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Classes of rings determined by a categorical property

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Contents

Preface

The core of the presented habilitation thesis consists of the following articles:

A. Simion Breaz, Jan Žemlička, *When every self-small module is finitely generated*, J. Algebra 315/2 (2007), 885–893.

B. Jan Žemlička, *When products of self-small modules are self-small*, Commun. Algebra 36/7 (2008), 2570–2576.

C. Jan Žemlička, *Steadiness is tested by a single module*, in: Kelarev, A. V. (ed.) et al., Abelian groups, rings and modules. Proceedings of the AGRAM 2000 conference, Perth, Australia, July 9-15, 2000. American Mathematical Society (AMS), Providence, RI. (2001), Contemp. Math. 273, 301–308.

D. Jan Žemlička, *Steadiness of regular semiartinian rings with primitive factors artinian*, J. Algebra 304/1 (2006), 500–509.

E. Simion Breaz, Jan Žemlička, *The defect functor of homomorphisms and direct unions*, Algebr. Represent. Theor. 19/1 (2016), 181–208.

F. Jan Žemlička, *Socle chains of abelian regular semiartinian rings*, J. Pure Appl. Algebra 217/6 (2013), 1018–1025.

G. Tomáš Penk, Jan Žemlička, Commutative tall rings, J. Algebra Appl., 13/4 (2014).

H. M. Tamer Kosan, Jan Žemlička, Mod-retractable rings, Commun. Algebra 42/3 (2014) 998-1010.

I. M. Tamer Kosan, Jan Žemlička, On modules and rings with restricted minimum condition, Colloq. Math., 140,1 (2015), 75–86.

Chapter 1

Introduction

This chapter contains a survey summarizing several particular concepts and tools useful in research of connections between a ring structure and a structure of categories of modules.

CLASSES OF RINGS DETERMINED BY A CATEGORICAL PROPERTY

There are many properties of the category of all modules over a ring which can be easily recognized from the structure of rings. This phenomenon can be illustrated by the classical theorem characterizing left perfect rings:

Theorem 0.1. [17, Theorem P] *The following statements are equivalent for a ring R:*

(1) *Every left module has a projective cover.*
(2) *$R/J(R)$ is semisimple and $J(R)$ is left T-nilpotent .*
(3) *$R/J(R)$ is semisimple and every non-zero left R module contains a maximal submodule.*
(4) *R satisfies the descending chain condition on principal right ideals.*
(5) *Every flat left module is projective.*

Note that the conditions (1) and (5) deals with the structure of the category of all modules over the ring, however the conditions (2), (3), and (4) are expressed in the language of ring structure.

Characterization by both ring-theoretical and categorical properties are known for various classical classes of rings such semisimple, hereditary, semihereditary or abelian regular ones. Recall for example a characterization of von Neumann regular rings [49, Theorem 1.1 and Corollary 1.13] which appears to be useful as a test class for ring theoretical characterization of some categorical properties.

Theorem 0.2. *The following statements are equivalent for a ring R:*

(1) *For every $x \in R$ there exists an element $y \in R$ such that $x = xyx$.*
(2) *Every principal left (right) ideal is generated by an idempotent.*
(3) *Every finitely generated left (right) ideal is generated by an idempotent.*
(4) *Every finitely generated submodule of a projective left R-module P is a direct summand of P.*
(5) *Every left (right) module is flat.*

Clearly, the first three condition are ring-theoretical and the last two module-theoretical. Furthermore, it is worth mentioning that the first condition quantifies only elements of a ring while the second and third ones are formulated in language of lattices of one-sided ideals.

Unfortunately, not all classes of rings defined by some natural condition on category of modules can be described by some nice ring-theoretical property. There are two reasons of such a lack. The first one is caused by our ignorance; the problem seems to be simply too hard for our imperfect tools and the goal of this thesis is to at least partially solve some problems of the described character. The second reason is fundamental, based on set theory. Such an example of a module-theoretical properties which cannot be describe in the ring structure example is the existence of Whitehead test module for projectivity. As it is proved in [96] it is independent of ZFC with Generalized Continuum Hypothesis over all right hereditary non-right perfect ring.

The main objective of this thesis is to partially summarize known results for several classes of rings determined by some categorical property. Four of the studied classes, namely of steady, strongly steady, tall, and mod-retractable rings represent typical examples of such rings, since their categorical definitions are natural an easily applicable, while other two classes of general semiartinian, and RM-rings can be defined by ring-theoretical property, nevertheless relevant structural questions coming from the context of module theory needs transfer categorical properties to the ring structure.

Let us remark that a module means right R-module over some unitary associative ring R within the whole text. For non-explained terminology we refer to standard monographs [11, 49, 90].

1. Compact objects

An object c of an abelian category closed under coproducts and products is said to be *compact* if the covariant functor $\mathrm{Hom}(c, -)$ commutes with all direct sums i.e. there is a canonical isomorphism in the category of abelian groups $\mathrm{Hom}(c, \bigoplus \mathcal{D}) \cong \bigoplus \mathrm{Hom}(c, \mathcal{D})$ for every system of objects $\mathcal{D}$. The concept of compactness presents an easy way to replace finitely generated modules in general abelian categories. Nevertheless, the clear form of the categorical definition is a reason why compact objects can be applied as a useful tool also in categories containing finitely generated objects in standard sense.

1.1. History.

The systematic research of compact objects in the context of module categories was started by Hyman Bass in 60's. His famous book [18] contains as an exercise a basic non-categorical characterization of the notion. Let us mention here the author's comment to the exercise that examples of compact objects in the category of all modules which are not finitely generated "are not easy to find" [18, p.54].

The introductory work on theory of compact modules is due to Rudolf Rentschler. However his PhD thesis [78] and the paper [79] contains a list of basic examples and several necessary and sufficient conditions of compactness, the core of these works is an attempt to answer the natural question over which rings coincide the classes of compact modules and of finitely generated ones. It should be mentioned that compact objects in categories of all modules have been studied under various terms: module of type Σ, dually slender, Σ-compact, or U-compact module. We will use the term *small* module here.

Further study of small modules has been motivated by progress of research in several different branches of algebra. One of the most important source of questions concerning smallness or more generally compactness comes from the context of representable equivalences of module categories. However $\star$-modules which appear to be an important notions in this branch of module theory were shown to be necessary finitely generated [32, 33], more general context deals with infinitely generated small modules [93, 94, 95]. Another important motivation for study of the notion has appeared in the structure theory of graded rings [75] and almost free modules [94]. Lattice theoretical approach to smallness is presented in the work [54]. Fruitful motivation of many questions in this theory comes from the dual context of so called slim and slender modules [38, 42, 45, 74], albeit the tools and ideas of research are far from being dual.

Commuting properties of functors Hom are studied in many cases only for modules from the category $\mathrm{Add}(M)$ of direct summands of direct sums of copies of some module M. Recall that a module M which is a compact object of the category $\mathrm{Add}(M)$ is called *self-small.* The notion was introduced in [12] as a tool for generalization of Baer's lemma [48, 86.5]. Self-small modules turn out to be important also in the study of splitting properties, [5, 21] and representable equivalences between subcategories of module categories in connection with tilting theory [32, 33]. The notion is very useful in structure theory of mixed abelian groups [9, 20].

The work [71] is devoted to study of compactness in stable categories, i.e. categories whose objects are all modules and groups of morphisms factorize through projective modules. Namely, it is proved that over right perfect rings compact of objects of the stable category can be represented by some standard finitely generated modules.

1.2. **Small modules.** As it is shown in [18] or in [79, 1^o], small modules can be described in natural way by language of systems of submodules.

Lemma 1.1. *The following conditions are equivalent for an arbitrary module M:*

(1) *M is small,*
(2) *if $M = \bigcup_{i<\omega} M_n$ for an increasing chain of submodules $M_n \subseteq M_{n+1} \subseteq M$, then there exists n such that $M = M_n$,*
(3) *if $M = \sum_{i<\omega} M_n$ for a system of submodules $M_n \subseteq M$, $n < \omega$, then there exists n such that $M = \sum_{i<\omega} M_n$.*

The condition (2) implies immediately that every finitely generated module is small. Moreover, it is clear from (3) that there is no infinitely countably generated small module. An another easy consequence of Lemma 1.1 is an observation that a union of strictly increasing chain of the length κ, for an arbitrary cardinal κ of uncountable cofinality, consisting of small submodules give us as well a small module. A small module can be constructed in such a way for instance as a κ-generated uniserial module, which is, indeed, the union of a chain of κ-many cyclic submodules.

This construction motivates definition of particular subclasses of small modules. For an arbitrary cardinal number λ we say that a module M is *λ-reducing* if for every submodule $N \subseteq M$ such that $gen(N) \leq \lambda$ there exists a finitely generated submodule F such that the inclusions $N \subseteq F \subseteq M$ holds.

For an arbitrary ring R let us denote by $\mathcal{SM}(R)$, $\mathcal{R}_\kappa(R)$, $\mathcal{FG}(R)$ and $\mathcal{FP}(R)$ respectively the classes of all small, κ-reducing, finitely generated and finitely presented right R-modules. It is easy to formulate the following hierarchy of these classes:

$$\mathcal{FP}(R) \subseteq \mathcal{FG}(R) \subseteq \mathcal{R}_\kappa(R) \subseteq \mathcal{R}_\lambda(R) \subseteq \mathcal{SM}(R)$$

where $\lambda < \kappa$ are infinite cardinals.

Note that all the inclusions are strict in general. Of course, every κ^+-generated ideal in valuation ring is a witness of the inequalities $\mathcal{R}_{\kappa^+}(R) \neq \mathcal{R}_\kappa(R) \neq \mathcal{FG}(R)$. Furthermore, it is proved in the paper [95, Theorem 2.8] that a ring power F^ω for each field F contains a small right ideal which is not ω-reducing. It is important to remark that classes $\mathcal{R}_\kappa(R)$ and $\mathcal{SM}(R)$ have similar class properties as the class $\mathcal{FG}(R)$. Namely, the classes $\mathcal{SM}(R)$ and $\mathcal{R}_\kappa(R)$ for each infinite cardinal are closed under taking homomorphic images, extensions and finite sums [104, Proposition 1.3].

There exist natural classes of rings over which each injective module is necessary small. Let us denote by $\mathcal{I}(R)$ the class of all injective modules over a ring R.

Theorem 1.2. [95, Theorem 1.6] *Let κ be an infinite cardinal and R a ring.*

1. *If there exists an embedding $R_R^{(\kappa)} \to R_R$, then $\mathcal{I}(R) \subseteq \mathcal{R}_\kappa(R)$.*
2. *If there exists an embedding $R_R^2 \to R_R$, then $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$.*

The hypothesis of (1) is satisfied by the endomorphism ring $End(V)$ for any κ-dimensional vector space V. Furthermore, any non-commutative domain which does not satisfy the right Ore condition (for example polynomials in two non-commuting variables $\mathbb{Z}\langle x, y\rangle$) satisfies the hypothesis of (2).

A similar observation as in Theorem 1.2 is made in the paper [33]:

Proposition 1.3. [33, Lemma 1.10] *Let R be a simple ring containing an infinite orthogonal set of idempotents. Then $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$.*

As a consequence we get that $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$ for every non-artinian simple von Neumann regular ring R. Indeed, it means that if all injective modules are small, then there exists a proper class of non-isomorphic small modules. Thus we have examples of rings over which small modules can be arbitrarily large.

1.3. **Steady rings.** Rings over which the class of all compact (or small) modules coincides with the class of all finitely generated ones are called *right steady*. It is well-known that the class of all right steady rings is closed under factorization [33, Lemma 1.9], finite products [94, Theorem 2.5], and Morita equivalence [43, Lemma 1.7].

Clearly, rings over which there exists proper class of non-isomorphic small modules (as those from Theorem 1.2 and Proposition 1.3) are not steady. On the other hand, several classes of rings satisfying some finiteness conditions are well-known to be right steady:

Theorem 1.4. *A ring R is right steady provided any of the following conditions holds true:*

(1) *R is right noetherian; ,*
(2) *R is right perfect;*
(3) *R is right semiartinian of finite socle length,*
(4) *R is a countable commutative ring,*
(5) *R is an abelian regular ring with countably generated ideals.*

Note that (1) has been established independently by several authors ([79, 7^0], [32, Proposition 1.9], [44, p.79], (2) is proved in [33, Corollary 1.6], (3) in [95, Theorem 1.5], (4) in [79, 11^0], and (5) in [113, Corollary 7].

As an easy consequence of the Theorem 1.4(1) we obtain a characterization of rings over which small modules are precisely finitely presented ones:

Theorem 1.5. [104, Theorem 1.4] *A ring R is right noetherian if and only if $\mathcal{SM}(R) = \mathcal{FP}(R)$.*

Although an existence of a general ring-theoretic criterion for steady rings is still an open problem, there is a construction of some kind of minimal example of an infinitely generated small module over a non-steady ring:

Theorem 1.6. [102, Theorem 1.4] *Let R be a ring, $\kappa = card(R)^+$, and denote by Simp the representative set of all simple right modules. Then R is not right steady if and only if $T = \prod_{S\in Simp} S^\kappa \oplus \bigoplus_{S\in Simp} E(S)$ contains an infinitely generated small submodule.*

Obviously Theorem1.6 can be reformulated to the claim that a ring R is right steady if and only if the module T (of cardinality bounded by $2^{2^{card(R)}}$) contains no infinitely generated small submodule.

Let us remark that for commutative regular rings a module-theoretical criterion of existence of an infinitely generated small module can be formulated in a more elegant form, that the representative class of small modules over a commutative regular ring is in general a set, and there is an estimate of the cardinality of each small module:

Theorem 1.7. [102, Theorem 2.7] *Let R be a commutative regular ring. Then R is steady if and only if the module $R^* = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ contains no infinitely generated small submodule.*

We have remarked that a ring-theoretical characterization of steadiness is an open problem, nevertheless criteria of steadiness are known for several particular classes of rings.

Remind that a ring is called *right semiartinian* if every non-zero cyclic module contains a simple submodule [72]. Large classes of examples both steady and non-steady abelian regular semiartinian rings are constructed in the paper [43]. The articles [83] and [105] characterize steadiness of abelian regular semiartinian rings [83, Theorem 3.4] and regular semiartinian rings with primitive factors artinian:

Theorem 1.8. [105, Theorem 3.5] *Let R be a regular semiartinian ring with primitive factors artinian. Then the following conditions are equivalent:*

1. *R is right steady;*
2. *R is left steady;*
3. *There exists no infinitely generated small right ideal of any factor of R.*
4. *There exists no infinitely generated small left ideal of any factor of R.*

In particular, an abelian regular semiartinian ring R is is not right steady if and only if there is an abelian regular factor-ring, $\bar{R}$, of R and a member, I, of the socle chain of $\bar{R}$ such that I is an infinitely generated dually slender right $\bar{R}$-module [83, Criterion A].

In the case of abelian regular rings the criterion of steadiness is formulated in the work [109] where $w(M) = \sup\{\dim_{R/I}(M/MI) |\ I \text{ maximal ideal}\}$:

Theorem 1.9. [109, Theorem 3.2] *Let R be an abelian regular ring. Then the following conditions are equivalent:*

1. *R is right steady,*
2. *$R/\bigcap_{n<\omega} I_n$ is right steady for every system of maximal ideals I_n, and there exists no small module M with finite $w(M)$ which is either ω_1-generated or contained in $\prod_{i<\omega} F_i$ where F_i are n-generated modules.*
3. *There exists no (ω_1-generated) $\omega_1$reducing module and no infinitely generated small submodule of $\prod_{n<\omega} R/J_n$ for any system of ideals J_n.*
4. *There exists no infinitely generated small submodule of $\prod_{n<\omega} R/J_n$ for any system of ideals J_n and every ω_1-generated module M with finite $w(M)$ contains a countable set C such that $M/\bigcap_{c\in C} MAnn(c)$ is infinitely generated.*

The fact that steady continuous regular rings are precisely semisimple rings is presented in [107, Theorem 4.7]. Furthermore, a necessary and sufficient condition of steadiness of valuation rings is given in [113, Theorem 13] and more general case of chain rings (i.e. rings with linearly ordered lattices of both right and left ideals) is characterized in the paper [103]:

Theorem 1.10. [103, Theorem 2.4] *For a chain ring R the following conditions are equivalent:*

(1) *R is right steady.*
(2) *There exists no ω_1-generated uniserial right module.*
(3) *$R/rad(R)$ contains no uncountable strictly decreasing chain of ideals, R contains no uncountably generated right ideal and for every ideal I and for every prime ideal $P \subseteq I$ there exists an ideal K such that $P \subset K \subset I$.*

However countable valuation rings are steady, it is known an example of a countable chain ring which is not right steady [103, Example 1.9]. The result of Theorem 1.10 can be generalized for a class of serial rings:

Theorem 1.11. [103, Theorem 3.5] *The following conditions are equivalent for a serial ring R with a complete set of orthogonal idempotents $\{e_i,\ i \le n\}$:*

(1) *R is right steady,*
(2) *e_iRe_i is right steady for every $i \le n$,*
(3) *there exists no ω_1-generated uniserial right R-module.*

It is an open problem whether some analogue of Hilbert basis theorem is valid for steadiness, i.e. whether a polynomial ring over a right steady ring is necessary right steady. It is known for example that polynomial rings in finitely many variables over right perfect ring [108, Proposition 2.6] and polynomial rings in countably many variables over commutative neotherian rings are right steady [79, 11^o], but the question whether polynomial rings in countably many variables over non-commutative neotherian rings are right steady waits for an answer. The strongest result concerning countably many variables is the following claim:

Theorem 1.12. [108, Theorem 2.7] *If X is a countable set of variables and R a right perfect ring such that $End_R(S)$ is finitely generated as a right module over its center for every simple module S, then $R[X]$ is right steady.*

On the other hand, polynomial rings in uncountably many variables are not steady as it is witnessed by the following example.

Example 1.13. [108, Example 3.1, Proposition 3.2] Let R be an arbitrary ring and consider the additive monoid $\mathbb{N}^{\omega_1}$. For every $\alpha < \beta \le \omega_1$ define $e_{\alpha\beta} \in \mathbb{N}^{\omega_1}$ by the rule $e_{\alpha\beta}(\gamma) = 1$ whenever $\gamma \in \langle \alpha, \beta)$ and $e_{\alpha\beta}(\gamma) = 0$ elsewhere. Moreover, E denotes the submonoid of $\mathbb{N}^{\omega_1}$ generated by $\{e_{\alpha\beta} |\ \alpha < \beta \le \omega_1\}$ and consider a monoid ring $S = R[E]$. Then the ideal $\bigcup_{\alpha<\omega} e_{0\alpha}S$ is ω_1-generated and ω_1-reducing as a right S-module, which proves that S is not right steady.

Let X be an uncountable set of variables. Since there exists a surjective map of X onto the monoid E, it can be extended to a surjective homomorphism from the free commutative monoid in free generators X to the monoid E and this homomorphism of monoids can be extended to a surjective homomorphism of the polynomial ring $R[X]$ onto $R[E]$. Thus $R[X]$ is not right steady.

1.4. **Self small modules.** Recall that a module M is self-small provided it is a compact object in the category $\mathrm{Add}(M)$. Similarly as in the case of non-small modules, non-self-small ones can be characterized by the condition that there exists a countable chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq \ldots,\ n < \omega$ of submodules of M such that $M = \bigcup_{n<\omega} M_n$ and for every $n < \omega$ there exists a non-zero endomorphism $f_n : M \to M$ such that $f_n(M_n) = 0$ [12, Proposition 1.1]. It is worth mentioning that the full endomorphism ring can serve as a tool for recognizing whether a module is self-small. In particular, if for a module M either $\mathrm{End}(M)$ is countable or the finite topology on $\mathrm{End}(M)$ is discrete, then M is self-small [12, Corollaries 1.4 and 2.1]. Nevertheless, endomorphism rings cannot detect self-smallness of a module in general:

Theorem 1.14. [106, Theorem 2.9] *Let R be a non-artinian abelian regular ring. Then there exists a pair of a self-small module M and a non-self-small module N such that* $\mathrm{End}_R(M) \cong \mathrm{End}_R(N)$.

The class of all self-small modules is closed under endomorphic images and direct summands but the following example shows that it is not closed under finite direct sums:

Example 1.15. [40, Example 4] The group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is self-small by [106, Example 2.7] as well as the group $\mathbb{Q}$. The product $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small by [40, Example 3].

Let us remark that the natural question which finite sums of self-small modules are as well self-small has an easy answer. Note that the hypothesis on Hom-groups in the condition (2) is satisfied if for example $\mathrm{Hom}(M_i, M_j) = 0$ whenever $i \neq j$.

Proposition 1.16. [40, Proposition 2.4] *The following conditions are equivalent for a finite system of self-small modules $(M_i|\ i \le k)$:*

1. *$\bigoplus_{i\le k} M_i$ is not self-small*
2. *there exist $i, j \le k$ and a chain $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n \subseteq \ldots$ of proper submodules of M_i such that $\bigcup_{n=1}^{\infty} N_n = M_i$ and $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ for each $n \in \mathbb{N}$.*

The case of infinite products of self-small modules is much more complicated and only particular results are known.

Proposition 1.17. [106, Proposition 1.6] *Let $(M_i|\ i \in I)$ be a system of self-small modules satisfying the condition $Hom_R(\prod_{j\in I\setminus\{i\}} M_j, M_i) = 0$ for each $i \in I$. Then $\prod_{j\in I} M_j$ is a self-small module.*

It is well-known that over semisimple rings as well as over local or commutative perfect rings the classes of small, self-small and finitely generated modules coincides. On the other hand, every generic module is an example of an infinitely generated self-small module over (of course artinian) Kronecker algebras.

It motivates the definition of *right strongly steady* rings as rings over which every right self-small module is finitely generated. Note that the ring $R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ is non-singular right artinian but it is not right strongly steady since its maximal right ring of quotients $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ serves as an example of an infinitely generated self-small R-module [22, Example 3.11]. On the other hand, every upper triangular matrix ring over a division ring (and, in particular, over a field) is right strongly steady [22, Example 3.13].

Closure properties of strongly steady rings are similar as in the case of steady ones; they include factorization, finite products and Morita equivalence [22, Lemmas 2.1-4]. However the commutativity simplifies the situation, ring theoretical characterization of strongly steady rings is an open problem even in this case. More clear is the (important) case of right non-singular rings:

Theorem 1.18. [22, Theorem 3.9] *Let R be a right non-singular right strongly steady ring. Then R is right artinian.*

This result allows to formulate a criterion for commutative non-singular rings:

Theorem 1.19. [22, Theorem 3.10] *A non-singular commutative ring is strongly steady if and only if it is semi-simple.*

Furthermore note that every right noetherian right strongly steady ring is right artinian by [22, Proposition 3.16].

Special attention is given to study of self-small abelian groups. It is easy to see that every self-small torsion group is finite [12, Proposition 3.1], but the question which mixed abelian groups are self-small seems to be very interesting and attractive for researchers [5, 7, 20, 21]. If A is a torsion free abelian group of finite rank, then the *R-typ of* A is the quasi-isomorphism class of A/F, where F is a free subgroup of A with A/F torsion. To conclude this section recall at least one basic result about self-smallness of mixed abelian group with finite rank torsion-free part:

Proposition 1.20. [12, Proposition 3.6] *Suppose that A is a mixed abelian group and that A/tA has finite rank. Then A is self-small if and only if*

(a) *for all primes, p, $(tA)_p$ is finite and*
(b) *the R-type of A/tA is p-divisible for all primes p with $(tA)_p \neq 0$.*

1.5. **Abelian categories.** However the definition of a compact object is categorical, we have discussed results in the category of modules which can be formulated just in the language of modules. Nevertheless, some particular questions of the theory can be easily formulated in language of abelian categories. Before we try to do it, let us start with needed categorical terminology and basic tools.

A category with a zero object is called *additive* if for every finite system of objects there exist product and coproduct which are canonically isomorphic, every Hom-set has the structure of an abelian group and the composition of morphisms is bilinear. An additive category is *abelian* if there exists kernel and a cokernel for each morphism, monomorphisms are exactly kernels of some morphisms and epimorphisms cokernels. A category is said to be *complete (cocomplete)* whenever it has all limits (colimits) of small diagrams exist.

We suppose in the sequel that $\mathcal{A}$ is an abelian category closed under arbitrary coproducts and products. By the term family or system we mean any discrete diagram, which can be formally described as a mapping from a set of indexes to a set of objects. Suppose that $\mathcal{N} \subseteq \mathcal{M}$ are two families of objects of the category $\mathcal{A}$. Then a corresponding coproducts are denoted by $(\bigoplus \mathcal{M}, (\nu_M | \ M \in \mathcal{M})), (\bigoplus \mathcal{N}, (\widetilde{\nu}_N | \ N \in \mathcal{N}))$ and a products by $(\prod \mathcal{M}, (\pi_M | \ M \in \mathcal{M})), (\prod \mathcal{N}, (\widetilde{\pi}_N | \ N \in \mathcal{N}))$. Note that there exists canonical morphisms $\nu_{\mathcal{N}} : \bigoplus \mathcal{N} \to \bigoplus \mathcal{M}$ and $\pi_{\mathcal{N}} : \prod \mathcal{M} \to \prod \mathcal{N}$ given by universal properties of colimit $\bigoplus \mathcal{N}$ and limit $\prod \mathcal{N}$, which satisfies $\nu_N = \nu_{\mathcal{N}} \widetilde{\nu}_N$ and $\pi_N = \widetilde{\pi}_N \pi_{\mathcal{N}}$ for each $N \in \mathcal{N}$.

For arbitrary $\varphi = (\varphi_N | \ N \in \mathcal{N}) \in \bigoplus \{\mathcal{A}(M, N) \mid N \in \mathcal{N}\}$ let us denote by $\mathcal{F}$ a finite subsystem such that $\varphi_N = 0$ whenever $N \notin \mathcal{F}$ and let $\tau : M \to \prod \mathcal{N}$ be the morphism given by the universal property of the product $(\prod \mathcal{N}, (\pi_N), N \in \mathcal{F})$ applied on the cone $(M, (\varphi_N | \ N \in \mathcal{N}))$ (i.e. $\pi_N \circ \tau = \varphi_N$). Then

$$\Psi_{\mathcal{N}}(\varphi) = \nu_{\mathcal{F}} \circ \nu^{-1} \circ \pi_{\mathcal{F}} \circ \tau$$

where $\nu : \bigoplus \mathcal{F} \to \prod \mathcal{F}$ denotes the canonical isomorphism. Note that the definition $\Psi_{\mathcal{N}}(\varphi)$ does not depend on choice of $\mathcal{F}$. Furthermore the mapping $\Psi_{\mathcal{N}}$ is a monomorphism in the category of abelian groups for every family of objects $\mathcal{N}$.

Now, we are ready to formulate precise general definition of the central notion. An object M is said to be $\mathcal{C}$-*compact* if $\Psi_{\mathcal{N}}$ is an isomorphism for every family $\mathcal{N} \subseteq \mathcal{C}$. Note that the class of all $\mathcal{C}$-compact objects is closed under finite coproducts and cokernels since the contravariant functor $\mathcal{A}(-, \bigoplus \mathcal{N})$ commutes with finite coproducts and it is left exact.

Now we are able to formulate an elementary criterion of compact object, which generalizes Lemma 1.1:

Lemma 1.21. *If M is an object and a class of objects $\mathcal{C}$, then it is equivalent:*

1. *M is $\mathcal{C}$-compact,*
2. *for every $\mathcal{N} \subseteq \mathcal{C}$ and every $f \in \mathcal{A}(M, \bigoplus \mathcal{N})$ there exists finite subsystem $\mathcal{F} \subseteq \mathcal{N}$ and a morphism $f' \in \mathcal{A}(M, \bigoplus \mathcal{F})$ such that $f = \nu_{\mathcal{F}} \circ f'$,*
3. *for every $\mathcal{N} \subseteq \mathcal{C}$ and every $f \in \mathcal{A}(M, \bigoplus \mathcal{N})$ there exists finite subsystem such that $\mathcal{F} \subseteq \mathcal{N}$, $f = \sum_{F \in \mathcal{F}} \nu_F \circ \rho_F \circ f$.*

Note that the commuting properties seems to play important role not only for Hom-functors. For example coherent functors introduced in [13] are characterized in the module categories in [36, Lemma 1] as exactly those covariant functors which commute with direct limits and direct products. The result was extended to locally finitely presented categories in [66, Chapter 9]. Commuting properties of covariant Ext^1-functors are studied in [24, 91, 50, 8, 86].

The defect functor $\mathrm{Dev}_\beta = \mathrm{CokerHom}(\beta, -)$ of a morphism β is a natural generalization of both covariant Hom and Ext^1 functors in an arbitrary locally finitely presented abelian category.

If $\beta : L \to P$ is a homomorphism in $\mathcal{C}$, then we have the following examples [23, Example 2]:

(1) If $\mathcal{C}$ is abelian, P is projective and β a monomorphism, then $\mathrm{Def}_\beta(-)$ is canonically equivalent to $\mathrm{Ext}^1(P/\beta(L), -)$.
(2) If $P = 0$, then $\mathrm{Def}_\beta(-)$ is canonically equivalent to $\mathrm{Hom}(L, -)$.
(3) If β is an epimorphism and $v : K \to L$ is the kernel of β then $\mathrm{Def}_\beta(-)$ represents the covariant defect functor associated to the exact sequence $0 \to K \xrightarrow{v} L \xrightarrow{\beta} P \to 0$.
(4) If R is a unital ring, $\mathcal{C} = \text{Mod-}R$, and L and P are finitely generated and projective then $\mathrm{Def}_\beta(R)$ represents the transpose of $P/\beta(L)$.

Furthermore, the following criteria are known for an arbitrary homomorphism $\beta : L \to P$ [23, Proposition 9]:

(1) Suppose that P is a compact object. The functor Def_β commutes with direct sums if and only if L is a compact object.
(2) Suppose that P is a finitely generated object. The functor Def_β commutes with direct unions if and only if L is finitely generated.
(3) Suppose that P is a finitely presented object. Then Def_β commutes with direct limits if and only if L is finitely presented.

As an analogue of [36, Lemma 1] in the case of direct unions the following result can be proven:

Theorem 1.22. [23, Theorem 10] *A functor $F : \mathcal{C} \to Ab$ commutes with respect direct products and direct unions if and only if it is naturally isomorphic to a defect functor* Def_β *associated to a homomorphism $\beta : L \to P$ with L and P finitely generated.*

As an consequence we obtain for any homomorphism $\beta : L \to P$ between projective modules equivalence of the three following properties:

(1) Def_β commutes with direct sums,
(2) Def_β commutes with direct unions
(3) Def_β commutes with direct limits

Moreover, if these conditions are valid, then $\mathrm{Def}_\beta(R)$ is a finitely presented left R-module [23, Proposition 11]. Let π_J denote the canonical projection. Then the commuting of Def_β with a direct sum of objects can be characterize in the following way:

Theorem 1.23. [23, Theorem 24] *If $\beta : L \to P$ is a homomorphism and $(M_i, i \in I)$ a family of objects, the following conditions are equivalent:*

(1) Def_β *commutes with the direct sum of* $(M_i, i \in I)$,

(2) *for every* $f \in \mathrm{Hom}(L, \bigoplus_{i\in I} M_i)$ *there exist finite subset* $F \subset I$, *and* $g \in \mathrm{Hom}(P, \bigoplus_{i\in I\setminus F} M_i)$ *such that* $\pi_{I\setminus F} f = g\beta$.

If κ is a cardinal less than the first ω-measurable cardinal and Def_β commutes with countable direct sums then Def_β commutes with direct sums of κ objects [23, Proposition 26]. Thus in the constructible universe $\mathrm{Def}_\beta(-)$ commutes with countable direct sums if and only if $\mathrm{Def}_\beta(-)$ commutes with all direct sums and, in particular, for each $M \in \mathcal{C}$ $\mathrm{Ext}^1_{\mathcal{C}}(M, -)$ commutes with countable direct sums if and only if $\mathrm{Ext}^1_{\mathcal{C}}(M, -)$ commutes with all direct sums.

2. Semiartinian rings

Recall that a module M is *semiartinian* provided each non-zero factor of M contains a simple submodule and a ring R is right semiartinian if R_R is a semiartinian module. Of course, a right semiartinian ring can be characterized by the module class conditions such that

(1) every module is semiartinian, or
(2) every non-zero module contains a simple submodule.

However the class of all semiartinian ring can be easily described by both ring-theoretical and categorical conditions, it seems to interesting the question how the structure of a semiartinian ring reflects some additional condition such as steadiness or strongly steadiness. This way of research motivates the definition the *right socle chain*, which is the uniquely defined strictly increasing chain of ideals $(S_\alpha \mid \alpha \leq \sigma+1)$ in a right semiartinian ring R satisfying $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(R/S_\alpha)$, $S_0 = 0$ and $S_{\sigma+1} = R$.

2.1. **History.** The notion generalizes the notion of a right artinian ring, which can be described precisely as a semiartinian ring with the socle chain of a finite socle length and finitely generated slices $\mathrm{Soc}(R/S_\alpha)$. Moreover, by [17, Theorem P] every non-zero module over a left perfect ring has a non-zero socle, hence every left perfect ring is right semiartinian. Basic structural results about general semiartinian rings are published in papers [26, 39, 48, 72, 85]. Furthermore, let us recall the important construction presented in the paper [16]:

Proposition 2.1. [16, Proposition 4.7] *Let* κ *be an infinite cardinal,* K *a field and* R_γ *a semiartinian* K*-algebra with primitive factors artinian and socle chain* $(S_{\alpha\gamma} \mid \alpha \leq \sigma_\gamma)$ *for each* $\gamma < \kappa$. *Let* $R = \bigoplus_{\gamma<\kappa} R_\gamma + K \subseteq \prod_{\gamma<\kappa} R_\gamma$ *and put* $\sigma = \sup_{\gamma<\kappa} \sigma_\gamma$. *If either* σ *is limit or* $\{\gamma \mid \sigma_\gamma = \sigma\}$ *is infinite, then:*

(1) $\bigoplus_{\gamma<\kappa} S_{\alpha\gamma}$ *is the* α*-th member of the socle chain* $\forall \alpha \leq \sigma$.
(2) *If each* R_γ *is right semiartinian, then* R *is right semiartinian with socle length* $\sigma + 1$.
(3) R *has primitive factors artinian.*

Note that if the algebras in the construction are supposed to be (abelian) regular, then the constructed ring R is so. Actually, the classical result claims that commutative semiartinian rings are close to abelian regular ones:

Theorem 2.2. [72, Théorème 3.1, Proposition 3.2] *Let R be a ring.*

(1) *R is left semiartinian if and only if $\mathrm{J}(R)$ is right T-nilpotent and $R/\mathrm{J}(R)$ is left semiartinian.*
(2) *Let R be a commutative semiartinian ring. Then $R/\mathrm{J}(R)$ is abelian regular and semiartinian.*

Properties and constructions of semiartinan rings close to von Neumann regular ones are studied in papers [18, 15, 39, 83, 43] while papers [2, 1] are focused to correspondence between the class of semiartinian rings and other interesting classes of rings defined by some property of module categories.

2.2. **Results.** The notion of a dimension sequence plays an important role in research of regular semiartinian ring with primitive factors artinian. Nevertheless, before the definition we need to formulate the following result:

Theorem 2.3. [83, Theorem 2.1] *Let R be a right semiartinian ring and $\mathcal{L} = (S_\alpha \mid \alpha \leq \sigma + 1)$ the right socle chain of R. Then the following conditions are equivalent:*

(1) *R is regular and all right primitive factor rings of R are right artinian,*
(2) *for each $\alpha \leq \sigma$ there are a cardinal λ_α, positive integers $n_{\alpha\beta}$, $\beta < \lambda_\alpha$, and skew-fields $K_{\alpha\beta}$, $\beta < \lambda_\alpha$, such that $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$, as rings without unit. The pre-image of $M_{n_{\alpha\beta}}(K_{\alpha\beta})$ coincides with the β-th homogeneous component of R/S_α and it is finitely generated as right R/S_α-module for all $\beta < \lambda_\alpha$. Moreover, λ_α is infinite if and only if $\alpha < \sigma$.*

If (1) holds true, then R is also left semiartinian, and $\mathcal{L}$ is the left socle chain of R.

Denote by $\mathcal{R}$ the class of all regular right semiartinian rings R such that all (right) primitive factor-rings of R are (right) artinian. If $R \in \mathcal{R}$, then the family

$$\mathcal{D}(R) = \{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$$

collecting data from the previous theorem is said to be the *dimension sequence* of R.

The dimension sequences of a regular semiartinian ring naturally reflects the structure of single semisimple slices. Note that structural theory of the notion is developed in [110, 112]. An application of combinatorial set theory [46, 92] allows to prove necessary conditions satisfied by this invariant:

Theorem 2.4. [110, Proposition 3.1 and Theorem 3.5] *Let $R \in \mathcal{R}$ be abelian regular, Generalized Continuum Hypothesis holds and α, δ be ordinals satisfying $\alpha + \delta \leq \sigma$. Then $|\langle \alpha, \sigma \rangle| \leq 2^{\lambda_\alpha}$. If $\mathrm{cf}(\lambda_\alpha) > \max(|\delta|, \omega)$, then $\lambda_{\alpha+\delta} \leq \lambda_\alpha$. Otherwise $\lambda_{\alpha+\delta} \leq \lambda_\alpha^+$.*

On the other hand, commutative regular semiartinian rings with a particular given rank of slices of the socle chain can be constructed:

Theorem 2.5. [110, Theorem 5.1] *Let σ be an ordinal, K a field, and $(\lambda_\alpha \mid \alpha \leq \sigma)$ a family of cardinals satisfying for every $\alpha \leq \beta \leq \sigma$ the conditions:*

(a) $\lambda_\beta \leq \lambda_\alpha^+$ *if* $\mathrm{cf}(\lambda_\alpha) = \omega$, *and* $\lambda_\beta \leq \lambda_\alpha$ *otherwise,*
(b) $\lambda_\alpha < \omega$ *iff* $\alpha = \sigma$,
(c) $|\langle \alpha, \sigma \rangle| \leq \lambda_\alpha$.

Then there exists a commutative regular semiartinian K-algebra with dimension sequence $\{(\lambda_\alpha, \{(1, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ where $K_{\alpha\beta} = K$ for all $\alpha \leq \sigma$ and $\beta < \lambda_\alpha$.

Furthermore, it is possible to generalize results on dimension sequences for a suitable subclass of regular right semiartinian rings R with primitive factors artinian, namely, for those R satisfying the condition that every ideal which is finitely generated as two-sided ideal is finitely generated as right ideal. It is proved in the paper [112, Theorem 3.4] that over these rings and under Generalized Continuum Hypothesis it holds that $\lambda_{\alpha+\delta}(n) \leq \lambda_\alpha(m)$, whenever $m \geq n$ and α, δ are ordinals such that $\alpha+\delta \leq \sigma$ and $\mathrm{cf}(\lambda_\alpha(\mathrm{n})) > \max(|\delta|, \omega)$, where $\{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ is dimension sequence and $\lambda_\alpha(n) = \mathrm{card}\{\beta < \lambda_\alpha \mid n_{\alpha\beta} \geq n\})$.

3. Tall rings

A module M which contains a non-noetherian submodule N such that the factor M/N is non-noetherian as well is studied first in the paper [84] under the term *tall*. The notion of a *right tall* ring is defined in the same paper as a ring over which every non-noetherian right module is tall. Note that this notion presents a "typical" example of a ring described by a module-class property.

3.1. **History.** It is not hard to see that the class of all right tall rings is closed under factors, finite products, and Morita equivalence. Although in [84] is presented a criterion of right tall rings using the notion of Krull dimension of all modules, an existence a general ring-theoretic necessary and sufficient condition remains to be an open problem.

Theorem 3.1. [84, Theorem 2.7] *The following statements are equivalent for a ring R:*

(1) *R is right tall,*
(2) *every non-noetherian module has a proper non-noetherian submodule,*
(3) *every module with Krull dimension is noetherian.*

Since every maximal submodule of a non-noetherian module is non-noetherian, the condition (2) implies that every right max ring, over which every nonzero right module contains a maximal submodule, is necessarily right tall [31, p. 31]. Nevertheless, the following example shows that the revers implication does not hold.

Example 3.2. [76, Example 3.2] Put $I = \sum_i x_i^2 F[\mathbf{X}]$ and $R = F[\mathbf{X}]/I$ for a field F and an infinite countable set of variables $\mathbf{X} = \{x_1, x_2, \dots\}$. Let $X_i = x_i + I$ and define an ideal $J = \sum_i X_i R$. Then J is a nil ideal, since $X_i^2 = 0$ and R is commutative. As $R/J \cong F$, R is tall ring by [76, Lemma 3.1]. Moreover, J is a nil maximal ideal of R, thus it is the Jacobson radical of R. Since $X_1 \cdot \dots \cdot X_n \neq 0$ for every n, J is not T-nilpotent, hence R is not a max ring.

No general ring-theoretical criterion characterizing max rings neither correspondence between the classes of all tall and all max rings is known. Nevertheless, max rings are studied by many authors from various points of view and with different motivations [25, 27, 31, 47, 53, 63, 97]. Among another results let us recall several classical module-theoretical necessary and sufficient conditions:

Theorem 3.3. [47, 53, 63] *The following conditions are equivalent for a ring R:*

(1) *R is a right max ring;*
(2) *$R/J(R)$ is a right max ring and $J(R)$ is right T-nilpotent,*
(3) *every non-zero submodule of injective envelops $E(S)$ contains a maximal submodule for every simple module S,*
(4) *there is a cogenerator for the category of right modules whose every non-zero submodule contains a maximal submodule.*

Much more is known about both commutative max rings and commutative tall rings. The most important fact from our point of view is ring-theoretical criteria of commutative max ring:

Theorem 3.4. [47, 53, 63] *The following conditions are equivalent for a commutative ring R:*

(1) *R is a max ring;*
(2) *$R/J(R)$ is a regular ring and $J(R)$ is left T-nilpotent,*
(3) *the localization at any maximal ideal of R is a max ring,*
(4) *the localization at any maximal ideal of R is a perfect ring.*

3.2. **Results.** For description of commutative tall rings are very useful to formulate necessary structural condition of non-tall rings:

Theorem 3.5. [76, Theorem 2.6] *Let R be a commutative non-tall ring. Then there exists a maximal ideal I and a sequence of ideals $I = J_1 \supset J_2 \supset \dots$ such that*

(1) *$IJ_i \subseteq J_{i+1}$ for each i,*
(2) *R/J_i is artinian for each i,*
(3) *$\bigcap_i J_i$ is a prime ideal,*
(4) *$R/\bigcap_n I^n$ is not a tall ring.*

Note that ideals J_n from the previous theorem cannot be replaced by the powers J_1^n in general [76, Example 3.7]. On the other hand, if R is tall, then for every non-idempotent maximal ideal I such that R/I^i is artinian for each i, the intersection $\bigcap_j I^j$ is not a prime ideal [76, Proposition 2.9]. As the consequence can be formulated the following criterion:

Theorem 3.6. [76, Theorem 2.12] *The following conditions are equivalent for a commutative ring R:*

(1) *R is not tall,*
(2) *there exists a non-noetherian artinian module,*
(3) *there exists an artinian module M, elements $x \in R$ and $m_j \in M$ such that $m_{j+1}x = m_j$ and $m_{j+1} \notin m_j R$ for each j and $M = \bigcup_j m_j R$.*
(4) *there exists a sequence of ideals J_j of R and elements $x_j \in R$ such that R/J_j is artinian, $J_{j+1} \subsetneq J_j$, $x_j r \in J_{j+1}$ iff $r \in J_j$ and the length of $S_j(R/J_j)$ is equal to the length of $S_j(R/J_k)$ for each $j \leq k < \omega$.*

Finally note that the previous criterion can be expressed in a very simple form in the case of a commutative noetherian ring R, namely, R is tall if and only if R is artinian [76, Theorem 2.10].

4. Retractability and coretractability

Both the central notions of this section, i.e. completely coretractable and mod-retractable rings present examples of rings naturally determined by a categorical property. A module M such that its every nonzero submodule contains a nonzero endomorphic image of M is called *retractable* and, dually, M is called *coretractable* if there exists a nonzero homomorphism of M/K to M for every proper submodule $K \subseteq M$. For example each finitely generated module over commutative ring is retractable [41].

4.1. **History.** The importance of the notions has emerged in research of Baer modules [81, 82], endomorphism rings of nonsingular modules [61, 62], compressible modules [87, 89] and module lattices [51, 101]. The works [10, 41, 52] are devoted to rings over which every module is retractable or coretractable.

Main results of [10] describe rings over which every right module is coretractable, such rings are called *right completely coretractable.* Dually, a ring R is said to be *right mod-retractable* provided every right R-module is retractable.

It is proved in [111, Theorem 2.4] that a ring is right (left) completely coretractable if and only if it is isomorphic to a finite product of matrix rings over right and left perfect rings. Furthermore, every cyclic right and left R-module is coretractable [111, Proposition 3.2].

The papers [41, 52] started to study mod-retractable rings. Note that the class of mod-retractable rings is closed under Morita equivalence, factorization, and finite products [41]. Moreover it is known that any right mod-retractable ring is an example of a right max ring [64, Theorem 3.3].

4.2. **Results.** However mod-retractable rings are precisely rings such that all their torsion theories are hereditary, the general ring-theoretical criterion of mod-retractability is not know. The characterization is available only for several particular classes of rings and for all commutative rings:

Theorem 4.1. [64, Theorem 3.3] *Let R be a left perfect ring. Then R is right mod-retractable if and only if $R \cong \prod_{i\leq k} M_{n_i}(R_i)$ for a system of a local rings R_i, $i \leq k$, which are both left and right perfect.*

As an easy consequence can be shown that every commutative perfect ring is mod-retractable.

A similar criterion is proved for the class of right noetherian rings:

Theorem 4.2. [64, Theorem 3.3] *Let R be a right noetherian ring. Then R is right mod-retractable if and only if $R \cong \prod_{i\leq k} M_{n_i}(R_i)$ for a local right artinian rings R_i.*

As it was mentioned a criterion of mod-retractability is known for the class of commutative rings:

Theorem 4.3. [64, Theorem 3.10] *Let R be a commutative ring. Then R is mod-retractable if and only if R is semiartinian.*

Finally note that from the previous result immediately follows that every commutative semiartinian ring is necessarily mod-retractable.

5. RM rings

First recall that a module M satisfies the *restricted minimum condition* if for every essential submodule N of M, the factor M/N is artinian. The class of all modules satisfying the restricted minimum condition is well-known to be closed under submodules, factors as well as finite direct sums. Note that a semiartinian module M satisfies the restricted minimum condition if and only if $M/\operatorname{Soc}(M)$ is artinian.

5.1. **History.** A ring R is called a right *RM-ring* if R_R satisfies restricted minimum condition as a right module. Obviously, the class of all right RM-rings contains all right artinian rings and principal ideal domains. This observation partially explains the historical motivation of research of these rings. Structure theory of RM-rings and domains was studied in the papers [28, 29, 34, 37, 73]. Among others let us recall the following result:

Theorem 5.1. [34, Theorem 1] *Let R be a noetherian domain. Then R has Krull dimension 1 if and only if it is an RM-domain*

However the definition of RM-rings has a ring-theoretical nature (it actually deals with cyclic modules), the correspondence between RM-rings and the classes of rings studied above is clarified in the context of results of the paper [6], which is devoted to structure research of classes of torsion modules over RM-domains. Namely, it seems to be interesting question here whether there exists nice categorical property which is equivalent to the ring-theoretical definition.

5.2. **Results.** Ring-theoretical results for non-commutative as well as commutative rings are proved in [113]. Recall a useful technical result which consists of several necessary conditions of modules over general RM-ring where $E(M)$ denotes the injective envelope of a module M:

Theorem 5.2. [113, Theorem 2.11] *Let R be a right RM-ring and M a right R-module.*

(1) *If M is singular, then M is semiartinian.*
(2) *$E(M)/M$ is semiartinian.*
(3) *If M is semiartinian, then $E(M)$ is semiartinian. In particular, $E(S)$ is semiartinian for every simple module S.*

As an consequence it can be obtained the observation for a right RM-ring R that R is a nonsingular ring of finite Goldie dimension whenever $\mathrm{Soc}(R) = 0$ [113, Theorem 2.12].

For a semilocal RM-rings can be proved the following criterion:

Theorem 5.3. [113, Theorem 2.17] *Let R be a semilocal RM-ring and* $\mathrm{Soc}(R) = 0$. *Then the following conditions are equivalent:*

(1) *R is noetherian,*
(2) *$J(R)$ is finitely generated,*
(3) *the socle length of $E(R/J(R))$ is at most ω.*

Recall characterization of commutative RM-domains from the paper [6] which motivates are research:

Theorem 5.4. [6, Theorem 6 and 9] *The following conditions are equivalent for a commutative domain R:*

(1) *R is an RM-ring,*
(2) *$M = \oplus_{P\in Max(R)} M_{[P]}$ for all torsion modules M,*
(3) *R is noetherian and every non-zero (cyclic) torsion R-module has an essential socle,*
(4) *R is noetherian and every self-small torsion module is finitely generated.*

The most important result of the article [113] describes commutative RM-ring in the language of module categories which generalizes the previous result:

Theorem 5.5. [113, Theorem 3.7] *The following conditions are equivalent for a commutative ring R:*

(1) *R is an RM-ring,*
(2) *$M = \oplus_{P\in Max(R)} M_{[P]}$ for all singular modules M,*

(3) $R/\operatorname{Soc}(R)$ *is Noetherian and every self-small singular module is finitely generated.*

The question whether a similar result is valid in the non-commutative case remains open.

6. Conclusion

Let us summarize the contribution of the present thesis:

(1) We describe the structure of rings belonging to classes determined by a particular categorical property. Namely, in [22], [105], [76], [64], and [65] respectively are characterized several subclasses of right strongly steady, steady, tall, mod-retractable, and RM-rings. The structural theory of abelian regular semiartinian rings is developed in [110].

(2) We answer several structural question on classes of compact objects, in particular, [106] is devoted to closure properties of the class of all self-small modules and [102] determines test modules for the class of all small modules.

(3) We contribute to the study of commuting properties of functors in [23].

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Chapter 2

Self-small modules and strongly steady rings

The main goal of this chapter is to describe structural properties of classes of self small modules. The text consists of two papers:

A. Simion Breaz, Jan Žemlička, *When every self-small module is finitely generated*, J. Algebra 315/2 (2007), 885–893.

   DOI: 10.1016/j.jalgebra.2007.01.037

B. Jan Žemlička, *When products of self-small modules are self-small*, Commun. Algebra 36/7 (2008), 2570–2576.

   DOI: 10.1080/00927870802070207

Chapter 3

Small modules and steady rings

The chapter presents two particular solutions of general problem on ring-theoretical description of steady rings. The first text provides instead a ring-theoretical property an idea of a test module of steadiness and the second one gives a ring-theoretical characterization in the particular class of regular semiartinian rings with primitive factors artinian:

C. Jan Žemlička, *Steadiness is tested by a single module*, in: Kelarev, A. V. (ed.) et al., Abelian groups, rings and modules. Proceedings of the AGRAM 2000 conference, Perth, Australia, July 9-15, 2000. American Mathematical Society (AMS), Providence, RI. (2001), Contemp. Math. 273, 301–308.

   DOI: 10.1090/conm/273

D. Jan Žemlička, *Steadiness of regular semiartinian rings with primitive factors artinian*, J. Algebra 304/1 (2006), 500–509.

   DOI: 10.1016/j.jalgebra.2005.12.011

Chapter 4

The defect functor of homomorphisms and direct unions

This chapter is constituted by the single article devoted to commuting properties of the defect functor which generalizes the notions of Hom and Ext functors:

E. Simion Breaz, Jan Žemlička, *The defect functor of homomorphisms and direct unions*, Algebr. Represent. Theor. 19/1 (2016), 181–208.
DOI: 10.1007/s10468-015-9569-0

Chapter 5

Reflection of categorical properties to a ring structure

This chapter summarizes results on four classes of rings with similar correspondence between ring structure and categories of modules. The work was originally published as the following four papers:

F. Jan Žemlička, *Socle chains of abelian regular semiartinian rings*, J. Pure Appl. Algebra 217/6 (2013), 1018–1025.

DOI: 10.1016/j.jpaa.2012.09.025

G. Tomáš Penk, Jan Žemlička, Commutative tall rings, J. Algebra Appl., 13/4 (2014).

DOI: 10.1142/S0219498813501296

H. M. Tamer Kosan, Jan Žemlička, Mod-retractable rings, Commun. Algebra 42/3 (2014) 998–1010.

DOI: 10.1080/00927872.2012.721430

I. M. Tamer Kosan, Jan Žemlička, On modules and rings with restricted minimum condition, Colloq. Math., 140,1 (2015), 75–86.

DOI: 10.4064/cm140-1-6