DÉPARTEMENT DE GÉNIE ÉLECTRIQUE
SECTION AUTOMATIQUE

Rapport Technique EP74-R-28

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PRACTICAL SENSITIVITY REDUCTION TESTS FOR LINEAR AND NONLINEAR SYSTEMS

par

Dr. Romano M. DeSantis
Professeur agrégé

28 mai 1974

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by

Romano M. DeSantis *

0. SUMMARY

This paper presents some graphical sensitivity reduction tests for time invariant linear and nonlinear feedback systems. These tests are useful in analyzing the sensitivity behavior of a multivariate plant which has been modified by the introduction of a 1-input - 1-output feedback compensator. They are based on Nyquist plot and describing function techniques and involve little extra effort in addition to that which is usually required to determine stability.

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+ This research was in part supported by the Canadian National Research Council, Grant No. A-8244.

1. INTRODUCTION

We will consider the 1 input - 2 output **linear** plant represented in Figure 1. This plant will be studied in terms of the following mathematical model

$$\begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} u \quad ; \quad \begin{bmatrix} \Delta_1^0 \\ \Delta_2^0 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \eta + \begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix} + \begin{bmatrix} \Delta_1^0 \\ \Delta_2^0 \end{bmatrix} .$$

The symbols T_1, T_2, ΔT_1, ΔT_2, F_1 and F_2 represent linear time invariant mappings on $L_2[0,\infty)$, the space of real valued square integrable function defined on the real line; y_{01}, y_{02}, u, η, Δ_1^0, Δ_2^0, y_1 and y_2 are elements of $L_2[0,\infty)$. T_1 and T_2 describe the **input-output behavior of the plant under** nominal operating

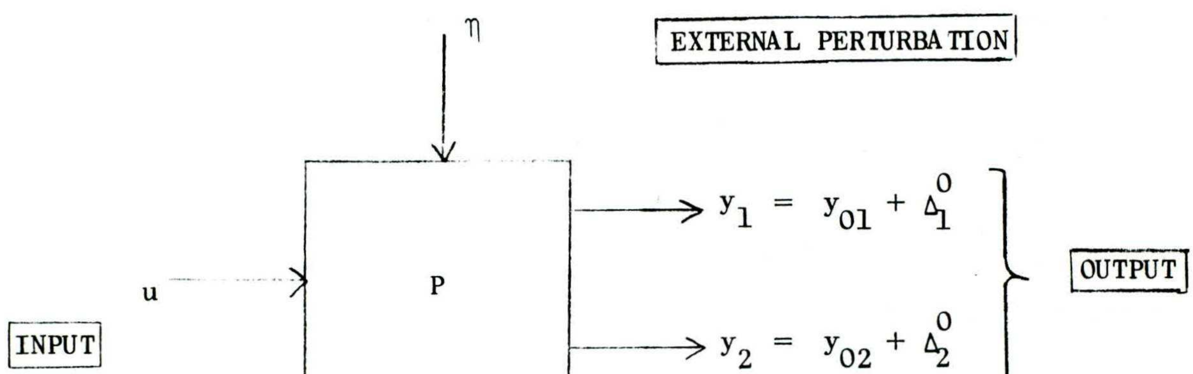

Figure 1: The Open loop plant

conditions; ΔT_1 and ΔT_2 take into account the variation of the plant parameters from their nominal values; F_1 and F_2 are used to evaluate the influence of an external perturbation. Finally, the elements y_{01}, y_{02}, u, η, $\overset{0}{\Delta_1}$, $\overset{0}{\Delta_2}$, y_1 and y_2 represent respectively the nominal output, the input, the external perturbation, the output error and the output of the plant.

To reduce the sensitivity of the outputs y_1 and y_2 to the presence of η, ΔT_1 and ΔT_2 it is natural to embed the plant of Figure 1 in a feedback configuration of the type represented in Figure 2. This configuration is such that under normal operating conditions, (that is: for $\eta = 0$, $\Delta T_1 = 0$, $\Delta T_2 = 0$), the compensated plant is equivalent to the original one. As a consequence the compensated plant can be modelled in terms of the following equation

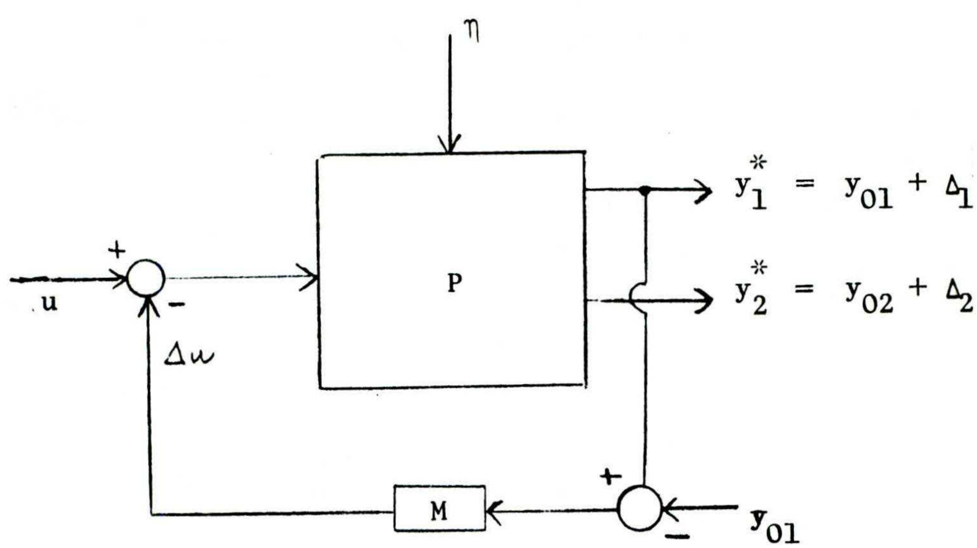

Figure 2: The compensated plant.

$$\begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} u \quad ; \quad \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \eta + \begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix} [u] - \begin{bmatrix} T_1 + \Delta T_1 \\ T_2 + \Delta T_2 \end{bmatrix} M(y_1^* - y_{01})$$

$$\begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix} = \begin{bmatrix} y_{01} \\ y_{02} \end{bmatrix} + \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix} . \qquad (2)$$

We will first consider the question of sensitivity in the case where M is linear and time invariant. Later we will also discuss the case where M is given by the cascade composition of a linear operator M' followed by a nonlinear memoryless operator N.

Our purpose is to present some practical graphical tests which simplify a discussion about the following questions:

- Determination of Sensitivity Reduction: Given ΔT_1, ΔT_2 and η, is the error in the compensated plant smaller than that in the original open loop plant ?
- Design Procedure for Sensitivity Reduction: What restrictions must be imposed on the compensator M in order to achieve sensitivity reduction ?
- Sensitivity Reduction under Normal Operating Conditions: Suppose that we can observe Δ_1^0 and Δ_2^0 in the original physical plant, while η, F_1 and F_2 are unknown. How can one determine whether the introduction of a given pair of compensators L and M reduce sensitivity ?

This type of questions have already been considered in relation to a 1 input - 1 output plant [5]; the present development extends the results in [5] to the case of a 1 input - 2 output plant. This extension is of interest in a number of practical situations where one has to analyse the sensitivity implications associated with the introduction of a 1 input - 1 output feedback compensator to improve the performance of a multivariate plant. A typical situation of this kind occurs for example when the control engineer has to analyze the improvement in the performance of an hydroelectric power plant which can be obtained by introducing a frequency deviation signal into the voltage control loop, (Cuenod, Gilles, et alias, [6]). As a motivation vehicle, it may be helpful to clarify the connections between our mathematical model and the model considered in [6]. To do this it is sufficient to compare our Figure 2 with Figure 5 in [6] and to subsequently interpret our symbols as follows: Δ_1 and Δ_2 are to be viewed as the frequency and the voltage variations of the synchronous generator (denoted respectively by u and v in [6]); Δu represents the output of the frequency-voltage regulator; η represents the percentage variation of the load (denoted by z in [6]). With this correspondence of notations, the reader should now have no difficulty to determine from Figure 5 in [6] the mappings T_1, T_2, F_1, F_2 and M which characterize the model in Figure 2. In doing this one would find that the physical problem considered in [6] can now be studied in terms of the mathematical setting proposed in the present development.

2. MAIN RESULTS

The analysis of the sensitivity questions under consideration is in general quite complex and usually calls for the utilization of advanced mathematical techniques [1], [2], [3]. Our graphical tests however turn out to be very simple. In the linear case, this simplicity follows from the assumption that the perturbation is represented by a sinusoidal function. In the nonlinear case we will further assume that our system satisfies those reasonable engineering conditions under which the describing function approach is usually applied (see [4], chapter 6). In this vein the errors $\Delta_1(t)$ and $\Delta_2(t)$ will be approximated in terms of their first harmonics $\tilde{\Delta}_1(t)$ and $\tilde{\Delta}_2(t)$. The following simplified sensitivity reduction criterion will be used: the compensated system reduce the sensitivity of the i-th output of the original system, if the amplitude of $\tilde{\Delta}_i(t)$ is smaller than the amplitude of $\Delta_i^0(t)$.

A direct approach to the analysis of sensitivity reduction would suggest the utilization of a setting of the type represented in Figure 3. In this setting η and u are applied to both the compensated and the open loop plants; the outputs of these plants are substracted from the output of the nominal plant. This makes it possible to compare Δ_1^0, Δ_2^0, with Δ_1 and Δ_2 in correspondance with various values of η, ΔT_1, ΔT_2, u and M. Though not very efficient, this approach would in principle allow one to answer the question about whether sensitivity reduction has been attained or not. To do this, however, both u and η are required to be accessible; a condition which is not usually satisfied. Thus the problem of determining sensitivity reduction under normal operating conditions remains unsolved. An additional and more relevant shortcoming of the direct approach is that it does not offer the insight which is needed to develope a well defined compensator's design strategy leading to sensitivity reduction. The following propositions alleviate in part these two shortcomings, and are at the basis of the alternative approach presented in the subsequent sections.

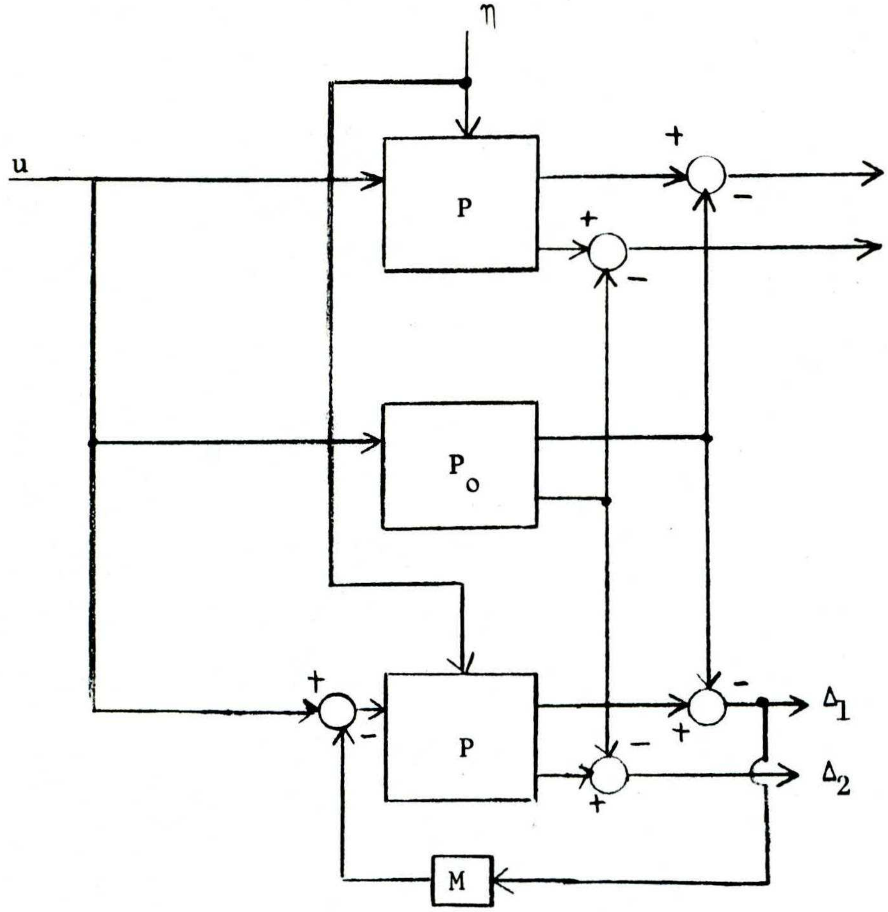

Figure 3: Setting used in the direct approach to sensitivity reduction (P_o represents the nominal plant).

Proposition 1. The closed and open loop errors Δ_1, Δ_2 and Δ_1^0, Δ_2^0 can be obtained the one from the other by using the sensitivity models represented in Figures 4 and 5.

Proof. It is sufficient to note that

$$\Delta_2 = \Delta_2^0 - \tilde{T}_2 M \Delta_1$$

where $\tilde{T}_2 = T_2 + \Delta T_2$ and that

$$\Delta_1 = (I+\tilde{T}_1 M)^{-1} \Delta_1^0$$

where $\tilde{T}_1 = T_1 + \Delta T_1$, (this was proved in [1]).

Proposition 2.* The sensitivity of the first output of the compensated plant is better than that of the original plant if $\|I+\tilde{T}_1 M\| > 1$. The sensitivity of the second output is better if

$$\|I-T_2 M(I+T_1 M)^{-1} F_1 F_2^{-1}\| < 1, \qquad \text{when } \Delta T_1 = \Delta T_2 = 0$$

or if

$$\|I-\tilde{T}_2 M(I+\tilde{T}_1 M)^{-1} \Delta T_1 \Delta T_2^{-1}\| < 1, \quad \text{when } \eta = 0.$$

Proof. The first part of the proposition is an immediate consequence of Proposition 1. With regard to the second part, it is sufficient to observe that if $\Delta T_1 = \Delta T_2 = 0$ then

$$\Delta_2 = (I-T_2 M(I+T_1 M)^{-1} F_1 F_2^{-1}) \Delta_2^0 ,$$

and if $\eta = 0$ then

$$\Delta_2 = (I-\tilde{T}_2 M(I+\tilde{T}_1 M)^{-1} \Delta T_1 \Delta T_2^{-1}) \Delta_2^0 .$$

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* Given an operator, T, the symbol $\|T\|$ denotes the norm of T.

Proposition 3.* Suppose that all the mappings under consideration are time invariant. If $\Delta T_1 = \Delta T_2 = 0$, then any one of the following conditions is sufficient to improve the sensitivity of the i-th output:

a) $\bar{P}_i(j\omega)$ is interior to the circle which has center **at** (-1,0) and passes through the point $MT_1(j\omega)$, where $\bar{P}_1(j\omega) = 0$ and $\bar{P}_2(j\omega) = M(T_1F_2-T_2F_1)F_2^{-1}(j\omega)$;

b) $P_i(j\omega)$ is interior to the circle which has center at $-\frac{1}{M(j\omega)}$ and passes through $T_1(j\omega)$ where $P_1 = 0$ and $P_2(j\omega) = (T_1F_2-T_2F_1)F_2^{-1}(j\omega)$;

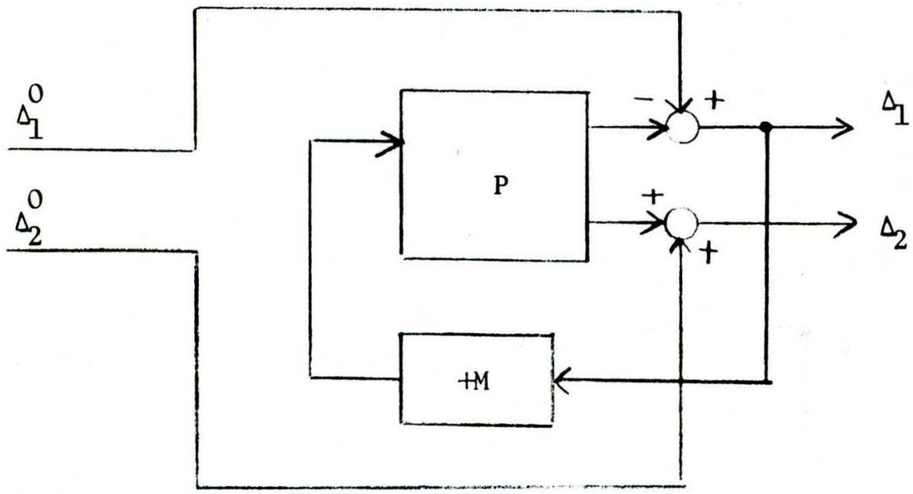


Figure 4: Simplified model for obtaining the compensated plant error from the open loop plant error.

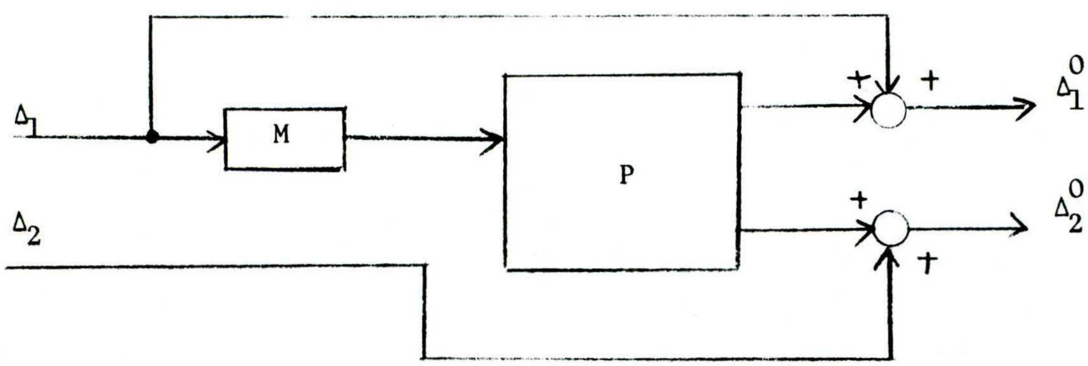


Figure 5: Simplified model for obtaining the open loop plant error from the compensated plant error.

* In this proposition the letters F_1, F_2, T_1, T_2, M, P_1 and P_2 are used to denote the frequency responses of the underlaying mappings.

c) $T_iM(I+T_1M)^{-1}(j\omega)$ is interior to the circle which passes through the origin and has center in $c_i = (c_i, 0)$, where $c_1 = 0$, and $c_2 = F_2F_1^{-1}(j\omega)$

d) $$20 \log \left| \frac{1/MT_1(j\omega)}{1 + 1/MT_1(j\omega)} \right| \leq 20 \log \left| \frac{1/\bar{P}_i(j\omega)}{1 + 1/\bar{P}_i(j\omega)} \right|$$

Proof. The case i = 1 is a direct consequence of Proposition 2. In regard to the case i = 2, observe that for an improvement of the sensitivity of the output y_2, we must have

$$|1 - T_2M(I+T_1M)^{-1}F_1F_2^{-1}| < 1$$

that is

$$|M(I+T_1M)^{-1}| \; \left| \frac{1+T_1M}{M} - T_2F_1F_2^{-1} \right| < 1.$$

This last inequality implies the following inequalities:

$$\left|1 + M \frac{T_1F_2 - T_2F_1}{F_2}\right| < |1 + T_1M| \; ;$$

$$\left|\frac{1}{M} + \frac{T_1F_2 - T_2F_1}{F_2}\right| < \left|\frac{1}{M} + T_1\right| \; ;$$

$$\left| \frac{F_2}{F_1} - T_2M(I+T_1M)^{-1} \right| < \left| \frac{F_2}{F_1} \right| \; ;$$

$$20 \log \left| \frac{|1/T_1M|}{|1+1/T_1M|} \right| < 20 \log \frac{1/\bar{P}_2}{1+1/\bar{P}_2}$$, which in turn imply a),b),c),and d).

Proposition 3'. If $\eta = 0$ while ΔT_1 and ΔT_2 are different from zero, then the statement of Proposition 3 is still valid, provided that the following change of notations is implemented

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \rightarrow \begin{bmatrix} \Delta T_1 \\ \Delta T_2 \end{bmatrix} .$$

3. APPLICATIONS

3.1 Determination of Sensitivity Improvement:

"It is proposed to embed the assigned plant P (Figure 1) in a feedback configuration adopting a specific compensator M (Figure 2). How can we verify whether this embedding improves the plant performance ?" Using the model of Figure 6, the proposed procedure to answer the above question can be described as follows:

1) Determine the frequency responses between the points (2) and (3) , and (2) and (4) . This step gives F_1 and F_2;

2) Determine the frequency responses between (1) and (3) and (1) and (5) . This gives us T_1M and $\bar{P}_2$;

3) Verify that the origin and $\bar{P}_2(j\omega)$ are interior to the circle which has center at (-1,0) and passes through $T_1M(j\omega)$.

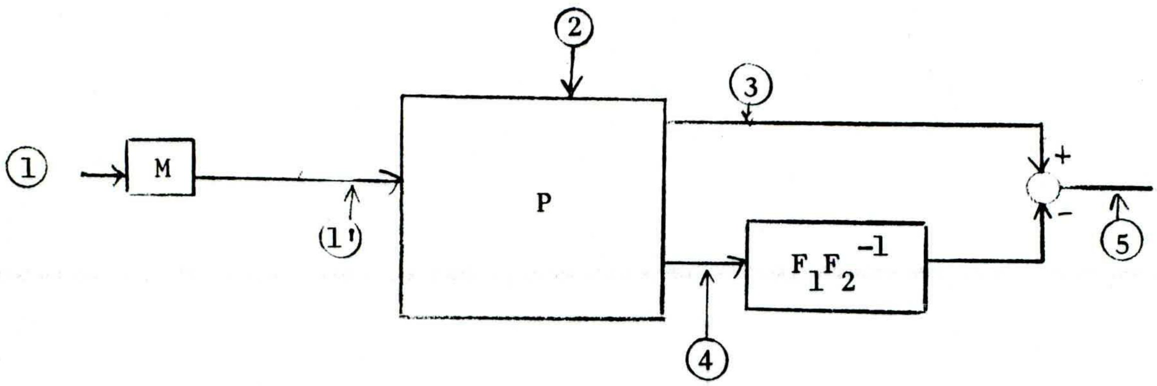


Figure 6: Open loop tests to be considered for sensitivity analysis.

If the plant P has already been embedded in the closed loop configuration and if $F_2(j\omega) / F_1(j\omega)$ is known, then with reference to Figure 7 , the following alternative procedure can be used:

1) Determine the frequency response between (1) and (2) , and between (1) and (3) . This gives us $G_1(j\omega) = T_1M(I+T_1M)^{-1}(j\omega)$ and $G_2(j\omega) = T_2M(I+T_1M)^{-1}(j\omega)$.

2) Verify that: i) $G_1(j\omega)$ is interior to the circle with center at (1,0) and passing through the origin;

ii) $G_2(j\omega)$ is interior to the circle with center at $\frac{F_2}{F_1}(j\omega)$ and passing through the origin.

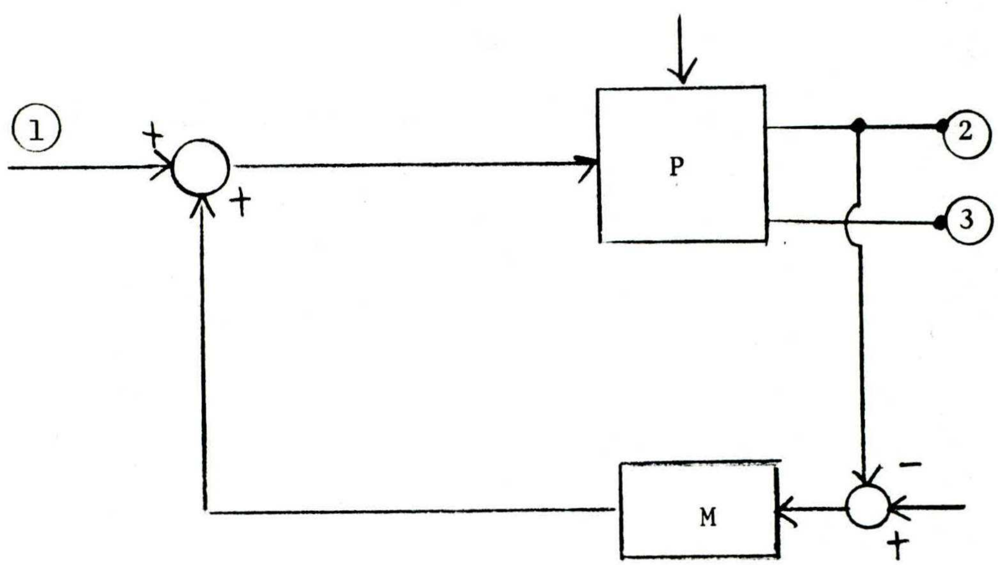


Figure 7 : Feedback tests to be considered for sensitivity analysis.

Example 1. Let the plant P be defined by the block diagram represented in Figure 8 and suppose that the perturbation η has a frequency spectrum contained in the interval $\omega \in [0,1]$. Does the addition of the compensator $M(j\omega) = \frac{1}{1+j\omega}$ improve the plant sensitivity, (Figure 9) ? Using the procedure proposed in 3.1 we represent the Nyquist plots of $T_1M(j\omega) = \frac{1}{(1+j\omega)^2}$, and $\bar{P}_2 = \frac{-1-j\omega+\omega^2}{(1+3j\omega-2\omega^2)(1+j\omega)}$. This is done in Figure 9. From these Nyquist plots we can see that if $\omega \in [0,1]$ then both the origin and the point $\bar{P}_2(j\omega)$ are interior to the circle with center at (-1,0) and passing by $T_1M(j\omega)$. We can then conclude that in the frequency interval of interest the compensator M does indeed improve the plant sensitivity.

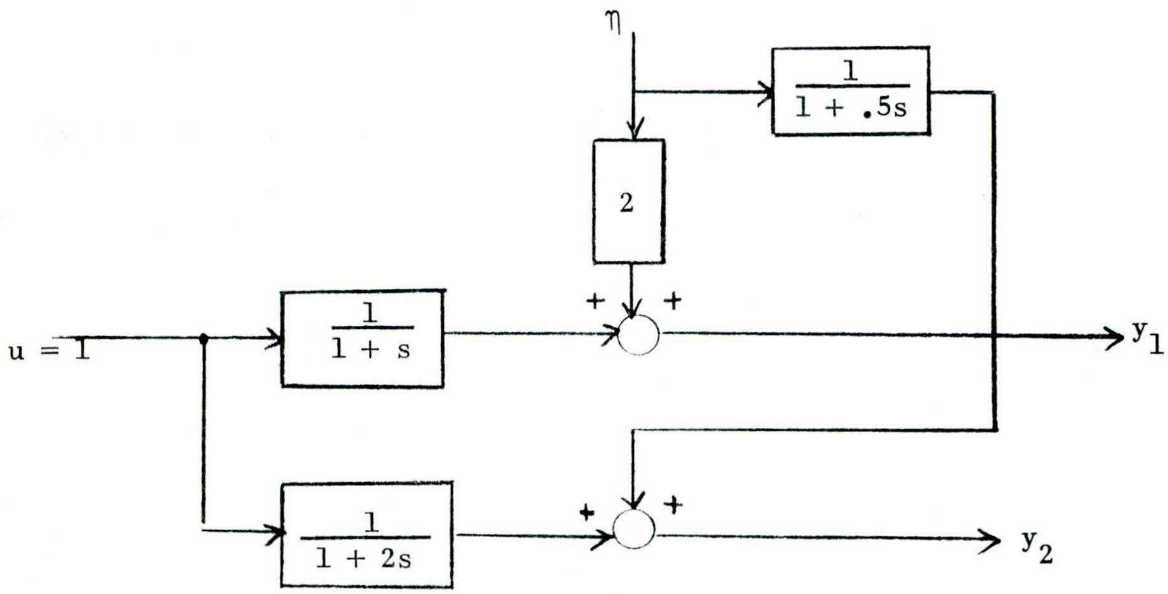


Figure 8: System considered in Example 1.

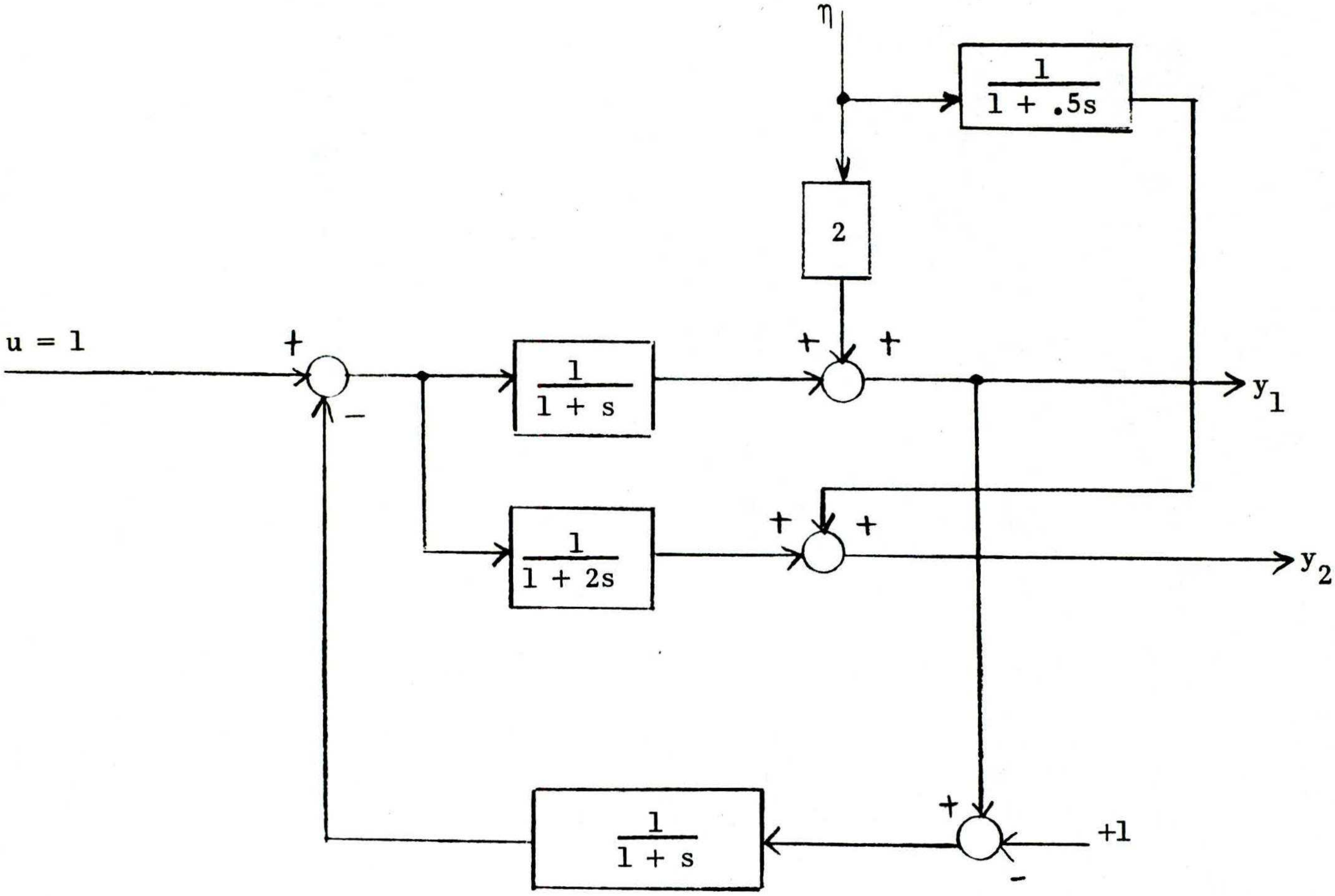


Figure 9: A compensation scheme for the system of Figure 8.

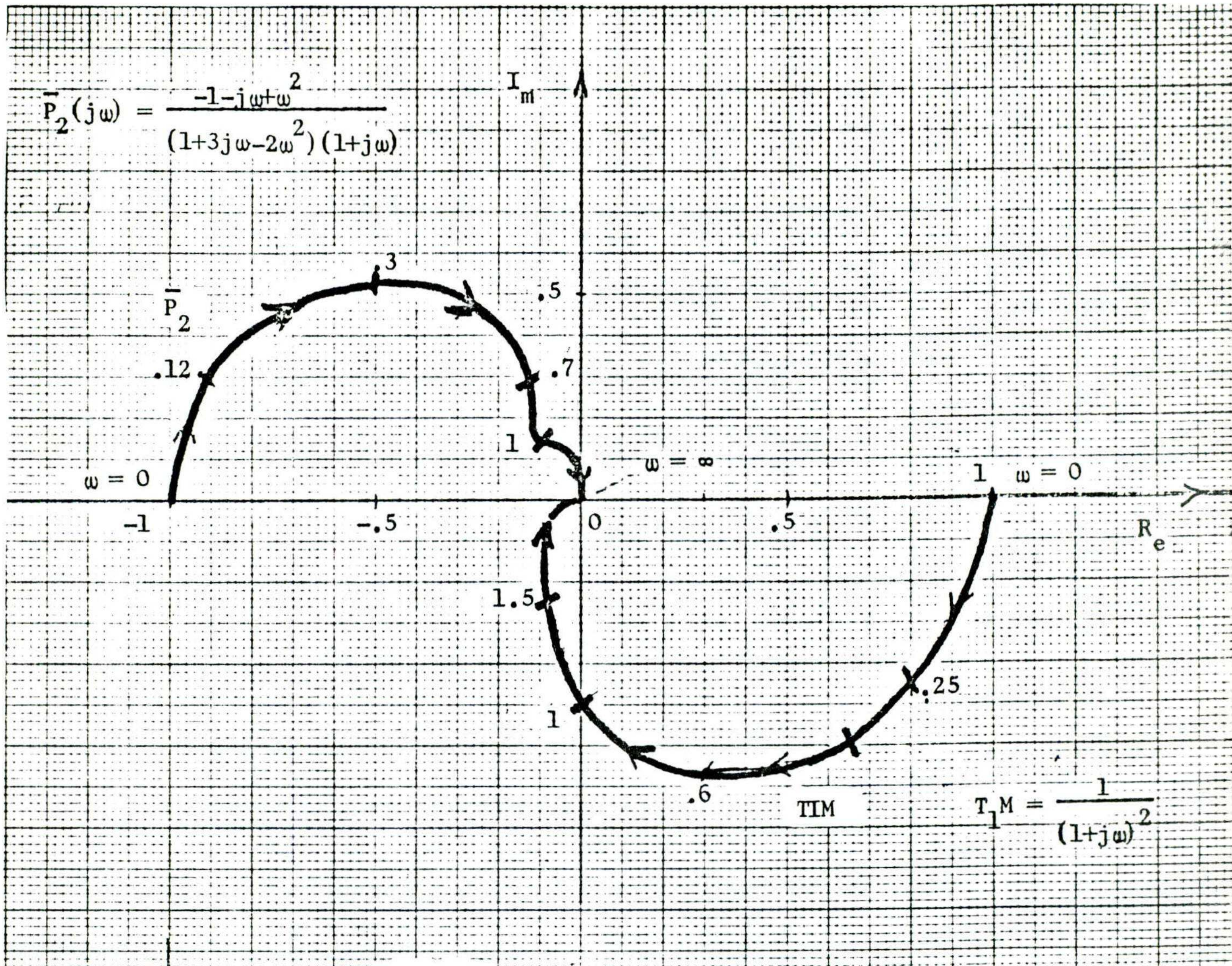


Figure 10: Frequency plots considered in Example 1.

3.2 Compensator Design Procedure:

Let the plant P be assigned (Figure 1) and let us consider the problem of designing a compensator M to improve the sensitivity (Figure 2). How do we proceed to select the appropriate M ?

With reference to Figure 6 , this question can be approached as follows:

1) Determine the frequency response between (2) and (3) and (2) and (4) . This gives us F_1 and F_2;

2) Determine the frequency responses between (1') and (3) and (1') and (5) . This gives us T_1 and P_2;

3) Choose $M(j\omega)$ in such a way that $P_2(j\omega)$ and the origin are interior to the circle with center at $-1/M(j\omega)$ and passing by $T_1(j\omega)$.

Example 2. Consider once again the plant represented in Figure 8. In this case we have $T_1(j\omega) = \frac{1}{1+j\omega}$ and $P_2(j\omega) = \frac{-1+\omega^2-j\omega}{+1-2\omega^2+3j\omega}$. The Nyquist plot of these two frequency responses are represented in Figure 10. From this figure we can see that to improve the plant sensitivity it is sufficient to choose a compensator $M(j\omega)$ in such a way that $-\frac{1}{M(j\omega)}$ is contained in the region $\mathcal{S}$. For example in the case of a proportional compensator, $M = K$, it is sufficient to choose $-\frac{1}{K} < -.5$, that is $0 < K < 2$.

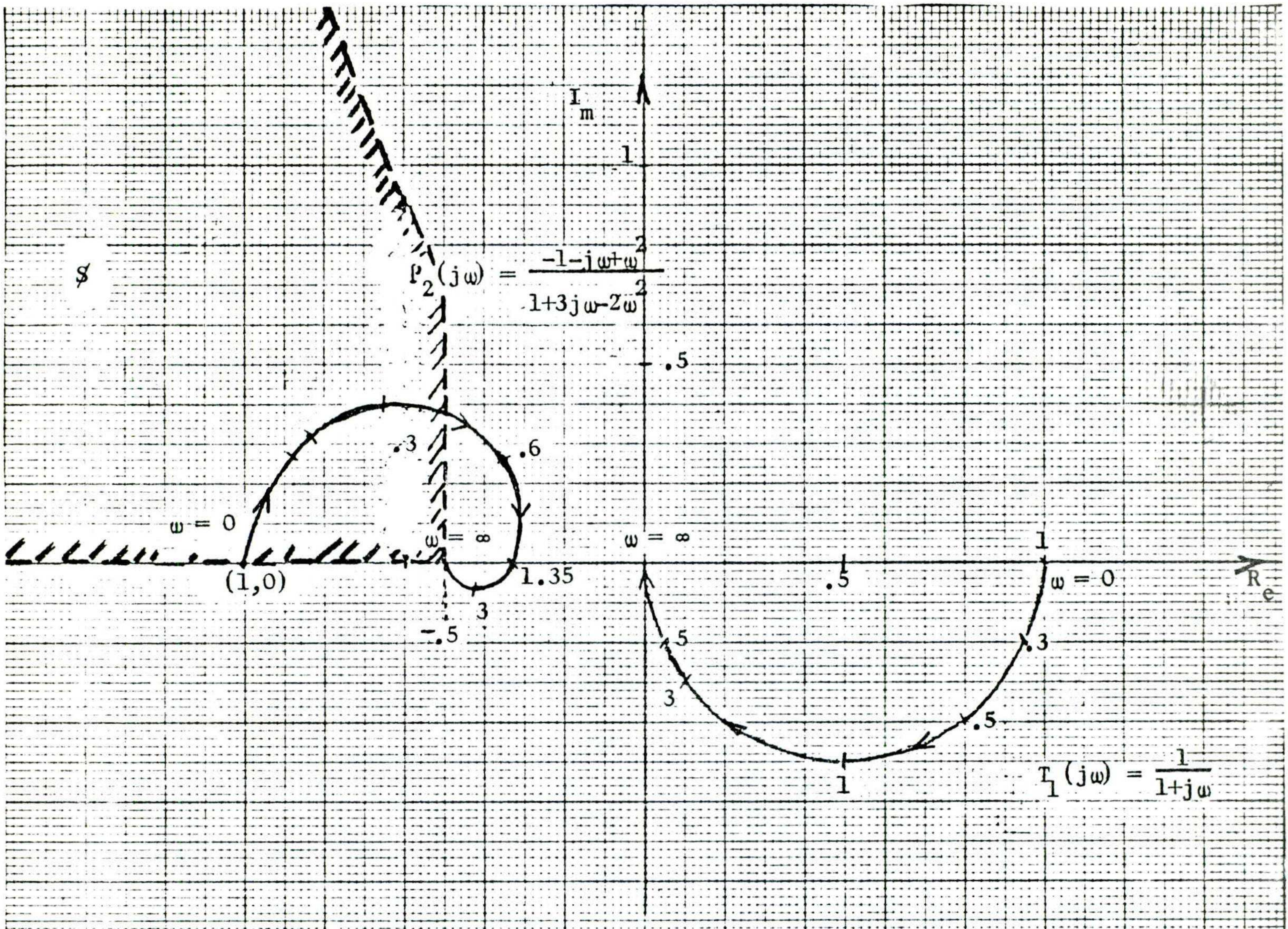


Figure 11: Frequency plots considered in Example 2.

3.3 Determination of Sensitivity Improvement from Normal Operating Data:

"Suppose that we are given the physical open loop plant P, (Figure 1), with T_1 and T_2 known, while F_1, F_2, u and η are unknown; let the errors Δ_1^0 and Δ_2^0 be accessible. It is proposed to embed this plant in a feedback configuration adopting a specific compensator M (Figure 2). Does this embedding improve the plant performance ?"

This question can be solved by adopting the diagram represented in Figure 12. The errors Δ_1^0 and Δ_2^0 are applied at the input of the mathematical model suggested by Proposition 1. The output of this model gives Δ_1 and Δ_2.

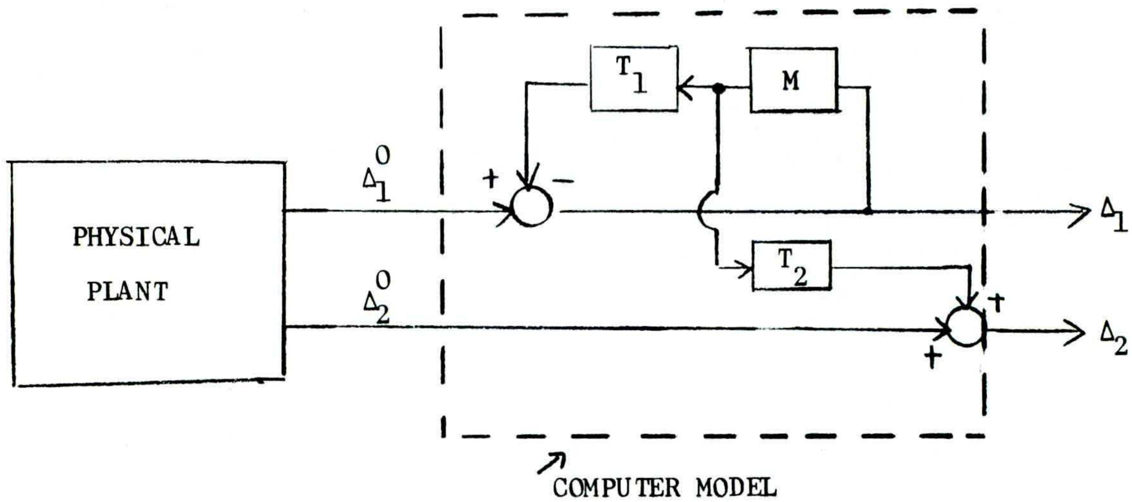


Figure 12: Configuration to be considered for sensitivity analysis using normal operating data.

4. THE CASE OF THE NONLINEAR COMPENSATOR

In this section we will suppose that the feedback compensator appearing in Figure 2 is given by the cascade composition of a nonlinear memoryless and time invariant system, N, followed by a linear and time invariant system M'. This case is more involved than that considered in section 3 because now for a sinusoidal perturbation of a given frequency, the compensated plant may have a smaller sensitivity than the original plant for some values of the amplitude and a larger sensitivity for other amplitude values. The test which distinguishes these two possible occurences is established by the following proposition. In stating this proposition, the symbol $\hat{N}(.)$ will denote the describing function associated with the nonlinearity N; the symbol δ_1 will represent the amplitude of the first harmonic of the error function $\Delta_1(t)$.

Proposition 1. For a sinusoidal perturbation of amplitude A and angular velocity ω_o, the compensated plant reduces the sensitivity of the i-th output if A and ω_o are such that $\bar{P}_i(j\omega)$ is interior to the circle which has center at $-\frac{1}{\hat{N}(\delta_1)}$ and passes through the point $M'T_1(j\omega)$, where $\bar{P}_1(j\omega) = 0$ and $\bar{P}_2(j\omega) = M'(T_1F_2-T_2F_1)F_2^{-1}(j\omega)$.

Proof: Let $\eta(t) = A \sin \omega_o t$. Note that Δ_1 and Δ^o_1 are related by the following equation

$$\Delta_1 = (I + T_1M'N)^{-1} \Delta^o_1 \qquad (1)$$

where $\Delta^{o}_{1}(t) = \delta^{o}_{1} \sin \omega_{o}t$, with $\delta^{o}_{1} = |F_{1}(j\omega_{o})|\ A$.

Assuming that the function Δ_1 can be approximated by its first harmonic $\tilde{\Delta}_1(t) = \delta_1 \sin(\omega_o t + \theta)$, and that δ_1 can be computed using the describing function approach, from (1) we obtain

$$\tilde{\delta}_1 = [1 + T_1(j\omega_o)\, M'(j\omega_o)\hat{N}(\delta_1)]^{-1}\delta^{o}_{1} \qquad (2)$$

where: $T_1(j\omega_o)$ and $M'(j\omega_o)$ denote the frequency response at T_1 and M'; $\hat{N}(.)$ is the describing function of N; and δ_1 is a complex number whose modulus is equal to δ_1 and whose phase is θ. From (2) it follows that we will have sensitivity improvement with respect to the first output if and only if

$$|1 + T_1(j\omega_o)\, M'(j\omega_o)\, N\,(\delta_1)| > 1,$$

that is if and only if

$$\left|\frac{1}{\hat{N}(\delta_1)} + T_1(j\omega_o)\, M'(j\omega_o)\right| > \left|\frac{1}{\hat{N}(\delta_1)}\right| .$$

This last inequality is equivalent to the geometrical condition considered by the proposition in the case $i = 1$.

Proceeding in a similar way we would also find that the condition for sensitivity improvement with respect to the second output is

$$\left| \frac{1}{\hat{N}(\delta_1)} + T_1(j\omega_o)M'(j\omega_o) \right| > \left| \frac{1}{\hat{N}(\delta_1)} + \bar{P}_2(j\omega) \right| .$$

This is equivalent to the geometrical condition proposed by the proposition in the case i = 2.

<u>Example 3:</u> Let us modify the plant represented in Figure 13 by introducing the nonlinear compensation scheme indicated in Figure 14. Let the characteristic curve of the nonlinear element be represented in Figure 15. To illustrate the applicability of Proposition 1, we sketch in Figure 16 the critical locus $-\frac{1}{\hat{N}(\lambda)}$, $\lambda \in (0,\infty)$, together with the Nyquist plot of

$$T_1M'(i\omega) = \frac{.625}{(s^2 + .6s + 1)\ (s + 1)}$$

and $$\bar{P}_2\ (i\omega) = M'(T_1F_2 - T_2F_1)F_2^{-1}(i\omega) = .25\ \frac{(-s^3 - 2.6s^2 + 2.8s + .5)}{(s^2 + .6s + 1)\ (2s + 1)\ (s + 1)} .$$

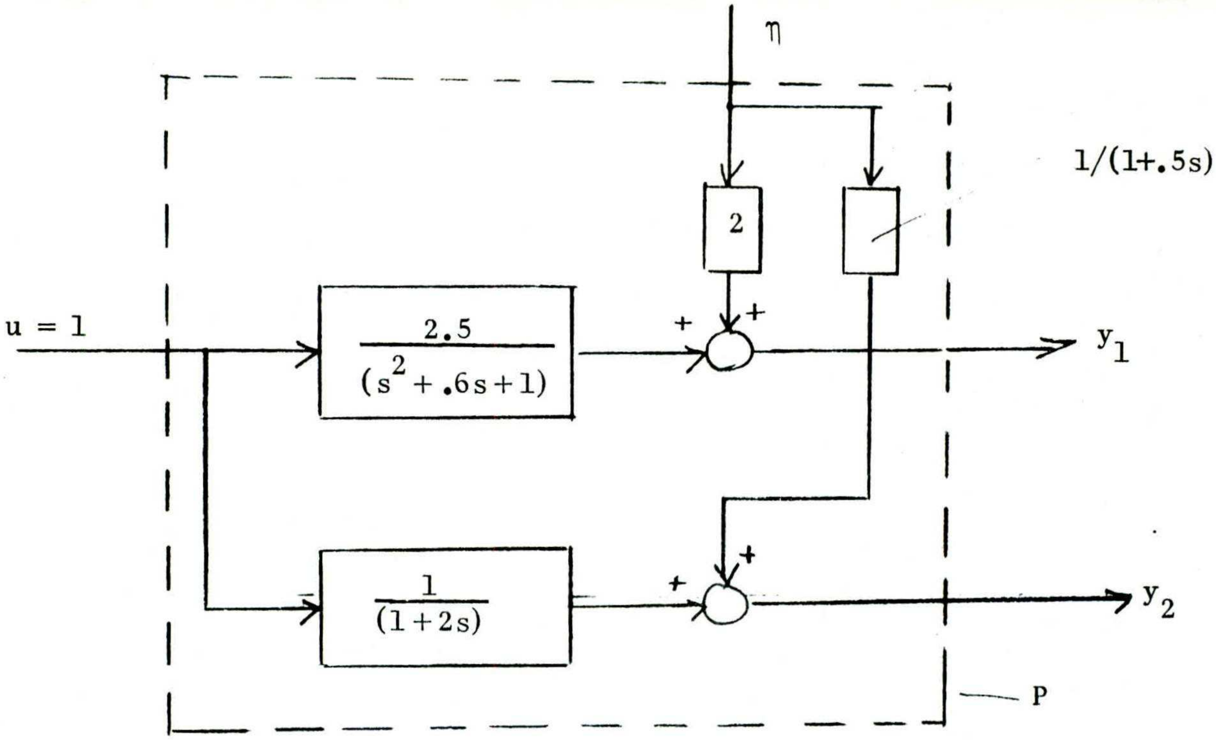


Figure 13: Open Loop Plant considered in Example 3.

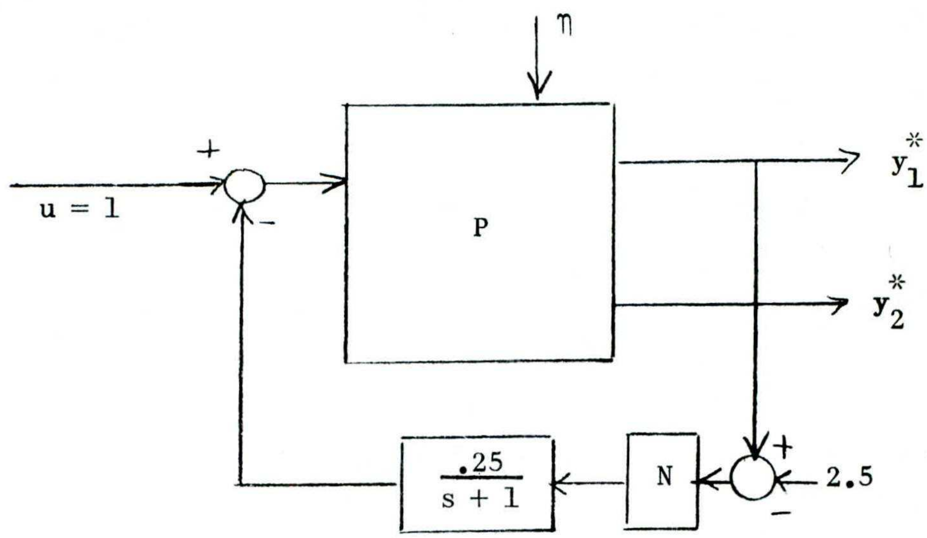


Figure 14: Nonlinear Feedback Compensation for the Plant of Figure 12.

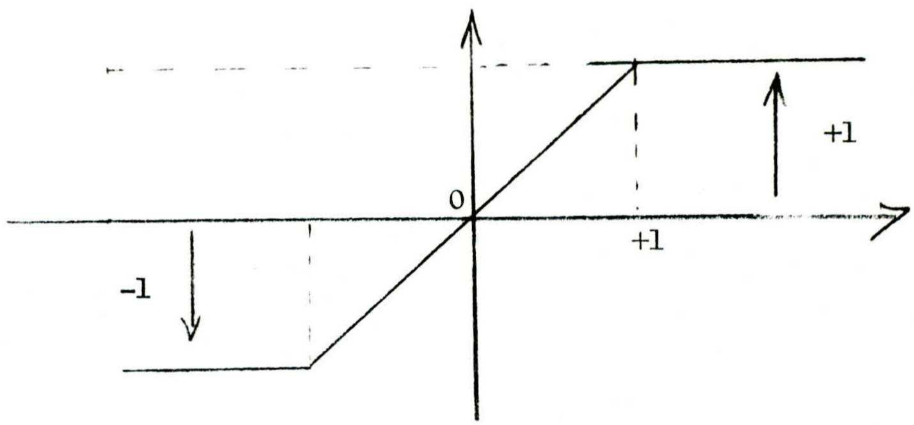
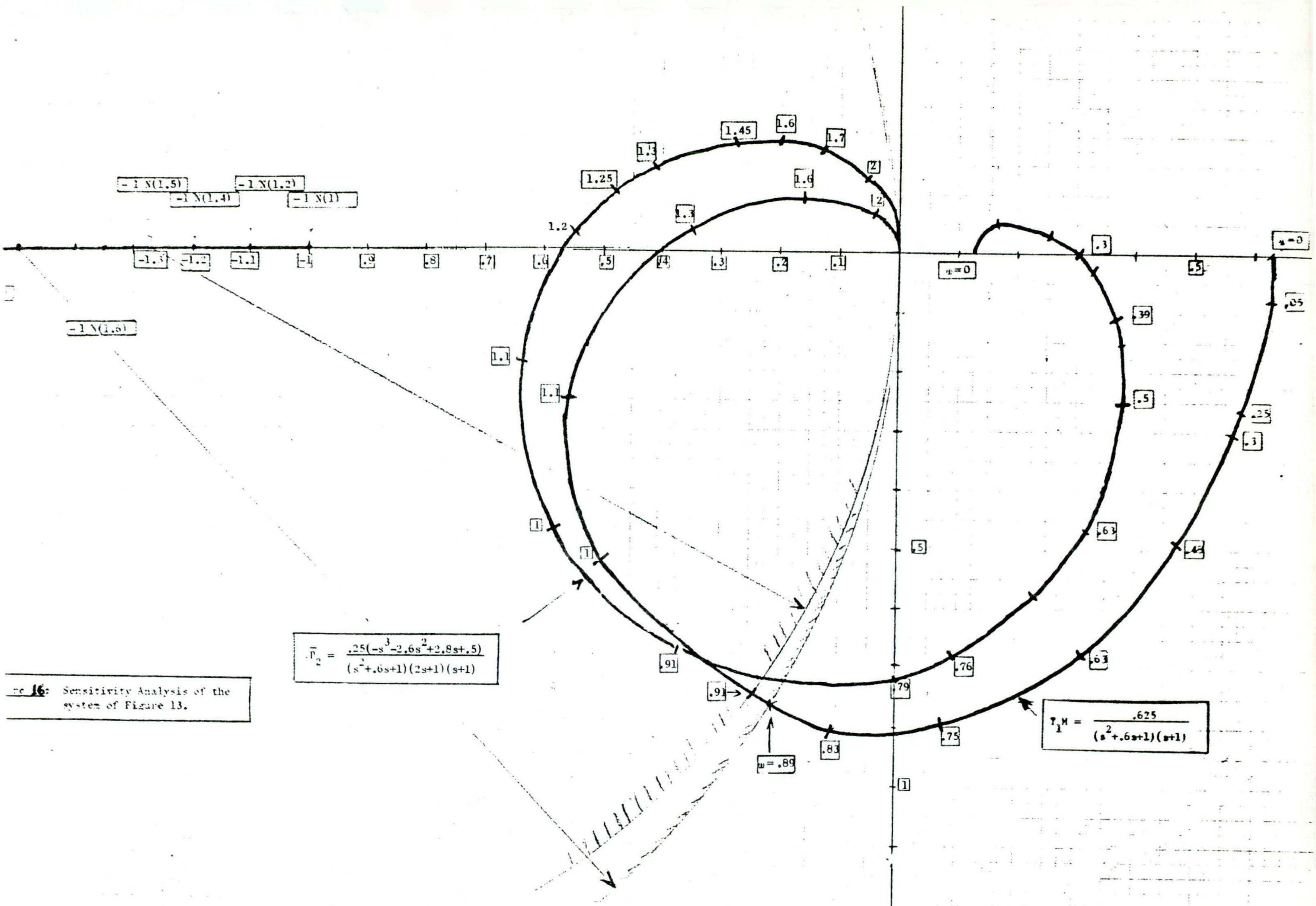


Figure 15: Characteristic curve associated with the nonlinearity of the system in Figure 14.

Suppose that the perturbation η is sinusoidal with amplitude A and frequency equal to $\omega_o = .91$ rad/sec. For what values of A is the sensitivity of the compensated plant better than that of the original plant? To answer this question we draw the circle which has center on the critical locus, passes by the origin and contains the point $T_1M'(.91j)$, (Figure 16). Looking at this circle and applying Proposition 1 we can then conclude that we have sensitivity improvement with respect to the first output if the amplitude of the perturbation is such that δ_1 is smaller than 1.5 that is if A is smaller than .75; if A is bigger than .75 then we will have sensitivity deterioration. In regard to the second output, y_2, we observe that no matter what value the amplitude of A is, the point $\bar{P}_2(.91j)$ is always interior to the circle with center at $-\frac{1}{N(\delta_1)}$ and passing by $T_1M'(.91j)$. This means that in correspondence to a perturbation with a frequency equal to .91 rad/sec the sensitivity of the second output of the compensated plant is always better than that of the open loop plant.



16: Sensitivity Analysis of the system of Figure 13.

Suppose now that the sinusoidal perturbation has an amplitude which is equal to .9. For what values of the frequency ω does the compensated system improve sensitivity? The answer is obtained as follows. We draw the circle which passes by the origin and has center at $-\frac{1}{\hat{N}(1.8)}$; we note that the frequency associated with the intersection between this circle and the Nyquist plot of $T_1M'(j\omega)$ is equal to .89 rad/sec. From the geometry of Figure 16 and Proposition 1, it then follows that, in relation to the first output, sensitivity improvement can be insured for $0 \leq \omega \leq .89$; if $\omega > .89$ we have sensitivity deterioration. With regard to the second output we have sensitivity improvement for all values of the frequency.

CLOSURE

The main results of this paper permit to analyze the sensitivity improvement which can be obtained by introducing a 1 input - 1 output linear or nonlinear feedback compensator into a 2 outputs - 1 input plant. In the linear case, Proposition 3.3 gives a simple procedure to establish sensitivity performance in terms of Nyquist plots conditions. Proposition 4.1 suggests that by considering the critical locus associated with the nonlinearity a similar procedure can be developed for the nonlinear case. These two procedures can both be implemented graphically and require little extra effort in addition to that which is usually undertaken to verify stability conditions with the Nyquist and describing fonction criteria.

It is of interest to note that the conditions described by Proposition 3.3 and 4.1 are only sufficient. In practical applications they may turn out to be more restrictive than necessary. If the perturbation is not completely unknown, however, or if it presents certain given characteristics then natural modifications of the graphical procedure can be implemented. Thus, if it is known, for example, that the frequency spectrum of the perturbation is mainly concentrated in a given frequency interval, then one would implement the sensitivity procedure by considering only that part of the Nyquist plot which is associated with the relevant frequency interval. This concept is illustrated in Example 1.

In the case where one deals with sinusoidal perturbations of a given frequency, then using a reasoning similar to that adopted in the proof of Proposition 3.3, it follows that one can easily evaluate a quantitative measure of sensitivity improvement. This measure can be established in terms of the sensitivity factors $\frac{|\Delta_i(j\omega)|}{|\Delta_i^{\circ}(j\omega)|}$, i = 1,2 and can be evaluated graphically in terms of the following ratios;

$$\frac{|\Delta_i(j\omega)|}{|\Delta_i^{\circ}(j\omega)|} = \frac{|1+\bar{P}_i(j\omega)|}{|1+T_1M(j\omega)|} = \frac{\left|\frac{1}{M(j\omega)}+P_i(j\omega)\right|}{\left|\frac{1}{M(j\omega)}+T_1(j\omega)\right|} = \frac{|c_i(j\omega)-T_iM(I+T_1M)^{-1}(j\omega)|}{|c_i(j\omega)|} ,$$

From these expressions one can see that our graphical approach can readily be applied to the case of a multivariate plant embedded into a multivariate feedback configuration (Figure 17). To do this, it would be sufficient to visualize the multivariate feedback compensator as given by a n-tuple of 1-input - 1 output elements; the analysis would then be conducted along the indicated lines, by introducing an element at the time and by considering the product of the various sensitivity improvement factors.

As a final remark, note that the results of the present development can be extended to time varying and sampled data systems by using a procedure which is similar to that discussed respectively in [3] and [4].

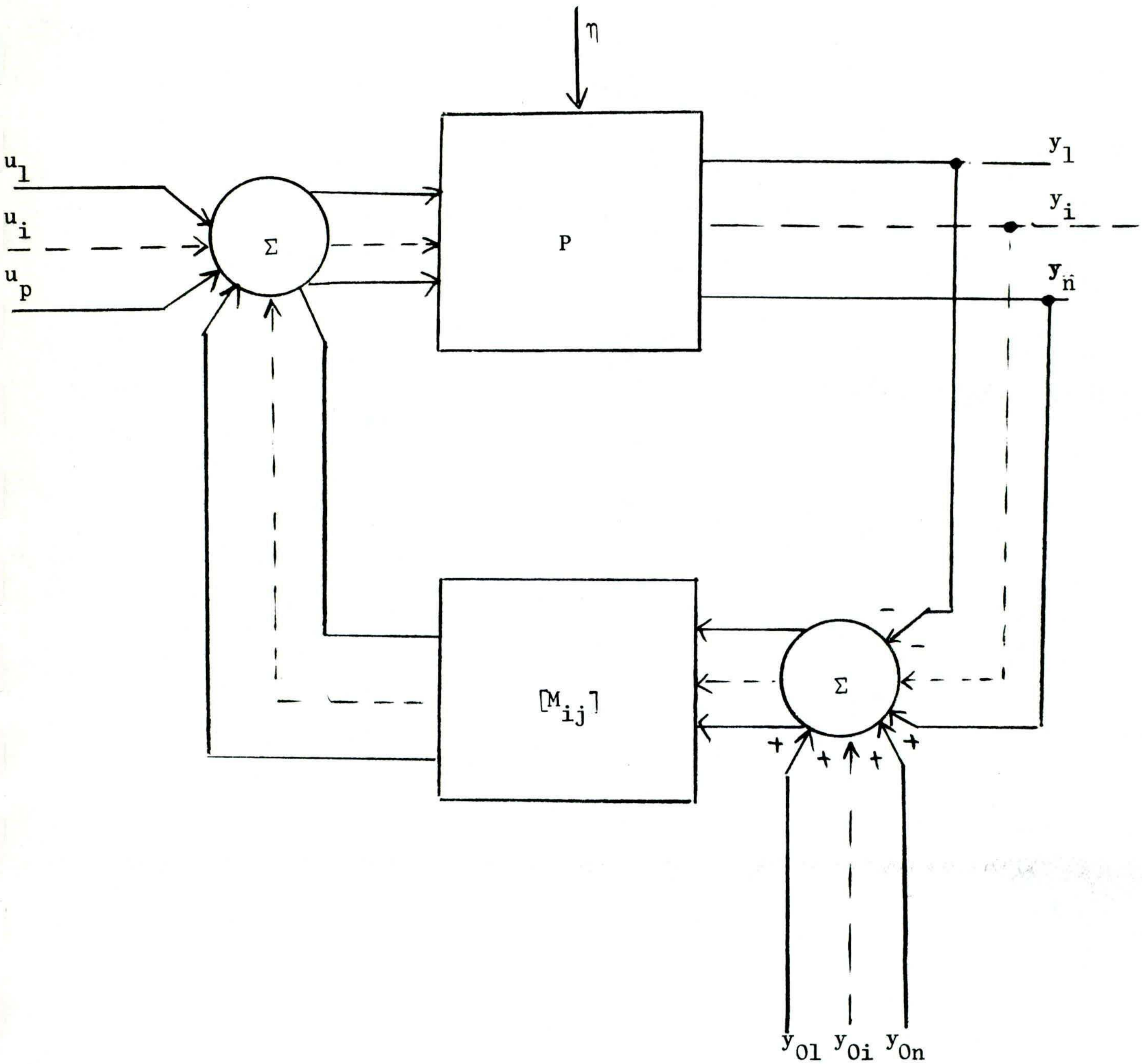


Figure 17: Multivariate Plant embedded in a Multivariate Feedback Configuration.

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ACKNOWLEDGEMENTS

Thanks are due to Professor William A. Porter and to Professor Louis Asselin for helpful comments. I am also grateful to Miss Carmen Gagnon for typing the manuscript.