DÉPARTEMENT DE GÉNIE INDUSTRIEL

Rapport Technique EP77-R-17
Classification: Library of Congress No....

SOME JOINT PROBABILITY DISTRIBUTION

PAR

Jean-Claude Warmoes

AVRIL 1977

Ecole Polytechnique de Montréal

Campus de l'Université
de Montréal
Case postale 6079
Succursale 'A'
Montréal, Québec
H3C 3A7

Rapport Technique EP77-R-17
Classification: Library of Congress No....

Dn

SOME JOINT PROBABILITY DISTRIBUTION

PAR

Jean-Claude Warmoes

AVRIL 1977

SOME JOINT PROBABILITY DISTRIBUTIONS

In the Technical Report EP76-R-4, it has been observed that the time to wait for the occurrence of an 'event' in a job network can be expressed in the generic form

$$Y \equiv [(X_1 + X_2) \wedge (X_3 + X_4 + \ldots)] \wedge [X_n \wedge X_{n+1}] \wedge [\ldots.]\ldots.$$

where the X_i are the random activity times. The formal rules of manipulating and expanding these expressions have also been laid down. It should be clear, then, that in order to calculate the expected value of Y, one must have some formulas for the expected value of 'suprema' and 'infima' of several random variables. This calls for not only a knowledge of their joint probability distributions but also, more concretely, some assumption regarding their mutual dependence or independence. Now, in the analysis of job networks that are equipped with a precedence relation, it has been customary to assume that service-times (i.e., completion-times for an individual, component task) are mutually independent, each being distributed exponentially.

The purpose of this and the following Chapter is to relax the assumption of independence. Accordingly, we present some preliminary results which will prepare the way for the substantive results.

§1 SOME RESULTS ON REPRESENTATION

Theorem 1 : If X and Y are arbitrary r.v.'s with a joint distribution function and if $E(Y|X = x)$ has a linear representation $ax + b$, then the coefficients in the representation are given by :

$$a = \rho \frac{\sigma_Y}{\sigma_X} \; ; \quad b = E(Y) - \rho \frac{\sigma_Y}{\sigma_X} E(X),$$

where ρ is the correlation coefficient.

Proof : From the definition, namely, $\rho = \dfrac{E(XY) - E(X)\,E(Y)}{\sigma_x \sigma_y}$, one can write

$$\iint xy\, f(xy)\, dx\, dy = \iint xy\, f(x)\, f(y|X=x)\, dy\, dx = \rho\sigma_x\sigma_y + E(X)E(Y)$$

which is equivalent to

$$\int x\, E(Y|X=x)\, f(x) dx = \rho\sigma_x\sigma_y + E(X)E(Y), \quad E(Y|X=x) \equiv g(x).$$

Suppose $g(x) = E(Y|X=x)$ has the form $ax + b$. So, it remains to determine a and b. Thus,

$$\int x(ax + b)\, f(x)\, dx = \rho\sigma_x\sigma_y + E(X)E(Y)$$

i.e.,
$$aE(X^2) + bE(X) = \rho\sigma_x\sigma_y + E(X)E(Y)$$

Applying the well-known formula $E(X^2) = E^2(X) + \sigma_x^2$, one has

$$a\sigma_x^2 + E(X)\,(aE(X) + b) = \rho\sigma_x\sigma_y + E(X)E(Y)$$ from which it follows that

$$a = \rho\,\frac{\sigma_y}{\sigma_x} \quad ; \quad b = E(Y) - \rho\,\frac{\sigma_y}{\sigma_x}\,E(X).$$

Therefore $g(x) = ax + b = E(Y) + \rho\,\dfrac{\sigma_y}{\sigma_x}\,(X - E(X))$.

Theorem 2 : Under the same assumption, the variance of $\{Y|X = x\}$ is $\sigma^2_{y|x} = \sigma_y^2\,(1 - \rho^2)$.

Proof : $\sigma^2_{y|X=x}$ can be written as :

$$\sigma^2_{y|X=x} = \iint \left[y - E(Y|X = x)\right]^2 f(xy)\, dxdy$$

which after expansion becomes,

$$\sigma^2_{y|X=x} = \iint y^2\, f(xy)\, dxdy - \iint 2y\, E(Y|X=x)\, f(xy)\, dxdy$$
$$+ \iint E^2\,(Y|X=x)\, f(xy)\, dxdy$$

The first integral on the right-hand side is seen to be

$$\iint y^2\, f(y|X=x)\, f(x)\, dydx = E(Y^2) \equiv \sigma_y^2 + E^2(Y)$$

The second integral can be evaluated by substituting $E(Y|X=x)$ by $E(Y) + \rho \frac{\sigma_y}{\sigma_x}(X-E(X))$ and a rearrangement of terms :

$$\iint 2y\, E(Y|X=x)\, f(xy)\, dxdy = 2\left[E^2(Y) + \rho^2 \sigma_y^2\right] ,$$

and the third integral, after expansion and simplification becomes,

$$\iint \left[E(Y) + \rho \frac{\sigma_y}{\sigma_x}(X-E(X))\right]^2 f(xy)\, dxdy = E^2(Y) + \rho^2 \frac{\sigma_y^2}{\sigma_x^2}\left[E(X^2) - E^2(X)\right].$$

Regrouping these three integrals, one has

$$\sigma^2_{y|X=x} = \sigma_y^2 (1-\rho^2) \quad \text{or} \quad \sigma_{y|X=x} = \sigma_y \sqrt{1-\rho^2} .$$

<u>Theorem 3</u> : The linear representation is unique in the class of polynomial representation.

<u>Proof</u> : Let us assume that $E(Y|X=x) = ax^2 + bx + c$; we will try to evaluate the parameter a, b and c by the same procedure used in Theorem 1.

Thus, $\int x(ax^2 + bx + c)\, f(x)\, dx = \rho\sigma_x\sigma_y + E(X)E(Y)$

the results obtained by expanding and evaluating the left-hand side of the equation shows that the parameters <u>a</u> and <u>b</u> are function of <u>c</u>. Which implies that the representation used for $E[Y|X=x]$ is not feasible.

Therefore we can see easily that the same type of results will be obtained for any representation of $E[Y|X=x]$ by any polynomial of higher degree.

§2 JOINT DISTRIBUTION OF EXPONENTIAL VARIABLES

The ideas developed in theorems 1, 2 and 3 will be used in the joint exponential distribution.

Theorem 4 :

$$f(x,y)=\lambda_1 e^{-\lambda_1 x} e^{\left[-\frac{E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} \cdot \frac{(1+\rho)}{\sigma_{y|X=x}} \cdot \left[\frac{y-E[Y|X=x]}{\sigma_{y|X=x}}\right]^{\rho} \cdot e^{-\left[\frac{y-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{1+\rho}}$$

is the joint exponential density distribution, written under the form $f(xy) = f(x) \cdot f(y|X=x)$ of the random variables X and Y.

Proof : The proof will be carried out by showing that the marginals are exponential and that f(x,y) is a density of distribution.

First, expand f(xy) using the regression formula

$$E(Y|X=x) = E(Y)+\rho \frac{\sigma_y}{\sigma_x}(x-E(X)) \ ; \ \sigma_{y|X=x} = \sigma_y \sqrt{1-\rho^2}$$

Replacing E(X), σ_x by $\frac{1}{\lambda_1}$ and E(Y), σ_y by $\frac{1}{\lambda_2}$ one obtains

$$E(Y|X=x) = \frac{1}{\lambda_2}\left[1+\rho \frac{\lambda_1}{\lambda_2} x - \rho\right] \ ; \ \sigma_{y|X=x} = \frac{1}{\lambda_2}\sqrt{1-\rho^2}$$

which gives

$$f(x,y)=\lambda_1 e^{-\lambda_1 x} e^{\left[-\frac{x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \cdot \frac{\lambda_2(1+\rho)}{\sqrt{1-\rho^2}} , \left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho}$$

$$\cdot e^{\left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}}$$

a) Proof that, the integration of f(xy) with respect to y gives an exponential distribution :

$$\int_0^\infty f(xy)dy = \int_0^\infty f(x) \cdot f(y|X=x)\ dy = \lambda_1\ e^{-\lambda_1 x}$$

$$\int_0^\infty \lambda_1 e^{-\lambda_1 x}\ e^{\left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \cdot \frac{\lambda_2(1+\rho)}{\sqrt{1-\rho^2}} \cdot \left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho}$$

$$\cdot\ e^{-\left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}}\ dy$$

$$= \lambda_1 e^{-\lambda_1 x} \cdot e^{\left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \cdot \left\{-e^{-\left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \Big|_0^\infty\right\}$$

$$= \lambda_1 e^{-\lambda_1 x}\ e^{\left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \left\{\ e^{-\left[\frac{-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}}\ \right\}$$

$$= \lambda_1 e^{-\lambda_1 x}$$

b) Proof that, the integration of f(xy) with respect to x gives an exponential distribution, $\int_0^\infty f(x,y)\ dx = \lambda_2\ e^{-\lambda_2 y}$. This can be carried out in an analogous manner, replacing f(x,y) by f(y) $\cdot$ f(x|Y=y).

c) Proof that $\int_0^\infty \int_0^\infty f(xy)\, dx\, dy = 1$: Replace f(xy) by $f(x) \cdot f(y|X=x)$;
then, the expression becomes $\int_0^\infty \int_0^\infty f(x)\, f(y|X=x)\, dydx = \int_0^\infty \lambda_1\, e^{-\lambda_1 x}\, dx = 1.$
Therefore, f(x,y) is the joint exponential distribution of the random variables X, Y.

§ 3 MEAN-VALUES AND VARIANCES OF XVY AND X∧Y

Consider the random variable $Z(\omega) \equiv X(\omega)\, VY(\omega)$, and suppose that $X(\omega)$ and $Y(\omega)$ are jointly distributed with the density $f_{XY}(xy)$. Then, the probability density function of $Z(\omega)$ is given by :

$$f_Z(\alpha) = \int_{-\infty}^{\alpha} f_{XY}\,(\alpha,y)\, dy + \int_{-\infty}^{\alpha} f_{XY}(x,\alpha)\, dx$$

$$= \int_{-\infty}^{\alpha} f_X(\alpha) \cdot f_Y(y|X=\alpha)\, dy + \int_{-\infty}^{\alpha} f_Y(\alpha) \cdot f_X(x|Y=y)\, dx \qquad (15)$$

The expected value of $Z(\omega)$, i.e., $E(Z) \equiv \int_{-\infty}^{\infty} zf\,(z)\, dz$ may be calculated in the following manner. Replace $f_Z(z)$ by the corresponding expression found in (15), to obtain

$$E(Z) = \int_{-\infty}^{\infty} \alpha \int_{-\infty}^{\alpha} f_X(\alpha)\, f_Y(y|X=\alpha) dy d\alpha + \int_{-\infty}^{\infty} \alpha \int_{-\infty}^{\alpha} f_Y(\alpha)\, f_X(x|Y=\alpha)\, dx d\alpha \qquad (16)$$

In the case where $f_{XY}(xy)$ is jointly exponential, the following theorem can be proved :

<u>Theorem 5</u> : When X and Y are jointly exponential, then the E Sup(X,Y) is given by :

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \int_0^\infty x\lambda_1 \exp\{ -\lambda_1 x + \left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} - \left[\frac{\lambda_2 x-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} \} \, dx$$

$$- \int_0^\infty y\lambda_2 \exp\{ -\lambda_2 y + \left[\frac{-y\rho\lambda_2+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} - \left[\frac{\lambda_1 y-\rho\lambda_2 y+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} \} \, dy$$

Proof : As it is well know , the expected value of Z is given by $E(Z) = \int_0^\infty x \int_0^x f(x,y)\, dydx + \int_0^\infty y \int_0^y f(x,y)\, dxdy$

Since f(x,y) is jointly exponential, therefore, the first integral on the right-hand side becomes

$$\int_0^\infty x \int_0^x \lambda_1 e^{-\lambda_1 x} e^{\left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \cdot \frac{\lambda_2(1+\rho)}{\sqrt{1-\rho^2}} \cdot \left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho}$$

$$\cdot\, e^{-\left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \, dydx$$

$$= \int_0^\infty x\lambda_1\, e^{-\lambda_1 x} e^{\left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \cdot \{ - e^{-\left[\frac{\lambda_2 y-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \Big|_0^x \} \, dx$$

$$= \int_0^\infty x\lambda_1\, e^{-\lambda_1 x} e^{\left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \{- e^{-\left[\frac{\lambda_2 x-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} + e^{-\left[\frac{-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}} \} \, dx$$

Now split this expression into two integrals, namely,

$$= \int x\lambda_1 e^{-\lambda_1 x} dx - \int_0^\infty x\lambda_1 \exp\left\{ -\lambda_1 x + \left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} - \left[\frac{\lambda_2 x-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}\right\} dx$$

which is simplified as

$$\frac{1}{\lambda_1} - \int_0^\infty x\lambda_1 \exp\left\{ - \lambda_1 x + \left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} - \left[\frac{\lambda_2 x-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}\right\} dx$$

The second integral on the right-hand side of E(Z) can be evaluated in a similar manner, giving

$$\frac{1}{\lambda_2} - \int_0^\infty y\lambda_2 \exp\left\{ -\lambda_2 y + \left[\frac{-y\rho\lambda_2+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} - \left[\frac{\lambda_1 y-\rho\lambda_2 y+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}\right\} dy \; *$$

The integrals can always be evaluated by numerical methods in the general case.

The expected value of Inf, using the relation (13) becomes :

$$E(U) = E(X) + E(Y) - E(Z)$$

or

$$E(U) = \int_0^\infty x\lambda_1 \exp\left\{ -\lambda_1 x + \left[\frac{-x\rho\lambda_1+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} - \left[\frac{\lambda_2 x-\rho\lambda_1 x+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}\right\} dx$$

$$\int_0^\infty y\lambda_2 \exp\left\{ -\lambda_2 y + \left[\frac{-y\rho\lambda_2+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1} - \left[\frac{\lambda_1 y-\rho\lambda_2 y+\rho-1}{\sqrt{1-\rho^2}}\right]^{\rho+1}\right\} dy$$

---

* Special case, $\rho = 0$; here, E(Z) becomes,

$$E(Z) = \frac{1}{\lambda_1}+\frac{1}{\lambda_2} - \int_0^\infty x\lambda_1 e^{-\lambda_1 x -\lambda_2 x} dx - \int_0^\infty y\lambda_2\, e^{-\lambda_2 y -\lambda_1 y}\, dy$$

$$= \frac{1}{\lambda_1}+\frac{1}{\lambda_2} - \frac{\lambda_1}{(\lambda_1+\lambda_2)^2} - \frac{\lambda_2}{(\lambda_1+\lambda_2)^2} = \frac{1}{\lambda_1}+\frac{1}{\lambda_2} - \frac{1}{(\lambda_1+\lambda_2)}$$

which corresponds to the results obtained in the case of independence.

The same method can be used to evaluate the variances of supremum and infinum of a joint exponential r.v.'s.

In certain classes of problems, the joint normal-exponential density may be useful.

§ 4 JOINTLY NORMAL-EXPONENTIAL RANDOM VARIABLES

Two random variables X and Y are said to be jointly normal-exponential, when their marginal densities are of the forms,

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-E(X))^2}{2\sigma_x^2}} \quad ; \; f(y) = \lambda e^{-\lambda y} .$$

First, write f(xy) as $f(x)\, f(y|X=x)$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-E(X))^2}{2\sigma_x^2}} e^{\left[\frac{-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} \cdot \frac{(\rho+1)}{\sigma_{y|X=x}} \cdot \left[\frac{y-E[Y|X=x]}{\sigma_{y|X=x}}\right]^{\rho} e^{-\left[\frac{y-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}}$$

and note that in this case, the regression formulas,

$$E(Y|X=x) = E(Y) + \rho \frac{\sigma_y}{\sigma_x}(x-E(X)) \text{ and } \sigma_{y|X=x} = \sigma_y \sqrt{1-\rho^2}$$

become, with the proper parameters,

$$E(Y|X=x) = \frac{1}{\lambda}\left[1 + \frac{\rho}{\sigma_x}(x-E(X))\right]; \; \sigma_{y|X=x} = \frac{1}{\lambda}\sqrt{1-\rho^2}$$

Next, verify that $\int_0^\infty f(xy)dy = \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-E(X))^2}{2\sigma_x^2}}$, observe that

$$\int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_x} e^{\frac{-(x-E(X))^2}{2\sigma_x}} e^{\left[\frac{-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} \cdot \frac{(\rho+1)}{\sigma_{y|X=x}} \cdot \left[\frac{y-E[Y|X=x]}{\sigma_{y|X=x}}\right]^{\rho} e^{-\left[\frac{y-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-E(X))^2}{2\sigma_x^2}} e^{\left[\frac{-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} \{-e^{-\left[\frac{y-E(y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} \Big|_0^\infty \}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-E(X))^2}{2\sigma_x^2}}$$, which is the marginal density of X.

On the other hand, $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-E(X))^2}{2\sigma_x^2}} dx = 1$ which shows that

$f(xy) = f(x) \cdot f(y|X=x)$ is a density distribution.

We may also write, $f(x,y)$ as $f(y) \cdot f(x|Y=y)$

$$= \lambda e^{-\lambda y} \frac{1}{\sigma_x \sqrt{2\pi(1-\rho^2)}} e^{-\frac{[x-E(X|Y=y)]^2}{2\sigma^2_{X|Y=y}}}$$

In that case, the regression formulas

$$E(X|Y=y) = E(X) + \rho \frac{\sigma_x}{\sigma_y} (y - E(Y)) \; ; \; \sigma_{x|Y=y} = \sigma_x \sqrt{1-\rho^2}$$

become, with the proper parameters,

$$E(X|Y=y) = E(X) + \rho \, \sigma_x \, (x - \tfrac{1}{\lambda}) \; ; \; \sigma_{x|Y=y} = \sigma_x \sqrt{1-\rho^2}$$

We now verify that $\int_{-\infty}^{\infty} f(xy)\, dx = \lambda e^{-\lambda y}$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda y} \cdot \frac{1}{\sigma_x \sqrt{2\pi(1-\rho^2)}} e^{-\frac{[x-E(x|Y=y)]^2}{2\sigma^2_{x|Y=y}}} dx$$

$= \lambda e^{-\lambda y}$ which is the marginal density of Y.

On the other hand, $\int_0^\infty \lambda \, e^{-\lambda y} dy = 1$ which shows that $f(xy) = f(y)\, f(\mathbf{x}|Y=y)$ is a density distribution.

Next, we shall evaluate the expected value of $Z \equiv X \vee Y$ for the case on hand. Now, $E(Z) = \int_{-\infty}^{\infty} x \int_{0}^{x} f(x) \cdot f(y|X=x)\, dydx + \int_{0}^{\infty} y \int_{0}^{y} f(y)\, f(x|Y=y)\, dxdy.$ The first integral becomes ,

$$\int_{-\infty}^{\infty} x \int_{0}^{x} \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-E(X))^2}{2\sigma_x^2}} e^{\left[\frac{-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} \frac{(\rho+1)}{\sigma_{y|X=x}} \cdot \left[\frac{y-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho}$$

$$\cdot\, e^{-\left[\frac{y-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1}} dy\, dx$$

$$= E(X) - \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma_x} \exp\{ -\frac{(x-E(X))^2}{2\sigma_x^2} + \left[\frac{-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1} - \left[\frac{x-E(Y|X=x)}{\sigma_{y|X=x}}\right]^{\rho+1} \} dx$$

(1)

While the second integral in the expression for $E(Z)$ becomes,

$$\int_{0}^{\infty} y \int_{-\infty}^{y} \lambda e^{-\lambda y} \cdot \frac{1}{\sigma_x \sqrt{2\pi(1-\rho^2)}} e^{-\frac{[x-E(X|Y=y)]^2}{2\sigma^2_{x|Y=y}}} dx\, dy,$$

$$= \int_{0}^{\infty} y\lambda\, e^{-\lambda y}\, F_{N(0,1)} \left(\frac{y-E(X|Y=y)}{\sigma_{x|Y=y}} \right) dy \ldots \qquad (2)$$

Therefore, $E(X \vee Y)$ in this particular case is equal to (1), (2), which can always be evaluated by numerical methods.

§ 5 PRACTICAL BOUNDS

In Technical Report, we have shown that the assumption of independence between two activities can be relaxed by the introduction of joint distributions. But the results are quite complicated and an extension to the case of more than two random variables is difficult and tedious. This leads us to search for a new formulation of the P.E.R.T. type problems.

§17 PRACTICAL BOUNDS

In this chapter, we have shown that the assumption of independence between two activities can be relaxed by the introduction of joint distributions. But the results are quite complicated and an extension to the case of more than two random variables is difficult and tedious. This leads us to search for a new formulation of the P.E.R.T. type problems, a task which we take up in the following Chapters.