Charles University in Prague

Faculty of Social Sciences
Institute of Economic Studies

RIGOROUS THESIS

My ventures are not in one bottom trusted

Comparative study to Modern Portfolio Theory and Black-Litterman portfolio formation

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Declaration of Authorship

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Abstract

This work uses Lagrange multiplier solution to Modern Portfolio Theory and Monte-Carlo simulation to explain large variations in Mean Variance optimized portfolios. Author also summarized main criticism of Modern Portfolio Theory and suggested a better solution of using Black-Litterman framework. Practical part of the thesis revealed a high significance of expected variance-covariance matrix for portfolio weights. Author compared unintuitive and sensitive weights of Mean Variance optimization to Black-Litterman portfolios based on implied returns and analysts' predictions. Essay gave an example of insensitivity of Black-Litterman portfolios to expected covariance and by using Monte Carlo simulation presented superiority of Black-Litterman to Markowitz's optimization.

JEL Classification	G11, G14, C58, B4
Keywords	Markowitz, Black-Litterman, Modern Portfolio Theory, Portfolio formation

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Teze RIGORÓZNÍ práce
Tyto teze tvoří přílohu „Přihlášky ke státní rigorózní zkoušce"

<table>
<tr><td colspan="2">TUTO ČÁST VYPLŇUJE ŽADATEL:</td></tr>
<tr><td>Informace o žadateli:</td><td rowspan="5">Razítko podatelny:

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Komparativní studie k Moderní teorii portfolia a Black-Littermanově tvorbě portfolia</td></tr>
<tr><td colspan="2">Předpokládaný název rigorózní práce v angličtině:
Comparative study to Modern Portfolio Theory and Black- Litterman portfolio formation</td></tr>
<tr><td colspan="2">Předpokládaný termín předložení práce:
10.9.2013</td></tr>
<tr><td colspan="2">Pedagog, s nímž byly teze konzultovány :
PhDr. Petr Gapko</td></tr>
<tr><td colspan="2">Charakteristika tématu a jeho dosavadní zpracování žadatelem (rozsah do 1000 znaků):
Autor se bude zabývat za pomoci simulace Monte-Carlo a řešení Markowitzovi teorie portfolia pomocí Lagrangeho multiplikátorů několika problematickými body v optimalitě nabídnutého portfolia a procesu, kterým Markowitzova teorie v praxi dosahuje výsledku. Zejména pak neintuitivními extrémními hodnotami některých pozic v portfoliu. V rigorózní práci by se chtěl zabývat vysokým vlivem očekávaných hodnot variančně-kovarianční matice na váhy v portfoliu, ale nízkým vlivem na celkovou výkonnost portfolia. Dále prozkoumá dlouhodou stabilitu variančně-kovarianční matice v čase. Aplikací Monte-Carlo simulace se pokusí získat dostatečné množství dat ke studiu, zda Black-Littermanova metoda vyřešila problematické body Markowitzovi optimalizace portfolia a zároveň dosáhla pozitivního vlivu díky možnosti aplikace vlastního názoru investora na budoucí výkonnost trhů. Dále doplní diskusi některých dalších problémů, které byly již diskutovány ve starší literatuře a pokusím se najít jejich spojitost k vlastním výsledkům a případně zda-li dojde k jejich odstranění využitím Black-Littermanovi metody.</td></tr>
<tr><td colspan="2">Předpokládaný cíl rigorózní práce, původní přínos autora ke zpracování tématu, případně formulace problému, výzkumné otázky nebo hypotézy (rozsah do 1200 znaků):
I přes nesporný teoretický přínos Markowitzovi teorie portfolia k modernímu pojetí skladby portfolia je jeho praktická použitelnost diskutabilní. V práci vysvětlíme problematické body, ukážeme příklady extrémních portfolií a jejich neintuitivní jednotlivé pozice. Hlavním cílem bude prokázat mnohem vyšší</td></tr>
</table>

praktickou využitelnost Black-Littermanovi metody za využití simulace Monte-Carlo, která umožní srovnání mnoha tržních situací a jejich výstupů.

Předpokládaná struktura práce (rozdělení do jednotlivých kapitol a podkapitol se stručnou charakteristikou jejich obsahu)**:**

1. Metodologie
   Vysvětlíme Moderní teorii portfolia i s jejím řešením pomocí matic a lagrangeovými multiplikátory. Druhé řešení je důležité pro umožnění aplikaci Monte-Carlo simulace. Dále vysvětlíme jaké Monte-Carlo simulace využijeme a jejich základní rámec.
2. Projdeme základní kritické články k teorii a shrneme s jakými problémy se potýkaly jiné studie na téma Moderní teorie portfola.
3. Stabilita kovarianční matice v čase
4. Test Markowitzovi teorie portfolia
   Za využití statistických nástrojů přidělíme jednotlivým proměným jejich teoretické distribuce a křížové korelace. To nám umožní vytvořit model náhodně generující tržní situace a analyzovat dopady jednotlivých proměných na pozice a výnosy portfolia.
5. Black-Littermanův model
   Vysvětlíme základní principy použití Black-Littermanova modelu pro formaci portfolia. Nastavíme statistický rámec pro test metody za využití simulace Monte-Carlo tak, aby výsledky byly porovnatelné se závěry z kapitoly 3.

Vymezení podkladového materiálu (např. analyzované tituly a období, za které budou analyzovány) **a metody (techniky) jeho zpracování:**
Jako základní data použijeme 6 významných akciových titulů z indexu S&P 500. Využijeme data od roku 1984 do roku 2011. Důležite pro řešení bude využití statistických metod jako Anderson-Darling test, Monte-Carlo simulace, regresní analýzu a testy autokorelace.

Základní literatura (nejméně 10 nejdůležitějších titulů k tématu a metodě jeho zpracování; u všech titulů je nutné uvést stručnou anotaci na 2-5 řádků)**:**

Black, F. & R. Litterman, 1990, Asset Allocation: Combining Investors Views with Market Equilibrium, Fixed Income Research, Goldman, Sachs (September 1990).
Základní literatura na Black-Littermanovu metodu. Výzkumníci v ní popisují problematičnost Markowitzovi teorie a nabízejí alterantivu. Důležitou částí je zahrnutí investorova názoru do praktického formováni portfolio.

Cochrane, J.H., 2005, Asset Pricing, Princeton University Press
Kniha se poměrně rigorózně zabývá oceňováním aktiv za využití stochastického diskotního faktoru. Na základě svého výzkumu Cochrane prosazuje strategii pasivního investování, která je základem pro použití modelů popisovaných v této práci.

Dunn, W.L., Shultis J.K., 2011, Exploring Monte Carlo Methods, Elsevier Science & Technology
V této literatuře autoři zevrubně popisují metodu Monte-Carlo. Popisují ji jako metodu táhání čísel z klobouku.

Haugh, M, The Monte Carlo Framework, Examples from Finance and Generating Correlated Random Variables, 2004, 10 p
Nabízí praktické příklady využití metody Monte-Carlo

Malkiel, B.G., 2003, Passive Investment Strategies and Efficient Markets, European Financial Management, Vol. 9, No. 1, 2003, 1-10

Autoři se zabývají přidanou hodnotou aktivního investování pro průměrného investora. Silně obhajují využití pasivního přístupu k portfoliu.

Markowitz, Harry, 1952, Portfolio Selection, The Journal of Finance, Vol. 7, No. 1. (Mar., 1952), pp. 77-91

Základ Moderní teorie portfolio. Harry Markowitz zde pokládá základy teorie, která je prvním rigorózním přístupem k formaci portfolia

Markowitz, Harry, 1991, Portfolio Selectio, Blackwell, Oxford.

Michaud, R.O., 1989, The Markowitz Optimization Enigma: Is Optimized Optimal? Financial Analysts Journal, vol. 45, no. 1 (January/February): 31-42.

Kritická literature k Markowitzovu dílu poukazující na mnohé nedostatky a problematické využití teorie v praxi.

T. W. Anderson and D. A. Darling, 1952, Asymptotic Theory of Certain "Goodness of Fit" Criteria Based on Stochastic Processes, Ann. Math. Statist. Volume 23, Number 2, 193-212

Pro tuto práci bude velmi důležité správně určit statistické rozdělení, ze kterých jsou zvolené proměnné distribuovány. Tato kniha se zabývá jednou ze statistických možností jak toto určit. Velmi efektivní pro fat-tails distribuce, které jsou běžné ve finančních trzích.

Murphy, J Michael, 1977, Efficient markets, index funds, illusion, and reality, Journal of Portfolio Management

Praktická studie na téma indexového investování, které je jednou z nejčastějších forem pasivního investování.

Diplomové a disertační práce k tématu (seznam bakalářských, magisterských a doktorských prací, které byly k tématu obhájeny na UK, případně dalších oborově blízkých fakultách či vysokých školách za posledních pět let)

Datum / Podpis studenta

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Návrhy na oponenta:		**Příjmení a jméno** ………………………	

Table of Contents

1. Introduction

Prior to Harry Markowitz's work portfolio formations were purely intuitive. He provided an effective and rigorous idea to form a portfolio using mathematical and statistical methods and to obtain a portfolio with higher return and lower risk then individual stocks. His theory was formed in 1952 and was heavily celebrated. Markowitz explained effect of covariance on total portfolio performance and introduced an idea of diversified portfolios performing better then individual shares. Although his idea was revolutionary and he has been awarded a Nobel Prize in the year 1990, practical use of his theory has been rare (Michaud, 1989).

There were many prior financial crises to the one of 2011. However in our research we'll use the one most recent since the data are easily accessible and its aftermath is still felt in the financial and banking sector around the world.

In august 2011 we saw significant decline in stock prices around the world. Once again there was a panic in capital markets around the world as we can see in then current newspaper articles as *"Dow plunges as $2.5tn erased from equities"*[1], *"Global Bonds Gain $132 Billion as Stock Rout Cuts $7.8 Trillion"*[2] and the shock was global *"Istanbul Stock Index Falls 5% In Early Trading, Extending Losses"*[3] and *"Japan follows Switzerland by weakening currency".* Effects on portfolios were large and wiped out significant values of investors' wealth. Even though we are sure investors used different investment strategies we believe there was a need for shift in strategy since there were large changes in markets around the world. Investors are mostly worried only about losing value. It is not only return that is changing during such an upheaval but there is a growth in volatility (Schwert, 1990) and growth in asset correlations (Forbes, Rigobon, 2002) as well. Investors should take all the new information into account and amend their portfolios since their portfolios can lose their optimality.

It's been a long time since Harry Markowitz introduced his revolutionary work on Modern Portfolio Theory (MPT) but it is still one of the most influential financial hypotheses. We will study its performance on the background of the recent market turmoil using a Monte-Carlo framework. We will compare MPT to

[1] *The Irish Times*. 9 August 2011. Retrieved 10 August 2011.
[2] McDonald, Sarah (9 August 2011). *Bloomberg Businessweek*. Retrieved 10 August 2011.
[3] Candemir, Yeliz (9 August 2011 *The Wall Street Journal* (Istanbul). Retrieved 9 August 2011.

Black-Litterman model in practical example and artificial Monte Carlo framework. We will decide which one is more efficient and more suitable for practical use.

The focus of this study will be on portfolio formation strategy and influences of variability of returns and variance-covariance matrices. We expect to show that optimal portfolio formation is time variable and it is difficult to keep an efficient portfolio for a long time without extensive trading. Since investor is maximizing return and minimizing transaction costs it is desirable to keep number of trades on the lowest possible number while maintaining portfolio efficiency.

2. Theoretical framework

Harry Markowitz (1952) work caused a revolution in investment management. Prior to Markowitz's work managers decided on structures of their portfolios based on their subjective feeling about future returns and risks of each individual stock. In the year 1952 professor Harry Markowitz laid basics of Modern Portfolio Theory and in connection with starting computerization it was possible to employ mathematics and statistics to financial markets. However it is important to mention that contrary to popular belief investors did diversify their portfolios prior Markowitz's work just without extensive rigorous attitude as he offered.

> *"My ventures are not in one bottom trusted, Nor to one place; nor is my whole estate. Upon the fortune of this present year; Therefore, my merchandise makes me not sad. "*
>
> William Shakespeare, "Merchant of Venice", *1598,* Act I, Scene 1

The Shakespeare cited above is a practical example how a Venice merchant Antonio diversified his risk intuitively between different assets and also inspired naming of this essay. Markowitz (1999) himself clarifies he is not the one who invented diversification. On the other hand he is definitively the one who hugely improved its understanding and practical use.

In this article we will cover the stability of portfolios towards changes in returns, expected returns and real and estimated covariance matrixes. We will run analysis to evaluate Modern Portfolio Theory's mean variance optimization performance and compare these portfolio formations with outcomes of modern Black-Litterman model that is an extension to original Markowitz's idea trying to overcome some of the original practical problems.

Portfolio formations of Markowitz and Black-Litterman are based on the idea of efficient market hypothesis (EMH). This theory states that markets are informationally efficient. Therefore no investor can consistently achieve excess returns over average risk-adjusted returns given the information known at the moment of investment. EMH usually takes three main forms. Weak, stating that prices of traded assets already reflect all past information. Semi-strong EMH states that asset prices reflect all past information and all publicly available information. Strong EMH claims that asset prices include all publicly and privately available information. Practical impact of this theory is that we cannot beat the market by stock picking.

Although there have been a lot of analysts trying to find a profitable active trading strategies, we will base our reasoning on articles of Malkiel (2003) and Cochrane (2001) who argued in favour of passive investment management. As Malkiel (2003, page 1) put it *"I conclude that the evidence strongly supports passive investment management in all markets… Recent attacks on the efficient market hypothesis do not weaken the case for indexing."* We believe that passive strategies are the most efficient for majority of investors since transaction costs are lowering total portfolio return without significant gain to overall performance. Consequently it is more profitable for majority of investors to form an efficient portfolio and not to trade excessively.

Harry Markowitz gave a revolutionary idea how to form an efficient portfolio and it deserves deeper study how this portfolio performs and how immune it is to changes in market conditions. In the practical part of this essay we will base our research on Monte-Carlo simulation of 10.000 random market conditions and optimum portfolios formed. The data collected will allow us to study effects of changes in investment environment and efficiency of Markowitz's portfolio formation. We will also compare it to a portfolio formed by a Black-Litterman method.

2.1. Modern Portfolio Theory

2.1.1. Historical perspective

As was already mentioned the foundation of MPT was established in 1952. It was in Markowitz's doctoral dissertation on statistics and the most persuasive aspect of his work was his description of influence of number of assets included in a portfolio on portfolio's total variance and inter-asset covariance relationships Megginson (1996). His dissertation findings were published in the year 1952 in The Journal of Finance as "*Portfolio Selection".* The book followed afterwards and in the year 1959 was published under the name of *Portfolio Selection: Efficient Diversification.* Finally in the year 1990 Markowitz was together with Merton Miller and William Sharpe awarded *The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel*[4] for his contributions to the fields of economics and corporate finance.

Obviously there were other contributors to the development of such an extensive theory. One of the most important was James Tobin. He in his essay *"Liquidity Preference as Behavior Toward Risk"* from the year 1958 and published in Review of Economic Studies introduced an idea of Efficient Frontier and Capital Market Line. These are now inseparable parts of Modern Portfolio Theory and are concepts based on Markowitz's prior work. Tobin's idea is that investor will maintain stock portfolios in the same structure as long as he maintains identical expectations regarding the future. Consequently the investor's portfolio will be different only in their relative proportion of stocks and bonds in accordance to individual risk aversion.

Another breakthrough contribution has been independently developed by three academics and is now known as Capital Asset Pricing Model (CAPM). The first to develop an idea of CAPM was Jack Treynor (1962) but his paper hadn't been published until 1999. The most notable contributor to CAPM developments and Nobel prize laureate is William Sharpe he published his work *Capital asset prices: A theory of market equilibrium under conditions of risk* in Journal of Finance in the year 1964. He introduced an idea of Sharpe ratio measuring risk premium per unit of risk. Sharpe also further developed Tobin's concepts of Capital Market Line and Efficient Frontier. Another notable theoretics in the field of CAPM were

[4] http://www.nobelprize.org/nobel_prizes/economics/laureates/1990/#

John Lintner (1965) and Jan Mossin (1966) who further developed the theory that evolved into incredibly important outgrowth of original Markowitz's work.

2.1.2. Theory framework

MPT is a theory and a tool for selection and construction of investment portfolio with properties of simultaneous minimization of investment risk and maximization of expected returns. The revolutionary part of the MPT framework is an ability to measure risk component of asset via various mathematical formulations and minimise it using the concept of diversification. This concept of not putting all eggs in one basket aims to properly select a weighted collection of assets that put together exhibit lower risk factors than isolated investment into each of the assets.

Therefore diversification is the core concept of MPT.

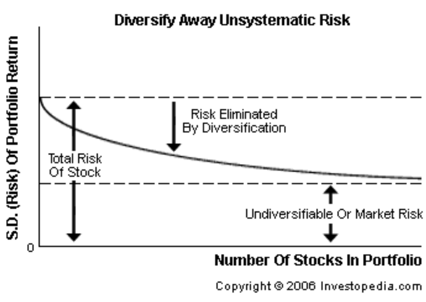
Markowitz's portfolio theory is generally regarded as being a normative theory. Fabozzi, Gupta, & Markowitz (2002) defined this as *" the one that describes a standard or norm of behaviour that investors should pursue in constructing a portfolio..." (p. 7).* Contrarily Sharpe's asset pricing theory is considered as positive theory hence studies investors' actual behaviour and bases its conclusions on observed market performance. Jointly these two theories provide an efficient framework to identify and measure an investment risk in connection with relationships between return and risk.

Markowitz demonstrated that investor's portfolio selection problem could be simplified to two critical dimensions. First is the expected return of a portfolio and second is the variance (used as a measurement of a risk) of a portfolio (Royal Swedish Academy of Sciences, 1990). Crucial for the MPT is the risk reduction potential allowed for by employing the concept of diversification. As McClure (2010) comments on diversification that there is a potential to reduce total portfolio risk since portfolio risk is defined by variances of individual assets and covariances of pairs of assets. In a word of Harry Markowitz (1952) the portfolio selection is to be based on total risk reward characteristics, opposed to simply betting on individual assets with attractive risk-reward characteristics.

2.1.3. Risk and return

One of the most important and discussed concepts of MPT is a risk and its measurement. For the MPT purposes is usually defined as *"deviation away from the mean historical returns during a particular time period. For example, U.S. stocks may average 11 percent returns over time. However, they may see a 33 percent gain one year and an 11 percent loss another year to arrive at that average"* (Bofah, 2013). Nevertheless Markowitz thinks about risk in the context of a portfolio. *"the essential aspect pertaining to the risk of an asset is not the risk of each asset in isolation, but the contribution of each asset to the risk of the aggregate portfolio"* (Royal Swedish Academy of Sciences, 1990). Risk of an asset can be studied in two very different ways. Firstly on a stand-alone basis, so the asset is studied as isolated and secondly on a portfolio basis. If considered on a portfolio basis we can split the risk into two basic components. Systematic risk, also called market or common risk, and unsystematic risk, or because of his characteristics called diversifiable risk (Lowering portfolio risk, 2013). MPT assumes that all portfolios are subjects to these two kinds of risk. Systematics risk is a macro-level risk that is difficult or better put impossible to diversify away. This type of risk affects large number of assets and good examples are e.g. inflation, interest rates, unemployment, exchange rates, global economic conditions, and gross national product. Some of these risks can be partially diversified away at higher costs to investor or swapped for another kind of risk but portfolio in general can't get rid of them completely. Contrarily diversifiable (unsystematic) risk is a micro-level form of risk that affects only single asset or just a small fraction of a market (Ross, Westerfield, & Jaffe, 2002).

Figure 1 Unsystemic and Systemic risk, 2006



It is the best explained by the example taken from the work of Myles (2013*) " the ill-received change in the announced consumer pricing structure of NetFlix resulted in extremely negative consumer response and defections, which resulted in lower earnings and lower stock prices for NetFlix. However, it did not impact the overall stock performance of the Dow Jones or S&P, or even that of entertainment and media industry companies for that matter—with the possible exception of its biggest rival Blockbuster Video, whose value increased significantly as a result of NetFlix' faltering market share. Other examples of unsystematic risk might include a firm's credit rating, negative press reports about a business, or a strike affecting a particular company".* It can be seen in the example given above that if investor held just a particular stock of Netflix he would suffer heavy losses. On the other hand if he held well diversified portfolio of media companies Blockbuster would probably offset the losses inflicted by Netflix on his portfolio. Nevertheless Blockbuster went into administration on January 2013[5] so it would be better to diversify on wider scale in a full spectrum of S&P index. In figure 1 we can see graphical explanation how adding stocks to portfolio lowers total risk until it converges to just market or systemic risk.

Although in reality unsystematic risk can be reduced significantly by adding securities within a portfolio (McClure, 2010) it can never be fully eliminated irrespective how many assets are added into portfolio. The reason is that returns on any asset are to at least some degree correlated. Consequently the absolute diversification takes a form of limit going to infinite and never reaching the level of just systematic risk (Royal Swedish Academy of Sciences, 1990).

It is important to mention even though it is not a part of the MPT that systematic risk can be also reduced. A pair of negatively correlated assets can be used to offset potential losses. For example when there is a global recession investors usually store they cash in gold and short-term treasury notes so their prices are soaring as money flow in from stocks whose prices are falling. This financial operation is called hedging and is usually used at expense of possible future returns. On the other hand diversification is ordinarily lowering riskiness of portfolio without negative effects on potential return.

[5] http://www.bbc.co.uk/news/business-21047652

Risk and return trade-off is the main hypothesis relating to Markowitz's basic concept of the riskier assets providing greater potential return in order to attract investors. Investors will hold a security only if their prediction of return sufficiently compensate them for the risk taken (Ross, Westerfield, Jaffe, 2002). In general risk is a probability that actual return of an investment will negatively differ from investor's expectations. It could be statistically measured by standard deviation. Therefore assets with higher standard deviation are expected to yield higher returns so the investors are sufficiently compensated and willing to buy and hold such investments. Sharpe and Markowitz use important term of risk premium, which is the return in excess of the risk-free rate of return that a risky investment is expected to yield. Since the future return is not guaranteed it is only a potential of excess yield that attracts investors. Riskier assets not always pay out risk premium over risk free assets and yield can even be negative. This is what makes them risky investments. Nevertheless historical analysis proved that for investors to earn higher returns it is necessary to invest into riskier assets (Bradford, Miller, 2009).

Markowitz in his work on Modern Portfolio Theory used volatility as a measure of risk. It is statistically defined as standard deviation, variance of returns, or in CAPM model as beta.

2.1.4. Basic assumptions and their violations for MPT framework

The MPT framework is using many assumptions about individuals and markets. The explicit one is normal distribution of daily returns. Also omission of taxes and transaction fees in his basic theory makes live easier for MPT critics. Here are few examples that were be discussed further.

(Mandelbrot and Hudson (2004)) pointed that extreme events occur far more frequent then normal distribution would predict that is questioning that assets are normally distributed.

Cadle (2011) discussed the problem of fat tails in return distribution. Linear correlation assumption is not credible when there is a significant probability of extreme events. *"The presence of fat tails in the distribution of stock returns implies that linear correlation coefficients do not correctly measure the covariation between stock returns."* Thus the discussion about correlation matrix stability is necessary.

The assumption of investors maximizing utility in terms of money, even though the key assumption of EHM, is criticised by modern behavioural economists. Investors sometimes show irrational decision-making and herd behaviour. This is more or less a problem of investors' rationality that is in connection with investors' risk aversion very important for EHM.

However modern research in behavioural economics points out that investors are commonly irrational. (Barberis and Thaler (2003)) in their work on Behavioral Finance list many examples of irrational investors' behaviour, e.g. excessive trading, naïve diversification, etc. Investors tend to go for popular sectors. The markets are driven by sentiment and we have a long history of booms and busts. Many people base their decision-making just on rumours and popular beliefs. Several centuries of Tulipomanias, Real Estate and Gold rushes, Junk bond busts, dotcom bubles and Asian crises have proved that markets are heavily affected by sentiment and politics. There is another dubiety of efficient access to information and their fair usage. Behavioural finance studies a possibility of information bias. We can read in the news some stories about fund managers being investigated by SEC for insider trading, thus using publicly unavailable information for trading. So there isn't equal access to all information and abuse has to be criminally punished. Practically this means society is enforcing efficient markets where there are none.

Consequently it is difficult for investors to estimate future returns that is crucial for portfolio formation in MPT framework. There is even a strong discussion about the method to be used. For example Amit Goyal and Ivo Welch (2006) argued that none of the regressions of variables could outperform out of sample estimation by historical average. On the other hand Campbell and Thompson (2007) opposed that many variables are predicting with better precision when correct restrictions are imposed.

Essentially for the MPT to be fully viable there mustn't be any barriers to trade such as taxes, limits, investors must be price takers and every market participant must be able to borrow and lend at risk-free rate. In reality trades are usually subject to transaction costs and taxes. Transaction costs have a major impact on markets. Correspondingly taxes and costs are important whether to be a short term or long-term investor. Every investor faces some limits. It could take a form of available cash or a form of institutional and legal barriers. Liquidity is a major reason for traders to keep out thinly traded assets. Usually only the governments and largest corporations are allowed to borrow at T-bill rates. Short selling has been massively discussed in recent market turmoil and is illegal or heavily restricted in several countries around the world.

Some of the more specific discussions follow.

2.1.4.1. Returns follow joint-normal distribution

Despite the normal distribution assumption we can observe market swings even 6 standard deviations from the mean far more frequently then would statistical normal distribution predict. This suggest that markets would be better described as some fat tail distribution and the MPT can be customized to use one. The problem with customized distribution is the need for symmetrical distribution and it is empirically proved that returns don't follow such pattern.

2.1.4.2. The efficient market hypothesis

The efficient market theory (EMH) is another drawback of the MPT. Of all three major versions mentioned earlier in the essay (they were weak, semi-strong, strong) we can discuss whether weak and semi-strong are valid but there is some strong evidence against validity of strong assumption of efficient markets hypothesis (Andrei, 2000) used in MPT framework. Intuitively we can feel that it is difficult for actual prices to reflect any even privately held information, assumption needed to fulfill strong version of EMH. For example there have been a strong criticism for late 2000s global financial markets performance based on markets being inefficient with tendency to create bubbles and crashes. On the other hand proponents argued that efficient doesn't mean we can foresee future but rather it is a simplification of the world with all his uncertainty and it renders markets efficient for practical investment purposes (Chamberman, 1983).

Anyway one of the strongest arguments against strong efficient market hypothesis is the report of illegal insider trading operations making huge profits as is the one cited below.

"The Securities and Exchange Commission alleged that $276 million in illegal profits or avoided losses were made by investment advisers and their hedge funds, by trading ahead of negative news in July 2008 on a clinical trial involving an Alzheimer's drug developed by Elan Corp. (US:ELN) and Wyeth, now a subsidiary of Pfizer Inc. (US:PFE)"[6]

Additional outcome of the 2000s crisis was the realization that assets tend to grow in correlation when there is a market upheaval and assets decline in value all at once. *We can see this behavior at graph 3 in the chapter of our empirical analysis of Markowitz's portfolio theory.* The main idea of Modern Portfolio Theory of diversification is thus failing since the assets are subject to global decline and losses in one asset are not offset by gains in others as predicted by Markowitz in 1952.

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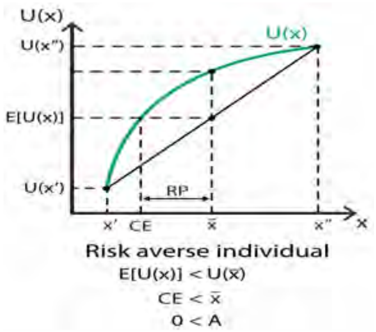
[6] http://articles.marketwatch.com/2012-11-20/economy/35223226_1_expert-network-firm-hedge-fund-manager-sec-complaint, 20. 11. 2012

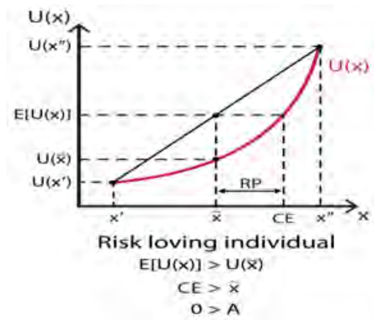
2.1.4.3. Utility maximization and rationality

Figure 2 The utility functions (Policonomics.com)

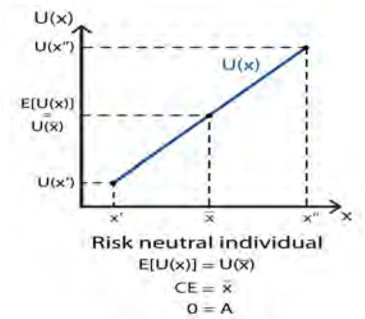
a) Risk averse UF



b) Risk taking UF



c) Risk neutral UF



Every investor is a subject to his individual utility maximization. Simply put everyone is trying to make as much money as possible. This is the key assumption of efficient markets hypothesis. MPT framework employs idea of utility (expressed in numerical form) to decide on combination of risk free asset and optimized portfolio. The basic utility takes three basic forms and distinguishes between risk averse, risk taking and risk neutral individual and their respective utility functions are shown in figure 2. Markowitz assumed that investors are risk averse and rational so MPT doesn't account for ideas of herd behaviour or investors accepting lower returns for higher risk as described by modern behaviourist.

2.1.4.4. Investors are perfectly foreseeing

The main problem that is also well theoretically covered is the problem of prediction of future return distributions. MPT framework originally didn't offer any solution for this and just assumed that investors' beliefs match true future distributions. This also reveals that MPT was more built as a theoretical concept then actual tool. The problem of estimation of future is not only about returns but also about variance-covariance matrix and its stability. Since MPT takes variance as a measure of risk and is trying to diversify away any unsystematic risk using correlation between assets a precise out of sample estimation is very important and almost impossible to obtain.

Since investors' expectations are generally biased by their own beliefs and inability to predict future returns the assets' prices doesn't offer the unbiased information they were supposed to.

2.1.4.5. Institutional restrictions

In the MPT framework there are no taxes and transaction costs, investors are price takers and are subject only to risk free rate when lending or borrowing unlimited amounts of money, all securities can be split to fractions of any size and traded so as well.

In reality every investor is limited by his individual budget and credit constraints. Therefore some portfolios suggested by MPT could not be feasible. Moreover only national governments and large corporation are usually allowed to borrow at or near risk free rate. The trades are usually subject to taxes and transaction costs and it is usually suggested to individual investors to keep amount of trades as low as possible. On the other hand large investors can buy or sell large bunches of stocks that could shift prices of individual stocks or certain markets. Consequently they are not price takers and their own action can forbid them from obtaining optimum portfolio since market reality is changing while they are trading.

Taxes and transaction costs are difficult to predict for use in long-term portfolios. Especially in less developed and efficient markets the change can be swift and have very large consequences. The optimum portfolio could be relocated by an action of government, trader, market maker or some other important market participant.

Splitting of assets is usually not possible and the smallest portion is usually given. There can also be a minimum order size given for some assets so investor is not able to obtain specific amount suggested by MPT.

There are many theoretical and practical models handling some of the above-mentioned difficulties. Some can be solved by sophistication and polishing of mathematical expression used to obtain optimum portfolio. It is easy to account for restraints and costs of trade. Contrarily other shortcomings as non-normality and fat tails are more difficult to solve and model would be probably too complex for practical use.

2.1.5. Criticism of Modern Portfolio Theory

"Now, under the whole theory of beta and modern portfolio theory, we would have been doing something riskier buying the stock for $40 million than we were buying it for $80 million, even though it's worth $400 million – because it would have had more volatility. With that, they've lost me."

Warren Buffett (Lecture of What Every Lawyer Should Know About Business, Stanford Law School, 1990)

Warren Buffett is one of the most well known public figures in constant criticism of efficient market hypothesis and Modern Portfolio Theory. The quotation above is an example of his investment into The Washington Post Company in 1974 of $40 million that he personally valued on $400 million. He pointed out that buying it for $80 million would show less variance and thus in MPT framework would be considered less risky. Mr. Buffett is one of the more moderate critics, there are some that suggest to deny the framework whatsoever, but he suggest caution and recommend business schools to teach the shortcomings of this framework so the graduates are aware of these.

2.1.5.1. Volatility

As we already discussed Modern Portfolio Theory uses the term of volatility as a measure of riskiness. The greater the volatility the greater is a risk of the asset. Standard deviation is statistically a measure of how much variation exists in data set from its mean. Buffett's note from his guest lecture at Stanford Law School reveals one of the most obvious problems of the standard deviation employed as measurement of risk: It doesn't fully distinguish between upward and downward movement of a value. Investors are very concerned by downward movement but do they get nervous if their stocks are going upwards? Volatility regards upwards and downwards movements equally bad. *"Suppose the price of a stock goes up 10 percent in one month, 5 percent the next, and 15 percent in the third month. The standard deviation would be five with a return of 32.8 percent. Compare this to a stock that declines 15 percent three months in a row. The standard deviation would be zero with a loss of 38.6 percent. An investor holding the falling stock might find solace knowing that the loss was incurred completely "risk-free"* (Keppler, 1990, p. 1). Subsequently in Buffett's example the widening of a gap between buying and selling price would be in terms of volatility considered a growth in risk even though it is just obviously a better deal.

On the other hand this definition of risk is simple, mathematically explainable and thus compelling to financiers. It is based on logical assumptions of investors and markets' rationality with prices being set according to risk adversity thus investors being paid accordingly for risk they have in their assets. So is really statistical risk analysis good enough to replace rigorous analysis of company's financial and even non-financial indicators?

Let us denote that in this work we will use a standard deviation as a measure of volatility.

2.1.5.2. Risk and return correlation

The main problem of the concept is that correlation between risk and return is actually weak. Murphy (1977) conducted a research on Efficient Markets and found *that realised returns appear to be higher than expected low low-risk securities and lower than expected for high-risk securities ... or that the [risk-reward] relationship was far weaker than expected. Other important studies have concluded that there is not necessarily any stable relationship between risk and return; that there often may be virtually no relationship between return achieved and risk taken; and that high volatility unit trusts were not compensated by greater returns*". This research was strongly disputing existence of strong positive correlation between risk and return. For investors this is needed so they are fairly awarded for risk taken and have incentives to hold riskier assets in their portfolios.

Eugen Fama, one of the foremost proponents of Efficient Market hypothesis, with K. R. French conducted an extensive research on risk and return.

Journal of Finance (p. 449,) published their paper in the year 1992 as *The Cross-Section of Expected Stock Returns*. They openly denounced the CAPM beta as a correct measure of risk saying, *"...we find that this simple relation between β and average return disappears during the more recent 1963-1990 period... In short, our tests do not support the central prediction of the Sharpe-Lintner-Black model, that average stock returns are positively related to market β."* Eugen Fama came to similar conclusions about CAPM beta as Murphy came to with risk and return correlation. The validity of concepts that investors are fairly rewarded was from the point of view of their papers very weak. Fama's research also recommended better predictors for future returns. E.g. companies provide highest return when they have low price to earnings ratio (P/E), low price to book ratio and smaller capitalization. These three attributes were, according to their research, better related to stock returns then beta.

2.1.5.3. Portfolio time variance

The above mentioned are not the only troubles with volatility concept. The fact well known to every option trader is that volatility is changing rapidly even in intraday trading. This can be seen in Figures 3 and 4 where is depicted the VIX index of Chicago Board Options exchange that measures implied volatility of S&P 500 index options. At 5-year horizon changes in values are tremendous. Nonetheless we can observe some significant swings even on average trading day, as was The 10th of February 2013. We don't think that real financial situation of companies included in the S&P 500 index could be changing so rapidly so the volatility index is a reflection of real market developments. Volatility is not only important for option traders and Black-Scholes option pricing model but also for Modern Portfolio Theory. First of all it is part of the variance-covariance matrix used to calculate weights for optimal portfolios. Second and even more obviously it is the beta in the CAPM model of Sharpe, Lintner and Mossin. Since stocks don't posses a fixed volatility it is important for forming of portfolios to have a reliable tool for its prediction.

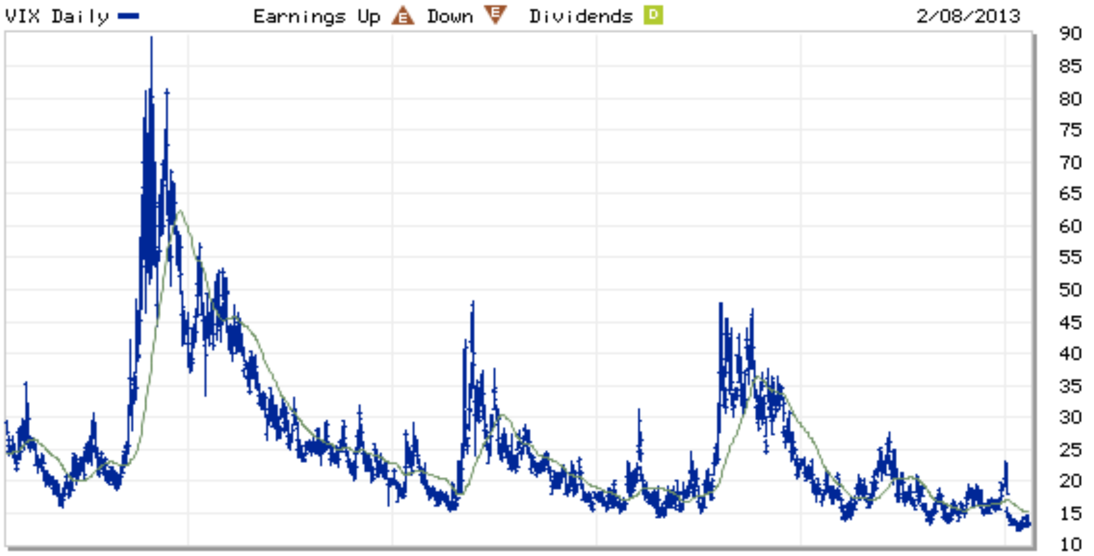


Figure 3 CBOE volatility index (VIX), 5 years, retrieved: 10. 2. 2013, http://www.marketwatch.com/investing/index/vix

Figure 4 CBOE volatility index (VIX), 1 day, retrieved: 10. 2. 2013, http://www.marketwatch.com/investing/index/vix

In figure 5 we can see a lot of noise in daily returns. Thus their predictability is tricky but there has been a lot of studies on this topic so we would just say there is high variability in daily returns and dependent on chosen method their prediction could be very time variable similarly as in case of volatility.

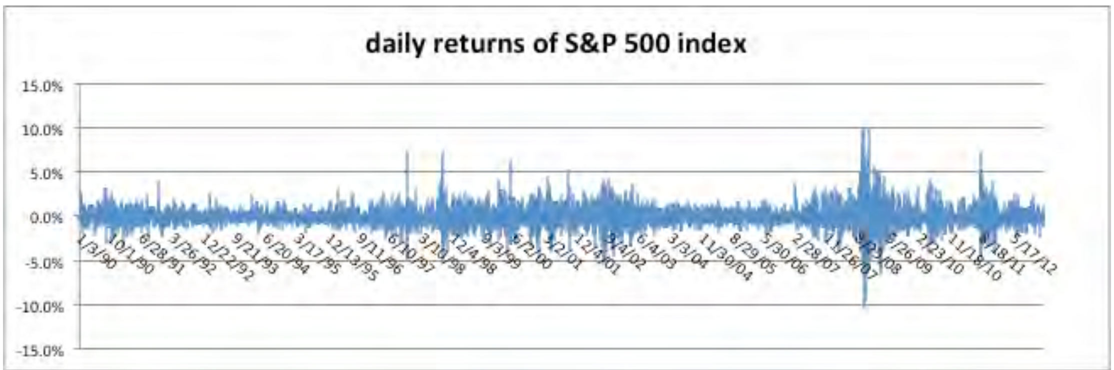


Figure 5 finance.yahoo.com, retrieved at: 11. 2. 2013

2.1.5.4. Statistical significance

The disputes about Efficient Markets Hypothesis, Modern Portfolio Theory and validity of volatility as a measure of risk are going to take long time to resolve. Supporters of EMH usually point out that no investor in history has ever turned in a statistically significant outperformance over a long time. Famous investors as Warren Buffett and John Templeton are considered just statistical outliers and their superior performance is accounted to pure luck. Lawrence Summers, Secretary of the Treasury of the United States under Clinton administration once proclaimed that it would require 50.000 years of data to disprove the EMH to the satisfaction of the Stalwarts. Statistical significance is just a weak tool when there is only little data available. Statistics is supposed to be used with large datasets and its performance when handling small samples is inconclusive. A manager with thirty years long career earns only 120 quarterly figures of his performance. He would need to outperform the market heavily for his performance to be statistically significant.

2.1.5.5. Stability

As we have shown before volatility and thus variance and covariance are very fluctuating variables. Practically the most important problem of Markowitz's mean variance optimization (MV), how is the practical solution to Modern Portfolio Theory sometimes called, is its instability and time variability. Since all input variables needed for successful use of MV optimization are varying heavily it is difficult to define optimal portfolio. It is not only a matter of correct out of sample estimation of values but also the Markowitz's model is very sensitive to changes and even small adjustment to value of input variables could cause very large changes in portfolio weights. Michaud (2008, p.5) comments on this as "*The procedure overuses statistically estimated information and magnifies the impact of estimation errors. It is not simply a matter of garbage in, garbage out, but rather a molehill of garbage in, a mountain of garbage out." This is one of the most important* shortcomings of a Mean Variance optimization as introduced by Markowitz. (Michaud, 1989) commented on rare usage of this method by professional portfolio managers and also offered explanation by pointing out to unintuitive portfolio weights offered by mean variance optimization. (Black and Litterman, 1992) added that Markowitz's optimization maximizes errors. Since model is overweighting stocks with higher expected return and lower expected variance the possible error in out of sample estimation and its impact on portfolio is maximized. The same is true for opposite. Assets with worse predicted performance are systematically underweighted in portfolios thus the loss of their potential if there is a mistake in estimation is maximized. According to Michaud

(2008) this was crucial reason for numbers of sophisticated institutional investors to abandon this statistical method of portfolio formation and rely on intuition.

2.2. Black-Litterman model

2.2.1. Developments

Fischer Black and Robert Litterman first introduced the Black-Litterman model (B-L model) in an internal Fixed Income document of Goldman Sachs in the year 1990 under a name of *Asset Allocation: Combining Investors Views with Market equilibrium*. The paper was introduced to academia in the Journal of Fixed Income in 1991. The extended version of the paper was published in 1992 in the Financial Analysts journal. As the name of the article suggests it offered a sophisticated method to overcome Markowitz's unintuitive and highly concentrated portfolios by including investors' market views. Markowitz's Mean Variance optimization is very input sensitive as we have shown in previous chapters with weights varying significantly with changes in input variables. These were the main reasons why majority of portfolio managers didn't use Markowitz's optimization of maximizing return for given level of risk. Black and Litterman created a model that allows for Bayesian approach to combine investors' opinions about expected returns with the prior distribution of expected returns. This gives a portfolio which is a combination of market equilibrium with investors opinions and portfolio weights are much more intuitive.

Main goal of Black-Litterman is to create a portfolio that is more stable, efficient and accounts for investor's believes. (Lee (2000)) also pointed out that B-L optimization is also reducing the problem of error-maximization as was described by (Michaud (1989)). This is overcome by spreading errors throughout the whole vector of expected returns. (Best and Grauer (1991)) showed that small changes in input variables of one asset can force half of the assets out of portfolio. (Black and Litterman (1992)) and further (He and Litterman (1999)) studied various possibilities to predict input variables. They demonstrated that most of historical based Mean-Variance optimized portfolios contain extremely large long and short positions.

2.2.2. Main contributions

The most significant contributions of Black-Litterman model to asset allocation problems are intuitive a prior portfolio and a clear way to employ investors views. The first is important since it allows for use of CAPM equilibrium market portfolio as an initial point of optimization. Prior work started with even uniform prior distribution or global minimum variance portfolio. Later method is based on Stein's estimator and takes form of $\mu_i = x + c_i (x_i - x)$ (Frost, Savarino, 1986) and takes shrinkage approach to shrink expected returns towards a common mean (Jorion, 1985). This would improve performance of Markowitz's Mean Variance optimized portfolios. Basic idea is that in mean and variance there is a connection between assets and thus it is more efficient to forecast their return in a group then individually. Nevertheless these methods were more precise then simple sample mean (Stein, 1955) they still lacked intuitive connection back to market. B-L model allowed for use of distribution of returns from the CAPM market portfolio as an initial point for portfolio optimization.

The second and maybe even more revolutionary was ability of the model to input investors views and believes and to blend them with prior information extracted from capital markets. The B-L model allows for use of partial or complete information spanning a whole market, a set of assets or just an individual stock. Black and Litterman provided a quantitative and effective tool to blend Bayesian and non-Bayesian processes of portfolio formation.

"When used as part of an asset allocation process, the Black-Litterman model leads to more stable and more diversified portfolios than plain mean-variance optimization." (Walters, 2009)

The interesting part of the Black-Litterman model is that benchmark portfolio is used as reference point for optimization. This means that portfolio manager is evaluated based on the same portfolio he bases his portfolio structure on. According to behavioural finance the actual utility of an investor is based on past reference and evaluates losses and gains in relation to a benchmark. This is the reason why portfolios structured by B-L model are considered more intuitive and logical.

2.2.3. Shortcomings of Black-Litterman model

The B-L model is "data consuming". It needs an investor to input a lot of data so the model functions properly. First an investor needs to define his universe of assets and find market capitalization of every asset included. Secondly it is also necessary to define variance-covariance matrix from historical returns. It is common to use a proxy for selected market (e.g. S&P 500 for US large capitalization companies). And the third is a need to find a value of a proxy for risk free rate.

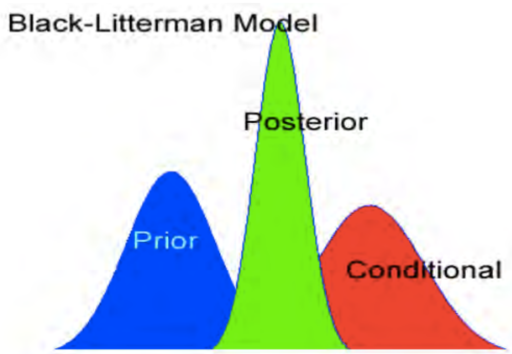


Figure 6 B-L information mixture

The market capitalization for liquid assets if widely available and thank to information technology it is easily obtainable even for individual investors. On the other hand for illiquid asset classes like private equity, commodities and real estates the capitalization data could be unreliable or unavailable even for professional investors.

Very important and distinguishing step of B-L portfolio formation is quantifying of investor's views. B-L allows for inputting of quantitative, qualitative, complete, incomplete and even conflicting opinions (Walters, 2007). See Figure 6[7] for graphic example of what the mixing might look like in a single dimension. Here we can perceive how combination of prior and conditional information forms posterior information. On the other hand however ideal it could seem obtaining of the correct market prediction is obviously difficult and in efficient markets shouldn't be even possible (as discussed before). "*With all the technical analysis tools that are currently available for use for this purpose it remains a relatively hard task to achieve. This is because of the markets volatility. Few professionals have been able to master and achieve accurate predictions for these markets. These predictions have enabled traders in the markets to make*

[7] http://www.blacklitterman.org/intro.html

decisions which would otherwise have taken them long."[8] However difficult it is to obtain correct prediction of future returns B-L model allows for not even setting a predicted value but also its reliability. Or put in different words how certain is the investor about his forecast. Consequently Investors' views take the form of conditional distribution (Black and Litterman, 1992).

One of the most confusing aspects of the B-L model is the tau variable (uncertainty ratio). This can be defined as a level of confidence in the CAPM distribution (Xu, Chen, Tsui, 2008). There is another problem in connection to tau. The Black-Litterman model assumes that assets follow the same probability distribution. At least it could be any distribution investor decides. Anyway Black and Litterman (1990) introduced this parameter to scale the variance of the expected return. He and Litterman used a value of 0.025 where Satchell and Scowcroft remark that many people use a value close to 1. Other authors as Meucci, Krishnan and Mains, completely eliminate τ. When He and Litterman (1999) proposed a version of the B-L model generating an estimate of the return as well as an estimate of variance. The variance of posterior distribution is largely affected by changes in the value of tau and omega (covariance matrix of views). If we are not interested in estimating variance we can eliminate t and change omega as required. Walters (2010) concludes in his work on tau as follows *"Most of the Black-Litterman literature makes use of the Alternative Reference model explicitly or implicitly, and most investors would be well served to explicitly use the Alternative Reference Model rather than struggling with τ."*

[8] http://www.forextokens.com/forex-strategy/22-forex.html

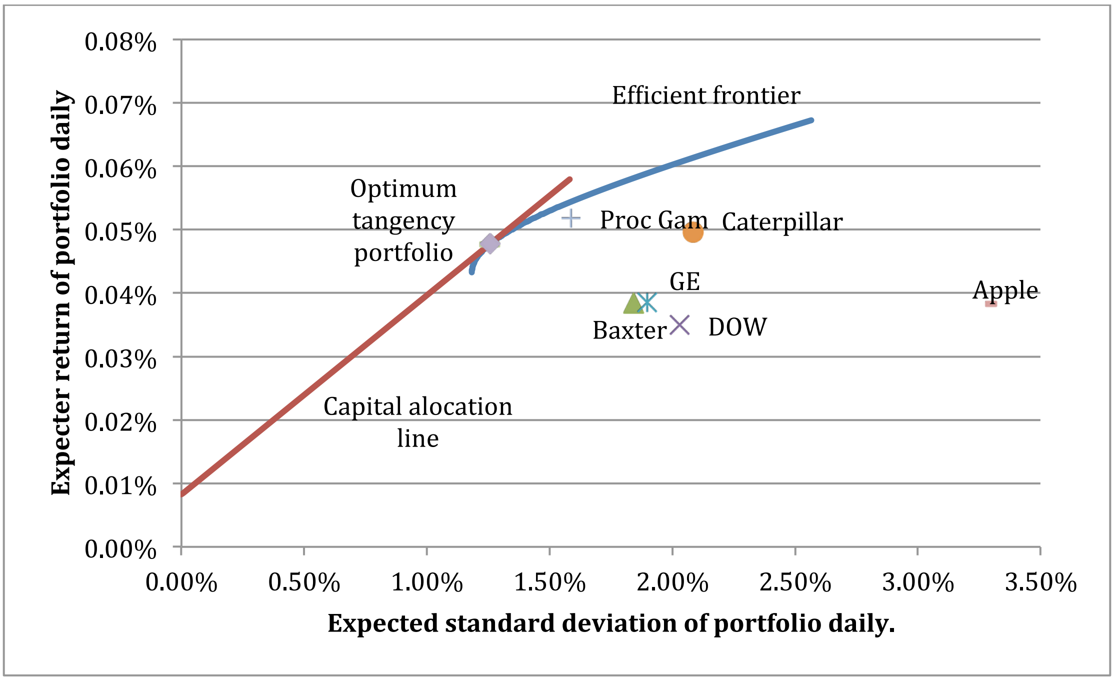
3. Research methodology

3.1. Modern Portfolio Theory (MPT)

MPT is a theory that explains maximization of expected return for given level of risk, or vice versa minimize risk for given expected return. It gives mathematical solution to originally intuitive problem of choosing a collection of assets that form a portfolio of lower risk than any individual asset. This is possible because of different dependencies between asset prices. The concept of the MPT is that selection of assets shouldn't be done limited only to assets' attributes. Assets should rather form a clever selection where change in a return of one asset is balanced by change in a price of another asset in the portfolio. This is a general idea of a correlation that is a statistical measure of interdependency between different assets. Since investing is in general considered to be a trade off between risks and returns, Markowitz in his revolutionary work argued that investor is able to form better portfolio by finding such asset classes that give the lowest variance for given return. This concept is called diversification.

Technically MPT takes asset's returns as normally distributed and defines risk as a standard deviation of return. It forms portfolio weights by combining assets in order to get weighted portfolio return of assets that are not perfectly positively correlated. Hence MPT seeks to reduce total variance of portfolio when maintaining a portfolio return.

As we have already discussed in previous chapters even though theory was very influential, recently it has been widely criticised. Criticism is mainly based on non-Gaussian distribution of asset returns and investors' irrationality causing market inefficiencies. Correlations between assets are not fixed but are influenced by external events. Even though we ignore those problems, most common solutions to MPT still give unintuitive and extreme portfolio weights that are difficult to be fully trusted by investors.



Graph 1 Graphical depiction of MPT

In order to make our research simpler and also to relieve are computational power a bit we decided to base our research on six major companies included in S&P index and widely covered by analytics. In the Graph 1 we can see an example of practical solution for these American companies based on data sample from the year 1984 to 2011. The important part is the Efficient Frontier. It is a line connecting combinations of assets with the lowest possible variance for given return. These portfolios are the most efficient. Interesting part is that all the individual stocks lay under this line, which means they are suboptimal in variance for their return when invested on their own compared to when formed into a portfolio.

3.1.1. Solutions to Mean Variance optimization

Investor can use many solutions to Mean Variance optimization to minimize risk and maximize return. The most accessible to common investor is the use of Excel Add-in Solver. It requires matrix algebra solution that is solved by random generating of numbers by computer. Another possible but not commonly used solution is to use mathematical optimization of function on given set by Lagrange Multipliers

3.1.1.1. Matrix algebra solution

Investor forming a portfolio can use many mathematical solutions. The simplest one is to use matrix algebra as follows. This is based on forming the problem as a combination of expected return vectors, expected covariance matrices and asset weights vector. This means to transform linear algebra solution to matrix algebra solution. We start with linear formulation of maximizing portfolio return and minimizing variance.

$$E(R_p) = W^T R$$

$$\sigma_p^2 = W^T V W$$

Where W = matrix of weights of assets in portfolio

R = returns of assets

V = variance-covariance matrix

σ = portfolio variance

Subscripts

mw = minimum variance portfolio

ef = efficient portfolio

sr = sharpe ratio portfolio

Matrix algebra solution can be easily performed in a statistical package. We can set the general solution for given portfolio weights and than use some random number generator to find the optimum portfolio weights minimizing portfolio return. The structure of such practical optimization follows:

1. Find the global minimum variance portfolio, compute its mean and variance

$$R_{p,mw} = W_{mw}^T * R$$

$$\sigma_{mw} = W_{mw}^T * V * W_{mw}$$

$$optimized\ to\ minimum\ portfolio\ variance$$

2. Determine an efficient portfolio with target return equal to the largest expected return of the given set of assets, compute its mean and variance

$$R_{p,ef} = W_{ef}^T * R$$

$$\sigma_{ef} = W_{ef}^T * V * W_{ef}$$

$Optimized\ to\ minimum\ Portfolio\ Variance\ given\ the\ Portfolio\ Return = Return\ of\ the\ best\ performing\ stock$

3. Compute the covariance between the returns on the global minimum variance portfolio and the returns on the efficient portfolio

$$\sigma_{mw,ef} = W_{mw}^T * V * W_{ef}$$

4. Compute portfolio frontier using result that any portfolio on the frontier is a convex combination of any two frontier portfolios

5. Get maximum Sharpe Ratio Portfolio

$$R_{p,sr} = W_{sr}^T * R$$

$$\sigma_{sr} = W_{sr}^T * V * W_{sr}$$

Subject to maximized sharpe ration $\frac{E(R)-R_f}{SE}$

6. Construct Capital Allocation Line by combining Weights of Maximum Sharpe Ratio Portfolio with Risk-free asset

7. Obtain optimal weighting of tangential portfolio and risk free asset with respect to individual utility function. Formula used $\frac{E(R_p)-R_f}{A*Var_P}$.

In the example above we can see that only necessary inputs are expected returns, expected variance and covariance, risk-free rate and a coefficient of individual risk aversion. In general there are more assumptions of the model. Although we need only these few numerical values, their correct estimation is more complicated.

3.1.1.2. Lagrange multipliers solution

To minimize or maximize desired functions the most commonly used method is a random procedure such as already mentioned Solver in MS Excel, or some other statistical package. In our case this method is not optimal since we will use Monte-Carlo simulation further in our work. This would result in double optimization problem where two dynamic random variable functions were running at the same time. For this reason we've decided to use Lagrange multiplier solution for finding the efficient portfolio on the Efficient Frontier. This solution is theoretically covered in Appendix 6. LM solution is static and allows to input random function for any of the variables. We will solve the part for finding a portfolio with minimum variance for given return, thus portfolio on Efficient Frontier and minimum variance portfolio. For general discussion we won't cover the utility optimizing portfolio and impacts of changes in risk free rate. Our research will focus on changes in inner assets variables such as expected return, variance and actual return and variance.

3.2. Black-Litterman model solution

To reduce extreme portfolio positions Black-Litterman optimization starts with equilibrium returns as a neutral starting point. Investor can use either CAPM or reverse optimization that allows for extraction of expected returns from known information. These returns are called implied expected equilibrium returns (Π) and we will use this method of estimation in our analysis. For estimation on matrix Π we use formula below.

$\boldsymbol{\Pi} = \boldsymbol{\delta} * \boldsymbol{\Sigma} * \boldsymbol{w}$,

$\boldsymbol{w}$ = Vector of market capitalization weights

$\boldsymbol{\Sigma}$ = Fixed covariance matrix

$\boldsymbol{\delta}$ = Risk-aversion coefficient

After defining market equilibrium returns we need to input investors' views into our model. Black and Litterman suggested use of following formula:

$$E[R] = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P \right]^{-1} \left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q \right]$$

E[R] = A posterior vector of returns that includes investors' views. Form of n x 1, where n stands for number of assets in the model.

$\boldsymbol{\tau}$ = A scalar that could be used for better calibration of the model. Walters (2010) argued that for most investors is beneficial to omit this variable.

$\boldsymbol{\Sigma}$ = covariance matrix (*n x n* form)

P = Matrix of actual stocks we have views on. It Identifies the assets involved in the views. Matrix takes form of k x n where k is a number of views.

$\boldsymbol{\Omega}$ = Represent level of confidence of individual to his expressed views (*k x k* matrix).

$\boldsymbol{\Pi}$ = Implied Equilibrium Return Vector mentioned before (*n x 1* column vector).

Q = Vector of actual expected returns of each of the views (*k x 1* column vector).

3.3. Monte Carlo methods

"Picking numbered pieces of paper from a hat can also be used to generate random numbers" (Dunn, Shultis, (2011), p. 69).

The quotation from the book Exploring Monte Carlo Methods simply describes basic idea of Monte Carlo simulation. We form a hypothesis or mathematical problem. Use the correct formulas to get a bottom line result. Than we substitute variables in our calculation by random numbers and study influences of these changes to final outcome. This gives us an opportunity to see all possible results, their probabilities and allows us for study of uncertainty or extreme events. This technique is widely used in risk management, finance, project management, engineering, manufacturing, insurance, oil and gas, and so on. Its wide use was allowed by development in computer technologies since this simulation is the best performed by computerized simulation.

More technically Monte Carlo is a class of algorithms simulating events by generating pseudorandom numbers. We have decided to use Latin Hypercube Sampling[9] for generating of our samples. *„Monte Carlo methods (or Monte Carlo experiments) are a class of computational algorithms that rely on repeated random sampling to compute their results. Monte Carlo methods are often used in simulating physical and mathematical systems."*[10]. Its name given after famous casino city isn't coincidental. Basic idea is to generate enough of random observation to test behaviour of some variable, function or equation. One of the first important usages was project Manhattan during the Second World War. Monte Carlo is a summary name for all methods using pseudorandom repeated sampling.

All of those methods follow this structure:

1. Define group, or distribution that we will generate random numbers from
2. Generate enough observations to so a hypothesis is evidential
3. Run deterministic calculations using generated values
4. Aggregate results and interpret as final solution
5. In a case of more variables in the model we need to define cross-correlations for model to give sensible results

[9] Guide to Using Palisade @Risk v. 5.5, p. 649, 02/2009

[10] princeton.edu, [31. August 2013], WWW: <http://www.princeton.edu/~achaney/tmve/wiki100k/docs/Monte_Carlo_method.html >

3.3.1. Latin Hypercube Sampling

For a method of Monte Carlo we need to deliver a series of random numbers. In our research we used software Palisade @Risk 5.5 that uses LHS[11]. McKay, Beckman and Conover have first described this method in the year 1979. The purpose of this statistical tool is to create acceptable sets of variables from multidimensional distribution. It is based on simple Latin Square Sampling, which generates numbers from matrix of a form n x n with n different values. Each of the values is used in one column only once. For example matrix can look like this 3 x 3 matrix with three different values.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

Hypercube is just a multidimensional extension to standard matrix. Maximum number of combinations is given by a formula

$$\prod_{n=0}^{N} (M - n)^{N-1}$$

N is a number of variables we generate and number of dimensions of matrix
M is a number of equal splits of hypercube to probability intervals
n is a number of unique numbers in each column

Advantage of this process is repeatability. It is a basic difference to purely random process that is impossible to repeat. Another difference from random sampling is a need of advance definition of number of generated values. In latin hypercube we need to define parameters of sampling and then we can generate numbers.

[11] Latin Hypercube Samplig

3.3.2. Anderson-Darling test

We have defined that we will deliver values for Monte Carlo simulation by Latin Hypercube Sampling. We need to discuss what values will the basic matrix acquire. We will need to define original statistical distributions that input variables follow. For our purposes we will use the Anderson-Darling test that tests which theoretical distribution is the closest match to estimated values. (Anderson and Darling (1952)) introduced this test in 1952 in Annals of Mathematical Statistics.

I have decided to use Anderson-Darling compared to other common tests such as Kolmogorov-Smirnov and Chi.squared for its better performance in fat tailed distributions. This is common for financial data since occurrence of extreme events is more common then most of the theoretical distributions predict.

T. D. Andersen a D. A. Darling has introduced this test in a year 1952. The basic test statistics takes a form of

$$A^2 = -n - S$$

$$S = \sum_{k=1}^{n} \frac{2k-1}{n} [\ln F(Y_k) + \ln(1 - F(Y_{n+1-k}))]$$

$$H_0 = estimated\ variables\ follow\ given\ distribution$$

$$H_1 = estimated\ variables\ don't\ follow\ given\ distribution$$

4. Research data

Name	Symbol	Industry
General Electric Company	GE	Conglomerate
The Dow Chemical Company	DOW	Chemical Industry
Baxter	Baxter	Health Care Supplies
Caterpillar	Caterpillar	Industrial Goods
Procter and Gamble	Proc Gam	Consumer Goods
Apple Computers	Apple	Personal Electronics

Table 1 List of selected companies

In previous chapters we introduced multiple steps theoretical procedure to obtain data sample for our study. This is based on defining original distributions, picking random numbers, generate portfolios and gather final data for our analysis. For better practical implementation we decided to base our research on a real market situation that every investor can face.

As already mentioned our research includes six major American companies. Reasons for those six are long historical data samples, wide coverage by analysts, which we will use when forming Black-Litterman model, and different industries they operate in. By using very liquid and well-covered companies we also expect to overcome some of the problems as inefficient markets and unreal pricing. Since companies are selected through different industries we expect to find low correlations and thus diversification should be very efficient. In Table 1 we can see the list of those companies with industry they operate in and symbols we use in our graphs and tables.

Although the vast discussion whether it is appropriate to use historical values as estimator of future performance, most of the investors still use it. The efficiency of different out of sample estimation is disputable and difficult to defend. If there is some "magical" tool it probably won't be available to academics. Efficient markets hypothesis revealed that making a strategy publicly available would also render it inefficient. Research by (Goyal and Welch (2007), p. 20) studied predictability of future returns by using dividend price ratios, dividend yields, earnings-price ratios, dividend payout ratios, net issuing ratios, book-market ratios, interest rates, and consumption based macroeconomic ratios. They dismissed practical efficiency of those in all historical periods. They noted in their work

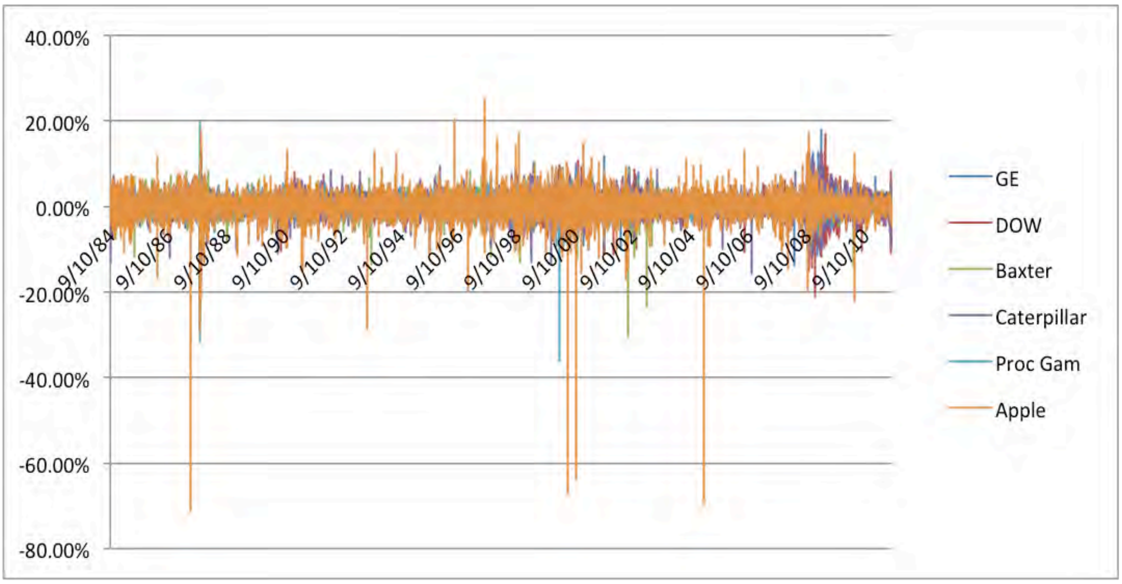
"Our paper has systematically investigated the empirical real-world out-of-sample performance of plain linear regressions to predict the equity premium. We find that none of the popular variables has worked—and not only post-1990. In our monthly tests, we can solidly reject regression model stability for all variables we examined".

On the other hand Markowitz never offered a solution how to obtain expected values needed for his optimization. Investors in general believe that Modern Portfolio Theory is a tool, but it is in general more an idea. Common usage of historical data isn't part of the original paper, but just a practical solution to a problem of obtaining reliable estimates of future returns and covariance matrix. (Markowitz (1991), p.14) even mentions this in his paper as he tries to dispel this common misbelief.

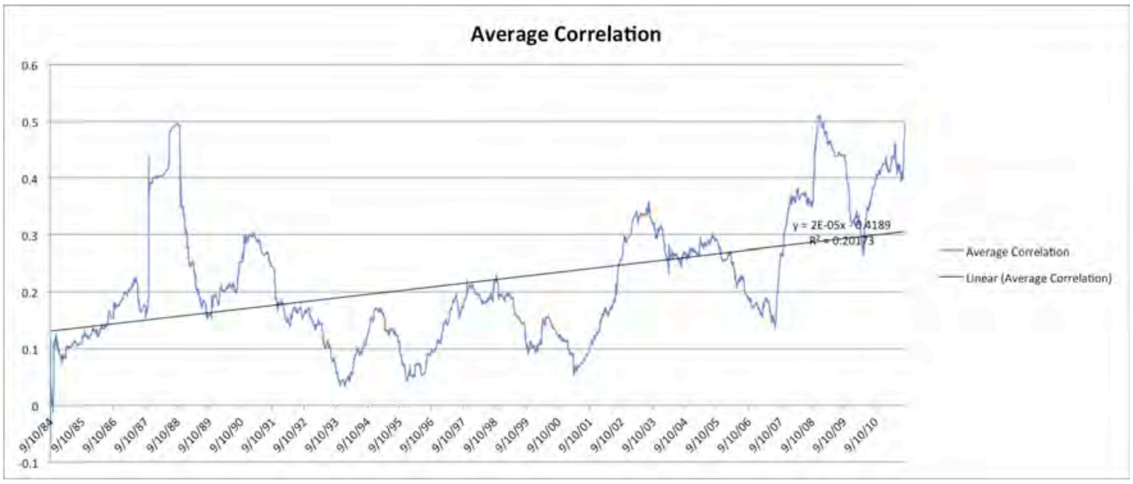
"Portfolio *selection should be based on reasonable beliefs about future returns rather than past performances per se. Choices based on past performances alone assume, in effect, that average returns of the past are good estimates of the 'likely' return in the future; and variability of return in the past is a good measure of the uncertainty of return in the future."*

4.1.Assumptions testing

Markowitz stated assumptions of the MPT. We will cover statistically testable properties of our time series. In the graph 2 we can see daily returns of all the stocks in our portfolio. We can clearly observe volatility clustering which is happening in several historical periods. This volatility clustering is in contrary to assumption of stability of variance-covariance matrix. Our data sample starts on 17/9/1984 and ends on 19/8/2011.



Graph 2 Daily returns of selected stocks



Graph 3 Average correlation of all six stocks

In the Graph 3 is an Average Correlation of all six stocks. There is observable that correlation between stocks is changing rapidly and is in general higher in the years of some economic turmoil. It is higher in 1987's Black Monday, 1990-1991's Savings & Loans crisis, 1994-1995's economic crisis in Mexico, 1997-

1998's Asian Financial crisis, 2001's dot com bubble and 2008-2010's financial crisis. This points to major changes in optimum portfolio during crisis. When a financial turmoil starts, portfolio formed in "better" times might be inefficient. If investor wants to face transaction costs he can set new weights for his portfolio's positions but how often should this be done? Another interesting outcome of this graph is a growing trend with pretty high R^2 of regression. However Augmented Dickey-Fuller test revealed first order autoregressive process in the correlation data, so we can't conclude much of this regression. The problem of this volatile correlation is that it is considered by portfolio managers to be stable. Correlation is a function of covariance and variance, so volatile correlation means volatile variance-covariance matrix as well. Pafka and Kondor (2002) based their research on covariance matrices and concluded that

"...covariance matrices determined from empirical financial time series appear to contain such a high amount of noise that their structure can essentially be regarded as random. This seems, however, to be in contradiction with the fundamental role played by covariance matrices in finance, which constitute the pillars of modern investment theory and have also gained industry-wide applications in risk management"

Table 2 Basic statistics and Jarque-Bera normality test

	APPLE	BAXTER	CATERPILLAR	DOW	GE	PROC_GAM
Mean	0.000384	0.000385	0.000496	0.000350	0.000384	0.000519
Median	0.000396	0.000000	0.000000	0.000000	0.000000	0.000000
Maximum	0.249488	0.127750	0.137240	0.168972	0.179921	0.197826
Minimum	-0.712692	-0.304933	-0.243050	-0.214960	-0.188677	-0.360283
Std. Dev.	0.032782	0.018976	0.020846	0.020288	0.018408	0.015855
Skewness	-5.534393	-1.500362	-0.382667	-0.491367	-0.139921	-2.690087
Kurtosis	117.0722	25.71587	10.25562	12.58691	11.73863	71.27047
Jarque-Bera	3720485.	148710.5	15077.33	26306.72	21652.18	1328387.
Probability	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000

In Table 2 are basic descriptive statistics. We can see that companies have experienced daily losses in terms of tens of per cent. But more importantly there are results of Jarque-Bera normality test. The statistic has a χ^2 distributions with 2 degrees of freedom under the null hypothesis of normally distributed errors. Normality has been rejected for all stocks since p-value is zero to six decimal places. Chris Brooks (2008) offers an explanation that even though normality was rejected, for large samples it is appropriate to appeal to a central limit theorem and expect test statistics to follow appropriate distribution even without error terms normality.

Table 3 Test for randomness

	GE	DOW	Baxter	Caterpillar	Proc Gam	Apple
Runs Test for Randomness	Data Set #1	Data Set #1	Data Set #1	Data Set #1	Data Set #1	Data Set #1
Observations	6798	6798	6798	6798	6798	6798
Below Mean	3622	3499	3585	3511	3523	3398
Above Mean	3176	3299	3213	3287	3275	3400
Number of Runs	3409	3360	3401	3215	3418	3159
Mean	0.00038	0.00035	0.00039	0.00050	0.00052	0.00038
E(R)	3385.3695	3397.0580	3389.8217	3396.3095	3395.4763	3399.9997
StdDev(R)	41.0445	41.1863	41.0985	41.1772	41.1671	41.2220
Z-Value	0.5757	-0.8998	0.2720	-4.4032	0.5471	-5.8464
P-Value (two-tailed)	0.5648	0.3682	0.7856	< 0.0001	0.5843	< 0.0001

On the other hand in the Table 3 we can observe that returns are randomly distributed. Only with exception of Apple and Caterpillar in their cases Runs test for randomness rejected entire randomness and there are some interdependencies. Actually using Akaike information criterion we found that Apple is AR (2) process and Caterpillar is ARMA (2,3) process. We have now covered statistically testable assumptions of Markowitz's Modern Portfolio Theory. Randomness is important for our research. Since we plan to run two different and independent processes. One random variable will be used for to obtain expected returns as prior information for portfolio formation and second as posterior information to test portfolio's outcome.

We run tests for unit root and autocorrelation in data series of daily values for selected stocks. The results can be seen in Appendix 4 for Augmented Dickey Fuller test for Unit root and in Appendix 5 for autocorrelation. We found out that for daily values the data were of first order autocorrelation and non-stationary. For these reasons we used a transformation in the form of

$$\ln E_i = \ln \left(\frac{V_{t-1,i}}{V_{t,i}}\right).$$

Where E –daily earnings of a stock i

$V_{t,i}$ – value of a stock i at a time t

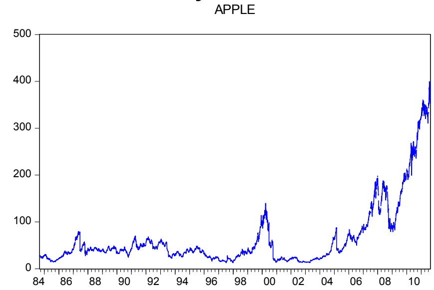
$V_{t-1,I}$ – value of a stock I at a time t-1

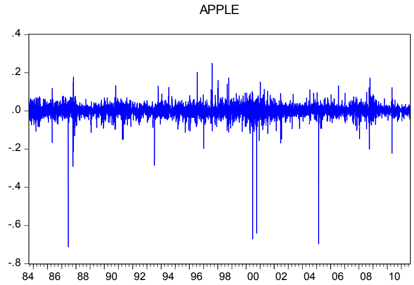
In the table 4 we can see the resulting graphs. In daily values it is possible to observe statistical problems with analysis only with a plain eye. On the other hand in a logarithmic expression is observable a very strong presence of volatility clustering. Even though some companies such as GE show this more then other. For this reason we used a Garch model to test for time variability of variance and covariance. The results are observable in the table 5 and the estimation outputs are in the Appendix 8. In general we can comment that the conditional covariance matrix of the assets returns is strongly autoregressive. This practically rejects the assumption of constant covariance matrix over time. Bollerslev, Engle and Wooldridge (1988) came with the similar conclusion but with wider application. In their research they also covered bond returns and they commented, *"The expected return or risk premia for the assets are significantly influenced by the conditional second moments of returns."* As commented above conditional standard deviation spikes coincide with financial crunches.

Table 4 Daily values and Logarithmic daily changes of analysed stocks

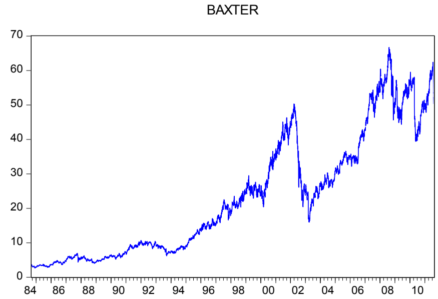
Daily values | Logarithmic daily changes

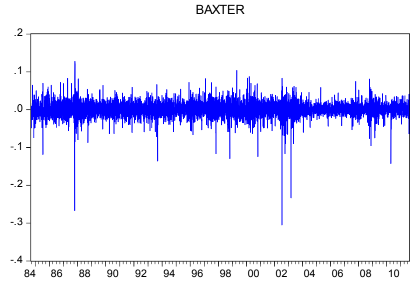
Apple



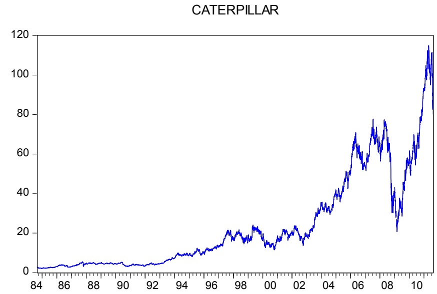


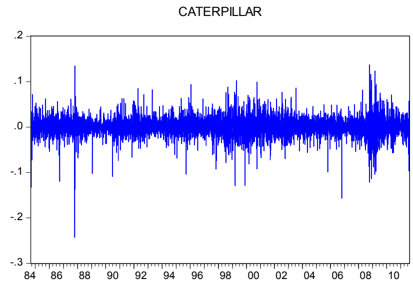
Baxter



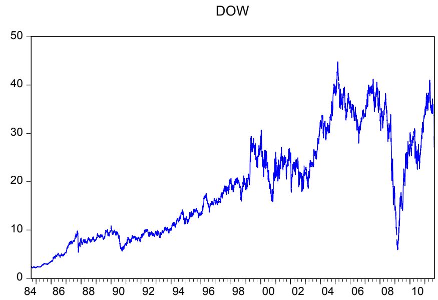


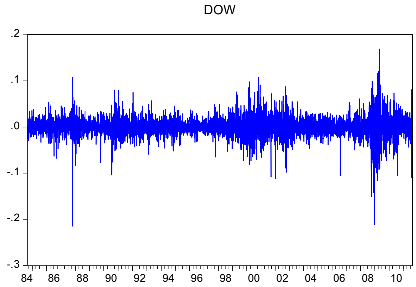
Caterpillar



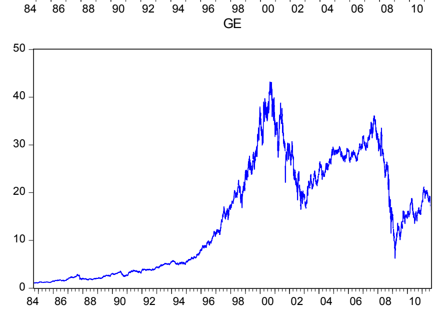


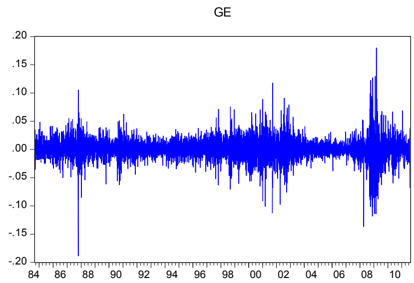
Dow



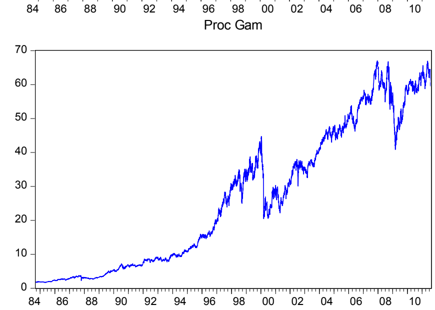


GE





Procter & Gamble



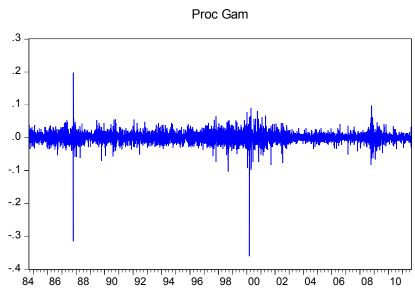
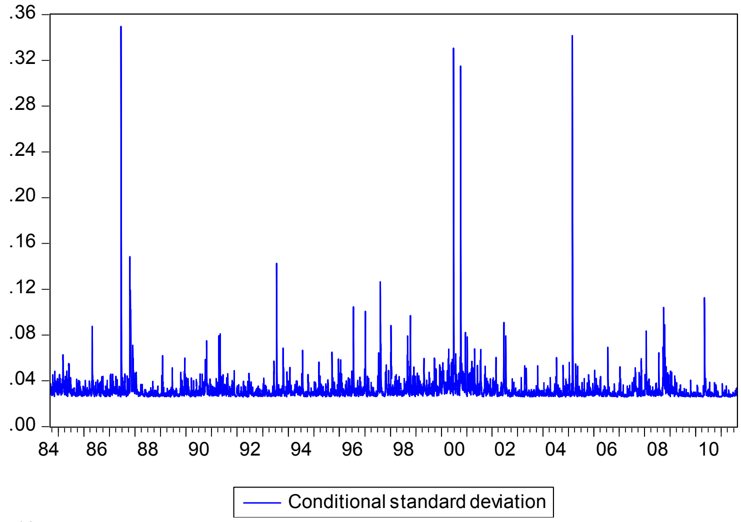
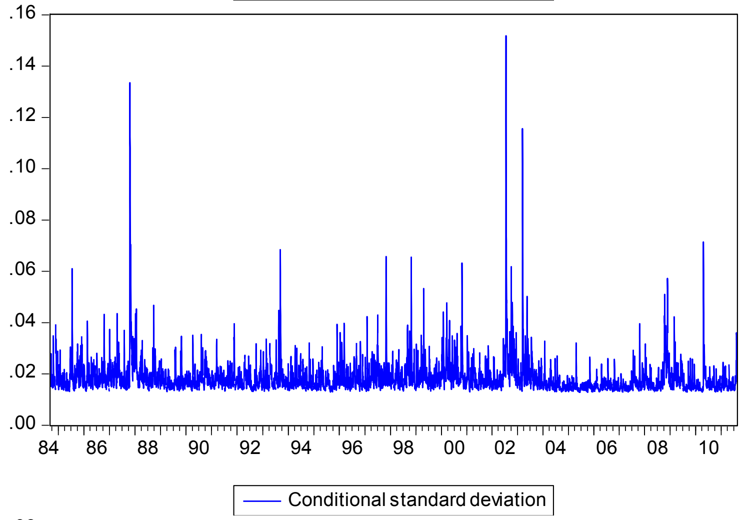


Table 5 Garch - Conditional standard deviation

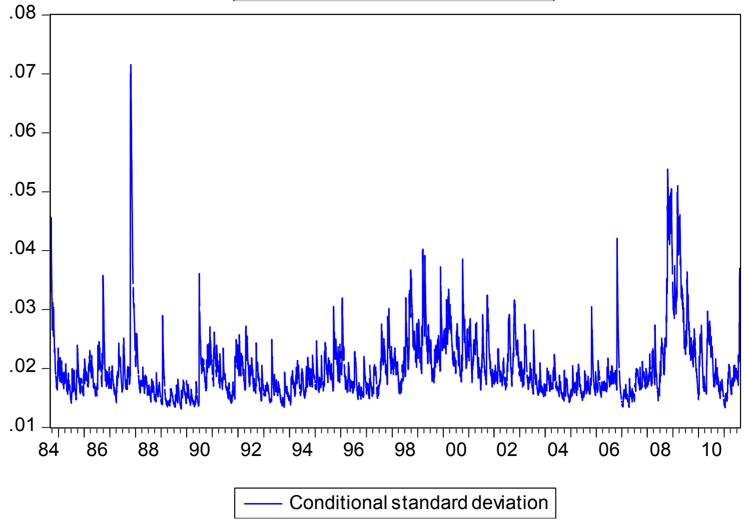
Apple



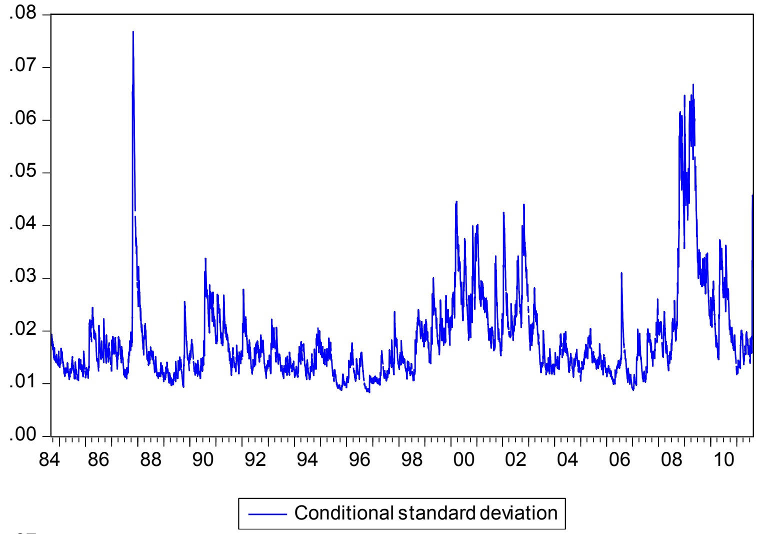
Baxter



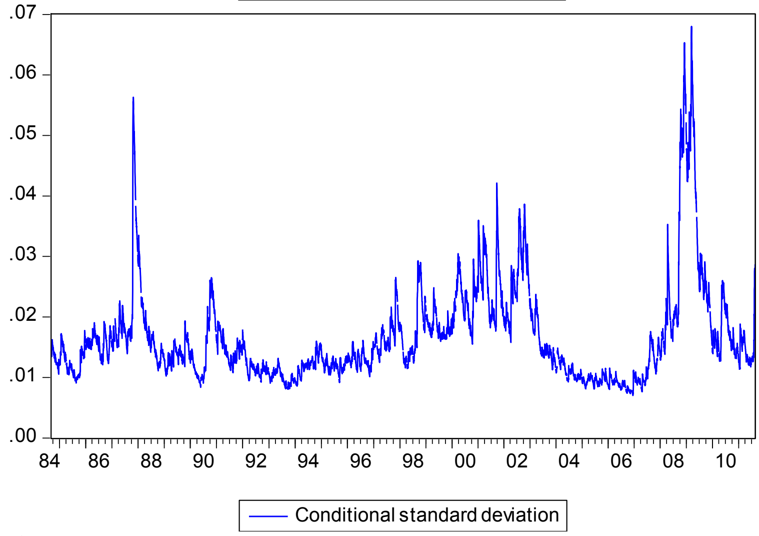
Caterpillar



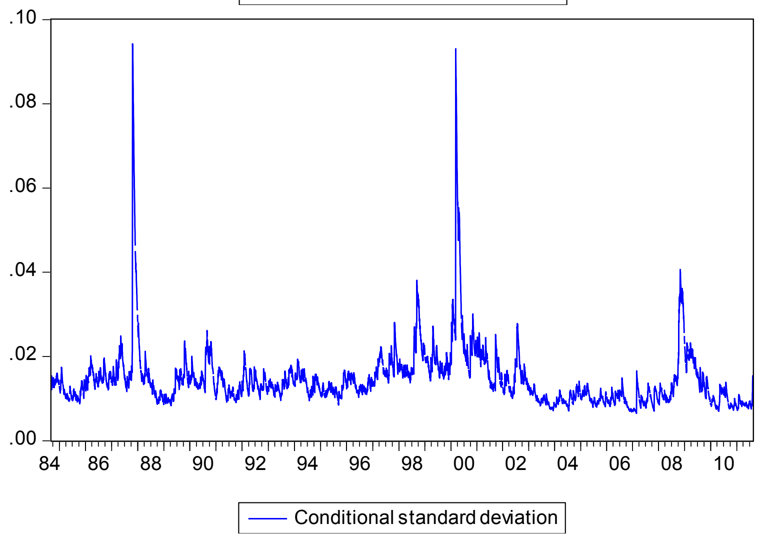
Dow



GE



Proc_Gam



4.2. Distribution fitting

As we mentioned before as an input to Monte Carlo simulation we needed to define source distribution that will provide numbers for Latin Hypercube Sampling. For our practical example we used the Anderson Darling test for its ability to recognize distributions with fat tails. We tested variables of our six stocks. All returns followed lognormal distribution. As explained above we tested daily values by Augmented Dickey Fuller and found strong autocorrelation, we transformed data to take a form of $\ln E_i = \ln\left(\frac{V_{t-1,i}}{V_{t,i}}\right)$. Hence lognormal distribution was expected. Distributions of variances and covariances were mostly Loglogistic or Pearson. The distribution fitting is one of shortcomings of our method. Since most of the theoretical distributions don't include fat tails we basically removed those from our tests. Even though Anderson Darling test accounted for fat tails, these are not usually included in theoretical distributions. However some extreme events in our simulations appeared anyway, but they were a result of coincident combination of extreme positions between different distributions and accounted for less then 1 % of total sample. After careful consideration we have removed these outliers in order to get better performance of OLS regressions.

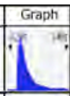
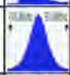
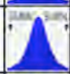
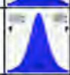
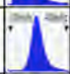
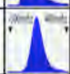
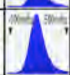
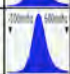
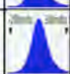
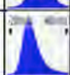
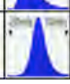
Name	Graph	Distribution	Min	Mean	Max
var GE		Pearson	-0.00119%	0.03499%	+∞
var DOW		Loglogistic	-0.00002%	0.03704%	+∞
var Baxter		Loglogistic	-0.00021%	0.03563%	+∞
var Caterpillar		Pearson	-0.00277%	0.04353%	+∞
var Proc Gam		Pearson	-0.00104%	0.02530%	+∞
var Apple		Pearson	-0.00406%	0.10529%	+∞
GE Estimated Return		Logistic	-∞	0.03914%	+∞
DOW estimated returns		Logistic	-∞	0.04121%	+∞
Baxter estimated return		Logistic	-∞	0.05214%	+∞
Caterpillar estimated return		Logistic	-∞	0.04947%	+∞
Proc Gam estimated return		Logistic	-∞	0.05299%	+∞
Apple estimated return		Logistic	-∞	0.10618%	+∞
covar Baxter/Caterpillar		Loglogistic	-0.00258%	0.00979%	+∞
covar Baxter/Apple		Loglogistic	-0.02415%	0.00510%	+∞
covar Baxter/Proc Gam		Pearson	-0.00513%	0.00886%	+∞
covar Caterpillar/Proc Gam		Loglogistic	-0.02415%	0.00510%	+∞
covar Caterpillar/Apple		Loglogistic	-0.02978%	0.01044%	+∞
covar DOW/Baxter		Loglogistic	-0.03356%	0.00236%	+∞
covar DOW/Caterpillar		Loglogistic	-0.07134%	0.00833%	+∞
covar DOW/Proc Gam		Loglogistic	-0.01990%	0.00265%	+∞
covar DOW/Apple		Loglogistic	-0.04547%	0.00678%	+∞
covar GE/DOW		Loglogistic	-0.01686%	0.00484%	+∞
covar GE/Baxter		Pearson	-0.00082%	0.01219%	+∞
covar GE/Catterpillar		Pearson	-0.00173%	0.01849%	+∞
covar GE/Proc Gam		Pearson	-0.00171%	0.01223%	+∞
covar GE/Apple		Loglogistic	-0.01467%	0.01069%	+∞
covar Proc Gam/Apple		Loglogistic	-0.03228%	0.00395%	+∞

Figure 7 Fitted distributions used for Monte Carlo simulation

4.3. Cross Correlation

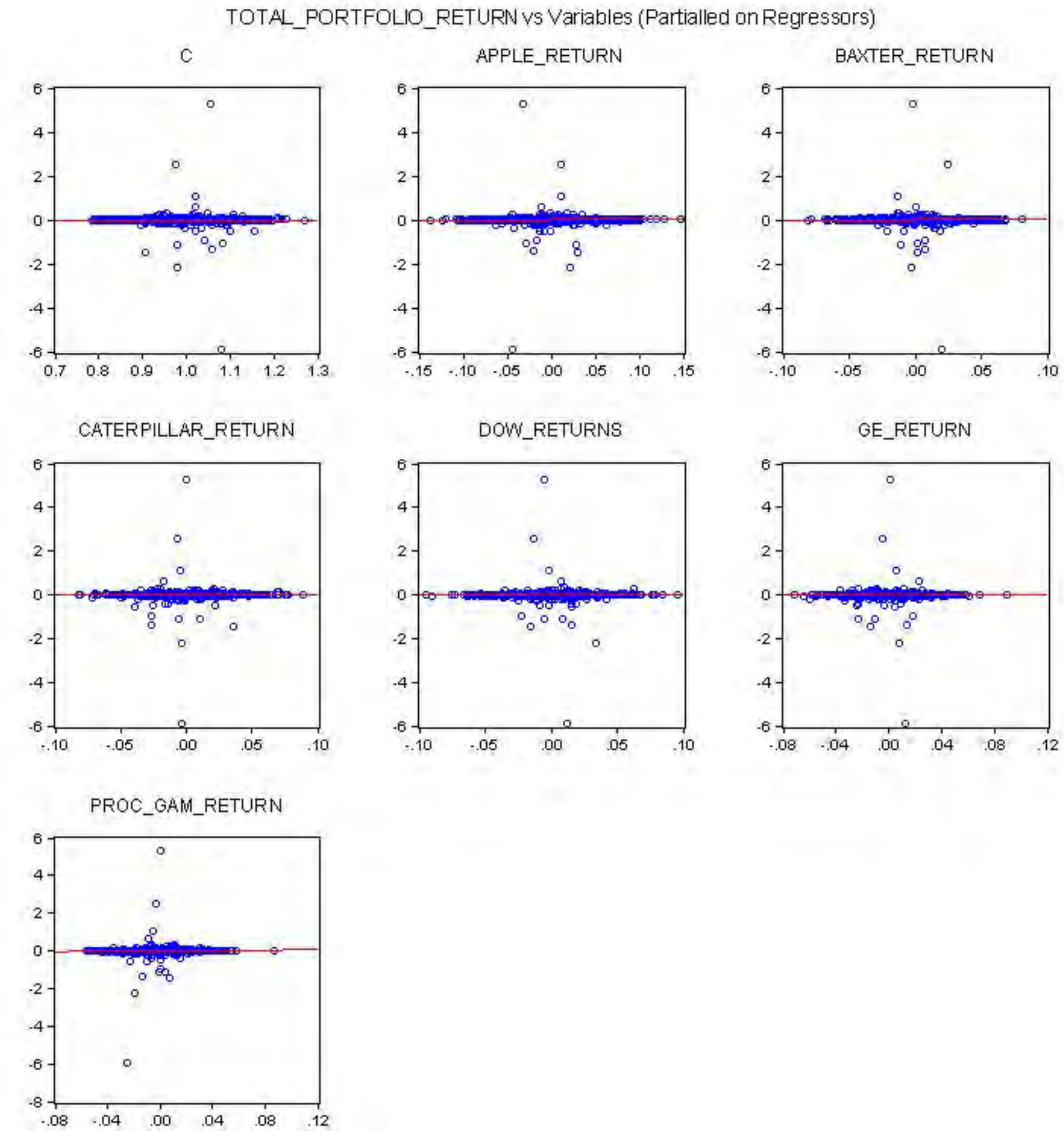
The reason for defining correlations between each of the variable is the best explained on an example. If we simulate agriculture production and amount of rain in each year to project our future revenue, it is not logical to have observations with high yield and low rain in the same year. In our case we have 27 variables that give us a matrix in form of 27x27. The table itself is in Appendix 1. We used the same correlations for actual and expected values but each process was run individually without defined correlations between actual and expected variables. This was allowed by testing for returns being uncorellated. Interesting outcome is a very low correlation between return and variance of individual stocks. (Murphy (1977), p.20) studied a relationship between risk and return, pointing out *"realised returns appear to be higher than expected for low-risk securities and lower than expected for high-risk securities ... or that the [risk-reward] relationship was far weaker than expected." The author continued on: "Other important studies have concluded that there is not necessarily any stable relationship between risk and return; that there often may be virtually no relationship between return achieved and risk taken; and that high volatility unit trusts were not compensated by greater returns"*. This is very serious for basic assumptions of Efficient Markets Hypothesis and Markowitz's portfolio theory itself. Since there is only weak correlation between riskier asset and higher return, investors might not be rewarded properly for holding riskier assets. (Fama and French (1992), p. 427) examined 9,500 stocks and Fama stated "*What we are saying is that over the last 50 years, knowing the volatility of an equity doesn't tell you much about the stock's return."* This provoked newspaper articles announcing, *"Beta as the sole variable in explaining returns on stocks ... is dead."*[12]

[12] http://finance.wharton.upenn.edu/~acmack/Chapter_10_app.pdf

4.4. Data trimming

First of all we used OLS regression on the full sample of 10.000 observations that were created in the Monte Carlo simulation. The resulting coefficients were highly insignificant and R^2 of the model was very low. We can observe this in Appendix 3, table 15. We run analysis for outliers in Eviews with total results shown in Appendix 9 and just return outliers for all stocks are graphically depicted in Table 9 below. Analysing this we realised that the most of the data is consistent but just a few outliers are strongly negatively influencing our model. Since we created all the data artificially in the Monte Carlo simulation using theoretical distributions of each of the stocks those outliers are the result of distribution's tails meeting at one data point. Even thought we realise extreme events occur in financial markets and they are even more common then predicted by theoretical distributions, our model is not made to simulate them and would perform rather poorly when predicting these. For this reason we have decided to trim our data to 99% interval. Thus we removed approximately 100 of observations out of 10.000. R^2 improved significantly for most of the models as is observable in Appendix 3. It also improved significance of our independent variables and enhanced our case of study.

Table 6 Stability diagnostics - Leverage Plots for return variables



4.5. Data sample periods

To illustrate a problem that investor faces when deciding which sample to use to run portfolio optimization we formed two different data samples. This problem is noted by Fabozzi, Gupta, & Markowitz in the year 2002 as *"A particularly glaring drawback of using the historical performance of returns to forecast expected returns is the uncertainty of the time-frame over which to sample…"* Thus we decided to use two different samples. One was taken from data between 7. 9. 1984 and 19. 8. 2011 (In our analysis is called FULL sample). Second was taken only for 500 observations from 27. 8. 2008 to 19. 8. 2010 (this sample is called 500 sample). All data were taken as logarithms of daily performances to avoid autocorrelation. For Markowitz's model in both both samples we formed a portfolio on efficient frontier for given daily return of 0.05 % by minimizing the variance of a portfolio. We evaluated performance of each portfolio on an investment period from 19. 8. 2010 to 19. 8. 2011.

5. Results

5.1. MPT framework

The solutions to MPT framework given above will give us an optimum portfolio weights for given expected returns, variances and covariances. These weights are changing as inputs vary. This means when our estimation of future returns and covariances changes we will form different portfolio. Monte-Carlo simulation allows us to input large number of random expected returns, variances and covariances into our calculations to get large number of different portfolios. We can afterwards analyse weights and return of each of these portfolios as well as other attributes. Regression analysis allows us to study dependencies of portfolio weights and actual returns on changes in input variables. This will give us a sensitivity analysis of final portfolio weights and returns on input variables.

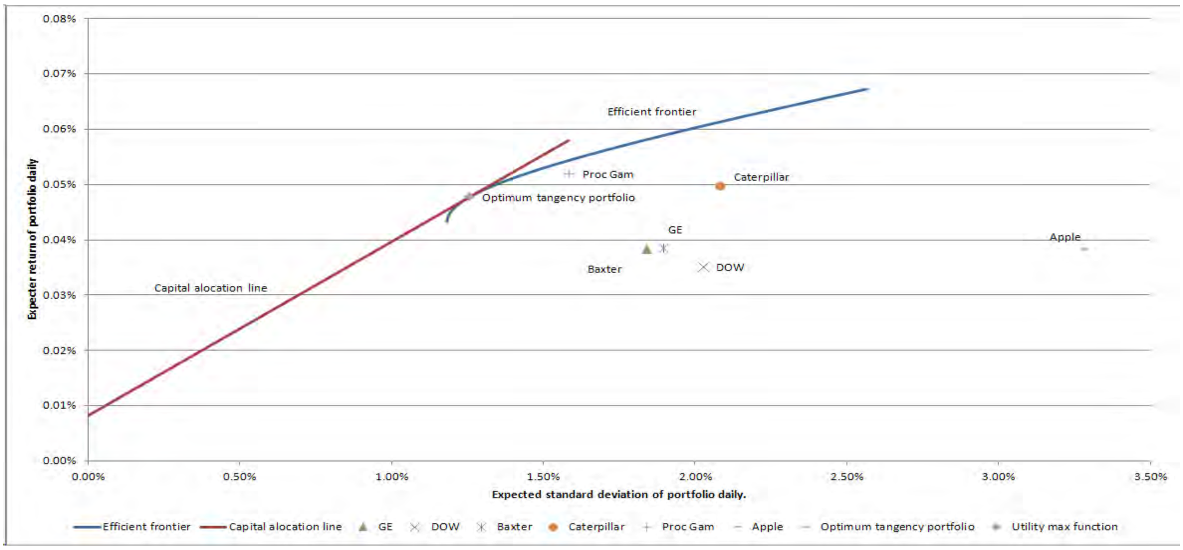
For a test of sensitivity of weights and returns we have formed a Monte-Carlo method based analysis based on following steps:

1. Solve Markowitz's portfolio optimization using Lagrange multipliers
2. Define source distributions for returns and variance-covariance matrix using historical data for selected stocks
3. Calculate cross-correlation matrix between returns, variances and covariances (define two identical correlation matrixes, one for actual and one for expected values)
4. Generate randomly expected returns, expected variances and expected covariances from defined distributions
5. Get optimum weights for Markowitz's minimum variance portfolio for generated variables
6. Get optimum weights for Markowitz's efficient portfolio for generated variances and given daily return of 0.1 %
7. Generate randomly actual returns, variances and covariances from defined distributions and calculate return of both portfolios
8. Repeat steps from 4 to 7 10.000 times
9. Collect data as an Eviews table of 10.000 observations.
10. Filter outcome to 99 % of original distribution to remove outliers to improve performance of OLS estimation

11. Run regression sensitivity and correlation analysis to reveal and describe possible dependencies

5.1.1. Static solution for data sample 1984-2011

Firstly we will present our results for static data. Portfolios presented are based on actual data observed in the market. This portfolio is based on large number of observations therefore should allow for very precise return prediction. Mean Variance Optimization based on this sample suggested forming portfolio with weights that could be considered reasonable. Investor should sell General Electrics for 17 % of his total portfolio, buy Dow Chemical Company for 3 %, buy Baxter for 4 %, invest 30 % in Caterpillar, 76 % in Procter and Gamble and only 4 % in Apple. The results could be observed in Table 4. In Graph 4 we can see the efficiency of Mean Variance Optimization since all the stocks have higher variance for given return then the portfolio they form.



	W(GE)	W(DOW)	W(BAXTER)	W(CAT)	W(PG)	W(APPLE)
efficient (max R)	-17%	3%	4%	30%	76%	4%

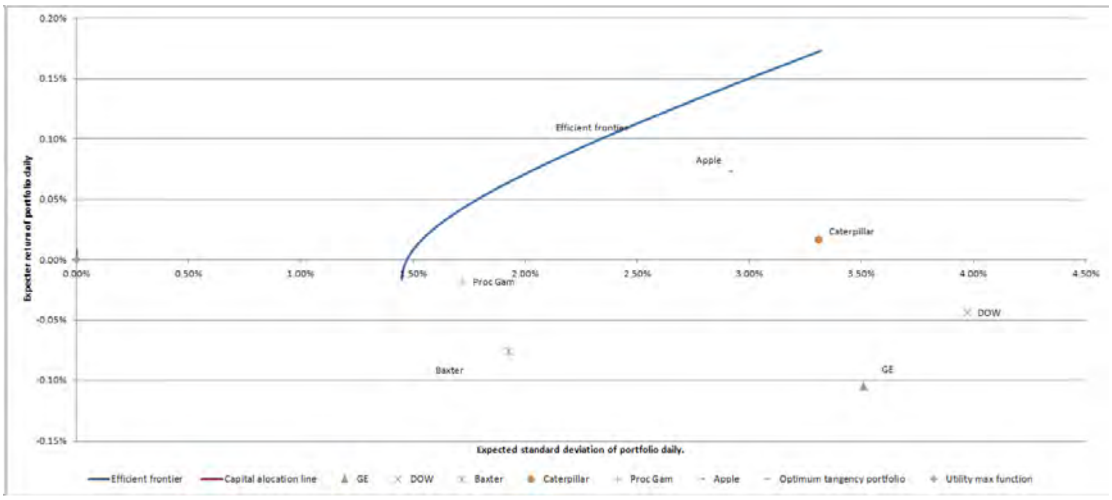
Graph 4 Depiction of individual stocks and Efficient Frontier for data sample of 1984 to 2011

5.1.2. Static solution for data sample 2008-2010

Second portfolio was based only on the recent data sample of 500 observations taken prior to the investment. The portfolio structure is very different and takes into account good performance of Apple, which was on the other hand one of the worst performing in the long-term sample. It also suggested short selling of General Electrics and Dow Chemical Company. This was caused by their bad performance during US recession of the year 2009. In general suggested portfolio is very different, apart from Procter and Gamble that takes major part in both portfolios.

	W(GE)	W(DOW)	W(BAXTER)	W(CAT)	W(PG)	W(APPLE)
efficient (max R)	-27%	-13%	-1%	15%	86%	40%

Table 7 Efficient portfolio based on data sample from 2008 to 2010

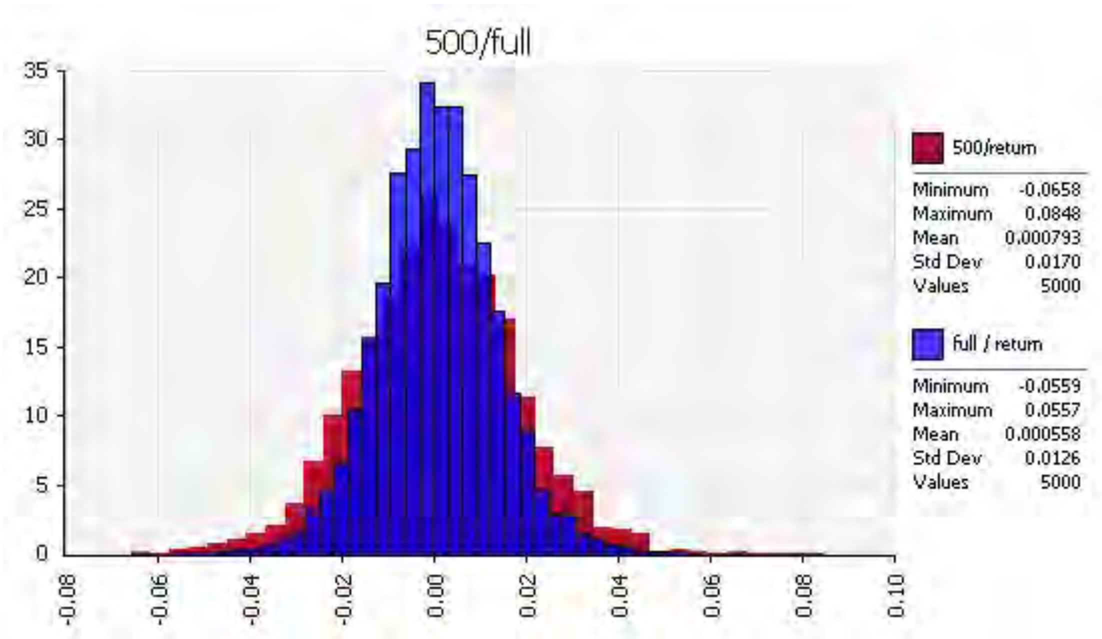


Graph 5 Depiction of individual stocks and Efficient Frontier for data sample of 2008 to 2010

5.1.3. MPT framework static solution total performance

We compared the performance of both data samples of the period of crisis year 2011. Despite the losses in stock markets at the end of summer 2011 both portfolios performed pretty well. Both delivered positive return and even reached lower level of variance then predicted by Markowitz model. The full sample portfolio delivered a return that was near to given value of 0.05 %. This portfolio could be considered more predictable. On the other hand second portfolio outperformed the first and more then doubled its Sharpe ratio with higher return for lower variance.

	Total return	Daily return	Est. var.	Real var.	Sharpe ratio
Full sample	10.92%	0.044%	0.021%	0.0091%	0.046
500 sample	22.22%	0.089%	0.033%	0.0078%	0.100



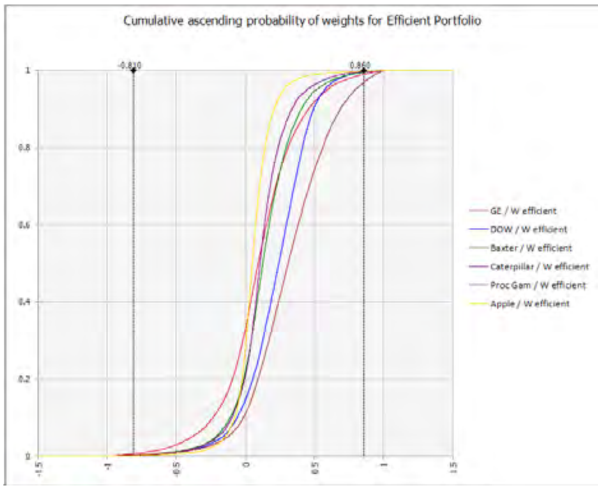
Graph 6 Monte-Carlo simulation for daily return of Full and 500 portfolios

Table 8 Comparison of both portfolios performance

By using Monte-Carlo simulation we found that full sample portfolio underperformed the 500 one in long-term as well. Full sample portfolio managed to maintain given level of return with lower variance. In general we can say that full sample portfolio addressed the given task better.

Why did one portfolio over performed second and Mean Variance Optimization failed to keep portfolio on efficient frontier and given return? We believe the answer lays in the precision of estimated returns and variances. The sensitivity of return and weight of each asset to changes in returns and covariance matrix are crucial to structure and return of a portfolio. In chapter 2.4.3 we have discussed time variance of variables. In table 4 and 5 we can observe a very different structures of portfolios that are based just on different time frames. The out of sample estimation obviously gives very different results for different time frames and since Mean Variance optimization is very input sensitive as pointed out by Michaud (2008) the given results for optimal portfolio weights.

5.1.4. MPT Sensitivity analysis



Graph 7 Cumulative ascending probability of weight of each stock in an efficient portfolio

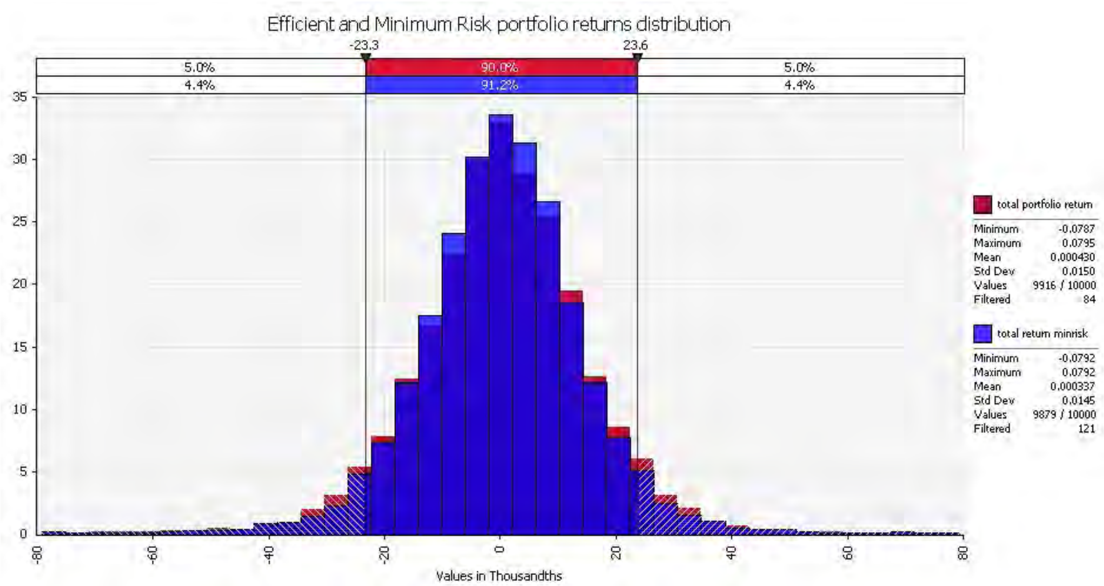
Based on our methodology introduced in prior chapters we run the analysis of effects of changes in expected and actual returns, variances and covariances. Since we conclude from the original Markowitz's formulas that there will be large effects of changes in expected values on final portfolio weights, we expect to find an explanation why these sensitivities are important for unintuitive differences in portfolio weights.

5.1.4.1. Portfolio weights

From our Monte-Carlo test we found out that Markowitz's mean variance optimization is systematically overweighting or underweighting some of the stocks in our portfolio. In Graph 7 we can see this practically on cumulative ascending probability function that shows the probability distribution of weight in each of the simulated portfolios. In appendix 2 we can see this more practically as we have attached each of the individual distributions. What is interesting is a regression of each of our randomized input on final weight of each individual stock in a portfolio. The results are graphically depicted in Appendix 2. It shows large importance of estimated covariance but low influence of estimated returns on final portfolio formation. Weights in Markowitz's portfolio are mostly defined by variance and covariance with individual stocks. This is given by portfolios based either on minimizing general variance or minimizing variance for given return, thus the role of expected stock return is low. Important is to mention that significant impact on weight of a stock in portfolio has a covariance between two different stocks. That practically means that Individual stock can be underweight or overweight in portfolio when variance or covariance changes for different stocks. The most used stock in our portfolio was Procter and Gamble. We can see in table 2 that it has the lowest variance from all six stocks. On the other hand the least used was the stock of Apple. Despite its very good performance in recent months its poor performance throughout the 27 years of data Markowitz's portfolio optimization would underweight this stock. Apart from using shorter sample of data investor could use some statistical tools like exponential smoothing to emphasize Apple's recent good performance. The most important conclusion from this chapter is large impact of expected variance covariance matrix and negligible effect of expected returns. This is surprising since covariance matrix is usually considered given and most portfolio managers predict only future returns. Since we defined cross-correlations between variance, covariance and returns, the reason might be that changes in expected returns appear in changes in covariance matrix as well. The final portfolio weights are strongly influenced by changes in covariance matrix so managers should take into consideration influences of stability of covariance matrix.

5.1.4.2. Portfolio performance

Graph 8 shows that performances of efficient portfolio and minimum variance portfolio are very similar. Originally the Efficient portfolio daily performance was -0.031 % (-7.75 % p.a.). It clearly missed daily return target of 0.1 %. Minimum variance portfolio performed on -0.236 % (-59 % p.a.). We trimmed data distribution on 99 % confidence level. That removed outliners and gave us more intuitive result of 0.043 % (10.75 % p.a.) of daily return for Efficient Portfolio and 0.0337 % (8.425 % p.a.) for minimum variance portfolio. It also shows lower variance for minimum variance portfolio. In untrimmed sample outliners caused higher variance in minimum risk portfolio then in efficient portfolio, this is clearly unfeasible. In Appendix 3 we can see that R^2 coefficient of determination is very low. This suggests that some outliers are far from original regression. In the second regression we reached an R^2 of 0.5152 proving that trimming was appropriate method of improving our model.



Graph 8 Daily performance distribution of MC simulated portfolios

For the test what influences return in Markowitz's portfolios we formed three regression equations.

R_p = actual return of a portfolio

R = actual return of stocks

R^{est} = prior estimation of future stocks' returns

R^{avg} = average return of stocks

V = actual variance-covariance matrix

V^{est}= prior estimation of variance covariance matrix

VAR^{avg} = average variance of stocks

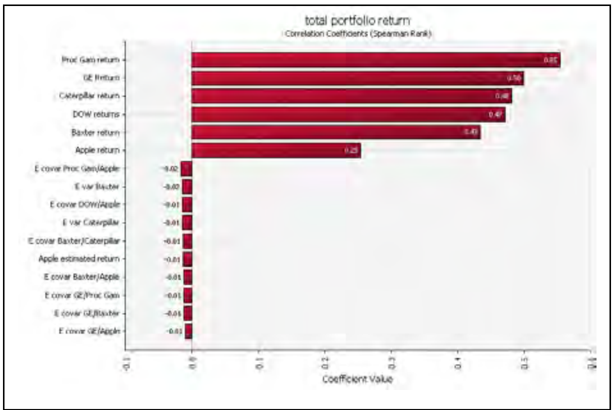
COV^{avg} = average covariance of stocks

1. $R_p = \alpha + \beta_1 * R + \beta_2 * R^{est} + \beta_3 * V + \beta_4 * V^{est} + \varepsilon$
2. $R_p = \alpha + \beta_1 * [R^{est} - R] + \beta_2 * [V^{est} - V] + \varepsilon$
3. $R_p = \alpha + \beta_1 * R^{avg} + \beta_2 * VAR^{avg} + \beta_3 * cov^{avg} + \varepsilon$

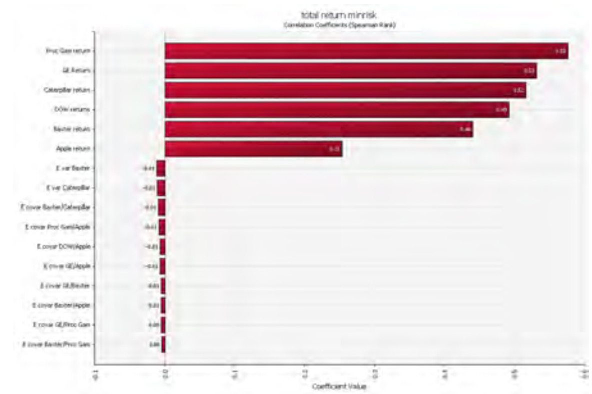
Results from regressions are in the Appendix 3 each of the tables denoted with number of the correspondent equation. Equation 1 shows that the most important for total portfolio performance are returns of each individual stock. The coefficients are approximately equal to: for Procter & Gamble 0.36, Dow Chemicals 0.26, Baxter 0.15, Caterpillar 0.11, GE 0.10 and Apple 0.05. This shows that 1 % change in return is positively correlated with total portfolio return and affects portfolio from 0.05 % to 0.36 %. What is interesting is very high significance and regression coefficient of some expected covariances. For example 1 % change in expected covariance of Baxter and Caterpillar would change return of our portfolio by 6.2 %. (Pagan and Schwert (1990)) used a series of tests to rule out the covariance stationarity in stock market data. Their conclusion is important because it might mean the importance of precise covariance matrix estimation for Markowitz's Mean Variance optimization. On the other hand our test in equation 1 revealed only few significant covariance variables. We have shown before that expected covariances are very significant for the weight of each individual stock in portfolio. We think this is the connection to high correlation between portfolio return and expected covariance of some particular stocks. Higher expected covariance means higher weight of a stock that could bring higher total performance to portfolio. We suppose that the most important conclusion is very low significance of actual covariance and variance. A priory estimated variables are significant due to their high impact on stock portfolio weights. Posterior variance and covariance non-significance means that return is not significantly influenced by changes in covariance matrix and thus is only dependent on future return of individual stocks.

Second equation allowed us to test for effect of estimation error. Coefficient of determination is lower in this model. On the other hand coefficients on variable defined as [estimated - actual return] are very significant. We can also see that coefficients are positive which means positive correlation between return and estimation error. We believe that explanation for this is what studied Jobson and Korkie (1980) as Markowitz's error maximizing optimization. Overestimation of return leads to overweight of a stock in portfolio. If actual return is lower the portfolio could be shifted towards higher risk thus lower Sharpe ratio but higher return. Return variables are strongly significant but actual values aren't high. The percentage change in portfolio return for 1 % error in estimation is for Apple value of 0.03, Baxter 0.08, Caterpillar 0.05, Dow 0.12, GE 0.05 and Procter & Gamble 0.16. Results also revealed significant impact of error in estimation of covariances on total return. This is mostly the case for stocks that failed randomness test in previous chapters.

Third equation aggregated all values as average. Equation became simpler and gave us more intuitive results. In this case we have included regressions of both portfolios. Results were almost identical in both previous examples, but in the third equation efficient portfolio and minimum variance portfolio performed a bit differently. Average return has significant positive impact on both portfolios of approximately 0.88. Expected average covariance has in case of minimum variance portfolio negative coefficient of -2.58 significant at 90 % level of confidence. For efficient portfolio is this coefficient strongly insignificant. The rest of coefficients like average variance and average covariance are again without real impact on portfolio performance thus insignificant.



Graph 9 Correlation coefficients of portfolio returns with individual variables



In Graph 9 is depicted practically what we have revealed by regression analysis. There exists strong correlation of portfolio return with actual future returns of individual stocks, low correlation with expected covariance matrix and no correlation with future correlation matrix. The last finding is particularly important. Although we have shown in previous chapters that covariance is unstable in time and difficult to predict. This finding was supported by findings of Jobson and Korkie (1980), our results showed that future variances and covariances are not important so they prediction is redundant.

The importance of these two variables resides in their importance for definition of initial portfolio. Since portfolio weights are insensitive to expected return, portfolio is mostly defined by variance covariance matrix. Although changes in covariance have high values of coefficients, actual changes are so small that these coefficients are also insignificant.

5.2. Black-Litterman framework

5.2.1. B-L static solution

We based our practical experiment of Black-Litterman model on the same stocks and data sample as in Markowitz's portfolio tests. For a value of coefficient of risk aversion we used the value of 2.5 suggested by (He and Litterman (1999)) as a World Average Risk Tolerance. As a risk free rate we used the value of 0 that allowed us to account excess return as a stock return. It is also in accordance with today's capital markets where US short-term government bonds yields are close to 0 (As of 19th of august 2011). To calculate implied expected returns we used data in table 7. We used our stocks as a closed universe of investments so their weights sum up to 100 %.

As a proxy for investors believes we used Mean value of analysts' target value. We set a view for each stock so our matrix P took a form of diagonal 6 x 6 representing discrete opinions about each stock without their mutual influences. This is very simplified attitude, but since stock analysis is not a part of this work it should be adequate for our analysis. We can see the form that matrix P took in Table 8. In table 9 are estimated excess returns. These were calculated as difference between today's value of a stock and mean target value of each stock. For matrix Ω representing confidence in estimated values we used a value of 0.5 for all views.

Table 9 Black-Litterman data input, retrieved on 9th of September 2011

	Capitalization (bil. USD)	Mean target value	Last trade value
GE	165.26	22.32	15.59
DOW	31.77	40.23	26.9
Baxter	31.16	65.8	54.84
Caterpillar	56.23	126.4	87.04
Proc Gam	172.86	70.2	62.91
Apple	356.13	493.2	384.14

Table 10 Matrix P of investors believes

1	0	0	0	0	0
0	1	0	0	0	0
0	0	1	0	0	0
0	0	0	1	0	0

0	0	0	0	1	0
0	0	0	0	0	1

	Analyst estimated return
GE	43.169%
DOW	49.554%
Baxter	19.985%
Caterpillar	45.221%
Proc Gam	11.588%
Apple	28.391%

Table 11 Matrix V, estimated excess return

We solved portfolio optimization using Black-Litterman and inputs given above. First we obtained Implied daily returns. In Table 10 we can compare implied return obtained from Black-Litterman optimization with Mean daily return from historical analysis. Implied returns obtained from Black-Litterman model are in comparison to Mean returns obtained by analysis of historical values. We used two data samples as in previous chapters. One is with data since 1984. Second is only last 500 observations, approximately for past 2 years. Implied returns are not as different on both samples as are in case of historical estimation. This is given by their definition by actual market capitalization. Impacts of differences between full sample and 500-sample variance-covariance matrix are small. On the other hand differences in historical Mean returns are very large and this points to difficulties with sample selection for prediction of returns from historical values.

Table 12 Implied and mean returns for Full and 500 observations samples

	GE	*DOW*	*Baxter*	*Caterpillar*	*Proc Gam*	*Apple*
Implied return 2010	0.136%	0.131%	0.040%	0.121%	0.054%	0.125%
Implied return FULL	0.042%	0.024%	0.024%	0.038%	0.028%	0.130%
Mean return 2010	-0.105%	-0.043%	-0.076%	0.016%	-0.018%	0.073%
Mean return FULL	0.038%	0.035%	0.039%	0.050%	0.052%	0.038%

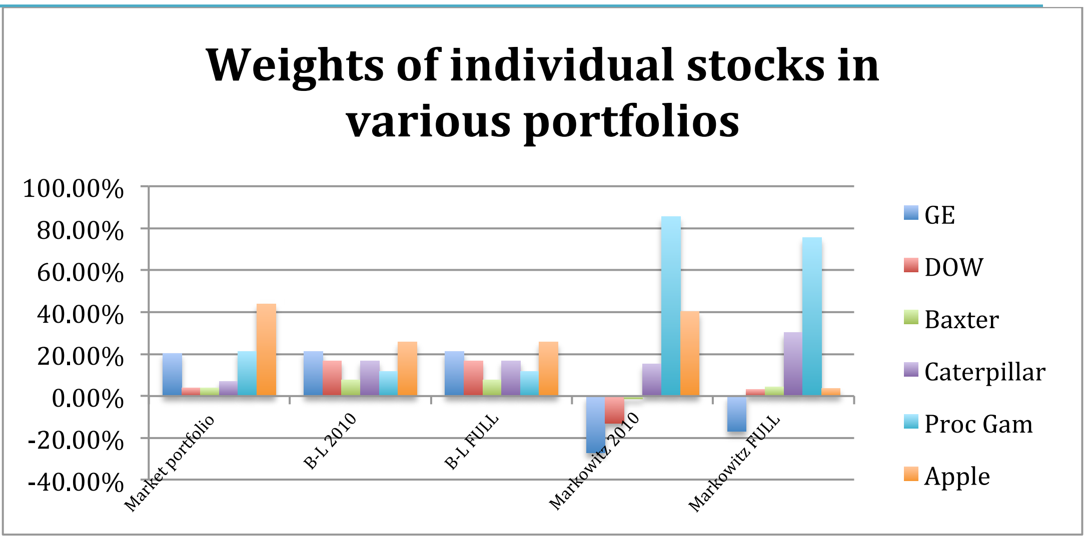
All our stocks are considered by analysts to be undervalued. Because of this if we account for they opinion, expected returns grow significantly. Weights stay stable and changes in portfolio are not that tremendous.

Table 13 Implied returns including analysts' opinions

	GE	DOW	Baxter	Caterpillar	Proc Gam	Apple
Returns with opinion 2010	0.41957%	0.47008%	0.12872%	0.39876%	0.16488%	0.27032%
Weights with opinion 2010	21.225%	16.809%	7.676%	16.652%	11.832%	25.805%
Returns with opinion FULL	0.11200%	0.09307%	0.06836%	0.11992%	0.06480%	0.22541%
Weights with opinion FULL	21.244%	16.865%	7.667%	16.678%	11.820%	25.725%

Table 14 Portfolio weights for different methods of optimization

	GE	DOW	Baxter	Caterpillar	Proc Gam	Apple
Market portfolio	20.32%	3.91%	3.83%	6.91%	21.25%	43.78%
B-L 2010	21.225%	16.809%	7.676%	16.652%	11.832%	25.805%
B-L FULL	21.244%	16.865%	7.667%	16.678%	11.820%	25.725%
Markowitz 2010	-27%	-13%	-1%	15%	86%	40%
Markowitz FULL	-17%	3%	4%	30%	76%	4%

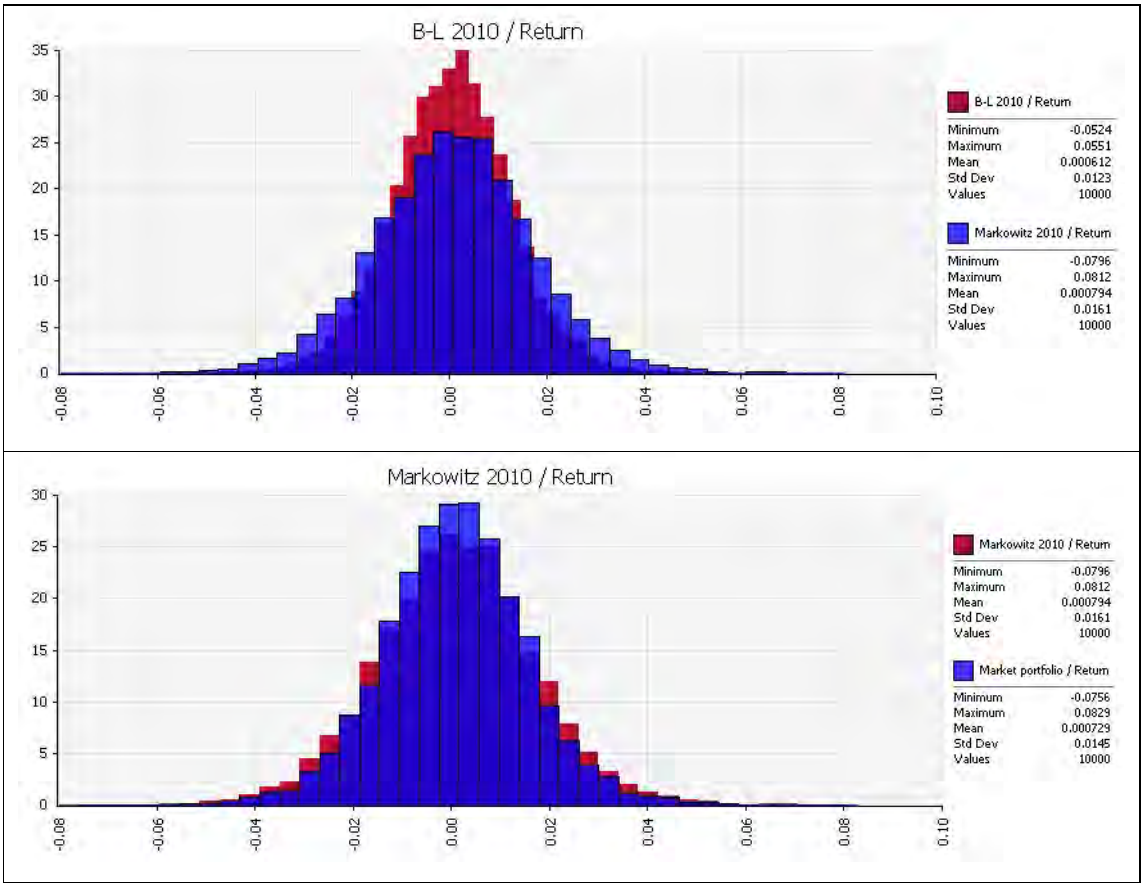


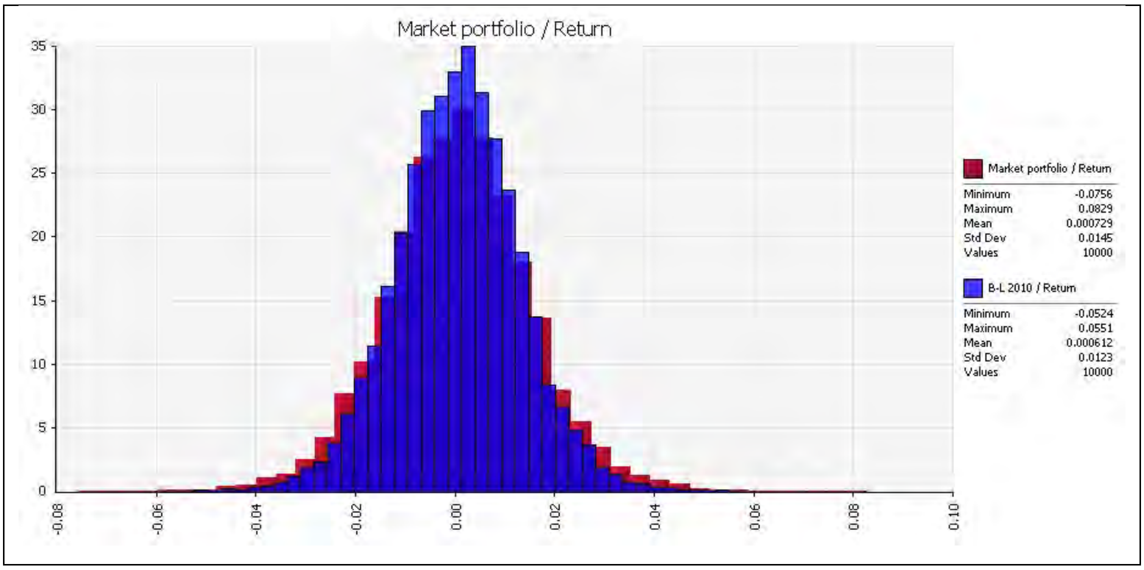
Graph 10 Portfolio weights with different methods of portfolio optimization

Table 12 and especially Graph 10 shows differences in portfolio weights when using different types of optimizations. Both Markowitz's portfolios show very large and unrealistic positions. Market portfolio on the other hand is emphasizing Apple above to DOW and Baxter. Markowitz also suggests large short positions in stocks of GE. If we look on suggested portfolio by Black-Litterman optimization, both portfolios aren't surprisingly extreme and most of investors would find them intuitive. We added Market portfolio that is based purely on individual capitalization. By comparing Market portfolio to Black-

Litterman reader can get better idea of influences of B-L optimization on final portfolio.

Graph 11 shows comparison of performance of portfolio optimized by Black-Litterman and Markowitz's methods. For a better comparison we added a performance of Market portfolio. Black-Litterman portfolios show lower variance of returns with probability distribution without too fat tails. On the contrary Markowitz's portfolio performance is flatter and thus exhibits higher probability of extreme events. It would be interesting to study impacts of changes in investor's opinions and confidence levels, but it is not a subject of this work.





Graph 11 Monte-Carlo simulation of Black-Litterman, Markowitz and Market portfolios performance

5.2.2. Black-Litterman dynamic solution

To compare Black-Litterman model to Markowitz and get better idea of its performance we decided to make our model dynamic. We still use a formula for BL model of $\boldsymbol{\Pi} = \boldsymbol{\delta} * \boldsymbol{\Sigma} * \boldsymbol{w}$ and $E[R] = \left[(\tau\Sigma)^{-1} + P'\Omega^{-1}P\right]^{-1}\left[(\tau\Sigma)^{-1}\Pi + P'\Omega^{-1}Q\right]$ but instead of static numbers derived from market we use theoretical distributions randomly representing those numbers.

For the variance-covariance matrix $\boldsymbol{\Sigma}$ and expected returns matrix Q we used theoretical distributions as defined in the section 4.3. For the matrix Ω representing a level of confidence in our expected returns we set a uniform distribution giving confidence levels from 10% to 100%. We omitted 0% confidence. First of all the biggest advantage of using BL model is the ability to input opinions about expected market situation. Second when Ω is set to 0 model gives extreme solutions since it is only defined by market capitalization, variance covariance matrix and individual risk aversion. Matrix P giving the stocks we have views on is set as singular diagonal matrix of 6x6. For simplicity we have only separated views for returns of each stock regardless the performance of other stocks in portfolio. Tau is set to 1 as in static solution and risk aversion is static and set to 0,4.

In figure 7 is observable the slight superiority of Black-Litterman model (red graph) to Markowitz's efficient portfolio return (blue graph) obtained in Monte Carlo simulation.

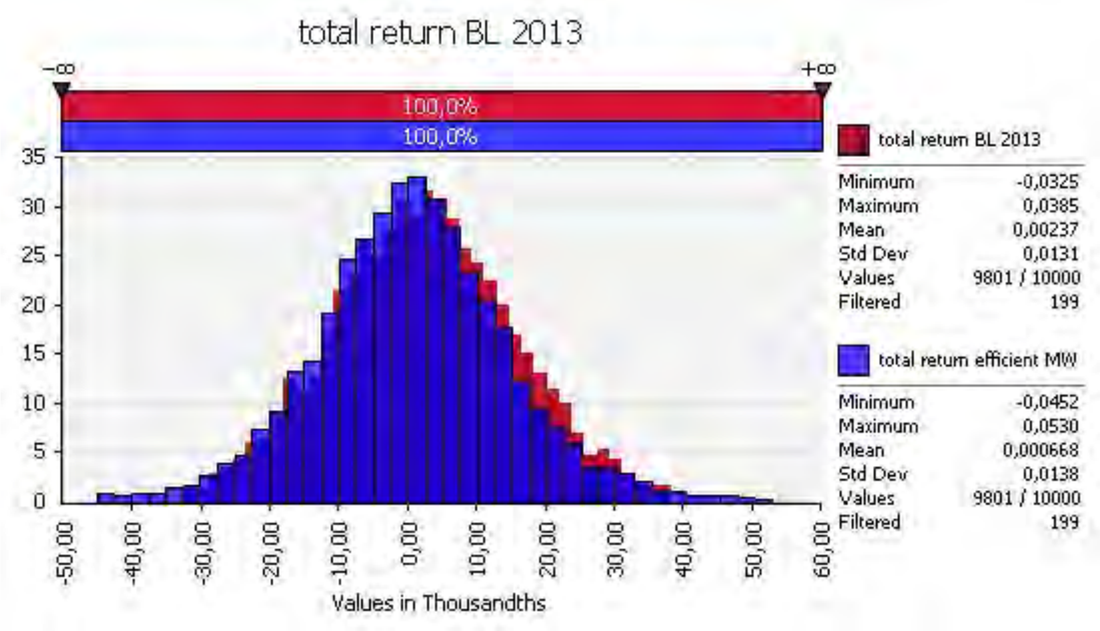
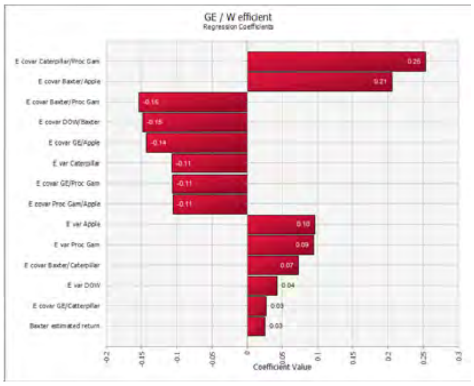
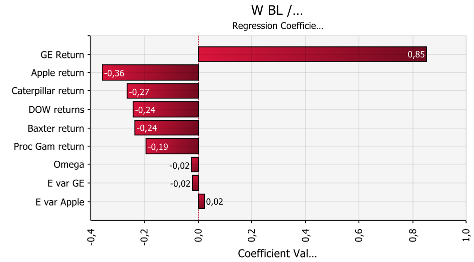
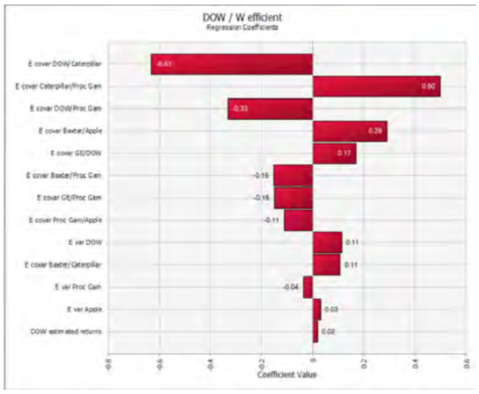
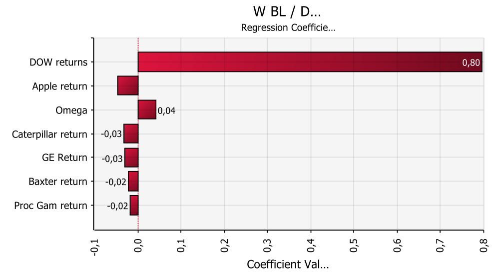
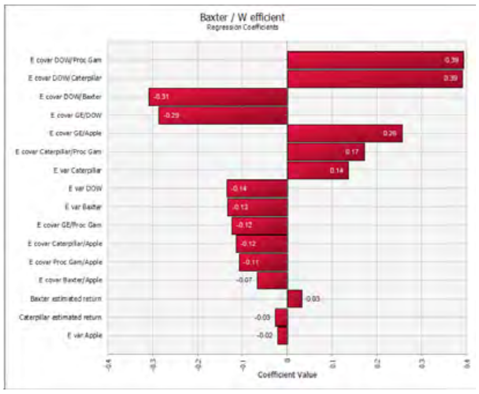
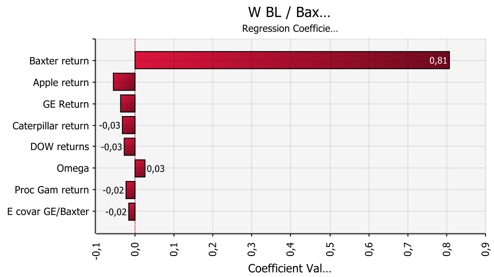
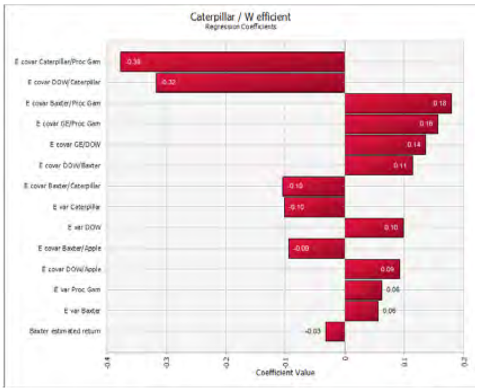
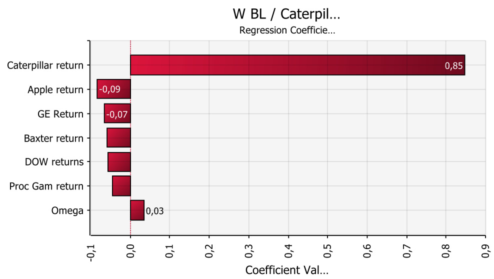
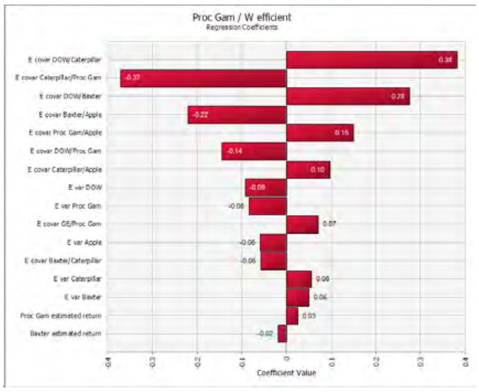


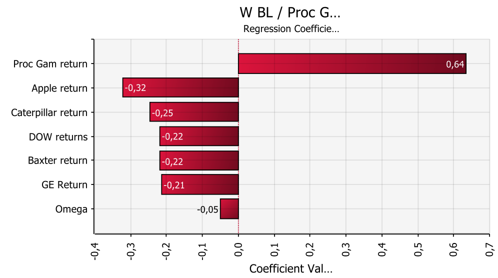
Figure 8 B-L and Markowitz simulated returns

Table 15 Regression coefficients for weights of Markowitz and B-L optimization

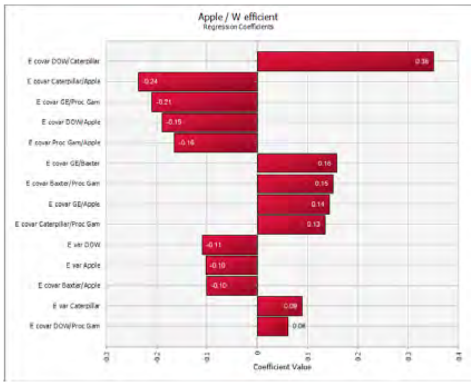
	Markowitz optimization	Black-Litterman optimization
GE		
DOW		
Baxter		
Caterpillar		

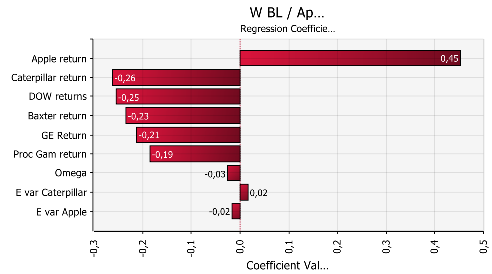
Proc Gam





Apple





The outcome of our simulation is in the table 16. The most important part is the comparison of regressors influencing resulting weight of a stock in a portfolio. In Markowitz's case there is a strong presence of expected variance covariance matrix and it is the most influential element. On the other hand positions in Black-Litterman model are almost exclusively influenced by predicted future returns and slightly by matrix Ω. We believe this is the most important reason for Black-Litterman superiority to Markowitz's optimization since it is not influenced by variables as variance and covariance that are difficult to control in classic MPT framework. The only important variables used for construction of Black-Litterman portfolio are those that are studied by investor prior to investment decision.

6. Conclusion

Based on a general arguing of researchers towards passive investment we have performed a comprehensive analysis of basic method of passive portfolio optimization. Namely we focused on Markowitz's Modern Portfolio Theory and its extension Black-Litterman model. Our analysis revealed some drawbacks of Modern Portfolio Theory and its practical use. Most of portfolio analysts take variance-covariance matrix as time invariant and thus fixed. We discussed that covariance actually changes significantly, especially in times of financial crisis. Weights of individual stocks in Markowitz's optimization are oversensitively influenced by changes in expected variances and covariance. Their standard deviation from mean portfolio weight is actually between 20 and 30 %. This points to large changes in portfolio weights dependent on changes in expected covariance matrix. The nature of our Monte-Carlo tests led to a finding that portfolio is very differently structured depending on which day is an analyst forming a portfolio. This is clearly inappropriate. On the other hand our Monte-Carlo simulation of random Markowitz's portfolio also revealed that actual covariance is insignificant for final return of a portfolio. This leads to a conclusion that portfolio formation is mostly based on expected variance-covariance matrix, although the return of portfolio is based almost purely on actual stocks return.

When we tested the impact of out of sample estimation error on return of portfolio we found low positive coefficients of very high significance. This says that positive error in estimation has positive impact on final return. Black and Litterman (1992) named a problem of error maximization in Markowitz, which we think is a connection to our coefficients in error test. Since the asset with positive out of sample estimation error tend to be outweighed in our portfolio it will scale down a risk free asset proportion from the portfolio while providing higher expected Sharpe ratio.

We have also taken two different portfolios. One based on short-term data sample based on the years 2008, 2009 and 2010. This portfolio gave a large position to Apple with its very good performance. On the other hand when we have taken a sample from 1984 to 2010 it gave to Apple a very low position. We showed that the selection of data sample is very important to portfolio formation.

To show the contrast we formed a portfolio based on Black-Litterman model and analysts' consensus target price recommendation. This portfolio showed much better stability. Positions are more intuitive and Monte-Carlo test revealed that actual returns distribution is less variable with fewer extreme events in comparison to Markowitz's optimized portfolios. Dynamic model of the Black-Litterman model revealed extended practicality and superiority to Markowitz's portfolio with strong connection between prior information processing and resultant portfolio. In particular it is much better performing in an environment of inconstant variance covariance matrix which we have proved is the case of our chosen stocks and we believe it applies to the whole financial market.

Markowitz's Modern Portfolio Theory and idea of diversifications are very important and efficient for portfolio formations. But actual portfolios are highly concentrated, positions are unstable and maintaining such a portfolio on efficient frontier would mean a lot of trading bringing excessive transaction costs. On the other hand Black-Litterman portfolios bring stability and possibility of inputting of analysts' views to portfolio formation. Black-Litterman model has some parameters that need to be defined and their proper definition can be difficult. These are drawbacks for practical use of B-L model, but its superior performance to Markowitz's portfolios justifies further studies in the field.

During this essay we realized that Black-Litterman approach to portfolio selection is simple, intuitive and results include private information as well as all information included in prices. This is very conforming when we realize that markets are not perfect and thus B-L framework allows amending investor's portfolio to imperfect markets.

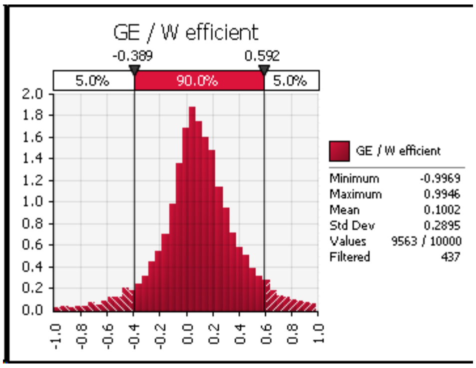
Appendix 1 Cross Correlation Table

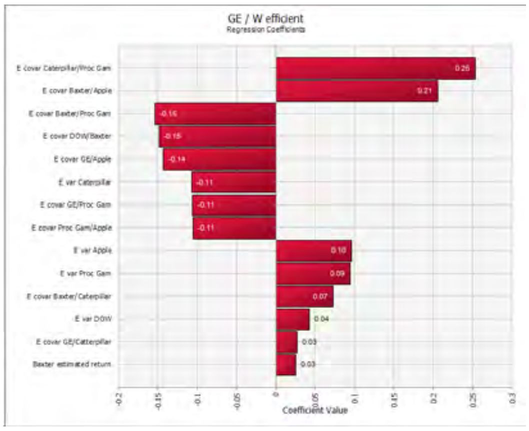
Correlation Table	GE	DOW	Baxter	Caterpillar	Proc Gam	Apple	var GE	var DOW	var Baxter
GE	1.000	0.270	0.343	0.468	0.418	0.201	-0.005	-0.002	0.002
DOW	0.270	1.000	0.120	0.317	0.141	0.130	0.018	0.023	0.017
Baxter	0.343	0.120	1.000	0.251	0.326	0.107	-0.012	-0.007	-0.013
Caterpillar	0.468	0.317	0.251	1.000	0.300	0.191	0.012	0.020	0.017
Proc Gam	0.418	0.141	0.326	0.300	1.000	0.099	-0.006	-0.006	-0.001
Apple	0.201	0.130	0.107	0.191	0.099	1.000	0.011	0.009	0.004
var GE	-0.005	0.018	-0.012	0.012	-0.006	0.011	1.000	0.947	0.428
var DOW	-0.002	0.023	-0.007	0.020	-0.006	0.009	0.947	1.000	0.347
var Baxter	0.002	0.017	-0.013	0.017	-0.001	0.004	0.428	0.347	1.000
var Caterpillar	0.001	0.019	-0.014	0.002	-0.013	0.006	0.861	0.843	0.367
var Proc Gam	-0.005	0.000	0.006	0.001	-0.013	-0.029	0.438	0.447	0.530
var Apple	-0.021	-0.002	0.005	0.012	-0.010	-0.037	0.232	0.249	0.312
covar GE/DOW	-0.003	0.019	-0.008	0.020	-0.005	0.017	0.840	0.847	0.175
covar GE/Baxter	-0.005	0.005	-0.012	0.002	-0.008	0.003	0.629	0.535	0.721
covar GE/Catterpillar	-0.006	0.018	-0.014	0.013	-0.005	0.013	0.970	0.929	0.456
covar GE/Proc Gam	-0.001	0.008	-0.008	0.001	-0.009	0.005	0.718	0.643	0.592
covar GE/Apple	-0.016	0.003	-0.011	0.006	-0.010	-0.004	0.744	0.709	0.598
covar DOW/Baxter	-0.015	0.006	-0.009	0.013	-0.011	0.013	0.785	0.724	0.271
covar DOW/Caterpillar	-0.005	0.017	-0.006	0.021	-0.008	0.017	0.874	0.903	0.254
covar DOW/Proc Gam	-0.007	0.014	-0.004	0.017	-0.011	0.009	0.828	0.818	0.094
covar DOW/Apple	-0.014	0.005	-0.011	0.010	-0.010	0.009	0.766	0.755	0.396
covar Baxter/Caterpillar	-0.008	-0.001	-0.010	0.000	-0.008	0.000	0.546	0.466	0.643
covar Baxter/Proc Gam	-0.006	0.001	-0.008	-0.001	-0.008	-0.005	0.486	0.434	0.593
covar Baxter/Apple	-0.005	-0.004	-0.015	-0.006	-0.009	0.008	0.253	0.205	0.530
covar Caterpillar/Proc Gam	-0.004	0.010	-0.005	0.007	-0.012	0.001	0.758	0.712	0.555
covar Caterpillar/Proc Gam	-0.013	0.009	-0.011	0.010	-0.006	0.008	0.651	0.617	0.549
covar Proc Gam/Apple	-0.004	0.001	-0.017	-0.003	-0.007	0.008	0.289	0.256	0.441

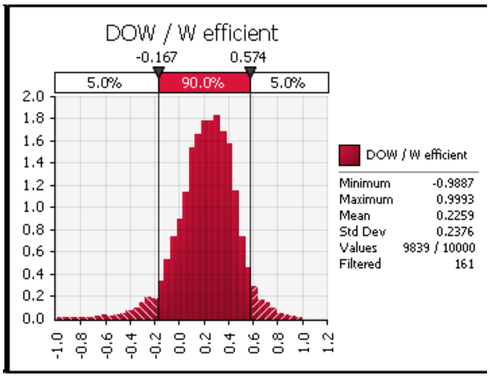
covar DOW/Apple	covar DOW/Proc	covar DOW/Caterpi	covar DOW/Baxter	covar GE/Apple	covar GE/Proc Gam	covar GE/Catterpill	covar GE/Baxter	covar GE/DOW	var Apple	var Proc Gam	var Caterpillar
-0.014	-0.007	-0.005	-0.015	-0.016	-0.001	-0.006	-0.005	-0.003	-0.021	-0.005	0.001
0.005	0.014	0.017	0.006	0.003	0.008	0.018	0.005	0.019	-0.002	0.000	0.019
-0.011	-0.004	-0.006	-0.009	-0.011	-0.008	-0.014	-0.012	-0.008	0.005	0.006	-0.014
0.010	0.017	0.021	0.013	0.006	0.001	0.013	0.002	0.020	0.012	0.001	0.002
-0.010	-0.011	-0.008	-0.011	-0.010	-0.009	-0.005	-0.008	-0.005	-0.010	-0.013	-0.013
0.009	0.009	0.017	0.013	-0.004	0.005	0.013	0.003	0.017	-0.037	-0.029	0.006
0.766	0.828	0.874	0.785	0.744	0.718	0.970	0.629	0.840	0.232	0.438	0.861
0.755	0.818	0.903	0.724	0.709	0.643	0.929	0.535	0.847	0.249	0.447	0.843
0.396	0.094	0.254	0.271	0.598	0.592	0.456	0.721	0.175	0.312	0.530	0.367
0.626	0.639	0.703	0.548	0.669	0.725	0.848	0.592	0.654	0.283	0.554	1.000
0.322	0.072	0.168	0.049	0.729	0.673	0.381	0.625	0.011	0.690	1.000	0.554
0.164	-0.055	0.015	-0.043	0.562	0.337	0.151	0.330	-0.110	1.000	0.690	0.283
0.731	0.960	0.963	0.913	0.376	0.427	0.855	0.331	1.000	-0.110	0.011	0.654
0.746	0.298	0.478	0.412	0.826	0.950	0.697	1.000	0.331	0.330	0.625	0.592
0.848	0.812	0.901	0.792	0.752	0.771	1.000	0.697	0.855	0.151	0.381	0.848
0.796	0.412	0.575	0.442	0.836	1.000	0.771	0.950	0.427	0.337	0.673	0.725
0.727	0.361	0.541	0.416	1.000	0.836	0.752	0.826	0.376	0.562	0.729	0.669
0.739	0.916	0.880	1.000	0.416	0.442	0.792	0.412	0.913	-0.043	0.049	0.548
0.844	0.917	1.000	0.880	0.541	0.575	0.901	0.478	0.963	0.015	0.168	0.703
0.691	1.000	0.917	0.916	0.361	0.412	0.812	0.298	0.960	-0.055	0.072	0.639
1.000	0.691	0.844	0.739	0.727	0.796	0.848	0.746	0.731	0.164	0.322	0.626
0.726	0.261	0.432	0.375	0.769	0.918	0.646	0.950	0.288	0.319	0.594	0.584
0.668	0.122	0.327	0.195	0.813	0.914	0.554	0.931	0.142	0.453	0.705	0.509
0.606	-0.049	0.202	0.088	0.643	0.785	0.383	0.846	0.021	0.193	0.436	0.292
0.816	0.500	0.641	0.503	0.839	0.975	0.806	0.905	0.497	0.384	0.701	0.760
0.765	0.299	0.536	0.389	0.917	0.814	0.706	0.827	0.360	0.430	0.541	0.557
0.654	0.031	0.263	0.127	0.624	0.757	0.427	0.799	0.095	0.116	0.335	0.303

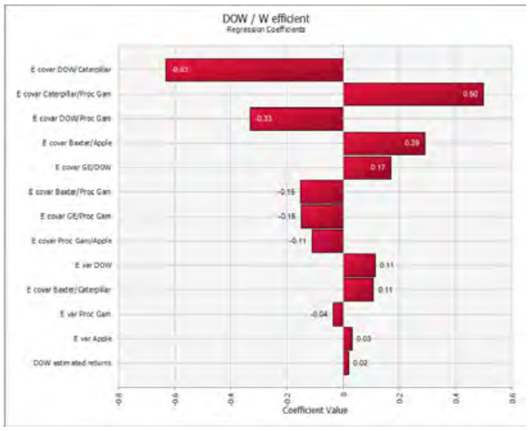
covar Proc Gam/Apple	covar Caterpillar/Pr	covar Caterpillar/Pr	covar Baxter/Apple	covar Baxter/Proc	covar Baxter/Cater
-0.004	-0.013	-0.004	-0.005	-0.006	-0.008
0.001	0.009	0.010	-0.004	0.001	-0.001
-0.017	-0.011	-0.005	-0.015	-0.008	-0.010
-0.003	0.010	0.007	-0.006	-0.001	0.000
-0.007	-0.006	-0.012	-0.009	-0.008	-0.008
0.008	0.008	0.001	0.008	-0.005	0.000
0.289	0.651	0.758	0.253	0.486	0.546
0.256	0.617	0.712	0.205	0.434	0.466
0.441	0.549	0.555	0.530	0.593	0.643
0.303	0.557	0.760	0.292	0.509	0.584
0.335	0.541	0.701	0.436	0.705	0.594
0.116	0.430	0.384	0.193	0.453	0.319
0.095	0.360	0.497	0.021	0.142	0.288
0.799	0.827	0.905	0.846	0.931	0.950
0.427	0.706	0.806	0.383	0.554	0.646
0.757	0.814	0.975	0.785	0.914	0.918
0.624	0.917	0.839	0.643	0.813	0.769
0.127	0.389	0.503	0.088	0.195	0.375
0.263	0.536	0.641	0.202	0.327	0.432
0.031	0.299	0.500	-0.049	0.122	0.261
0.654	0.765	0.816	0.606	0.668	0.726
0.816	0.793	0.895	0.857	0.918	1.000
0.843	0.842	0.881	0.889	1.000	0.918
0.951	0.762	0.701	1.000	0.889	0.857
0.679	0.798	1.000	0.701	0.881	0.895
0.755	1.000	0.798	0.762	0.842	0.793
1.000	0.755	0.679	0.951	0.843	0.816

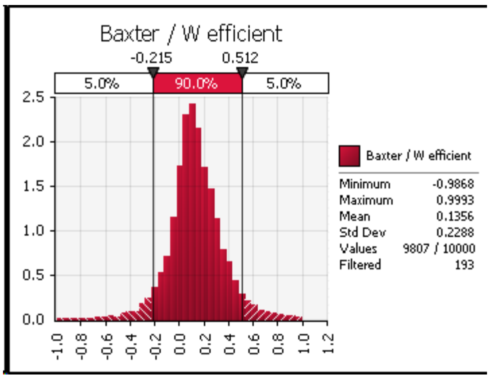
Appendix 2 Weight Distributions for Efficient Portfolio and regression coefficients for each weight

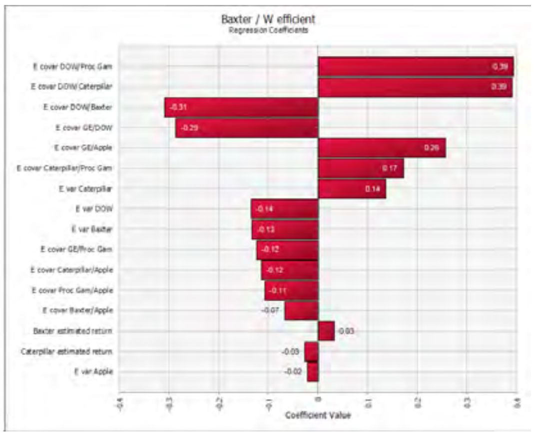


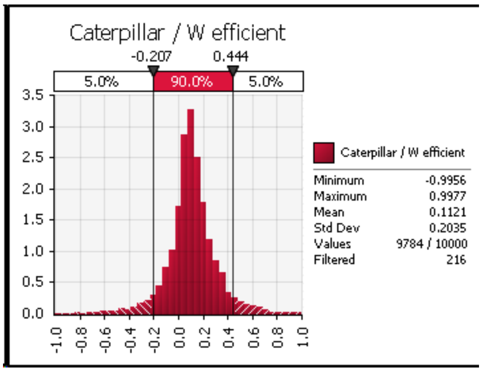


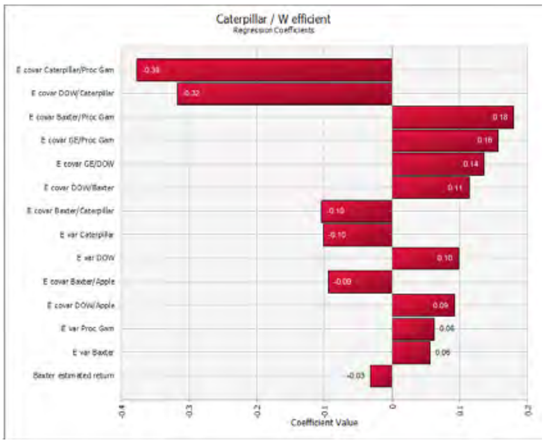


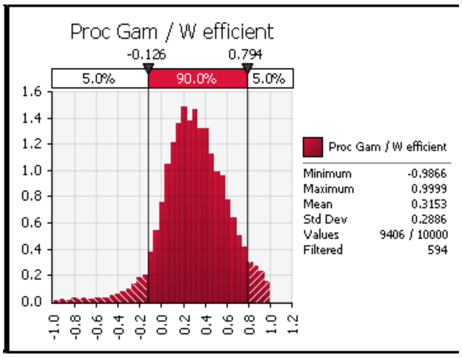


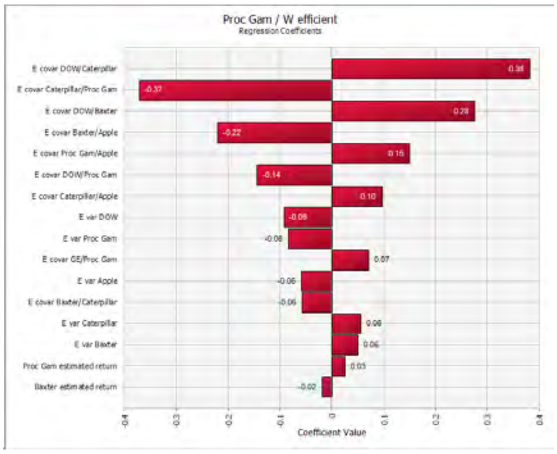


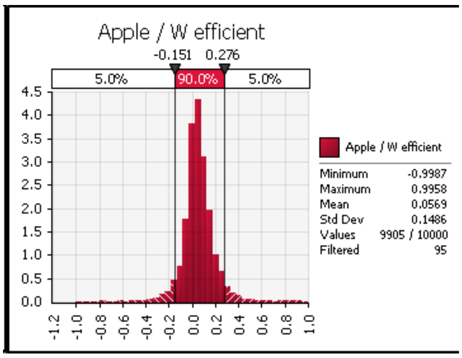


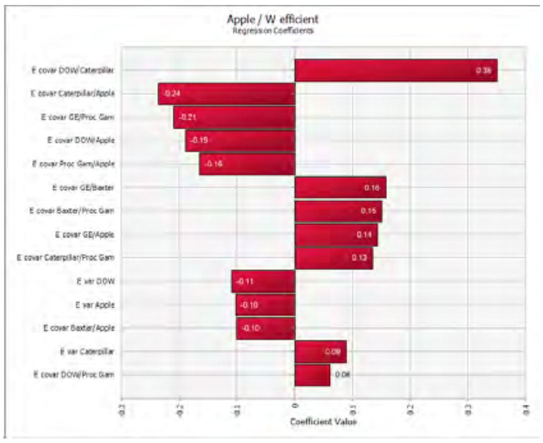

Caterpillar / W efficient
-0.207
0.444
5.0%
90.0%
5.0%
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Caterpillar / W efficient
Minimum -0.9956
Maximum 0.9977
Mean 0.1121
Std Dev 0.2035
Values 9784 / 10000
Filtered 216


Caterpillar / W efficient
Regression Coefficients
E covar Caterpillar/Proc Gam -0.38
E covar DOW/Caterpillar -0.32
E covar Baxter/Proc Gam 0.18
E covar GE/Proc Gam 0.16
E covar GE/DOW 0.14
E covar DOW/Baxter 0.11
E covar Baxter/Caterpillar -0.10
E var Caterpillar -0.10
E var DOW 0.10
E covar Baxter/Apple -0.09
E covar DOW/Apple 0.09
E var Proc Gam 0.06
E var Baxter 0.06
Baxter estimated return -0.03
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
Coefficient Value


Proc Gam / W efficient
-0.126
0.794
5.0%
90.0%
5.0%
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Proc Gam / W efficient
Minimum -0.9866
Maximum 0.9999
Mean 0.3153
Std Dev 0.2886
Values 9406 / 10000
Filtered 594


Proc Gam / W efficient
Regression Coefficients
E covar DOW/Caterpillar 0.38
E covar Caterpillar/Proc Gam -0.37
E covar DOW/Baxter 0.28
E covar Baxter/Apple -0.22
E covar Proc Gam/Apple 0.15
E covar DOW/Proc Gam -0.14
E covar Caterpillar/Apple 0.10
E var DOW -0.09
E var Proc Gam -0.08
E covar GE/Proc Gam 0.07
E var Apple -0.06
E covar Baxter/Caterpillar -0.06
E var Caterpillar 0.06
E var Baxter 0.06
Proc Gam estimated return 0.03
Baxter estimated return -0.02
-0.4
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
Coefficient Value


Apple / W efficient
-0.151
0.276
5.0%
90.0%
5.0%
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
-1.2
-1.0
-0.8
-0.6
-0.4
-0.2
0.0
0.2
0.4
0.6
0.8
1.0
Apple / W efficient
Minimum -0.9987
Maximum 0.9958
Mean 0.0569
Std Dev 0.1486
Values 9905 / 10000
Filtered 95


Apple / W efficient
Regression Coefficients
E covar DOW/Caterpillar 0.35
E covar Caterpillar/Apple -0.24
E covar GE/Proc Gam -0.21
E covar DOW/Apple -0.19
E covar Proc Gam/Apple -0.16
E covar GE/Baxter 0.16
E covar Baxter/Proc Gam 0.15
E covar GE/Apple 0.14
E covar Caterpillar/Proc Gam 0.13
E var DOW -0.11
E var Apple -0.10
E covar Baxter/Apple -0.10
E var Caterpillar 0.09
E covar DOW/Proc Gam 0.06
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
0.4
Coefficient Value

Appendix 3 OLS regressions of portfolio returns on theoretical variables

Table 16 Full sample regression, 10000 observations, equation 1

Dependent Variable: TOTAL_PORTFOLIO_RETURN
Method: Least Squares
Date: 09/01/13 Time: 22:34
Sample: 1 10000
Included observations: 10000

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.002590	0.005259	0.492436	0.6224
COVAR_BAXTER_APPLE	58.70529	52.07807	1.127255	0.2597
COVAR_BAXTER_CATERPILLAR	2.397932	19.24206	0.124619	0.9008
COVAR_BAXTER_PROC_GAM	-52.25624	48.29058	-1.082121	0.2792
COVAR_CATERPILLAR_APPLE	35.49807	28.10062	1.263249	0.2065
COVAR_CATERPILLAR_PROC_G	-15.37868	45.89502	-0.335084	0.7376
COVAR_DOW_APPLE	4.836414	48.53340	0.099651	0.9206
COVAR_DOW_BAXTER	7.103086	49.84154	0.142513	0.8867
COVAR_DOW_CATERPILLAR	-38.29909	34.07450	-1.123981	0.2610
COVAR_DOW_PROC_GAM	57.14868	68.46130	0.834759	0.4039
COVAR_GE_APPLE	3.935485	21.58992	0.182283	0.8554
COVAR_GE_BAXTER	-6.347560	16.25673	-0.390457	0.6962
COVAR_GE_CATTERPILLAR	0.129934	0.843871	0.153974	0.8776
COVAR_GE_DOW	14.76020	36.13293	0.408497	0.6829
COVAR_GE_PROC_GAM	11.56421	18.23863	0.634050	0.5261
COVAR_PROC_GAM_APPLE	-40.35005	50.70263	-0.795818	0.4262
VAR_APPLE	-0.933241	1.392434	-0.670223	0.5027
VAR_BAXTER	0.055878	6.488514	0.008612	0.9931
VAR_CATERPILLAR	-3.927365	6.798708	-0.577663	0.5635
VAR_DOW	-6.193604	6.795678	-0.911403	0.3621
VAR_GE	0.902770	4.706948	0.191795	0.8479
VAR_PROC_GAM	2.978336	5.663653	0.525868	0.5990
APPLE_RETURN	0.075398	0.036780	2.049997	0.0404
BAXTER_RETURN	0.104258	0.059432	1.754225	0.0794
CATERPILLAR_RETURN	0.141770	0.056479	2.510112	0.0121
DOW_RETURNS	0.177337	0.054619	3.246780	0.0012
GE_RETURN	0.075349	0.068878	1.093953	0.2740
PROC_GAM_RETURN	0.492402	0.076247	6.458022	0.0000
E_VAR_APPLE	-0.425157	1.360619	-0.312473	0.7547
E_VAR_BAXTER	1.044870	6.154877	0.169763	0.8652
E_VAR_CATERPILLAR	2.916496	7.142357	0.408338	0.6830
E_VAR_DOW	-3.618580	4.054709	-0.892439	0.3722
E_VAR_GE	6.531472	5.749587	1.135990	0.2560
E_VAR_PROC_GAM	5.389546	5.438950	0.990917	0.3218
APPLE_ESTIMATED_RETURN	0.017203	0.036693	0.468854	0.6392
BAXTER_ESTIMATED_RETURN	0.098672	0.058822	1.677468	0.0935
CATERPILLAR_ESTIMATED_RE	-0.057452	0.056942	-1.008953	0.3130
DOW_ESTIMATED_RETURNS	-0.020360	0.054904	-0.370835	0.7108
GE_ESTIMATED_RETURN	-0.088031	0.069934	-1.258772	0.2081
PROC_GAM_ESTIMATED_RETU R	-0.004166	0.076138	-0.054722	0.9564
E_COVAR_BAXTER_APPLE	-58.73131	52.15539	-1.126083	0.2602
E_COVAR_BAXTER_CATERPILL	1.813892	17.61643	0.102966	0.9180
E_COVAR_BAXTER_PROC_GAM	-58.79795	47.52331	-1.237245	0.2160
E_COVAR_CATERPILLAR_APPL	0.896368	27.29276	0.032843	0.9738
E_COVAR_CATERPILLAR_PROC	20.08264	46.00715	0.436511	0.6625
E_COVAR_DOW_APPLE	52.36029	48.27816	1.084554	0.2781

E_COVAR_DOW_BAXTER	-3.555710	47.38172	-0.075044	0.9402
E_COVAR_DOW_CATERPILLAR	34.68307	31.17734	1.112445	0.2660
E_COVAR_DOW_PROC_GAM	37.90794	69.40236	0.546205	0.5849
E_COVAR_GE_APPLE	-26.75496	21.65884	-1.235291	0.2168
E_COVAR_GE_BAXTER	0.714380	21.40014	0.033382	0.9734
E_COVAR_GE_CATTERPILLAR	-9.013437	8.550691	-1.054118	0.2919
E_COVAR_GE_DOW	-67.88495	31.51295	-2.154192	0.0312
E_COVAR_GE_PROC_GAM	2.049395	19.08284	0.107395	0.9145
E_COVAR_PROC_GAM_APPLE	86.25067	50.05473	1.723127	0.0849

R-squared	0.019163	Mean dependent var	-0.000310
Adjusted R-squared	0.013837	S.D. dependent var	0.093992
S.E. of regression	0.093340	Akaike info criterion	-1.899661
Sum squared resid	86.64349	Schwarz criterion	-1.860005
Log likelihood	9553.307	Hannan-Quinn criter.	-1.886238
F-statistic	3.598107	Durbin-Watson stat	1.994916
Prob(F-statistic)	0.000000		

Table 17 Trimmed sample regression, 9916 observations, equation 1

Dependent Variable:
Efficient portfolio return
Method: Least Squares
Date: 09/08/11 Time: 16:00
Sample: 1 9916
Included observations: 9916

Variable	Coefficient	Std. Error	t-Statistic	Prob.
PROC_GAM_RETURN	0.357249	0.008576	41.65806	0
DOW_RETURN	0.238523	0.006145	38.8188	0
BAXTER_RETURN	0.146801	0.006677	21.98596	0
CATERPILLAR_RETURN	0.109375	0.006368	17.17653	0
GE_RETURN	0.102812	0.007758	13.25235	0
APPLE_RETURN	0.054428	0.004134	13.16553	0
E_COVAR_BAXTER_CATERPILL	6.207749	1.979202	3.136491	0.0017
E_COVAR_GE_APPLE	5.627881	2.44528	2.301529	0.0214
E_COVAR_PROC_GAM_APPLE	-11.70707	5.661023	-2.068012	0.0387
E_VAR_APPLE	-0.314955	0.152894	-2.059952	0.0394
PROC_GAM_ESTIMATED_RETUR	0.0176	0.008547	2.059334	0.0395
E_VAR_PROC_GAM	-1.085989	0.610986	-1.777438	0.0755
E_COVAR_GE_PROC_GAM	-3.521847	2.149421	-1.63851	0.1013
VAR_BAXTER	1.019526	0.728442	1.399599	0.1617
COVAR_PROC_GAM_APPLE	7.693223	5.698685	1.35	0.177
COVAR_CATERPILLAR_APPLE	4.248734	3.153283	1.3474	0.1779
E_COVAR_GE_BAXTER	-3.168406	2.413617	-1.312721	0.1893
E_COVAR_GE_CATTERPILLAR	-1.241484	0.95979	-1.293496	0.1959
COVAR_BAXTER_APPLE	-7.207329	5.85825	-1.230287	0.2186
E_VAR_GE	0.787213	0.645569	1.219409	0.2227
COVAR_GE_PROC_GAM	2.443018	2.044492	1.194926	0.2321
E_COVAR_BAXTER_APPLE	6.54052	5.900498	1.108469	0.2677
CATERPILLAR_ESTIMATED_RE	-0.006882	0.006388	-1.077321	0.2814
E_COVAR_DOW_BAXTER	-5.591955	5.353615	-1.044519	0.2963
COVAR_GE_APPLE	-2.484279	2.421712	-1.025836	0.305

E_COVAR_BAXTER_PROC_GAM	-5.464756	5.403313	-1.011371	0.3119
VAR_CATERPILLAR	-0.715198	0.764346	-0.935698	0.3495
COVAR_DOW_APPLE	-4.90251	5.454699	-0.898768	0.3688
E_COVAR_DOW_APPLE	4.682888	5.438915	0.860997	0.3893
E_COVAR_CATERPILLAR_PROC	4.152804	5.199371	0.798713	0.4245
COVAR_GE_BAXTER	-1.400979	1.822515	-0.768706	0.4421
E_COVAR_DOW_PROC_GAM	5.959536	7.836453	0.760489	0.447
COVAR_GE_CATTERPILLAR	-0.069544	0.09449	-0.735989	0.4618
COVAR_BAXTER_PROC_GAM	-3.921986	5.427757	-0.72258	0.47
COVAR_CATERPILLAR_PROC_G	3.576798	5.154891	0.693865	0.4878
C	0.000395	0.000594	0.664837	0.5062
APPLE_ESTIMATED_RETURN	-0.002725	0.004122	-0.661129	0.5085
E_COVAR_GE_DOW	-2.278465	3.552526	-0.641365	0.5213
COVAR_DOW_CATERPILLAR	2.195891	3.835123	0.572574	0.5669
E_VAR_DOW	-0.243546	0.454334	-0.53605	0.5919
VAR_GE	0.285736	0.535256	0.53383	0.5935
DOW_ESTIMATED_RETURNS	0.003139	0.00617	0.508713	0.611
BAXTER_ESTIMATED_RETURN	0.002914	0.006612	0.440725	0.6594
VAR_DOW	-0.2995	0.768075	-0.389936	0.6966
COVAR_GE_DOW	-1.513031	4.06102	-0.372574	0.7095
E_VAR_CATERPILLAR	0.281619	0.803391	0.350538	0.7259
VAR_PROC_GAM	0.182839	0.63536	0.287773	0.7735
COVAR_DOW_BAXTER	-1.469983	5.597313	-0.262623	0.7928
E_COVAR_CATERPILLAR_APPL	-0.77039	3.080993	-0.250046	0.8026
VAR_APPLE	0.032624	0.15674	0.208139	0.8351
E_VAR_BAXTER	-0.10656	0.699053	-0.152435	0.8788
E_COVAR_DOW_CATERPILLAR	-0.514189	3.510737	-0.146462	0.8836
COVAR_DOW_PROC_GAM	0.623179	7.688407	0.081054	0.9354
GE_ESTIMATED_RETURN	-0.000532	0.007845	-0.067868	0.9459
COVAR_BAXTER_CATERPILLAR	0.135384	2.156764	0.062772	0.9499

R-squared	0.515259	Mean dependent var	0.00043
Adjusted R-squared	0.512604	S.D. dependent var	0.014956
S.E. of regression	0.010442	Akaike info criterion	-6.280491
Sum squared resid	1.075132	Schwarz criterion	-6.240545
Log likelihood	31193.68	Hannan-Quinn criter.	-6.266964
F-statistic	194.1076	Durbin-Watson stat	0.868992
Prob(F-statistic)	0		

Table 18 Regression of Estimation Errors, trimmed, 9916 observations, equation 2

Dependent Variable: TOTAL_PORTFOLIO_RETURN
Method: Least Squares
Date: 09/08/11 Time: 17:38
Sample: 1 9900
Included observations: 9900

Variable	Coefficient	Std. Error	t-Statistic	Prob.

DIFF__RET__APPLE	0.03071	0.003522	8.720609	0
DIFF__RET__BAXTER	0.07864	0.005615	14.0046	0
DIFF__RET__CATERPILLAR	0.054173	0.005445	9.94879	0
DIFF__RET__DOW	0.118629	0.005281	22.46209	0
DIFF__RET__GE	0.050064	0.006674	7.501592	0
DIFF__RET__PROC_GAM	0.166313	0.00722	23.03655	0
C	0.000369	0.000127	2.91103	0.0036
DIFF__COVAR_PROC_GAM_APP	-9.667363	4.860756	-1.98886	0.0467
DIFF__COVAR_GE_APPLE	3.927872	2.068061	1.899302	0.0576
DIFF__VAR_APPLE	-0.193928	0.132044	-1.468669	0.142
DIFF__COVAR_BAXTER_CATER	1.917126	1.722331	1.113099	0.2657
DIFF__COVAR_DOW_BAXTER	-5.073281	4.638727	-1.09368	0.2741
DIFF__VAR_PROC_GAM	-0.580451	0.534146	-1.086689	0.2772
DIFF__COVAR_GE_DOW	-2.899512	3.131475	-0.925925	0.3545
DIFF__COVAR_GE_PROC_GAM	-1.523219	1.762256	-0.864357	0.3874
DIFF__COVAR_BAXTER_PROC_	-3.681753	4.621142	-0.796719	0.4256
DIFF__VAR_BAXTER	-0.456898	0.611258	-0.747471	0.4548
DIFF__COVAR_GE_CATTERPIL	0.076337	0.106175	0.718967	0.4722
DIFF__COVAR_GE_BAXTER	1.254746	1.748389	0.717659	0.473
DIFF__COVAR_CATERPILL01	-1.846247	2.646805	-0.697538	0.4855
DIFF__COVAR_CATERPILLAR_	3.050912	4.378642	0.696771	0.486
DIFF__COVAR_DOW_CATERPIL	1.912034	2.975073	0.642685	0.5204
DIFF__COVAR_BAXTER_APPLE	2.665835	5.030384	0.529947	0.5962
DIFF__COVAR_DOW_PROC_GAM	2.97858	6.576341	0.452924	0.6506
DIFF__VAR_GE	-0.075685	0.256953	-0.29455	0.7683
DIFF__COVAR_DOW_APPLE	1.053758	4.573691	0.230396	0.8178
DIFF__VAR_CATERPILLAR	-0.017722	0.656674	-0.026987	0.9785
DIFF__VAR_DOW	0.007707	0.451423	0.017073	0.9864

R-squared	0.262991	Mean dependent var	0.000372
Adjusted R-squared	0.260975	S.D. dependent var	0.014668
S.E. of regression	0.012609	Akaike info criterion	-5.905915
Sum squared resid	1.569632	Schwarz criterion	-5.885551
Log likelihood	29262.28	Hannan-Quinn criter.	-5.899019
F-statistic	130.4696	Durbin-Watson stat	0.501283
Prob(F-statistic)	0		

Table 19 Regression of average variables, 9824 observations, equation 3

Dependent Variable: TOTAL_RETURN_MINRISK
Method: Least Squares
Date: 09/08/11 Time: 16:23
Sample: 1 9824
Included observations: 9824

Variable	Coefficient	Std. Error	t-Statistic	Prob.
RETURN_AVG_	0.877633	0.009375	93.61478	0
E_COVAR_AVG_	-2.577602	1.600904	-1.610091	0.1074
E_VAR_AVG_	0.614897	0.46845	1.31262	0.1893
C	-0.0003	0.000245	-1.222988	0.2214
E_RET_AVG_	0.006184	0.009373	0.659808	0.5094
COVAR_AVG_	-0.356497	0.941189	-0.378773	0.7049
VAR_AVG_	0.027013	0.402316	0.067143	0.9465

R-squared	0.471882	Mean dependent var	0.000302
Adjusted R-squared	0.47156	S.D. dependent var	0.014337
S.E. of regression	0.010422	Akaike info criterion	-6.289114
Sum squared resid	1.066271	Schwarz criterion	-6.283989
Log likelihood	30899.13	Hannan-Quinn criter.	-6.287378
F-statistic	1461.943	Durbin-Watson stat	0.766212
Prob(F-statistic)	0		

Table 20 Regression of average variables, 9916 observations, equation 3

Dependent Variable: TOTAL_PORTFOLIO_RETURN
Method: Least Squares
Date: 09/08/11 Time: 16:20
Sample: 1 9916
Included observations: 9916

Variable	Coefficient	Std. Error	t-Statistic	Prob.
RETURN_AVG_	0.890921	0.009952	89.52144	0
E_COVAR_AVG_	-1.185832	1.688939	-0.702117	0.4826
COVAR_AVG_	-0.65493	1.002721	-0.653153	0.5137
VAR_AVG_	0.246073	0.427937	0.575022	0.5653
E_RET_AVG_	0.002532	0.009956	0.25429	0.7993
E_VAR_AVG_	-0.104193	0.494735	-0.210604	0.8332
C	-9.18E-06	0.00026	-0.035259	0.9719

R-squared	0.447346	Mean dependent var	0.00043
Adjusted R-squared	0.447011	S.D. dependent var	0.014956
S.E. of regression	0.011122	Akaike info criterion	-6.159056
Sum squared resid	1.225759	Schwarz criterion	-6.153972

Log likelihood	30543.6	Hannan-Quinn criter.	-6.157334
F-statistic	1336.806	Durbin-Watson stat	0.777641
Prob(F-statistic)	0		

Table 21 Total return of minimized risk portfolio and its independent variables

Dependent Variable:
TOTAL_RETURN_MINRISK
Method: Least Squares
Date: 09/08/11 Time: 16:04
Sample: 1 9916
Included observations: 9824

Variable	Coefficient	Std. Error	t-Statistic	Prob.
E_COVAR_DOW_PROC_GAM	-32.88687	7.326981	-4.488461	0
APPLE_RETURN	0.043292	0.003857	11.22529	0
BAXTER_RETURN	0.127286	0.006238	20.40388	0
CATERPILLAR_RETURN	0.135306	0.00594	22.77869	0
DOW_RETURNS	0.232297	0.005732	40.523	0
GE_RETURN	0.107484	0.007255	14.81615	0
PROC_GAM_RETURN	0.342485	0.008001	42.80651	0
E_COVAR_GE_CATTERPILLAR	-2.765399	0.892376	-3.098917	0.0019
E_VAR_GE	1.705417	0.600292	2.84098	0.0045
E_COVAR_CATERPILLAR_PROC	12.78349	4.871972	2.623884	0.0087
E_VAR_PROC_GAM	-1.235779	0.569729	-2.169066	0.0301
E_COVAR_DOW_BAXTER	10.17764	5.012867	2.030302	0.0424
E_COVAR_GE_PROC_GAM	3.915374	2.006706	1.951145	0.0511
PROC_GAM_ESTIMATED_RETUR	0.015059	0.007972	1.888885	0.0589
DOW_ESTIMATED_RETURNS	0.010095	0.005762	1.751951	0.0798
APPLE_ESTIMATED_RETURN	-0.005549	0.003839	-1.445457	0.1484
E_COVAR_CATERPILLAR_APPL	-3.998152	2.885715	-1.385498	0.1659
E_COVAR_GE_DOW	4.377627	3.316696	1.319876	0.1869
COVAR_GE_BAXTER	-2.028269	1.694396	-1.197045	0.2313
E_COVAR_DOW_CATERPILLAR	3.927119	3.281499	1.196746	0.2314
E_COVAR_BAXTER_CATERPILL	-2.043441	1.847667	-1.105957	0.2688
COVAR_CATERPILLAR_APPLE	3.054203	2.941265	1.038398	0.2991
E_COVAR_GE_BAXTER	-2.253102	2.25193	-1.00052	0.3171
GE_ESTIMATED_RETURN	-0.007154	0.007327	-0.976409	0.3289
E_COVAR_BAXTER_PROC_GAM	-4.678606	5.067455	-0.923266	0.3559
COVAR_GE_PROC_GAM	1.560902	1.902874	0.820287	0.4121
VAR_CATERPILLAR	-0.578149	0.711214	-0.812905	0.4163
COVAR_GE_APPLE	-1.829103	2.257807	-0.810124	0.4179
E_VAR_DOW	-0.334746	0.422335	-0.792608	0.428
COVAR_GE_DOW	-2.783281	3.79141	-0.734102	0.4629
E_COVAR_PROC_GAM_APPLE	-3.327394	5.287969	-0.629239	0.5292
C	0.000342	0.000554	0.616538	0.5376
E_VAR_BAXTER	0.394541	0.651354	0.605725	0.5447
E_VAR_APPLE	0.086335	0.145188	0.594643	0.5521

COVAR_BAXTER_PROC_GAM	-2.953604	5.057672	-0.583985	0.5592
E_COVAR_DOW_APPLE	2.746199	5.0913	0.539391	0.5896
VAR_GE	0.255567	0.497744	0.513451	0.6076
VAR_DOW	0.325305	0.714473	0.455307	0.6489
E_COVAR_BAXTER_APPLE	-2.475241	5.524412	-0.448055	0.6541
VAR_APPLE	0.064367	0.146	0.440869	0.6593
BAXTER_ESTIMATED_RETURN	0.002233	0.006166	0.36221	0.7172
COVAR_DOW_PROC_GAM	2.556168	7.170228	0.356498	0.7215
COVAR_BAXTER_APPLE	1.639744	5.457653	0.300449	0.7638
COVAR_DOW_APPLE	-1.36814	5.085415	-0.269032	0.7879
COVAR_DOW_BAXTER	1.383014	5.228544	0.264512	0.7914
E_COVAR_GE_APPLE	0.591984	2.285173	0.259054	0.7956
COVAR_CATERPILLAR_PROC_G	1.213116	4.80264	0.252594	0.8006
VAR_BAXTER	-0.156693	0.678306	-0.231007	0.8173
COVAR_DOW_CATERPILLAR	0.352478	3.57452	0.098609	0.9215
COVAR_PROC_GAM_APPLE	0.231191	5.317283	0.043479	0.9653
E_VAR_CATERPILLAR	-0.018916	0.748502	-0.025272	0.9798
VAR_PROC_GAM	-0.014811	0.591255	-0.02505	0.98
CATERPILLAR_ESTIMATED_RE	9.39E-05	0.005965	0.015736	0.9874
COVAR_BAXTER_CATERPILLAR	-0.026093	2.00535	-0.013012	0.9896
COVAR_GE_CATTERPILLAR	0.000209	0.087811	0.00238	0.9981

R-squared	0.545072	Mean dependent var	0.000302
Adjusted R-squared	0.542558	S.D. dependent var	0.014337
S.E. of regression	0.009696	Akaike info criterion	-6.428523
Sum squared resid	0.9185	Schwarz criterion	-6.388255
Log likelihood	31631.91	Hannan-Quinn criter.	-6.414881
F-statistic	216.7545	Durbin-Watson stat	1.806937
Prob(F-statistic)	0		

Appendix 4 Unit root tests for selected stocks

Table 22 Unit root test for daily values

Null Hypothesis: APPLE has a unit root
Exogenous: Constant
Lag Length: 0 (Automatic - based on SIC, maxlag=34)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		1.649123	0.9996
Test critical values:	1% level	-3.431134	
	5% level	-2.861771	
	10% level	-2.566935	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(APPLE)
Method: Least Squares
Date: 09/01/13 Time: 13:23
Sample (adjusted): 9/10/1984 8/19/2011
Included observations: 6798 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
APPLE(-1)	0.000679	0.000412	1.649123	0.0992
C	0.004504	0.039241	0.114791	0.9086

R-squared	0.000400	Mean dependent var	0.048507
Adjusted R-squared	0.000253	S.D. dependent var	2.372621
S.E. of regression	2.372321	Akaike info criterion	4.565909
Sum squared resid	38247.25	Schwarz criterion	4.567917
Log likelihood	-15517.52	Hannan-Quinn criter.	4.566602
F-statistic	2.719608	Durbin-Watson stat	1.942492
Prob(F-statistic)	0.099169		

Null Hypothesis: BAXTER has a unit root
Exogenous: Constant
Lag Length: 2 (Automatic - based on SIC, maxlag=34)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-0.788162	0.8219
Test critical values:	1% level	-3.431134	
	5% level	-2.861771	
	10% level	-2.566935	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(BAXTER)
Method: Least Squares
Date: 09/01/13 Time: 13:27
Sample (adjusted): 9/12/1984 8/19/2011
Included observations: 6796 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
BAXTER(-1)	-0.000274	0.000347	-0.788162	0.4306
D(BAXTER(-1))	-0.015770	0.012117	-1.301526	0.1931
D(BAXTER(-2))	-0.056006	0.012128	-4.617892	0.0000
C	0.013970	0.010123	1.380065	0.1676

R-squared	0.003468	Mean dependent var	0.007069
Adjusted R-squared	0.003028	S.D. dependent var	0.502507
S.E. of regression	0.501745	Akaike info criterion	1.459139
Sum squared resid	1709.874	Schwarz criterion	1.463156
Log likelihood	-4954.156	Hannan-Quinn criter.	1.460526
F-statistic	7.879801	Durbin-Watson stat	2.002457
Prob(F-statistic)	0.000030		

Null Hypothesis: CATERPILLAR has a unit root
Exogenous: Constant
Lag Length: 0 (Automatic - based on SIC, maxlag=34)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-0.126898	0.9448
Test critical values:	1% level	-3.431134	
	5% level	-2.861771	
	10% level	-2.566935	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(CATERPILLAR)
Method: Least Squares
Date: 09/01/13 Time: 13:28
Sample (adjusted): 9/10/1984 8/19/2011
Included observations: 6798 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
CATERPILLAR(-1)	-4.29E-05	0.000338	-0.126898	0.8990
C	0.012372	0.011419	1.083488	0.2786

R-squared	0.000002	Mean dependent var	0.011361
Adjusted R-squared	-0.000145	S.D. dependent var	0.674265
S.E. of regression	0.674314	Akaike info criterion	2.050053
Sum squared resid	3090.139	Schwarz criterion	2.052061
Log likelihood	-6966.130	Hannan-Quinn criter.	2.050746
F-statistic	0.016103	Durbin-Watson stat	2.032502
Prob(F-statistic)	0.899025		

Null Hypothesis: DOW has a unit root
Exogenous: Constant
Lag Length: 0 (Automatic - based on SIC, maxlag=34)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-1.858565	0.3524
Test critical values:	1% level	-3.431134	
	5% level	-2.861771	
	10% level	-2.566935	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(DOW)
Method: Least Squares
Date: 09/01/13 Time: 13:28
Sample (adjusted): 9/10/1984 8/19/2011
Included observations: 6798 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
DOW(-1)	-0.000868	0.000467	-1.858565	0.0631
C	0.019633	0.010045	1.954630	0.0507

R-squared	0.000508	Mean dependent var	0.003505
Adjusted R-squared	0.000361	S.D. dependent var	0.417177
S.E. of regression	0.417102	Akaike info criterion	1.089320
Sum squared resid	1182.325	Schwarz criterion	1.091328
Log likelihood	-3700.599	Hannan-Quinn criter.	1.090013
F-statistic	3.454264	Durbin-Watson stat	2.065722
Prob(F-statistic)	0.063132		

Null Hypothesis: GE has a unit root
Exogenous: Constant
Lag Length: 2 (Automatic - based on SIC, maxlag=34)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-1.461415	0.5533
Test critical values:	1% level	-3.431134	
	5% level	-2.861771	
	10% level	-2.566935	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(GE)
Method: Least Squares
Date: 09/01/13 Time: 13:29
Sample (adjusted): 9/12/1984 8/19/2011
Included observations: 6796 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
GE(-1)	-0.000543	0.000371	-1.461415	0.1439
D(GE(-1))	-0.035205	0.012117	-2.905420	0.0037
D(GE(-2))	-0.051795	0.012123	-4.272628	0.0000
C	0.010313	0.007008	1.471515	0.1412

R-squared	0.004144	Mean dependent var	0.002059
Adjusted R-squared	0.003704	S.D. dependent var	0.356818
S.E. of regression	0.356156	Akaike info criterion	0.773693
Sum squared resid	861.5462	Schwarz criterion	0.777710
Log likelihood	-2625.010	Hannan-Quinn criter.	0.775080
F-statistic	9.420930	Durbin-Watson stat	1.999426
Prob(F-statistic)	0.000003		

Null Hypothesis: PROC_GAM has a unit root
Exogenous: Constant
Lag Length: 4 (Automatic - based on SIC, maxlag=34)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-0.363756	0.9129
Test critical values:	1% level	-3.431134	
	5% level	-2.861771	
	10% level	-2.566935	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(PROC_GAM)
Method: Least Squares
Date: 09/01/13 Time: 13:29
Sample (adjusted): 9/14/1984 8/19/2011
Included observations: 6794 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
PROC_GAM(-1)	-9.83E-05	0.000270	-0.363756	0.7161
D(PROC_GAM(-1))	-0.068542	0.012122	-5.654143	0.0000
D(PROC_GAM(-2))	-0.072048	0.012149	-5.930151	0.0000
D(PROC_GAM(-3))	0.027490	0.012149	2.262725	0.0237
D(PROC_GAM(-4))	-0.052495	0.012125	-4.329408	0.0000
C	0.012781	0.009096	1.405109	0.1600

R-squared	0.013272	Mean dependent var	0.008715
Adjusted R-squared	0.012545	S.D. dependent var	0.463415
S.E. of regression	0.460499	Akaike info criterion	1.287870
Sum squared resid	1439.458	Schwarz criterion	1.293897
Log likelihood	-4368.895	Hannan-Quinn criter.	1.289950
F-statistic	18.25973	Durbin-Watson stat	2.001460
Prob(F-statistic)	0.000000		

Table 23 Apple stock as an example of unit root test for logarithmic daily returns

Null Hypothesis: APPLEDF has a unit root
Exogenous: Constant
Lag Length: 0 (Automatic - based on SIC, maxlag=34)

		t-Statistic	Prob.*
Augmented Dickey-Fuller test statistic		-78.38917	0.0001
Test critical values:	1% level	-3.431134	
	5% level	-2.861771	
	10% level	-2.566935	

*MacKinnon (1996) one-sided p-values.

Augmented Dickey-Fuller Test Equation
Dependent Variable: D(APPLEDF)
Method: Least Squares
Date: 09/01/13 Time: 13:44
Sample (adjusted): 9/11/1984 8/19/2011
Included observations: 6797 after adjustments

Variable	Coefficient	Std. Error	t-Statistic	Prob.
APPLEDF(-1)	-0.949758	0.012116	-78.38917	0.0000
C	0.000366	0.000397	0.922319	0.3564

R-squared	0.474878	Mean dependent var	-5.73E-08
Adjusted R-squared	0.474801	S.D. dependent var	0.045184
S.E. of regression	0.032745	Akaike info criterion	-3.999829
Sum squared resid	7.285906	Schwarz criterion	-3.997821
Log likelihood	13595.42	Hannan-Quinn criter.	-3.999136
F-statistic	6144.862	Durbin-Watson stat	1.996924
Prob(F-statistic)	0.000000		

Appendix 5: Autocorrelations test

Table 24 Apple daily values autocorrelation

Date: 09/01/13 Time: 13:52
Sample: 9/07/1984 8/19/2011
Included observations: 6799

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
|*******	|*******	1	0.998	0.998	6776.5	0.000
|*******	| |	2	0.996	-0.023	13528.	0.000
|*******	| |	3	0.994	-0.028	20252.	0.000
|*******	| |	4	0.992	0.005	26949.	0.000
|*******	| |	5	0.990	-0.008	33620.	0.000
|*******	| |	6	0.988	0.011	40264.	0.000
|*******	| |	7	0.986	0.031	46884.	0.000
|*******	| |	8	0.984	-0.004	53479.	0.000
|*******	| |	9	0.982	0.027	60051.	0.000
|*******	| |	10	0.981	0.006	66600.	0.000
|*******	| |	11	0.979	-0.033	73126.	0.000
|*******	| |	12	0.977	-0.037	79625.	0.000
|*******	| |	13	0.975	-0.017	86098.	0.000
|*******	| |	14	0.972	-0.019	92542.	0.000
|*******	| |	15	0.970	-0.005	98959.	0.000
|*******	| |	16	0.968	0.014	105348	0.000
|*******	| |	17	0.966	-0.019	111709	0.000
|*******	| |	18	0.964	-0.014	118041	0.000
|*******	| |	19	0.961	-0.008	124344	0.000
|*******	| |	20	0.959	0.019	130620	0.000
|*******	| |	21	0.957	-0.001	136868	0.000
|*******	| |	22	0.955	0.013	143089	0.000
|*******	| |	23	0.953	-0.003	149283	0.000
|*******	| |	24	0.951	0.026	155452	0.000
|*******	| |	25	0.949	0.012	161596	0.000
|*******	| |	26	0.947	0.017	167716	0.000
|*******	| |	27	0.945	-0.003	173812	0.000
|*******	| |	28	0.943	-0.003	179884	0.000
|*******	| |	29	0.941	0.016	185932	0.000
|*******	| |	30	0.939	-0.005	191957	0.000
|*******	| |	31	0.937	0.004	197959	0.000
|*******	| |	32	0.935	-0.005	203937	0.000
|*******	| |	33	0.934	0.019	209892	0.000
|*******	| |	34	0.932	0.001	215826	0.000
|*******	| |	35	0.930	0.017	221738	0.000
|*******	| |	36	0.928	-0.006	227629	0.000

Table 25 Apple daily logarithmic values autocorrelation

Date: 09/01/13 Time: 13:53
Sample: 9/10/1984 8/19/2011
Included observations: 6798

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
| |	| |	1	0.050	0.050	17.167	0.000
| |	| |	2	-0.025	-0.028	21.574	0.000
| |	| |	3	-0.010	-0.008	22.304	0.000

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
| |	| |	4	0.012	0.012	23.265	0.000
| |	| |	5	0.023	0.021	26.789	0.000
| |	| |	6	0.007	0.006	27.142	0.000
| |	| |	7	0.010	0.011	27.865	0.000
| |	| |	8	-0.014	-0.014	29.164	0.000
| |	| |	9	-0.015	-0.014	30.775	0.000
| |	| |	10	-0.001	-0.000	30.779	0.001
| |	| |	11	-0.017	-0.019	32.863	0.001
| |	| |	12	0.030	0.032	39.125	0.000
| |	| |	13	0.001	-0.003	39.128	0.000
| |	| |	14	0.019	0.022	41.709	0.000
| |	| |	15	0.002	0.001	41.727	0.000
| |	| |	16	0.001	0.002	41.737	0.000
| |	| |	17	0.010	0.008	42.376	0.001
| |	| |	18	-0.019	-0.020	44.740	0.000
| |	| |	19	0.016	0.016	46.410	0.000
| |	| |	20	-0.015	-0.018	47.986	0.000
| |	| |	21	-0.012	-0.009	48.929	0.001
| |	| |	22	-0.002	-0.001	48.949	0.001
| |	| |	23	-0.008	-0.006	49.351	0.001
| |	| |	24	0.015	0.014	50.866	0.001
| |	| |	25	-0.008	-0.007	51.282	0.001
| |	| |	26	0.023	0.024	54.939	0.001
| |	| |	27	-0.005	-0.007	55.133	0.001
| |	| |	28	0.015	0.018	56.771	0.001
| |	| |	29	0.025	0.021	60.962	0.000
| |	| |	30	-0.013	-0.013	62.070	0.001
| |	| |	31	0.003	0.003	62.134	0.001
| |	| |	32	0.003	0.004	62.192	0.001
| |	| |	33	0.004	0.002	62.313	0.002
| |	| |	34	0.016	0.016	64.146	0.001
| |	| |	35	0.007	0.008	64.500	0.002
| |	| |	36	-0.004	-0.006	64.616	0.002

Table 26 Baxter daily values autocorrelation

Date: 09/01/13 Time: 13:54
Sample: 9/07/1984 8/19/2011
Included observations: 6799

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
|*******	|*******	1	0.999	0.999	6792.6	0.000
|*******	| |	2	0.999	0.004	13577.	0.000
|*******	| |	3	0.998	0.015	20353.	0.000

Table 27 Baxter daily logarithmic values autocorrelation

Date: 09/01/13 Time: 13:55
Sample: 9/10/1984 8/19/2011
Included observations: 6798

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
| |	| |	1	0.001	0.001	0.0034	0.954
| |	| |	2	-0.063	-0.063	27.380	0.000
| |	| |	3	-0.040	-0.041	38.518	0.000

Appendix 6 Lagrange multiplier solution to Markowitz model

We face the problem of minimizing

$$\sigma^2 = \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

Subject to

$$\sum_{i=1}^{n} w_i r_i = r_p$$

$$\sum_{i=1}^{n} w_i = 1$$

Where w_i is weight of i-th asset

σ_{ij} is a covariance between i-th and j-th asset

r_i is a return of i-th asset

r_p is a desired portfolio return

So we form the Lagrangian

$$L(w,\lambda,\gamma) = \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j\, \sigma_{ij} + \lambda\left(r_p - \sum_{i=1}^{n} w_i r_i\right) + \gamma\left(1 - \sum_{i=1}^{n} w_i\right)$$

Melicharčík, Olšarová and Úradníček (2005) proved that matrix σ_{ij} is positive-definite, thus $L(w,\lambda,\gamma)$ reaches its minimum when its first partial derivatives equal to 0.

Thus

$$\frac{\partial L}{\partial w_i} = \sum_{i,j=1}^{n} w_j\, \sigma_{ij} - \lambda r_i - \gamma = 0$$

Lets define w as a column vector of weights w_i, r as a column vector of expected returns r_i, V as variance-covariance matrix of σ_{ij} and I as a column vector of ones.

Partial derivatives of L with respect to w, r, λ gives us a system of n+2 linear equations with n+2 variables.

$$\frac{\partial L}{\partial w} = Vw - \lambda r - \gamma I = 0$$

$$\frac{\partial L}{\partial \lambda} = r_p - w^T = 0$$

$$\frac{\partial L}{\partial \gamma} = 1 - w^T I = 0$$

Let w_p be a column vector of solutions to previous system of equation, then we get

$$w_p = \lambda V^{-1} r + \gamma V^{-1} I$$

Thus

$$r_p = \lambda r^T V^{-1} + \gamma I^T V^{-1} r$$

$$1 = \lambda r^T V^{-1} I + \gamma I^T V^{-1} I$$

Define

$$A = I^T V^{-1} r = r^T V^{-1} I$$

$$B = r^T V^{-1} r > 0$$

$$C = I^T V^{-1} I > 0$$

$$D = BC - A^2$$

Gives a simplified system of equations

$$r_p = \lambda B + \gamma A$$

$$1 = \lambda A + \gamma C$$

System of equations has just one solution when determinant of $\mathrm{D} \neq 0$. Since vector r doesn't have all variables equal$\Longrightarrow$

$$Ar - BI \neq 0$$

Since matrix $V^{-1}$is positive-definite we can write

$$(Ar - BI)^T V^{-1} (Ar - BI) > 0$$

$$\Longrightarrow A^2 B - BA^2 - A^2 B + B^2 C > 0$$

$$B(BC - A^2) > 0$$

$$BC - A^2 > 0$$

Directly expressing λ and γ we get just one solution

$$\lambda = \frac{1}{D}(r_p C - A)$$

$$\gamma = \frac{1}{D}(B - Ar_p)$$

Thus we get a solution to Markowitz problem in form of

$$w_p = g + hr_p$$

$$g = \frac{1}{D}[B(V^{-1}I - A(V^{-1}r]$$

$$h = \frac{1}{D}[C(V^{-1}r - A(V^{-1}I]$$

For global minimum variance portfolio we face similar problem that forms Lagrangian

$$L(w, \gamma) = \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \, \sigma_{ij} + \gamma\left(1 - \sum_{i=1}^{n} w_i\right)$$

We minimize this equation by taking partial derivatives with respect to w_i and equalize to 0

$$\frac{\partial L}{\partial w_i} = \sum_{i,j=1}^{n} w_j \, \sigma_{ij} - \gamma = 0$$

We define again all the variables as column vectors

$$\frac{\partial L}{\partial w} = Vw - \gamma I = 0$$

$$\frac{\partial L}{\partial \gamma} = 1 - w^T I = 0$$

$$w_p = \gamma V^{-1} I$$

$$1 = \gamma I^T V^{-1} I$$

$$C = I^T V^{-1} I > 0$$

Thus $1 = C\gamma$

So at the end we get the formula for weights of Global Minimum Variance portfolio

$$w_p = \frac{1}{C} V^{-1} I$$

Appendix 7: Garch test results

Dependent Variable: APPLE
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 09/01/13 Time: 16:25
Sample: 9/10/1984 8/19/2011
Included observations: 6798
Convergence achieved after 36 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000922	0.000189	4.875241	0.0000
		Variance Equation		
C	0.000311	3.38E-06	91.79840	0.0000
RESID(-1)^2	0.238745	0.003735	63.92162	0.0000
GARCH(-1)	0.530812	0.004869	109.0170	0.0000

R-squared	-0.000270	Mean dependent var	0.000384
Adjusted R-squared	-0.000270	S.D. dependent var	0.032782
S.E. of regression	0.032787	Akaike info criterion	-4.089895
Sum squared resid	7.306541	Schwarz criterion	-4.085879
Log likelihood	13905.55	Hannan-Quinn criter.	-4.088509
Durbin-Watson stat	1.898943		

Dependent Variable: BAXTER
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 09/01/13 Time: 16:30
Sample: 9/10/1984 8/19/2011
Included observations: 6798
Convergence achieved after 15 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(1) + C(2)*RESID(-1)^2 + C(3)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
		Variance Equation		
C	6.34E-05	2.68E-06	23.62563	0.0000
RESID(-1)^2	0.240958	0.005915	40.73794	0.0000
GARCH(-1)	0.608333	0.011733	51.84631	0.0000

R-squared	-0.000413	Mean dependent var	0.000385
Adjusted R-squared	-0.000265	S.D. dependent var	0.018976
S.E. of regression	0.018978	Akaike info criterion	-5.220709
Sum squared resid	2.448452	Schwarz criterion	-5.217698
Log likelihood	17748.19	Hannan-Quinn criter.	-5.219670
Durbin-Watson stat	1.997142		

Dependent Variable: CATERPILLAR
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 09/01/13 Time: 16:36
Sample: 9/10/1984 8/19/2011
Included observations: 6798
Convergence achieved after 19 iterations

Presample variance: backcast (parameter = 0.7)
GARCH = C(1) + C(2)*RESID(-1)^2 + C(3)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
		Variance Equation		
C	1.34E-05	1.08E-06	12.34463	0.0000
RESID(-1)^2	0.061522	0.002597	23.68896	0.0000
GARCH(-1)	0.907106	0.004739	191.3992	0.0000

R-squared	-0.000567	Mean dependent var	0.000496
Adjusted R-squared	-0.000420	S.D. dependent var	0.020846
S.E. of regression	0.020851	Akaike info criterion	-5.040025
Sum squared resid	2.955477	Schwarz criterion	-5.037014
Log likelihood	17134.05	Hannan-Quinn criter.	-5.038986
Durbin-Watson stat	1.942122		

Dependent Variable: DOW
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 09/01/13 Time: 16:37
Sample: 9/10/1984 8/19/2011
Included observations: 6798
Convergence achieved after 19 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(2) + C(3)*RESID(-1)^2 + C(4)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	0.000635	0.000183	3.473702	0.0005
		Variance Equation		
C	2.77E-06	3.58E-07	7.731768	0.0000
RESID(-1)^2	0.067143	0.002835	23.68096	0.0000
GARCH(-1)	0.927842	0.002822	328.8425	0.0000

R-squared	-0.000198	Mean dependent var	0.000350
Adjusted R-squared	-0.000198	S.D. dependent var	0.020288
S.E. of regression	0.020290	Akaike info criterion	-5.306612
Sum squared resid	2.798146	Schwarz criterion	-5.302597
Log likelihood	18041.18	Hannan-Quinn criter.	-5.305227
Durbin-Watson stat	2.028145		

Dependent Variable: GE
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 09/01/13 Time: 16:38
Sample: 9/10/1984 8/19/2011
Included observations: 6798
Convergence achieved after 11 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(1) + C(2)*RESID(-1)^2 + C(3)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
		Variance Equation		
C	1.08E-06	1.93E-07	5.571686	0.0000
RESID(-1)^2	0.048877	0.002222	22.00024	0.0000

GARCH(-1)	0.948955	0.002302	412.2234	0.0000

R-squared	-0.000435	Mean dependent var	0.000384
Adjusted R-squared	-0.000288	S.D. dependent var	0.018408
S.E. of regression	0.018410	Akaike info criterion	-5.521847
Sum squared resid	2.304122	Schwarz criterion	-5.518836
Log likelihood	18771.76	Hannan-Quinn criter.	-5.520808
Durbin-Watson stat	2.034545		

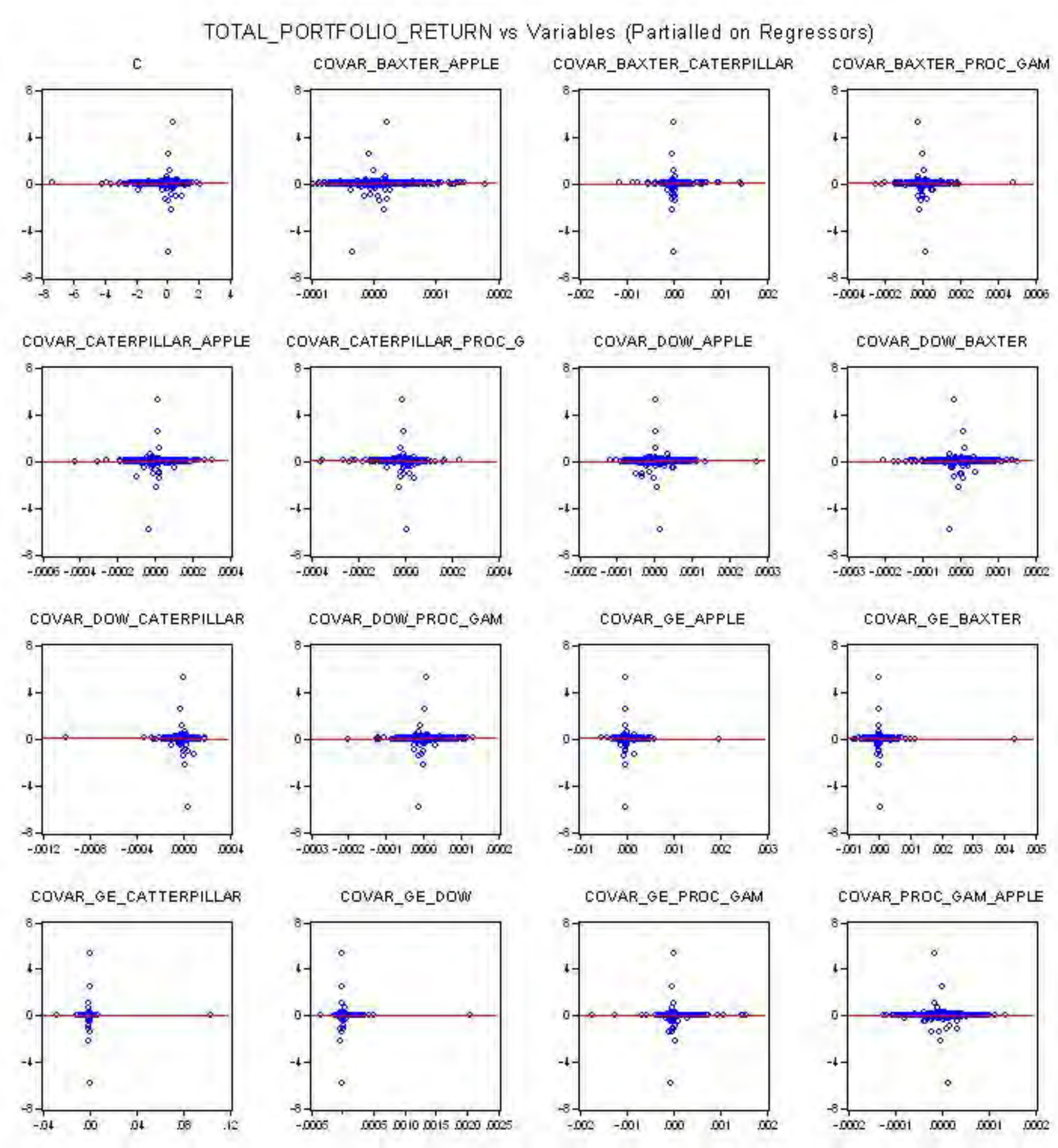
Dependent Variable: PROC_GAM
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 09/01/13 Time: 16:40
Sample: 9/10/1984 8/19/2011
Included observations: 6798
Convergence achieved after 17 iterations
Presample variance: backcast (parameter = 0.7)
GARCH = C(1) + C(2)*RESID(-1)^2 + C(3)*GARCH(-1)

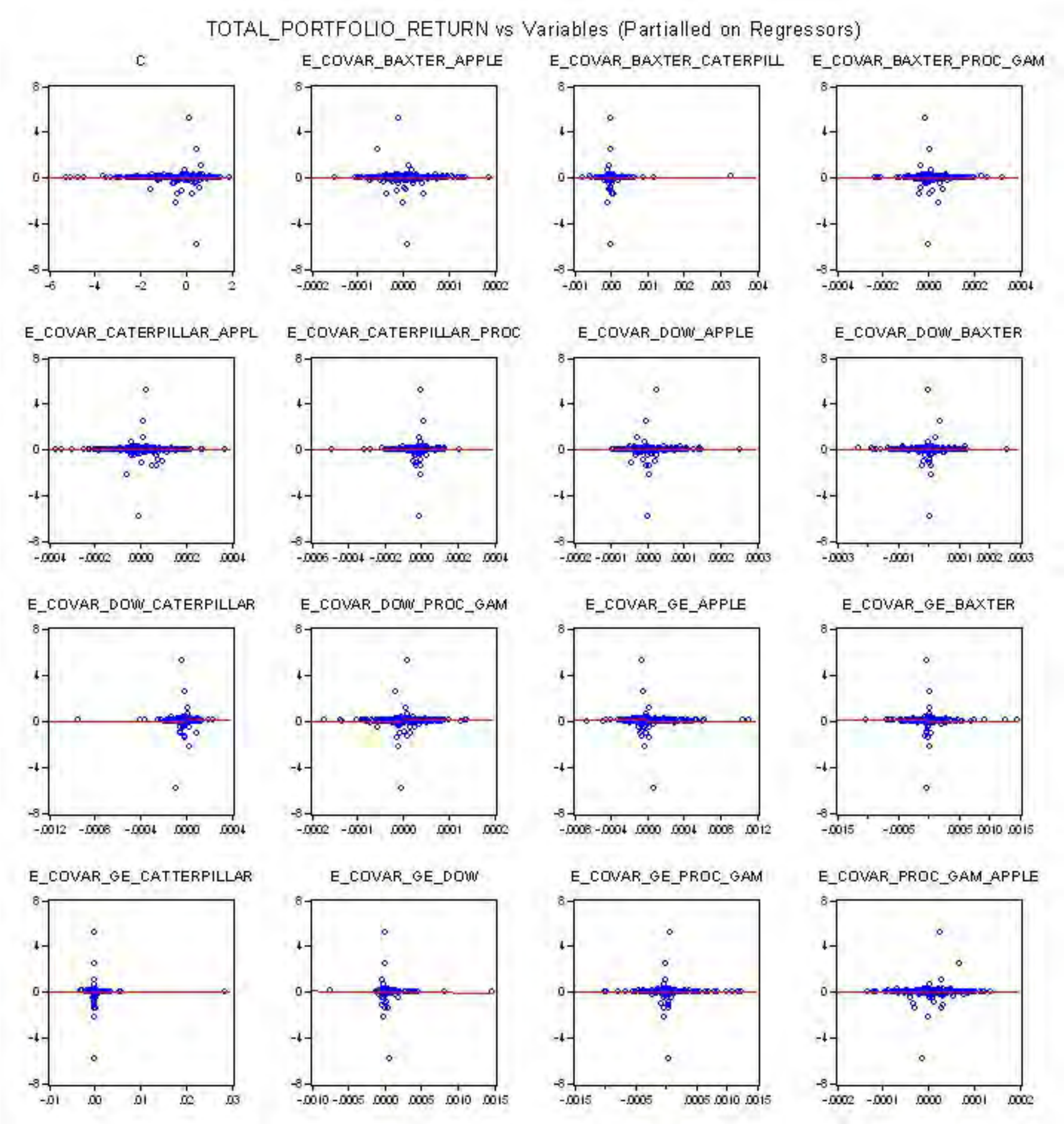
Variable	Coefficient	Std. Error	z-Statistic	Prob.
		Variance Equation		
C	1.53E-06	8.70E-08	17.58149	0.0000
RESID(-1)^2	0.062640	0.001221	51.28368	0.0000
GARCH(-1)	0.935089	0.001153	811.1139	0.0000

R-squared	-0.001072	Mean dependent var	0.000519
Adjusted R-squared	-0.000924	S.D. dependent var	0.015855
S.E. of regression	0.015862	Akaike info criterion	-5.760006
Sum squared resid	1.710412	Schwarz criterion	-5.756994
Log likelihood	19581.26	Hannan-Quinn criter.	-5.758967
Durbin-Watson stat	2.080084		

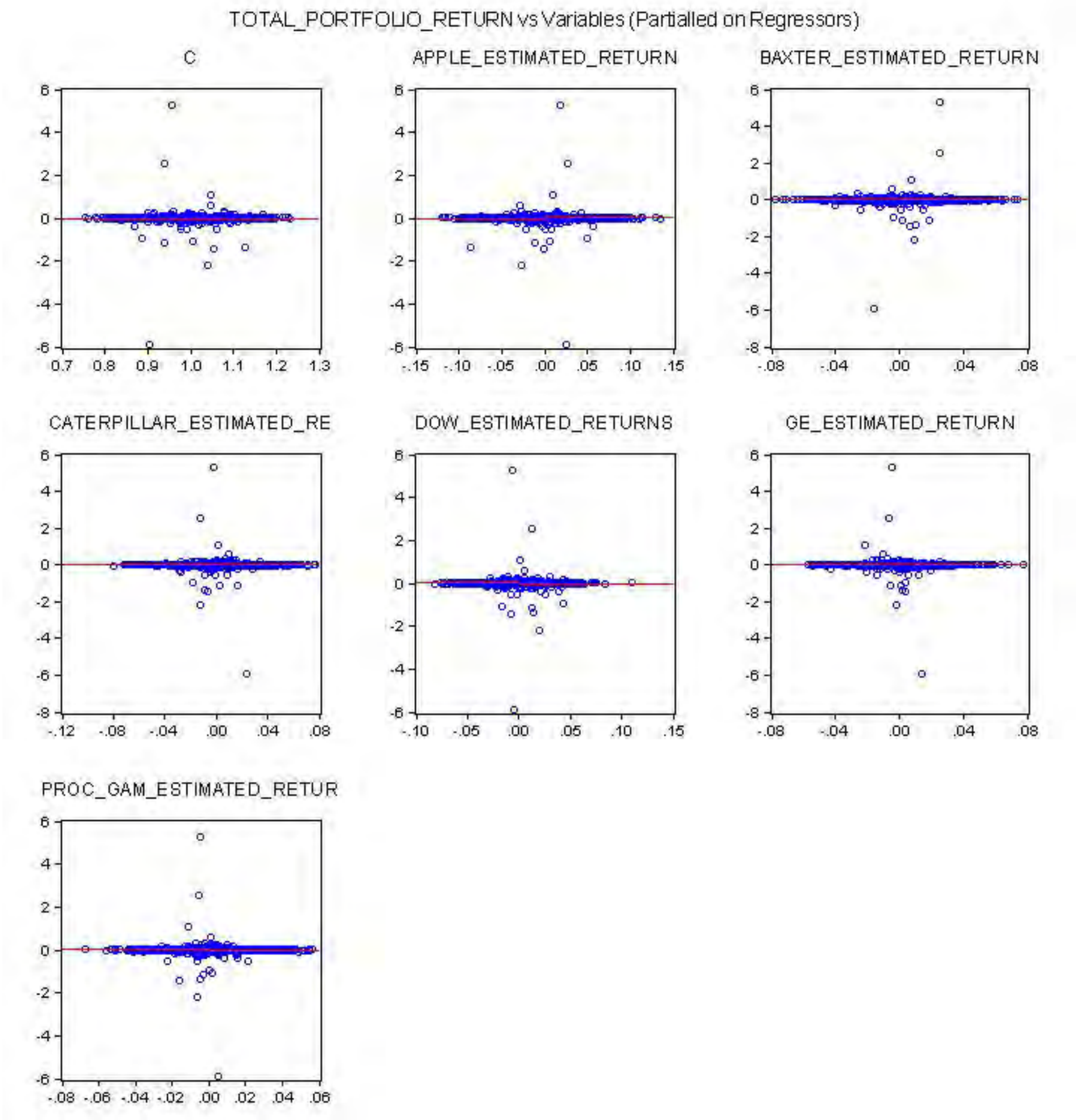
Appendix 8: Outliers analysis

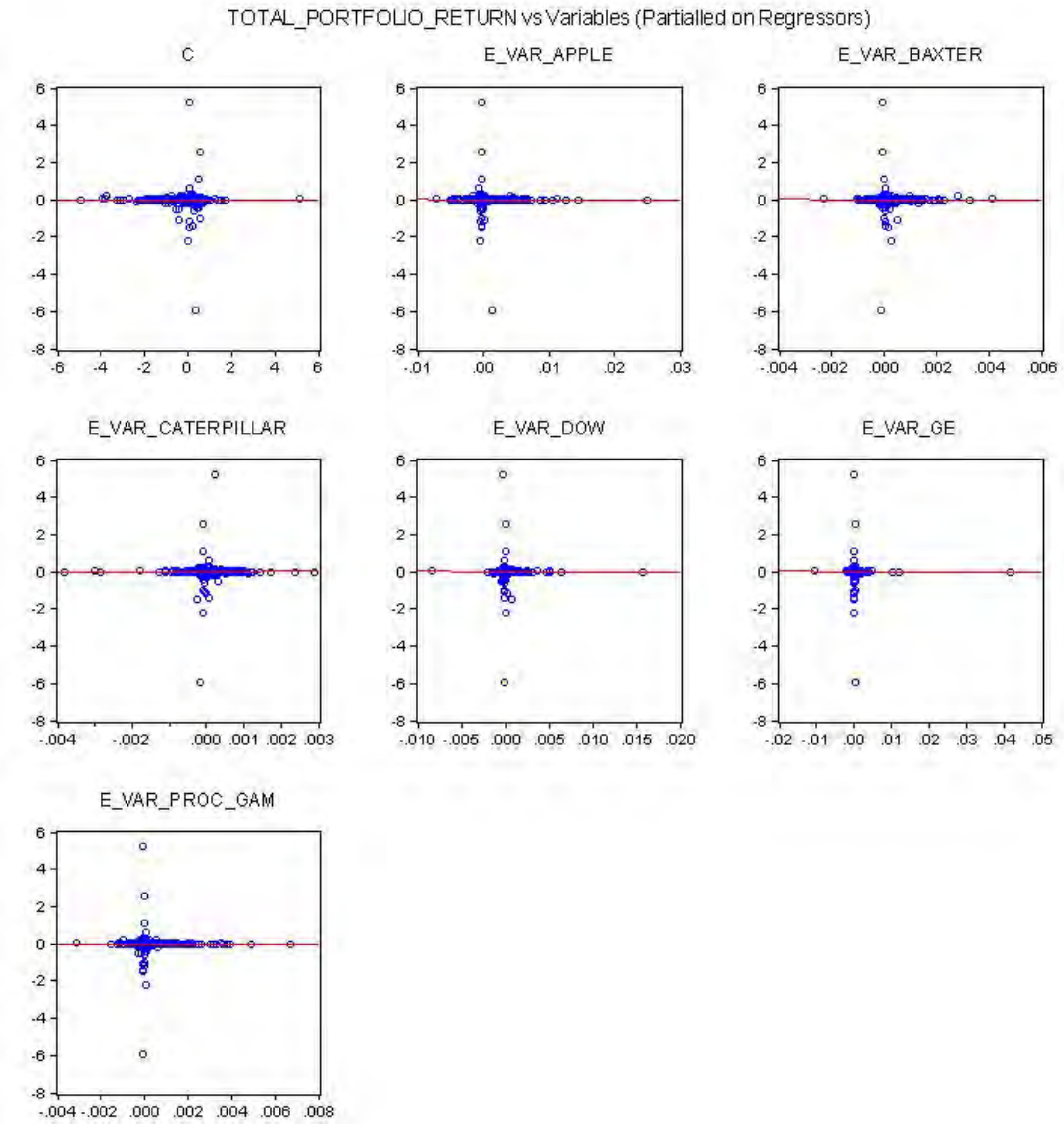
Table 28 Stability diagnostics - Leverage Plots for all used variables

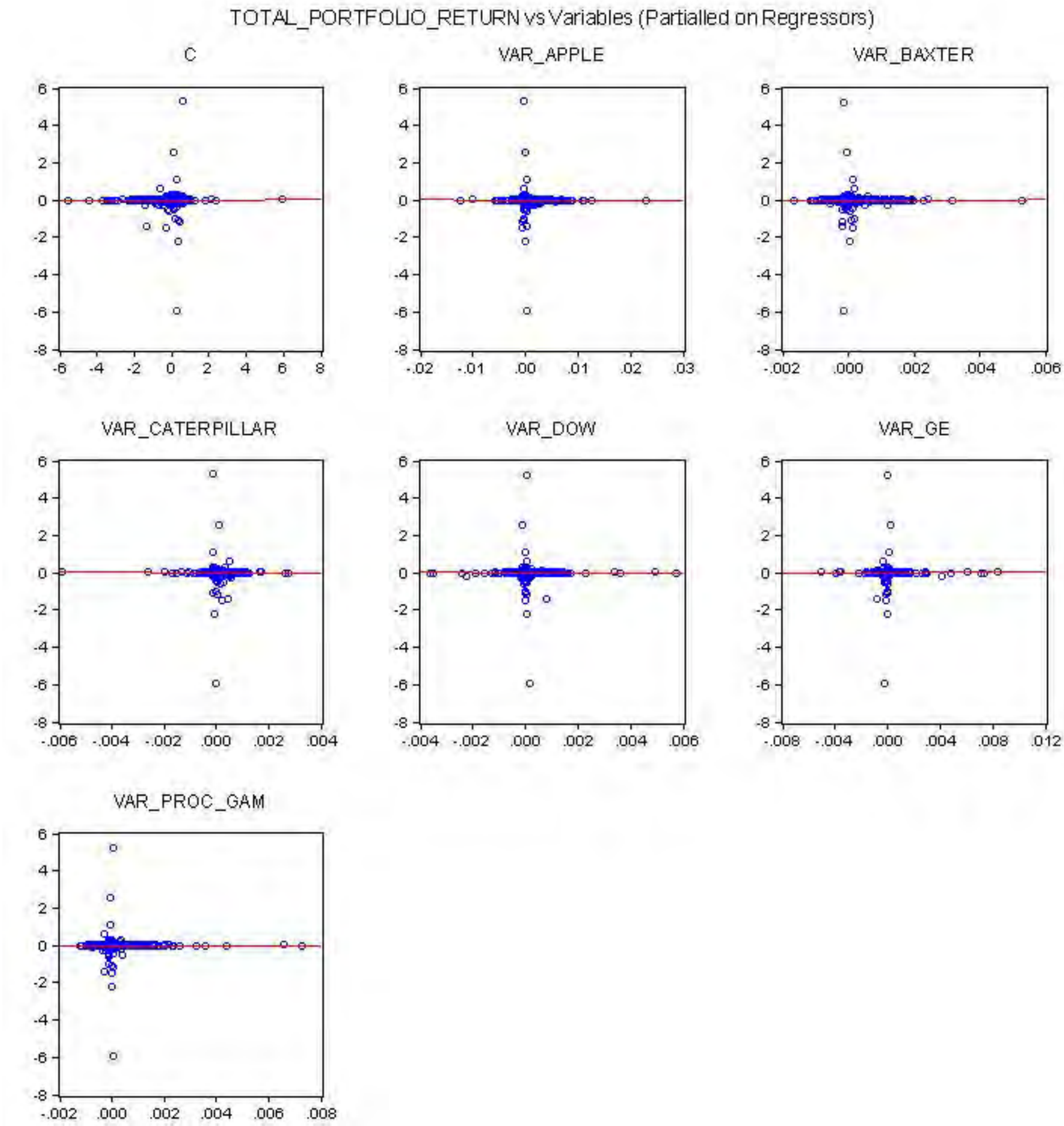




TOTAL_PORTFOLIO_RETURN vs Variables (Partialled on Regressors)
C
E_COVAR_BAXTER_APPLE
E_COVAR_BAXTER_CATERPILL
E_COVAR_BAXTER_PROC_GAM
E_COVAR_CATERPILLAR_APPL
E_COVAR_CATERPILLAR_PROC
E_COVAR_DOW_APPLE
E_COVAR_DOW_BAXTER
E_COVAR_DOW_CATERPILLAR
E_COVAR_DOW_PROC_GAM
E_COVAR_GE_APPLE
E_COVAR_GE_BAXTER
E_COVAR_GE_CATTERPILLAR
E_COVAR_GE_DOW
E_COVAR_GE_PROC_GAM
E_COVAR_PROC_GAM_APPLE


TOTAL_PORTFOLIO_RETURN vs Variables (Partialled on Regressors)
C
APPLE_ESTIMATED_RETURN
BAXTER_ESTIMATED_RETURN
CATERPILLAR_ESTIMATED_RE
DOW_ESTIMATED_RETURNS
GE_ESTIMATED_RETURN
PROC_GAM_ESTIMATED_RETUR


TOTAL_PORTFOLIO_RETURN vs Variables (Partialled on Regressors)
C
E_VAR_APPLE
E_VAR_BAXTER
E_VAR_CATERPILLAR
E_VAR_DOW
E_VAR_GE
E_VAR_PROC_GAM

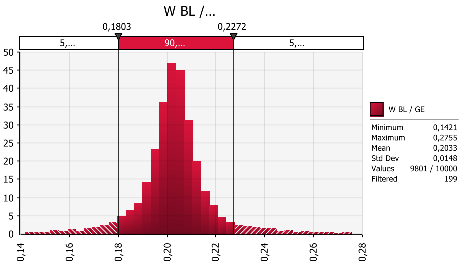

TOTAL_PORTFOLIO_RETURN vs Variables (Partialled on Regressors)
C
VAR_APPLE
VAR_BAXTER
VAR_CATERPILLAR
VAR_DOW
VAR_GE
VAR_PROC_GAM

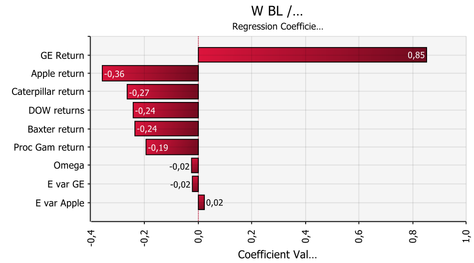
Appendix 9: Portfolio weights and regression coefficients from B-L Monte Carlo

B-L portfolio weight

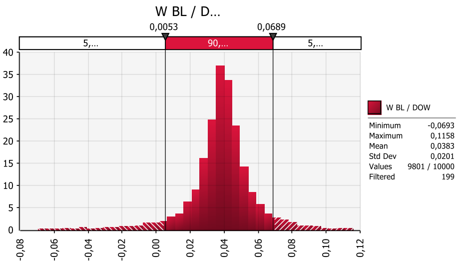
Regression coefficients for weight

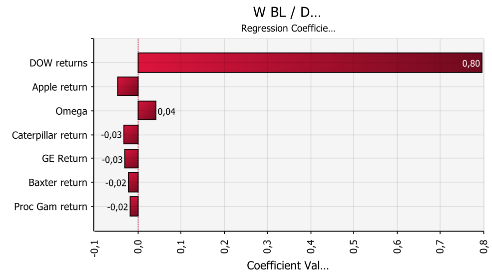
GE



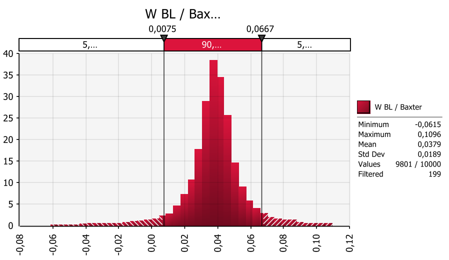


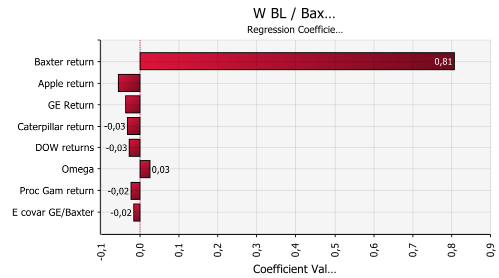
DOW



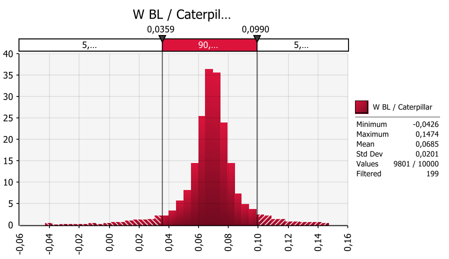


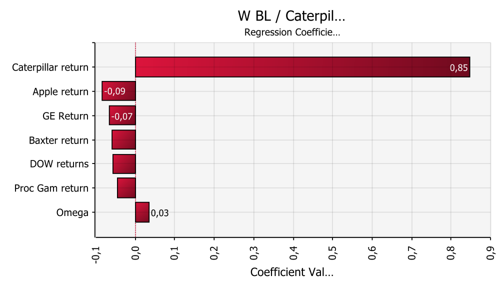
Baxter



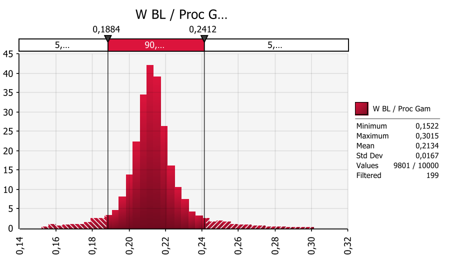


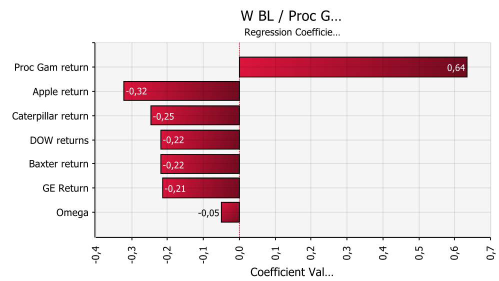
Caterpillar



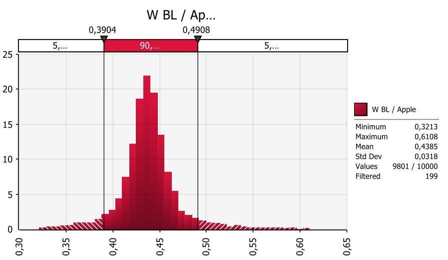


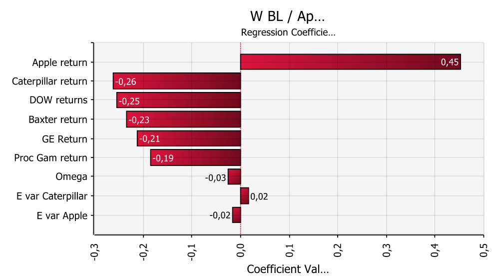
Proc Gam





Apple





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Eviews 7.2

Eviews 8

Palisade Decision Tools Suite 5.5

Spearian 1.9