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The bounds for the distance two labelling and radio labelling of nanostar tree dendrimer

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ABSTRACT

The distance two labelling and radio labelling problems are applicable to find the optimal frequency assignments on AM and FM radio stations. The *distance two labelling,* known as L(2,1)-labelling of a graph A, can be defined as a function, k, from the vertex set $V(A)$ to the set of all non-negative integers such that $d(c,s)$ represents the distance between the vertices c and s in A where the absolute values of the difference between $k(c)$ and $k(s)$ are greater than or equal to both 2 and 1 if $d(c,s)$=1 and $d(c,s)=2,$ respectively. The L(2,1)-*labelling number* of A, denoted by $\lambda_{2,1}(A)$, can be defined as the smallest number j such that there is an $L(2,1)-$labeling with maximum label j. A *radio labelling* of a connected graph A is an injection k from the vertices of A to N such that $d(c,s)+|k(c)-k(s)| \geq 1+d \quad \forall\, c,s \in V(A)$, where d represents the diameter of graph A. The *radio numbers* of k and A are represented by $rn(k)$ and $rn(A)$ which are the maximum number assigned to any vertex of A and the minimum value of $rn(k)$ taken over all labellings k of A, respectively. Our main goal is to obtain the bounds for the distance two labelling and radio labelling of nanostar tree dendrimers.

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1. INTRODUCTION

In the field of communication engineering, the radio frequencies are commonly used in communication devices such as radio transmitters, computers, televisions, and mobile phones due to the fact that the frequency and energy of radio waves are very low. Researchers and engineers are working on optimizing the usage of the allotted bandwidth for a specified communication system due to the high cost of spectrum. In 1992, Griggs and Yeh [1] optimized the number of channels for the amplitude modulation (AM) radio stations in the stipulated bandwidth with the help of a graph labelling technique, known as distance two labelling. Motivated by the distance two labelling concept, Chartrand *et al.* [2] introduced in the early 21st century the radio labelling concept for the frequency modulation (FM) radio stations. This type of channel allocation concerns with the maximum number of channels in a particular geographical area such that all the stations can receive the distinct frequencies. Since the distance between transmitters and their difference in frequency has played a vital role in assigning the maximum number of channels, the distance two labelling and radio labelling can be defined as follows: The distance two labelling, denoted by *L(2,1)*-labelling of a graph A, is a function, k, from the vertex set $V(A)$ to the set of all non-negative integers such that $d(c,s)$ represents the distance between the vertices c and s in A; therefore, we have $|k(c)-k(s)| \geq 2$ and

$|k(c) - k(s)| \geq 1$ if $d(c, s) = 1$ and $d(c, s) = 2$, respectively. The L(2,1)-labelling number of A, denoted by $\lambda_{2,1}(A)$, can be defined as the smallest number j such that there is a $L(2,1)$ −labeling with maximum label j. On one hand, Fotakis *et al.* [3] proved the NP-hardness of the radio coloring problem for graphs with diameter 2. On the other hand, Fiala *et al.* [4] investigated the NP-completeness for series-parallel graphs. Havet *et al.* [5] established the optimal exact algorithm for L(2,1)-labelling as $O(3.8730^n)$ via dynamic programming. However, Szaniawski *et al.* [6] improved this bound by O^* (3.5616^n). By using the algorithm proposed by Chang ada Kuo [7], the upper bound $\lambda_{2,1}(A) \leq \Delta^2 + \Delta - 2$ was determined by Goncalves [8]. Bodlaender *et al.* [9] showed that, for a given permutation graph A, $\lambda_{2,1}(A) \leq 5\Delta - 2\lambda_{2,1}(A) \leq 5\Delta - 2$ is obtained. In addition, Sakai [10] obtained the distance two labelling of chordal graphs. Smitha and Thirusangu [11] proved the results for the quadrilateral snake Q_n as 8 and for the alternate quadrilateral snake graph Q_n as 5 for $n \geq 2$. Kujur *et al.* [12] proved that $\lambda_{2,1}(B_{m,n}) \leq 13$, where the bloom graph is $B_{m,n}(n, m > 2)$. Furthermore, Yenoke *et al.* [13] found the bounds for silicate and oxide networks as $\lambda_{2,1}(OX(n)) \leq 8$ and $\lambda_{2,1}(SL(n)) \leq 10$, respectively.

For a connected graph A, radio labelling is an injection, k, from the vertices of A to N such that d represents the diameter of a graph A, the result, $d(c, s) + |k(c) - k(s)| \geq 1 + d \ \forall \ c, s \in V(A)$, is obtained. The radio numbers of k and A are represented by $rn(k)$ and $rn(A)$ which are the maximum number assigned to any vertex of A and the minimum value of $rn(k)$ taken over all labellings k of A, respectively. The following Figure 1 depicts the definition of radio number.

For the last two decades, several authors studied the radio labelling problem for general graphs and certain interconnection networks. The radio number of the total path of graphs were determined by Vaidya and Bantva [14]. Cada *et al.* [15] obtained the radio number of distance graphs. The same problem for trees was studied by Liu [16]. Kim *et al.* [17] presented the product of graphs namely P_l $(l \geq 4)$ and $K_l(l \geq 2)$. Bharati and Yenoke [18] determined both upper and lower bounds for the hexagonal mesh as $n(3n^2 - 4n - 1) + 3$ and $3n^2 - 3n + 2 + \sum_{i=0}^{n-2} i(n - i - 1)$, respectively. Bantva [19] slightly improved the lower bound that was established in [20]. In addition, Yenoke *et al.* [21] proved that $r_n(EN\ (n, n)) \leq (n - 2)(4n^2 - 9n + 8) + 2(n - 1)^2 + (n + 1)$, where $EN\ (n, n)$ is the enhanced mesh, $n \geq 4$. This paper is divided as follows: In section 2, we discuss the methodology of our research work. In section 3, our main results are obtained by studying the bounds for the *L(2,1)*-labelling number and radio number of the general tree dendrimer $T_{n,p}$. Our research work is concluded in section 4.

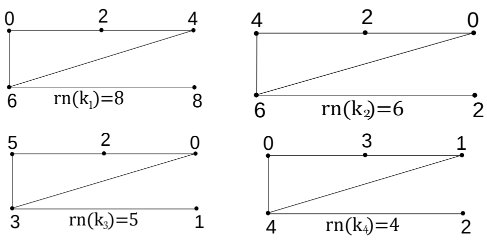

Figure 1. Different radio labellings and radio number of a graph A, $rn(A) = \min\{8,6,5,4\} = 4$

2. RESEARCH METHOD

The author studied in [18], [21] the same problem for the networks that contains the number of vertices in the n^{th} dimension as $3n^2 - 3n + 1$ and n^2, respectively. In addition, the authors in [14], [15], [17] studied the same problem for the graphs with n vertices. Since the vertices have been increased in terms of the high order for a network, especially of order n^n; therefore, finding a good solution is very complicated. The authors are trying to find a solution for such networks. However, a good bound is obtained in this paper for the tree dendrimer chemical network which grows in the order of generation. Further, most of the networks were studied separately for the L(2,1) labelling or radio labelling. Due to the exponential growth of communication technology, we are today in need to estimate the lower and upper bounds for the graphs that are growing in higher order to compete with the consumers' demand. By taking this into our account and according to the best of our knowledge, for the first time ever, we have estimated in this research work the bounds for both L(2,1)-labelling and radio labelling numbers for an n^n (number of vertices) expanding chemical structure, known as nanaostar tree dendrimer. This research study provides a detailed analysis of the growth of such graph in terms of diameter and vertices for $n, p > 2$, and its bounds have been obtained separately for $L(2,1)$-labelling number and radio labelling number. Therefore, all obtained results in this study are novel and worthy.

2.1. Nanostar tree dendrimer

Nanostar is a star-looking type of nanoparticle that contains a spherical core with many branches. Dendrimers have very complex chemical structures and hyper-branched macromolecules with a star-shaped architecture. In addition, dendrimers are classified by a generation which represents the repeated branching cycles number that are performed during its synthesis. The structure of these materials has a huge impact on the physical and chemical properties of dendrimers due to the uniqueness of dendrimers' behavior which makes them very suitable for various biomedical and industrial applications [22]-[24]. Yang and Xia [24] defined a tree dendrimer graph, denoted by $T_{n,p}$, as follows: The center vertex of the graph $T_{n,p}$ is represented by v_1^0 which is a p −regular graph except the pendant vertices. In addition, the distance from the center vertex v_1^0 to every pendant vertex is exactly n. Moreover, n signifies here the n^{th} generation of the tree dendrimer. The diameter and radius of a tree dendrimer graph are $2n$ and n, respectively.

In this research work, we have named the n generation vertices of the tree dendrimer $T_{n,p}$ as follows: First, we name the p vertices in the first generation which are adjacent to the center vertex v_1^0 as $v_1^1, v_2^1 \dots v_p^1$ in the clockwise sense. Next, we name the $p(p-1)$ vertices in the second generation as $v_1^2, v_2^2 \dots v_{p(p-1)}^2$ in the same order as we did the previous step. Similarly, we name the vertices of $3^{rd}, 4^{th} \dots (n-1)^{th}$ generations. Finally, the $p(p-1)^{n-1}$ vertices in the n^{th} generation are named as $v_1^n, v_2^n \dots v_{p(p-1)^{n-1}}^n$ as shown in Figure 2.

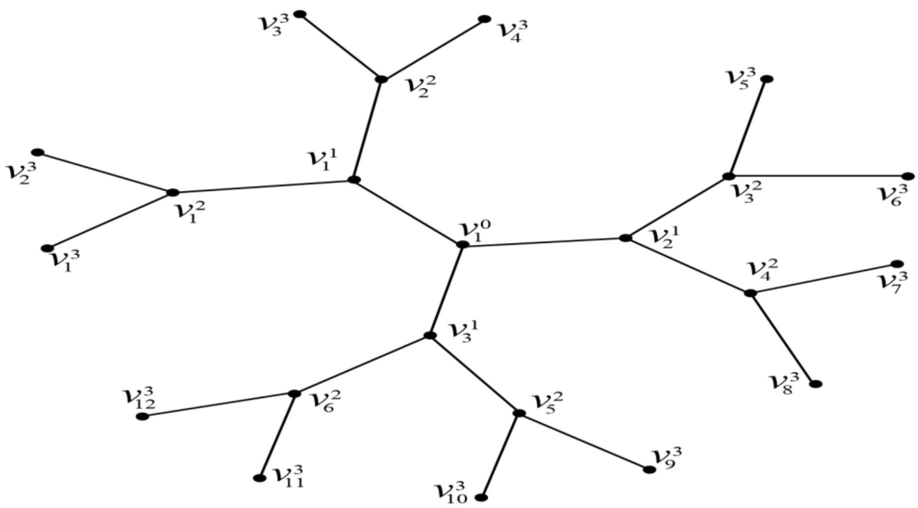

Figure 2. Vertices' Naming in $T_{3,3}$

3. RESULTS AND ANALYSIS

The bounds for the *L(2,1)*-labelling number and radio number of the general tree dendrimer $T_{n,p}$ are determined in this section.

Proposition 3.1: For any connected simple graph A of diameter 2, $\lambda_{2,1}(A) + 1 = rn(A)$.

Proof: The proof is directly derived from the definitions of *L(2,1)*-labelling and radio labelling.

a) Theorem 3.1: For $p > 2$, $\lambda_{2,1}(T_{1,p}) + 1 = rn(T_{1,p}) = p + 2$.

Proof: It is known that the tree dendrimer $T_{1,p}$ is an ordinary star graph S_{p+1}. Rajan *et al.* [25] showed that the radio number of a star graph is $rn(S_{p+1}) = p + 2, p > 2$. Since the diameter of $T_{1,p}$ is 2, from Proposition 3.1, we obtain $\lambda_{2,1}(T_{1,p}) + 1 = rn(T_{1,p}) = p + 2$, $p > 2$.

b) Theorem 3.2: For $n = 2$ and $p > 2$, the L(2,1) labelling number of $T_{n,p}$ satisfies, $\lambda_{2,1}(T_{2,p}) \leq 2p$.

Proof: By defining a mapping $k: V(T_{2,p}) \to N$, we have the following:

$$k(v_1^0) = 1$$

$$k(v_{(j-1)(p-1)+i}^2) = i + 1, i = 1, 2 \dots p - 1, j = 1, 2 \dots p$$

$$k(v_i^1) = p + 1 + i, i = 1, 2 \dots p.$$

Next, we claim that k is a valid radio 2-chromatic labelling.
Let c and s be any two vertices in $T_{2,p}$.

1) Case 1: Assume c and s are any two second generation vertices.

If $c = v^2_{(z-1)(p-1)+j}$ and $s = v^2_{(z-1)(p-1)+l}$, $1 \leq j \neq l \leq p-1$, then we have $d(c,s) = 2$ and $|k(c) - k(s)| = 1$. Therefore, we have: $d(c,s) + |k(c) - k(s)| \geq 3$.
In addition, if $c = v^2_{(j-1)(p-1)+i}$ and $c = v^2_{(l-1)(p-1)+i}$, $1 \leq j \neq l \leq p$; hence, the distance between them is 4. Therefore, the condition is trivially satisfied.

2) Case 2: Suppose c and s are the first-generation vertices, then $d(c,s) = 2$ and $k(c) = p+1+j, k(s) = p+1+l$, $1 \leq j \neq l \leq p$. Therefore, we obtain $d(c,s) + |k(c) - k(s)| \geq 2 + |j - l| \geq 3$.
3) Case 3: Suppose c is a second-generation vertex and s is a first-generation vertex, then we have: $d(c,s) \geq 1$ and $k(c) = j+1, k(s) = p+1+l$, $1 \leq j \leq p-1$, $1 \leq l \leq p$. Therefore, $d(c,s) + |k(c) - k(s)| \geq 1 + |p| \geq 3$.
4) Case 4: If c is the center vertex and $v = v^2_{(k-1)(p-1)+i}$, then $d(c,s) = 2$ and $|k(c) - k(s)| \geq 1$. Therefore, we obtain: $d(c,s) + |k(c) - k(s)| \geq 3$.
5) Case 5: If $c = v^0_1$ and $v = v^1_i$, then $k(c) = 1$, $k(s) \geq p+2$. Therefore, $d(c,s) + |k(c) - k(s)| \geq p + 3 > 3$.

Hence, the $L(2,1)$ labelling condition is satisfied, and the vertex v^1_p attains the maximum value $2p+1$. Since we start labelling from 1, we get $\lambda_{2,1}(T_{2,p}) + 1 \leq 2p+1$, which implies that $\lambda_{2,1}(T_{2,p}) \leq 2p$.

c) Theorem 3. 3: Let A be a tree dendrimer $T_{n,p}$, where $p, n > 2$, then the L(2,1) labelling number of A satisfies $\lambda_{2,1}(A) \leq 3p+1$.

Proof: Define a mapping $k: V(T_{n,p}) \to N$ as follows:

$$k(v^0_1) = 1.$$

$$k(v^1_i) = i + 2, i = 1, 2 \ldots p.$$

$$k\left(v^{3j-1}_{(z-1)(p-1)+i}\right) = p + 3 + i, i = 1, 2 \ldots p-1, z = 1,2 \ldots (p-1)^{3(j-1)}p, j = 1,2 \ldots \left\lceil\frac{n}{3}\right\rceil.$$

$$k\left(v^{3j}_{(z-1)(p-1)+i}\right) = 2p + 3 + i, i = 1, 2 \ldots p-1, z = 1,2 \ldots (p-1)^{3j-2}p, j = 1,2 \ldots \left\lceil\frac{n}{3}\right\rceil.$$

$k\left(v^{3j+1}_{(z-1)(p-1)+i}\right) = i + 2, i = 1, 2 \ldots p-1, z = 1,2 \ldots (p-1)^{3j}p, j = 1,2 \ldots \left\lceil\frac{n}{3}\right\rceil - 1$ as shown in Figure 3.

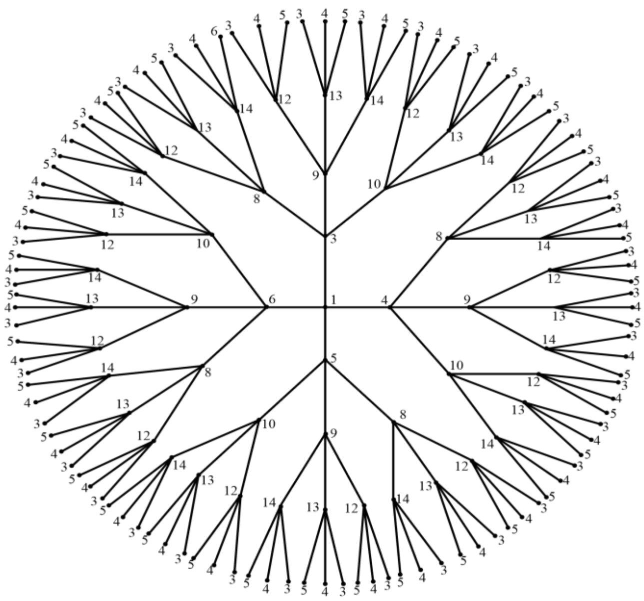

Figure 3. Radio 2-chromatic labelling of $T_{n,p}$ with $n = p = 4$ which attains the upper bound

Next, we verify the radio 2-chromatic labelling condition $d(c,s)+|k(c)-k(s)|\geq 3\ \ \forall c,s\in V(T_{n,p})$. By letting $c,s\in V(T_{n,p})$, we have the following:

1) Case 1: Suppose c and s are any two vertices in $(3j-1)^{th}$ generation, where $j=1,2\ldots\left\lceil\frac{n}{3}\right\rceil$.
   - Case 1.1: If $c=v^{3j-1}_{(p-1)(z-1)+i}$ and $s=v^{3l-1}_{(p-1)(z-1)+i}$, $1\leq j\neq l\leq\left\lceil\frac{n}{3}\right\rceil$, then we have: $d(c,s)\geq 3$ and $|k(c)-k(s)|\geq 0$. Therefore, we obtain: $d(c,s)+|k(c)-k(s)|\geq 3$.
   - Case 1.2: If $c=v^{3j-1}_{(p-1)(w-1)+i}$ and $s=v^{3j-1}_{(p-1)(t-1)+i}$, $1\leq w\neq t\leq(p-1)^{3(j-1)}p$, then we have: $d(c,s)\geq 4$ and $|k(c)-k(s)|\geq 0$. Therefore, we obtain $d(c,s)+|k(c)-k(s)|>3$
   - Case 1.3: If $c=v^{3j-1}_{(p-1)(z-1)+w}$ and $s=v^{3j-1}_{(p-1)(z-1)+t}$, $1\leq w\neq t\leq p-1$, then we have:
   - $d(c,s)=2$ and $|k(c)-k(s)|=|(p+3+w)-(p+3+t)|=|w-t|\geq 1$, since $w\neq t$. Hence, we have $d(c,s)+|k(c)-k(s)|\geq 2+1=3$.
2) Case 2: Let us assume that c and s lie in the $(3j)^{th}$ generation, where j varies from 1 to $\left\lceil\frac{n}{3}\right\rceil$.
   - Case 2.1: If c and s are of the form $v^{3j}_{(p-1)(z-1)+i}$ and $v^{3l}_{(p-1)(z-1)+i}$, $1\leq j\neq l\leq\left\lceil\frac{n}{3}\right\rceil$, then the distance between them is at least 3, which directly verifies the radio 2-chromatic labelling condition.
   - Case 2.2: If $c=v^{3j}_{(p-1)(w-1)+i}$ and $s=v^{3k}_{(p-1)(t-1)+i}$, $1\leq w\neq t\leq(p-1)^{3j-2}p$, then $d(c,s)\geq 4$. Therefore, $d(c,s)+|k(c)-k(s)|>3$.
   - Case 2.3: If $c=v^{3j}_{(p-1)(z-1)+w}$ and $s=v^{3j}_{(p-1)(z-1)+t}$, $1\leq w\neq t\leq p-1$, then $d(c,s)=2$ and $|k(c)-k(s)|=|(2p+3+w)-(2p+3+t)|$. Hence, $d(c,s)+|k(c)-k(s)|\geq 2+|w-t|\geq 3$, since $w\neq t$.
3) Case 3: Assume that c and s are $(3j+1)^{th}$ generation vertices, where $j=1,2\ldots\left\lceil\frac{n}{3}\right\rceil$.
   - Case 3.1: If $c=v^{3j+1}_{(p-1)(z-1)+i}$ and $s=v^{3l+1}_{(p-1)(z-1)+i}$, $1\leq j\neq l\leq\left\lceil\frac{n}{3}\right\rceil-1$, then $|k(c)-k(s)|\geq 0$ and the distance between them is greater than 2. Therefore, $d(c,s)+|k(c)-k(s)|\geq 3$.
   - Case 3.2: If $c=v^{3j+1}_{(p-1)(w-1)+i}$ and $s=v^{3j+1}_{(p-1)(t-1)+i}$, $1\leq w\neq t\leq(p-1)^{3j}p$, then we have: $d(c,s)\geq 4$, which trivially verifies the labelling condition.
   - Case 3.3: If $c=v^{3j-1}_{(p-1)(z-1)+w}$ and $s=v^{3j-1}_{(p-1)(z-1)+t}$, then we have $k(c)=w+2$ and $k(s)=t+2$, $1\leq w\neq t\leq p-1$. In addition, the distance between them is exactly 2. Hence, we obtain the following: $d(c,s)+|k(c)-k(s)|\geq 3$.
4) Case 4: Suppose c and s are first generation vertices then, $k(c)=w+2$ and $k(s)=t+2$, $1\leq w\neq t\leq p$. In addition, in this case $d(c,s)=2$. Therefore, $d(c,s)+|k(c)-k(s)|\geq 2+|w-t|\geq 3$ since $w\neq t$.
5) Case 5: Suppose $c=v^1_i$ and $c=v^{3j-1}_{(p-1)(z-1)+i}$, then $k(c)=i+2, 1\leq i\leq p$ and $k(s)=p+3+i, 1\leq i\leq p-1$. In addition, $d(c,s)\geq 1$. Hence, $d(c,s)+|k(c)-k(s)|\geq 1+|(p+3+1)-(p+2)|=3$.
6) Case 6: Suppose $c=v^1_i, 1\leq i\leq p$ and $c=v^{3j}_{(p-1)(z-1)+i}$, $1\leq i\leq p-1$, $1\leq z\leq(p-1)^{3(j-1)}p, 1\leq j\leq\left\lceil\frac{n}{3}\right\rceil$, then $k(c)=2+i$, and $k(s)=2p+3+i$. Also, the distance between them is at least 2. Therefore, $d(c,s)+|k(c)-k(s)|\geq 2+|(2p+3+1)-(p+2)|=p+4>3$.
7) Case 7: Suppose $c=v^1_i, 1\leq i\leq p$ and $c=v^{3j+1}_{(p-1)(z-1)+i}$, $1\leq i\leq p-1$, $1\leq z\leq(p-1)^{3j}p, 1\leq j\leq\left\lceil\frac{n}{3}\right\rceil-1$, then $|k(c)-k(s)|\geq 0$. However, the distance between them is at least 3. Hence, we obtain: $d(c,s)+|k(c)-k(s)|\geq 3$.
8) Case 8: Suppose $c=v^{3j-1}_{(p-1)(z-1)+i}$ and $c=v^{3j}_{(p-1)(z-1)+i}$, $1\leq i\leq p-1$, then we have $k(c)=p+3+i, k(s)=2p+3+i$ and $d(c,s)\geq 1$. Therefore, we obtain: $d(c,s)+|k(c)-k(s)|\geq 1+|(2p+3+1)-(2p+2)|=3$.
9) Case 9: Suppose $c=v^{3j-1}_{(p-1)(z-1)+i}$ and $c=v^{3j+1}_{(p-1)(z-1)+i}$, $1\leq i\leq p-1$, then $k(c)=p+3+i, k(s)=i+2$ and $d(c,s)\geq 2$. Therefore, we have: $d(c,s)+|k(c)-k(s)|\geq 2+|(p+4)-(p+1))|>3$.
10) Case 10: If $c=v^{3j}_{(p-1)(z-1)+i}$ and $c=v^{3j+1}_{(p-1)(z-1)+i}$, $1\leq i\leq p-1$, then we get: $k(c)=2p+3+i, k(s)=i+2$ and $d(c,s)\geq 2$. Therefore, $d(c,s)+|k(c)-k(s)|\geq 2+|(2p+4)-(p+1))|=p+5>3$.
11) Case 11: If c is the center vertex v^0_1, and s is any other vertex in the graph. Then, $d(c,s)\geq 1$ and $|k(c)-k(s)|\geq 2$. Therefore, we obtain: $d(c,s)+|k(c)-k(s)|\geq 3$.

Hence, $d(c,s)+|k(c)-k(s)|\geq 3$ for every pair of vertices c and s in $V(T_{n,p})$.

In addition, the vertices $v_{z(p-1)}^{3j}$, $z = 1,2 \dots (p-1)^{3j-2}p, j = 1,2 \dots \left\lfloor\frac{n}{3}\right\rfloor$ attains the maximum value $3p + 2$, which implies that $\lambda_{2,1}(A) + 1 \leq 3p + 2$.
Thus, $\lambda_{2,1}(A) \leq 3p + 1$.

Remark 1: The branches of the tree dendrimer which are connected to the center vertex v_1^0 by a single edge are called the main branches of the tree dendrimer graph. We denote the p main branches in $T_{n,p}$ as B^j, $j = 1,2 \dots p$.

Lemma 3.1: Let $T_{n,p}$ be a tree dendrimer graph of n generations with each vertex of degree p expects the pendant vertices, then the number of vertices in each main branch B^j $(1 \leq j \leq p)$ is $\frac{(p-1)^n-1}{p-2}$.

Proof: From the construction of the tree dendrimer, the first generation contains $p-1$ vertices. Since the root vertex of a main branch is a vertex of first generation, there is only a single vertex in the first generation. Therefore, the number of vertices in a second generation is $p-1$. In general, the number of vertices in the n^{th} generation is $(p-1)^{n-1}$. Hence, the total number of vertices in a main branch is calculated as follows:

$$1 + (p-1) + (p-1)^2 + \cdots (p-1)^{n-1} = \frac{(p-1)^{n-`1}}{p-1-1} = \frac{(p-1)^n-1}{p-2}.$$

d) Theorem 3.4: Let $T_{n,p}$ $(n, p > 2)$ be a tree dendrimer graph of diameter $2n$. Then, an upper bound for the radio number of $T_{n,p}$ is given by $rn(T_{n,p}) \leq n + (2n-1)p + 1 + \sum_{l=1}^{n-1}(2l-1)p((p-1)^{n-l-1}))(p-2) + (2l-1)p(p)^{n-l-1} - 1$, whenever $p \geq 2n-3$.

Proof: Define a mapping h from the vertex set of $T_{n,p}$ to the natural numbers as follows:
First, we label the center vertex v_1^0 as $h(v_1^0) = 1$. Since the pendant vertices are at a distance n from the center vertex, we label the n^{th} generation pendant vertices in $B^j, j = 1,2 \dots p$ as $k\left(v_{(p-1)^{n-1}(j-1)+z+(p-1)(i-1)}^{n}\right) = (p(p-1)^{n-2})(z-1) + pi - 1$, $i = 1,2 \dots (p-1)^{n-2}, z = 1,2 \dots p-1$.
Next, we label the $(n-1)^{th}$ generation vertices in $B^j, j = 1,2 \dots p$ as
$k\left(v_{(p-1)^{n-2}(j-1)+z+(p-1)(i-1)}^{n-1}\right) = (p(p-1)^{n-2})(p-2) + p((p-1)^{n-2}) - 1 + (3p((p-1)^{n-3}))(z-1) + 3pi - 1$, $i = 1,2 \dots (p-1)^{n-3}, z = 1,2 \dots p-1$.
In general, $(n-l)^{th}$ $(1 \leq l \leq n-1)$ generation vertices in $B^j, j = 1,2 \dots p$ are labelled as $k\left(v_{(p-1)^{n-l}(j-1)+z+(p-1)(i-1)}^{n-l}\right) = k\left(v_{(p-1)^{n-l+1}+p-1+(p-1)((p-1)^{n-l-3}-1)}^{n-l+1}\right) + (2l-1)p((p-1)^{n-l-2}))(z-1) + (2l-1)pi - 1, i = 1,2 \dots (p-1)^{n-l-2}, z = 1,2 \dots p-1$.
Finally, let us label the first-generation vertices of $T_{n,p}$ as $k(v_i^1) = k\left(v_{p(p-1)}^2\right) + i(2n-1) - 1$, $i = 1,2 \dots p$ as shown in Figure 4.

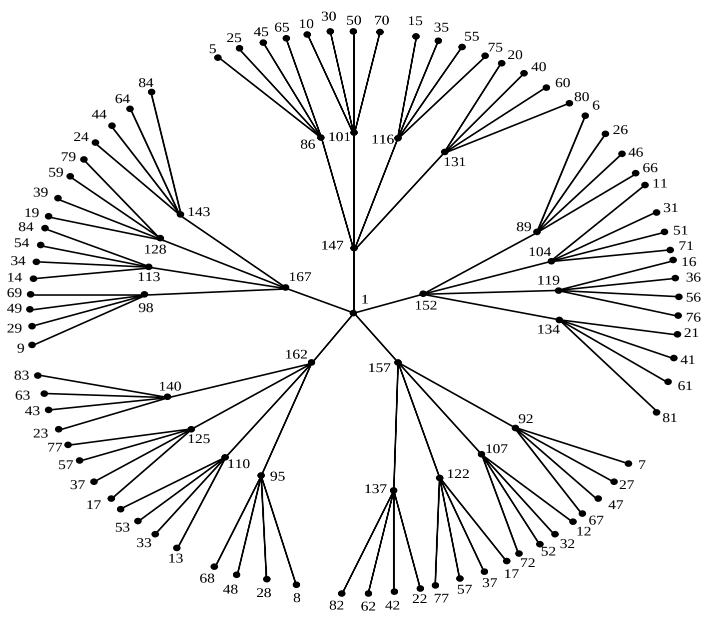

Figure 4. A radio labelling of $T_{3,5}$ which attains the upper bound

Given the diameter of the graph is $2n$, to prove k is a valid radio labelling, we must verify the condition $d(c,s) + |k(c) - k(s)| \geq 2n + 1$ for every pair of vertices c and s in $T_{n,p}$.

1) Case 1: Assume c and s are any two vertices in the same B^j $(1 \leq j \leq p)$, then we have: $d(c,s) = 2i, i = 1,2 \dots n-1$. Therefore, for such cases, $|k(c) - k(s)| \geq 2n - 2i + 1$ since $p \geq 2n - 3$. Hence, we obtain: $d(c,s) + |k(c) - k(s)| \geq |2i - (2n - 2i + 1)| = 2n + 1$.
2) Case 2: Assume c is a vertex of B^w and s is a vertex of B^t, $1 \leq w \neq t \leq p$, then the distance between them is exactly $2n$, and $|k(c) - k(s)| \geq n$. Therefore, we have: $d(c,s) + |k(c) - k(s)| \geq 2n + 1$.
3) Case 3: Assume c is the center vertex and s is any other vertex in $T_{n,p}$.
   - Case 3.1: If c is a pendant vertex, then $d(c,s) = n$ and $k(c) = 1$, $k(s) \geq n + 1$. Therefore, in this sub case, $d(c,s) + |k(c) - k(s)| \geq 2n + 1$.
   - Case 3.2: If s is not a pendant vertex, then $d(c,s) \geq 1$ and $k(c) = 1$, $k(s) \geq (p-1)^2 + n + 1$. Since $p \geq 2n - 3$, we obtain: $d(c,s) + |k(c) - k(s)| \geq 2n + 1$.

Thus, k is a valid radio labelling and the vertex $k(v_p^1)$ attains the maximum value $n + (2n-1)p + 1 + \sum_{l=1}^{n-1}(2l-1)p((p-1)^{n-l-1}))(p-2) + (2l-1)p(p)^{n-l-1} - 1$.
Hence, the radio number of $T_{n,p}$ $(n,p > 2)$ satisfies $rn(T_{n,p}) \leq n + (2n-1)p + 1 + \sum_{l=1}^{n-1}(2l - 1)p((p-1)^{n-l-1}))(p-2) + (2l-1)p(p)^{n-l-1} - 1$, whenever $p \geq 2n - 3$.
Hence, the theorem is proven.
Next, we determine the lower bound for the radio number of $T_{n,p}$ $(n,p > 2)$ by using the following theorem which was proven by Bharati and Yenoke [18]:
Theorem 3.5 (As Theorem 2 in [18]): Let A be a simple connected graph of order m. Let $m_0, m_1 \dots m_j$ be the number of vertices that have eccentricities $e_0, e_1 \dots e_j$, where $diam(A) = d = e_0 > e_1 > \dots > e_j = rad(A)$. Then, we obtain the following:

$$rn(A) \geq \begin{cases} m - 2(d - e_j) + \sum_{i=1}^{k} 2(d - e_i)m_i, & if\ m_j > 1 \\ m - (d - e_j) - (d - e_{j-1}) + \sum_{i=1}^{j} 2(d - e_i)m_i, & if\ m_j = 1 \end{cases}.$$

Lemma 3.2. For the tree dendrimer graph $T_{n,p}$, the eccentricities $e_0, e_1 \dots e_j$ are given by:

$$e_{i-1} = 2n - i - 1, i = 1,2 \dots n+1.$$

Proof: It is obvious that the diameter of the graph is $2n$, and there also exists at least one path $v_1^n, v_1^{n-1} \dots v_1^2, v_1^1, v_1^0, v_2^2 \dots v^n_{\frac{(p-1)^n - 1}{p-2}1}$ which passes through the center of the graph. Hence, the consecutive eccentricates are different by exactly 1. Therefore, the eccentricities of $T_{n,p}$ are $e_0, e_1 \dots e_n$. That is, $e_{i-1} = 2n - i - 1, i = 1,2 \dots n+1$.
Lemma 3.3. For the tree dendrimer graph $T_{n,p}$, the number of vertices that have eccentricities $e_0, e_1 \dots e_{j=n}$ are given by $m_n = 1$ and $m_{i-1} = p(p-1)^{n-i}, i = 1,2 \dots n$.
Proof: We know that every pendant vertex in $T_{n,p}$ has $(p-1)$ diametrically opposite vertices. Moreover, the number of pendant vertices in $T_{n,p}$ is $p(p-1)^{n-1}$. Therefore, the number of vertices that have eccentricity $e_0 = 2n$ is $m_0 = p(p-1)^{n-1}$. Similarly, we obtain the number of vertices with eccentricities $e_i, i = 1,2 \dots n-1$ as $m_i = p(p-1)^{n-i-1}, i = 1,2 \dots n-1$. Finally, the centre vertex is the only vertex of radius n; hence, we have: $m_n = 1$ and $m_{i-1} = p(p-1)^{n-i}, i = 1,2 \dots n$.

e) Theorem 3.6: Let A be a tree dendrimer graph, $T_{n,p}$ $(n,p > 2)$, of order $p\left(\frac{(p-1)^n - 1}{p-2}\right) + 1$. Then, the radio number of A satisfies $rn(A) \geq p\left(\frac{(p-1)^n - 1}{p-2}\right) + n + \left(\sum_{i=1}^{n-1} 2(2n - (2n - i))p(p-1)^{n-i-1}\right)$.

Proof**:** From Lemma 3.3, we have: $m_J = 1$; hence, we must apply the second part of the result in Theorem 3.5 as follows:

$$rn(A) \geq m - (d - e_j) - (d - e_{j-1}) + \sum_{i=1}^{k} 2(d - e_i)m_i$$

$$= p\left(\frac{(p-1)^n - 1}{p-2}\right) + 1 - (2n - n) - (2n - (2n-1)) + \left(\sum_{i=1}^{n-1} 2(2n - (2n - i))p(p-1)^{n-i-1}\right) + 2(2n - (2n - n)) = p\left(\frac{(p-1)^n - 1}{p-2}\right) + n + \left(\sum_{i=1}^{n-1} 2(2n - (2n - i))p(p-1)^{n-i-1}\right)$$

Hence, the theorem is proven.

By combining Theorem 3.4 with Theorem 3.6, the following theorem is obtained:

f) Theorem 3.7: For $p \geq 2n-3$, the radio number of tree dendrimer graph $T_{n,p}$ $(n,p > 2)$ satisfies $p\left(\frac{(p-1)^n-1}{p-2}\right) + n + \left(\sum_{i=1}^{n-1} 2\big(2n-(2n-i)\big)p(p-1)^{n-i-1}\right) \leq rn\big(T_{n,p}\big) \leq n + (2n-1)p + 1 + \sum_{l=1}^{n-1}(2l-1)p((p-1)^{n-l-1}))(p-2) + (2l-1)p(p)^{n-l-1} - 1.$

4. CONCLUSION

The upper bound for the $L(2,1)$ labelling number of $T_{n,p}$ as $3p+1$ for $p,n > 2$ has been obtained in this research study. The upper and lower bounds have also been determined for the radio labelling for $T_{n,p}$. For $p < 2n-3$, the radio labelling problem for $T_{n,p}$ is still an open problem. Further research can be extended to identify more chemical structures and study their properties due to their various applications in the field of communication engineering.

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