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ABSTRACT

This paper presents a characterization of all minimum cuts, separating a source from a sink in a network. A binary relation is associated with any maximum flow in this network, and minimum cuts are identified with closures for this relation. As a consequence, finding all minimum cuts reduces to a straightforward enumeration. Applications of this result arise in sensitivity and parametric analysis of networks, algorithms for minimum cost flow, the vertex packing and maximum closure problems, in unconstrained boolean optimization and project selection, as well as in other areas of application of minimum cuts.

I. INTRODUCTION

Consider a finite directed network with positive arc capacities, and two special vertices, a source s and sink t. The problem of finding a cut separating s from t, with minimum capacity can be solved by applying any maximum flow algorithm and using the maximum flow-minimum cut theorem of Ford and Fulkerson. Here we consider the problem of finding all the minimum cuts.

It appears that this is only the problem of finding all optimum solutions to a linear programming problem. However, this is not a simple task . Consider for instance a network with n vertices and 2n-4 arcs, namely (s,i) and (i,t) for all vertices $i \neq s$ and t, all with equal capacities (see Figure I): this network admits 2^{n-2} cuts separating s from t, all being minimum cuts. It follows that we cannot expect a polynomial algorithm for finding all minimum cuts.

In the next section, we show that we can associate a binary relation with every network, such that finding all minimum cuts reduces to finding all closures for this relation. There exist efficient enumerative methods for generating all closures, thus producing all minimum cuts. In addition this associated binary relations provides more insight into the structure of minimum cuts in a network.

In the last section, we mention several applications in which it is useful to know all the minimum cuts in a network or at least all the arcs which belong to some minimum cut. In these applications, finding all the minimum cuts allows to better solve the problem considered, or at least helps reducing the computational burden for a subsequent algorithm.

2. STRUCTURE OF MINIMUM CUTS

We are given a finite directed network N=(V,A,c), with vertex set V, including a source s and a sink t, arc set A and positive arc capacities c_{ij} defined on A. Given two disjoint subsets S and T of V, we denote by (S,T) the set of all arcs in A with tail in S and head in T. When a function f is defined on A, we denote by f(S,T) the sum of the values of f on the arcs in (S,T). A cut separating s from t is any arc set $(S,\bar{S})$ where $s \in S$, $\bar{S}=V-S$ is the complement of S and $t \in \bar{S}$. By a minimum cut we mean a cut separating s from t with minimum capacity.

Given a binary relation R defined on V, whenever iRj we say that i is a predecessor of j and j is a successor of i. A subset $C \subseteq V$ is a closure [18] for R iff for all vertices $i, j \in V$, the conditions $i \in C$ and iRj imply $j \in C$. (This is sometimes called an hereditary subset for R, [7]).

Consider any maximum flow f in N. From the maximum flow-minimum cut theorem of Ford and Fulkerson [5], we know that such a flow exists and has a value equal to the minimum capacity of a cut. We assume that such a maximum flow is given, since it can be computed by efficient algorithms (see [22] for a recent review).

THEOREM I:

Let f be any maximum flow in N. Define a relation R on the vertex set V as follows:

$$iRj \Longleftrightarrow ((i,j) \in A \text{ and } f_{ij} < c_{ij}) \text{ or } ((j,i) \in A \text{ and } f_{ji} > 0) \quad (1)$$

Then a cut $(S,\bar{S})$ separating s from t is a minimum cut if and only if S is a closure for R containing s and not t.

PROOF:

Consider a cut $(S,\bar{S})$ separating s from t. For any feasible flow f in N, we have.

$$c(S,\bar{S}) \geqslant f(S,\bar{S}) - f(\bar{S},S) \quad (2)$$

and equality holds if and only if both f is a maximum flow and $(S,\bar{S})$ is a minimum cut. Thus if $(S,\bar{S})$ is a minimum cut then for all arcs $(i,j) \in (S,\bar{S})$ we have $f_{ij} = c_{ij}$ and for all arcs (j,i) with $i \in S$ and $j \in \bar{S}$ we have $f_{ji}=0$. This implies that S is a closure for R, containing s and not t, for otherwise there would exist two vertices i and j such that $i \in S$, $j \in \bar{S}$ and either $f_{ij} < c_{ij}$ or $f_{ji} > 0$, a contradiction. Conversely, consider a closure S for R, containing s and not t. For every arc (i,j) in $(S,\bar{S})$ we must have $f_{ij} = c_{ij}$, and for every arc (j,i) in $(\bar{S},S)$ we must have $f_{ji} = 0$. It follows that equality holds in (2) and thus $(S,\bar{S})$ is a minimum cut.

#

This theorem gives more insight into the structure of minimum cuts in N. The following proposition is immediate from the definition of a closure:

PROPOSITION 2:

Given a binary relation R on a set, if C and C' are closures for R, then $C \cup C'$ and $C \cap C'$ are also closures for R.

Hence the following corollary [5], a proof of which requires two pages in [10]:

COROLLARY 3:

If $(S,\bar{S})$ and $(S',\bar{S}')$ are minimum cuts in a network N, then $(S \cup S', \overline{S \cup S'})$ and $(S \cap S', \overline{S \cap S'})$ are also minimum cuts in N.

Given a maximum flow, the corresponding relation R can be deduced by a simple examination of all the arcs in A. Distinct maximum flows may produce different relations but the set of closures remains the same. Define the transitive closure $\tilde{R}$ of a binary relation R as the smallest binary relation on the same set, containing R. The following proposition is easily proven:

PROPOSITION 4:

A subset C is a closure for R if and only if it is a closure for $\tilde{R}$. A bit more difficult to prove is the following:

PROPOSITION 5 [24]:

If R and R' are transitive relations defined on the same set, such that any subset C is a closure for R if and only if it is also a closure for R', then R = R'.

Thus the different binary relations defined by different maximum flows have the same transitive closure, which we call the preorder R associated with the network N. This transitive closure can be obtained by an algorithm in $O(|V|^3)$ of Warshall [30], or slightly better, in $O(|V|^{2.81})$ by using the fast matrix multiplication [4], [2]. It can also be obtained in linear expected time by using the algorithm of Schnorr [26].

Once this transitive closure is obtained, we can shrink its strong connected components to single vertices. The resulting relation $\bar{R}$ on the reduced set $\bar{V}$ is acyclic, that is a precedence relation (or a partial order). After eliminating the component T containing the sink t, and all its predecessors (which cannot belong to a closure not containing T) and the component S containing the source s, and all its successors (which must belong to a closure containing S) we are left with a further reduced relation, every closure of which induces (after addition of S and all its successors) a minimum cut in N.

AN EXAMPLE:

Consider the network given by Figure 2. A maximum flow is given in Figure 3. The associated relation R appears on Figure 4, where an arc (i,j) represents iRj and a bidirected arc (i,j) stands for both iRj and jRi (when the corresponding arc has flow strictly between zero and its capacity). The strong connected components are

S = {s,2}, T = {t,8,12}, V1 = {1}, V3 = {3,7}

V4 = {4}, V5 = {5,9} and V6 = {6,10,11}, and after shrinking these to a single vertex, the resulting relation $\bar{R}$ is given by Figure 5. Here V3 is a successor of S and V6 is a predecessor of T. The other components V1, V4 and V5 are all predecessors of S and successors of T, and they induce the relation given in Figure 6. This relation admits six closures C, each one defining a minimum cut $(X,\bar{X})$, as follows:

C = ∅ and X = S ∪ V3

C = {V1} and X = S ∪ V3 ∪ V1

C = {V1, V4} and X = S ∪ V3 ∪ V1 ∪ V4

C = {V1, V4, V5} and X = S ∪ V3 ∪ V1 ∪ V4 ∪ V5

C = {V1, V5} and X = S ∪ V3 ∪ V1 ∪ V5

C = {V4} and X = S ∪ V3 ∪ V4

3. APPLICATIONS AND EXTENSIONS

The main result of the previous section provides more insight into the structure of minimum cuts in a network. In this section we mention several domains of applications for this result.

The structure revealed by the preorder associated with the network can be used to simplify sensitivity and parametric analyses of the maximum flow. In sensitivity analysis, it is required to find all the arcs such that a modification (increase or decrease) of the capacity of one of them implies a modification of the maximum value of a flow. It is clear that only saturated arcs are to be considered, and that any reduction in the capacity of an arc which belongs to some minimum cut implies a reduction in the flow value. These arcs are identified as follows:

COROLLARY 6:

A saturated arc belongs to some minimum cut if and only if its ends do not lie in the same connected component of the relation R.

On the other hand, an increase in the capacity of an arc allows an increase in the flow value if and only if this arc has its tail in the connected component containing the source (or some successor of it) and its head in the component containing the sink (or some predecessor of it). Similar results apply to various parametric analyses such as adding new arcs or nodes [29], finding the most vital arcs [13], [28], [31] or nodes [9] and in the analysis of dynamic maximum flow [15, pp. 128-151]. One practical application of dynamic maximum flow is the modeling of building evacuation [6]: given the minimum evacuation time, it is desired to detect all evacuation bottlenecks which may cause delays and to which special attention must be given; these are precisely the arcs which belong to some minimum cut.

Another application arises in computations of minimum cost flows. In the primal-dual method [5], only one dual variable change is necessary between flow augmentations if the cheapest set of arcs required to connect the source component to the sink component is identified; using the related binary relation is an alternative to shortest path computations, taking advantage of zero reduced costs, which might be of practical interest. For the primal method Cycle [12] T.C. Hu has noted that, using any minimum cut, "we can split the network into two parts and find negative cycles in each of these parts" [10, p.173]; indeed we can further decompose the network and restrict the search for negative cycles to each connected component of the associated preorder.

Consider now the problem of finding all minimum cuts in a network. This is equivalent to enumerating all the closures for the associated binary relation R, and we can apply a procedure of Gutjahr and Nemhauser [8], or of Schrage and Baker [27]. The Schrage-Baker procedure appears very efficient, requiring very little bookkeeping effort for every closure generation. Identifying all minimum cuts is useful whenever a problem is reduced to finding a minimum cut in a network satisfying additional constraints. Consider for example the vertex packing problem in a vertex-weighted undirected graph [16]: solving a linear programming relaxation of one integer programming formulation can be achieved by finding a minimum cut in a related bipartite network, producing a solution with values 0,1 or $\frac{1}{2}$ and it is desired to find a solution with the maximum number of 0,1 components [20]; this can be achieved by classical sensitivity analysis [16], or by a specialized algorithm [19] and also by identifying all minimum cuts and retaining the one producing the most integral solution.

Another problem amenable to a minimum cut solution, which has significant practical implications is the maximum closure problem [18], a generalization of the selection problem [25], [3]. In investment application, or in mining engineering, it is desirable to obtain all solutions with maximum weight, from which a "best" one is selected on the basis of ill-formulated constraints or objectives (e.g. [14]). In mathematical programming, the unconstrained maximization (or minimization) of a boolean polynomial can be approached by solving a related maximum closure problem [21]; the corresponding solution may be overestimated, by omission of some nonlinear terms with negative costs which cannot be covered by other positive terms (see [21] for further details) and identification of all optimal closures may be useful by producing several tentative solutions from which the best one can be retained as an incumbent in a subsequent branch-and-bound algorithm. There are several other applications of minimum cuts and maximum closures, which may benefit from identification of all optimal solutions and the reader is refered to [22] for a more detailed survey.

The results of this paper can be extended to undirected networks and to networks with lower capacities. Any undirected network can be converted to a directed network by arbitrarily directing its edges and adding some source and sink-arcs, such that the relative capacities of the cuts remain unchanged [23]. Hence all the minimum cuts of an undirected networks can be found after this reduction by applying the previous results. Among possible applications are a layout problem of electrical connexions on a line [1] and the design of optimum communication networks [11]. The results of this paper also extend to networks with lower capacities [5], and this is left to the reader as an exercise. The project time/cost tradeoff problem of critical path analysis can be approached by finding minimum cuts in the project network, which includes both lower and upper capacities [17]. The authors note that the minimum cut is not necessary unique and state: "The practical significance of this fact is that a decision based on other than cost must be rendered to select a minimal cut set" [17, p.396]. Clearly, this selection process is best performed when all minimum cuts have been identified.

REFERENCES

[1] ADOLPHSON, D. and HU, T.C., "Optimal Linear Ordering", SIAM J. Appl. Math. 25, 403-423, (1973).

[2] AHO, A., HOPCROFT, J., and ULLMAN, J., The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading, Mass., (1974).

[3] BALINSKI, M.L., "On a Selection Problem", Management Science 17, 230-231, (1970).

[4] FISCHER, M.J. and MEYER, A.R., "Boolean Matrix Multiplication and Transitive Closure", Twelfth Annual IEEE Symp. on Switching and Automata Theory, (East Lansing, MI, 1971), 129-131, (1971).

[5] FORD, L.R. and FULKERSON, D.R., Flows in Networks, Princeton University Press, Princeton, N.J., (1962).

[6] FRANCIS, R., "Networks Optimization Models of Building Evacuation", private communication, (1978).

[7] GRATZER, G., Lattice Theory: First Concepts and Distributive Lattices, W.H. Freeman and Co., San Francisco, Cal., (1971).

[8] GUTJAHR, A.L. and NEMHAUSER, G.L., "An Algorithm for the Line Balancing Problem", Management Science, 11, 308-315, (1964).

[9] CHANG, Han, "Finding the n Most Vital Nodes in a Flow Network", doctoral dissertation, the University of Texas at Arlington, (1972).

[10] HU, T.C., Integer Programming and Network Flows, Addison-Wesley, Reading, Mass., (1970).

[11] HU, T.C., "Optimum Communication Spanning Trees", SIAM J. Comput. 3, 188-195, (1974).

[12] KLEIN, M., "A Primal Method for Minimal Cost Flows", Management Science, 14, 205-220, (1967).

[13] LUBORE, S.H., RATLIFF, H.D. and SICILIA, G.T., "Determining the Most Vital Link in a Flow Network", Naval Res. Log. Quart, 18, 497-502, (1971).

[14] McGINNIS, L.F. and NUTTLE, H.L.W., "The Project Coordinator's Problem", Omega 6, 325-330, (1978).

[15] MINIEKA, E., Optimization Algorithms for Networks and Graphs, Marcel Dekker Inc., New Yorkm N.Y., (1978).

[16] NEMHAUSER, G.L. and TROTTER, L.E., "Vertex Packings: Structural Properties and Algorithms", Math. Programming 8, 232-248, (1975).

[17] PHILLIPS, S., Jr. and DESSOUKY, M.I., "Solving the Project Time/Cost Tradeoff Problem Using the Minimal Cut Concept", Management Science 24, 393-400, (1977).

[18] PICARD, J.-C., "Maximal Closure of a Graph and Application to Combinatorial Problems", Management Science 22, 1268-1272, (1976).

[19] PICARD, J.-C. and QUEYRANNE, M., "Vertex-Packings: (VLP)-Reductions through Alternate Labeling", Techn. Rep. EP75-R-47, Ecole Polytechnique de Montréal, Canada, (September, 1975).

[20] PICARD, J.-C. and QUEYRANNE, M., "On the Integer-Valued Variables in the Linear Vertex Packing Problem", Mathematical Programming 12, 97-101, (1977).

[21] PICARD, J.-C. and QUEYRANNE, M., "Networks, Graphs and Some Nonlinear 0-1 Programming Problems", Techn. Rep. EP77-R-32, Ecole Polytechnique de Montréal, Canada, (1977).

[22] PICARD, J.-C. and QUEYRANNE, M., "A Survey of Applications of the Maximum Flow and Minimum Cut Problems", INFOR (to appear).

[23] PICARD, J.-C. and RATLIFF, H.D., "Minimum Cuts and Related Problems", Networks 5, 357-370, (1975).

[24] QUEYRANNE, M., "Anneaux achevés d'ensembles et préordres", Techn. Rep. EP77-R-14, Ecole Polytechnique de Montréal, Canada, (1977).

[25] RHYS, J.M.W., "A Selection Problem of Shared Fixed Costs and Network Flows", Management Science 17, 200-207, (1970).

[26] SCHNORR, C.P., "An Algorithm for Transitive Closure with Linear Expected Time", SIAM J. Comput 7, 127-133, (1978).

[27] SCHRAGE, L. and BAKER, K.R., "Dynamic Programming Solution of Sequencing Problems with Precedence Constraints", Operations Research 26, 444-449, (1978).

[28] SICILIA, G.T., "Finding the n Most Vital Links in a Network", doctoral dissertation, University of Florida, Gainesville, (1970).

[29] TOPKIS, D.M., "Monotone Minimum Node-Cuts in Capacitated Networks", Res. Rept. ORC 70-39, University of California, Berkeley, (1970).

[30] WARSHALL, S., "A Theorem on Boolean Matrices", J. Assoc. Comput. Mach. 9, 11-12, (1962).

[31] WOLLMER, R., "Sensitivity Analysis in Networks", Techn. Rep. ORC 65-8, University of California, Berkeley, (1965).

ANNEXE I

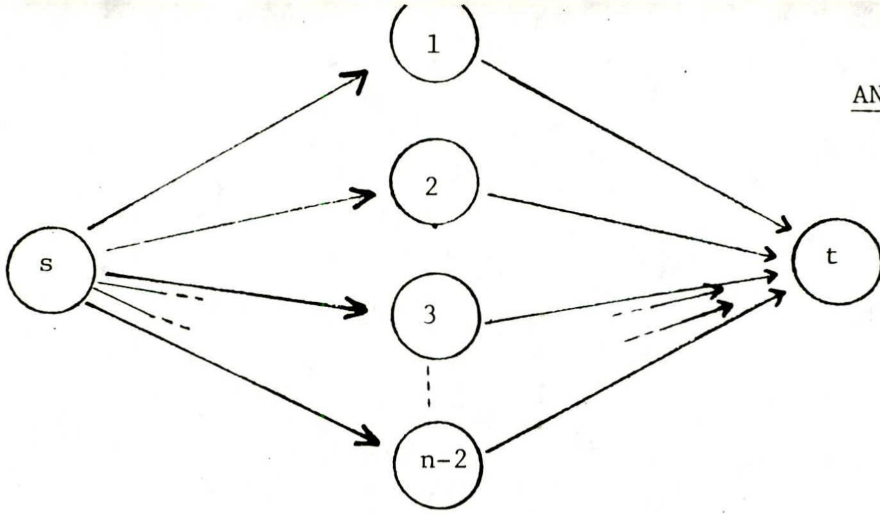

FIGURE I

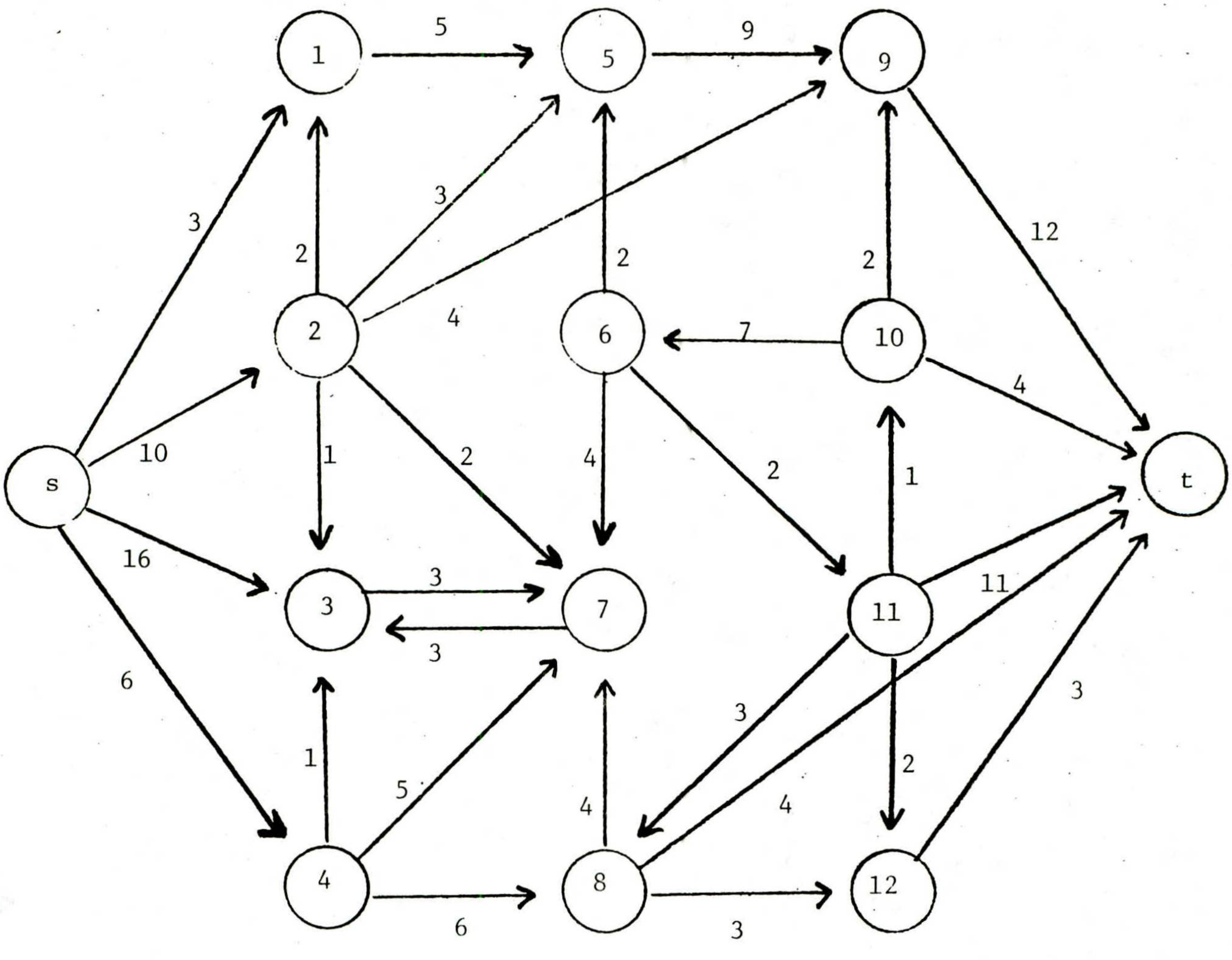

FIGURE 2

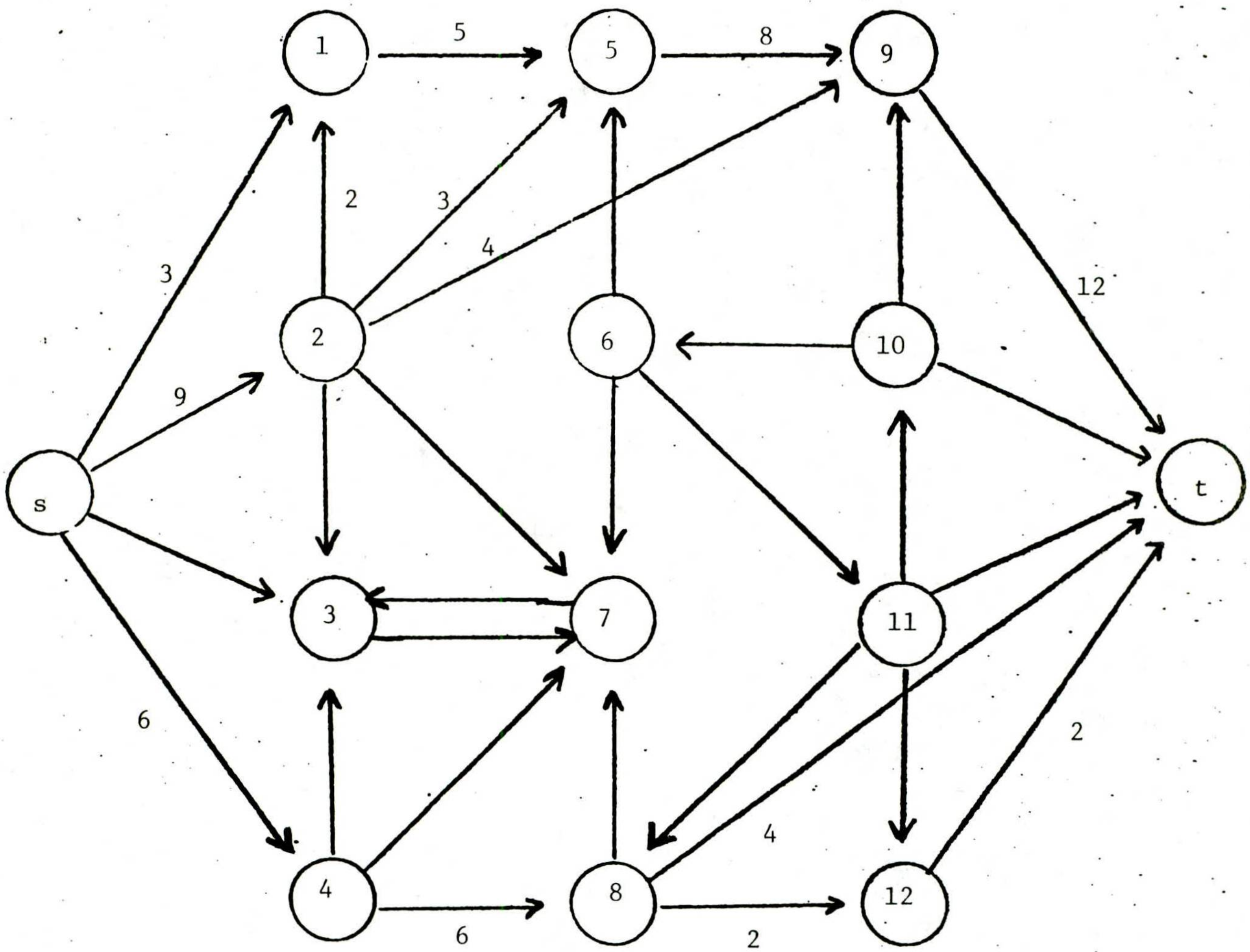

FIGURE 3

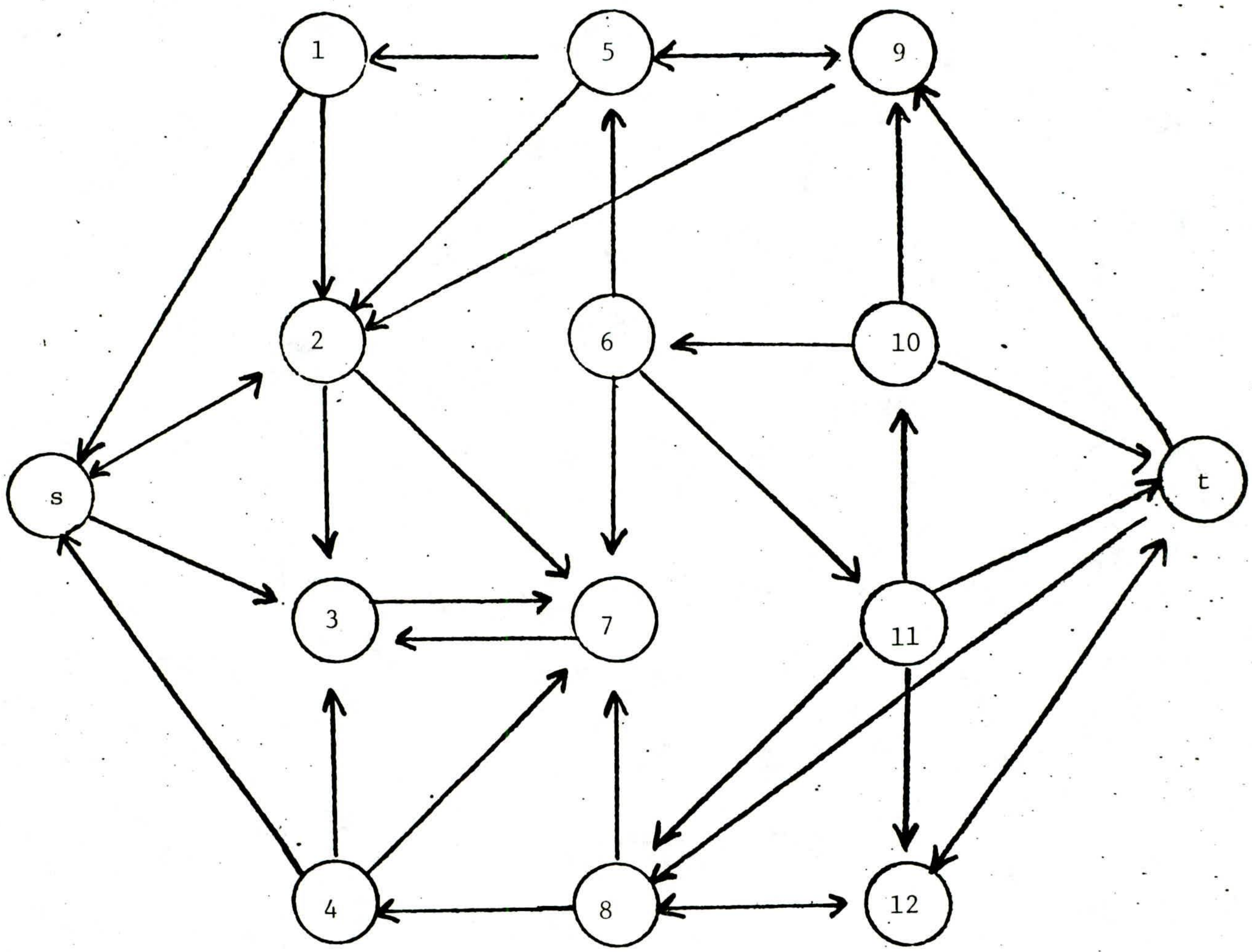

FIGURE 4

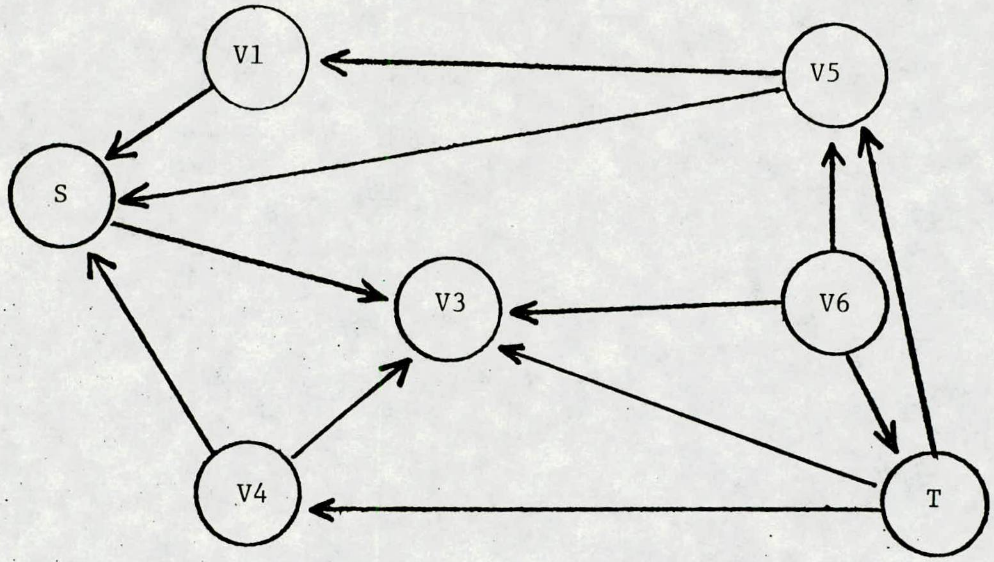

FIGURE 5

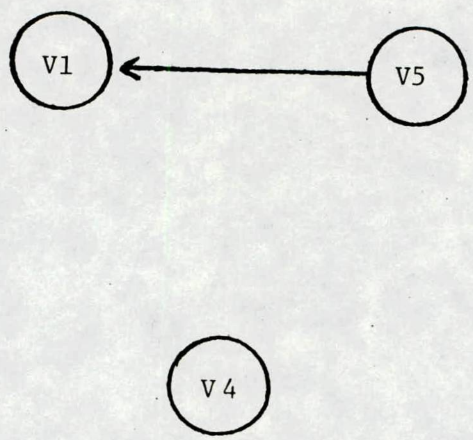

FIGURE 6