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Dynamic Analysis of the Symbiotic Model of Commensalism and Parasitism with Harvesting in Commensal Populations

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ABSTRACT

This article discussed about a dynamic analysis of the symbiotic model of commensalism and parasitism with harvesting in the commensal population. This model is obtained from a modification of the symbiosis commensalism model. This modification is by adding a new population, namely the parasite population. Furthermore, it will be investigated that the three populations can coexist. The analysis carried out includes the determination of all equilibrium points along with their existence and local stability along with their stability requirements. From this model, it is obtained eight equilibrium points, namely three population extinction points, two population extinction points, one population extinction point and three extinction points can coexist. Of the eight points, only two points are asymptotically stable if they meet certain conditions. Next, a numerical simulation will be performed to illustrate the model's behavior. In this article, a numerical simulation was carried out using the RK-4 method. The simulation results obtained support the results of the dynamic analysis that has been done previously.

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A. INTRODUCTION

Yukalov et al., (2012) described the symbiosis between organisms. The term symbiosis describes the relationships that occur between living things with one another. For example, symbiosis of parasitism, mutualism, commensalism, neutralism, and symbiosis of amensalism. The symbiosis of commensalism is the interaction between living things that is one gain (commensal), while the others are neither benefited nor disadvantaged (host). The symbiosis of parasitism is an interaction between living things where one benefits (parasites) while the other is harmed (host).. An example of a symbiotic interaction of commensalism and parasitism is shown in Figure 1 as follows.

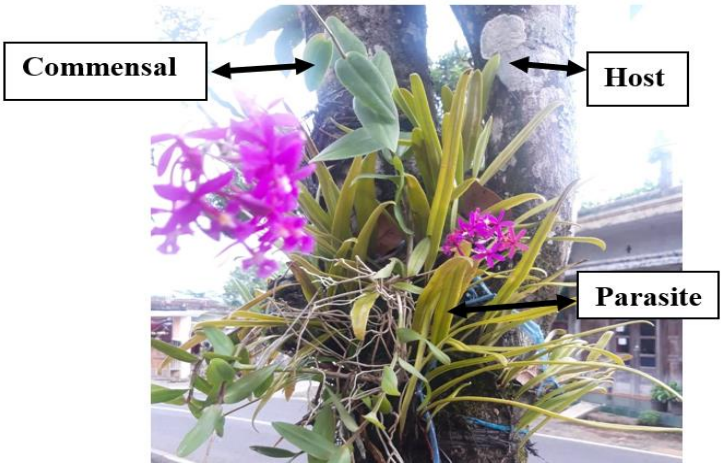

Figure 1. Symbiotic Interaction of Comensalism and Parasitism

Figure 1 describes the relationship between the mango tree (host) and orchids (commensals) and parasites (parasites) attached to the trunk of the mango tree. These interactions can be formed in the mathematical model called a symbiotic mathematical model. This model studies in mathematics how symbiotic processes in an ecosystem can be formed in systems of ordinary and nonlinear ordinary differential equations (Kenassa Edessa, 2018).

Research on the mathematical model of symbiosis has been widely carried out, for example in a two-population and three-population symbiosis model. In the symbiosis model, two populations include the symbiotic commensalism model (Prasad & Ramacharyulu, 2012), (Sun, G. C & Sun, 2013), (B. Ravindra Reddy, 2013), (XIE et al., 2015), (Wu et al., 2016), (Chen, Jinghuang & Wu, 2017), (Chen, 2018a), (Chen, 2018b), (Lin, 2018), (Zhao et al., 2018), (Chen, 2019), amensalism symbiotic model (Xie et al., 2016), (Chen, 2018c), (Liu et al., 2018), (Wu, 2018), (Guan & Chen, 2019), (Su & Chen, 2019), (Wei et al., 2020) and a mutualism symbiotic model (Ahmad, 2017). Some of these studies include harvesting in one (Chen, 2019) or both populations (Chen, Jinghuang & Wu, 2017), (Chen, 2018c), (Lin, 2018), (Su & Chen, 2019), (Ahmad, 2017). In the symbiosis model, there are three populations, which discusses predator-prey with symbiosis of commensalism (Kumar, N. P. & Ramacharyulu, 2010). Furthermore, the model is modified by adding an enemy to the predator and making the third population the host, thus forming a symbiotic interaction of amensalism (Kiran, D. R.& Reddy, 2012). Other research on the three-population symbiosis model is studying a three-population mathematical model between symbiosis of commensalism and parasitism using the Monod response function (Kenassa Edessa, 2018).

Based on the description above, this study will modify the symbiotic commensalism model by harvesting Michaelis-Menten in commensal populations (Chen, 2019) by adding a new population, namely the parasite population. The results obtained from this article are the equilibrium point, the conditions of presence and local stability at the equilibrium point. Furthermore, a numerical simulation is used to verify the results of the dynamic analysis.

B. METHODS

The research method used in this study consists of several stages as follows.

1. Specifying the Model

The model studied in this study is derived from the symbiosis commensalism model with Michaelis-Menten harvesting in the commensal population conducted by (Chen, 2019). The model is as follows.

$$\begin{aligned}\frac{dx}{dt} &= r_1 x\left(1-\frac{x}{k_1}+a\frac{y}{k_1}\right)-\frac{qEx}{m_1E+m_2x},\\ \frac{dy}{dt} &= r_2 y\left(1-\frac{y}{k_2}\right),\end{aligned} \tag{1}$$

where $x(t)$ and $y(t)$ denote the commensal population and host population, respectively. All parameters used in this model are positive. The definitions of these parameters are: r_1 and r_2 represent the intrinsic growth of x and y. k_1 and $k_2$show the carrying capacities of x and y. The parameter a is the interaction parameter between x and y. The parameter E is the fishing effort parameter used to harvest, q is the catching power coefficient and m_1, m_2 are the suitable constants. The model will be modified. Modification of the model is by adding a new population, namely the parasite population. This parasite population is detrimental to the host population.

2. Dynamic Analysis and Numerical Simulation

In dynamic analysis, it uses the definition and theorem as follows.

Definition 1. (Trahan et al., 1979) The point $\vec{x}^*$ is said to be the equilibrium point of the equation $\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}), \vec{x} \in \mathbb{R}^n$ if it satisfies the $\frac{d\vec{x}}{dt} = \vec{0}$ condition.

Theorem 1. (Trahan et al., 1979) Suppose that the eigenvalue of the Jacobi matrix $Df(\vec{x}^*)$ are $(\lambda_1,\ \lambda_2,$ dan $\lambda_3)$, that the stability criterion is:

a. Asymtotically stable, if all the eigenvalue of the Jacobi matrix $Df(\vec{x}^*)$ have a negative real part or $Re(\lambda_i) < 0, \forall i = 1, 2, 3.$
b. Unstable, if there is an eigenvalue in the Jacobi matrix $Df(\vec{x}^*)$ has a positive real part or $Re(\lambda_i) > 0, \exists i = 1, 2, 3.$

The numerical simulation section in this article used the Runge Kutta order 4 (RK-4) method and the Matlab software (R2015b).

C. RESULT AND DISCUSSION

1. Dynamical Analysis

Mathematical model investigated in this paper is a modification of model (1) by adding one species, namely a parasite species, resulting in the following model.

$$\begin{aligned}\frac{dx}{dt} &= r_1 x\left(1-\frac{x}{k_1}+a\frac{y}{k_1}\right)-\frac{qEx}{m_1E+m_2x},\\ \frac{dy}{dt} &= r_2 y\left(1-\frac{y}{k_2}-b\frac{z}{k_2}\right),\\ \frac{dz}{dt} &= r_3 z\left(1-\frac{z}{k_3}+c\frac{y}{k_3}\right),\end{aligned} \tag{2}$$

where r_3 represents the intrinsic growth of z and k_3 show the carrying capacities of z. The parameter b *and* c is the interaction parameter between y and z.

Based on Definition 1, the equilibrium point of the system of equation (2) can be obtained by solving this equation

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 0,$$

so that system (2) becomes

$$r_1 x\left(1 - \frac{x}{k_1} + a\frac{y}{k_1}\right) - \frac{qEx}{m_1 E + m_2 x} = 0, \tag{3}$$

$$r_2 y\left(1 - \frac{y}{k_2} - b\frac{z}{k_2}\right) = 0, \tag{4}$$

$$r_3 z\left(1 - \frac{z}{k_3} + c\frac{y}{k_3}\right) = 0. \tag{5}$$

Equation (3) can be written in the form

$$x = 0 \tag{3.a}$$

or

$$r_1\left(1 - \frac{x}{k_1} + a\frac{y}{k_1}\right) - \frac{qE}{m_1 E + m_2 x} = 0. \tag{3.b}$$

The solution of equation (3.b) is

$$x^* = \frac{-B_1 \pm \sqrt{{B_1}^2 - 4A_1C_1}}{2A_1}.$$

Equation (4) can be written in the form

$$y = 0 \tag{4.a}$$

or

$$1 - \frac{y}{k_2} - b\frac{z}{k_2} = 0. \tag{4.b}$$

The solution of equation (4.b) is

$$y^* = k_2 - bz^*.$$

Equation (5) can be written in the form

$$z = 0 \tag{5.a}$$

or

$$1 - \frac{z}{k_3} + c\frac{y}{k_3} = 0. \tag{5.b}$$

The solution of equation (5.b) is

$$z^* = k_3 + cy^*.$$

Theorem 2. The system of equation (2) has eight solutions, namely T_i, $i = 1,2, \ldots ,8$, then $T_i \geq 0$ is the equilibrium point of the system of equation (2) if it meets the conditions of existence. The eight points of equilibrium are as follows.

a. If $x^* = y^* = z^* = 0$ then you will get the equilibrium point $T_0 = (0,0,0)$.
b. If $x^* = y^* = 0$ then you will get the equilibrium point $T_1 = (0,0,k_3)$.
c. If $x^* = z^* = 0$ then you will get the equilibrium point $T_2 = (0,k_2,0)$.
d. If $y^* = z^* = 0$ then you will get the equilibrium point $T_3 = ({x_3}^*,0,0)$.
e. If $x^* = 0$ then you will get the equilibrium point $T_4 = \left(0, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}\right)$.
f. If $y^* = 0$ then you will get the equilibrium point $T_5 = ({x_5}^*, 0, k_3)$.
g. If $z^* = 0$ then you will get the equilibrium point $T_6 = ({x^*}_6, k_2, 0)$.

h. If $x^* = x^*_7, y^* = y^*_7$ *and* $z^* = z^*_7$ then you will get the equilibrium point

$T_7 = (x^*_7, y^*_7, z^*_7)$.

Proof.

a. $T_0 = (0,0,0)$ is obtained from a combination of equations (3.a), (4.a) and (5.a).

b. $T_1 = (0,0,k_3)$ is obtained from a combination of equations (3.a), (4.a) and (5.b), so that will be found is the value of $z^* = k_3$.

c. $T_2 = (0,k_2,0)$ is obtained from a combination of equations (3.a), (4.b) and (5.a), so that will be found is the value of $y^* = k_2$.

d. $T_3 = (x_3{}^*, 0,0)$ is obtained from a combination of equations (3.b), (4.a) and (5.a). If $D_1 = B_1{}^2 - 4A_1C_1 \geq 0$, then you will get $x^* = x_3{}^*$ which is real positive. Where, $A_1 = r_1m_2$, $B_1 = (m_1E - m_2k_1)r_1$ and $C_1 = (q - r_1m_1)k_1E$. There are several possible values for $x_3{}^*$ to be positive real as follows.

i. If $D_1 = 0$, then T_3 is the two positive twin roots, i.e.

$$T_{3+}{}^* = \frac{-B_1}{2A_1},$$

with the value $B_1 < 0$ or it can be written $\frac{m_1}{m_2} < \frac{k_1}{E}$.

ii. If $D_1 > 0$, then T_3 is the one positive roots, i.e.

$$T_{3+}{}^* = \frac{-B_1+\sqrt{D_1}}{2A_1},$$

with the value $C_1 < 0$ or it can be written $r_1 > \frac{q}{m_1}$.

iii. If $D_1 > 0$, then T_3 is the one positive roots, i.e.

$$T_{3+}{}^* = \frac{-B_1}{A_1},$$

with the value $C_1 = 0$ and $B_1 < 0$ or it can be written $r_1 = \frac{q}{m_1}$ and $\frac{m_1}{m_2} < \frac{k_1}{E}$.

iv. If $D_1 > 0$, then T_3 is the two positive roots, i.e.

$$T_{3\pm}{}^* = \frac{-B_1\pm\sqrt{D_1}}{2A_1},$$

with the value $C_1 > 0$ and $B_1 < 0$ or it can be written $r_1 > \frac{q}{m_1}$ and $\frac{m_1}{m_2} < \frac{k_1}{E}$.

e. $T_4 = \left(0, \frac{k_2-bk_3}{1+bc}, \frac{k_3+ck_2}{1+bc}\right)$ is obtained from a combination of equations (3.a), (4.b) and (5.b), so that will be found is the value of $y^* = \frac{k_2-bk_3}{1+bc}$ and $z^* = \frac{k_3+ck_2}{1+bc}$ with the condition of existence $b < \frac{k_2}{k_3}$.

f. $T_5 = (x_5{}^*, 0, k_3)$ is obtained from a combination of equations (3.b), (4.a) and (5.b) so that will be found is the value of $x^* = x_5{}^*$ and $z^* = k_3$. If $D_2 = B_2{}^2 - 4A_2C_2 \geq 0$, then you will get $x^* = x_5{}^*$ which is real positive. Where, $A_2 = r_1m_2$, $B_2 = (m_1E - m_2k_1)r_1$ and $C_2 = (q - r_1m_1)k_1E$. There are several possible values for $x_5{}^*$ to be positive real as follows.

i. If $D_2 = 0$, then T_5 is the two positive twin roots, i.e.

$$T_{5+}{}^* = \frac{-B_2}{2A_2},$$

with the value $B_2 < 0$ or it can be written $\frac{m_1}{m_2} < \frac{k_1}{E}$.

ii. If $D_2 > 0$, then T_5 is the one positive roots, i.e.

$$T_{5+}{}^{*} = \frac{-B_2+\sqrt{D_2}}{2A_2},$$

with the value $C_2 < 0$ or it can be written $r_1 > \frac{q}{m_1}$.

iii. If $D_2 > 0$, then T_5 is the one positive roots, i.e.

$$T_{5+}{}^{*} = \frac{-B_2}{A_2},$$

with the value $C_2 = 0$ and $B_2 < 0$ or it can be written $r_1 = \frac{q}{m_1}$ and $\frac{m_1}{m_2} < \frac{k_1}{E}$.

iv. If $D_2 > 0$, then T_5 is the two positive roots, i.e.

$$T_{5\pm}{}^{*} = \frac{-B_2\pm\sqrt{D_2}}{2A_2},$$

with the value $C_2 > 0$ and $B_2 < 0$ or it can be written $r_1 > \frac{q}{m_1}$ and $\frac{m_1}{m_2} < \frac{k_1}{E}$.

g. $T_6 = (x^*{}_6, k_2, 0)$ is obtained from a combination of equations (3.b), (4.b) and (5.a) so that will be found is the value of $x^* = x_6{}^*$ and $y^* = k_2$. If $D_3 = B_3{}^2 - 4A_3C_3 \geq 0$, then you will get $x^* = x_6{}^*$ which is real positive. Where, $A_3 = r_1m_2$, $B_3 = (m_1E - m_2k_1 - am_2k_2)r_1$ and $C_3 = \big((q - r_1m_1)k_1 - ar_1m_1k_2\big)E$. There are several possible values for $x_6{}^*$ to be positive real as follows.

i. If $D_3 = 0$, then T_6 is the two positive twin roots, i.e.

$$T_{6+}{}^{*} = \frac{-B_3}{2A_3},$$

with the value $B_3 < 0$ or it can be written $\frac{m_1}{m_2} < \frac{k_1}{E}$.

ii. If $D_3 > 0$, then T_6 is the one positive roots, i.e.

$$T_{6+}{}^{*} = \frac{-B_3+\sqrt{D_3}}{2A_3},$$

with the value $C_3 < 0$ or it can be written $r_1 > \frac{q}{m_1}$.

iii. If $D_3 > 0$, then T_6 is the one positive roots, i.e.

$$T_{6+}{}^{*} = \frac{-B_3}{A_3},$$

with the value $C_3 = 0$ and $B_3 < 0$ or it can be written $r_1 = \frac{q}{m_1}$ and $\frac{m_1}{m_2} < \frac{k_1}{E}$.

iv. If $D_3 > 0$, then T_6 is the two positive roots, i.e.

$$T_{6\pm}{}^{*} = \frac{-B_3\pm\sqrt{D_3}}{2A_3},$$

with the value $C_3 > 0$ and $B_3 < 0$ or it can be written $r_1 > \frac{q}{m_1}$ and $\frac{m_1}{m_2} < \frac{k_1}{E}$.

h. $T_7 = \left(x^*{}_7, \frac{k_2-bk_3}{1+bc}, \frac{k_3+ck_2}{1+bc}\right)$ is obtained from a combination of equations (3.b), (4.b) and (5.b) so that will be found is the value of $x^* = x^*{}_7$, $y^* = \frac{k_2-bk_3}{1+bc}$ and $y^* = \frac{k_3+ck_2}{1+bc}$. If $D_4 = B_4{}^2 - 4A_4C_4 \geq 0$, then you will get $x^* = x_7{}^*$ which is real positive. Where, $A_4 = (1 + bc)r_1m_2 > 0$, $B_4 = \big((1 + bc)(m_1E - m_2k_1) + (bk_3 - k_2)am_2\big)r_1$ and $C_4 = \big((1 + bc)(q - r_1m_1)k_1 + (bk_3 - k_2)ar_1m_1\big)E$. There are several possible values for $x_7{}^*$ to be positive real as follows.

i. If $D_4 = 0$, then T_7 is the two positive twin roots, i.e.

$$T_{7+}{}^{*} = \frac{-B_4}{2A_4},$$

with the value $B_4 < 0$ or it can be written $\frac{m_1E-m_2k_1}{am_2} < \frac{k_2-bk_3}{1+bc}$.

ii. If $D_4 > 0$, then T_7 is the one positive roots, i.e.

$$T_{7+}{}^* = \frac{-B_4+\sqrt{D_4}}{2A_4},$$

with the value $C_4 < 0$ or it can be written $\frac{k_1(q-r_1m_1)}{ar_1m_1} < \frac{k_2-bk_3}{1+bc}$.

iii. If $D_4 > 0$, then T_7 is the one positive roots, i.e.

$$T_{7+}{}^* = \frac{-B_4}{A_4},$$

with the value $C_4 = 0$ and $B_4 < 0$ or it can be written $\frac{k_1(q-r_1m_1)}{ar_1m_1} = \frac{k_2-bk_3}{1+bc}$ and $\frac{m_1E-m_2k_1}{am_2} < \frac{k_2-bk_3}{1+bc}$.

iv. If $D_4 > 0$, then T_7 is the two positive roots, i.e.

$$T_{7\pm}{}^* = \frac{-B_4\pm\sqrt{D_4}}{2A_4},$$

with the value $C_4 > 0$ and $B_4 < 0$ or it can be written $\frac{m_1E-m_2k_1}{am_2} < \frac{k_2-bk_3}{1+bc} < \frac{k_1(q-r_1m_1)}{ar_1m_1}$.

The local stability analysis is performed by using Jacobian matrices of (2) evaluated at the equilibrium points, namely

$$J(x^*,y^*,z^*) = \begin{bmatrix} r_1 - \frac{2r_1x^*}{k_1} + \frac{ar_1y^*}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x^*)^2} & \frac{ar_1x^*}{k_1} & 0 \\ 0 & r_2 - \frac{2r_2y^*}{k_2} & \frac{-br_2y^*}{k_2} \\ 0 & \frac{cr_3z^*}{k_3} & r_3 - \frac{2r_3z^*}{k_3} \end{bmatrix}.$$

Jacobian matrices at $T_0 = (0,0,0)$ is

$$J(T_0) = J(0,0,0) = \begin{bmatrix} r_1 - \frac{q}{m_1} & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix},$$

which has eigenvalues $\lambda_1 = r_1 - \frac{q}{m_1}$, $\lambda_2 = r_2 > 0$, and $\lambda_3 = r_3 > 0$. Based on Theorem 1, T_0 is unstable.

Jacobian matrices at $T_1 = (0,0,k_3)$ is

$$J(T_1) = \begin{bmatrix} r_1 - \frac{q}{m_1} & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & cr_3 & -r_3 \end{bmatrix},$$

with eigenvalues $\lambda_1 = r_1 - \frac{q}{m_1}$, $\lambda_2 = r_2 > 0$, and $\lambda_3 = r_3 > 0$. Based on Theorem 1, T_1 is unstable.

Jacobian matrices at $T_2 = (0,k_2,0)$ is

$$J(T_2) = \begin{bmatrix} r_1 - \frac{q}{m_1} & 0 & 0 \\ 0 & -r_2 & -br_2 \\ 0 & 0 & r_3 \end{bmatrix},$$

with eigenvalues $\lambda_1 = r_1 - \frac{q}{m_1}$, $\lambda_2 = -r_2 < 0$, and $\lambda_3 = r_3 > 0$. Based on Theorem 1, T_2 is unstable.

Jacobian matrices at $T_3 = (x^*_3, 0,0)$ is

$$J(T_3) = \begin{bmatrix} r_1 - \frac{2r_1 x^*_3}{k_1} - \frac{qm_1E^2}{(m_1E + m_2x^*_3)^2} & \frac{ar_1x^*_3}{k_1} & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix},$$

with eigenvalues $\lambda_1 = r_1 - \frac{2r_1x^*_3}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x^*_3)^2}$, $\lambda_2 = r_2 > 0$, and $\lambda_3 = r_3 > 0$. Based on Theorem 1, T_3 is unstable.

Jacobian matrices at $T_4 = \left(0, \frac{k_2 - bk_3}{1+bc}, \frac{k_3 + ck_2}{1+bc}\right)$ is

$$J(T_4) = \begin{bmatrix} r_1 + \frac{ar_1(k_2 - bk_3)}{k_1(1+bc)} - \frac{q}{m_1} & 0 & 0 \\ 0 & r_2 - \frac{2r_2(k_2 - bk_3)}{k_2(1+bc)} & \frac{-br_2(k_2 - bk_3)}{k_2(1+bc)} \\ 0 & \frac{cr_3(k_3 + ck_2)}{k_3(1+bc)} & r_3 - \frac{2r_3(k_3 + ck_2)}{k_3(1+bc)} \end{bmatrix},$$

which resulted in the following characteristic equation.

$[\lambda^2 + B\lambda + C]\left(r_1 + \frac{ar_1(k_2 - bk_3)}{k_1(1+bc)} - \frac{q}{m_1} - \lambda\right) = 0.$

where

$$B = \frac{1}{k_2k_3(1+bc)}\left(k_2k_3(r_2 + r_3) - cr_3k_2(bk_3 - 2k_2) - br_2k_3(ck_2 + 2k_3)\right)$$

and

$$C = \frac{r_2r_3}{k_2k_3(1+bc)^2}\left((1 - 5bc)k_2k_3 + (2 - bc)ck_2{}^2 + (bc - 2)bk_3{}^2\right).$$

Since existence condition of T_4 needs $r_1 + \frac{ar_1(k_2 - bk_3)}{k_1(1+bc)} - \frac{q}{m_1} < 0$, it is clear that both B and C are positive. Hence all of the eigenvalue are negative. It lead to the conclusion that T_4 is asymptotically stable.

Jacobian matrices at $T_5 = (x^*_5, 0, k_3)$ is

$$J(T_5) = \begin{bmatrix} r_1 - \frac{2r_1x^*_5}{k_1} - \frac{qm_1E^2}{(m_1E + m_2x^*_5)^2} & \frac{ar_1x^*_5}{k_1} & 0 \\ 0 & r_2 & 0 \\ 0 & cr_3 & -r_3 \end{bmatrix},$$

with eigenvalues $\lambda_1 = r_1 - \frac{2r_1x^*_5}{k_1} - \frac{qm_1E^2}{(m_1E+m_2x^*_5)^2}$, $\lambda_2 = r_2 > 0$, and $\lambda_3 = -r_3 < 0$. Based on Theorem 1, T_5 is unstable.

Jacobian matrices at $T_6 = (x^*_6, k_2, 0)$ is

$$J(T_6)=\begin{bmatrix} r_1-\frac{2r_1x^*{}_6}{k_1}+\frac{ar_1k_2}{k_1}-\frac{qm_1E^2}{(m_1E+m_2x^*{}_6)^2} & \frac{ar_1x^*{}_6}{k_1} & 0 \\ 0 & -r_2 & -br_2 \\ 0 & 0 & r_3 \end{bmatrix},$$

with eigenvalues $\lambda_1=r_1-\frac{2r_1x^*{}_6}{k_1}+\frac{ar_1k_2}{k_1}-\frac{qm_1E^2}{(m_1E+m_2x^*{}_6)^2}$, $\lambda_2=-r_2<0$, and $\lambda_3=r_3>0$. Based on Theorem 1, T_6 is unstable.

Finally, so that jacobian matrices at $T_7=\left(x^*{}_7,\frac{k_2-bk_3}{1+bc},\frac{k_3+ck_2}{1+bc}\right)$ is

$$J(T_7)=\begin{bmatrix} r_1-\frac{2r_1x^*{}_7}{k_1}+\frac{ar_1(k_2-bk_3)}{k_1(1+bc)}-\frac{qm_1E^2}{(m_1E+m_2x^*{}_7)^2} & \frac{ar_1x^*{}_7}{k_1} & 0 \\ 0 & r_2-\frac{2r_2(k_2-bk_3)}{k_2(1+bc)} & \frac{-br_2(k_2-bk_3)}{k_2(1+bc)} \\ 0 & \frac{cr_3(k_3+ck_2)}{k_3(1+bc)} & r_3-\frac{2r_3(k_3+ck_2)}{k_3(1+bc)} \end{bmatrix},$$

with the following characteristic equation.

$$[\lambda^2+B\lambda+C]\left(r_1-\frac{2r_1x^*{}_7}{k_1}+\frac{ar_1(k_2-bk_3)}{k_1(1+bc)}-\frac{qm_1E^2}{(m_1E+m_2x^*{}_7)^2}-\lambda\right)=0,$$

where

$$B=\frac{1}{k_2k_3(1+bc)}\left(k_2k_3(r_2+r_3)-cr_3k_2(bk_3-2k_2)-br_2k_3(ck_2+2k_3)\right)$$

and

$$C=\frac{r_2r_3}{k_2k_3(1+bc)^2}\left((1-5bc)k_2k_3+(2-bc)ck_2{}^2+(bc-2)bk_3{}^2\right).$$

Since existence condition of T_7 needs $r_1-\frac{2r_1x^*{}_7}{k_1}+\frac{ar_1(k_2-bk_3)}{k_1(1+bc)}-\frac{qm_1E^2}{(m_1E+m_2x^*{}_7)^2}<0$, it is clear that both B and C are positive. Hence all of the eigenvalues are negative. Based on Theorem 1, T_7 is asymptotically stable.

2. Numerical Simulation

To illustrate the dynamical analysis, we perform two simulations, by choosing the parameter values as folows. $r_1=r_2=r_3=a=c=E=k_1=k_3=m_2=1$, $b=0.1$, and $k_2=m_1=2$. In the first simulation we take $q=1$, while in the second simulation we take $q=7$. The first set of parameters satisfies condition $k_2>bk_3$, while the second one satisfies condition $k_2>bk_3$ *and* $\frac{k_1(q-r_1m_1)}{ar_1m_1}<\frac{k_2-bk_3}{1+bc}$.

The first simulation is addressed to show situation when equilibrium point $T_4=\left(0,\frac{k_2-bk_3}{1+bc},\frac{k_3+ck_2}{1+bc}\right)=(0,1.7,2.7)$ is asymptotically stable while T_7 does not exist since $\frac{k_1(q-r_1m_1)}{ar_1m_1}>\frac{k_2-bk_3}{1+bc}$. The second simulation is intended to show a situation when the equilibrium point $T_7=(2.5,1.7,2.7)$ exists and is stable while T_4 is unstable because $\frac{k_1(q-r_1m_1)}{ar_1m_1}<\frac{k_2-bk_3}{1+bc}$.

Starting from several initial values, it can be shown in Figure 2 that all paths in the simulation go to the equilibrium point T_4. These numerical results are consistent with the results of the analysis. The results of the second numerical simulation presented in Figure 3 show that when $\frac{k_1(q-r_1m_1)}{ar_1m_1}<\frac{k_2-bk_3}{1+bc}$ both equilibrium points $T_4$and T_7 exist but T_4 is no longer stable because all

paths starting from several initial values tend to be T_7. This means that these results are in accordance with the results of the analysis which states that T_7 is locally asymptotically stable.

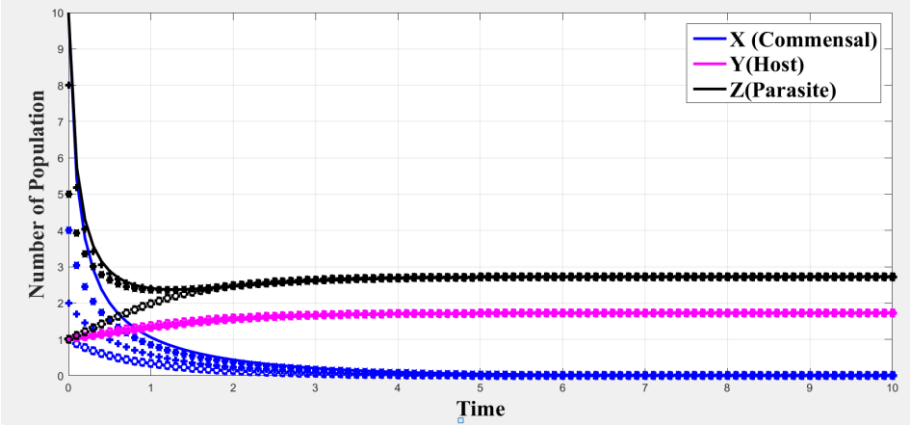

Figure 2. Numeric Simulations when $\frac{k_1(q-r_1m_1)}{ar_1m_1} > \frac{k_2-bk_3}{1+bc}$

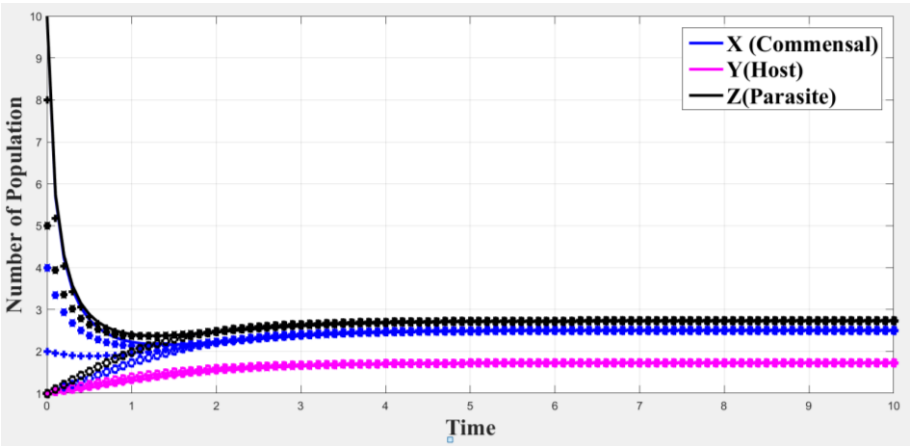

Figure 3. Numeric Simulations when $\frac{k_1(q-r_1m_1)}{ar_1m_1} < \frac{k_2-bk_3}{1+bc}$

D. CONCLUSION AND SUGGESTIONS

The model consist of three population, namely commensal population, host population, and parasite population. The dynamic analysis in this research produced eight equilibrium points with their existenceand stability properties. T_4 and T_7 are asymptotically stable if they meet the predetermined stability conditions, while the other points are always unstable. From T_4, it can be interpreted that the host species population and parasites will never become extinct, while T_7 can be interpreted that the three populations can coexist. Based on the results of the numerical simulation performed, it showed a supportive behavior with the analysis carried out. From the first simulation with the parameter values used, it showed that the graph is converging towards T_4, while from the second simulation with the parameter values used, it showed that the graph is converging towards T_7. For further research, the researchers advise to analyse this model using a discrete dynamic system and then compare the both results.

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