\frac{130}{27}E_n+\frac{7}{3}E_{n+1}-\frac{35}{27}E_{n+2}-\frac{164}{27}E_{n+3}+\frac{29}{27}E_{n+4})^2+3E_{2n}(\frac{130}{27}E_{2n}+\frac{7}{3}E_{2n+1}-\frac{35}{27}E_{2n+2}-\frac{164}{27}E_{2n+3}+\frac{29}{27}E_{2n+4})+E_n(\frac{130}{27}E_n+\frac{7}{3}E_{n+1}-\frac{35}{27}E_{n+2}-\frac{164}{27}E_{n+3}+\frac{29}{27}E_{n+4})^3+2E_n(\frac{130}{27}E_{3n}+\frac{7}{3}E_{3n+1}-\frac{35}{27}E_{3n+2}-\frac{164}{27}E_{3n+3}+\frac{29}{27}E_{3n+4})+3E_n(\frac{130}{27}E_{2n}+\frac{7}{3}E_{2n+1}-\frac{35}{27}E_{2n+2}-\frac{164}{27}E_{2n+3}+\frac{29}{27}E_{2n+4})(\frac{130}{27}E_n+\frac{7}{3}E_{n+1}-\frac{35}{27}E_{n+2}-\frac{164}{27}E_{n+3}+\frac{29}{27}E_{n+4})).$

In fact, we use the following identities in the last three corollaries which can be obtained by using Remark 2.

$$\begin{aligned}
Q_n &= -P_{n+4}+P_{n+2}+8P_{n+1}-P_n,\ \text{(Pentanacci case)},\\
Q_n &= -P_{n+4}+P_{n+3}+2P_{n+2}+9P_{n+1}-5P_n,\ \ \text{(fifth-order Pell case)},\\
Q_n &= \frac{1}{5}E_{n+4}-\frac{6}{5}E_{n+3}-\frac{2}{5}E_{n+2}+\frac{37}{5}E_{n+1}+\frac{6}{5}E_n,\ \text{(modified fifth-order Pell case)},\\
H_n &= -\frac{7}{11}G_{n+4}+\frac{69}{11}G_{n+3}-\frac{67}{11}G_{n+2}-\frac{64}{11}G_{n+1}-\frac{61}{11}G_n,\ \text{(5-primes case)},\\
H_n &= -\frac{29}{27}E_{n+4}+\frac{164}{27}E_{n+3}+\frac{35}{27}E_{n+2}-\frac{7}{3}E_{n+1}-\frac{130}{27}E_n,\ \text{(modified 5-primes case)},
\end{aligned}$$

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