이학박사 학위논문

Generalization of continued fraction: its number-theoretical, geometrical, and combinatorial properties

(연분수의 일반화: 정수론적, 기하학적, 조합론적 성질)

2020년 8월

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Generalization of continued fraction: its number-theoretical, geometrical, and combinatorial properties

A dissertation
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Abstract

Generalization of continued fraction: its number-theoretical, geometrical, and combinatorial properties

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Continued fraction is a formal expression of the iterated fraction which is investigated in various perspectives; metrical number theory, hyperbolic geometry, and combinatorics on words. In this thesis, we consider three topics related to continued fractions.

One of the important properties of continued fraction is that the classical continued fraction gives an algorithm to generate the best approximation of every irrational as the principal convergents. We define a new continued fraction which we call odd-odd continued fraction. We prove that the odd-odd continued fraction gives best-approximations among the rationals whose denominators and numerators are both odd.

The second topic is Lévy constants of real numbers whose continued fraction expansions are Sturmian words. Lévy constant is the exponential growth rate of denominators of principal convergents of a continued fraction. We prove the existence of a real number whose continued fraction is a quasi-Sturmian word. Also, we show that the set of the Lévy constants of real numbers whose continued fractions are Sturmian words or periodic words is the whole spectrum of the Lévy constants.

The last topic is about quasi-Sturmian colorings of trees. We characterize quasi-Sturmian colorings of regular trees by its quotient graph and its recurrence functions. We find an induction algorithm of quasi-Sturmian colorings which is similar to the continued fraction algorithm of Sturmian words.

Key words: Continued fractions, Diophantine approximation, Symbolic dynamics, Sturmian words, Lévy constants, Colorings of trees
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Chapter 1

Introduction

The regular continued fraction is a formal expression of the iterated fraction:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{}{a_n + \cfrac{1}{\ddots}}}}}. \tag{1.1}$$

The investigation of the continued fractions has evolved from various perspectives, such as metrical number theory, hyperbolic geometry, and combinatorics on words. We focus on a generalization of continued fractions and related problems with the above aspects.

I. Odd-odd continued fractions

Gauss map G is defined by

$$G(x) = \left\{ \frac{1}{x} \right\} \qquad \text{for } x \in (0, 1]$$

where $\{t\}$ is the fractional part of t. Its invariant measure is $\frac{dx}{\log 2(1-x)}$ which is called Gauss measure. A continued fraction expansion is itself a sequence of nonnegative integers and we denote the continued fraction expansion of (1.1) by

$$[a_0; a_1, a_2, \cdots, a_n, \cdots].$$

Since G is the left shift map of the continued fraction expansion, we call G the continued fraction map of the regular continued fraction.

Bowen and Series defined a map associated to a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ on the boundary of Poincaré disk called Bowen-Series map by edge identifications of the fundamental domain of Γ. Bowen-Series map is used to define the expansions of boundary points [15]. Bowen-Series map associated to $\mathrm{SL}_2(\mathbb{Z})$ is related to a slow down of Gauss map, i.e., the Farey map. In other words, Gauss map is related to the cuspidal acceleration of Bowen-Series map associated to $\mathrm{SL}_2(\mathbb{Z})$ [50] (see Section 2.3 for the definitions of the Bowen-Series map and its cuspidal acceleration).

The group Θ is an index-3 subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Bowen-Series map associated to Θ is related to the Romik dynamical system. Romik examined a map on the first quadrant $\mathbf{Q}$ of the unit circle $\mathbb{S}^1$, which we call Romik map, to study Pythagorean triples [57]. We conjugate Romik map via the stereographic projection of $\mathbf{Q}$ from $(-1, 0)$ onto the y-axis. Then, we obtain the following map

$$R(x) = \begin{cases} \dfrac{x}{1-2x}, & \text{if } 0 < x \le \dfrac{1}{3}, \\ \dfrac{1}{x} - 2, & \text{if } \dfrac{1}{3} < x \le \dfrac{1}{2}, \\ 2 - \dfrac{1}{x}, & \text{if } \dfrac{1}{2} < x \le 1. \end{cases}$$

Romik map is related to Bowen-Series map associated to Θ.

Schweiger introduced the continued fraction with the even partial quotients, which we call even integer continued fraction, and proved that its corresponding continued fraction map is ergodic [59, 60]. Short and Walker showed that the convergent of the even integer continued fraction is the best approximation with the orbit $\Theta(\infty)$ consisting of the rationals of the form $\frac{\text{even}}{\text{odd}}$ and $\frac{\text{odd}}{\text{even}}$, and the converse is also true [65]. Even integer continued fraction corresponds to the cuspidal acceleration of Bowen-Series map associated to Θ

with respect to ∞.

In Chapter 2, we define a continued fraction whose convergents are in the orbit $\Theta(1)$ which is the set of rationals of the form $\frac{\text{odd}}{\text{odd}}$. We call the continued fraction odd-odd continued fraction. Odd-odd continued fraction is a counterpart of the even integer continued fraction in the sense that odd-odd continued fraction corresponds to the cuspidal acceleration of Bowen-Series map associated to Θ with respect to 1. We investigate properties of odd-odd continued fraction expansions.

A traditional question in Diophantine approximation is to find rationals a/b which minimize $|bx - a|$ for bounded b. We call a rational number p/q a best approximation if, for any $a/b \neq p/q$ with $0 < b \leq q$,

$$|qx - p| < |bx - a|.$$

It is known that the regular continued fraction gives the best approximations.

Theorem (Theorem 16 and 17 in [36]). *Every best approximation of x is a principal convergent of the regular continued fraction of x, and if the fractional part of x is not $1/2$, then a principal convergent of x is its best approximation.*

We prove that odd-odd continued fraction gives the best approximation in $\Theta(1)$, i.e., $p/q \in \Theta(1)$ such that, for any $a/b \neq p/q$ with $a/b \in \Theta(1)$ and $0 < b \leq q$,

$$|qx - p| < |bx - a|.$$

Theorem (Theorem 3.3.22). *The convergents of odd-odd continued fractions are the best approximations of an irrational with rationals whose numerators and denominators are odd, and vice versa.*

II. Lévy constants of Sturmian continued fractions

Words are sequences of finite or infinite letters. For a word with finite letters, factor complexity is a function counting the number of distinct factors (or subwords) of each length. Coven and Hedlund showed that a word is eventually periodic if and only if its factor complexity is bounded [23]. The least complexity of aperiodic words is $n + 1$. We call a word a Sturmian word if its factor complexity is $n + 1$. Sturmian words have some dynamical properties since Sturmian words can be defined by codings of orbits of irrational rotations and cutting sequences of the billiard of the square.

On the other hand, certain words in a family of Sturmian words, which we call Characteristic words, are constructed by an algorithm related to continued fraction expansions. Characteristic words represent all Sturmian words since there is exactly one characteristic word whose factor set is the same as the factor set of a fixed Sturmian word.

In Chapter 5, we deal with Lévy constants. For a continued fraction expansion $x = [a_0; a_1, a_2, \cdots]$, the nth principal convergent of x is

$$\frac{P_n(x)}{Q_n(x)} := [a_0; a_1, \cdots, a_n].$$

Theorem (Lévy [47]). *For almost every $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{\log Q_n(x)}{n} = \frac{\pi^2}{12\log 2}.$$

The limit of the equation above is called Lévy constant of x. Euler and Lagrange proved that x is a quadratic irrational if and only if x has an eventually periodic continued fraction expansion. By using Euler and Lagrange's theorem, Jager-Liardet [34] found formulas of the Lévy constants for all quadratic irrationals. Since Sturmian words have the lowest complexity of aperiodic words, Lévy constant of a number whose continued fraction is Sturmian is a reasonable next object to calculate Lévy constant.

Theorem (Theorem 5.2.8). *There exists Lévy constant of a real number if its continued fraction expansion is a Sturmian word.*

Jun Wu investigated the spectrum of Lévy constants of quadratic irrationals. Wu proved that the set of Lévy constants of quadratic irrationals is dense in $\left[\log \frac{\sqrt{5}+1}{2}, \infty\right)$ [66]. We examine the spectrum of Lévy constants of real numbers whose continued fractions are Sturmian or periodic.

Theorem (Theorem 5.2.2). *The set of Lévy constants of real numbers whose continued fraction expansions are Sturmian words or periodic words is the same as*

$$\left[\log \frac{\sqrt{5}+1}{2}, \infty\right).$$

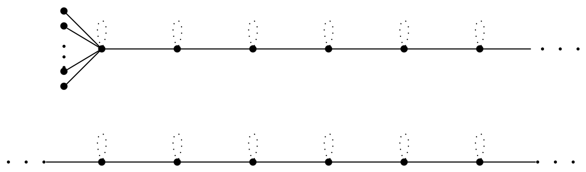


Figure 1.1: Quotient graphs of quasi-Sturmian colorings

III. Quasi-Sturmian colorings of trees

Dong Han Kim and Seonhee Lim studied vertex colorings of undirected regular trees which are the maps from the vertex set of a tree to a finite set of letters. They defined factor complexity $b_\phi(n)$ of a coloring ϕ, which is the number of colored balls of radius n up to the isomorphisms preserving ϕ. They proved the analog of Coven-Hedlund theorem and generalized Sturmian words to Sturmian colorings on a regular tree [41]. Furthermore, they examined the continued fraction algorithm of Sturmian colorings [42].

Let $\mathrm{Aut}(\mathcal{T})$ be the group of automorphisms of a regular tree $\mathcal{T}$ and Γ_ϕ be the group of color-preserving automorphisms of ϕ. Quasi-Sturmian words, which are infinite words with factor complexity eventually $n+c$, share many properties with Sturmian words. There are analogs of quasi-Sturmian words for colorings of trees. We characterize the quotient graph $\Gamma_\phi \backslash \mathcal{T}$ of a quasi-Sturmian coloring ϕ.

Theorem (Theorem 6.2.10)**.** *The quotient graph of a quasi-Sturmian coloring is a union of a finite graph and a geodesic ray or a bi-infinite geodesic as in Figure 1.1.*

The nth factor graph is the graph of which vertices are the non-equivalent colored balls of radius n and edges connecting two factors whose centers are adjacent in the underlying tree. Thus, factor graphs represent the relations of colored balls of the same radius and we can see a pattern of a coloring by observing the growth of factor graphs.

Theorem (Theorem 6.2.20)**.** *For a quasi-Sturmian colorings without cycles on factor graphs, the factor graphs evolve as*

$$\text{(I)} \to \text{(II)} \to \cdots \to \text{(II)} \to \text{(I)} \text{ or}$$
$$\text{(I)} \to \text{(II)} \to \cdots \to \text{(II)} \to \text{(III)} \to \text{(I)}$$

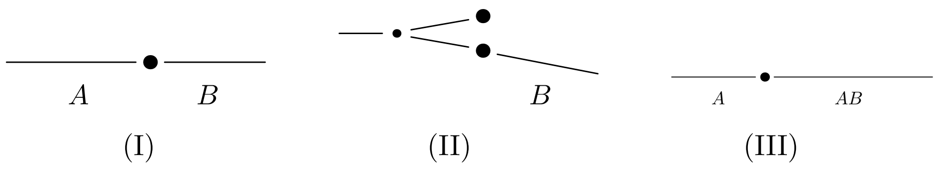


Figure 1.2: Factor graphs of a quasi-Sturmian coloring

where (I), (II), (III) looks like the figures in Figure 1.2, respectively:

The thesis is organized as follows. In Chapter 2, we review some definitions and properties of the regular continued fraction and the Bowen-Series map. In Chapter 3, we consider the Romik dynamical system and even integer continued fractions. We define the odd-odd continued fraction and investigate its properties, following the paper [39]. This is joint work with Dong Han Kim and Lingmin Liao.

In Chapter 4, we give some preliminaries of combinatorics on words and Sturmian words. In Chapter 5, we study Lévy constants. We deal with the history of Lévy constants in Section 5.1. Then, we examine the existence and the spectrum of Lévy constants of a real number whose continued fraction expansion is a Sturmian word in Section 5.2, which are from the paper [16]. This is joint work with Yann Bugeaud and Dong Han Kim.

In Chapter 6, we survey basic definitions of colorings of trees and known results. We give the proof of the main theorems about the quasi-Sturmian coloring. The contents are partly from the paper [40] which is the joint work with Dong Han Kim, Seonhee Lim, and Deokwon Sim.

Chapter 2

Generalization of continued fractions

In this chapter, we introduce the regular continued fraction and their properties. Then, we focus on a generalization of the continued fraction defined by the Bowen-Series maps and their cuspidal accelerations.

2.1 Regular continued fraction

In this section, we recall some definitions and properties of the regular continued fraction, following [36], [56] and [31].

A regular continued fraction is an iterated fraction of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \ddots}}}} \tag{2.1}$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N}$ for all $n \geq 1$. We call a_n *the nth partial quotient* or *the nth digit*. If the continued fraction is finite, then we write the continued fraction as

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}. \tag{2.2}$$

We denote a continued fraction as in (2.1) and (2.2) by sequences

$$[a_0; a_1, a_2, \cdots, a_n, \cdots] \quad \text{and} \quad [a_0; a_1, a_2, \cdots, a_n], \tag{2.3}$$

respectively, not only to save space but also to see the continued fraction as a sequence of integers. More precisely, the set of the sequences determined by the continued fractions is

$$\Omega := \left(\mathbb{Z} \times \mathbb{N}^{\mathbb{N}}\right) \cup \left(\mathbb{Z} \times \bigcup_{n=1}^{\infty} \mathbb{N}^n\right) \tag{2.4}$$

where the former is the set of the infinite continued fractions and the latter is the set of the finite continued fractions.

2.1.1 Basic properties of continued fractions

We define an approximated sequence of a continued fraction. We refer to the readers to [36] and [56] for more details.

Definition 2.1.1. The principal convergents of $[a_0; a_1, a_2, \cdots, a_n, \cdots]$ *are the truncated continued fractions*

$$\frac{P_n}{Q_n} := [a_0; a_1, a_2, \cdots, a_n] \quad \textit{for all} \quad n \geq 0.$$

The principal convergents P_n/Q_n, $n \in \mathbb{N}$ are given by

$$\begin{pmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.5}$$

where $P_{-1} = 1$ and $Q_{-1} = 0$ by convention. Thus, there are recursive relations

$$\begin{cases} P_n = a_n P_{n-1} + P_{n-2}, \\ Q_n = a_n Q_{n-1} + Q_{n-2}, \end{cases} \quad \text{for } n \geq 1. \tag{2.6}$$

The determinant of the matrix of (2.5) is $P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n+1}$ and hence

$$\frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} = \frac{(-1)^{-n+1}}{Q_n Q_{n-1}}. \tag{2.7}$$

Thus $[a_0; a_1, a_2, \cdots, a_n, \cdots]$ have the value as the limit of principal convergents $\lim_{n\to\infty} P_n/Q_n$. If $x = \lim_{n\to\infty} P_n/Q_n$, then we say that x has *a continued fraction expansion* $[a_0; a_1, a_2, \cdots, a_n, \cdots]$. We denote a_n by $a_n(x)$ and P_n/Q_n by $P_n(x)/Q_n(x)$ when we want to emphasize the value x.

Each real number $x \in \mathbb{R}$ has a continued fraction expansion. The process to find the digits of the continued fraction expansion of x is related to the Euclidean algorithm. The algorithm is to find the greatest common divisor of two integers m_0 and n_0. We recall briefly the process to find the continued fraction expansion of a rational. Without loss of generality, we assume that $|m_0| \geq n_0 > 0$. Then there are two integers a_0 and r_0 such that $m_0 = a_0 n_0 + r_0$ with $0 \leq r_0 < n_0$. We set $m_1 = n_0$ and $n_1 = r_0$, then we find a_1 and r_1 such that $m_1 = a_1 n_1 + r_1$ with $0 \leq r_1 < n_1$. We can repeat the process by setting $m_i = n_{i-1}$ and $n_i = r_{i-1}$. We end the process when $r_i = 0$. By the above expressions, we deduce the following form:

$$\frac{m_0}{n_0} = a_0 + \frac{1}{\frac{m_1}{n_1}} = a_0 + \frac{1}{a_1 + \frac{1}{\frac{m_2}{n_2}}} = \cdots$$

We note that the sign of a_0 is the same as the sign of m_0 and $a_i > 0$ for all $i \geq 1$. Thus, we have $a_i = \lfloor m_i/n_i \rfloor$, which is inductively defined by

$$\frac{m_i}{n_i} = \frac{1}{\frac{m_{i-1}}{n_{i-1}} - a_{i-1}}. \tag{2.8}$$

The process is represented by an alternative sequence of the inversion and the translation by a_i.

The above process is only for the rationals, but it can be generalized to the real numbers. Let $t = t_0$ be a real number. Then, we take the integral part of t as the 0th partial quotient a_0. As the recurrence formula in (2.8), we define

t_i by

$$t_i = \frac{1}{t_{i-1} - a_{i-1}}, \tag{2.9}$$

where a_{i-1} is the integral part of t_{i-1}. We finish the process when t_i is an integer. Note that if t is a rational, then the process is finished in a finite time, but if t is an irrational, then the process does not stop. The continued fraction expansion that we obtain by the above process is the expression of the real number t since

$$t = a_0 + \frac{1}{t_1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{t_2}} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{t_3}}} = \cdots .$$

We define a map $\Phi : \Omega \to \mathbb{R}$ by

$$\Phi(\{a_n\}_{n\geq 0}) := [a_0; a_1, \cdots, a_n, \cdots]. \tag{2.10}$$

Since $0 < [0; a_{n+1}, a_{n+2}, a_{n+3}, \cdots] < 1$ for all $n \geq 1$, the nth partial quotient a_n is the integral part of t_n. In other words, if t is an irrational, then the process in (2.9) is a unique way to find the continued fraction expansion of t. On the other hand, if t is a rational, then it has two continued fraction expansions $[a_0; a_1, \cdots, a_n]$ and $[a_0; a_1, \cdots, a_n - 1, 1]$. Then, Φ is a 1-1 map on $\mathbb{Z} \times \mathbb{N}^{\mathbb{N}}$ and it is a 2-1 map on $\mathbb{Z} \times \bigcup_{n\in\mathbb{Z}} \mathbb{N}^n$. We remark the following property.

Proposition 2.1.2. *The continued fraction is finite if and only if the corresponding value is a rational.*

Continued fractions, as we discussed above, is a tool to investigate the properties of a real number. It can be compared with the decimal (or any n-ary) system. A number with a finite decimal expansion is a rational, and whose denominator of the irreducible form has only factors 2, 5. A number with an infinite periodic decimal expansion is a rational, and vice versa. For continued fraction expansions, Euler and Lagrange showed a property of the periodic continued fraction expansions.

We denote by $[a_0; a_1, \cdots, a_n, \overline{a_{n+1}, \cdots, a_{n+k}}]$ *a periodic continued fraction* with the periodic block $a_{n+1}, \cdots, a_{n+k}$. We call the length of the minimal repeated block *the period.*

Theorem 2.1.3 (Euler)**.** *If x has a periodic continued fraction expansion, then x is* a quadratic irrational*, i.e., an irrational solution of a quadratic equation.*

In order to show the Euler's theorem, we need a relation between x and the principal convergents P_n/Q_n. We define *the nth complete quotient*

$$x_n := [a_n; a_{n+1}, a_{n+2}, \cdots].$$

Since $x_n = a_n + 1/x_{n+1}$, we infer the following relation

$$\frac{P_n x_{n+1} + P_{n-1}}{Q_n x_{n+1} + Q_{n-1}} = \frac{P_n(a_{n+1} + 1/x_{n+2}) + P_{n-1}}{Q_n(a_{n+1} + 1/x_{n+2}) + Q_{n-1}} = \frac{P_{n+1}x_{n+2} + P_n}{Q_{n+1}x_{n+2} + Q_n}.$$

By induction, we can express x by its complete quotients as

$$x = \frac{P_n x_{n+1} + P_{n-1}}{Q_n x_{n+1} + Q_{n-1}} \quad \text{for } n \geq 0. \tag{2.11}$$

If x has a periodic continued fraction expansion, then $x_{i+1} = x_{j+1}$ for some i and j. By (2.11), we have

$$x_{i+1} = x_{j+1} = \frac{Q_{i-1}x - P_{i-1}}{Q_i x - P_i} = \frac{Q_{j-1}x - P_{j-1}}{Q_j x - P_j}.$$

The above equation is equivalent to a quadratic equation and its discriminant is not a square number. The converse of Theorem 2.1.3 holds.

Theorem 2.1.4 (Lagrange)**.** *A quadratic irrational has a periodic continued fraction expansion.*

Later, Charves gave a shorter proof. Let us consider a real number x and the complete quotients $x_0, x_1, x_2, \cdots$. The idea of the Charves' proof is that the quadratic equations of x_n, $n \geq 0$ have bounded coefficients. Similarly, in the Lagrange's proof, he showed that there are finitely many complete quotients by using the fact that the discriminants of complete quotients are the same. See Theorem 28 in [36] and Theorem 1-3 in [56, Chapter III, §1] for more details of Euler and Lagrange theorem.

In the rest of the section, we consider Diophantine approximation. Diophantine approximation problem is to find a rational which is "close" to a fixed irrational number. More precisely,

Definition 2.1.5. *We call a rational* p/q a best approximation of $x \in \mathbb{R}$ *if*

$$|qx - p| < |bx - a| \text{ for all } a/b \neq p/q \text{ such that } 0 < q \leq b.$$

In other words, $|qx - p|$ is the distance $||qx||$ of qx mod 1 from 0 on the unit circle $\mathbb{S}^1$ whose circumference is normalized by 1. Thus, if p/q is a best approximation of x, then $||qx|| < ||bx||$ holds for any $0 < q \leq b$. Lagrange established a connection between the best approximations and regular continued fractions.

Theorem 2.1.6. *Every best approximation of x is a principal convergent of the regular continued fraction of x, and if the fractional part of x is not* $1/2$*, then a principal convergent of x is its best approximation.*

The theorem tells us that the continued fraction gives an algorithm to find the best approximations. There are several versions of proofs of the theorem. In his monograph, Khinchin gave an arithmetic proof (see Theorem 16 and Theorem 17 in [36]). Irwin proved the theorem using plane lattices in [32]. Short gave the proof using Ford circles in [64] (see Definition 3.3.10 for the definition of Ford circles).

2.1.2 Gauss map and related dynamical systems

In this subsection, we deal with a dynamical system related to the regular continued fraction.

Gauss map or *The continued fraction map* $G : [0, 1] \to [0, 1]$ is defined by

$$G(x) = \begin{cases} \dfrac{1}{x} - \left\lfloor \dfrac{1}{x} \right\rfloor & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $\lfloor t \rfloor$ is the greatest integer not exceeding t. The Gauss map is piecewise invertible and has infinitely many branches as in Figure 2.1. In (2.8) and (2.9) of the previous section, we see that $n_i/m_i = G(n_{i-1}/m_{i-1})$ and $1/t_i = G(1/t_{i-1})$. More precisely, the Gauss map G is the left shift map of continued fraction expansions:

$$G([0; a_1, a_2, \cdots, a_n, \cdots]) = [0; a_2, a_3, \cdots, a_{n+1}, \cdots].$$

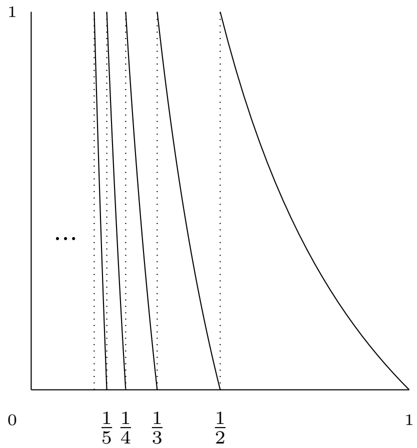


Figure 2.1: Gauss map

Using Gauss map, we obtain continued fraction expansions of real numbers in $(0,1)$: the nth partial quotient $a_n(x)$ is $\lfloor (G^{n-1}(x))^{-1} \rfloor$.

The Gauss map G has an invariant measure $\mu = (\log 2(1+t))^{-1} dt$. Using

$$G^{-1}([0,x)) = \bigcup_{n=1}^{\infty} \left[\frac{1}{x+n}, \frac{1}{n} \right],$$

it follows that the measure $\mu(G^{-1}([0,x)))$ is equal to $\mu([0,x))$:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \int_{\frac{1}{x+n}}^{\frac{1}{n}} \frac{1}{\log 2(1+t)} dt = \frac{1}{\log 2} \sum_{n=1}^{\infty} \log \frac{n+1}{n} \cdot \frac{x+n}{x+n+1} \\
&= \frac{1}{\log 2} \sum_{n=1}^{\infty} \log \frac{\frac{x}{n}+1}{\frac{x}{n+1}+1} = \frac{1}{\log 2} \sum_{n=1}^{\infty} \int_{\frac{x}{n+1}}^{\frac{x}{n}} \frac{1}{t+1} dt = \frac{1}{\log 2} \int_0^x \frac{1}{t+1} dt.
\end{aligned}$$

We denote again by Φ as the restriction of Φ in (2.10) to $\mathbb{N}^{\mathbb{N}} \cong \{0\} \times \mathbb{N}^{\mathbb{N}}$. Let σ be the left shift map on $\mathbb{N}^{\mathbb{N}}$. Then, the following diagram commutes:

$$\begin{array}{ccc}
\mathbb{N}^{\mathbb{N}} & \xrightarrow{\ \sigma\ } & \mathbb{N}^{\mathbb{N}} \\
\Phi \big\downarrow & \circlearrowleft & \big\downarrow \Phi \\
(0,1)\backslash\mathbb{Q} & \xrightarrow{\ G\ } & (0,1)\backslash\mathbb{Q}
\end{array}$$

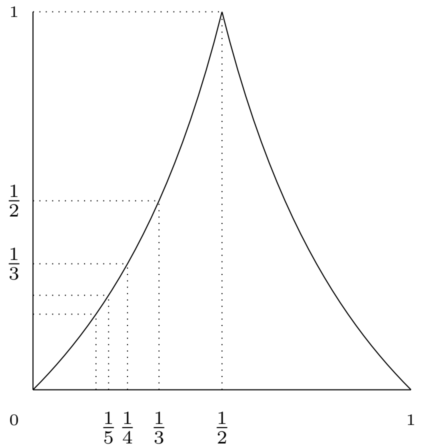


Figure 2.2: Farey map

We can give the pullback measure $\Phi^*\mu$ on $\mathbb{N}^{\mathbb{N}}$. Then two measure preserving dynamical systems $(\mathbb{N}^{\mathbb{N}}, \sigma, \Phi^*\mu)$ and $((0,1)\backslash\mathbb{Q}, G, \mu)$ are equivalent.

Ito introduced the Farey map to find intermediate convergents in [33] (see Definition 3.3.26 for the definition of intermediate convergents). Let us denote by F *Farey map* which is defined by

$$F(x) = \begin{cases} \dfrac{x}{1-x} & \text{if } 0 \le x < \dfrac{1}{2}, \\ \dfrac{1-x}{x} & \text{if } \dfrac{1}{2} \le x \le 1. \end{cases} \tag{2.12}$$

The Farey map is the same as the Gauss map on the subinterval $[1/2, 1]$ and it has a fixed point at 0 at which the derivative is 1 (see Figure 2.2). The Farey map is a slow down of the Gauss map in the sense that

$$F^{a_1(x)}(x) = G(x). \tag{2.13}$$

Schweiger introduced the notion of a jump transformation as an acceleration of a transformation in [61, Chapter 19]. For a transformation T of a domain X, let us define *the first return time of* $x \in X$ *to a subset* $Y \subset X$ by

$$n_Y(x) := \min\{n \ge 1 : T^n(x) \in Y\}.$$

Definition 2.1.7. *We call* $J(x) = T^{n_Y(x)+1}(x)$ the jump transformation associated to T with respect to Y.

The Farey map sends the intrerval $(\frac{1}{n+1}, \frac{1}{n})$ to the interval $(\frac{1}{n}, \frac{1}{n-1})$ for $n \geq 2$. Since $a_1(x) = n$ for $x \in (\frac{1}{n+1}, \frac{1}{n})$, $a_1(x) - 1$ is the first return time of x to the subinterval $(\frac{1}{2}, 1)$ in the orbit of x under F. It means that the Gauss map G is the jump transformation associated to the Farey map F with respect to the subinterval $(\frac{1}{2}, 1)$ by (2.13).

2.2 Coding of geodesics on the modular surface

In this section, we investigate a connection between geodesics on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ and continued fractions, following [24], [52] and [62].

2.2.1 Hyperbolic surface

The hyperbolic plane $\mathbb{H}$ is a 2-dimensional Riemannian manifold with the constant curvature -1. The hyperbolic plane can be represented by *the upper-half plane model* consisting of complex numbers whose imaginary parts are positive:

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}.$$

We identify its boundary with $\mathbb{R} \cup \{\infty\}$. We consider *the Poincaré metric* $ds^2 = \frac{dx^2+dy^2}{y^2}$ on the unit tangent bundle

$$T^1\mathbb{H} = \{(z, \mathbf{v}) : z \in \mathbb{H},\ \mathbf{v} \cdot \mathbf{v} = 1\}.$$

The special linear group $\mathrm{SL}_2(\mathbb{R})$ acts on $\mathbb{H}$ via *the Möbius transformations* which are defined by

$$g(z) = \frac{az+b}{cz+d} \quad \text{for } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

The action extends to $T^1\mathbb{H}$ as $g(z, \mathbf{v}) = (g(z), g'(z)\mathbf{v})$. *A geodesic* is the shortest path between two points. Geodesics on $\mathbb{H}$ are vertical lines or half circles perpendicular to the real line. We refer the readers to Chapter 9 in [24] for the details.

The modular surface M is obtained from $\mathbb{H}$ by quotienting it using the

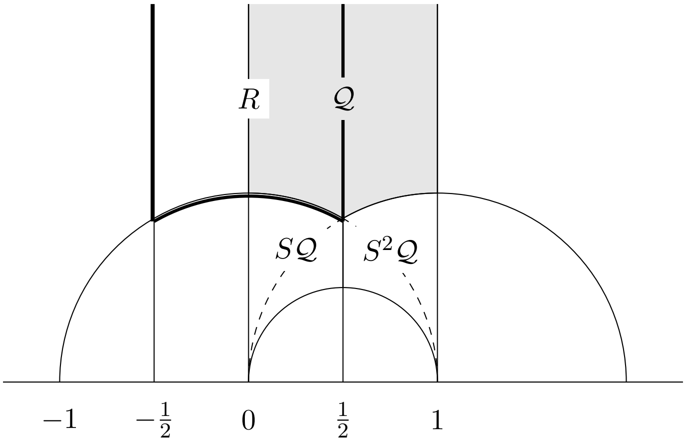


Figure 2.3: The standard fundamental domain R with the thick boundary and the modified fundamental domain $\mathcal{Q}$ with the shaded face of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$.

isometries in $\mathrm{SL}_2\mathbb{Z}$. The group $\mathrm{SL}_2\mathbb{Z}$ is generated by the matrices

$$\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \iota = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.14}$$

A fundamental domain R, which is often called "the standard fundamental domain", is the region

$$R = \left\{ z \in \mathbb{H} : -\frac{1}{2} < \operatorname{Re} z < \frac{1}{2} \text{ and } |z| > 1 \right\}. \tag{2.15}$$

See Figure 2.3.

The Möbius transformations corresponding to the generators in (2.14) are

$$z \mapsto z + 1 \text{ and } z \mapsto -\frac{1}{z}.$$

The first map maps the line from $-1/2$ to the line from $1/2$. The second map identifies the arc connecting i and $(-1+\sqrt{3}i)/2$ with the arc connecting i and $(1+\sqrt{3}i)/2$. We can obtain another fundamental domain by modifying the standard fundamental domain. After cutting R by the middle line and pasting the left piece to the right-hand side, we have the quadrilateral $\mathcal{Q}$ (see Figure 2.3).

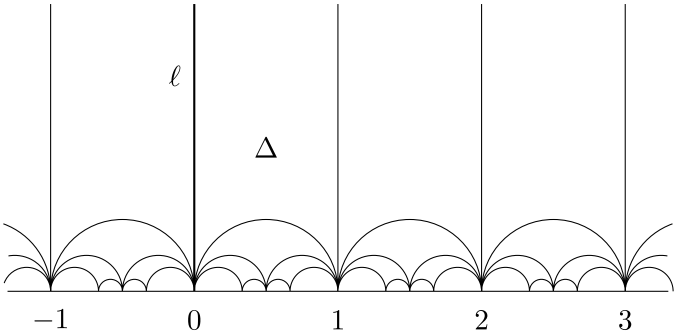


Figure 2.4: Farey tessellation. The geodesic line ℓ connecting 0 and ∞.

Now, let us consider a matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \tag{2.16}$$

which sends z to $-(z-1)^{-1}$. The map S fixes $1/2 + \sqrt{3}/2i$ and sends ∞ to 0, 0 to 1 and 1 to ∞. It means that S rotates the modified fundamental domain $\mathcal{Q}$ centered at $1/2 + \sqrt{3}/2i$. Then, the union of $\mathcal{Q}$, $S\mathcal{Q}$ and $S^2\mathcal{Q}$ makes the ideal triangle Δ whose vertices are 0, 1 and ∞. We will see that the regular continued fraction corresponds to the coding of the geodesic associated to the tessellation of $\mathbb{H}$ with the images of Δ under $\mathrm{SL}_2(\mathbb{Z})$.

2.2.2 Cutting sequences with Farey tessellation

Farey tessellation is the tessellation consisting of the images of Δ under $\mathrm{SL}_2\mathbb{Z}$ (see Figure 2.4). Denote by ℓ the geodesic joining 0 and ∞. We call each triangle *an elementary triangle* and each geodesic of the Farey tessellation *an elementary edge*. The endpoints of $\gamma(\ell)$ are p/q and p'/q' where $\gamma = \begin{pmatrix} p & p' \\ q & q' \end{pmatrix}$. Thus for any endpoint p/q and p'/q' of $\gamma(\ell)$, we know that $|pq' - p'q| = 1$. The map τ in (2.14) and S in (2.16) generate $\mathrm{SL}_2(\mathbb{Z})$. The $\mathrm{SL}_2(\mathbb{Z})$-action preserves the Farey tessellation since τ and S preserve the Farey tessellation.

Farey tessellation is named after a geologist J. Farey since the tessellation is related to the Farey sequences. *The Farey sequence of order* n is the collection of all rationals whose denominators and numerators are at most n. For

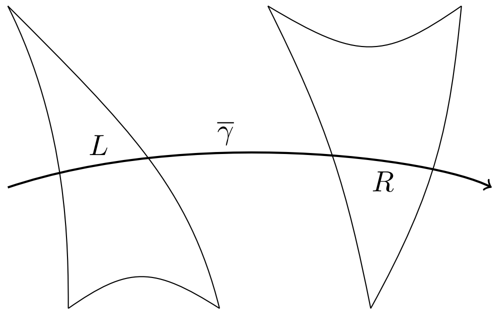


Figure 2.5: Segments of the type L (left) and the type R (right).

example, the nonnegative entries of the Farey sequences are:

$$F_1 : 0,\ 1,\ \infty, \quad F_2 : 0,\ \frac{1}{2},\ 1,\ 2,\ \infty, \quad F_3 : 0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1,\ \frac{3}{2},\ 2,\ 3,\ \infty, \quad \cdots.$$

Two rationals p/q and p'/q' are adjacent to each other in a Farey sequence if and only if $|pq' - p'q| = 1$. In other words, two consecutive numbers in a Farey sequence are connected by an elementary edge. In Section 3.3.1, we deal with a relation between continued fractions and the collection of the elementary edges which is called *Farey graph* (see Definition 3.3.8 in Section 3.3.1).

Let γ be an oriented geodesic of the modular surface M. Let us denote by π the canonical projection from $T^1\mathbb{H}$ to T^1M (see Figure 2.6). Let $\overline{\gamma}$ be a lift of γ, i.e., $\gamma = \pi(\overline{\gamma})$. Let us denote by $\overline{\gamma}_{-\infty}$ the backward endpoint of $\overline{\gamma}$ and by $\overline{\gamma}_{\infty}$ the forward endpoint of $\overline{\gamma}$. We can find a lift $\overline{\gamma}$ of γ

$$-1 < \overline{\gamma}_{-\infty} < 0, \overline{\gamma}_{\infty} > 1, \quad \text{or} \quad -1 < \overline{\gamma}_{-\infty} < 0, \overline{\gamma}_{\infty} > 1.$$

By the condition, $\overline{\gamma}$ intersects $i\mathbb{R}$ at a point, say $\xi_{\overline{\gamma}}$. We recall that the set of the pure imaginary numbers $i\mathbb{R}$ is the y-axis of $\mathbb{H}$.

The Farey tessellation divides $\overline{\gamma}$ into segments. For each segment, we give a type L or R as follows. If there is only one vertex of the elementary triangle on the left side of the segment, then we give *the type* L on the segment. Otherwise, we give *the type* R on the segment (see Figure 2.5). See the geodesic $\overline{\gamma}$ in Figure 2.6 for an example. The segment of $\overline{\gamma}$ between 0 and 1 is of type L and the next segment is of type L also. The preceding segment is of type R.

Let us consider the sequence of the types of the segements of $\overline{\gamma}$. We denote by R^n (or L^n, respectively) the block repeating R (or L, respectively) n times. Note that the types of the segment before $\xi_{\overline{\gamma}}$ and the segment after $\xi_{\overline{\gamma}}$ are

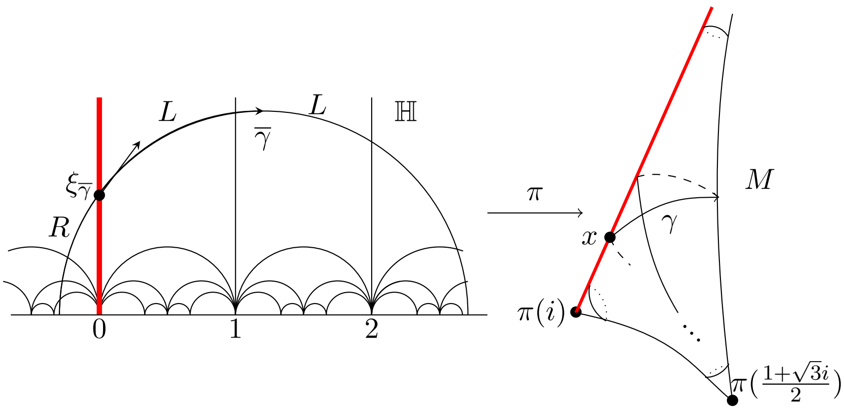


Figure 2.6: Projection from $\mathbb{H}$ to M. Corresponding Labels of an oriented geodesic $\overline{\gamma}$.

distinct. We can indicate $\xi_{\overline{\gamma}}$ between the types of segment before and after $\xi_{\overline{\gamma}}$ in the sequence of the types.

Definition 2.2.1. The cutting sequence of $\overline{\gamma}$ *is defined by the sequence of the types of segments as*

$$\cdots L^{n_{-2}} R^{n_{-1}} \xi_{\overline{\gamma}} L^{n_0} R^{n_1} L^{n_2} \cdots \text{ or } \cdots R^{n_{-2}} L^{n_{-1}} \xi_{\overline{\gamma}} R^{n_0} L^{n_1} R^{n_2} \cdots ,$$

where n_i is the number of repetitions of each type.

This labeling is invariant under $\mathrm{SL}_2(\mathbb{Z})$-action. It means that we can associate γ with the cutting sequence of $\overline{\gamma}$. Let $x := \xi_{\overline{\gamma}}$. Then, we have the following cutting sequence of γ:

$$\cdots L^{n_{-2}} R^{n_{-1}} x L^{n_0} R^{n_1} L^{n_2} \cdots \text{ or } \cdots R^{n_{-2}} L^{n_{-1}} x R^{n_0} L^{n_1} R^{n_2} \cdots .$$

Series clarified a connection between the cutting sequences and the regular continued fractions of the endpoints of geodesics on M [62].

Theorem 2.2.2 ([62], Theorem A)**.** *Let γ be a geodesic on M. If γ has a cutting sequence $\cdots L^{n_{-2}} R^{n_{-1}} x L^{n_0} R^{n_1} \cdots$, then*

$$\gamma_\infty = [n_0; n_1, n_2, \cdots] \text{ and } \gamma_{-\infty} = -[0; n_{-1}, n_{-2}, \cdots].$$

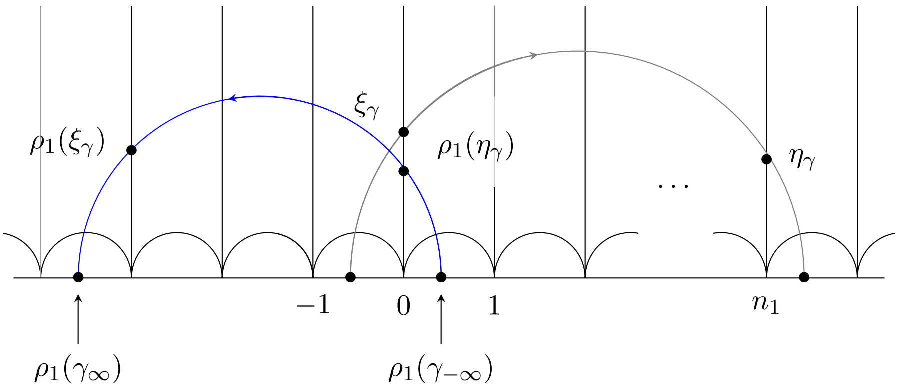


Figure 2.7: Geodesics $\overline{\gamma}$ (black) and $\rho_0(\overline{\gamma})$ (blue)

On the other hand, if γ *has a cutting sequence* $\cdots R^{n_{-2}}L^{n_{-1}}xR^{n_0}L^{n_1}\cdots$, *then*

$$\gamma_\infty = -[0; n_0, n_1, n_2, \cdots] \text{ and } \gamma_{-\infty} = [n_{-1}; n_{-2}, n_{-3}, \cdots].$$

Sketch of the proof. See Figure 2.7 for the procedure of the proof. Since the segments of the type L after $\xi_{\overline{\gamma}}$ is iterated n_0 times, the forward endpoint is between n_0 and $n_0 + 1$. It means that the integral part of $\overline{\gamma_\infty}$ is n_0 which is the 0th partial quotient.

The first partial quotient of $\overline{\gamma_\infty}$ is the integral part of $\frac{1}{\overline{\gamma_\infty} - n_0}$. Consider a Möbius transformation $\rho_0(z) = -1/(z - n_0)$. It is enough to show that $\lfloor -\rho_0(\overline{\gamma_\infty}) \rfloor$ is equal to n_1. Let $\eta_{\overline{\gamma}}$ be the point on $\overline{\gamma}$ such that $\mathrm{Re}(\overline{\gamma}(t)) = n_0$. Then $\rho_0(\eta_{\overline{\gamma}})$ lies in $i\mathbb{R}$. The cutting sequence after $\eta_{\overline{\gamma}}$ of $\overline{\gamma}$ is the same as the cutting sequence after $\rho_0(\eta_{\overline{\gamma}})$ of $\rho_0(\overline{\gamma})$ which starts from R^{n_1}. Thus the integral part of $-\rho_0(\overline{\gamma_\infty})$ is n_1. As a similar argument, we can show that $\overline{\gamma_\infty} = [n_0; n_1, n_2, \cdots]$ inductively.

For the backward endpoint, we also use the previous argument. For the inversion map ι in (2.14), the cutting sequence of $\iota(\overline{\gamma})$ is $\cdots R^{n_{-1}}\iota(\xi_{\overline{\gamma}})L^{n_0}R^{n_1}\cdots$. If we reverse the direction of the geodesic $\iota(\gamma)$, then its cutting sequence is $\cdots L^{n_1}R^{n_0}\iota(\xi_\gamma)L^{-1}\cdots$. By the previous argument, $-1/\overline{\gamma_{-\infty}}$ has the continued fraction expansion $[n_{-1}; n_{-2}, n_{-3}, \cdots]$.

We can prove the second assertion by the above arguments exchanging R and L. ☐

2.3 Bowen-Series map

In this section, we deal with a generalization of the discussion in the previous section to *Fuchsian groups*, i.e., discrete subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

In this section, we use the Poincaré disk model

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$$

for the hyperbolic plane instead of the upper-half plane model. We can identify $\mathbb{H}$ with $\mathbb{D}$ by

$$\omega := \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} : z \mapsto \frac{z - i}{-iz + 1}.$$

Since $\mathrm{Isom}^+(\mathbb{H}) = \mathrm{PSL}_2^+(\mathbb{R})$, $\mathrm{Isom}^+(\mathbb{D}) = \omega \mathrm{PSL}_2^+(\mathbb{R})\omega^{-1}$ which is a subgroup of $\mathrm{PSL}_2^+(\mathbb{C})$ consisting of oriented preserving maps preserving $\mathbb{D}$. Let Γ be a subgroup of $\mathrm{PSL}_2^+(\mathbb{R})$ and $z \in \mathbb{H}$. By abuse of notation, we denote by Γ the conjugation $\omega\Gamma\omega^{-1}$ of the subgroup Γ and by z the image $\omega(z)$ of z for simplification.

The limit set of a group is the set of the accumulation points of its orbits. The limit set of a Fuchsian group is contained in the boundary $\partial\mathbb{D}$. We say that a Fuchsian group is of the *first kind* if its limit set is the same as whole $\partial\mathbb{D}$. Bowen and Series established a boundary map on $\partial\mathbb{D}$ by edge identifications of the fundamental domain of a finitely generated Fuchsian group of the first kind [15], which we briefly recall.

From now on, let Γ be a finitely generated Fuchsian group of the first kind. Let $\mathcal{F}$ be a fundamental domain of Γ. Let us denote by a set of greek letters $\mathcal{A} = \{\alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \gamma^{-1}, \cdots\}$ the set of elements of Γ identifying two sides of $\mathcal{F}$. We note that $\mathcal{A}$ is a finite set since $\mathcal{F}$ has finite sides. Let us consider a pair of sides (s, s') of $\mathcal{F}$ which are identified to each other by an element of Γ. We denote by $s_\alpha := s$ if $\alpha^{-1}(s) = s'$.

The *isometric circle* of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|cz + d| = 1$, i.e., the set of the points z such that $|\gamma'(z)| = 1$ in $\mathbb{H}$. In the Poincaré disk $\mathbb{D}$, the isometric circle of $\gamma \cong \omega\gamma\omega^{-1}$ is the image of $|cz + d| = 1$ in $\mathbb{H}$ under ω. We denote by C_α the isometric circle of α in $\mathbb{D}$. We label endpoints of C_α with v_α and w_α on $\partial\mathbb{D}$ as in Figure 2.8. Bowen and Series constructed a fundamental domain $\mathcal{F}$ of Γ such that

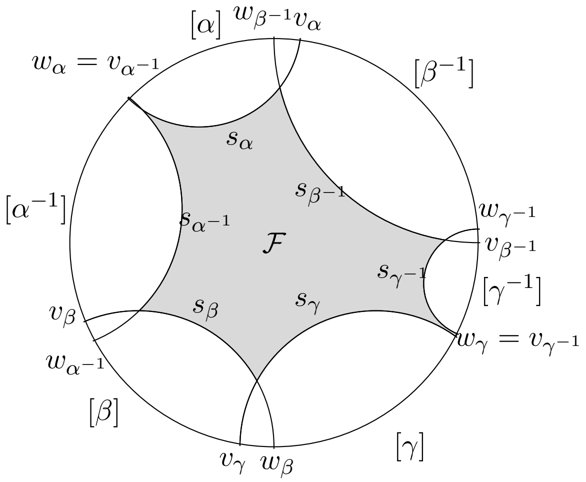


Figure 2.8: Labeling of the sides of a fundamental domain $\mathcal{F}$ and endpoints of its isometric circles, the corresponding partition of the boundary.

$$\text{the geodesic containing } s_\alpha \text{ is } C_\alpha, \text{ and} \tag{2.17}$$
$$C_\alpha \text{ is contained in the tessellation consisting of } \bigcup_{\gamma\in\Gamma} \gamma(\partial\mathcal{F}).$$

We denote by the interval notations (v, w), $[v, w)$, $(v, w]$, $[v, w]$ the arc of $\partial\mathbb{D}$ between v and w. We define a partition $\{[\alpha]\}_{\alpha\in\mathcal{A}}$ of $\partial\mathbb{D}$ by $[\alpha] := [v_\alpha, v_{\alpha'})$ such that $s_{\alpha'}$ is the next side of s_α in an anti-clockwise direction. Bowen and Series defined a map associated to Γ by

$$f_\Gamma(x) = \alpha^{-1}(x) \quad x \in [\alpha]. \tag{2.18}$$

The boundary expansion of $x \in \partial\mathbb{D}$ is defined by

$$[\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots] \text{ such that } \alpha_n := \alpha \text{ if } f_\Gamma^{n-1}(x) \in [\alpha].$$

Then, f_Γ is a left shift map of the boundary expansion. From now on, we denote by the finite expansion form $[\alpha_0, \alpha_1, \cdots, \alpha_n]$ the subarc of $\partial\mathbb{D}$ whose elements have the expansions starting with $\alpha_0, \alpha_1, \cdots, \alpha_n$. Then, the arc is exactly

$$[\alpha_0, \alpha_1, \cdots, \alpha_n] = \alpha_0 \circ \alpha_1 \circ \cdots \circ \alpha_{n-1}([\alpha_n]).$$

Example 2.3.1. *For example, let us consider the congruence subgroup of level*

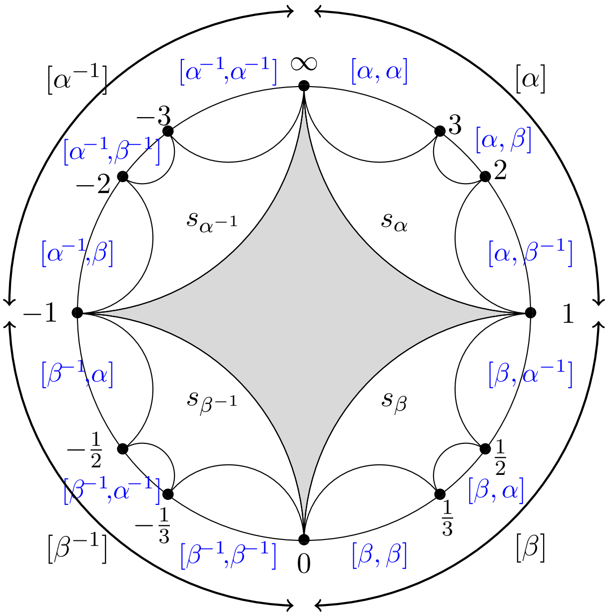


Figure 2.9: Fundamental domain $\mathcal{D}$ of $\Gamma(2)$ in $\mathbb{D}$.

2

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{2} \right\}.$$

Then the fundamental domain $\mathcal{D}$ of $\Gamma(2)$ satisfying the condition 2.17 is as in Figure 2.9. Let

$$\alpha = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}.$$

The group $\Gamma(2)$ is generated by $\{\alpha, \beta\}$ and α, β identify the sides of $\mathcal{D}$. The arcs are $[\alpha^{-1}] = [\infty, -1)$, $[\beta^{-1}] = [-1, 0)$, $[\beta] = [0, 1)$ and $[\alpha] = [1, \infty)$.

The subarcs corresponding to the words of length 2 are

$$\begin{aligned} &[\alpha, \alpha] = [3, \infty),\ [\beta, \alpha] = \left[\frac{1}{3}, \frac{1}{2}\right),\ [\beta^{-1}, \alpha] = \left[-1, -\frac{1}{2}\right), \\ &[\alpha^{-1}, \alpha^{-1}] = [\infty, -3),\ [\beta, \alpha^{-1}] = \left[\frac{1}{2}, 1\right),\ [\beta^{-1}, \alpha^{-1}] = \left[-\frac{1}{2}, -\frac{1}{3}\right), \end{aligned} \tag{2.19}$$

$$
\begin{aligned}
&[\alpha, \beta^{-1}] = [1,2),\ [\alpha^{-1}, \beta^{-1}] = [-3,-2),\ [\beta^{-1}, \beta^{-1}] = \left[-\frac{1}{3}, 0\right), \\
&[\alpha, \beta] = [2,3),\ [\alpha^{-1}, \beta] = [-2,-1),\ [\beta, \beta] = \left[0, \frac{1}{3}\right).
\end{aligned} \tag{2.20}
$$

We can modify the choice of the partition of $[\alpha]$'s as long as $[\alpha] \subset [v_\alpha, w_\alpha)$ and the disjoint union of $[\alpha]$'s is $\partial\mathbb{D}$. Let us consider the case of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. For the standard fundamental domain R in (2.15), $\omega(R)$ looks like the region in Figure 2.10.

We will compare the following two choices of a partition of $\partial\mathbb{D}$:

(i) $[\tau^{-1}] = [\infty, -1),\ [\tau] = [1, \infty),\ [\iota^{-1}] = [-1, 1)$ and

(ii) $[\tau^{-1}] = [\infty, -\frac{1}{2}),\ [\tau] = [\frac{1}{2}, \infty),\ [\iota] = [-\frac{1}{2}, \frac{1}{2})$.

Example 2.3.2 (Farey map)**.** *We denote by f_1 the Bowen-Series map associated to $f_{SL_2(\mathbb{Z})}$ with the partition (i). By* (2.18)*, f_1 is explicitly the following map*

$$
f_1 := \begin{cases} x+1 & \text{if } x \in (-\infty, -1), \\ x-1 & \text{if } x \in [1, \infty), \\ -\dfrac{1}{x} & \text{if } x \in [-1, 1). \end{cases}
$$

The map f_1 is related to the Farey map F in (2.12) *since*

$$
F(x) = \begin{cases} \iota \circ f_1^2(x) & \text{if } x \in [0, \frac{1}{2}), \\ |f_1^2(x)| & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}
$$

We observe that $[\frac{1}{n+1}, \frac{1}{n}) = [\iota, \tau^{-n}] = [\iota, \tau^{-1}, \tau^{-1}, \cdots, \tau^{-1}]$ for $n \in \mathbb{N}$ where the number of τ^{-1} is n. For $x \in [1, \infty]$, the boundary expansion of x is of the form

$$
[\tau^{-n_1}, \iota, \tau^{n_2}, \iota, \tau^{-n_3}, \cdots].
$$

We can see that the cutting sequence of a geodesic is $\cdots L^{n_1} R^{n_2} L^{n_3} \cdots$ if the forward endpoint is x (see Definition 2.2.1).

Example 2.3.3. *The nearest integer continued fraction map is*

$$
T(x) = \left| \frac{1}{x} - \left\lfloor \frac{1}{x} + \frac{1}{2} \right\rfloor \right| \text{ for } x \in \left(0, \frac{1}{2}\right),
$$

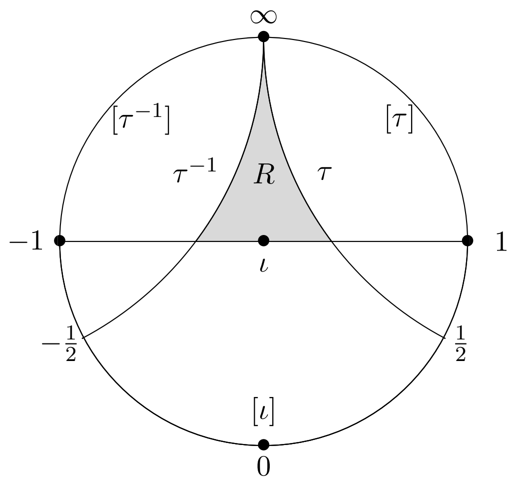


Figure 2.10: Fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathbb{D}$.

here, $\left\lfloor \frac{1}{x} + \frac{1}{2} \right\rfloor$ *is the nearest integer of* $\frac{1}{x}$*. For* $x \in \mathbb{R}$*, the nearest integer continued fraction is*

$$x = c_0 + \cfrac{\kappa_0}{c_1 + \cfrac{\kappa_1}{c_2 + \cfrac{\kappa_2}{\ddots}}}$$

where c_0 *is the nearest integer of* x*, the* 0*th numerator* κ_0 *is the sign of* $x - c_0$*, each digit* c_n *is the nearest integer of* $\frac{1}{T^{n-1}(x-c_0)}$ *and* κ_n *is the sign of* $\frac{1}{T^{n-1}(x-c_0)} - c_n$*. Then,* T *is a left shift map of the nearest continued fraction expansions.*

The Bowen-Series map associated to $SL_2(\mathbb{Z})$ *with the partition (ii) is*

$$f_2(x) := \begin{cases} x+1 & \text{if } x \in \left(-\infty, -\dfrac{1}{2}\right), \\ x-1 & \text{if } x \in \left[\dfrac{1}{2}, \infty\right), \\ -\dfrac{1}{x} & \text{if } x \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right). \end{cases}$$

By iterating of f_2, we can induce T such that

$$T(x) = -f_2^{n(x)}(x) \text{ for } x \in \left(0, \frac{1}{2}\right)$$

where $n(x)$ is the first return time of $x \in (0, 1/2)$ to $(-1/2, 0)$ as an orbit under T. We note that $c_1 = n(x) - 1$. In this sense, we can say that f_2 is a slow down map of T.

By the above discussion, we can say that a continued fraction map corresponds to an acceleration of the Bowen-Series map in some sense. We deal with such an acceleration of the Bowen-Series map.

Artigiani, Marchese and Ulcigrai investigated Lagrange spectra related to Fuchsian groups in [5] and [6]. Marchese studied about bad sets for non-uniform Fuchsian groups in [50]. In this investigation, they defined the cuspidal acceleration of the Bowen-Series map. In the rest of this section, we introduce the definition of the cuspidal acceleration, following [5], [6] and [50].

For a fixed Bowen-Series map f_Γ, we call a finite word $\alpha_1\alpha_2\cdots\alpha_n$ over $\mathcal{A}$ *an admissible word* if there is a corresponding arc $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ on $\partial\mathbb{D}$. We say that an infinite word $\alpha_1\alpha_2\cdots\alpha_n\cdots$ is *admissible* if there is $x \in \partial\mathbb{D}$ whose boundary expansion is $[\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots]$. A finite word $\alpha_1\alpha_2\cdots\alpha_n$ has *no backtracking* if $\alpha_{i+1} \neq \alpha_i^{-1}$ for any $1 \leq i < n$. We can show that a word is admissible if and only if a word has no backtracking. We say that a word $\alpha_1\alpha_2\cdots\alpha_n$ is *a cuspidal word* if one of the endpoints of the corresponding arc $[\alpha_1, \alpha_2, \cdots, \alpha_n]$ is a cusp of $\Gamma\backslash\mathbb{H}$. We note that all words of length 1 are cuspidal. For example, the cuspidal word of length 2 of $\Gamma(2)$ are

$$\alpha^{-1}\alpha^{-1},\ \alpha^{-1}\beta,\ \alpha\alpha,\ \alpha\beta^{-1},\ \beta^{-1}\beta^{-1},\ \beta^{-1}\alpha,\ \beta\beta. \text{ and } \beta\alpha^{-1}$$

since one of the endpoints of their corresponding arcs is 0, 1, -1 or ∞ (see (2.19) and (2.20)).

Consider an admissible word $\alpha_1\alpha_2\cdots\alpha_n\cdots$. We can decompose a word by cuspidal words such that

$$\alpha_1\alpha_2\cdots\alpha_n\cdots = W_1W_2W_3\cdots$$

where $W_k = \alpha_{n(k)}\cdots\alpha_{n(k+1)-1}$ is defined inductively by the maximal cuspidal word from $\alpha_{n(k)}$ for $k \in \mathbb{N}$.

Definition 2.3.4. The cuspidal acceleration *of f_Γ is a map $\mathcal{C}_\Gamma : \partial\mathbb{D} \to \partial\mathbb{D}$*

defined by

$$\mathcal{C}_\Gamma(x) = (\alpha_0 \circ \alpha_1 \circ \cdots \circ \alpha_n)^{-1}(x) \text{ if } W_1 = \alpha_0\alpha_1 \cdots \alpha_n.$$

In Example 2.3.2, the cuspidal words are ι and τ^n for all $n \in \mathbb{Z}$. A word $\iota\tau^{-n_1}\iota\tau^{n_1}\iota\tau^{-n_3}$ is decomposed by

$$(\iota)(\tau^{-n_1})(\iota)(\tau^{n_1})(\iota)(\tau^{-n_3}) \cdots .$$

For the cuspidal acceleration $\mathcal{C}_1$ induced by f_1, the Gauss map G is

$$G(x) = \mathcal{C}_1 \circ \iota(x) = \mathcal{C}_1^2(x) \text{ for } x \in [0,1).$$

Similarly, for the cuspidal acceleration $\mathcal{C}_2$ induced by f_2 in Example 2.3.3, we can see that

$$T(x) = \mathcal{C}_2 \circ \iota(x) = \mathcal{C}_2^2(x) \text{ for } x \in \left[-\frac{1}{2}, \frac{1}{2}\right).$$

Chapter 3

Continued fraction related to Θ-group

In this chapter, we focus on the Bowen-Series map associated to Θ-group. The group Θ is an index-3 subgroup of $\mathrm{SL}_2(\mathbb{Z})$ generated by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

The group Θ is explicitly

$$\Theta = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ : \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pmod 2 \right\}.$$

3.1 Romik dynamical system

Romik introduced a dynamical system on the unit interval $[0, 1]$ as follows

$$R(x) = \begin{cases} \dfrac{x}{1-2x}, & 0 \le x \le \dfrac{1}{3}, \\ \dfrac{1}{x} - 2, & \dfrac{1}{3} \le x \le \dfrac{1}{2}, \\ 2 - \dfrac{1}{x}, & \dfrac{1}{2} \le x \le 1. \end{cases} \tag{3.1}$$

See [57] for the details. The graph of the Romik map is as in Figure 3.1. We refer the reader to [19] and [20] for the investigation of number-theoretical

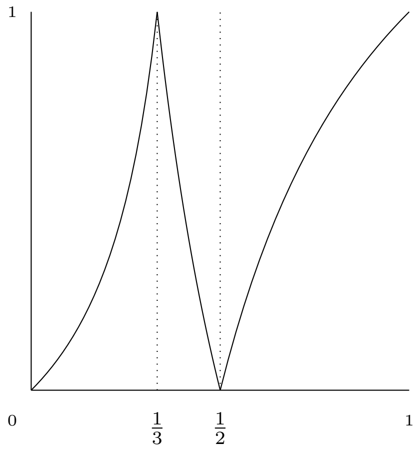


Figure 3.1: Romik map R.

properties and the Lagrange spectrum of the Romik map.

We remark that, for the Farey map F, a real number x between 0 and 1 is rational if and only if $F^n(x) = 0$ for some $n \geq 0$. It is related to the fact that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ has the only one cusp and the cusp corresponding to $\mathbb{Q} \cup \{\infty\}$.

The Romik map has two *indifferent fixed points* 0 and 1, i.e., the fixed point with differential 1. By using the fixed points, we can classify rational numbers into two classes such that

$$\{x : R^n(x) = 0 \ \text{ for some } n \geq 0\} \ \text{ or } \{x : R^n(x) = 1 \ \text{ for some } n \geq 0\}.$$

Actually, the first set is $\Theta(\infty) \cap [0,1]$ and the second set is $\Theta(1) \cap [0,1]$, thus, each of them is related to the two cusps of the surface $\Theta\backslash\mathbb{H}$ corresponding to ∞ and 1.

Definition 3.1.1. *We call a rational in the orbit* $\Theta(\infty)$ an ∞-rational *and a rational in the orbit* $\Theta(1)$ a 1-rational*.*

Note that an ∞-rational is of the form $\frac{\text{even}}{\text{odd}}$ or $\frac{\text{odd}}{\text{even}}$, but a 1-rational is of the form $\frac{\text{odd}}{\text{odd}}$.

Romik originally introduced the Romik dynamical system $\widehat{R}$ as a dynamical system on the first quadrant

$$\mathbf{Q} = \{(x, y) : x^2 + y^2 = 1, x \geq 0, y \geq 0\}$$

of $\mathbb{S}^1$ to investigate an algorithm generating the Pythagorean triples by multi-

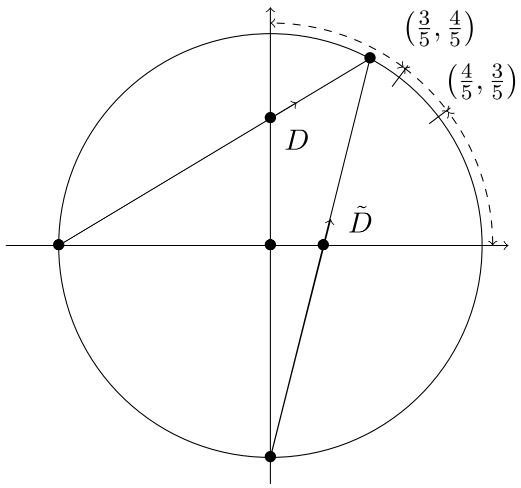


Figure 3.2: Romik map on the quadrant **Q**.

plying matrices (see [12] and also [4], [7], [22], [21]). More precisely, the Romik map $\widehat{R}$ on **Q**

$$\widehat{R}(x,y) = \left(\frac{|2-x-2y|}{3-2x-2y}, \frac{|2-2x-y|}{3-2x-2y} \right).$$

Let us define $D : (0,1) \to \mathbf{Q}$ and $\tilde{D} : (0,1) \to \mathbf{Q}$ by

$$D(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \quad \text{and} \quad \tilde{D}(t) = \left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2} \right).$$

The maps D and $\tilde{D}$ are the inverses of the stereographic projections of **Q** from $(-1,0)$ and $(0,-1)$ onto y-axis and x-axis, respectively (see Figure 3.2). Then the Romik map R is a conjugation of $\widehat{R}$ such that

$$R = D^{-1} \circ \widehat{R} \circ D = \tilde{D}^{-1} \circ \widehat{R} \circ \tilde{D}. \tag{3.2}$$

See Theorem 4 in [57] for more details.

3.2 Even integer continued fraction

In this section, we introduce even integer continued fraction and the related results, following [59], [60], [45], [65] and [14].

Schweiger investigated the continued fraction with even partial quotients which we call *the even integer continued fraction* in [59, 60]. We abbreviate the even integer continued fraction to *EICF*. The EICF is related to the cuspidal acceleration of Bowen-Series map associated to Θ with respect to ∞. The cuspidal acceleration in here is not exactly the cuspidal acceleration of the previous section, but it is obtained by accelerating for a class of cuspidal words which corresponds to the arcs adjacent to ∞.

For $b_0 \in 2\mathbb{Z}$, $b_n \in 2\mathbb{N}$, $n \geq 1$ and $\eta_i \in \{-1, 1\}$, an EICF expansion is

$$x = b_0 + \cfrac{\eta_0}{b_1 + \cfrac{\eta_1}{b_2 + \cfrac{\eta_2}{\ddots}}}.$$

We write an EICF expansion as a sequence in a double angle bracket:

$$\langle\!\langle (b_0, \eta_0), (b_1, \eta_1), \cdots, (b_i, \eta_i), \cdots \rangle\!\rangle \,. \tag{3.3}$$

The left shift map of EICF expansions is the EICF map T_{e} on the unit interval $[0, 1]$ defined by

$$T_{\mathrm{e}}(0) = 0, \quad \text{and} \quad T_{\mathrm{e}}(x) = \begin{cases} \dfrac{1}{x} - 2k, & \text{if } \dfrac{1}{2k+1} \leq x \leq \dfrac{1}{2k}, \\ 2k - \dfrac{1}{x}, & \text{if } \dfrac{1}{2k} \leq x \leq \dfrac{1}{2k-1}, \end{cases} \quad \text{for all } k \in \mathbb{N}.$$

See Figure 3.3 for the graph of T_{e}.

As the Gauss map G is the jump transformation associated to the Farey map F with respect to $[1/2, 1]$, the EICF map T_{e} is the jump transformation associated to the Romik map R with respect to $[1/3, 1)$, i.e.,

$$T_{\mathrm{e}}(z) = R^{n_{[1/3,1)}(x)+1}(x)$$

where $n_{[1/3,1)}(x) = \min\{i \geq 1 : R^i(x) \in [\frac{1}{3}, 1)\}$ (see Definition 2.1.7). Note that T_{e} has an invariant measure $\mu_e = (1 - x^2)^{-1} dx$ which is absolutely continuous with respect to Lebesgue measure. Schweiger showed that μ_e is an ergodic measure of T_{e} (see Theorem 2 in [59] for the proof).

The nth EICF principal convergent, denoted by p_n^e/q_n^e, is the truncated

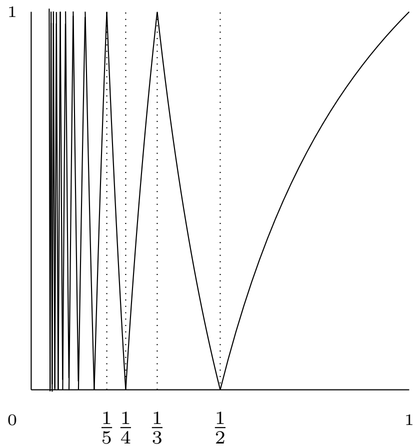


Figure 3.3: Even integer continued fraction map T_{e}.

continued fraction

$$\frac{p_n^e}{q_n^e} := \langle\!\langle (b_0, \eta_0), (b_1, \eta_1), \cdots, (b_n, \eta_n) \rangle\!\rangle .$$

Before introducing the recursive formula for the EICF principal convergents, let us see the following lemma for general continued fraction forms (see p.3 in [44] for details).

Lemma 3.2.1. *Let us consider a general continued fraction of the form*

$$x = \mathfrak{a}_0 + \cfrac{\mathfrak{b}_0}{\mathfrak{a}_1 + \cfrac{\mathfrak{b}_1}{\mathfrak{a}_2 + \cfrac{\mathfrak{b}_2}{\mathfrak{a}_3 + \ddots}}},$$

where $\mathfrak{a}_n$, $\mathfrak{b}_n$ are integers. Let $\mathfrak{r}_n/\mathfrak{s}_n$ be the finite continued fraction of the form

$$\mathfrak{a}_0 + \cfrac{\mathfrak{b}_0}{\mathfrak{a}_1 + \cfrac{\mathfrak{b}_1}{\mathfrak{a}_2 + \cfrac{\mathfrak{b}_2}{\ddots + \cfrac{\mathfrak{b}_{n-1}}{\mathfrak{a}_n}}}}.$$

Then the following matrix relation holds:

$$\begin{pmatrix} \mathfrak{r}_n & \mathfrak{b}_n\mathfrak{r}_{n-1} \\ \mathfrak{s}_n & \mathfrak{b}_n\mathfrak{s}_{n-1} \end{pmatrix} = \begin{pmatrix} \mathfrak{a}_0 & \mathfrak{b}_0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{a}_1 & \mathfrak{b}_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} \mathfrak{a}_n & \mathfrak{b}_n \\ 1 & 0 \end{pmatrix}.$$

Consequently, we deduce the following recursive formulas:

$$\begin{cases} \mathfrak{r}_n = \mathfrak{a}_n\mathfrak{r}_{n-1} + \mathfrak{b}_{n-1}\mathfrak{r}_{n-2}, \\ \mathfrak{s}_n = \mathfrak{a}_n\mathfrak{s}_{n-1} + \mathfrak{b}_{n-1}\mathfrak{s}_{n-2}, \end{cases}$$

where $\mathfrak{r}_{-1} = 1$, $\mathfrak{s}_{-1} = 0$, $\mathfrak{r}_0 = \mathfrak{a}_0$ *and* $\mathfrak{s}_0 = 1$.

Thus, for EICF, the following matrix relation holds:

$$\begin{pmatrix} q_n & \varepsilon_n q_{n-1} \\ p_n & \varepsilon_n p_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & \varepsilon_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & \varepsilon_1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & \varepsilon_n \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

Kraaikamp-Lopes characterized the finite and eventually periodic EICF.

Proposition 3.2.2 ([45], Proposition 2 and Proposition 3)**.** *The following statements hold:*

(1) A rational p/q is an ∞-rational, i.e., $p \not\equiv q \pmod 2$, if and only if the EICF expansion of p/q is finite.

(2) A real number x is a quadratic irrational or a 1-rational if and only if its EICF expansion is eventually periodic.

Boca and Merriman gave geometrical proofs of the above propositions by using the Farey graph in [14] (see Definition 3.3.8 for the definition of the Farey graph). Kraaikamp and Lopes obtained the asymptotic growth number of geodesics on $\Theta\backslash\mathbb{H}$ in [45].

Since each matrix in (3.4) belongs to Θ, we can easy to conclude that each EICF principal convergent p_n^e/q_n^e is an ∞-rational. Short and Walker showed that an EICF principal convergent is a best approximation among the rationals in the orbit $\Theta(\infty)$ consisting of $\frac{\text{even}}{\text{odd}}$ and $\frac{\text{odd}}{\text{even}}$, and also the converse holds.

Definition 3.2.3. A best ∞-rational approximation p/q of x *is a rational* $p/q \in \Theta(\infty)$ *such that*

$$|qx - p| < |bx - a| \quad \textit{for any } \infty\textit{-rational } \frac{a}{b} \neq \frac{p}{q} \text{ such that } 0 < b \leq q.$$

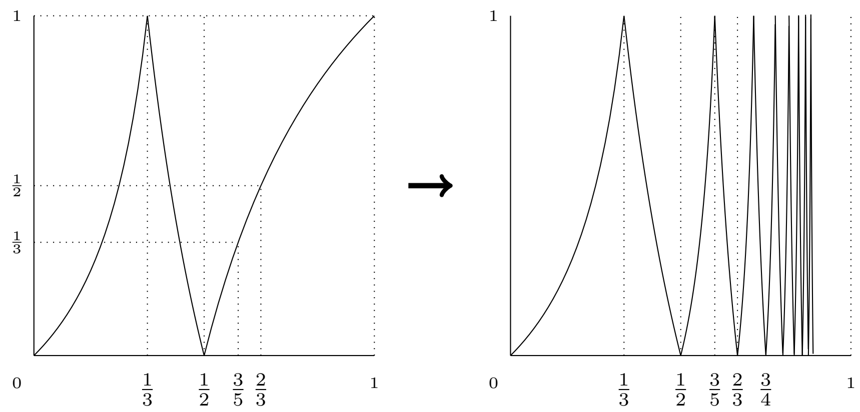


Figure 3.4: The graph of the Romik map R (left) and the graph of the OOCF map T_{o} (right)

Proposition 3.2.4 ([65], Theorem 5)**.** *An ∞-rational is a best ∞-rational approximation of x if and only if it is an EICF principal convergent of x.*

They showed the proposition by using Ford circles (see Definition 3.3.10).

3.3 Odd-odd continued fraction

We define a continued fraction whose convergents are in the orbit $\Theta(1)$ which is the set of rationals of the form $\frac{\text{odd}}{\text{odd}}$. We call the continued fraction *the odd-odd continued fraction*. In this section, we define the odd-odd continued fraction and properties.

3.3.1 Continued fraction with $\frac{\textbf{odd}}{\textbf{odd}}$ convergents

Let $E = [1/3, 1]$ and $E' = [0, \frac{1}{2}]$. *The odd-odd continued fraction map T_{o}* is defined by the jump transformation associated to R with respect to E' such that

$$T_{\mathrm{o}}(1) = 1, \quad \text{and} \quad T_{\mathrm{o}}(x) = R^{n_{E'}(x)+1}(x)$$

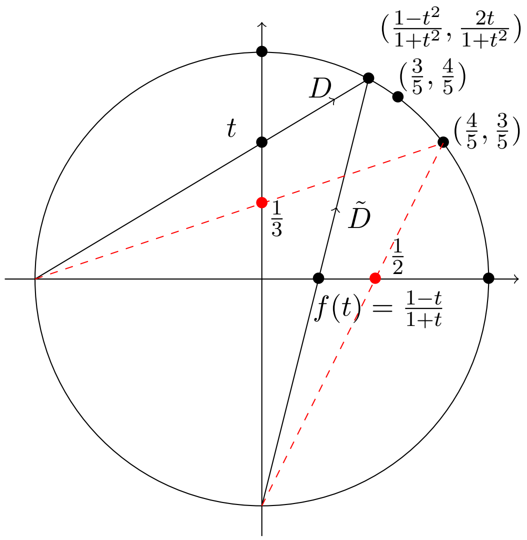


Figure 3.5: Continued fraction maps T_{o} and T_{e} are conjugate via $f(t) = \frac{1-t}{1+t}$.

where $n_{E'}(x) = \min\{i \geq 0 : R^i(x) \in E'\}$. A simple computation shows that $T_{\mathrm{o}}(1) = 1$ and

$$(3.5) \qquad T_{\mathrm{o}}(x) = \begin{cases} \dfrac{kx-(k-1)}{k-(k+1)x}, & \text{if } x \in \left[\dfrac{k-1}{k}, \dfrac{2k-1}{2k+1}\right), \\ \dfrac{k-(k+1)x}{kx-(k-1)}, & \text{if } x \in \left[\dfrac{2k-1}{2k+1}, \dfrac{k}{k+1}\right), \end{cases} \quad \text{for all } k \geq 1.$$

We abbreviate the odd-odd continued fraction to *OOCF*. The map T_{o} has an indifferent fixed point 0 (see Figure 3.4).

Let

$$(3.6) \qquad f(x) = \tilde{D}^{-1} \circ D(x) = \frac{1-x}{1+x}.$$

We note that $f = f^{-1}$. As in Figure 3.5,

$$D(E) = \left\{(x, y) \in \mathbf{Q} : 0 < x < \frac{4}{5}\right\} = \tilde{D}(E'),$$

where $\mathbf{Q}$ is the first quadrant of the unit circle.

Since the conditions $R^j(f^{-1}(t)) \in E$ and $R^j(t) \in E'$ are equivalent to $\widehat{R}^j \tilde{D}(t) \in D(E) = \tilde{D}(E')$ by (3.2), we can show that $n_E(f^{-1}(t))$ and $n_{E'}(t)$

are the same. Thus, T_{o} is conjugate to T_{e} since

$$f \circ T_{\mathrm{o}} \circ f^{-1}(s) = \tilde{D}^{-1} \circ \widehat{R}^{n_E(f^{-1}(s))+1} \circ \tilde{D}(s) = R^{n_{E'}(s)+1}(s) = T_{\mathrm{e}}(s).$$

The above arguments are summarized as the following theorem.

Theorem 3.3.1. *The OOCF map T_o is conjugate to T_e. More precisely, let f be the function on the interval $[0,1]$ defined by $x \mapsto \frac{1-x}{1+x}$. Then*

$$f \circ T_o \circ f^{-1} = T_e.$$

To find an invariant measure, let $y = f(x)$. Since $dy = |f'(x)|dx = \frac{2dx}{(1+x)^2}$,

$$\frac{dx}{1-x^2} = \frac{(1+x)^2 dy}{2(1-x^2)} = \frac{(1+x)dy}{2(1-x)} = \frac{dy}{2y}.$$

Hence, we find an absolutely continuous invariant measure of T_{o} as follows:

Proposition 3.3.2. *The map $T_o : [0,1] \to [0,1]$ preserves the measure*

$$\frac{1}{x} dx.$$

Remark 3.3.3. *Schweiger proved that T_e admits an ergodic invariant measure $d\mu := \frac{dx}{1-x^2}$ [59, Theorem 2]. By Theorem 3.3.1, we deduce that the measure $f_*^{-1}\mu$ is ergodic invariant with respect to T_{OOCF}. By simple computation, one can find that this measure $f_*^{-1}\mu$ is nothing but the measure in Proposition 3.3.2.*

By using T_{o}, we induce an expansion of a real number in $[0,1]$. Denote by

$$\mathcal{U} := \bigcup_{k \geq 1} \left\{ \frac{2k-1}{2k+1} \right\}, \quad \mathcal{V} := \bigcup_{k \geq 1} \left\{ \frac{k}{k+1} \right\}. \tag{3.7}$$

By (3.5), for $k \geq 1$, for $x \in (\frac{k-1}{k}, \frac{2k-1}{2k+1})$,

$$1 - x = \frac{T_{\mathrm{o}}x + 1}{(k+1)T_{\mathrm{o}}x + k} = \cfrac{1}{(k+1) + \cfrac{-1}{2 - (1 - T_{\mathrm{o}}x)}}, \tag{3.8}$$

and, for $x \in (\frac{2k-1}{2k+1}, \frac{k}{k+1})$,

$$1 - x = \frac{T_o x + 1}{(k+1) + kT_o x} = \cfrac{1}{k + \cfrac{1}{2 - (1 - T_o x)}}. \tag{3.9}$$

Thus, for all $x \in [0,1] \setminus \bigcup_{n \geq 0} T_o^{-n}(\mathcal{U} \cup \mathcal{V} \cup \{0\})$,

$$1 - x = \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{2 - \cfrac{1}{\ddots \, \cfrac{\ddots}{a_n + \cfrac{\varepsilon_n}{2 - (1 - T_o^n x)}}}}},$$

where

$$(a_n, \varepsilon_n) = \begin{cases} (k+1, -1), & \text{if } T_o^{n-1}(x) \in (\frac{k-1}{k}, \frac{2k-1}{2k+1}), \\ (k, 1), & \text{if } T_o^{n-1}(x) \in (\frac{2k-1}{2k+1}, \frac{k}{k+1}). \end{cases} \tag{3.10}$$

Hence we deduce an OOCF expansion of any $x \in [0,1] \setminus \bigcup_{n \geq 0} T_o^{-n}(\mathcal{U} \cup \mathcal{V} \cup \{0\})$:

$$x = 1 - \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{2 - \cfrac{1}{a_2 + \cfrac{\varepsilon_2}{2 - \ddots}}}},$$

where $a_n \in \mathbb{N}$; and $\varepsilon_n \in \{1, -1\}$ for $a_n \geq 2$ and $\varepsilon_n = 1$ for $a_n = 1$. We denote the OOCF expansion of x by a sequence in a double bracket

$$x = [\![(a_1, \varepsilon_1), (a_2, \varepsilon_2), \cdots, (a_n, \varepsilon_n), \cdots]\!].$$

We call (a_n, ε_n) the *nth digit* of x in its OOCF expansion.

Remark 3.3.4. *While for* $x \in \bigcup_{n \geq 0} T_o^{-n}(\mathcal{U} \cup \mathcal{V} \cup \{0\})$, *the situation is more* *complicated. First, note that for any* $x \in \mathcal{U}$, $T_o(x) = 1$ *and if* $x = \frac{2k-1}{2k+1}$, *we deduce*

$$1 - x = \cfrac{1}{(k+1) + \cfrac{-1}{2-(1-T_o x)}} = \cfrac{1}{(k+1) + \cfrac{-1}{2}},$$

or

$$1 - x = \cfrac{1}{k + \cfrac{1}{2-(1-T_o x)}} = \cfrac{1}{k + \cfrac{1}{2}}.$$

Thus $x = \frac{2k-1}{2k+1}$ *has two finite OOCF expansions:*

$$x = [\![(k+1, -1)]\!] \text{ and } x = [\![(k, 1)]\!].$$

Further, for $x \in [0, 1]$ *such that* $T_o^n(x) \in \mathcal{U}$, *for some* $n \geq 1$, *we can apply the iteration (3.8) or (3.9)* n *times, and then we can write* $1 - T_o^n(x)$ *in two different ways. Therefore, any* $x \in \bigcup_{n \geq 0} T_o^{-n}\mathcal{U}$ *has two finite OOCF expansions which differ at the last digit.*

If $x = 0$, *then* $T_o(0) = 0$ *and we deduce*

$$1 - x = \cfrac{1}{2 + \cfrac{-1}{2-(1-T_o x)}}.$$

Thus 0 *has a unique infinite OOCF expansion:*

$$0 = [\![(2, -1), (2, -1), \cdots]\!] = [\![(2, -1)^\infty]\!].$$

For $x \in \mathcal{V}$, $T_o(x) = 0$ *and if* $x = \frac{k}{k+1}$, *we know*

$$1 - x = \cfrac{1}{(k+2) + \cfrac{-1}{2-(1-T_o x)}} = \cfrac{1}{(k+2) + \cfrac{-1}{2-(1-0)}},$$

or

$$1 - x = \cfrac{1}{(k+1) + \cfrac{1}{2-(1-T_o x)}} = \cfrac{1}{(k+1) + \cfrac{1}{2-(1-0)}}.$$

Hence $x = \frac{k}{k+1}$ *has two infinite OOCF expansions:*

$$x = [\![(k+2,-1),(2,-1)^\infty]\!] \text{ and } x = [\![(k+1,1),(2,-1)^\infty]\!].$$

Similarly, any $x \in \bigcup_{n\geq 0} T_o^{-n}\mathcal{V}$ *has two infinite OOCF expansions.*

We consider three forms of truncated continued fractions which give us three types of convergents. We investigate the basic properties of such convergents and give geometrical interpretations.

Definition 3.3.5. *We define three types of convergents. For* $n \geq 1$*, the* nth OOCF principal convergent *is defined by*

$$\frac{p_n}{q_n} = [\![(a_1,\varepsilon_1),(a_2,\varepsilon_2),\cdots,(a_n,\varepsilon_n)]\!] = 1 - \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{2 - \cfrac{1}{\ddots \cfrac{\varepsilon_{n-1}}{2 - \cfrac{1}{a_n + \cfrac{\varepsilon_n}{2}}}}}}.$$

We denote

$$\frac{p'_n}{q'_n} := 1 - \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{2 - \cfrac{1}{\ddots \cfrac{\varepsilon_{n-1}}{2 - \cfrac{1}{a_n}}}}} \quad \text{and} \quad \frac{p''_n}{q''_n} := 1 - \cfrac{1}{a_1 + \cfrac{\varepsilon_1}{2 - \cfrac{1}{\ddots \cfrac{\varepsilon_{n-1}}{2 - \cfrac{1}{a_n + \varepsilon_n}}}}},$$

and call them the nth sub-convergent *and* nth pseudo-convergent*, respectively.*

Applying Lemma 3.2.1, we infer recursive relations of three types of convergents.

Lemma 3.3.6. *Let* $p'_0 = 1$*,* $q'_0 = 0$*,* $p_0 = 1$ *and* $q_0 = 1$*. We deduce the following recursive formulas for* $n \geq 1$*:*

$$\begin{cases} p'_n = a_n p_{n-1} - p'_{n-1}, \\ q'_n = a_n q_{n-1} - q'_{n-1}, \end{cases} \quad \begin{cases} p''_n = p'_n + \varepsilon_n p_{n-1}, \\ q''_n = q'_n + \varepsilon_n q_{n-1}, \end{cases} \quad \text{and} \quad \begin{cases} p_n = 2p'_n + \varepsilon_n p_{n-1}, \\ q_n = 2q'_n + \varepsilon_n q_{n-1}. \end{cases}$$

By the above recursive formulas, we further see

$$\begin{cases} p_n = p'_n + p''_n, \\ q_n = q'_n + q''_n, \end{cases} \quad \text{and} \quad \begin{cases} p_{n-1} = \varepsilon_n(p''_n - p'_n), \\ q_{n-1} = \varepsilon_n(q''_n - q'_n). \end{cases} \tag{3.11}$$

Proof. We see the lemma by plugging $\mathfrak{a}_0 = 1$, $\mathfrak{b}_0 = -1$, $\mathfrak{a}_{2n} = 2$, $\mathfrak{a}_{2n-1} = a_n$, $\mathfrak{b}_{2n} = -1$, $\mathfrak{b}_{2n-1} = \varepsilon_n$ in the formula in Lemma 3.2.1. □

The recursive formulas for the principle convergents are given by the following lemma.

Lemma 3.3.7. *We deduce*

$$\begin{cases} p_n = (2a_n + \varepsilon_n - 1)p_{n-1} + \varepsilon_{n-1}p_{n-2}, \\ q_n = (2a_n + \varepsilon_n - 1)q_{n-1} + \varepsilon_{n-1}q_{n-2}, \end{cases}$$

where $p_{-1}/q_{-1} = -1/1$, $p_0/q_0 = 1/1$ *and* $\varepsilon_0 = 1$.

Proof. By Lemma 3.3.6,

$$\begin{aligned} p_n = p'_n + p''_n &= 2(a_n p_{n-1} - p'_{n-1}) + \varepsilon_n p_{n-1} \\ &= (2a_n + \varepsilon_n - 1)p_{n-1} + (p_{n-1} - 2p'_{n-1}) \\ &= (2a_n + \varepsilon_n - 1)p_{n-1} + \varepsilon_{n-1}p_{n-2} \end{aligned}$$

and

$$\begin{aligned} q_n = q'_n + q''_n &= 2(a_n q_{n-1} - q'_{n-1}) + \varepsilon_n q_{n-1} \\ &= (2a_n + \varepsilon_n - 1)q_{n-1} + (q_{n-1} - 2q'_{n-1}) \\ &= (2a_n + \varepsilon_n - 1)q_{n-1} + \varepsilon_{n-1}q_{n-2}. \end{aligned}$$

□

We use the same notations in Chapter 2. Let us consider the hyperbolic plane $\mathbb{H}$ as the upper half-plane model. The boundary of $\mathbb{H}$ is $\mathbb{R}_\infty = \mathbb{R} \cup \{\infty\}$. Let ℓ be the vertical line whose endpoints are 0 and ∞.

Definition 3.3.8. Farey graph $\mathscr{G}$ *is defined by*

$$\mathscr{G} = \bigcup_{\gamma \in SL_2(\mathbb{Z})} \gamma(\ell).$$

The Farey graph $\mathscr{G}$ *is a graph on* $\mathbb{H} \cup \mathbb{R}_\infty$ *whose endpoints are all rationals or* ∞ *in* $\mathbb{R}_\infty$ *(see Figure 3.6).*

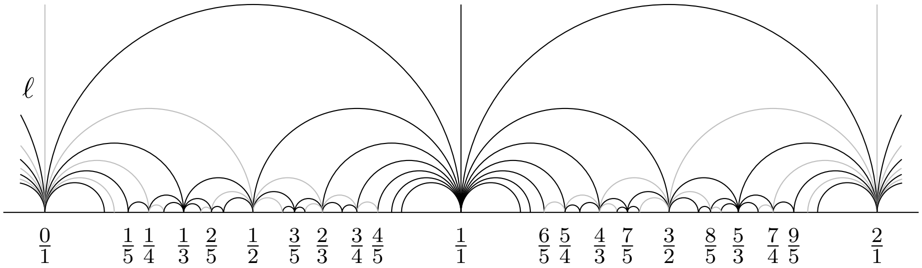


Figure 3.6: Farey graph

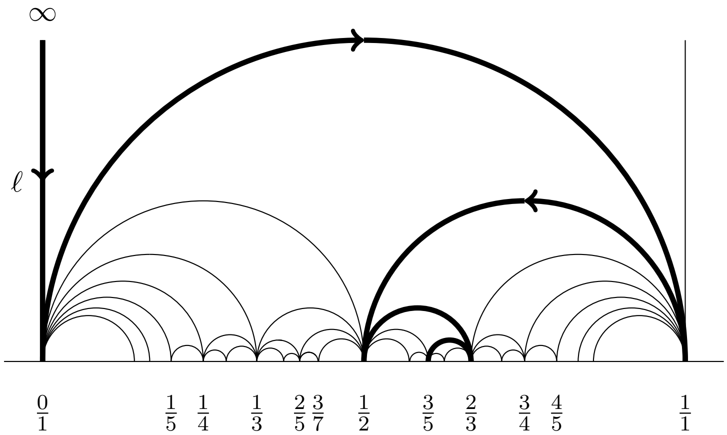


Figure 3.7: The corresponding path of $\frac{1+\sqrt{5}}{2}$ in the Farey graph for the regular continued fraction

Let $\gamma = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Since $\gamma(\infty) = a/c$ and $\gamma(0) = b/d$, two rationals a/b and c/d are adjacent to each other in $\mathscr{G}$ if and only if $|ad-bc| = 1$. It is known that a regular continued fraction is related to a path on the Farey graph. By (2.5), $|P_{n+1}Q_n - P_nQ_{n+1}| = 1$ where P_n/Q_n the nth principal convergent of the regular continued fraction. For each $x \in \mathbb{R}$, there is a path corresponding to the sequence $\{P_n(x)/Q_n(x)\}_n$ on $\mathscr{G}$ starting from $\lfloor x \rfloor$, passing through all P_n/Q_n consecutively.

Example 3.3.9. *For example, let* $\frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, \cdots]$. *The principal convergents of the regular continued fraction are*

$$\frac{P_1}{Q_1} = 1,\ \frac{P_2}{Q_2} = \frac{1}{2},\ \frac{P_3}{Q_3} = \frac{2}{3},\ \frac{P_4}{Q_4} = \frac{3}{5},\ \cdots.$$

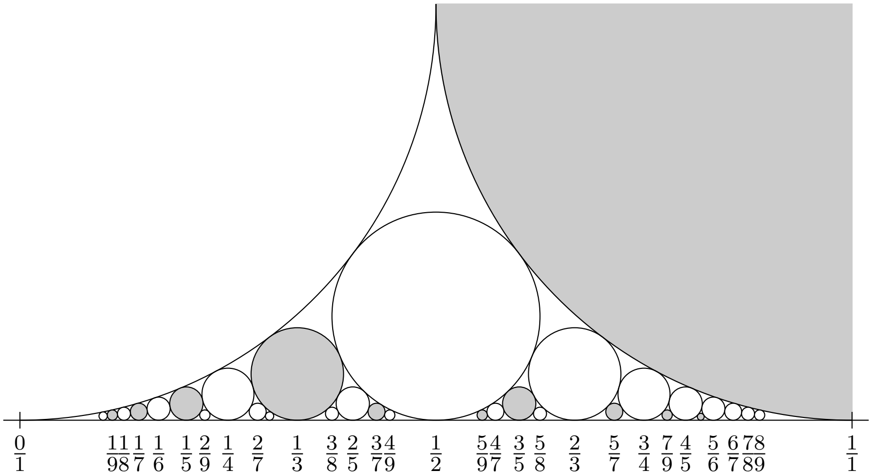


Figure 3.8: Ford circles: white circles are based at ∞-rationals and gray circles are based at 1-rationals

The corresponding path on Farey graph is as in Figure 3.7.

Definition 3.3.10. A Ford circle $C_{a/b}$ *is a horocycle of* $\mathbb{H}$ *whose base point is* a/b *and Euclidean radius is* $(2b^2)^{-1}$. *We define* C_∞ *as the line* $y = 1$ *(see Figure 3.8).*

We note that the collection of Ford circles is a dual of Farey graph in the sense that two Ford circles are tangent to each other if and only if their base points are adjacent to each other in the Farey graph.

Short and Walker examined a similar relation between EICF and a subtree of the Farey graph and the Ford circles [65]. The *Farey tree* $\mathscr{F}$ is a subtree of $\mathcal{G}$ defined by

$$\bigcup_{\gamma \in \Theta} \gamma(\ell).$$

The shaded lines in Figure 3.6 represent the Farey tree. Every vertex of $\mathscr{F}$ on $\mathbb{R}_\infty$ is ∞-rationals. We will see that each OOCF corresponds to a path on $\mathscr{G} - \mathscr{F}$. The following lemma is on relations between the three distinct convergents of OOCF.

Lemma 3.3.11. *Each* p_{n-1}/q_{n-1} *(and also each* p_n/q_n*) is adjacent to* p'_n/q'_n *and* p''_n/q''_n *in Farey graph. Moreover,* p'_n/q'_n *and* p''_n/q''_n *are adjacent to each other.*

Proof. Recall that $p_0'/q_0' = 0/1$, $p_0''/q_0'' = 1/0$ and $p_0/q_0 = 1/1$. The first convergents are

$$\frac{p_1'}{q_1'} = \frac{a_1 - 1}{a_1}, \quad \frac{p_1''}{q_1''} = \frac{a_1 + \varepsilon_1 - 1}{a_1 + \varepsilon_1} \quad \text{and} \quad \frac{p_1}{q_1} = \frac{2a_1 + \varepsilon_1 - 2}{2a_1 + \varepsilon_1}.$$

Clearly, p_0/q_0 and p_1/q_1 are adjacent to p_1'/q_1' and p_1''/q_1''. Let

$$A_{(a_n,\varepsilon_n)} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_n \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_n - 1 & a_n + \varepsilon_n - 1 \\ a_n & a_n + \varepsilon_n \end{pmatrix}. \tag{3.12}$$

We see that

$$p_0/q_0 = A_{(a_1,\varepsilon_1)} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad p_1'/q_1' = A_{(a_1,\varepsilon_1)} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$p_1''/q_1'' = A_{(a_1,\varepsilon_1)} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad p_1/q_1 = A_{(a_1,\varepsilon_1)} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By Lemma 3.3.6, we infer that

$$\begin{cases} p_n' = (a_n - 1)p_{n-1}' + a_n p_{n-1}'' \quad \text{and} \quad p_n'' = (a_n - 1 + \varepsilon_n)p_{n-1}' + (a_n + \varepsilon_n)p_{n-1}'', \\ q_n' = (a_n - 1)q_{n-1}' + a_n q_{n-1}'' \quad \text{and} \quad q_n'' = (a_n - 1 + \varepsilon_n)q_{n-1}' + (a_n + \varepsilon_n)q_{n-1}''. \end{cases}$$

Thus we have

$$\begin{pmatrix} p_{n-1} & p_n' & p_n'' & p_n \\ q_{n-1} & q_n' & q_n'' & q_n \end{pmatrix} = A_{(a_1,\varepsilon_1)} \cdots A_{(a_n,\varepsilon_n)} \begin{pmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}. \tag{3.13}$$

Since $|\det(A_{(a_n,\varepsilon_n)})| = |-\varepsilon_n| = 1$, the matrix $A_{(a_n,\varepsilon_n)}$, which is a linear fractional map or an anti-linear fractional map on $\mathbb{H} \cup \mathbb{R}_\infty$, preserves the adjacency of the vertices in the Farey graph. □

Example 3.3.12. *Let x be in the subinterval $[3/7, 1/2]$ which is the blue segment in Figure 3.9 and Figure 3.10. The principal convergents and subconvergents of x are*

$$\frac{p_0'}{q_0'} = \frac{1}{0}, \quad \frac{p_0}{q_0} = \frac{1}{1}, \quad \frac{p_1'}{q_1'} = \frac{0}{1}, \quad \frac{p_1}{q_1} = \frac{1}{3}, \quad \frac{p_2'}{q_2'} = \frac{2}{5}, \quad \frac{p_2}{q_2} = \frac{3}{7}, \quad \cdots.$$

The corresponding path of the OOCF expansion of x is the branch starting

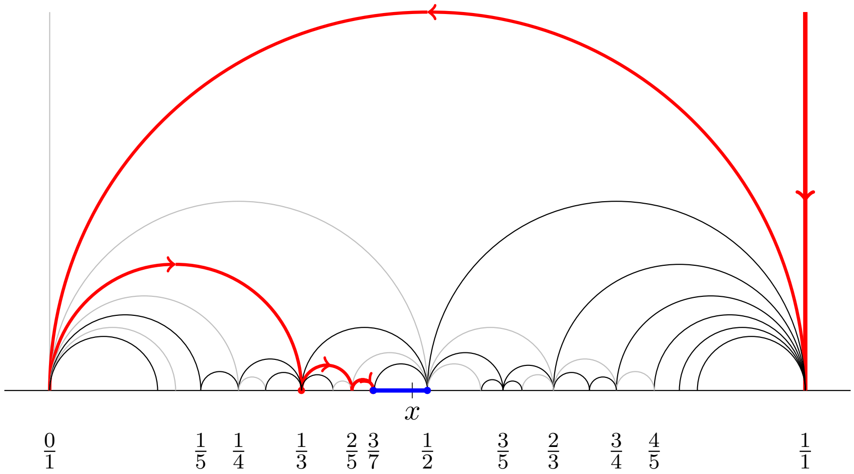


Figure 3.9: The collection of the red geodesics is the path corresponding to the OOCF of x on $\mathscr{G} - \mathscr{F}$.

from ∞*, going down to* 1 *and then going along with the arcs connecting* p'_n/q'_n *and* p_n/q_n*, then the arcs connecting* p_n/q_n *and* p'_{n+1}/q'_{n+1}*, repeatedly. In Figure 3.9, the red arrows follow the convergents*

$$p_{n-1}/q_{n-1} \to p'_n/q'_n \to p_n/q_n \to \cdots .$$

From the duality between the Farey graph and the collection of the Ford circles, there are Ford circles tangent to each other when their base points are p'_n/q'_n*,* p_n/q_n *or* p_n/q_n*,* p'_{n+1}/q'_{n+1}*. The Ford circles numbered from 1 to 5 in Figure 3.10 are the first five Ford circles corresponding to the OOCF expansion of* x*.*

For each OOCF digit (a, ε), we partition $[0, 1]$ into the subintervals $B(a, \varepsilon)$ defined by

$$B(k+1, -1) = \left[\frac{k-1}{k}, \frac{2k-1}{2k+1}\right], \quad B(k, 1) = \left[\frac{2k-1}{2k+1}, \frac{k}{k+1}\right]. \tag{3.14}$$

Note that the set of endpoints of $B(a, \varepsilon)$ is $\mathcal{U} \cup \mathcal{V}$ as in (3.7). The first OOCF digit of $x \in B(a, \varepsilon)$ is (a, ε) and the restriction of T_o to $B(a, \varepsilon)$ is monotone

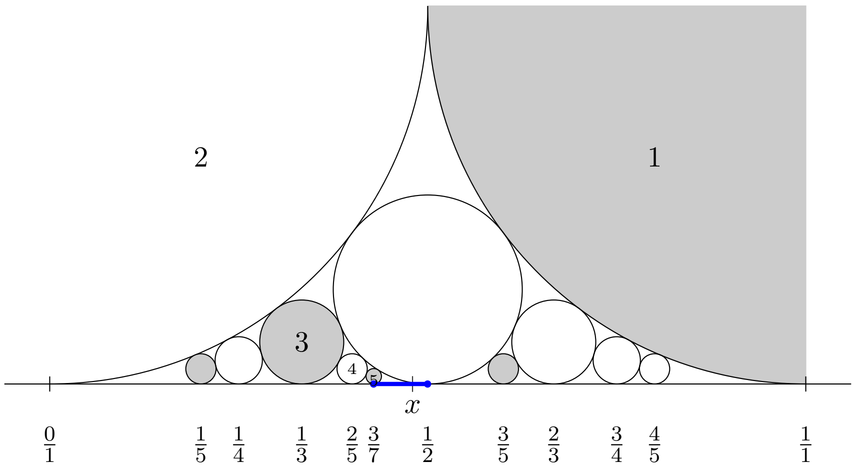


Figure 3.10: The Ford circles numbered from 1 to 5 are Ford circles based at principal convergents and sub-convergents of x consecutively.

(see Figure 3.4). Writing $f_{(a,\varepsilon)} = (T_{\mathrm{o}}|_{B(a,\varepsilon)})^{-1}$, we have

$$f_{(a,\varepsilon)}(t) = 1 - \cfrac{1}{a + \cfrac{\varepsilon}{1+t}}. \tag{3.15}$$

Lemma 3.3.13. *For all $x \in (0,1)$, x is a point between p_n/q_n and p''_n/q''_n.*

Proof. By (3.10), we deduce that $1 - T_{\mathrm{o}}^{n-1}(x)$ is between $\frac{1}{a_n+(\varepsilon_n/2)}$ and $\frac{1}{a_n+\varepsilon_n}$. Let

$$g := f_{(a_1,\varepsilon_1)} \circ \cdots \circ f_{(a_n,\varepsilon_n)}.$$

Since g is monotone, g does not change the relative positions of points. From this, it follows that $g(1 - T_{\mathrm{o}}^{n-1}(x)) = x$ is a point between $g(\frac{1}{a_n+(\varepsilon_n/2)}) = p_n/q_n$ and $g(\frac{1}{a_n+\varepsilon_n}) = p''_n/q''_n$, which is the desired conclusion. □

By (3.12), it is easy to check that either $A_{(a_n,\varepsilon_n)}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A_{(a_n,\varepsilon_n)}$ belongs to the Θ-group according to $\det(A_{(a_n,\varepsilon_n)})$. Applying (3.13), we obtain a property of the OOCF convergents.

Theorem 3.3.14. *All principal convergents p_n/q_n are 1-rationals. All sub-convergents p'_n/q'_n and all pseudo-convergents p''_n/q''_n are ∞-rationals.*

Now, we will discuss the relative positions of the different convergents. We define *the nth complete quotient* by $\zeta_n = T_{\mathrm{o}}^{n-1}(x)$, i.e.,

$$\zeta_n = 1 - \cfrac{1}{a_n + \cfrac{\varepsilon_n}{2 - \cfrac{1}{a_{n+1} + \cfrac{\varepsilon_{n+1}}{\ddots}}}}.$$

The following equality holds:

$$x = \frac{p''_{n-1} + p'_{n-1}\zeta_n}{q''_{n-1} + q'_{n-1}\zeta_n}. \tag{3.16}$$

By (3.11), we have

$$\begin{cases} p''_{n-1} = \frac{1}{2}(p_{n-1} + \varepsilon_{n-1}p_{n-2}), \\ p'_{n-1} = \frac{1}{2}(p_{n-1} - \varepsilon_{n-1}p_{n-2}). \end{cases} \quad \text{and} \quad \begin{cases} q''_{n-1} = \frac{1}{2}(q_{n-1} + \varepsilon_{n-1}q_{n-2}), \\ q'_{n-1} = \frac{1}{2}(q_{n-1} - \varepsilon_{n-1}q_{n-2}). \end{cases}$$

An easy computation shows that

$$x = \frac{p_{n-1}(1+\zeta_n) + \varepsilon_{n-1}p_{n-2}(1-\zeta_n)}{q_{n-1}(1+\zeta_n) + \varepsilon_{n-1}q_{n-2}(1-\zeta_n)}$$

and

$$\zeta_n = -\frac{(q_{n-1}x - p_{n-1}) + \varepsilon_{n-1}(q_{n-2}x - p_{n-2})}{(q_{n-1}x - p_{n-1}) - \varepsilon_{n-1}(q_{n-2}x - p_{n-2})}.$$

Lemma 3.3.15. *The following statements hold.*

(1) The nth principal convergent p_n/q_n is between p'_n/q'_n and p''_n/q''_n.

(2) The $(n-1)$th principal convergent p_{n-1}/q_{n-1} is not between p'_n/q'_n and p''_n/q''_n.

(3) The three distinct convergents p_n/q_n, p'_n/q'_n and p''_n/q''_n are in the half closed interval whose endpoints are p_{n-1}/q_{n-1} and p''_{n-1}/q''_{n-1}. The interval contains p''_{n-1}/q''_{n-1}, but does not contain p_{n-1}/q_{n-1}.

We denote by I_n the half closed interval whose endpoints are p_n/q_n and p''_n/q''_n.

Proof. (1) By Lemma 3.3.6, $p_n = p'_n + p''_n$ and $q_n = q'_n + q''_n$. Hence, p_n/q_n is between p'_n/q'_n and p''_n/q''_n.

(2) There are only two elementary triangles that share the arc connecting p'_n/q'_n and p''_n/q''_n. By Lemma 3.3.11, the other endpoints of the two elementary triangles are p_{n-1}/q_{n-1} and p_n/q_n. By (1), the $(n-1)$th principal convergent p_{n-1}/q_{n-1} is not between p'_n/q'_n and p''_n/q''_n since two elementary triangles of the Farey graph do not overlap each other.

(3) By Lemma 3.3.6, we deduce that

$$\begin{cases} p'_n = (a_n - 1)p_{n-1} + p''_{n-1}, \\ q'_n = (a_n - 1)q_{n-1} + q''_{n-1}, \end{cases} \quad \text{and} \quad \begin{cases} p''_n = (a_n - 1 + \varepsilon_n)p_{n-1} + p''_{n-1}, \\ q''_n = (a_n - 1 + \varepsilon_n)q_{n-1} + q''_{n-1}. \end{cases}$$

Since $a_n - 1 \geq 0$ and $a_n + \varepsilon_n - 1 \geq 0$, both p'_n/q'_n and p''_n/q''_n are in I_{n-1}. By (1), p_n/q_n is also in I_{n-1}. □

Theorem 3.3.16. *The infinite OOCF expansions converge.*

Proof. Let x be a number whose OOCF expansion is infinite. By Lemma 3.3.15 (3), the intervals I_n are shrinking. By Lemma 3.3.13, we infer that

$$\left| x - \frac{p_n}{q_n} \right| < \left| \frac{p_n}{q_n} - \frac{p''_n}{q''_n} \right| \to 0 \quad \text{as} \quad n \to \infty.$$ □

Lemma 3.3.17. *The denominators of principal convergents are increasing.*

Proof. Since $2a_{n+1} + \varepsilon_{n+1} \geq 3$, we deduce that

$$q_{n+1} = (2a_{n+1} + \varepsilon_{n+1} - 1)q_n + \varepsilon_n q_{n-1} > q_n + (q_n - q_{n-1}).$$

Since $q_0 - q_{-1} = 1 > 0$, inductively, the sequence $\{q_n\}_n$ is increasing. □

Example 3.3.18. *As discussed above, any non-zero rational has exactly two OOCF expansions. For example, a* 1*-rational* 1/3 *has such two finite OOCF expansions:*

$$\frac{1}{3} = 1 - \cfrac{1}{1 + \cfrac{1}{2}} = 1 - \cfrac{1}{2 + \cfrac{-1}{2}}.$$

For the ∞-rational $2/3$, there are two infinite OOCF expansions:

$$\frac{2}{3} = 1 - \cfrac{1}{4 + \cfrac{-1}{2 - \cfrac{1}{2 + \cfrac{-1}{2 - \ddots}}}} = 1 - \cfrac{1}{2 + \cfrac{1}{2 - \cfrac{1}{2 + \cfrac{-1}{2 - \ddots}}}}.$$

Theorem 3.3.19. *The following properties hold.*

(1) Any finite OOCF is a 1-rational.

(2) Each 1-rational has exactly two finite OOCF expansions which differ only in the last digit.

(3) Any non-zero ∞-rational has exactly two infinite OOCF expansions ending with $(2, -1)^\infty$.

(4) An eventually periodic OOCF expansion converges to an ∞-rational or a quadratic irrational.

(5) A quadratic irrational has an eventually periodic OOCF expansion.

(6) Every irrational has a unique infinite OOCF expansion.

Proof. (1) It follows directly from Theorem 3.3.14.

(2) & (3) The OOCF map T_o sends a 1-rational to a 1-rational and an ∞-rational to an ∞-rational. For an irreducible rational a/b, the denominator of $T_o(a/b)$ is strictly less than b. Thus if a/b is a 1-rational, then $T_o^n(a/b) = 1$ for some n. We know that

$$T_o^{n-1}\left(\frac{a}{b}\right) \in T_o^{-1}(1) = \left\{\frac{2k-1}{2k+1} : k \geq 1\right\}.$$

Thus, a/b has exactly two finite OOCF expansions which differ only at the last digit.

If a/b is a non-zero ∞-rational, then $T_o^n(a/b) = 0$ for some $n \geq 1$. Since the OOCF expansion of 0 is $[\![(2, -1)^\infty]\!]$, the OOCF expansion of x ends with $(2,-1)^\infty$. Similar to the case of the 1-rationals, we know that

$$T_o^{n-1}\left(\frac{a}{b}\right) \in \left\{\frac{k}{k+1} : k \geq 1\right\}.$$

Then a/b has exactly two infinite OOCF expansions ending with $(2,-1)^\infty$.

(4) If x has an eventually periodic OOCF, then there exist n and m such that $\zeta_{n+1} = \zeta_{m+1}$. By (3.16), we have

$$x = \frac{p''_n + p'_n\zeta_{n+1}}{q''_n + q'_n\zeta_{n+1}} = \frac{p''_m + p'_m\zeta_{m+1}}{q''_m + q'_m\zeta_{m+1}}. \tag{3.17}$$

Thus we know that either $x = p''_n/q''_n$, i.e, $\zeta_{n+1} = 0$, or

$$\frac{1}{\zeta_{n+1}} = \frac{p'_n - q'_n x}{q''_n x - p''_n} = \frac{p'_m - q'_m x}{q''_m x - p''_m} = \frac{1}{\zeta_{m+1}}.$$

In the first case, $x = p''_n/q''_n$ is an ∞-rational by Theorem 3.3.14. In the second case, (3.17) implies

$$(q'_n q''_m - q'_m q''_n)x^2 + (q'_m p''_n + p'_m q''_n - q'_n p''_m - p'_n q''_m)x + (p'_n p''_m - p''_n p'_m) = 0.$$

Since $q'_n q''_m - q'_m q''_n \neq 0$, x is a quadratic irrational.

(5) Let x be a quadratic irrational between 0 and 1 such that

$$a_1 x^2 + b_1 x + c_1 = 0.$$

By (3.16), we have

$$a_1(p''_n + p'_n\zeta_{n+1})^2 + b_1(p''_n + p'_n\zeta_{n+1})(q''_n + q'_n\zeta_{n+1}) + c_1(q''_n + q'_n\zeta_{n+1})^2 = 0.$$

We derive the following quadratic equation of ζ_{n+1}:

$$(a_1 p'^2_n + b_1 p'_n q'_n + c_1 q'^2_n)\zeta^2_{n+1} + (2a_n p''_n p'_n + b_1(p''_n q'_n + p'_n q''_n) + 2c_1 q''_n q'_n)\zeta_{n+1} \\ + (a_1 p''^2_n + b_1 p''_n q''_n + c_1 q''^2_n) = 0.$$

Denote the coefficients of the above equation by

$$\begin{aligned} a_{n+1} &= a_1(p'_n)^2 + b_1 p'_n q'_n + c_1(q'_n)^2, \\ b_{n+1} &= 2a_1 p''_n p'_n + b_1(p''_n q'_n + p'_n q''_n) + 2c_1 q''_n q'_n \text{ and} \\ c_{n+1} &= a_1(p''_n)^2 + b_1 p''_n q''_n + c_1(q''_n)^2. \end{aligned} \tag{3.18}$$

Then $a_{n+1}\zeta_{n+1}^2 + b_{n+1}\zeta_{n+1} + c_{n+1} = 0$. Combining (3.13) and (3.13), we have $|q'_n p''_n - q''_n p'_n| = 1$. By direct calculation, we have

$$b_{n+1}^2 - 4a_{n+1}c_{n+1} = b_1^2 - 4a_1c_1, \tag{3.19}$$

and also we have

$$\left|\left(x - \frac{p'_n}{q'_n}\right) + \left(\frac{p''_n}{q''_n} - x\right)\right| = \frac{1}{q'_n q''_n}.$$

By Lemma 3.3.13 and the fact that p_n/q_n is between p'_n/q'_n and p''_n/q''_n, two distances $x - p'_n/q'_n$ and $x - p''_n/q''_n$ have opposite signs. Thus,

$$|q'_n x - p'_n| < \frac{1}{q''_n} \text{ and } |p''_n - q''_n x| < \frac{1}{q'_n}.$$

There are $|\alpha| < 1$ and $|\beta| < 1$ such that

$$p'_n = q'_n x + \alpha/q''_n \text{ and } p''_n = q''_n x + \beta/q'_n. \tag{3.20}$$

By plugging (3.20) in (3.18), we derive that

$$\begin{aligned} a_{n+1} &= \alpha\left(\frac{q'_n}{q''_n}(2a_1x + b_1) + a_1\frac{\alpha}{(q''_n)^2}\right), \\ b_{n+1} &= (2a_1x + b_1)(\alpha + \beta) + 2a_1\frac{\alpha\beta}{q'_n q''_n} \text{ and} \\ c_{n+1} &= \beta\left(\frac{q''_n}{q'_n}(2a_1x + b_1) + a_1\frac{\beta}{(q'_n)^2}\right). \end{aligned}$$

Since

$$|b_{n+1}| \le 2(|2a_1| + |b_1|) + |2a_1|,$$

the coefficient b_{n+1} is bounded. If $q''_n \ge q'_n$, then a_{n+1} is bounded since

$$|a_{n+1}| < 2|a_1| + |b_1| + |a_1|.$$

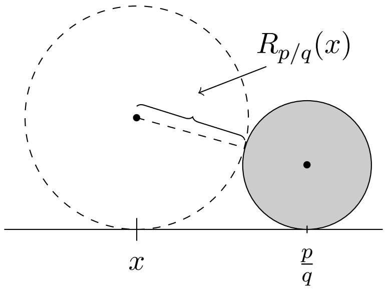


Figure 3.11: The horocycle based at x tangent to $C_{p/q}$.

By (3.19), c_{n+1} is bounded. Similarly, if $q''_n < q'_n$, then c_{n+1} is bounded since

$$|c_{n+1}| < 2|a_1| + |b_1| + |a_1|.$$

Then a_{n+1} is also bounded. Thus the sequence $\{\zeta_n\}$ has only finitely many values, which means that $\zeta_n = \zeta_m$ for some n and m. Therefore, the OOCF expansion of x is eventually periodic.

(6) Note that

$$\bigcup_{n=0}^{\infty} T_o^{-n}(\mathcal{U} \cup \mathcal{V} \cup \{0\}) = \bigcup_{n=1}^{\infty} T_o^{-n}(\{0,1\}) = \mathbb{Q}.$$

By the discussion in Remark 3.3.4, every irrational has a unique infinite OOCF expansion. □

Remark 3.3.20. *From the proof of Theorem 3.3.19 (4), we see that if x is an ∞-rational, then its OOCF expansion ends with $(2,-1)^\infty$. Thus there exists $n_0 \geq 0$ such that $\zeta_{n+1} = 0$ for all $n \geq n_0$. Hence, by (3.16), we have $x = p''_n/q''_n$ for all $n \geq n_0$.*

3.3.2 Diophantine properties of odd-odd continued fraction

In this section, we prove that the OOCF gives "the best 1-rational approximations".

Definition 3.3.21. *Let $x \notin \mathbb{Q}$.* A best 1-rational approximation of x *is a rational p/q such that, for each 1-rational $a/b \neq p/q$,*

$$|qx - p| \leq |bx - a|.$$

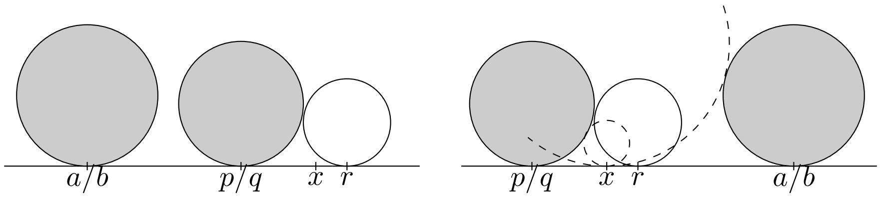


Figure 3.12: Two possible relative locations of x, p/q, a/b and r in the proof of Theorem 3.3.22. The dashed circles are the horocycles based at x tangent to $C_{p/q}$ and $C_{a/b}$.

Theorem 3.3.22. *All principal convergents p_n/q_n are the best 1-rational approximations, and vice versa.*

Proof. Let $R_{a/b}(x)$ be the Euclidean radius of the horocycle based at x which is externally tangent to $C_{a/b}$ (see Figure 3.11). A simple computation gives

$$R_{a/b}(x) := \frac{1}{2}|bx - a|^2.$$

Let x be an irrational number and p/q be an OOCF convergent of x. Fix a 1-rational a/b such that $1 \leq b \leq q$. Combining Lemma 3.3.11, Lemma 3.3.13 and Theorem 3.3.14, we can assert that there is an ∞-rational r which is adjacent to p/q and x is between p/q and r. The radius $R_{p/q}(x)$ is at most the radius of C_r. Since the radius of $C_{a/b}$ is at least the radius of $C_{p/q}$, the 1-rational a/b is outside of the interval $[p/q, r]$ (as shown in Figure 3.12). Thus $R_{a/b}(x)$ is at least the radius of C_r and the equality holds if and only if $a/b = \infty$. We conclude that

$$R_{p/q}(x) \leq R_{a/b}(x).$$

Conversely, let us assume that a/b is a 1-rational which is not a principal convergent of the OOCF expansion of x. There are the consecutive principal convergents p_n/q_n and p_{n+1}/q_{n+1} such that $q_n \leq b < q_{n+1}$. Both are adjacent to p'_{n+1}/q'_{n+1} and p''_{n+1}/q''_{n+1} in the Farey graph, i.e., C_{p_n/q_n} and $C_{p_{n+1}/q_{n+1}}$ are tangent to $C_{p'_{n+1}/q'_{n+1}}$ and $C_{p''_{n+1}/q''_{n+1}}$. Since the radius of $C_{a/b}$ is larger than the radius of $C_{p_{n+1}/q_{n+1}}$, the 1-rational a/b is not between p'_{n+1}/q'_{n+1} and p''_{n+1}/q''_{n+1}. Without loss of generality, as in Figure 3.13, we assume that

$$p_n/q_n < p'_{n+1}/q'_{n+1} < p''_{n+1}/q''_{n+1}.$$

(1) If $a/b < p_n/q_n$, then obviously

$$R_{a/b}(x) > R_{p_n/q_n}(x).$$

(2) Let us consider the case of $a/b > p''_{n+1}/q''_{n+1}$. We note that

$$|x - t| > |x' - t| \text{ implies that } R_t(x) > R_t(x').$$

Thus, $R_{a/b}(x) > R_{a/b}(p''_{n+1}/q''_{n+1})$ and $R_{p_n/q_n}(x)$ is proportional to the distance of x and p_n/q_n.

The radius $R_{a/b}(p''_{n+1}/q''_{n+1})$ is at least the radius of $C_{p''_{n+1}/q''_{n+1}}$ (they are the same when $C_{p''_{n+1}/q''_{n+1}}$ is tangent to $C_{a/b}$). Since x is between p'_{n+1}/q'_{n+1} and p''_{n+1}/q''_{n+1}, the radius $R_{p_n/q_n}(x)$ is between

$$R_{p_n/q_n}(p'_{n+1}/q'_{n+1}) = 1/2(q'_{n+1})^2 \text{ and } R_{p_n/q_n}(p''_{n+1}/q''_{n+1}) = 1/2(q''_{n+1})^2.$$

Thus we have $R_{a/b}(x) > R_{p_n/q_n}(x)$.

(3) The last case is of $a/b \in (p_n/q_n, p'_{n+1}/q'_{n+1})$. Let us consider a horocycle C based at x which is tangent to $C_{a/b}$. The thick dashed arc in Figure 3.13 is a part of C. The circle C intersects C_{p_n/q_n} since the tangent point of C and $C_{a/b}$ is an interior point of the shape bounded by C_{p_n/q_n}, $C_{p'_{n+1}/q'_{n+1}}$ and the real line.

Thus, for all the cases, $R_{a/b}(x) > R_{p_n/q_n}(x)$, i.e., a/b is not a 1-rational best approximation. □

3.3.3 Relation with EICF and the regular continued fraction

Now, we discuss the relation between the OOCF convergents of a number x and the EICF convergents of $1-x$. Remark that $p^e_n(x)/q^e_n(x)$ is an EICF convergent of x. Observe that if $p^e_n(x)/q^e_n(x)$ is of the form $\frac{\text{even}}{\text{odd}}$, then $1 - p^e_n(x)/q^e_n(x)$ is 1-rational. If $p^e_n(x)/q^e_n(x)$ is of the form $\frac{\text{odd}}{\text{even}}$, then $1 - p^e_n(x)/q^e_n(x)$ is also of the form $\frac{\text{odd}}{\text{even}}$.

Theorem 3.3.23. *Let $x \in (0, 1)$. All 1-rationals in $\{1 - p^e_n(1 - x)/q^e_n(1 - x)\}_{n\in\mathbb{N}}$ are best 1-rational approximations of x, and hence they are OOCF principal convergents of x.*

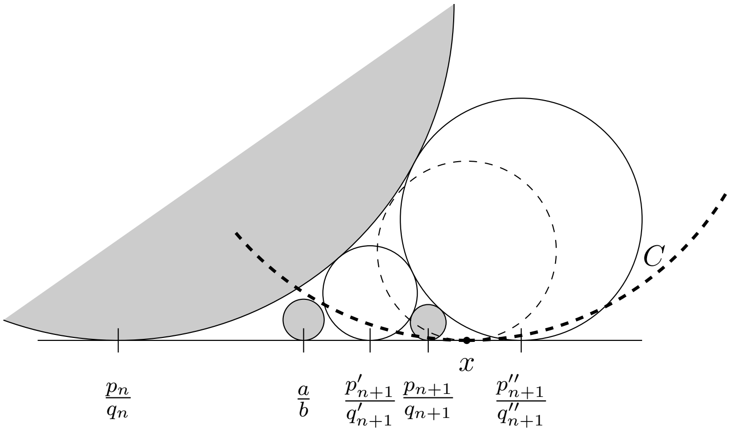


Figure 3.13: A possible relative position of x, a/b and the convergents. The dashed circles are horocycles based at x tangent to C_{p_n/q_n} and $C_{a/b}$.

Proof. For each $n \geq 0$, denote by $A_n/B_n := 1 - p_n^e(1-x)/q_n^e(1-x)$. Since $A_n = q_n^e(1-x) - p_n^e(1-x)$ and $B_n = q_n^e(1-x)$, for any ∞-rational a/b such that $1 \leq b \leq q_n^e(1-x)$ and $a/b \neq p_n^e(1-x)/q_n^e(1-x)$,

$$\begin{aligned} |A_n - B_n x| &= |q_n^e(1-x) - p_n^e(1-x) - x \cdot q_n^e(1-x)| \\ &= |(1-x) \cdot q_n^e(1-x) - p_n^e(1-x)| < |(1-x)b - a|. \end{aligned} \tag{3.21}$$

For any 1-rational c/d such that $1 \leq d \leq B_n$ and $c/d \neq A_n/B_n$, we deduce that

$$\frac{d-c}{d} = 1 - \frac{c}{d} \in \Theta(\infty) \text{ and } \frac{d-c}{d} \neq \frac{p_n^e(1-x)}{q_n^e(1-x)}.$$

By (3.21), we conclude that $|A_n - B_n x| < |(1-x)d - (d-c)| = |c - dx|$, which completes the proof. □

Example 3.3.24. *Let $x = 10 - \pi^2 = 0.1303955989\cdots$. The OOCF convergents of x are:*

$$\mathbf{\frac{1}{3}}, \mathbf{\frac{1}{5}}, \mathbf{\frac{1}{7}}, \mathbf{\frac{3}{23}}, \frac{287}{2201}, \frac{577}{4425}, \frac{867}{6649}, \frac{1157}{8873}, \mathbf{\frac{1447}{11097}}, \mathbf{\frac{6945}{53261}}, \cdots. \tag{3.22}$$

The sequence of $1 - p_n^e/q_n^e$ where p_n^e/q_n^e is the EICF convergent of $1-x$ is:

$$\frac{1}{2}, \mathbf{\frac{1}{3}}, \frac{1}{4}, \mathbf{\frac{1}{5}}, \frac{1}{6}, \mathbf{\frac{1}{7}}, \frac{1}{8}, \mathbf{\frac{3}{23}}, \frac{145}{1112}, \mathbf{\frac{1447}{11097}}, \frac{2749}{21082}, \mathbf{\frac{6945}{53261}}, \cdots. \tag{3.23}$$

The subsequece consisting of rationals of the form $\frac{odd}{odd}$ *in (3.23) is a (strict) subsequence of (3.22). However, the rational* 287/2201 *in (3.22) is not in (3.23) since*

$$\begin{aligned}|1112x-145| = 0.0000940113\cdots < |2201x-287| = 0.00071320\cdots \\ < |23x-3| = 0.00090122\cdots .\end{aligned}$$

Theorem 3.3.25. *The map* $f(x) = \frac{1-x}{1+x}$ *is in* (3.6). *The EICF principal convergents of* $f(x)$ *is* f *of the OOCF principal convergents of* x, *i.e.,*

$$\frac{p_n^e(f(x))}{q_n^e(f(x))} = f\left(\frac{p_n(x)}{q_n(x)}\right).$$

Proof. We write an EICF expansion by

$$\langle\langle (b_1,\eta_1),(b_2,\eta_2),\cdots,(b_n,\eta_n),\cdots\rangle\rangle = \langle\langle (b_n,\eta_n)\rangle\rangle_{n\in\mathbb{N}}$$

as in (3.3). Denote by $\mathcal{A}_o$ and $\mathcal{A}_e$, the sets of digits of OOCF and EICF respectively. Explicitly, we have

$$\begin{aligned}\mathcal{A}_o &= \{(a,\varepsilon) : a\in\mathbb{N}, \varepsilon=\pm 1\}\setminus\{(1,-1)\}, \text{ and} \\ \mathcal{A}_e &= \{(b,\eta) : b'\in 2\mathbb{N}, \eta=\pm 1\}.\end{aligned}$$

Recall the partition $\{B(a,\varepsilon)\}_{(a,\varepsilon)\in\mathcal{A}_o}$ in (3.14). Then we have

$$f(B(k+1,-1)) = \left[\frac{1}{2k},\frac{1}{2k-1}\right] \text{ and } f(B(k,1)) = \left[\frac{1}{2k+1},\frac{1}{2k}\right].$$

Thus, $f(B(k+1,-1))$ and $f(B(k,1))$ are subintervals which correspond to the EICF digits $(2k,1)$, $(2k,-1)\in\mathcal{A}_e$. In other words, there is a natural correspondence φ between $\mathcal{A}_o$ and $\mathcal{A}_e$ by

$$\varphi : (k+1,-1)\mapsto(2k,1), \quad (k,1)\mapsto(2k,-1).$$

Inductively, we define a subinterval corresponding to an OOCF expansion $[\![(a_1,\varepsilon_1),\cdots,(a_n,\varepsilon_n))]\!]$ by

$$B((a_1,\varepsilon_1),\cdots,(a_n,\varepsilon_n)) = \bigcap_{i=1}^{n} T_o^{-(i-1)}B(a_i,\varepsilon_i).$$

Since f is 1-1 and onto,

$$\begin{aligned} f(B((a_1,\varepsilon_1),\cdots,(a_n,\varepsilon_n))) &= \bigcap_{i=1}^{n} f(T_{\mathrm{o}}^{-(i-1)}B(a_i,\varepsilon_i)) \\ &= \bigcap_{i=1}^{n} T_{\mathrm{e}}^{-(i-1)} f(B(a_i,\varepsilon_i)) \quad \text{(by Theorem 3.3.1).} \end{aligned} \tag{3.24}$$

Since $f(B(a_i,\varepsilon_i))$ is a subinterval corresponding to $\varphi(a_i,\varepsilon_i)$, the interval in (3.24) is the subinterval corresponding to $\langle\!\langle\, \varphi(a_i,\varepsilon_i)\,\rangle\!\rangle_{i=1}^{n}$. In other word, if $x = [\![(a_i,\varepsilon_i)]\!]_{i\in\mathbb{N}}$, then the EICF expansion of $f(x)$ is $\langle\!\langle\, \varphi(a_i,\varepsilon_i)\,\rangle\!\rangle_{i\in\mathbb{N}}$ and

$$\frac{p_n^e(f(x))}{q_n^e(f(x))} = \langle\!\langle\, \varphi(a_i,\varepsilon_i)\,\rangle\!\rangle_{i\in\mathbb{N}}^{n} = f([\![(a_i,\varepsilon_i)]\!]_{i\in\mathbb{N}}^{n}) = f\left(\frac{p_n(x)}{q_n(x)}\right). \qquad \square$$

In their monograph [56], Rockett and Szüsz introduced "the best approximation of the first kind" and "the best approximation of the second kind". Our definition of the best approximation is the best approximation of the second kind. *The best approximation of the first kind of x* is a rational p/q such that $|x - p/q| < |x - a/b|$ for any $a/b \neq p/q$ and $q \leq b$. Every best approximation of the first kind is a form of

$$\frac{P_{n,k}}{Q_{n,k}} = \frac{P_{n-2} + kP_{n-1}}{Q_{n-2} + kQ_{n-1}}$$

for $1 \leq k \leq a_n$, $n \geq 1$ (see [36, Section 6] and [56, p.36] for more details).

Definition 3.3.26. *For the principal convergents P_n/Q_n,* The intermediate convergents *are*

$$\frac{P_{n,k}}{Q_{n,k}} = \frac{P_{n-2} + kP_{n-1}}{Q_{n-2} + kQ_{n-1}}$$

for $1 \leq k \leq a_n$ and for $n \geq 1$.

Kraaikamp and Lopes showed that the EICF convergents are intermediate convergents of the regular continued fraction (see [45] for the proof). We will show that the OOCF principal convergents are also intermediate convergents of the regular continued fraction.

Recall that $[d_0; d_1, d_2, \cdots]$ denotes a regular continued fraction as in (2.3). We explain how the piecewise inverses $f_{(a,\varepsilon)}$ in (3.15) change the regular continued fraction expansions.

Lemma 3.3.27. *Let* $x = [0; d_1, d_2, \cdots]$*. Then, the regular continued fraction expansion of* $f_{(a,\varepsilon)}(x)$ *is as follows:*

$$f_{(a,\varepsilon)}(x) = \begin{cases} [2, d_1, d_2, \cdots] & \text{if } \varepsilon = 1,\ a = 1, \\ [1, (a-1), 1, d_1, d_2, \cdots] & \text{if } \varepsilon = 1,\ a \geq 2, \\ [(d_1+2), d_2, \cdots] & \text{if } \varepsilon = -1,\ a = 2, \\ [1, (a-1), (d_1+1), d_2, \cdots] & \text{if } \varepsilon = -1,\ a \geq 3. \end{cases}$$

Proof. If $\varepsilon = 1$, then

$$f_{(a,\varepsilon)}(x) = 1 - [a, 1, d_1, d_2, \cdots] = \begin{cases} [2, d_1, d_2, \cdots] & \text{if } a = 1, \\ [1, a-1, 1, d_1, \cdots] & \text{if } a \geq 2. \end{cases}$$

If $\varepsilon = -1$, then

$$f_{(a,\varepsilon)}(x) = 1 - \frac{1}{a - [1, d_1, d_2, \cdots]} = 1 - \frac{1}{(a-1) + 1 - [1, d_1, d_2, \cdots]}$$

$$= 1 - [a-1, d_1+1, d_2, \cdots] = \begin{cases} [d_1+2, d_2, \cdots] & \text{if } a = 2, \\ [1, a-1, 1, d_1+1, d_2, \cdots] & \text{if } a \geq 3. \end{cases}$$ □

Theorem 3.3.28. *The OOCF principal convergents of* x *are intermediate convergents of* x*.*

Proof. Let $x = [d_1, d_2, \cdots] = [\![(a_1, \varepsilon_1), (a_2, \varepsilon_2), \cdots]\!]$. Note that

$$x = f_{(a_1,\varepsilon_1)} \circ f_{(a_2,\varepsilon_2)} \circ \cdots \circ f_{(a_k,\varepsilon_k)}([\![(a_{k+1}, \varepsilon_{k+1}), \cdots]\!])$$

and

$$\frac{p_k}{q_k} = f_{(a_1,\varepsilon_1)} \circ f_{(a_2,\varepsilon_2)} \circ \cdots \circ f_{(a_k,\varepsilon_k)}(1).$$

By Lemma 3.3.27, x and p_k/q_k have the same prefix in their regular continued fraction expansions, except for the last digit of p_k/q_k. Thus, p_k/q_k is an intermediate convergent of x. □

Recall that G is the Gauss map in Section 2.1.2. We use the following notations:

$$\text{(3.25)} \qquad \begin{cases} x = [0; d_1, \cdots, d_n, \alpha] & \text{if } G^n(x) = \alpha, \\ x = [\![(a_1, \varepsilon_1), (a_2, \varepsilon_2), \cdots, (a_n, \varepsilon_n), \gamma]\!] & \text{if } T_o^n(x) = \gamma. \end{cases}$$

Note that

$$\frac{1}{1+\alpha} = 1 - \frac{1}{1+\frac{1}{\alpha}}. \tag{3.26}$$

Theorem 3.3.29. *There is a conversion from the regular continued fractions into the OOCFs such that*

$$\begin{aligned} x &= [0; d_1, d_2, \alpha] \\ &= \begin{cases} [\![(2,-1)^{\frac{d_1-1}{2}}, (d_2+1, 1), F(\alpha)]\!] & \text{if } d_1 \text{ is odd and } \alpha \in [\frac{1}{2}, 1), \\ [\![(2,-1)^{\frac{d_1-1}{2}}, (d_2+2, -1), F(\alpha)]\!] & \text{if } d_1 \text{ is odd and } \alpha \in [0, \frac{1}{2}), \\ [\![(2,-1)^{\frac{d_1}{2}-1}, (1,1), G(x)]\!] & \text{if } d_1 \text{ is even.} \end{cases} \end{aligned}$$

Proof. By (3.26), we have

$$[0; 1, d_2, \alpha] = \frac{1}{1+\frac{1}{d_2+\alpha}} = 1 - \frac{1}{(d_2+1)+\alpha}.$$

Since $\alpha = \frac{1}{[\alpha^{-1}]+G(\alpha)}$, if $\alpha \in [\frac{1}{2}, 1)$, then $\alpha = \frac{1}{1+G(\alpha)}$. If $\alpha \in [0, \frac{1}{2})$, then

$$\alpha = 1 - \frac{1}{1+\frac{1}{[\alpha^{-1}]-1+G(\alpha)}}.$$

If $\alpha \in [0, \frac{1}{2})$, $F(\alpha) = \frac{1}{[\alpha^{-1}]-1+G(\alpha)}$, or otherwise, $F(\alpha) = G(\alpha)$. Thus we have

$$[0; d_1, d_2, \alpha] = \begin{cases} [\![(d_2+1, 1), F(\alpha)]\!], & \text{if } \alpha \in [\frac{1}{2}, 1), \\ [\![(d_2+2, -1), F(\alpha)]\!], & \text{if } \alpha \in [0, \frac{1}{2}). \end{cases}$$

Similarly, we have

$$x = [0; 2, G(x)] = \frac{1}{2+G(x)} = 1 - \frac{1}{1+\frac{1}{1+G(x)}} = [\![(1,1), G(x)]\!].$$

If $d_1 \geq 3$, then

$$\begin{aligned} x = [0; d_1, G(x)] &= \frac{1}{d_1+G(x)} = 1 - \frac{1}{1+\frac{1}{1+(d_1-2)+G(x)}} \\ &= 1 - \frac{1}{2-\frac{1}{1+\frac{1}{(d_1-2)+G(x)}}} = [\![(2,-1), \gamma]\!] \end{aligned}$$

where $\gamma = [0; d_1 - 2, G(x)] = R(x)$. Thus, by induction, we complete the proof. □

By Theorem 2.1.6, if P_n/Q_n is a 1-rational, then P_n/Q_n is a 1-rational best approximation. Thus P_n/Q_n is an OOCF convergent by Theorem 3.3.22. Now we check when an intermediate convergent is an OOCF convergent. Keita [35] proved the following proposition.

Proposition 3.3.30 ([35], Proposition 1.2)**.** *We have*

$$Q_{n,0} = Q_{n-2} < Q_{n-1} \leq Q_{n,1} < \cdots < Q_{n,d_n} = Q_n \text{ and}$$

$$\begin{aligned}|Q_{n,d_n}x - P_{n,d_n}| &= |Q_n x - P_n| < |Q_{n-1}x - P_{n-1}| \\ &\leq |Q_{n,d_n-1}x - P_{n,d_n-1}| < \cdots < |Q_{n,0}x - P_{n,0}| = |Q_{n-2}x - P_{n-2}|.\end{aligned}$$

By the above proposition and Theorem 3.3.28, if P_{n-1}/Q_{n-1} is a 1-rational, then $P_{n,j}/Q_{n,j}$ is not an OOCF principal convergent for any $1 \leq j < d_n$. If P_{n-1}/Q_{n-1} is an ∞-rational and $P_{n,j}/Q_{n,j}$ is a 1-rational, then $P_{n,j}/Q_{n,j}$ is an OOCF principal convergent.

Chapter 4

Combinatorics on words

One of the strategies to deal with mathematical objects is to encode the objects and to find some connections between properties of codings and the properties of the original objects. Continued fraction and coding of geodesics are good examples as we discussed in Section 2.2.2.

In this chapter, we will see some preliminaries of combinatorics on words and an important object, which is a Sturmian word, following [48] and [27].

4.1 Factor complexity

Let us consider a finite or countably infinite set $\mathcal{A}$. We call $\mathcal{A}$ *the alphabet* and the elements of $\mathcal{A}$ *the letters*. *A word* is a finite or an infinite sequence of letters. By convention, we define the empty word ε. We define the set of finite words of length n, for $n \in \mathbb{N}$, by

$$\mathcal{A}^0 = \{\varepsilon\} \text{ and } \mathcal{A}^n = \{\alpha_1\alpha_2\cdots\alpha_n \,|\, \alpha_i \in \mathcal{A} \text{ for } 1 \le i \le n\}.$$

The collection of all finite words over $\mathcal{A}$ is $\mathcal{A}^* = \bigcup_{n=0}^{\infty} \mathcal{A}^n$. We denote the set of one-sided infinite words over $\mathcal{A}$ by $\mathcal{A}^{\mathbb{N}}$ and the set of bi-infinite words over $\mathcal{A}$ by $\mathcal{A}^{\mathbb{Z}}$.

A factor (or a subword) of a word $\mathbf{u} = (u_i)$ is $u = u_iu_{i+1}\cdots u_{i+k}$ for some i, k. We call a factor starting the first letter of $\mathbf{u}$ *a prefix* and a factor ending the last letter of $\mathbf{u}$ *a suffix*. The set of factors of $\mathbf{u}$ of length n is denoted by

$F_n(\mathbf{u})$. The set of every factor of $\mathbf{u}$ is

$$F(\mathbf{u}) = \bigcup_{i=0}^{\infty} F_n(\mathbf{u}).$$

We denote by $|u|$ the length of u.

Definition 4.1.1. *Let $\mathbf{u} = (u_i)$ be a word.*

(1) A word $\mathbf{u} = (u_i)$ is purely periodic *if there exists $\ell \in \mathbb{N}$ (or $\mathbb{Z}$) such that $u_i = u_{i+\ell}$ for all i.*

(2) (a) For a one-sided infinite word, if there is n such that $u_i = u_{i+\ell}$ for all $i \geq n$, then we say that $\mathbf{u}$ is eventually periodic*.*

(b) For a bi-infinite word, if there is n such that $u_i = u_{i+\ell}$ for all $i \geq n$ and $i \leq -n$, then $\mathbf{u}$ is eventually periodic*.*

We call a word u *admissible* in $\mathbf{u}$ if u is a factor of $\mathbf{u}$. For a word $u = u_1u_2 \cdots u_{n-1}u_n$, *the reverse of* u is $\bar{u} = u_nu_{n-1} \cdots u_2u_1$. If $u = \bar{u}$, then we call u *a palindrome*. We denote the collection of the reverse of the factors by

$$\overline{F(\mathbf{u})} := \{\bar{u} : u \in F(\mathbf{u})\}.$$

Definition 4.1.2. *We say that a word $\mathbf{u}$ is* reversible *if $\overline{F(\mathbf{u})} = F(\mathbf{u})$.*

For a word over finite letters, factor complexity is a function counting the number of distinct factors of each length.

Definition 4.1.3. *Let $\mathbf{u}$ be an infinite word. The* factor complexity *of $\mathbf{u}$ is the function $p_{\mathbf{u}}$ assigning to each positive integer n, the number of distinct subwords of $\mathbf{u}$ of length n, i.e.,*

$$p_{\mathbf{u}}(n) = |F_n(\mathbf{u})|.$$

If $v \in F_n(\mathbf{u})$, then there is $va \in F_{n+1}(\mathbf{u})$ for some $a \in \mathcal{A}$. Thus, $p_{\mathbf{u}}(n) \leq p_{\mathbf{u}}(n+1)$ and thus $p_{\mathbf{u}}$ is a non-decreasing function. The factor complexity is an invariant measuring the randomness of words.

Theorem 4.1.4 ([23], Theorem 2.14 and 2.15)**.** *The following statements hold.*

(1) A one-sided infinite word is eventually periodic if and only if it has bounded factor complexity.

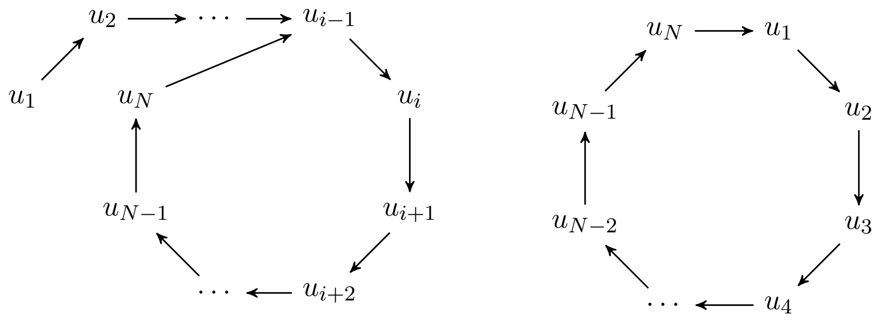


Figure 4.1: Graph composed with $F_n(\mathbf{u})$ in the proof of Theorem 4.1.4 (Rauzy graph) of a one-sided infinite word (left) and a bi-infinite word (right).

(2) A bi-infinite word is purely periodic if and only if it has bounded factor complexity.

For a word $\mathbf{u}$, the factor set $F_n(\mathbf{u}) = \{\mathbf{u}_i\}_{1 \le i \le p_{\mathbf{u}}(n)}$ of level n composes a graph whose vertices are $\mathbf{u}_i$'s. We give an edge between $\mathbf{u}_i$ and $\mathbf{u}_j$ if the length $n-1$ suffix of $\mathbf{u}_i$ is the same as the length $n-1$ prefix of $\mathbf{u}_j$. The graph is called *Rauzy gaph.* The theorem can be shown by the fact that the Rauzy graph has a cycle if $p_{\mathbf{u}}$ is bounded (see Figure 4.1). See also Proposition 1.3.13 in [48] for more detail.

Remark 4.1.5. *(1) If $p_{\mathbf{u}}(n) = p_{\mathbf{u}}(n+1)$, then $p_{\mathbf{u}}(n) = p_{\mathbf{u}}(n+k)$ for all k.*

(2) For the second assertion of the above theorem, we remark that there are many eventually periodic words with unbounded factor complexity. The word

$$\mathbf{u} = \cdots\ b\ b\ b\ a\ b\ b\ b\ \cdots$$

is one of the simplest examples which has $p_{\mathbf{u}}(n) = n+1$.

If $p_{\mathbf{u}}(n+1) > p_{\mathbf{u}}(n)$, then there is a factor of $\mathbf{u}$ of length n which is extended to two distinct factors of length $n+1$.

Definition 4.1.6. *Let $\mathbf{u}$ be an infinite word over $\mathcal{A}$. A factor u is* a right special word *(or* a left special word*, respectively) if there are distinct letters $a, b \in \mathcal{A}$ such that both ua and ub (or au and bu, respectively) are admissible.*

4.2 Sturmian words

Note that if $\mathbf{u}$ is aperiodic, i.e., non-eventually periodic, then

$$p_{\mathbf{u}}(n) \geq n+1.$$

Definition 4.2.1. A Sturmian word *is a word* $\mathbf{u}$ *with* $p_{\mathbf{u}}(n) = n+1$.

Sturmian words are composed with only two letters, say a and b. From now on, let $\mathcal{A} = \{a, b\}$. We will see that Sturmian words are related to a dynamical system and the continued fraction.

Example 4.2.2. *Let us define a sequence of finite words* $\mathbf{f}_n$ *by* $\mathbf{f}_n = \mathbf{f}_{n-1}\mathbf{f}_{n-2}$ *where* $\mathbf{f}_{-1} = b$ *and* $\mathbf{f}_0 = a$. *Let* $\mathbf{f} = \lim_{n\to\infty} \mathbf{f}_n$. *Then, we have*

$$\mathbf{f} = abaababaabaababaababaabaababaabaab\cdots.$$

We call $\mathbf{f}$ Fibonacci word. *The Fibonacci word is Sturmian.*

The first characterization of Sturmian words is that the factors of the same length contain a similar number of b. We define *the height* $h(u)$ of a finite word u by the number of b in u.

Definition 4.2.3. *We call* $\mathbf{u} \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}} \cup \mathcal{A}^{\mathbb{Z}}$ a balanced word *if*

$$|h(u) - h(v)| \leq 1$$

for all $u, v \in F(\mathbf{u})$ *such that* $|u| = |v|$.

Recall that a word is reversible if the reverses of all factors are also its factora (see Definition 4.1.2).

Proposition 4.2.4 ([51]; see also Theorem 2.1.19. in [48])**.** *Sturmian words are reversible.*

We define *the slope* $\pi(u)$ of a finite word u by

$$\pi(u) = \frac{h(u)}{|u|}.$$

The slope of an infinite word $\mathbf{u} = (u_i)_{i\in\mathbb{N}}$ is defined by the limit of the slopes of the prefixes such that

$$\pi(\mathbf{u}) = \lim_{i\to\infty} \pi(u_1 \cdots u_{i-1}u_i).$$

Lemma 4.2.5 ([48], Proposition 2.1.11)**.** *Let* $\mathbf{u}$ *be an infinite balanced word. Then,* $\mathbf{u}$ *is eventually periodic if and only if* $\pi(\mathbf{u})$ *is rational.*

Another characterization of Sturmian words is in terms of the coding of irrational rotations.

Definition 4.2.6. *For an irrational* $\theta \in [0,1]$ *and a real number* c*, let*

$$s_n = \begin{cases} a \ \text{ if } \lfloor \theta(n+1)+c \rfloor - \lfloor \theta n + c \rfloor = 0, \\ b \ \text{ if } \lfloor \theta(n+1)+c \rfloor - \lfloor \theta n + c \rfloor = 1, \end{cases}$$

and let

$$s'_n = \begin{cases} a \ \text{ if } \lceil \theta(n+1)+c \rceil - \lceil \theta n + c \rceil = 0, \\ b \ \text{ if } \lceil \theta(n+1)+c \rceil - \lceil \theta n + c \rceil = 1, \end{cases}$$

We call $\mathbf{s}_{\theta,c} = (s_n)$ a lower mechanical word*. We call* $\mathbf{s}_{\theta,c} = (s'_n)$ an upper mechanical words*.*

Morse and Hedlund characterized Sturmian words [53].

Proposition 4.2.7 ([53]; see also Theorem 2.1.13. in [48])**.** *For an infinite word* $\mathbf{u}$*, the following statements are equivalent:*

(1) $\mathbf{u}$ *is Sturmian,*

(2) $\mathbf{u}$ *is balanced and aperiodic, and*

(3) $\mathbf{u}$ *is irrational mechanical.*

Irrational mechanical words are generated by irrational rotations. Let $\mathbb{S}^1$ be a circle identified with $[0,1]/_{0\sim 1}$. Let $\mathcal{R}_\theta : \mathbb{S}^1 \to \mathbb{S}^1$ be a rotation defined by

$$x \mapsto x + \theta \pmod 1.$$

The following proposition tells us that a Sturmian word is a coding of an orbit of a point on $\mathbb{S}^1$ under $\mathcal{R}_\theta$.

Proposition 4.2.8 ([48], p.56)**.** *If* $\mathbf{s}_{\theta,c} = (s_n)$ *and* $\mathbf{s}'_{\theta,c} = (s'_n)$*, then*

$$s_n = \begin{cases} a, & \text{if } \mathcal{R}_\theta^n(c) \in [0, 1-\theta), \\ b, & \text{if } \mathcal{R}_\theta^n(c) \in [1-\theta, 1), \end{cases} \quad \text{and} \quad s'_n = \begin{cases} a, & \text{if } \mathcal{R}_\theta^n(c) \in (0, 1-\theta], \\ b, & \text{if } \mathcal{R}_\theta^n(c) \in (1-\theta, 1]. \end{cases}$$

Now, we introduce an algorithm to attain a class of Sturmian words.

Definition 4.2.9. A characteristic word $\mathbf{c}_\theta$ *with a slope* θ *is defined by*

$$\mathbf{c}_\theta := \mathbf{s}_{\theta,\theta} = \mathbf{s}'_{\theta,\theta}.$$

A characteristic word is a representation of Sturmian words with the same slope because the factor sets of two Sturmian words are the same if they have the common slope.

Proposition 4.2.10 ([48], Proposition 2.1.18)**.** *Let* $\mathbf{s}$ *and* $\mathbf{t}$ *be Sturmian words with the same slope* θ*. Then* $F(\mathbf{s}) = F(\mathbf{t})$*.*

Proposition 4.2.11 ([29]; see also Proposition 2.2.24. in [48])**.** *Let* $\theta = [0; 1 + d_1, d_2, \cdots, d_i, \cdots]$*. We define a sequence of words* M_n *by*

$$M_n = M_{n-1}^{d_n} M_{n-2}$$

where $M_{-1} = b$ *and* $M_0 = a$*. Then*

$$\mathbf{c}_\theta = \lim_{n \to \infty} M_n.$$

Definition 4.2.12. *We define two functions* Γ *and* Δ *from* $\mathcal{A}^* \times \mathcal{A}^*$ *to itself by*

$$\Gamma(u, v) = (u, uv) \text{ and } \Delta(u, v) = (vu, v).$$

Starting with (a, b)*, we can generate pairs of finite words by iterating* Γ *and* Δ *such that*

$$(u, v) = \Gamma^{n_1} \circ \Delta^{n_2} \circ \Gamma^{n_3} \circ \cdots \circ \Delta^{n_i}(a, b)$$

where each n_j*, for* $1 \leq j \leq i$*, is a positive integer. We call the pairs* the standard pairs*. We call any component of a standard pair* a standard word.

We can see that each M_n in Proposition 4.2.11 is a standard word (see Section 2.2.1 in [48] for more details).

Chapter 5

Lévy constants of Sturmian continued fraction expansions

Recall that $P_n(x)/Q_n(x)$, for $n \in \mathbb{N}$, is a regular continued fraction principal convergent of x as in Definition 2.1.1. Paul Lévy showed that

$$\lim_{n\to\infty} \frac{\log Q_n(x)}{n} = \frac{\pi^2}{12\log 2}$$

for almost every x [47]. The above limit value is called *the Lévy constant of* x.

In this section, we give some historical remarks of the Lévy constants in Section 5.1. In Section 5.2, we show our main result of Lévy constants of real numbers whose continued fraction expansions are Sturmian words.

5.1 History

An important property of continued fraction is that the convergents of the regular continued fraction give the best Diophantine approximations of a real number. We define a norm of $t \in \mathbb{R}$ by $||t|| := \inf\{|t-n| : n \in \mathbb{Z}\}$, which is the distance between 0 and t (mod 1) on the unit circle $\mathbb{S}^1$ indentified with the unit interval. From Definition 2.1.5 and Theorem 2.1.6, we recall that

$$||Q_n x|| < ||bx|| \text{ for any } 0 < q \le b.$$

Let us choose a non-increasing function $\psi : \mathbb{N} \to \mathbb{R}_{>0}$. We say that x is ψ-*approximable* if $||qx|| < \psi(q)$ for infinitely many positive integers q. Dirichlet showed that every irrational number is $1/q$-approximable. Khintchine showed that a stronger result hold.

Theorem 5.1.1 ([37]). *The following statements hold.*

(1) If $\sum_{n\in\mathbb{N}} \psi(n)$ *diverges, then* x *is* ψ*-approximable for almost every* $x \in \mathbb{R}$*.*

(2) If $\sum_{n\in\mathbb{N}} \psi(n)$ *converges, then* x *is not* ψ*-approximable for almost every* $x \in \mathbb{R}$*.*

This theorem is called Khintchine theorem or Khintchine-Groshev Theorem since Groshev proved a higher dimensional version of this theorem. In order to show the theorem, Khintchine proved that there exists a constant C such that, for almost every x,

$$\frac{\log Q_n(x)}{n} < C,$$

asymptotically. Later, he showed there exists γ such that

$$\lim_{n\to\infty} \frac{\log Q_n(x)}{n} = \gamma \quad \text{for almost every } x \in \mathbb{R} \text{ [38].}$$

We denote by

$$\mathcal{L}(x) := \lim_{n\to\infty} \frac{\log Q_n(x)}{n}.$$

Lévy found that

$$\mathcal{L}(x) = \frac{\pi^2}{12\log 2} \quad \text{for almost every } x \in \mathbb{R} \text{ [47].}$$

Thus, $\mathcal{L}(x)$ is called *Lévy constant* or *Khintchine-Lévy constant* of x. We can see the expression using Birkhoff ergodic theorem. Moreover, this theorem implies that, for almost every $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{P_n(x)}{Q_n(x)} \right| = -\frac{\pi^2}{6\log 2}.$$

By using the Euler-Lagrange theorem (see Theorem 2.1.3 and 2.1.4), Jager-Liardet found formulas of the Lévy constants for all quadratic irrationals [34].

Theorem 5.1.2 ([34], [11]). *If x is quadratic irrational whose continued fraction expansion is $[a_0; a_1, \cdots, a_i, \overline{a_{i+1}, \cdots, a_{i+n}}]$. Then,*

$$\mathcal{L}(x) = \frac{1}{n} \log \frac{t + \sqrt{t^2 - (-1)^n 4}}{2}.$$

where t is the trace of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & a_{i+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{i+2} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_{i+n} \end{pmatrix}. \tag{5.1}$$

We note that $\frac{t+\sqrt{t^2-(-1)^n 4}}{2}$ is the spectral radius of the matrix in (5.1), i.e., the largest eigenvalue of the matrix. Belova and Hazard showed the same formula by using generating functions whose coefficients are P_n and Q_n [11].

Queffelec showed the existence of Lévy constant of a real number whose continued fraction is a fixed point of a primitive morphism [55].

Let $\mathbb{L}$ be the set of Lévy constants. The minimum of $\mathbb{L}$ is

$$\mathcal{L}([0; \overline{1}]) = \log \frac{1 + \sqrt{5}}{2}.$$

Faivre showed that, for all $\beta \geq \log \frac{1+\sqrt{5}}{2}$, there exists $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $\mathcal{L}(x) = \beta$, i.e., $\mathbb{L} = \left[\log \frac{1+\sqrt{5}}{2}, \infty\right)$ [26]. Let

$$\mathbb{B} := \{\mathcal{L}(x) : x \text{ is a quadratic irrational}\}.$$

Golubeva showed that $\frac{\pi^2}{12 \log 2}$ is a limit point of $\mathbb{B}$ [30]. Jun Wu proved that $\mathbb{B}$ is dense in $\left[\log \frac{1+\sqrt{5}}{2}, \infty\right)$ [66]. Baxa proved the same theorem by showing a slightly stronger statement [10]: the following set

$$\{\mathcal{L}(x) : x \text{ is a quadratic irrational with partial quotients in } \{a, b\}\}$$

is dense in $\left[\mathcal{L}([0; \overline{a}]), \mathcal{L}([0; \overline{b}])\right] = \left[\log \frac{a+\sqrt{a^2+4}}{2}, \log \frac{b+\sqrt{b^2+4}}{2}\right]$ for two distinct integers a, b such that $a < b$.

5.2 Lévy constants of Sturmian continued fraction

By Euler-Lagrange theorem (Theorem 2.1.4), we know that a quadratic irrational has an eventually periodic continued fraction. Coven-Hedlund theorem (Theorem 4.1.4) tells us that an eventually periodic continued fraction has a bounded factor complexity. Combining above areguments, we can say that Jager-Liardet theorem (Theorem 5.1.2) means that $[0; w_1, w_2, \cdots]$ has a Lévy constant if $\mathbf{w} = w_1 w_2 \cdots$ has bounded factor complexity.

Let $\mathbf{w} = w_1 w_2 \cdots$ be a word. A natural question arises: how slow should grow the sequence $(p_{\mathbf{w}}(n))_{n\geq 1}$ to ensure that $[0; w_1, w_2, \cdots]$ has a Lévy constant? In particular, for a Sturmian word $\mathbf{w}$, does the real number $[0; w_1, w_2, \cdots]$ have a Lévy constant? The following result answers positively the second question.

Theorem 5.2.1. *Let $\mathbf{w} = w_1 w_2 \cdots$ be an infinite word over the positive integers. If there exists an integer k such that*

$$p_{\mathbf{w}}(n) \leq n + k, \quad \text{for } n \geq 1,$$

then the real number $[0; w_1, w_2, \cdots]$ has a Lévy constant.

Secondly, we show the following refinement of Faivre's result. A Sturmian (resp., mechanical) continued fraction is a continued fraction whose sequence of partial quotients is a Sturmian (resp., mechanical) sequence. Recall that any mechanical continued fraction is either Sturmian, or represents a quadratic number.

Theorem 5.2.2. *Let a, b be integers with $1 \leq a < b$. The set of Lévy constants of mechanical continued fractions with intercept 0 and written over the alphabet $\{a, b\}$ is equal to the whole interval $[\mathcal{L}([0; \overline{a}]), \mathcal{L}([0; \overline{b}])]$.*

5.2.1 Existence: Proof of Theorem 5.2.1.

Let $\mathbf{w}$ be an infinite word. Recall the definition of the characteristic words $\mathbf{c}_\theta$ in Definition 4.2.9.

Notation 5.2.3. *We use the following notations.*

(1) *The words are written over the alphabet $\{a, b\}$, where a, b are distinct integers.*

(2) For $\mathbf{w} = w_1 w_2 \cdots$ *with* w_j *in* $\mathbb{N}$ *for* $j \geq 1$*, we set*

$$x = [0; \mathbf{w}] = [0; w_1, w_2, \cdots, w_j, \cdots].$$

(3) We denote by $x = [0; \mathbf{s}]$ *the real number whose sequence of partial quotients is given by the Sturmian word* $\mathbf{s} = s_1 s_2 \cdots$*. The slope of the Sturmian word* $\mathbf{s}$ *is the irrational real number* $\theta = [0; 1 + d_1, d_2, \cdots]$*.*

We denote the principal convergents of $x = [0; s_1, s_2, \cdots]$ *by* P_n/Q_n *and the principal convergents of* $\theta = [0; 1 + d_1, d_2, \cdots]$ *by* $\mathfrak{p}_n/\mathfrak{q}_n$*.*

(4) For a finite word $M = b_1 b_2 \cdots b_n$*, we denote by* M^- *its prefix* $b_1 \cdots b_{n-1}$ *of length* $n-1$*.*

Let $K(a_1, \cdots, a_n)$ be the denominator of the rational number $[0; a_1, a_2, \cdots, a_n]$. Then we have

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} K(a_1, \cdots, a_n) & K(a_1, \cdots, a_{n-1}) \\ K(a_2, \cdots, a_n) & K(a_2, \cdots, a_{n-1}) \end{pmatrix}.$$

Therefore, we have

$$\begin{aligned} &K(a_1, \cdots, a_{n+m}) \\ &= K(a_1, \cdots, a_n) K(a_{n+1}, \cdots, a_{n+m}) + K(a_1, \cdots, a_{n-1}) K(a_{n+2}, \cdots, a_{n+m}). \end{aligned}$$

Thus,

$$\begin{aligned} K(a_1, \cdots, a_n) K(a_{n+1}, \cdots, a_{n+m}) &< K(a_1, \cdots, a_{n+m}) \\ &\leq 2K(a_1, \cdots, a_n) K(a_{n+1}, \cdots, a_{n+m}). \end{aligned}$$

Let us write $K(M) = K(b_1, b_2, \cdots, b_n)$ for a word $M = b_1 b_2 \cdots b_n$.

Remark 5.2.4. *We claim that, for* $k \geq 2$*,*

$$M_k M_{k-1}^{--} = M_{k-1} M_k^{--}.$$

For $k = 1$*,*

$$M_1 M_0^{--} = a^{d_1} = M_0 M_1^{--}$$

(a^0 *is the empty word* ε*). By the induction hypothesis,*

$$M_k M_{k-1}^{--} = M_{k-1}^{d_k} M_{k-2} M_{k-1}^{--} = M_{k-1}^{d_k} M_{k-1} M_{k-2}^{--} = M_{k-1} M_k^{--}.$$

Let $M \in F_{\mathfrak{q}_k-1}(\mathbf{c}_\theta)$. By Proposition 4.2.11, the characteristic word $\mathbf{c}_\theta$ can be written as the concatenation of words M_k and M_{k-1}, without two consecutive copies of M_{k-1}. Combined with the claim, we have the fact that

(5.2) *any factor M of length $\mathfrak{q}_k - 1$ of $\mathbf{s}$ is a factor of $M_k(M_k)^{--}$.*

Lemma 5.2.5. *Let M be a factor of $\mathbf{s}$ of length n with $\mathfrak{q}_k \le n \le \mathfrak{q}_{k+1} - 1$. Then,*

(a) M is a factor of $M_k M_k \cdots M_k M_{k-1}$, or

(b) $M = UV$, where U is a suffix of M_{k+1} and V is a prefix of M_{k+1} with $|V| \ge \mathfrak{q}_k - 1$.

Proof. By (5.2), M is a factor of M_{k+1} or $M = UV$ where U is a suffix of M_{k+1} and V is a prefix of M_{k+1}. If $|V| \le \mathfrak{q}_k - 2$, then V is a prefix of M_k^{--}. Thus, $M = UV$ is a factor of $M_{k+1}{M_k}^{--} = M_k {M_{k+1}}^{--}$ and also a factor of $M_k M_k \cdots M_k M_{k-1}$. □

We recall that $F(\mathbf{c}_\theta) = F(\mathbf{s}_{\theta,\rho}) = F(\mathbf{s}'_{\theta,\rho})$ for any intercept ρ from Proposition 4.2.10.

Proposition 5.2.6. *Let $\mathbf{s}$ be a Sturmian word of slope $\theta = [0; 1 + d_1, d_2, \cdots]$, where $d_1 \ge 1$. Let k be a nonnegative integer. Let n be an integer with $\mathfrak{q}_k \le n \le \mathfrak{q}_{k+1} - 1$. Let $c = \max\{b/a, a/b\}$. For any factor M, M' of $\mathbf{s}$ of length n, we have*

$$K(M) \le 2^{2k} c K(M').$$

Proof. Suppose first that $k = 0$. Recall that $\mathfrak{q}_0 = 1$ and $\mathfrak{q}_1 = d_1 + 1$. Let n be an integer with $\mathfrak{q}_0 \le n \le \mathfrak{q}_1 - 1$. A factor of $\mathbf{s}$ of length n is a factor of $M_1 M_1^{--} = a^{d_1} b a^{d_1 - 1}$. Then any factor of $\mathbf{s}$ of length n is either a^n, or $a^{n_1} b a^{n_2}$ with $n_1 + n_2 = n - 1$.

Assume that $a < b$. Clearly, $K(a^n) < K(a^{n_1} b a^{n_2})$. Since

$$\begin{aligned} aK(a^{n_1} b) &= a(bK(a^{n_1}) + K(a^{n_1 - 1})) \\ &\le b(aK(a^{n_1}) + K(a^{n_1-1})) = bK(a^{n_1+1}), \end{aligned}$$

it is easy to check that,

$$aK(a^{n_1} b a^{n_2}) \le bK(a^n).$$

Thus, $aK(a^{n_1}ba^{n_2}) \le bK(a^n) \le bK(a^{n'_1}ba^{n'_2})$ with $n'_1 + n'_2 = n-1$. Similarly, for the case of $a > b$, we check that $bK(a^{n_1}ba^{n_2}) \le bK(a^n) \le aK(a^{n'_1}ba^{n'_2})$. Hence, the proposition holds for every positive integer n at most equal to $\mathfrak{q}_1 - 1$.

We argue by induction. Let k be a positive integer and suppose that the proposition holds for every positive integer n at most equal to $\mathfrak{q}_k - 1$. Let n be an integer with $\mathfrak{q}_k \le n \le \mathfrak{q}_{k+1} - 1$. Let M, M' be two factors of $\mathbf{s}$ of length n. By Lemma 5.2.5, we distinguish the following cases:
Case (i): Both of M, M' are factors of $M_k M_k \cdots M_k M_{k-1}$.

Since $M_k M_k \cdots M_k M_{k-1}$ is a periodic word with period $\mathfrak{q}_k$, there exist factors N, N' of $\mathbf{s}$ such that $NM = M'N'$ and $|N| = |N'| < \mathfrak{q}_k$. Therefore,

$$\begin{aligned} K(M) = \frac{K(N)K(M)}{K(N)} &\le \frac{K(NM)}{K(N)} \\ &\le c2^{2(k-1)}\frac{K(M'N')}{K(N')} \\ &\le c2^{2k-1}\frac{K(M')K(N')}{K(N')} \le c2^{2k-1}K(M'). \end{aligned}$$

Case (ii): Let $M = UV$ and $M' = U'V'$ where U, U' are (possibly empty) suffixes of M_{k+1} and V, V' are (possibly empty) prefixes of M_{k+1}. We may assume that $|U| < |U'|$. Define the words N, N' by $U' = NU$ and $V = V'N'$. It follows from Case (i) that

$$K(N') \le c \cdot 2 \cdot 4^{k-1} K(N).$$

Therefore,

$$\begin{aligned} K(M) = \frac{K(N)K(UV)}{K(N)} \le \frac{K(NUV)}{K(N)} &\le c2^{2k-1}\frac{K(U'V'N')}{K(N')} \\ &\le c2^{2k-1}\frac{2K(U'V')K(N')}{K(N')} = c2^{2k}K(M'). \end{aligned}$$

Case (iii): Assume that M is a factor of $M_k \cdots M_k M_{k-1}$ and $M' = UV$, where U is a suffix of M_{k+1} and V is a prefix of M_{k+1} with $|V| \ge \mathfrak{q}_k - 1$. Write $M = NM_k \cdots M_k N'$ where N is a nonempty prefix of M_k and N' is a possibly empty suffix of M_k.

(1) If $|N'| \ge \mathfrak{q}_{k-1}$, then $N' = M_{k-1}N''$ is a prefix of $(M_k M_{k-1})^{--} =$

$(M_{k-1}M_k)^{--}$. Hence,

$$M = NM_k \cdots M_k N' = NM_k \cdots M_k M_{k-1} N'',$$

where $NM_k \cdots M_k M_{k-1}$ is a suffix of M_{k+1} and N'' is a prefix of M_k, which is a prefix of M_{k+1}. We apply the argument of Case (ii).

(2) If $|N'| \leq \mathfrak{q}_{k-1} - 2$, then $M_k N'$ is a prefix of $(M_k M_{k-1})^{--} = (M_{k-1}M_k)^{--}$. Define N'' by

$$M_k N' = M_{k-1} N''.$$

Since $n \geq \mathfrak{q}_k$, we get that N'' is a suffix of M. We write $M = WN''$, where W is a suffix of $M_k \cdots M_k M_{k-1}$. Since N'' is a prefix of M_k, which is a prefix of M_{k+1}, we apply the argument of Case (ii).

(3) Suppose that $|N| \geq 2$. Put $M_k = V'N$. Then V' is a prefix of V. Let V'', W be the words defined by $V = V'V''$ and $M = V''W$. Since UV' is a factor of $M_{k+1}M_k^{--} = M_k M_{k+1}^{--}$ and $M_k M_{k+1}^{--}$ is periodic with period $\mathfrak{q}_k$, we have

$$K(W) \leq c2^{2k-1} K(UV'),$$

in a similar way as in Case (i). Thus,

$$\begin{aligned} K(M) \leq 2K(V'')K(W) &\leq 2K(V'') \cdot c2^{2k-1} K(UV') \\ &\leq c2^{2k} K(UV'V'') = c2^{2k} K(M'). \end{aligned}$$

(4) The remaining case is the case where $|N| = 1$ and $|N'| = \mathfrak{q}_{k-1} - 1$. Then, for some $d \geq 0$,

$$M = \begin{cases} a(M_k)^{d+1}(M_{k-1})^{-}, & \text{if } k \text{ is even}, \\ b(M_k)^{d+1}(M_{k-1})^{-}, & \text{if } k \text{ is odd}, \end{cases}$$

since M_k is ending with a if k is even and with b otherwise. Note that $|M'| = |M| = (d+1)\mathfrak{q}_k + \mathfrak{q}_{k-1}$. Since U is a suffix of $(M_k)^{d_{k+1}} M_{k-1}$ and V is a prefix of $(M_k)^{d_{k+1}}(M_{k-1})^{--}$, we have

$$VU = (M_k)^{d+1} M_{k-1}.$$

If k is even, then

$$\begin{aligned} K(M) &\le 4K(a)K(V)K(U^-) \\ &\le 4cK(b)K(V)K(U^-) \le 4cK(U^-bV) = 4cK(M'). \end{aligned}$$

The case of odd k is symmetric. □

Before concluding the proof, we recall Fekete's lemma.

Lemma 5.2.7 (Fekete)**.** *If a sequence $(a_n)_{n\ge1}$ of positive real numbers is subadditive, that is, if it satisfies $a_{n+m} \le a_n + a_m$ for any integers n and m, then the sequence $(a_n/n)_{n\ge1}$ converges and*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\ge1} \frac{a_n}{n}.$$

We have now all the material to establish the following theorem.

Theorem 5.2.8. *Let $x = [0; s_1, s_2, \cdots]$ be a Sturmian continued fraction. Then, x has a Lévy constant $\mathcal{L}(x)$.*

Proof. We apply Proposition 5.2.6. Let $\theta = [0; 1 + d_1, d_2, \cdots]$, where $d_1 \ge 1$, denote the slope of x. For $k \ge 1$, let $\mathfrak{q}_k$ denote the denominator of θ. Let k be a nonnegative integer. Let n be an integer with $\mathfrak{q}_k \le n \le \mathfrak{q}_{k+1} - 1$. Let M, M' be factors of $\mathbf{s}$ of length n. Since $\mathfrak{q}_k \ge 2^{k/2-1}$, we have

$$K(M) \le c4^{k-1}K(M') \le 4c(\mathfrak{q}_k)^4K(M') \le 4cn^4K(M'). \tag{5.3}$$

Set $A_n = 2^7cn^4K(s_1, s_2, \cdots, s_n)$. Then for $m \le n$ we have

$$\begin{aligned} A_{n+m} &= 2^7c(n+m)^4K(s_1, s_2, \cdots, s_{n+m}) \\ &\le 2^7c(2n)^4 2K(s_1, s_2, \cdots, s_n)K(s_{n+1}, s_{n+2}, \cdots, s_{n+m}) \\ &\le 2^7cn^4K(s_1, s_2, \cdots, s_n)2^7cm^4K(s_1, s_2, \cdots, s_m) = A_nA_m. \end{aligned}$$

By Fekete's lemma, the following limits exist and are equal

$$\lim_{n\to\infty} \frac{1}{n} \log A_n = \lim_{n\to\infty} \frac{1}{n} \log\left(2^7cn^4Q_n(x)\right) = \lim_{n\to\infty} \frac{1}{n} \log Q_n(x).$$

This proves that x has a Lévy constant, which, by (5.3) and the fact that two Sturmian words with the same slope have the same set of factors, does not depend on the intercept. □

Completion of the proof of Theorem 5.2.1. Let $\mathbf{w} = w_1 w_2 \cdots$ be an infinite word defined over the positive integers such that the sequence $(p_{\mathbf{w}}(n) - n)_{n \geq 1}$ is bounded and $\mathbf{w}$ is not ultimately periodic. Since the function $n \mapsto p_{\mathbf{w}}(n)$ is non-decreasing, the sequence $(p_{\mathbf{w}}(n) - n)_{n \geq 1}$ of positive integers is eventually constant. Thus, there exist positive integers k and n_0 such that

$$p_{\mathbf{y}}(n) = n + k, \qquad \text{for } n \geq n_0. \tag{5.4}$$

Infinite words satisfying (5.4) are called *quasi-Sturmian words.* It follows from a result of Cassaigne [17, Proposition 8] that there are a finite word W, a Sturmian word $\mathbf{s}$ defined over $\{a, b\}$ and a morphism φ from $\{a, b\}^*$ into the set of positive integers such that $\varphi(ab) \neq \varphi(ba)$ and

$$\mathbf{w} = W\varphi(\mathbf{s}).$$

We briefly explain that Proposition 5.2.6 can be suitably extended to the word $\mathbf{w}$.

Put $c_\varphi = \max\{K(\varphi(a))/K(\varphi(b)), K(\varphi(b))/K(\varphi(a))\}$. For any nonnegative integers n_1, n_2, n_1', n_2' with $n_1 + n_2 = n_1' + n_2' = n - 1$, we have

$$\begin{aligned} K(\varphi(a^{n_1} b a^{n_2})) &\leq 4K(\varphi(a^{n_1}))K(\varphi(b))K(\varphi(a^{n_2})) \\ &\leq 4c_\varphi K(\varphi(a^n)) \leq 4^2 c_\varphi K(\varphi(a^{n_1'} b a^{n_2'})) \end{aligned}$$

or

$$K(\varphi(a^{n_1} b a^{n_2})) \leq 4K(\varphi(a^n)) \leq 4^2 c_\varphi K(\varphi(a^{n_1'} b a^{n_2'})),$$

depending on the fact that $K(\varphi(a)) \leq K(\varphi(b))$ or $K(\varphi(b)) \leq K(\varphi(a))$. Therefore, by replacing $K(M), K(M')$ with $K(\varphi(M))$ and $K(\varphi(M'))$ in the proof of Proposition 5.2.6, we conclude that, for any factors N, N' of $\mathbf{s}$ with $\mathfrak{q}_k \leq |N| = |N'| \leq \mathfrak{q}_{k+1} - 1$, we get

$$K(\varphi(N)) \leq 4^{k+2} c_\varphi K(\varphi(N')).$$

Set $h = \max\{|\varphi(a)|, |\varphi(b)|\}$. Let M, M' be factors of the same length of $\varphi(\mathbf{s})$. Let L (resp., N) be the word of minimal (resp., maximal) length such that M is a factor of $\varphi(L)$ (resp., $\varphi(N)$ is a factor of M'). Since $\mathbf{s}$ is a balanced word (see Proposition 4.2.7), we have $|L| - |N| \leq 6$. Setting $\tilde{c} = \max\{K(M) \,|\, M =$

$\varphi(N)$ for $|N| = 6\}$, we get

$$K(M) \le K(\varphi(L)) \le 2\tilde{c} \cdot 4^{k+2} c_{\varphi} K(\varphi(N)) \le 2\tilde{c} \cdot 4^{k+2} c_{\varphi} K(M'),$$

and we conclude as in the proof of Theorem 5.2.8. We observe that the Lévy constant of $[0; w_1, w_2, \cdots]$ depends only on the slope of the Sturmian word $\mathbf{s}$. □

We end this section with an example that the real number has no Lévy constant.

Example 5.2.9. *We remark that there is a real number that does not have Lévy constant. Let us consider a word* $\mathbf{w} = abaabbbbaa\cdots$ *whose* $(2^n + 1)th$ *up to* $(2^{n+1})th$ *letters are* b *if* n *is even and* a *if* n *is odd. Let* $x = [0; \mathbf{w}]$*. By the definition of* $\mathbf{w}$*, we have*

$$\begin{cases} Q_{2^m}(x) K(a^{2m}) \le Q_{2^{m+1}}(x) \le 2 Q_{2^m}(x) K(a^{2m}), & \text{if } m \text{ is odd,} \\ Q_{2^m}(x) K(b^{2m}) \le Q_{2^{m+1}}(x) \le 2 Q_{2^m}(x) K(b^{2m}), & \text{if } m \text{ is even.} \end{cases}$$

If $\mathcal{L}(x)$ *exists, then* $2\mathcal{L} = \mathcal{L} + \mathcal{L}([0; \overline{a}]) = \mathcal{L} + \mathcal{L}([0; \overline{b}])$*. It contradicts to* $a \neq b$*. Then* $x = [0; \mathbf{w}]$ *has no Lévy constant.*

The real numbers $x = [0; \mathbf{w}]$ *defined above show that we cannot hope for a much better result than Theorem 5.2.1. Indeed, it is easy to see that the factor complexity of the infinite word* $\mathbf{w}$ *formed by the concatenation of its partial quotients satisfies* $2n \le p_{\mathbf{w}}(n) \le 3n$*, for* $n \ge 1$*.*

Note that the set of real numbers which do not have a Lévy constant has full Hausdorff dimension [54, Theorem 3]; see also [8].

5.2.2 Spectrum: Proof of Theorem 5.2.2.

In this section, we will show that the set of Lévy constants of Sturmian continued fractions and quadratic irrationals is $\left[\log \frac{1+\sqrt{5}}{2}, \infty\right)$. From now on, the words are written over the alphabet $\{a, b\}$, where a, b are integers with $1 \le a < b$. For brevity, we define some notations about the trace of matrices.

Notation 5.2.10. *(1) For positive integers $a_1, \cdots, a_n$, let*

$$\begin{aligned} T(a_1, \cdots, a_n) &= \operatorname{Tr}\left(\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}\right) \\ &= \operatorname{Tr}\left(\begin{pmatrix} K(a_1, \cdots, a_n) & K(a_1, \cdots, a_{n-1}) \\ K(a_2, \cdots, a_n) & K(a_2, \cdots, a_{n-1}) \end{pmatrix}\right) \\ &= K(a_1, \cdots, a_n) + K(a_2, \cdots, a_{n-1}). \end{aligned}$$

(2) We define polynomials $\mathbf{T}_n(x)$ for $n \geq 1$ by

$$\mathbf{T}_n(x) = \operatorname{Tr}(X^n) = T(x^n)$$

where

$$X = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe that

$$\mathbf{T}_1(x) = x, \quad \mathbf{T}_2(x) = x^2 + 2, \quad \mathbf{T}_3(x) = x^3 + 3x, \cdots$$

Since all coefficients of $\mathbf{T}_n(x)$ is positive, $\mathbf{T}_n'(x) > 0$ for all positive x. If n is even, then $T(a_1, \cdots, a_n) > 2$ and $\mathbf{T}_n(0) = 2$. If n is odd, then $T(a_1, \cdots, a_n) > 0$ and $\mathbf{T}_n(0) = 0$. Thus, there is a unique positive μ such that $\mathbf{T}_n(\mu) = T(a_1, \cdots, a_n)$. This real number μ can be seen as being a mean of $a_1, \cdots, a_n$.

Proposition 5.2.11. *If α is the quadratic irrational whose continued fraction expansion is given by $[a_0; a_1, \cdots, a_r, \overline{a_{r+1}, \cdots, a_{r+s}}]$, then*

$$\mathcal{L}(\alpha) = \log \frac{\mu + \sqrt{\mu^2 + 4}}{2},$$

where μ is the positive real number such that

$$\mathbf{T}_s(\mu) = T(a_{r+1}, \cdots, a_{r+s}).$$

Proof. For a nonnegative integer n, define the polynomials $\mathbf{A}_n$ and $\mathbf{B}_n$ by

$$\left(\frac{x + \sqrt{x^2 + 4}}{2}\right)^n = \frac{\mathbf{A}_n(x) + \mathbf{B}_n(x)\sqrt{x^2 + 4}}{2} \tag{5.5}$$

where $\mathbf{A}_0 = 2$, $\mathbf{A}_1 = x$, $\mathbf{B}_0 = 0$, $\mathbf{B}_1 = 1$ and $\mathbf{B}_2 = x$. We have

$$\begin{pmatrix} \mathbf{A}_{n+1}(x) \\ \mathbf{B}_{n+1}(x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x & x^2+4 \\ 1 & x \end{pmatrix} \begin{pmatrix} \mathbf{A}_n(x) \\ \mathbf{B}_n(x) \end{pmatrix}.$$

Then

$$\begin{pmatrix} \mathbf{A}_{n-1}(x) \\ \mathbf{B}_{n-1}(x) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -x & x^2+4 \\ 1 & -x \end{pmatrix} \begin{pmatrix} \mathbf{A}_n(x) \\ \mathbf{B}_n(x) \end{pmatrix}.$$

Since

$$\mathbf{A}_{n+1}(x) - \frac{x}{2}\mathbf{A}_n(x) = \frac{x^2+4}{2}\mathbf{B}_n(x) = \frac{x}{2}\mathbf{A}_n(x) + \mathbf{A}_{n-1}(x)$$

and

$$\mathbf{B}_{n+1}(x) - \frac{x}{2}\mathbf{B}_n(x) = \frac{1}{2}\mathbf{A}_n(x) = \frac{x}{2}\mathbf{B}_n(x) + \mathbf{B}_{n-1}(x),$$

we have the following recurrence forms

$$\mathbf{A}_{n+1}(x) = x\mathbf{A}_n(x) + \mathbf{A}_{n-1}(x), \quad \mathbf{B}_{n+1}(x) = x\mathbf{B}_n(x) + \mathbf{B}_{n-1}(x).$$

Since $\mathbf{A}_0(x) = \mathbf{T}_0(x)$ and $\mathbf{A}_1(x) = \mathbf{T}_1(x)$, we have $\mathbf{A}_n(x) = \mathbf{T}_n(x)$.

Recall that the norm of a quadratic irrational is equal to the product of itself by its Galois conjugate. The norm of $(\mathbf{A}_n(x) + \mathbf{B}_n(x)\sqrt{x^2+4})/2$ is $(\mathbf{A}_n(x)^2 - \mathbf{B}_n(x)^2(x^2+4))/4$ and the norm of $x + \sqrt{x^2+4}/2$ is -1. By (5.5),

$$\mathbf{B}_n(x)\sqrt{x^2+4} = \sqrt{\mathbf{A}_n(x)^2 - (-1)^n 4}.$$

It follows from Theorem 5.1.2 and the definition of μ that

$$\begin{aligned} \mathcal{L}(\alpha) &= \frac{1}{s}\log\frac{T(a_{r+1},\cdots,a_{r+s}) + \sqrt{T(a_{r+1},\cdots,a_{r+s})^2 - (-1)^s 4}}{2} \\ &= \frac{1}{s}\log\frac{\mathbf{T}_s(\mu) + \sqrt{\mathbf{T}_s(\mu)^2 - (-1)^s 4}}{2}. \end{aligned}$$

Since $\mathbf{A}_s(\mu) = \mathbf{T}_s(\mu)$ and $\mathbf{B}_s(\mu)\sqrt{x^2+4} = \sqrt{\mathbf{T}_s(\mu)^2 - (-1)^s 4}$, this proves the proposition. □

Definition 5.2.12. *We define* the lower Christoffel word *with slope p/q by a prefix of $\mathbf{s}_{p/q,0}$ of length q. Let $w_{p/q}$ be the lower Christoffel word with slope p/q.*

For example,

$$w_{0/1} = a,\ w_{1/1} = b,\ w_{1/2} = ab,\ w_{1/3} = aab,\ w_{3/8} = aabaabab.$$

We have

$$|w_{p/q}| = q, \quad |w_{p/q}|_a = q - p, \qquad |w_{p/q}|_b = p$$

where $|\cdot|_a$ is the number of a and $|\cdot|_b$ is the number of b. Note that, for p/q in $[0,1)$, the word $w_{p/q}$ can be written as

$$w_{p/q} = aub, \text{ where } u \text{ is a palindrome.}$$

We refer the reader to [13] and [2] for additional results on Christoffel words.

For shorten the notation, for a finite word $v = v_1 \cdots v_n$ over the positive integers, we write

$$[0; \overline{v}] := [0; \overline{v_1, \cdots, v_n}] \quad \text{and} \quad T(v) := T(v_1, \cdots, v_n).$$

We define α_θ by

$$\alpha_\theta := \begin{cases} [0; \overline{w_{p/q}}] \text{ if } \theta = p/q, \\ [0; \mathbf{c}_\theta] \text{ if } \theta \text{ is irrational.} \end{cases}$$

Then $\mathcal{L}(\alpha_{p/q})$ is uniquely defined for each rational p/q. Note that each rational p/q has a unique continued fraction expansion if we allow the last partial quotient is bigger than 1.

To show Theorem 5.2.2, we need more statements. We define the function f on $[0,1]$ by

$$f(\theta) := \mathcal{L}(\alpha_\theta). \tag{5.6}$$

Proposition 5.2.11 and Theorem 5.1.2 imply that f is well-defined. We aim to prove that f is monotone and continuous.

We introduce further notation.

Notation 5.2.13. *(1) We denote by $x_{p/q}$ the positive real solution of*

$$\mathbf{T}_q(x_{p/q}) = T(w_{p/q}). \tag{5.7}$$

It has been shown just above Proposition 5.2.11 that $x_{p/q}$ is well-defined.

(2) *We define a function* $\varphi(x)$ *and* $\varphi'(x)$ *by*

$$\varphi(x) := \frac{x+\sqrt{x^2+4}}{2} \quad\text{and}\quad \varphi'(x) := \frac{x-\sqrt{x^2+4}}{2}. \tag{5.8}$$

Proposition 5.2.11 shows that

$$\mathcal{L}(\alpha_{p/q}) = \log\varphi(x_{p/q}).$$

(3) *Let*

$$X_{p/q} = \begin{pmatrix} x_{p/q} & 1 \\ 1 & 0 \end{pmatrix}.$$

By (5.7)*, we have*

$$\mathrm{Tr}(X_{p/q}^q) = T(w_{p/q}).$$

Example 5.2.14. (1) *For* $n = 1$*, we have that*

$$\mathcal{L}([0;\overline{a}]) = \log\frac{a+\sqrt{a^2+4}}{2} = \log\varphi(a).$$

(2) *For* $n = 2$*. we have that*

$$\mathcal{L}([0;\overline{a,b}]) = \frac{1}{2}\log\frac{ab+2+\sqrt{(ab+2)^2-4}}{2} = \frac{1}{2}\log\frac{ab+2+\sqrt{ab(ab+4)}}{2}.$$

Since

$$\begin{aligned}\log\varphi(\sqrt{ab}) &= \log\frac{\sqrt{ab}+\sqrt{ab+4}}{2} = \frac{1}{2}\log\left(\frac{\sqrt{ab}+\sqrt{ab+4}}{2}\right)^2 \\ &= \frac{1}{2}\log\frac{2ab+4+2\sqrt{ab(ab+4)}}{4},\end{aligned}$$

We have

$$\mathcal{L}([0;\overline{a,b}]) = \log\varphi(\sqrt{ab})$$

and $x_{1/2} = \sqrt{an}$*.*

Lemma 5.2.15. *Let* U, V *be* 2×2 *matrices. If* $U = WV$ *or* VW*, then we have*

$$\mathrm{Tr}(VU) = \mathrm{Tr}(UV) = \mathrm{Tr}(U)\,\mathrm{Tr}(V) - \det(V)\,\mathrm{Tr}(W).$$

In particular, for any integers q and q' such that $q \geq q' > 0$, we have the following relation:

$$\mathrm{Tr}(X^{q+q'}) = \mathrm{Tr}(X^q)\,\mathrm{Tr}(X^{q'}) + (-1)^{q'+1}\,\mathrm{Tr}(X^{q-q'}). \tag{5.9}$$

Proof. By a direct calculation, we get

$$\mathrm{Tr}(UV) = \mathrm{Tr}(U)\,\mathrm{Tr}(V) + \det(U - V) - \det(U) - \det(V)$$

and

$$\det(W - I) = \det(W) - \mathrm{Tr}(W) + 1$$

where I is the 2×2 identity matrix. Therefore, if $U = WV$, we have

$$\begin{aligned}\mathrm{Tr}(UV) &= \mathrm{Tr}(U)\,\mathrm{Tr}(V) + \det(WV - V) - \det(WV) - \det(V) \\ &= \mathrm{Tr}(U)\,\mathrm{Tr}(V) + \det(W - I)\det(V) - \det(W)\det(V) - \det(V) \\ &= \mathrm{Tr}(U)\,\mathrm{Tr}(V) - \det(V)\,\mathrm{Tr}(W).\end{aligned}$$

If $U = VW$, then $\mathrm{Tr}(UV) = \mathrm{Tr}(VWV) = \mathrm{Tr}(VU')$, with $U' = WV$, and we use the previous calculation. Finally, taking $U = X^q$ and $V = X^{q'}$, we immediately derive (5.9). □

By the previous lemma, we have the following relation between the traces of matrices associated to Chritoffel words.

Lemma 5.2.16. *Let p/q and p'/q' be rational numbers in $[0,1]$ with $q > q'$ and*

$$\det \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \pm 1.$$

Then the following relation holds:

$$T(w_{(p+p')/(q+q')}) = T(w_{p/q})T(w_{p'/q'}) + (-1)^{q'+1}T(w_{(p-p')/(q-q')}). \tag{5.10}$$

Example 5.2.17. *The lower Christoffel word with slope $0/1$, $1/4$, $1/3$, $2/7$ are*

$$w_{0/1} = a, \quad w_{1/4} = aaab, \quad w_{1/3} = aab, \quad w_{2/7} = aaabaab,$$

respectively. Their corresponding traces are

$$T(w_{0/1}) = a, \quad T(w_{1/4}) = a^3b + 2a^2 + 2ab + 2, \quad T(w_{1/3}) = a^2b + 2a + b,$$

$$T(w_{2/7}) = a^5b^2 + 4a^4b + 3a^3b^2 + 4a^3 + 8a^2b + 2ab^2 + 5a + 2b.$$

By direct calculation, we can check that

$$T(w_{2/7}) = T(w_{1/4})T(w_{1/3}) + T(w_{0/1}).$$

Lemma 5.2.18. *Let p/q and p'/q' be rational numbers in $[0,1]$ such that $q > q'$ and*

$$\det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \pm 1.$$

Then we have the following four relations:

$$\begin{aligned}\frac{\mathrm{Tr}(X_{p/q}^{q+q'}) - \mathrm{Tr}(X_{(p+p')/(q+q')}^{q+q'})}{\mathrm{Tr}(X_{p/q}^{q'}) - \mathrm{Tr}(X_{p'/q'}^{q'})} \\ = \mathrm{Tr}(X_{p/q}^{q}) + (-1)^{q'+1}\frac{\mathrm{Tr}(X_{p/q}^{q-q'}) - \mathrm{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\mathrm{Tr}(X_{p/q}^{q'}) - \mathrm{Tr}(X_{p'/q'}^{q'})},\end{aligned} \tag{5.11}$$

$$\begin{aligned}\frac{\mathrm{Tr}(X_{p'/q'}^{q+q'}) - \mathrm{Tr}(X_{(p+p')/(q+q')}^{q+q'})}{\mathrm{Tr}(X_{p'/q'}^{q}) - \mathrm{Tr}(X_{p/q}^{q})} \\ = \mathrm{Tr}(X_{p'/q'}^{q'}) + (-1)^{q'+1}\frac{\mathrm{Tr}(X_{p'/q'}^{q-q'}) - \mathrm{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\mathrm{Tr}(X_{p'/q'}^{q}) - \mathrm{Tr}(X_{p/q}^{q})},\end{aligned} \tag{5.12}$$

$$\begin{aligned}\frac{\mathrm{Tr}(X_{(p+p')/(q+q')}^{q'}) - \mathrm{Tr}(X_{p'/q'}^{q'})}{\mathrm{Tr}(X_{p/q}^{q}) - \mathrm{Tr}(X_{(p+p')/(q+q')}^{q})} = \frac{\mathrm{Tr}(X_{(p+p')/(q+q')}^{q'})}{\mathrm{Tr}(X_{p/q}^{q})} \\ + (-1)^{q'+1}\frac{\mathrm{Tr}(X_{(p+p')/(q+q')}^{q-q'}) - \mathrm{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{(\mathrm{Tr}(X_{(p+p')/(q+q')}^{q}) - \mathrm{Tr}(X_{p/q}^{q}))\,\mathrm{Tr}(X_{p/q}^{q})}\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}\frac{\mathrm{Tr}(X_{(p+p')/(q+q')}^{q}) - \mathrm{Tr}(X_{p/q}^{q})}{\mathrm{Tr}(X_{p'/q'}^{q'}) - \mathrm{Tr}(X_{(p+p')/(q+q')}^{q'})} = \frac{\mathrm{Tr}(X_{(p+p')/(q+q')}^{q})}{\mathrm{Tr}(X_{p'/q'}^{q'})} \\ + (-1)^{q'+1}\frac{\mathrm{Tr}(X_{(p-p')/(q-q')}^{q-q'}) - \mathrm{Tr}(X_{(p+p')/(q+q')}^{q-q'})}{(\mathrm{Tr}(X_{p'/q'}^{q'}) - \mathrm{Tr}(X_{(p+p')/(q+q')}^{q'}))\,\mathrm{Tr}(X_{p'/q'}^{q'})}.\end{aligned} \tag{5.14}$$

Proof. By Lemma 5.2.16, we have

$$\mathrm{Tr}(X^{q+q'}_{(p+p')/(q+q')}) = \mathrm{Tr}(X^{q}_{p/q})\,\mathrm{Tr}(X^{q'}_{p'/q'}) + (-1)^{q'+1}\,\mathrm{Tr}(X^{q-q'}_{(p-p')/(q-q')}). \tag{5.15}$$

By applying (5.9) with $X = X_{p/q}$, $X = X_{p'/q'}$, and $X = X_{(p+p')/(q+q')}$, we get three equalities. By combining these four equalities, we derive the four equations (5.11), (5.12), (5.13) and (5.14). $\square$

Proposition 5.2.19. *Let p/q and p'/q' be rational numbers $[0,1]$ with $q > q'$ and* $\det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = \pm 1$. *Then we have*

$$x_{p/q} < x_{(p+p')/(q+q')} < x_{p'/q'} \quad \text{or} \quad x_{p'/q'} < x_{(p+p')/(q+q')} < x_{p/q}.$$

Proof. Since $\mathbf{T}_n(x)$ is an strictly increasing function, for all $n \geq 1$ and every rational numbers p/q and r/s between 0 and 1,

$$\mathrm{Tr}(X^n_{p/q}) < \mathrm{Tr}(X^n_{r/s}) \text{ if and only if } x_{p/q} < x_{r/s}.$$

We will show the proposition inductively. Suppose that $x_{p/q}$ is a point between $x_{p'/q'}$ and $x_{(p-p')/(q-q')}$.

(i) Assume that q' is even. Since $x_{p/q}$ is between $x_{p'/q'}$ and $x_{(p-p')/(q-q')}$, we have

$$\frac{\mathrm{Tr}(X^{q-q'}_{p/q}) - \mathrm{Tr}(X^{q-q'}_{(p-p')/(q-q')})}{\mathrm{Tr}(X^{q'}_{p/q}) - \mathrm{Tr}(X^{q'}_{p'/q'})} < 0.$$

By (5.11),

$$\frac{\mathrm{Tr}(X^{q+q'}_{p/q}) - \mathrm{Tr}(X^{q+q'}_{(p+p')/(q+q')})}{\mathrm{Tr}(X^{q'}_{p/q}) - \mathrm{Tr}(X^{q'}_{p'/q'})} > \mathrm{Tr}(X^{q}_{p/q}) > 0. \tag{5.16}$$

Assume that q' is odd. If $p/q < p'/q'$, then $(p-p')/(q-q') < p/q < p'/q'$. Note that the traces are integers. Thus

$$\frac{\mathrm{Tr}(X^{q-q'}_{p/q}) - \mathrm{Tr}(X^{q-q'}_{(p-p')/(q-q')})}{\mathrm{Tr}(X^{q'}_{p'/q'}) - \mathrm{Tr}(X^{q'}_{p/q})} < \mathrm{Tr}(X^{q-q'}_{p/q}) < \mathrm{Tr}(X^{q}_{p/q}). \tag{5.17}$$

If $p'/q' < p/q$ and $q - q' < q'$, then

$$\frac{2p'-p}{2q'-q} < \frac{p'}{q'} < \frac{p}{q} < \frac{p-p'}{q-q'},$$

thus by (5.13),

$$\frac{\operatorname{Tr}(X_{p/q}^{q-q'}) - \operatorname{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'}) - \operatorname{Tr}(X_{p/q}^{q'})}$$

is equal to

$$\frac{\operatorname{Tr}(X_{p/q}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'})} + (-1)^{q-q'+1}\frac{\operatorname{Tr}(X_{p/q}^{2q'-q}) - \operatorname{Tr}(X_{(2p'-p)/(2q'-q)}^{2q'-q})}{(\operatorname{Tr}(X_{p/q}^{q'}) - \operatorname{Tr}(X_{p'/q'}^{q'}))\operatorname{Tr}(X_{p'/q'}^{q'})}.$$

For an odd q, we have

(5.18)
$$\begin{aligned}\frac{\operatorname{Tr}(X_{p/q}^{q-q'}) - \operatorname{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'}) - \operatorname{Tr}(X_{p/q}^{q'})} &= \frac{\operatorname{Tr}(X_{p/q}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'})} - \frac{\operatorname{Tr}(X_{p/q}^{2q'-q}) - \operatorname{Tr}(X_{(2p'-p)/(2q'-q)}^{2q'-q})}{(\operatorname{Tr}(X_{p/q}^{q'}) - \operatorname{Tr}(X_{p'/q'}^{q'}))\operatorname{Tr}(X_{p'/q'}^{q'})}\\ &< \frac{\operatorname{Tr}(X_{p/q}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'})} < \operatorname{Tr}(X_{p/q}^{q}).\end{aligned}$$

For an even q, we have

(5.19)
$$\begin{aligned}\frac{\operatorname{Tr}(X_{p/q}^{q-q'}) - \operatorname{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'}) - \operatorname{Tr}(X_{p/q}^{q'})} &= \frac{\operatorname{Tr}(X_{p/q}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'})} + \frac{\operatorname{Tr}(X_{p/q}^{2q'-q}) - \operatorname{Tr}(X_{(2p'-p)/(2q'-q)}^{2q'-q})}{(\operatorname{Tr}(X_{p/q}^{q'}) - \operatorname{Tr}(X_{p'/q'}^{q'}))\operatorname{Tr}(X_{p'/q'}^{q'})}\\ &< \operatorname{Tr}(X_{p/q}^{q-q'}) + \operatorname{Tr}(X_{p/q}^{2q'-q})\\ &< \operatorname{Tr}(X_{p/q}^{q'})\operatorname{Tr}(X_{p/q}^{q-q'}) + (-1)^{q-q'+1}\operatorname{Tr}(X_{p/q}^{2q'-q}) = \operatorname{Tr}(X_{p/q}^{q}).\end{aligned}$$

If $p'/q' < p/q$ and $q - q' > q'$, then

$$\frac{p'}{q'} < \frac{p}{q} < \frac{p-p'}{q-q'} < \frac{p-2p'}{q-2q'}.$$

By (5.14),

(5.20)

$$\begin{aligned}
&\frac{\operatorname{Tr}(X_{p/q}^{q-q'}) - \operatorname{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'}) - \operatorname{Tr}(X_{p/q}^{q'})} \\
&\qquad = \frac{\operatorname{Tr}(X_{p/q}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'})} + \frac{\operatorname{Tr}(X_{(p-2p')/(q-2q')}^{q-2q'}) - \operatorname{Tr}(X_{p/q}^{q-2q'})}{(\operatorname{Tr}(X_{p'/q'}^{q'}) - \operatorname{Tr}(X_{p/q}^{q'}))\operatorname{Tr}(X_{p'/q'}^{q'})} \\
&\qquad < \frac{\operatorname{Tr}(X_{p/q}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q'})} < \operatorname{Tr}(X_{p/q}^{q}).
\end{aligned}$$

Therefore, using (5.17), (5.18), (5.19), (5.20) and (5.11), we have

$$\frac{\operatorname{Tr}(X_{p/q}^{q+q'}) - \operatorname{Tr}(X_{(p+p')/(q+q')}^{q+q'})}{\operatorname{Tr}(X_{p/q}^{q'}) - \operatorname{Tr}(X_{p'/q'}^{q'})} > 0.$$

Thus, we have established that

$$\begin{aligned}
x_{p/q} < x_{(p+p')/(q+q')} \text{ if } x_{p/q} < x_{p'/q'}, \quad &\text{and} \\
x_{p/q} > x_{(p+p')/(q+q')} \text{ if } x_{p/q} > x_{p'/q'},
\end{aligned}$$

regardless of the parity of q'.

(ii) Suppose that q' is odd. Since $x_{p/q}$ is between $x_{p'/q'}$ and $x_{(p-p')/(q-q')}$, we have

$$\frac{\operatorname{Tr}(X_{p'/q'}^{q-q'}) - \operatorname{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\operatorname{Tr}(X_{p'/q'}^{q}) - \operatorname{Tr}(X_{p/q}^{q})} > 0.$$

By (5.12)

$$\frac{\operatorname{Tr}(X_{p'/q'}^{q+q'}) - \operatorname{Tr}(X_{(p+p')/(q+q')}^{q+q'})}{\operatorname{Tr}(X_{p'/q'}^{q}) - \operatorname{Tr}(X_{p/q}^{q})} > \operatorname{Tr}(X_{p'/q'}^{q'}) > 0.$$

Suppose that q' is even. Using (5.12), we have to show that

$$\frac{\mathrm{Tr}(X_{p'/q'}^{q-q'}) - \mathrm{Tr}(X_{(p-p')/(q-q')}^{q-q'})}{\mathrm{Tr}(X_{p'/q'}^{q}) - \mathrm{Tr}(X_{p/q}^{q})} < \mathrm{Tr}(X_{p'/q'}^{q'}).$$

Let $m \geq 1$ be the integer satisfying that $mq' < q < (m+1)q'$. Note that q is odd. By (5.11),

$$\begin{aligned}
&\frac{\mathrm{Tr}(X_{p'/q'}^{q-(m-1)q'}) - \mathrm{Tr}(X_{(p-(m-1)p')/(q-(m-1)q')}^{q-(m-1)q'})}{\mathrm{Tr}(X_{p'/q'}^{q-mq'}) - \mathrm{Tr}(X_{(p-mq')/(q-mq')}^{q-mq'})} \\
&\qquad = \mathrm{Tr}(X_{p'/q'}^{q'}) - \frac{\mathrm{Tr}(X_{((m+1)p'-p)/((m+1)q'-q)}^{(m+1)q'-q}) - \mathrm{Tr}(X_{p'/q'}^{(m+1)q'-q})}{\mathrm{Tr}(X_{p'/q'}^{q-mq'}) - \mathrm{Tr}(X_{(p-mq')/(q-mq')}^{q-mq'})} \\
&\qquad < \mathrm{Tr}(X_{p'/q'}^{q'}).
\end{aligned}$$

Here we use the fact that p'/q' is between $\frac{(m+1)p'-p}{(m+1)q'-q}$ and $\frac{p-mp'}{q-mq'}$. By (5.12), for each $0 \leq n < m$,

$$\begin{aligned}
&\frac{\mathrm{Tr}(X_{p'/q'}^{q-(n-1)q'}) - \mathrm{Tr}(X_{(p-(n-1)p')/(q-(n-1)q')}^{q-(n-1)q'})}{\mathrm{Tr}(X_{p'/q'}^{q-nq'}) - \mathrm{Tr}(X_{(p-nq')/(q-nq')}^{q-nq'})} \\
&\qquad = \mathrm{Tr}(X_{p'/q'}^{q'}) - \frac{\mathrm{Tr}(X_{p'/q'}^{q-(n+1)q'}) - \mathrm{Tr}(X_{(p-(n+1)p')/(q-(n+1)q')}^{q-(n+1)q'})}{\mathrm{Tr}(X_{p'/q'}^{q-nq'}) - \mathrm{Tr}(X_{(p-nq')/(q-nq')}^{q-nq'})}.
\end{aligned}$$

Therefore, inductively we have

$$\begin{aligned}
&\frac{\mathrm{Tr}(X_{p'/q'}^{q+q'}) - \mathrm{Tr}(X_{(p+p')/(q+q')}^{q+q'})}{\mathrm{Tr}(X_{p'/q'}^{q}) - \mathrm{Tr}(X_{p/q}^{q})} \\
&\qquad > \mathrm{Tr}(X_{p'/q'}^{q'}) - \cfrac{1}{\mathrm{Tr}(X_{p'/q'}^{q'}) - \cfrac{1}{\mathrm{Tr}(X_{p'/q'}^{q'}) - \ddots}} > 0.
\end{aligned}$$

Thus, we have established that

$$\begin{aligned}
x_{p'/q'} < x_{(p+p')/(q+q')} \text{ if } x_{p'/q'} < x_{p/q}, \quad \text{and} \\
x_{p'/q'} > x_{(p+p')/(q+q')} \text{ if } x_{p'/q'} > x_{p/q},
\end{aligned}$$

regardless of the parity of q'. We conclude that

$$x_{p'/q'} < x_{(p+p')/(q+q')} < x_{p/q} \quad \text{or} \quad x_{p/q} < x_{(p+p')/(q+q')} < x_{p'/q'}$$

holds. □

Proposition 5.2.20. *Let p/q and p'/q' be rational numbers $[0,1]$ with $p/q < p'/q'$. Then*

$$\mathcal{L}(\alpha_{p/q}) < \mathcal{L}(\alpha_{p'/q'}).$$

Proof. It is enough to show the monotonicity for rationals p/q and p'/q' with $\det\begin{pmatrix} p & p' \\ q & q' \end{pmatrix} = -1$ and $p/q < p'/q'$. By Proposition 5.2.11,

$$\mathcal{L}(\alpha_{p/q}) < \mathcal{L}(\alpha_{(p+p')/(q+q')}) < \mathcal{L}(\alpha_{p'/q'}),$$

is equivalent to

$$x_{p/q} < x_{(p+p')/(q+q')} < x_{p'/q'},$$

which is established by Proposition 5.2.19. □

The above proposition shows that f is monotone increasing on the rationals. From now on, we will discuss the continuity of f.

Lemma 5.2.21. *For a given rational $p/q \in [0,1]$, there exists a sequence of rational numbers (r_n) which converges to p/q such that x_{r_n} converges to $x_{p/q}$.*

Proof. Let p'/q' be a rational such that $p'q - pq' = \pm 1$. Let

$$r_n := \frac{p' + np}{q' + nq}.$$

We observe that r_n tends to p/q as n tends to infinity, and $r_n \neq p/q$, for $n \geq 1$. Since $\mathbf{T}_m(x) = x\mathbf{T}_{m-1}(x) + \mathbf{T}_{m-2}(x)$ for $m \geq 3$, we can show inductively that

$$\mathbf{T}_m(x) = \varphi(x)^m + \varphi'(x)^m, \quad \text{for } m \geq 1, \tag{5.21}$$

where φ and φ' are defined in (5.8).

By Lemma 5.2.16, we have

$$T(w_{r_n}) = T(w_{p/q})T(w_{r_{n-1}}) + (-1)^{q+1}T(w_{r_{n-2}}) \text{ for } n \geq 3. \tag{5.22}$$

Thus, the sequence $(T(w_{r_n}))_{n\geq 1}$ is a binary recurrence sequence and there exist constants C_1, C_2 such that

$$T(w_{r_n}) = C_1 u^n + C_2 v^n, \quad n \geq 1,$$

where

$$u=\frac{T(w_{p/q})+\sqrt{T(w_{p/q})^2+(-1)^{q+1}4}}{2} \text{ and } v=\frac{T(w_{p/q})-\sqrt{T(w_{p/q})^2+(-1)^{q+1}4}}{2}.$$

Since the integer $T(w_{p/q})^2 + (-1)^{q+1}4$ cannot be a positive perfect square, we deduce that C_1 and C_2 are nonzero.

For any $\varepsilon > 0$, we have

$$\frac{1}{q}\log u = \mathcal{L}(\alpha_{p/q}) = \log\varphi(x_{p/q}) \in \left(\log\varphi(x_{p/q}-\varepsilon), \log\varphi(x_{p/q}+\varepsilon)\right).$$

Thus,

$$0 < \varphi(x_{p/q}-\varepsilon)^q < u < \varphi(x_{p/q}+\varepsilon)^q.$$

Since $|\varphi'(x_{p/q}-\varepsilon)| < \varphi(x_{p/q}-\varepsilon)$ and $|v| < u$, by combining (5.21) and (5.22), we deduce that

$$\frac{\mathbf{T}_{q'+nq}(x_{p/q}-\varepsilon)}{\mathbf{T}_{q'+nq}(x_{r_n})} \to 0 \text{ and } \frac{\mathbf{T}_{q'+nq}(x_{p/q}+\varepsilon)}{\mathbf{T}_{q'+nq}(x_{r_n})} \to \infty,$$

as $n \to \infty$. There exists N depending on ε such that for any $n > N$,

$$\mathbf{T}_{q'+nq}(x_{p/q}-\varepsilon) < \mathbf{T}_{q'+nq}(x_{r_n}) < \mathbf{T}_{q'+nq}(x_{p/q}+\varepsilon).$$

Since $\mathbf{T}_{q'+nq}$ is monotone increasing, we conclude that

$$x_{p/q}-\varepsilon < x_{r_n} < x_{p/q}+\varepsilon.$$

Since ε can be taken arbitrarily small, this shows that x_{r_n} goes to $x_{p/q}$ as n goes to infinity. □

Lemma 5.2.22. *Let $\alpha = [0; s_1, s_2, \cdots, s_n, \cdots]$ where $s_1 s_2 \cdots s_n \cdots$ is a characteristic Sturmian word with slope θ. Then*

$$\mathcal{L}(\alpha) = \lim_{k\to\infty} \mathcal{L}(\alpha_{\mathfrak{p}_k/\mathfrak{q}_k}),$$

where $\mathfrak{p}_k/\mathfrak{q}_k$ *is the principal convergnet of* θ*.*

To show the lemma, we use the following result of Baxa [10].

Lemma 5.2.23 ([10])**.** *For all* $\alpha = [0; b_1, b_2, \cdots] \in \mathbb{R} \setminus \mathbb{Q}$*, we have*

$$\overline{\lim_{n\to\infty}} \frac{\log Q_n(\alpha)}{n} = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \log\left([b_i; b_{i+1}, \cdots]\right)$$

and

$$\underline{\lim_{n\to\infty}} \frac{\log Q_n(\alpha)}{n} = \underline{\lim_{n\to\infty}} \frac{1}{n} \sum_{i=1}^{n} \log\left([b_i; b_{i+1}, \cdots]\right)$$

where $Q_n(\alpha)$ *is the denominator of the nth principal convergent of* α*.*

Proof of Lemma 5.2.22. By Theorem 5.2.1 and the above lemma,

$$\mathcal{L}(\alpha) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \log\left([s_i; s_{i+1}, \cdots]\right).$$

The prefix $s_1 \cdots s_{\mathfrak{q}_k - 1} s_{\mathfrak{q}_k}$ is a standard word. It is known that a standard word is a permutation of a lower Christoffel word. More precisely, if $s_{\mathfrak{q}_k-1}s_{\mathfrak{q}_k} = ba$, then $s_{\mathfrak{q}_k}s_1 \cdots s_{\mathfrak{q}_k-1}$ is $w_{\mathfrak{p}_k/\mathfrak{q}_k}$ and if $s_{\mathfrak{q}_k-1}s_{\mathfrak{q}_k} = ab$, then $s_{\mathfrak{q}_k-1}s_{\mathfrak{q}_k-2} \cdots s_2 s_1 s_{\mathfrak{q}_k}$ is $w_{\mathfrak{p}_k/\mathfrak{q}_k}$. Thus,

$$\mathcal{L}(\alpha_{\mathfrak{p}_k/\mathfrak{q}_k}) = \frac{1}{\mathfrak{q}_k} \sum_{i=1}^{\mathfrak{q}_k} \log\left([\overline{s_i; s_{i+1}, \cdots, s_{i+\mathfrak{q}_k-1}}]\right) = \mathcal{L}([0; \overline{s_1 \cdots s_{\mathfrak{q}_k}}]).$$

Let $r \geq 3$ and $\gamma = [0; c_r, c_{r+1}, \cdots]$, $\delta = [0; d_r, d_{r+1}, \cdots]$ be continued fractions with partial quotients in $\{a, b\}$. Let $b_0, \cdots, b_{r-1}$ be integers in $\{a, b\}$. We denote by

$$R_{r-2}/T_{r-2} = [b_0; b_1, \cdots, b_{r-2}], \quad R_{r-1}/T_{r-1} = [b_0; b_1, \cdots, b_{r-1}].$$

Then, we infer

$$\begin{aligned}
&|[b_0; b_1, \cdots, b_{r-1}, c_r, c_{r+1}, \cdots] - [b_0; b_1, \cdots, b_{r-1}, d_r, d_{r+1}, \cdots]| \\
&= \left| \frac{R_{r-1}\gamma + R_{r-2}}{T_{r-1}\gamma + T_{r-2}} - \frac{R_{r-1}\delta + R_{r-2}}{T_{r-1}\delta + T_{r-2}} \right| = \frac{|\gamma - \delta|}{(R_{r-1}\gamma + R_{r-2})(T_{r-1}\delta + T_{r-2})} \\
&\leq \frac{\mathcal{L}([0; \overline{b}]) - \mathcal{L}([0; \overline{a}])}{2^{r-2}}.
\end{aligned}$$

Since $|\log x - \log y| < |x - y|$ if $x, y > 1$,

$$\left| \frac{1}{\mathfrak{q}_k} \sum_{i=1}^{\mathfrak{q}_k} \log\left([s_i; s_{i+1}, \cdots]\right) - \frac{1}{\mathfrak{q}_k} \sum_{i=1}^{\mathfrak{q}_k} \log\left([\overline{s_i; s_{i+1}, \cdots, s_{i+\mathfrak{q}_k-1}}]\right) \right|$$
$$\leq \frac{\mathcal{L}([0;\overline{b}]) - \mathcal{L}([0;\overline{a}])}{2^{\mathfrak{q}_k-2}} \to \infty \text{ as } k \to \infty.$$

Thus, $\mathcal{L}(\alpha) = \lim_{n\to\infty} \mathcal{L}(\alpha_{\mathfrak{p}_k/\mathfrak{q}_k})$. □

Proof of Theorem 5.2.2. By Proposition 5.2.19, the function f is monotone increasing on the rationals in $[0, 1]$. By Lemma 5.2.22, f is monotone increasing on $[0, 1]$. By Lemmas 5.2.21 and 5.2.22, $f([0, 1])$ is dense in

$$[\mathcal{L}([0;\overline{a}]), \mathcal{L}([0;\overline{b}])].$$

Therefore, f has no jump discontinuities, thus f is continuous on $[0, 1]$. □

Chapter 6

Colorings of trees

6.1 Preliminaries

6.1.1 Colorings of trees

In this section, we give definitions and properties of graphs and trees, following [63]. Then, we introduce colorings of trees, following [41] and [42].

A graph $\mathcal{G}$ is a structure consisting of a set of vertices $V\mathcal{G}$ and a set of edges $E\mathcal{G}$. The vertex set $V\mathcal{G}$ is a finite or countably infinite set and the edge set $E\mathcal{G}$ is a subset of ordered pairs of two distinct vertices

$$V\mathcal{G} \times V\mathcal{G} - \{(v, v) : v \in V\mathcal{G}\}$$

where (v, w) is the edge starting from v ending to w. For each edge $e \in E\mathcal{G}$, we denote by $\partial_0(e)$ *the initial vertex* of e and denote by $\partial_1(e)$ *the terminal vertex* of e. *The reversed edge* $\bar{e}$ of e is an edge such that and $\partial_0(\bar{e}) = \partial_1(e)$ and $\partial_1(\bar{e}) = \partial_0(e)$.

The degree of a vertex v is the number of edges starting from v. *A k-regular graph* is a graph whose vertices have the same degree k. *A tree* is a graph without cycles. In this section, let $\mathcal{T}$ be a k-regular tree.

We identify an edge e with the unit interval $[0, 1]$. One can identify $\mathcal{G}$ with

$$V\mathcal{G} \sqcup (E\mathcal{G} \times [0, 1]) \,/ \sim$$

with the equivalece relation generated by

$$(e, t) \sim (\bar{e}, 1 - t),\ (e, 0) \sim \partial_0(e),\ (e, 1) \sim \partial_1(e)\ \text{ for }\ e \in E\mathcal{G},\ t \in [0, 1].$$

Then there is an induced metric $\mathbf{d}$ on $\mathcal{G}$.

A coloring of a tree is a vertex coloring which is defined by a map

$$\phi : V\mathcal{T} \to \mathcal{A},$$

where $\mathcal{A}$ is an alphabet. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be subtrees of $\mathcal{T}$. *A color-preserving map* $f : \mathcal{T}_1 \to \mathcal{T}_2$ is a map such that $\phi(v) = \phi(f(v))$ for $v \in V\mathcal{T}$. Let $\mathrm{Aut}(\mathcal{T})$ be the group of automorphisms of $\mathcal{T}$. Let Γ_ϕ be the set of color-preserving automorphisms of $\mathcal{T}$. Since the composition of two color-preserving automorphisms is also a color-preserving automorphism, Γ_ϕ is a subgroup of $\mathrm{Aut}(\mathcal{T})$. We say that $v, w \in V\mathcal{T}$ are *in the same class* if there is a color-preserving automorphism g such that $g(v) = w$.

We define a quotient graph $\Gamma_\phi \backslash \mathcal{T}$. The vertex set and the edge set of $\Gamma_\phi \backslash \mathcal{T}$ are

$$V(\Gamma_\phi \backslash \mathcal{T}) = \{\Gamma_\phi.v : v \in V\mathcal{T}\} \text{ and } E(\Gamma_\phi \backslash \mathcal{T}) = \{\Gamma_\phi.e : e \in E\mathcal{T}\}.$$

The induced graph structure is defined by $\partial_i(\Gamma_\phi.v) = \Gamma_\phi.\partial_i(v)$ for $i = 0, 1$. Then, we have a graph which we call *the quotient graph* $\Gamma_\phi \backslash \mathcal{T}$. The covering map is denoted by

$$\pi : \mathcal{T} \to \Gamma_\phi \backslash \mathcal{T}.$$

Let $e \in E\mathcal{T}$. If $g(e) = \bar{e}$ for some $g \in \Gamma_\phi$, then $\pi(e)$ is a loop of $\Gamma_\phi \backslash \mathcal{T}$.

We call $(\mathcal{G}, i)$ *an edge-indexed graph* equipped with *an edge index map*

$$i : E\mathcal{G} \to \mathbb{N}.$$

In order to construct the universal cover of $(\mathcal{G}, i)$, let us start with a vertex $\tilde{x}$ which will be a lift of $x \in V\mathcal{G}$. Choose $\ell \in E\mathcal{G}$ such that $\partial_0(\ell) = x$. We attach $i(\ell)$ edges ℓ_j, $j = 1, \cdots, i(\ell)$ at $\tilde{x}$. Each ℓ_j is considered as a lift of ℓ. For $\ell' \in \mathcal{G}$ such that $\partial_0(\ell') = \partial_1(\ell)$, we attach again $i(\ell')$ edges ℓ'_j, $j = 1, \cdots, i(\ell')$ at ℓ_j. Repeating this process, we have a tree which is the universal cover of $(\mathcal{G}, i)$.

The quotient graph $\Gamma_\phi \backslash \mathcal{T}$ is equipped with an edge index map. Let ℓ be an edge of $\Gamma_\phi \backslash \mathcal{T}$. Then there is $e \in E\mathcal{T}$ such that $\Gamma_\phi.e = \ell$. Define $i(\ell)$ by the number of edges contained in the orbit $\Gamma_\phi.e$ emitting the same vertex of $\mathcal{T}$. Then, the universal cover of $(\Gamma_\phi \backslash \mathcal{T}, i)$ is $\mathcal{T}$.

Dong Han Kim and Seonhee Lim defined the *factor complexity (or subword complexity)* of colorings of trees [41]. The factor complexity is a generalization of the factor complexity of words in Section 4.1.

Definition 6.1.1. *(1) An* n-ball around x *is defined by*

$$\mathcal{B}_n(x) := \{y \in V\mathcal{T} \cup E\mathcal{T} : \mathbf{d}(x, y) \le n\}.$$

(2) An n-sphere around y *is defined by*

$$\mathcal{S}_n(x) := \{y \in V\mathcal{T} : \mathbf{d}(x, y) = n\}.$$

We say that two balls $\mathcal{B}_n(x)$ and $\mathcal{B}_n(y)$ are *equivalent* if there exists a color-preserving isomorphism $f : \mathcal{B}_n(x) \to \mathcal{B}_n(y)$. We call such an equivalence class *a colored n-ball* and denote it by $[\mathcal{B}_n(x)]$. The set of colored n-balls is denoted by $\mathbb{B}_\phi(n)$.

Definition 6.1.2. *The* factor complexity b_ϕ *of a coloring* ϕ *is a function that assigns each nonnegative integer* n *to the number of non-equivalent colored* n*-balls in the tree colored by* ϕ*, i.e.,*

$$b_\phi(n) := |\mathbb{B}_\phi(n)|.$$

As a special word in Definition 4.1.6, we define a special colored n-ball.

Definition 6.1.3. *We call a colored n-ball* $[\mathcal{B}_n(x)]$ a special n-ball *if there are distinct vertices* x *and* y *such that*

$$[\mathcal{B}_n(x)] = [\mathcal{B}_n(y)], \text{ but } [\mathcal{B}_{n+1}(x)] \ne [\mathcal{B}_{n+1}(y)].$$

In this case, we call $[\mathcal{B}_{n+1}(x)]$ *and* $[\mathcal{B}_{n+1}(y)]$ extensions *of* $[\mathcal{B}_n(x)]$.

The *type set* Λ_u of a vertex $u \in V\mathcal{T}$ is the set of integers n such that $[\mathcal{B}_n(u)]$ is special. A vertex u is said to be of *bounded type* if Λ_u is a finite set. For a vertex u of bounded type, the *maximal type* $\tau(u)$ of u is the maximum of elements in Λ_u. We say that a coloring ϕ is of *bounded type* if each vertex (or equivalently a vertex) of $\mathcal{T}$ is of bounded type. Otherwise, we say that a coloring ϕ is of *unbounded type*.

One of the motivations for studying colorings of trees is to investigate tree lattices. Let Γ be a subgroup of $\mathrm{Aut}(\mathcal{T})$. If $\Gamma\backslash\mathcal{T}$ has a finite volume, then we say that Γ is *a tree-lattice* (see [9] for more detail). Note that Γ_ϕ may or may not be a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$ even if $\Gamma_\phi\backslash\mathcal{T}$ is a finite graph.

Another motivation is that factor complexity gives an invariant of an automorphism of a tree as a Cayley graph. Let

$$\Gamma := \langle a_1, \cdots, a_k \,|\, a_i^2 = 1 \text{ for } 1 \leq i \leq k \rangle.$$

Then, the Cayley graph of Γ is a k-regular tree $\mathcal{T}$ and Γ is a discrete subgroup of $\mathrm{Aut}(\mathcal{T})$. Lubotzky, Mozes and Zimmer defined a coloring ϕ_g associated to $g \in \Gamma$ and they proved that ϕ_g has bounded factor complexity if and only if g is a commensurator of Γ [49].

6.1.2 Sturmian colorings of trees

Let $(\mathcal{T}, \phi)$ be a coloring of a tree and $\Gamma := \Gamma_\phi$ is the group of color-preserving automorphism.

Definition 6.1.4. *We say that ϕ is* periodic *if $\Gamma \backslash \mathcal{T}$ is a finite graph.*

Dong Han Kim and Seonhee Lim studied colorings of trees with factor complexity. They proved the analogous theorem of Coven-Hedlund theorem (Theorem 4.1.4) and generalized Sturmian words to Sturmian colorings on a regular tree [41].

Theorem 6.1.5 ([41], Theorem 2.7)**.** *The following statements are equivalent.*

(1) The coloring ϕ is periodic.

(2) The factor complexity satisfies $b_\phi(n) = b_\phi(n+1)$ for some n.

(3) The factor complexity $b_\phi(n)$ is bounded.

Assume that b_ϕ is unbounded. Then, $b_\phi(0)$ is at least 2 and it is strictly increasing. Thus the minimal factor complexity of non-periodic colorings is $n + 2$.

Definition 6.1.6. *A coloring is called a* Sturmian coloring *if $b(n) = n + 2$.*

Theorem 6.1.7 ([41], Theorem 3.9)**.** *The quotient graph of a Sturmian coloring is a graph that looks like a half-line or a 2-regular tree with possibly attached loops on the vertices (see Figure 6.1).*

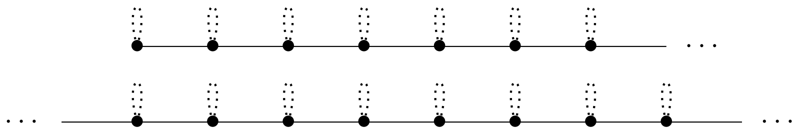


Figure 6.1: The quotient graph of a Sturmian coloring in Theorem 6.1.7.

The factor graph $\mathcal{G}_n$ is defined as the graph whose vertices are the colored n-balls. Its edges are pairs of colored n-balls whose centers are adjacent to each other, i.e., (D_n, E_n) such that

$$[\mathcal{B}_n(v)] = D_n \text{ and } [\mathcal{B}_n(w)] = E_n \text{ for some } v, w \in V\mathcal{T} \text{ with } \mathbf{d}(v,w) = 1.$$

By definition of Sturmian coloring, there is a unique special n-ball for each n. We denote by S_n the special n-ball. We denote by C_n the centered colored n-ball of S_{n+1}. The special ball S_n has exactly two extensions to $(n+1)$-balls. Denote by A_{n+1} and B_{n+1} the extensions of S_n. Then we can choose $\{A_n\}$ and $\{B_n\}$ such that A_{n+1} contains more A_n than B_n as a factor. We define subgraphs $\mathcal{G}_n^A$ and $\mathcal{G}_n^B$ of $\mathcal{G}_n$. The subgraphs $\mathcal{G}_n^A$, $\mathcal{G}_n^B$ are composed by the colored n-balls connected with S_n in A_{n+1}, B_{n+1}, respectively.

The following theorem explains the evolution of $\mathcal{G}_n^A$ and $\mathcal{G}_n^B$. In the theorem, $\bowtie$ is a concatenation defined as follows: If $C_n \neq S_n$, then

$$V(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = V\mathcal{G}_n^A \cup V\mathcal{G}_n^B \text{ and } E(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = E\mathcal{G}_n^A \cup E\mathcal{G}_n^B$$

where C_n in $\mathcal{G}_n^A$ and C_n in $\mathcal{G}_n^B$ are identified and the loops at C_n in $\mathcal{G}_n^A, \mathcal{G}_n^B$ are identified. If $C_n = S_n$, then we consider C_n in $\mathcal{G}_n^A$ and C_n in $\mathcal{G}_n^B$ as the distinct vertices such that

$$V(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = V\mathcal{G}_n^A \sqcup V\mathcal{G}_n^B \text{ and } E(\mathcal{G}_n^A \bowtie \mathcal{G}_n^B) = E\mathcal{G}_n^A \sqcup E\mathcal{G}_n^B \cup \{e\}$$

where e is the edge connecting C_n's.

Theorem 6.1.8 ([42], Theorem 1.2)**.** *Let ϕ be a Sturmian coloring.*

(1) If ϕ is such that $\mathcal{G}_n$ does not have any cycle for all n, then there exists

$K \in [0,\infty]$ and a sequence $(n_k)_k$ such that $n_k = k$ for $0 \geq k \geq K$ and

$$\begin{array}{ll}
\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A,\ \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, & \text{if } 0 \leq n < K, \\
\left.\begin{array}{l}\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B,\ \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B, \text{ or} \\ \mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B,\ \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B,\end{array}\right\} & \text{if } 0 \leq n = K, \\
\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A,\ \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B, & \text{if } n \neq n_k,\ n > K, \\
\left.\begin{array}{l}\mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B,\ \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^B, \text{ or} \\ \mathcal{G}_n^A \cong \mathcal{G}_{n-1}^A,\ \mathcal{G}_n^B \cong \mathcal{G}_{n-1}^A \bowtie \mathcal{G}_{n-1}^B,\end{array}\right\} & \text{if } n = n_k,\ n > K.
\end{array}$$

(2) *If a factor graph $\mathcal{G}_n$ of ϕ has a cycle, for some n, then ϕ is of bounded type. The coloring ϕ is of bounded type if and only if either $\mathcal{G}_n^A$ or $\mathcal{G}_n^B$ eventually stabilizes.*

6.1.3 Linear, intermediate and exponential complexities

In this section, we introduce constructions of colorings with linear, intermediate and exponential factor complexities, following [46].

For a bi-infinite word $\mathbf{w} = (w_n)$, we can induce the natural coloring of a 2-regular tree. Let X be a 2-regular tree. We label VX with v_n, $n \in \mathbb{Z}$. Then the coloring $\phi_{\mathbf{w}} : V\mathcal{T} \to \mathcal{A}$ defined by $v_n \mapsto w_n$ is a natural coloring induced by $\mathbf{w}$.

For a given one-sided sequence $\mathbf{v} = (v_n)_{n\in\mathbb{N}}$, we define a bi-infinite sequence $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ such that

$$w_n := v_n \text{ and } w_{-n+1} := v_n \text{ for } n \in \mathbb{N}.$$

Then $\mathbf{v}$ and $\mathbf{w}$ has the same asymptotic growth type since

$$p_{\mathbf{v}}(n) \leq p_{\mathbf{w}}(n) \leq 2p_{\mathbf{v}}(n) + n - 2.$$

Note that a ball of radius n in X corresponds to a subword of length $2n+1$ of a word, and two distinct subwords u and v of length $2n+1$ can be the same as an n-ball in X if $u = \overline{v}$. Thus, we have

$$\frac{p_{\mathbf{v}}(2n+1)}{2} \leq b_{\phi_{\mathbf{w}}}(n) \leq p_{\mathbf{v}}(2n+1) + n. \tag{6.1}$$

Let ϕ_0 be a coloring of X and let i be an index map of X. Let $\mathcal{T}$ be the universal covering of (X, i) and $\pi : \mathcal{T} \to X$ be the covering map. We can obtain the coloring $\phi = \phi_0 \circ \pi$ of a tree $\mathcal{T}$.

Theorem 6.1.9 ([46], Theorem 3.6). *Let ϕ_0 be a coloring of X. There exists an index map i of X such that*

(1) the universal covering $\mathcal{T}$ of (X, i) is a k-regular tree and

(2) for any n, $b_\phi(n) = b_{\phi_0}(n)$ where $\phi = \phi_0 \circ \pi$.

Rote investigated a class of words with $p(n) = 2n$. The word $\mathbf{w} = (w_n)_{n\in\mathbb{N}}$ is generated by

$$w_n = \begin{cases} 1, & \text{if } (c + n\theta) \bmod 1 \in [0, \lambda), \\ 0, & \text{if } (c + n\theta) \bmod 1 \in [\lambda, 1), \end{cases} \tag{6.2}$$

where $c,\ \lambda \in \mathbb{R}$ and $\theta \in \mathbb{R}\backslash\mathbb{Q}$ such that

$$0 < \lambda < 1,\ 0 < \theta < \min\{\lambda, 1 - \lambda\} \text{ and } m\theta \not\equiv \lambda \pmod 1$$

for any $m \in \mathbb{Z}$ [58]. We define a bi-infinite word $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ by the same process as in (6.2). Then the factor complexity is $p_{\mathbf{w}}(n) = 2n$ and $\mathbf{w}$ is non-eventually periodic. Note that the word $\mathbf{w}$ is reversible. The factor set $F_{2n+1}(\mathbf{w})$ has exactly two palindrome words. Thus, $b_{\phi_{\mathbf{w}}}(n) = 2n + 2$.

Corollary 6.1.10 ([46], Theorem 4.9). *There are colorings of k-regular trees with factor complexity $2n + 2$.*

We say that a function has an *intermediate* growth if it grows faster than any polynomial and slower than any exponential function. One-sided infinite words with intermediate factor complexity have been constructed (see [18], [43]). By (6.1), a coloring of a 2-regular tree induced by a word of intermediate factor complexity has intermediate factor complexity.

Corollary 6.1.11 ([46], Corollary 3.7). *There are colorings of k-regular trees with intermediated factor complexity.*

On the other hand, we define an index map that establishes a coloring with exponential factor complexity.

Theorem 6.1.12 ([46], Theorem 5.1). *Let (X, ϕ_0) be a non-periodic coloring of a 2-regular tree whose 1-balls colored by $[aaa]$ occur only finitely many times*

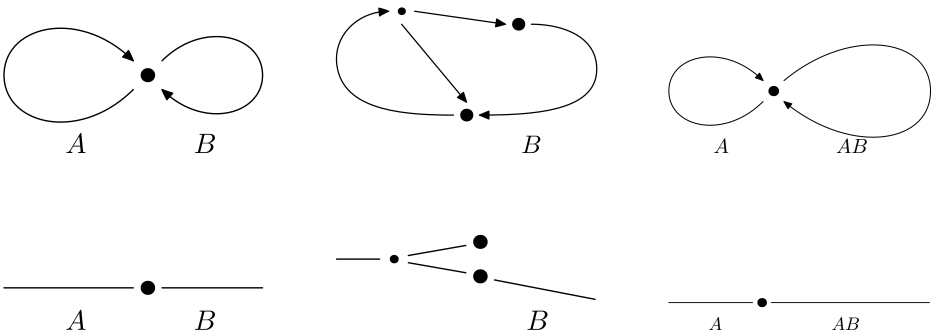


Figure 6.2: The evolution of Rauzy graphs of a quasi-Sturmian word (above) and the evolution of $\mathcal{G}_n$ of a quasi-Sturmian coloring on a tree (below)

for any $a \in \mathcal{A}$. Then, there is an edge index map i of X such that its universal cover $\mathcal{T}$ is a k-regular tree and $b_\phi(n)$ grows exponentially where $\phi = \phi_0 \circ \pi$.

There are many classes of words on which aaa occurs finitely many times for all $a \in \mathcal{A}$ including Sturmian words and the words in (6.2).

Corollary 6.1.13. *There are colorings of k-regular trees with exponential factor complexity.*

6.2 Quasi-Sturmian colorings

Quasi-Sturmian words, which are infinite words with factor complexity eventually $n + c$, share many properties with Sturmian words. In this section, we will study quasi-Sturmian colorings of trees.

Definition 6.2.1. *We say that a coloring is* quasi-Sturmian *if there exists a pair of integers c and N_0 such that $b(n) = n + c$ for $n \geq N_0$, i.e.,*

$$b(n+1) - b(n) = 1 \text{ for each } n \geq N_0. \tag{6.3}$$

We assume that N_0 is the minimal integer satisfying (6.3). Similar to the Sturmian colorings, a quasi-Sturmian coloring has a unique special n-ball for all $n \geq N_0$ which we denote by S_n.

For a coloring of bounded type, we define the subgraph G of X as the graph consisting of the vertices whose lifts are of maximal type less than or equal to N_1 (see the equation (6.4) for the definition).

6.2.1 Quotient graphs of quasi-Sturmian colorings

In this section, we characterize the quotient graphs of quasi-Sturmian colorings.

For $u \in V\mathcal{T}$, $\tau(u) \leq m$ if and only if $[\mathcal{B}_{m+1}(u)] = [\mathcal{B}_{m+1}(v)]$ implies that u and v are in the same class. If two vertices u and v are in the same class, then u and v have the same maximal type. Kim and Lim proved that the converse is also true in the case of a Sturmian coloring (see Proposition 3.2 in [41]). We observe that the same proof holds in quasi-Sturmian colorings as long as $b(n+1) - b(n) = 1$.

Lemma 6.2.2. *Suppose that $b(n)$ is a strictly increasing function. If*

$$b(n+1) - b(n) = 1$$

and two vertices u and v have maximal type n, then u and v are in the same class.

Proof. Suppose that $b(n+1) - b(n) = 1$ and there exist two vertices u and v not in the same class such that $\tau(u) = \tau(v) = n$. Since the alphabet $\mathcal{A}$ is finite, there is a number N such that $\mathcal{B}_N(w)$ contains a special n-ball for each $w \in V\mathcal{T}$.

Fix a vertex w and let z be the center of a special n-ball contained in $\mathcal{B}_N(w)$. Since the special n-ball is unique and it has only two extensions of radius $n+1$, either $[\mathcal{B}_{n+1}(z)] = [\mathcal{B}_{n+1}(u)]$ or $[\mathcal{B}_{n+1}(z)] = [\mathcal{B}_{n+1}(v)]$, thus z is in the same class of u or v. Since $w \in \mathcal{B}_N(z)$, the tree $\mathcal{T}$ is covered by N-balls whose centers are in the same class of u or v. Thus, the maximal types of vertices of $\mathcal{T}$ is bounded by $M = \max\{\tau(p) : p \in \mathcal{B}_N(u) \cup \mathcal{B}_N(v)\}$. It contradicts that $b(n)$ is strictly increasing. □

Corollary 6.2.3. *Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of bounded type with factor complexity $b(n) = n + c$ for $n \geq N_0$. If two vertices u and v of $(\mathcal{T}, \phi)$ have the same maximal type greater than or equal to N_0, then u and v are in the same class.*

Lemma 6.2.4. *If a vertex u of a quasi-Sturmian coloring $(\mathcal{T}, \phi)$ is of maximal type m, then the following hold.*

(1) If $m \geq N_0$, its neighboring vertices are of maximal type $m-1$, m, $m+1$. If $m = N_0 - 1$, its neighboring vertices are of maximal type at most N_0.

If $m \leq N_0 - 2$, its neighboring vertices are of maximal type at most $N_0 - 1$.

(2) *If $m \geq N_0$, one of its neighboring vertices is of maximal type $m+1$.*

(3) *If $m \geq N_0$ is not minimum among maximal types of vertices, one of its neighboring vertices is of maximal type $m-1$.*

Proof. Let $\{u_i\}_{i=1,\cdots,d}$ be the neighboring vertices of u, where d is the degree of T.

(1) Let $\tau = \max\{\tau(u_i)\}_{i=1,\cdots,d}$. Choose u_k such that $\tau(u_k) = \tau$. There is a vertex v such that $[\mathcal{B}_\tau(u_k)] = [\mathcal{B}_\tau(v)]$ but $[\mathcal{B}_{\tau+1}(u_k)] \neq [\mathcal{B}_{\tau+1}(v)]$. Let $f : \mathcal{B}_\tau(u_k) \to \mathcal{B}_\tau(v)$ be a color-preserving isometry. Let $w = f(u)$. Suppose that $\tau > m+1$. Since $\mathcal{B}_{m+1}(u) \subset \mathcal{B}_\tau(u_k)$, $[\mathcal{B}_{m+1}(u)] = [\mathcal{B}_{m+1}(w)]$. Thus, u and w are in the same class. Since $\mathbf{d}(w, v) = 1$, u_j and v are in the same class for some j. We have

$$[\mathcal{B}_\tau(u_j)] = [\mathcal{B}_\tau(v)] = [\mathcal{B}_\tau(u_k)] \text{ and } [\mathcal{B}_{\tau+1}(u_j)] = [\mathcal{B}_{\tau+1}(v)] \neq [\mathcal{B}_{\tau+1}(u_k)],$$

thus $\tau(u_j) \geq \tau$. By the maximality of τ, $\tau(u_j) = \tau$. By Corollary 6.2.3, if $\tau \geq N_0$, then u_k and u_j are in the same class. It contradicts $[\mathcal{B}_{\tau+1}(u_k)] \neq [\mathcal{B}_{\tau+1}(u_j)]$. Hence, $\tau < N_0$.

We conclude that $\tau > m+1$ implies $\tau < N_0$. If $m \geq N_0 - 1$, then $\tau \leq m+1$. If $m < N_0 - 1$, then $\tau \leq N_0 - 1$. In other words, for u, v such that $\mathbf{d}(u,v)=1$, if $|\tau(u) - \tau(v)| \geq 2$, then $\tau(u), \tau(v) \leq N_0 - 1$. Thus if $m \geq N_0$, then $\tau(u_i) \geq m-1$.

(2) Let $m \geq N_0$. Suppose that there is no u_i such that $\tau(u_i) = m+1$. By (1), $m - 1 \leq \tau(u_i) \leq m$ for each i. If $\tau(u_i) = m-1$, then there is no vertices on $\mathcal{B}_1(u_j)$ of maximal type greater than m. Even if $\tau(u_i) = m$, since u and u_i are in the same class by Corollary 6.2.3, we have the same conclusion. Thus, there is no vertex on $\mathcal{B}_2(u)$ of maximal type greater than m. Inductively, every vertex is of maximal type less than $m+1$. It contradicts the fact that $b(n)$ is strictly increasing.

(3) We can show it by a similar argument of the proof of (2). □

For a quasi-Sturmian coloring of bounded type, we define

$$N_1 := \max\{N_0,\ \min\{\tau(x) : x \in V\mathcal{T}\}\}. \tag{6.4}$$

For a coloring of bounded type, we define the subgraph G of X as the graph consisting of the vertices of maximal type less than or equal to N_1. The next proposition follows from Corollary 6.2.3 and Lemma 6.2.4.

Proposition 6.2.5. *For the quotient graph $\mathcal{X} = (X, i)$ of a quasi-Sturmian coloring ϕ of bounded type, the quotient graph X is a union of G and a geodesic ray (see Figure 6.3). The quotient graph X is linear from the vertex of maximal*

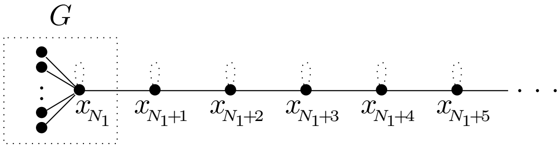
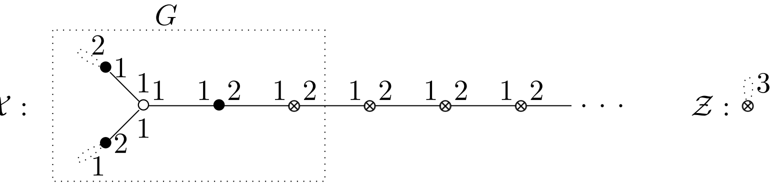


Figure 6.3: Quotient graphs of quasi-Sturmian colorings of bounded type.

type $N_1 + 1$. In the figure, the vertex labeled by x_k is of maximal type k.

In the rest of the section, we provide examples of quasi-Sturmian colorings. The following examples are quasi-Sturmian colorings. The alphabet $\mathcal{A}$ is {$\bullet$, $\circ$, $\otimes$} for the following examples.

Example 6.2.6 (The quotient graph is not a geodesic ray and $N_0 \neq 0$)**.**

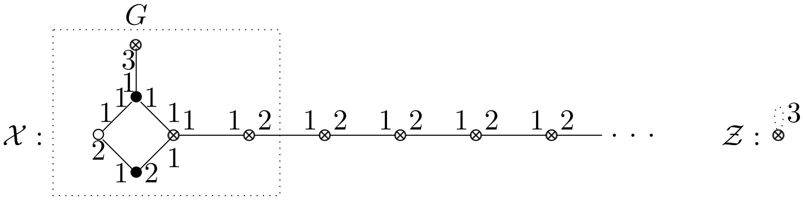


The factor complexity is

$$b(n) = \begin{cases} 3, & \text{if } n = 0 \\ n + 5, & \text{if } n \geq 1 \end{cases}$$

and $N_0 = N_1 = 1$

Example 6.2.7 (A cycle in the compact part G)**.**

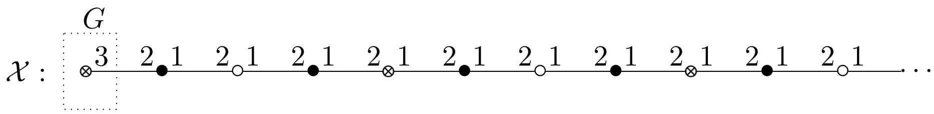


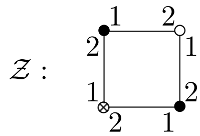
It has the factor complexity

$$b(n) = \begin{cases} 3, & \text{if } n = 0 \\ 5, & \text{if } n = 1 \\ n+5, & \text{if } n \geq 2 \end{cases}$$

and $N_0 = N_1 = 2$.

Example 6.2.8 (an example with $N_0 \neq N_1$).





The factor complexity is

$$b(n) = \begin{cases} 1, & \text{if } n = -1 \\ n+3, & \text{if } n \geq 0 \end{cases}$$

and $N_0 = 0,\ N_1 = 1$.

The quotient graph of a Sturmian coloring of unbounded type is a geodesic ray or an infinite geodesic (see Theorem 6.1.7). In this section, we show that a similar property holds for quasi-Sturmian colorings of unbounded type.

Proposition 6.2.9. *For a quasi-Sturmian coloring of unbounded type, the vertices of a* 1*-ball have at most three distinct type sets.*

Proof. Let us assume that there are three neighboring vertices u_1, u_2, u_3 of u such that the type sets of u, u_1, u_2, u_3 are all distinct. Since each special n-ball is unique for $n \geq N_0$, if there is $n \in \Lambda_u \cap \Lambda_v$ such that $n \geq N_0$, then $[\mathcal{B}_n(u)] = [\mathcal{B}_n(v)]$. Thus, if $\Lambda_u \cap \Lambda_v$ is infinite, then $\Lambda_u = \Lambda_v$. Let $N = \max \Lambda_u \cap \Lambda_v$. Note that $\Lambda_u \cap \Lambda_v$ is non-empty since every type set contains -1. Choose such N for each pair of vertices from different classes in $\mathcal{B}_2(u)$ and let M be the maximum of such N's. Then, the type sets of two non-equivalent vertices in $\mathcal{B}_2(u)$ intersected with $\{M+1, M+2, \cdots\}$ are all mutually disjoint.

Now let $l > M+1$ be in the type set Λ_u. Such l exists since the coloring is of unbounded type. At least one of u_1, u_2, u_3 has a type set disjoint from $\{l-1, l, l+1\}$, say u_i. Since $l \in \Lambda_u$, there is v such that $[\mathcal{B}_l(u)] = [\mathcal{B}_l(v)]$ but $[\mathcal{B}_{l+1}(u)] \neq [\mathcal{B}_{l+1}(v)]$. Let $f : \mathcal{B}_l(u) \to \mathcal{B}_l(v)$ be a color-preserving isometry. Then $[\mathcal{B}_{l-1}(u_i)] = [\mathcal{B}_{l-1}(f(u_i))]$.

Let $p = \min\{k \geq l-1 : k \in \Lambda_{u_i}\}$. Since $p > l+1$, $[\mathcal{B}_{l-1}(u_i)]$ has a unique extension to $[\mathcal{B}_p(u_i)]$. Thus, $[\mathcal{B}_p(u_i)]$ and $[\mathcal{B}_p(f(u_i))]$ are equivalent by a color-preserving isometry g. Since $[\mathcal{B}_{p-1}(g^{-1}(v))] = [\mathcal{B}_{p-1}(v)]$ and $p - 1 > l$, $[\mathcal{B}_l(g^{-1}(v))] = [\mathcal{B}_l(v)] = [\mathcal{B}_l(u)]$ and $[\mathcal{B}_{l+1}(g^{-1}(v))] = [\mathcal{B}_{l+1}(v)] \neq [\mathcal{B}_{l+1}(u)]$. Thus, $g^{-1}(v) \neq u$ and $\Lambda_{g^{-1}(v)} \cap \Lambda_u$ contains $l > M+1$. However, since $\mathbf{d}(g^{-1}(v), u) \leq 2$, it contradicts that $\Lambda_{g^{-1}(v)} \cap \Lambda_u \cap \{M+1, M+2, \cdots\}$ is empty. □

Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of a tree and $\mathcal{X} = (X, i)$ be its quotient graph. If two vertices u, v have the same type set, they have the same colored n-balls for every n, i.e. u, v are equivalent (see Lemma 2.4 in [41]). By Proposition 6.2.9, there are at most 2 adjacent vertices of each vertex $x \in VX$.

For a quasi-Sturmian coloring of unbounded type, we define G as the set of vertices that have only one adjacent vertex in X. Since factor complexity of ϕ is unbounded, X is an infinite graph. Since X is connected, G is empty or G has a single element. Thus, we obtain the quotient graphs of quasi-Sturmian colorings of trees.

Theorem 6.2.10. *If ϕ is a quasi-Sturmian coloring, then its quotient graph is one of graphs in Figure 6.4. More precisely, the quotient graph of a coloring*

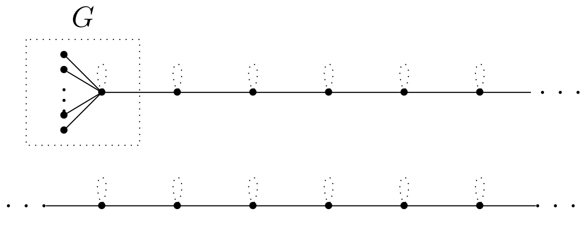


Figure 6.4: Quotient graphs of quasi-Sturmian colorings

of bounded type is the first graph, where the quotient graph of a coloring of unbounded type is a geodesic ray or a biinfinite geodesic.

6.2.2 Evolution of factor graphs

In this section, we look into quasi-Sturmian colorings of unbounded type in detail. Let us begin by explaining an induction algorithm for quasi-Sturmian colorings of bounded type. For $n \geq N_0$, S_n denotes a unique special n-ball, C_n denotes a centered n-ball of S_{n+1}, and A_{n+1}, B_{n+1} denote two types of extensions of S_n. For a class of n-balls $B = [\mathcal{B}_n(x)]$, denote the class of $[\mathcal{B}_{n+1}(x)]$ by $\overline{B}$ and the class of $[\mathcal{B}_{n-1}(x)]$ by $\underline{B}$. Note that if B is not special, then $\overline{B}$ is well-defined.

Recall from the introduction that for a given quasi-Sturmian coloring ϕ, for $n \geq N_0 + 1$, the *factor graph* $\mathcal{G}_n$ has $\mathbb{B}_\phi(n)$ as its vertex set. There is an edge between two colored n-balls D, E if there exist n-balls centered at x, y in the classes D, E, respectively, such that $\mathbf{d}(x, y)$=1.

Cyclic quasi-Sturmian colorings

We gather preliminaries of cyclic quasi-Sturmian colorings.

Definition 6.2.11. *We say that D is* weakly adjacent *to E if there exist $v, w \in V\mathcal{T}$ such that $\mathbf{d}(v, w) = 1$ and $[\mathcal{B}_n(v)] = D$ and $[\mathcal{B}_m(w)] = E$ for some n, m.*

We also say that D is strongly adjacent *to E if for any $\mathcal{B}_n(x)$ in the class D, there exists a vertex y such that $\mathcal{B}_m(y) \in E$ and $\mathbf{d}(x, y) = 1$. If D is strongly adjacent to E and vice versa, then we say that D and E are strongly adjacent.*

We remark the following fact. If $[\mathcal{B}_{n+1}(u)] = [\mathcal{B}_{n+1}(v)]$ and $[\mathcal{B}_{n+2}(u)] \neq [\mathcal{B}_{n+2}(v)]$, then there exist neighboring vertices u' and v' of u and v, respectively, such that $[\mathcal{B}_n(u')] = [\mathcal{B}_n(v')]$ and $[\mathcal{B}_{n+1}(u')] \neq [\mathcal{B}_{n+1}(v')]$ (see Lemma 2.11 in [41] for details). Thus, S_{n+1} is strongly adjacent to S_n for $n \geq N_0$.

Lemma 6.2.12. *Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring and $n \geq N_0$.*

(1) We can choose $\{A_n\}_{n \geq N_0+1}$, $\{B_n\}_{n \geq N_0+1}$ so that A_{n+1}, B_{n+1} are strongly adjacent to A_n, B_n, respectively. Moreover, A_{n+1}, B_{n+1} are uniquely determined if we give the condition that A_{n+1} contains more balls of the class A_n than B_{n+1} does.

(2) For each vertex x in $\mathcal{T} - \widetilde{G}$ and $n \geq N_0 + 1$, the n-balls with centers adjacent to x belong to at most two classes of n-balls apart from $[\mathcal{B}_n(x)]$.

Thus, for any class $D \neq S_n$ of n-balls with centers in $\mathcal{T} - \widetilde{G}$, each vertex of $\mathcal{G}_n$ has degree at most 2.

(3) *If $A_n \neq S_n$ (respectively $B_n \neq S_n$), then A_n (respectively B_n) is strongly adjacent to S_n.*

(4) *The two classes S_n, C_n are strongly adjacent.*

We will specify the choice of A_{N_0+1} from the two extensions of S_{N_0} for acyclic quasi-Sturmian colorings later.

Lemma 6.2.13. *Let ϕ be a quasi-Sturmian coloring and n be greater than N_0. Let D be a colored n-ball other than A_n, B_n and S_n. Assume that S_n and D are weakly adjacent. Then, we have that*

(1) *the special ball S_n and D are strongly adjacent, and*

(2) *if $D \neq C_n$, then $S_n \neq C_n$.*

Proposition 6.2.14. *If there are two vertices of degree at least three in $\mathcal{G}_n$ for some $n > N_0$, then the quasi-Sturmian coloring $(\mathcal{T}, \phi)$ is of bounded type.*

Proof. If ϕ is of unbounded type, S_n is the unique vertex adjacent to the distinct three classes of n-balls in $\mathcal{G}_n$ by Lemma 6.2.12 (2). Thus, there is at most one vertex of degree at least three in $\mathcal{G}_n$. □

Definition 6.2.15. *A quasi-Sturmian coloring is* cyclic *if there is a cycle containing S_n in $\mathcal{G}_n$ for some $n > N_0$. If not, we say that a quasi-Sturmian coloring is* acyclic.

Lemma 6.2.16. *Suppose that $\mathcal{G}_n$ has a cycle whose lift in X is not contained in G for some $n \geq N_0 + 1$. The following statements hold.*

(1) *The special ball S_n is in the cycle.*

(2) *If $D \neq A_n, B_n, C_n, S_n$, then D is not weakly adjacent to S_n.*

Lemma 6.2.17. *For $n > N_0$, suppose that $\mathcal{G}_n$ has a cycle whose lift in X is not contained in G.*

(1) *If C_n is not contained in the cycle, then $\mathcal{G}_{n+l}$ has a cycle containing C_{n+l} for some $l \geq 1$.*

(2) *If $C_n = S_n$, then $\mathcal{G}_{n+1}$ has a cycle containing C_{n+1} and $C_{n+1} \neq S_{n+1}$.*

Proposition 6.2.18. *(1) Let $n \geq N_0 + 1$. If there is a ball D which is weakly adjacent to S_n and different from A_n, B_n, C_n, and S_n, then $\mathcal{G}_{n+1}$ has a cycle containing $\overline{D}$.*

(2) Any cyclic quasi-Sturmian coloring is of bounded type.

Acyclic quasi-Sturmian colorings

Lemma 6.2.19. *Let ϕ be an acyclic quasi-Sturmian coloring. If $A_N = S_N = C_N$ for some $N > N_0 + 1$, then $A_n = S_n = C_n$ for all $N_0 + 1 \leq n < N$.*

We choose A_n as $S_n = C_n = A_n$ if there exists $n > N_0$ such that $S_n = C_n$ is identical to A_n or B_n. Define

$$K = \min\{n > N_0 : A_n, S_n, C_n \text{ are not all identical}\}$$

as in [42]. Note that K may be infinity.

For an acyclic quasi-Sturmian coloring, for each $n \geq K$, neither A_n, S_n, C_n nor B_n, S_n, C_n are identical. Therefore, the colored n-balls S_n, A_n, B_n, C_n satisfy one of the following conditions.

(I) S_n, C_n are distinct, but one of S_n, C_n is identical to A_n or B_n.

(II) S_n, A_n, B_n, C_n are all distinct.

(III) S_n, A_n, B_n are distinct, but $S_n = C_n$.

Case (I) is divided into three subcases:

(I-a) A_n, B_n, S_n are distinct and $C_n = A_n$ or B_n,

(I-b) A_n, B_n, C_n are distinct and $S_n = A_n$ or B_n,

(I-c) $A_n = S_n$, $B_n = C_n$ are distinct,

By Lemma 6.2.16 and Lemma 6.2.18, we deduce that S_n is a vertex of degree 3 in $\mathcal{G}_n$ for Case (II), but for Case (I) and (III), $\mathcal{G}_n$ is a linear graph and S_n is of degree 1 or 2.

Theorem 6.2.20. *Suppose that $\mathcal{G}_n$ corresponds to Case (I). Then S_n is a vertex of degree 2 or 1 in $\mathcal{G}_n$. Thus $\mathcal{G}_n$ is a linear graph. Let m be the number of vertices connected to S_n through C_n. Note that $m \geq 1$ since C_n is not identical to S_n. Then we have $\mathcal{G}_{n+k}$ belongs to Case (II) for all $0 < k < m$ and either $\mathcal{G}_{n+m}$ belongs to Case (I) or $\mathcal{G}_{n+m}$ belongs to Case (III) and $\mathcal{G}_{n+m+1}$ belongs to Case (I).*

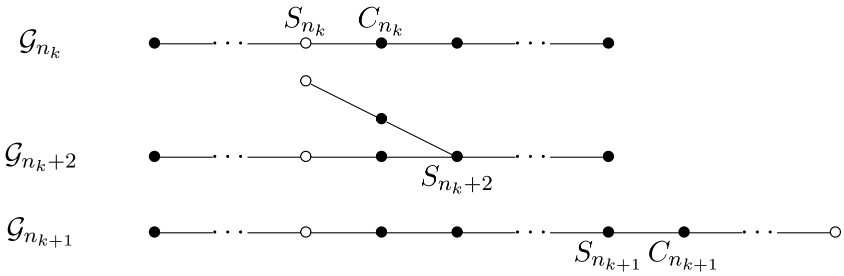


Figure 6.5: The evolution of $\mathcal{G}_{n_k}$ along the path (I) $\to$ (II) $\to \cdots \to$ (II) $\to$ (I) where the vertex $\circ$ represents either S_{n_k} or the extensions of S_{n_k}

Proof. If S_n and C_n are distinct, then $\mathcal{G}_n$ belongs to Case (I) or (II). We deduce that S_{n+1}, A_{n+1}, B_{n+1} are distinct. If C_n is of degree 2, then there exists D neighboring C_n which is not S_n. Thus $\overline{D}$ is weakly adjacent to S_{n+1} but different from $S_{n+1}, A_{n+1}, B_{n+1}$, which implies that $\overline{D} = C_{n+1}$, which corresponds Case (II). In this case, the number of vertices connected to S_{n+1} through C_{n+1} decreases by 1.

If C_n is of degree 1, then $m = 1$. In this case, S_{n+1} is connected to only two extensions A_{n+1}, B_{n+1} of S_n in $\mathcal{G}_{n+1}$, which implies that $C_{n+1} = S_{n+1}$, i.e. Case (III) or $C_{n+1} = A_{n+1}$ or B_{n+1}, i.e. Case (I-a).

If $\mathcal{G}_n$ belongs to Case (III), then $S_n = C_n$, thus we have either $S_{n+1} = A_{n+1}$ or $S_{n+1} = B_{n+1}$, say $S_{n+1} = A_{n+1}$. Since $\overline{A_n}$ is weakly adjacent to $A_{n+1} = S_{n+1}$ and $\overline{A_n}$ cannot be A_{n+1} nor B_{n+1}, we deduce that $C_{n+1} = \overline{A_n}$. Therefore, $\mathcal{G}_{n+1}$ belongs to the Case (I-b).

We remark that Case (I-c) can happen only for $n = K$. □

We denote by (n_k) the subsequence for which $\mathcal{G}_{n_k}$ is of Case (I). The evolution of $\mathcal{G}_n$ from $n = n_k$ to $n = n_{k+1}$ is shown in Figure 6.5. Compare with Sturmian words (see Figure 6.2): there are infinitely many n's such that the Rauzy graph has disjoint two cycles starting from a common bi-special word (see e.g. [1]). It corresponds to the factor graph $\mathcal{G}_n$ belongs to Case (I).

6.2.3 Quasi-Sturmian colorings of bounded type

In this section, we investigate a necessary and sufficient condition for a quotient graph to be a quotient graph of a quasi-Sturmian coloring of bounded type.

Let x be a vertex of the quotient graph X. For the two lifts $\tilde{x}$ and $\tilde{x}'$ of x, $[\mathcal{B}_n(\tilde{x})] = [\mathcal{B}_n(\tilde{x}')]$ for all n. Then, $\tau(\tilde{x}) = \tau(\tilde{x}')$. By abuse of notation, define $[\mathcal{B}_n(x)]$ as a class $[\mathcal{B}_n(\tilde{x})]$. Define the maximal type $\tau(x)$ of x as $\tau(\tilde{x})$.

Recall the examples in Section 6.2.1. Let $\mathcal{X} = (X, i)$ be the quotient graph for each of them. We obtain a periodic edge-indexed subgraph X' of X by removing a finite subgraph G in Proposition 6.2.5. Then, a lift of $(X', i|_{EX'})$ can be extended to a periodic coloring of a tree. It is natural to guess that the property holds for every quasi-Sturmian coloring.

From now on, let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of bounded type. By Proposition 6.2.5, the quotient graph X of $(\mathcal{T}, \phi)$ is the graph in Figure 6.3. Let $\widetilde{G}$ be the union of lifts of G. A connected component of $\mathcal{T} - \widetilde{G}$ is a lift of $(X - G, i|_{E(X-G)})$. Thus, all connected components of $\mathcal{T} - \widetilde{G}$ are equivalent to each other. Let Y be a connected component of $\mathcal{T} - \widetilde{G}$.

Lemma 6.2.21. *If u, v are vertices of Y with $[\mathcal{B}_{N_1}(u)] = [\mathcal{B}_{N_1}(v)]$, where N_1 is as in (6.4), then we have $[\mathcal{B}_{N_1+1}(u)] = [\mathcal{B}_{N_1+1}(v)]$.*

Proof. It suffices to consider the case of $[\mathcal{B}_{N_1}(u)] = S_{N_1}$. Every vertex of maximal type N_1 is the center of either A_{N_1+1} or B_{N_1+1}, say A_{N_1+1}. Since vertices of $X - G$ are of maximal type bigger than N_1, if u is a vertex of Y and $[\mathcal{B}_{N_1}(u)] = S_{N_1}$, then $[\mathcal{B}_{N_1+1}(u)] = B_{N_1+1}$. □

We define an edge-indexed graph $\mathcal{Z} = (Z, i_Z)$ as follows: the vertices of Z are of the form $[\mathcal{B}_{N_1}(u)]$ for a vertex u in Y or $X - G$, and any two vertices D, E of Z are adjacent if D and E are weakly adjacent. The index $i_Z(D, E)$ is the number of E which are adjacent to D. The indices are well-defined by Lemma 6.2.21. Since any vertex in $X - G$ is adjacent to at most two vertices besides itself, the graph Z is a line segment or a cycle.

Lemma 6.2.22. *A restriction of ϕ on any connected component of $\mathcal{T} - \widetilde{G}$ has a periodic extension to $\mathcal{T}$.*

Proof. Let u be the vertex of Y. Define a coloring ψ_k on $\mathcal{B}_k(u)$ with the alphabet $VZ = \{[\mathcal{B}_{N_1}(v)] \,|\, v \in Y\}$ recursively: Put $\psi_0(u) = [\mathcal{B}_{N_1}(u)] \in VZ$. Define $\psi_{k+1}(v) = \psi_k(v)$ for $v \in \mathcal{B}_k(u)$. Choose $w \in V\mathcal{T}$ with $\mathbf{d}(u, w) = k$ and let w_α $(\alpha = 0, \cdots d-1)$ be the neighboring vertices of w with $\mathbf{d}(u, w_\alpha) = k+1$ for $\alpha \geq 1$ and $\mathbf{d}(u, w_0) = k-1$. We define $\psi_{k+1}(w_\alpha)$ for $\alpha \geq 1$ in the following ways.

If $w \notin Y$, then $w_\alpha \notin Y$ for all $\alpha \geq 1$. Let $D_0 = \psi_k(w_0)$ and D_j be a colored N_1-ball satisfying $i_Z(\psi_k(w), D_j) > 0$ with $j = 0, 1, 2$ or $j = 0, 1$. We assign

$\psi_{k+1}(w_\alpha)$ as D_0 for $1 \leq \alpha < i_Z(\psi_k(w), D_0)$ and, for $\ell \neq 0$,

$$\psi_{k+1}(w_\alpha) = D_\ell \text{ for } \sum_{j=0}^{\ell-1} i_Z(\psi_k(w), D_j) \leq \alpha \leq \sum_{j=0}^{\ell} i_Z(\psi_k(w), D_j) - 1.$$

Then we have

$$i_Z(\psi_{k+1}(w), D) = \#\{0 \leq \alpha \leq d \mid \psi_{k+1}(w_\alpha) = D\} \tag{6.5}$$

for each $D \in VZ$.

If $w \in Y$, then we put $\psi_{k+1}(w_\alpha) = [\mathcal{B}_{N_1}(w_\alpha)]$ for all $\alpha \geq 1$. Using the fact that Y is an infinite subgraph of T, Lemma 6.2.21 implies that there exists a vertex v such that $\mathcal{B}_{N_1+1}(v) \subset Y$ and $[\mathcal{B}_{N_1+1}(v)] = [\mathcal{B}_{N_1+1}(w)]$, thus $\psi_{k+1}(w_\alpha) = [\mathcal{B}_{N_1}(w_\alpha)] \in VZ$ and (6.5) is satisfied.

Since $\psi_{k+\ell}|_{\mathcal{B}_k(u)} = \psi_k$ for $\ell \geq 1$, the coloring $\psi = \lim_{k\to\infty} \psi_k$ on $\mathcal{T}$ with alphabet VZ exists. By (6.5), we deduce that $\mathcal{Z}$ is the quotient graph of ψ. Since $\psi(u) = [\mathcal{B}_{N_1}(u)]$ on Y, by the coloring which gives the color of the center of $\psi(u)$, we complete the proof. □

Theorem 6.2.23. *Let $\mathcal{X} = (X, i)$ be the quotient graph of a coloring $(\mathcal{T}, \phi)$. The following statements are equivalent.*

(1) The coloring ϕ is a quasi-Sturmian coloring of bounded type.

(2) There is a finite connected subgraph G of the quotient graph X such that $X - G$ is a connected infinite ray and any connected component of $\mathcal{T} - \widetilde{G}$ has a periodic extension to $\mathcal{T}$ where $\widetilde{G}$ is the union of lifts of G.

Proof. By Lemma 6.2.21 and Lemma 6.2.22, (1) implies (2). Now we assume (2) holds. Let $\mathcal{A}$ be the alphabet of ϕ. Let $\tilde{x}$ be a lift of $x \in VX$. Define a new coloring ψ with an alphabet $\mathcal{A} \sqcup VG$ as

$$\psi(v) = \begin{cases} x & \text{if } v = \tilde{x} \text{ for some } x \in VG, \\ \phi(v) & \text{otherwise.} \end{cases}$$

Denote by $[\mathcal{B}_n(u)]_\psi$ a ψ-colored n-ball. As ever $[\mathcal{B}_n(u)]$ means a ϕ-colored n-ball. A map $\mathbb{B}_\psi(n) \to \mathbb{B}_\phi(n)$ which defined by $[\mathcal{B}_n(x)]_\psi \mapsto [\mathcal{B}_n(x)]$ is surjective. It implies $b_\phi(n) \leq b_\psi(n)$. Since X is not a finite graph, $b_\phi(n)$ is strictly increasing. Thus, it is enough to show that b_ψ is linear.

Let us denote by $\mathbf{d}(x, G) = \min\{\mathbf{d}(x, g) : g \in VG\}$ for $x \in VX$. Fix a positive integer n. If $\mathbf{d}(x, G) \leq n$, then $[\mathcal{B}_n(x)]_\psi \neq [\mathcal{B}_n(y)]_\psi$ for any other $y \in VX$. If x is a vertex such that $\mathbf{d}(x, G) > n + 1$, then $[\mathcal{B}_{n+1}(x)]_\psi = [\mathcal{B}_{n+1}(x)]$. Thus, $[\mathcal{B}_n(x)]$ has a unique extension to a colored $(n + 1)$-ball. Since X is not finite, ψ has at least one special n-ball for each n. Thus, for x such that $\mathbf{d}(x, G) = n + 1$, $[\mathcal{B}_n(x)]$ is the unique special n-ball and it has exactly two extensions to colored $(n + 1)$-balls. It means that $b_\psi(n) = n + |\mathcal{A}| + |VG|$ for all n. □

6.2.4 Recurrence functions of colorings of trees

In this section, we will extend the notion of recurrence functions $R(n), R''(n)$ for words to colorings of trees. We will show that the quasi-Sturmian colorings of trees satisfy a certain inequality between $R''(n)$ and $b(n)$. We also explain that the existence of $R(n)$ is related to the unboundedness of the quasi-Sturmian colorings of trees.

Let us briefly recall recurrence functions of words (see Section 10.9 in [3] for definitions and details). Recurrence functions are important objects related to symbolic dynamics. We recall that $\mathcal{A}^*$ be the set of finite words and $\mathcal{A}^{\mathbb{N}}$ be the set of infinite words over $\mathcal{A}$. For $\mathbf{u} \in \mathcal{A}^* \cup \mathcal{A}^{\mathbb{N}}$, we denote by $F_n(\mathbf{u})$ the set of factors of length n.

A recurrence function $R_{\mathbf{u}}(n)$ is defined as the smallest integer $m \geq 1$ such that every factor of length m contains all factors of length n. It is known that such an integer $R_{\mathbf{u}}(n)$ exists for all n if and only if the word is *uniformly recurrent*, i.e. any subword of the word infinitely occurs with bounded gaps. Another recurrence function $R''_{\mathbf{u}}(n)$ is defined by

$$R''_{\mathbf{u}}(n) = \min\{m \in \mathbb{N} \mid F_n(\mathbf{u}) = F_n(\mathbf{v}) \text{ for some } \mathbf{v} \in F_m(\mathbf{u})\},$$

i.e. it is the length of the smallest factor of $\mathbf{u}$ that contains all factors of length n of $\mathbf{u}$. From the definition, the following fact immediately holds.

Remark 6.2.24. *For all $n \geq 0$, $R''_{\mathbf{u}}(n) \geq p_{\mathbf{u}}(n) + n - 1$ for any word $\mathbf{u}$.*

Recall that a word $\mathbf{u}$ is said to have *grouped factors* if, for all $n \geq 0$, it satisfies $R''_{\mathbf{u}}(n) = p_{\mathbf{u}}(n) + n - 1$. If there is n_0 such that the equality holds for all $n \geq n_0$, we say that $\mathbf{u}$ has *ultimately grouped factors*. Cassaigne suggested some conditions that guarantee equality.

Theorem 6.2.25 ([17]). *A word* $\mathbf{u}$ *is Sturmian if and only if* $R''_{\mathbf{u}}(n) = 2n$ *for every* $n \geq 0$. *A uniformly recurrent word on a binary alphabet has ultimately grouped factors if and only if it is periodic or quasi-Sturmian.*

We want analogous results for quasi-Sturmian colorings of trees. Let $(\mathcal{T}, \phi)$ be a quasi-Sturmian coloring of a tree and $\mathcal{X} = (X, i)$ be the quotient graph of $(\mathcal{T}, \phi)$. We define $R_\phi(n)$ as the smallest radius m such that every colored n-ball of ϕ occurs in $[\mathcal{B}_m(x)]$ for all $x \in V\mathcal{T}$. We define $R''_\phi(n)$ as the smallest radius m such that every colored n-ball of ϕ occurs in $[\mathcal{B}_m(x)]$ for some $x \in V\mathcal{T}$.

Definition 6.2.26. *A coloring of a tree* $(\mathcal{T}, \phi)$ *is said to be* recurrent *if, for any compact subtree* $\mathcal{T}'$, *every colored ball appears in* $\mathcal{T} - \mathcal{T}'$. *A coloring of a tree is said to be* uniformly recurrent *if* $R_\phi(n) < \infty$ *for all* n.

Proposition 6.2.27. *Let* $(\mathcal{T}, \phi)$ *be a quasi-Sturmian coloring of a tree. The following conditions are equivalent.*

(1) $(\mathcal{T}, \phi)$ *is of unbounded type.*

(2) $(\mathcal{T}, \phi)$ *is uniformly recurrent.*

(3) For any colored ball, it appears in $\mathcal{T} - \pi^{-1}(S)$ *for any finite set* $S \subset X$.

Proof. (1) implies (2) : Suppose $(\mathcal{T}, \phi)$ is of unbounded type. Let $n \geq N_0$. For each colored n-ball $E = [\mathcal{B}_n(w)]$, we define m_E to be the smallest element of $\Lambda_w \cap \{n, n+1, \cdots\}$ which is not empty since Λ_w is infinite. Note that m_E depends only on E and not on w.

Choose a vertex $v \in V\mathcal{T}$ and a colored n-ball E which is distinct from $[\mathcal{B}_n(v)]$. Let $m = m_E$. Denote $F^1 = [\mathcal{B}_m(v)]$ which is not S_m. Let $[F^1 - F^2 - \cdots - F^l - S_m]$ be the shortest path from F^1 to S_m in $\mathcal{G}_m$. For arbitrary colored m-balls F and F', if $F \neq S_m$, then F has the unique extension. Thus, if F is weakly adjacent to F', then F is strongly adjacent to F'. Therefore, there is a path $[v - v_2 - v_3 - \cdots - v_l - w']$ in $\mathcal{T}$ such that $[\mathcal{B}_m(v_i)] = F^i$, $i = 2, \cdots, l$, and $[\mathcal{B}_m(w')] = S_m$.

Since S_m occurs in $[\mathcal{B}_{m+l}(v)]$, E occurs in $[\mathcal{B}_{n+l}(v)]$. Since $l \leq |V\mathcal{G}_m| = m + c$, E occurs in $[\mathcal{B}_{n+m+c}(v)]$. Every colored n-ball occurs in $[\mathcal{B}_{n+M+c}(v)]$ where $M = \max\{m_E : E \in \mathbb{B}_\phi(n)\}$. Thus, $R_\phi(n) \leq n + M + c$.

(2) implies (3) : Suppose that $R_\phi(n)$ exists for all n. Since the quotient graph X is infinite, for any finite $S \subset X$, there is x such that $\mathcal{B}_{R_\phi(n)}(x) \subset \mathcal{T} - \pi^{-1}(S)$.

(3) implies (1) : Assume that $(\mathcal{T},\phi)$ is of bounded type. Let v be a vertex of maximal type N_1. By Proposition 6.2.5, all vertices in $X-G$ are of maximal type larger than N_1. Therefore, $[\mathcal{B}_{N_1+1}(v)]$ does not appear in $\mathcal{T}-\pi^{-1}(G)$. $\square$

Recall that we denote by $\mathcal{Z}$ the quotient graph of $\mathcal{T}-\widetilde{G}$ with respect to the coloring ϕ. By abuse of notation, let $\mathbf{d}$ be the metric on X or $\mathcal{G}_n$ induced by $\mathbf{d}$ on T. Let us denote by

$$\mathbf{r}(x,G) := \max\{\mathbf{d}(x,y) : y \in VG\}.$$

Proposition 6.2.28. *Let $(\mathcal{T},\phi)$ be a quasi-Sturmian coloring.*

(1) Let ϕ be of unbounded type. As in Theorem 6.2.20, the factor graph $\mathcal{G}_n$ is of Case (I) on $n=n_k$. Then, we have

$$R''_\phi(n) = n + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor \quad \text{for } \ n_{k-1} < n \le n_k.$$

(2) Let ϕ be of bounded type. Let x_{N_1} be the vertex of X which is of maximal type N_1.

(a) If Z is acyclic, then we have

$$R''_\phi(n) = n+\left\lfloor \frac{1}{2}(b_\phi(n_k)-|VG|+\mathbf{r}(x_{N_1},G)+1)\right\rfloor \quad \text{for } \ n_{k-1} < n \le n_k.$$

(b) If Z is cyclic, then we have

$$R''_\phi(n) = n + \left\lfloor \frac{1}{2}(b_\phi(n) - |VG| + \mathbf{r}(x_{N_1},G)+1)\right\rfloor \quad \text{for all } \ n \ge N_1.$$

Proof. (1) In the case of a quasi-Sturmian coloring of unbounded type, the evolution of the factor graph follows Theorem 6.2.20. Let D and E be n_k-balls that are weakly adjacent. If $D \neq S_{n_k}$ or if $D = S_{n_k}$, $E = C_{n_k}$, then D and E are strongly adjacent by Lemma 6.2.12 (3), (4). If $D = S_{n_k}$ and $E \neq C_{n_k}$, then there exist vertices v, u and w in T with $\mathbf{d}(v,u)=\mathbf{d}(v,w)=1$ such that $D=[\mathcal{B}_{n_k}(v)]$, $E=[\mathcal{B}_{n_k}(u)]$ and $C_{n_k}=[\mathcal{B}_{n_k}(w)]$. Therefore, we can take a path with length $b_\phi(n_k)-1$ consisting of centers of all the colored n_k-balls in $\mathcal{T}$. Thus, we have

$$R''_\phi(n_k) \le n_k + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor.$$

Let D_{n_k}, E_{n_k} be the colored n_k-balls which are the endpoints of the graph $\mathcal{G}_{n_k}$. The distance between D_{n_k} and E_{n_k} in $\mathcal{G}_{n_k}$ is $b_\phi(n_k)-1$, thus for any vertices z, z' in T such that $[\mathcal{B}_{n_k}(z)] = D_{n_k}$ and $[\mathcal{B}_{n_k}(z')] = E_{n_k}$, we have $\mathbf{d}(z,z') \geq b_\phi(n_k)-1$. Therefore, it follows that

$$R''_\phi(n_k) = n_k + \Big\lfloor \frac{b_\phi(n_k)}{2} \Big\rfloor.$$

Now, let us consider the case $n_{k-1} < n < n_k$, then $\mathcal{G}_n$ is of Case (II) or Case (III). We define D_n, E_n and F_n as the colored n-balls which are the vertices of degree 1 and connected to S_n through A_n, B_n, C_n in $\mathcal{G}_n$, respectively. Note that if $S_n = C_n$, then we define $F_n = C_n$. Any vertex of the center of special ball S_n in T is adjacent to either centers of A_n and C_n or centers of B_n and C_n. Thus, the distance between the centers of D_n and E_n in T is at least $\mathbf{d}(D_n, F_n) + \mathbf{d}(E_n, F_n)$.

If $\mathcal{G}_n$ is of Case (II) for all $n_{k-1} < n < n_k$, then $\mathbf{d}(D_n, F_n) + \mathbf{d}(E_n, F_n) = b_\phi(n_k) - 1$. Otherwise, $\mathcal{G}_n$ is of Case (III) for $n = n_k - 1$ and $\mathcal{G}_n$ is of Case (II) for $n_{k-1} < n < n_k - 1$. Then, $\mathbf{d}(D_n, F_n) + \mathbf{d}(E_n, F_n) = b_\phi(n_k - 1) - 1$. However, on T, a path from a center of D_n to a center of E_n has at least two vertices which are centers of F_n where they are extended to two distinct colored n_k-balls C_{n_k} and S_{n_k}. It means that the length of the path is at least $b_\phi(n_k-1)-1+1 = b_\phi(n_k)-1$. Thus,

$$R''_\phi(n) \geq n + \Big\lfloor \frac{b_\phi(n_k)}{2} \Big\rfloor \quad \text{for } n_{k-1} < n < n_k.$$

On the other hand, since each n-ball is the restriction of an n_k-ball, there exists a path with length $b_\phi(n_k)-1$ consisting of centers of all the colored n_k-balls in $\mathcal{T}$. Thus we have a conclusion.

(2)-(a) If Z is acyclic and $n \geq N_1$, then the evolution of the factor graph $\mathcal{G}_n$ also follows Theorem 6.2.20. Hence, we apply the argument similar to the argument in (1). The difference between (1) and (2)-(a) is the existence of the compact part G of the quotient graph X. Take a finite graph G' in $\mathcal{G}_{n_k}$ isomorphic to G. Since every vertex in $\mathcal{G}_{n_k} - G'$ has at most degree 2, the maximal distance between any two vertices in $\mathcal{G}_{n_k}$ is $b_\phi(n_k) - |VG| + \mathbf{r}(x_{N_1}, G)$. Thus, by the similar argument with (1), we have for $n_{k-1} < n \leq n_k$

$$R''_\phi(n) = n + \Big\lfloor \frac{1}{2}(b_\phi(n_k) - |VG| + \mathbf{r}(x_{N_1}, G) + 1) \Big\rfloor.$$

(2)-(b) Let Z be cyclic and assume that $n \geq N_1$. Let G' be the subgraph of $\mathcal{G}_n$, which is isomorphic to G. Then $\mathcal{G}_n - G'$ consists of a cyclic graph isomorphic to Z and a finite linear graph with a common vertex S_n which is the unique vertex of degree 3 in $\mathcal{G}_n - G'$. We may assume that A_n belongs to the cycle in $\mathcal{G}_n - G'$. Consider the path $P = [A_n - \cdots - C_n - S_n - B_n - \cdots - [\mathcal{B}_n(\tilde{x}_{N_1})]]$ in $\mathcal{G}_n$, where a vertex $\tilde{x}_{N_1}$ is a lifting of x_{N_1} in $\mathcal{T}$. Since a vertex in T which is the center of B_{n+1} is a center of S_n and adjacent to centers of B_n, C_n (Lemma 6.2.12), there exists a lifting of a path P in T. Since the length of the path P is $b_\phi(n) - |VG|$, the maximal distance between any two vertices in $\mathcal{G}_n$ is also $b_\phi(n) - |VG| + \mathbf{r}(x_{N_1}, G)$. By a similar argument before, we have the third assertion. $\square$

We note that the converse of the proposition does not hold. Consider a sequence of words

$$X_k = \begin{cases} aL_kaL_kbL_ka, & \text{if } k \text{ is odd,} \\ bL_kaL_kbL_kb, & \text{if } k \text{ is even,} \end{cases}$$

where L_k is given by $L_1 = \varepsilon$, the empty word and $L_{k+1} = L_kaL_k$ for odd k, $L_{k+1} = L_kbL_k$ for even k recursively. Then L_k is a palindrome and we get

$$X_1 = aaba, \qquad X_2 = baaabab, \qquad X_3 = aabaaabababaa, \qquad \cdots$$

Since X_k is a factor of X_{k+1}, we have a coloring ϕ of a 2-regular tree by the limit of X_k. Let $n_k = |L_ka_kL_k| = 2^k - 1$. Then we can check that for $n_{k-1} < n \leq n_k$, we have

$$R''_\phi(n) - n = \left\lfloor \frac{|X_k|}{2} \right\rfloor$$

and

$$b_\phi(n_k) = |X_k|.$$

Thus, we have

$$R''_\phi(n) = n + \left\lfloor \frac{b_\phi(n_k)}{2} \right\rfloor \quad \text{for } n_{k-1} < n \leq n_k.$$

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국문초록

연분수는 무한히 반복되는 분수 꼴로서 측도론적 정수론, 쌍곡 기하, 문자열 조합론과 같은 수학의 다양한 학문적 관점에서 연구되어 왔다. 본 연구는 연분수와 관련된 다음 세 가지 주제에 대해서 다룬다.

연분수의 중요한 성질 중 하나는 고전적인 연분수가 모든 무리수에 대해서 가장 좋은 유리수 근사를 생성하는 알고리즘을 준다는 것이다. 이는 연분수의 근사 분수라는 형태로 표현된다. 우리는 분모와 분자가 모두 홀수인 유리수 중에서 가장 좋은 근사를 만들어내는 새로운 연분수인 홀수-홀수 연분수를 정의하고, 이의 성질에 대해서 다룬다.

두 번째 연구 주제는 스터미안 단어를 연분수 전개로 가지는 실수인 스터미안 연분수의 레비 상수에 대한 것이다. 근사 분수의 분모가 지수적으로 얼마나 빠르게 증가하는지 그 지수적 증가율을 레비 상수라고 한다. 우리는 스터미안 연분수의 레비 상수가 존재한다는 것을 증명하고, 그들의 스펙트럼이 무엇인지에 대해서 규명한다.

마지막 연구 주제는 정규 나무 위에서의 준-스터미안 채색의 성질이다. 정규 나무 위에서의 준-스터미안 채색을 그것의 몫 그래프와 재귀 함수로 어떻게 특징지을 수 있는가에 대해서 다룬다. 또, 스터미안 단어의 연분수 알고리즘과 유사한 준-스터미안 채색의 귀납적 알고리즘을 제시한다.

주요어휘: 연분수, 디오판틴 근사, 기호 동역학, 스터미안 단어, 레비 상수, 나무의 채색

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