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Ultramocninová konstrukce v teorii množin

Ultrapower constructions in set theory

Praha 2011 vedoucí práce: Mgr. Radek Honzík, PhD.

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Abstrakt: Předložená práce obsahuje historii vzniku míry, její souvislost s měřitelnými kardinály a shrnutí všech základních definic a pojmů potřebných k zobecnění ultramocninové konstrukce v teorii modelů pro vlastní třídy. Součástí uvedené teorie je i důkaz základních vlastností potřebných pro aplikaci ultramocninové konstrukce na měřitelné kardinály. Využitím všech předchozích výsledků poté dokážeme Teorém Dany Scotta o souvislosti mezi existencí měřitelného kardinálu a velikostí univerza.

Klíčová slova: κ-aditivita, κ-kompletnost, nedosažitelný kardinál, měřitelný kardinál, vnitřní model, ultramocnina, elementární vnoření, relativizace.

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Abstract: The presented work contains the history of origin of measure, its connection with measurable cardinals and summary of all elementary definitions and notions needed for the generalization of ultrapower construction in model theory for proper classes. One of the parts of the presented theory is the proof of elementary properties needed for the application of ultrapower construction to measurable cardinals. Using all previous results we prove the Theorem of Dana Scott about the connection between existence of a measurable cardinal and the size of the universe.

Keywords: κ-additivity, κ-completeness, inaccessible cardinal, measurable cardinal, inner model, ultrapower, elementary embedding, relativization.

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Preface

The presented thesis is based on the work of many mathematicians from the field of set theory and the theory of large cardinals. In order to connect the work of those mathematicians and their various approaches and notations, we are forced to choose one particular notation and reformulate the rest to be consistent with this notation.

In this thesis we collect all the relevant definitions from various sources, even though some of the authors may consider them redundant. In spite of that, we assume that the reader is familiar with the basics of logic and set theoretic terminology, as some of the symbols appear undefined throughout the thesis.

The first chapter contains the introduction to measure theory and a summary of some basic set theoretic knowledge, i.e. equivalence, filter and ultrafilter properties and cardinal properties, whose use is essential in the third chapter. The rest of this chapter is devoted to consistence of theories and formalization.

The second chapter introduces the axiom of constructibility and some basic facts about it. The other part of this chapter defines inner models and connects them to the Gödel's constructible universe $\mathbf{L}$.

In the final chapter we first construct the rest of the theory needed to formulate the Theorem of Łoś, which is an important theorem concerning ultraproducts. Next we extend the theory to proper classes, which is needed for the second important theorem called Mostowski's Collapsing Theorem. Using all previous results we prove Dana Scott's Theorem and show that the existence of a measurable cardinal implies existence of a nontrivial embedding of the universe.

The main contribution of this thesis is to help understand the basics about ultrapower construction and the technique of relativization in set theory and their use in large cardinal theory.

1 Preliminaries

Definitions used in this chapter mostly come from [2].

1.1 Defining the measure

Measure theory was initially created to deal with the notion of length of subsets of the real line.

Modern measure theory was founded by well known French mathematician Henri Lebesgue. He wanted to construct a function that would yield a non-negative real number to every bounded set of reals. Thus he defined a function m with the following properties:

(a) m is not identically zero,
(b) m is translation-invariant, i.e., $m(X) = m(Y)$ whenever there is a real r such that $Y = \{x + r \mid x \in X\}$,
(c) m is countably additive, i.e., if $\{X_n \mid n \in \omega\}$ is a pairwise disjoint collection whose union is a bounded set of reals, then $m(\bigcup_n X_n) = \sum_n m(X_n)$.

Unfortunately, assuming Axiom of Choice (we will further use just AC), Giuseppe Vitali was able to construct a bounded set X of reals from a well-ordering of reals, such that m does not map X to any non-negative real number.

Before we review the construction of Vitali, we formulate definitions concerning equivalence and the definition of cosets:

Definition 1.1 An *equivalence relation* on a set X is a binary relation $\equiv$ which is *reflexive*, *symmetric* and *transitive*, i.e., for all $x, y, z \in X$ we have

(i) $x \equiv x$,
(ii) $x \equiv y$ implies $y \equiv x$,
(iii) if $x \equiv y$ and $y \equiv z$ then $x \equiv z$.

Let $\equiv$ be an equivalence relation on X. Then for every $x \in X$,

$$[x] = \{y \in X \mid y \equiv x\}$$

is the *equivalence class* of x and the set

$$X/{\equiv} = \{[x] \mid x \in X\}$$

is the *quotient* of X by $\equiv$.

Definition 1.2 Let $(G, *)$ be a group and $(H, *)$ a subgroup. A subset of G of the form $\{g * h \mid h \in H\}$, where g is a fixed element of G, is abbreviated to $g * H$ and called a (*left*) *coset* of H in G.

The following is taken from [5]:

Lemma 1.3 *Let m be a measure satisfying conditions* (a)–(c) *above. We show that m does not measure all subsets of* $\mathbb{R}$.

Proof. This construction will use rational numbers as a subgroup of reals with group operator $+$. We shall take one element from each coset of $\mathbb{Q}$ in $\mathbb{R}$ and collect all choices in a set X. Let $\equiv$ be an equivalence relation on $\mathbb{R}$, where $a \equiv b$ if and only if $a - b \in \mathbb{Q}$. We claim X is non-measurable.

Consider $m(X) = 0$: First consider for each $q \in \mathbb{Q}$ the translates $X_q = X + q$. From the definition of X_q's we get $\bigcup X_q = \mathbb{R}$. We also get $m(X_q) = 0$ for every $q \in \mathbb{Q}$ from translation-invariance. Because there is only countable amount of the translates of X, we get $0 = \sum m(X_q) = m(\bigcup X_q) = m(\mathbb{R}) = 0$ from countable additivity. But $m(\mathbb{R}) > 0$ (because of (a)). This means $m(X) > 0$.

But then, there exists a positive integer N such that $m(X \cap [-N, N]) = a > 0$. Let A be a set of $(2N+2)/a$ many distinct rational numbers from $(0, 1)$(there is infinite of them, thus such A exists). For each element $b \in A$ consider the translates $X_b = (X \cap [-N, N]) + b$. All X_b's are disjoint by the definition and $\bigcup X_b \subseteq [-N, N+1]$.

By countable additivity we get

$$m(\bigcup X_b) = ((2N+2)/a) \cdot a = 2N + 2 > 2N + 1,$$

where $2N+1$ corresponds to the measure of interval $[-N, N+1]$. Because we have $m(\bigcup X_b) \subseteq [-N, N+1]$ it would mean that a subset of $[-N, N+1]$ has bigger measure than $[-N, N+1]$, which is a contradiction. □

Vitali's construction can be percieved in two ways. For Lebesgue, existence of Vitali's set gave him a reason to doubt AC, which was needed in the construction. If we agree to AC, we may ask if there is a way to extend the measure to be defined over the whole $\mathbb{R}$ again. We want to maintain both condition (a) and condition (c) so the only way is to give up the translation-invariance. Banach did just that in [6] and replaced the translation-invariance with a minimal condition that avoids trivial solutions. That is: $m(\{x\}) = 0$ for every singleton $x \in \mathbb{R}$.

Banach realised that his generalization of (b) also removed geometric considerations for m, which resulted in replacement of unit interval $[0, 1]$ with an arbitrary set S. In this case if m is not zero on the powerset of S denoted $P(S)$, then obviously $m(S) > 0$. On $P(S)$, $m(S)$ should have the highest possible value. For the generalization established by Banach we let $m(S) = 1$. The measure problem was then adjusted as follows: Is there a nonempty uncountable set S and a function $m : P(S) \to [0, 1]$ such that:

(i) $m(S) = 1$,
(ii) $m(\{x\}) = 0$ for every singleton $x \in S$,
(iii) for pairwise disjoint $\{X_n \mid n \in \omega\} \subseteq P(S)$, $m(\bigcup_n X_n) = \sum_n m(X_n)$.

Definition 1.4 Let m be a measure over a set S. We say that m is λ-*additive* if

$$\text{for any } \gamma < \lambda \text{ and pairwise disjoint } \{X_\alpha | \alpha < \gamma\} \subseteq P(S),$$
$$m(\textstyle\bigcup_\alpha X_\alpha) = \sum_\alpha m(X_\alpha).$$

For our purposes it is enough to consider a two valued measure m:

$$m : P(S) \to \{0, 1\}$$

Definition 1.5 Let S be a set. U is a *filter* over S if and only if

(i) $U \subseteq P(S)$,
(ii) $S \in U$,
(iii) $X \supseteq Y, Y \in U \to X \in U$,
(iv) $X, Y \in U \to X \cap Y \in U$.

U is *proper* if and only if $\emptyset \notin U$.
U is *principal* if and only if $U = \{X \mid Y \subseteq X \subseteq S\}$ for some $Y \subseteq S$.
U is an *ultrafilter* if and only if U is a proper filter and, for all $X \subseteq S$, $X \notin U \leftrightarrow -X \in U$.

For m a two valued measure we define ultrafilter U by

$$U = \{S \subseteq \kappa \mid m(S) = 1\}.$$

Then U is clearly a non-principal ultrafilter ($m(\{x\}) = 0$ for every singleton $x \in S$ and thus no singleton is in U) over S and we say U is associated with the measure m.

Definition 1.6 Let U be an ultrafilter. We say U is λ-*complete* if

$$\text{for any } \gamma < \lambda \text{ and } \{X_\alpha \mid \alpha < \gamma\} \subseteq U, \bigcap_{\alpha<\gamma} X_\alpha \in U.$$

For m a two valued measure we state as follows:

Proposition 1.7: Let κ be a cardinal number. A two valued measure m over κ is κ-additive if and only if the associated ultrafilter U over κ is κ-complete.

Proof. We first show that κ-completeness is equivalent to the following:

For every $\xi < \kappa$, let $\{X_\alpha \mid \alpha < \xi\}$ be a family of subsets not in U, then $\bigcup_{\alpha<\xi} X_\alpha \notin U$.

Assuming κ-completeness let $\{X_\alpha \mid \alpha < \xi\}$ be a family of subsets not in U. Then from the definiton of ultrafilter $\{-X_\alpha \mid \alpha < \xi\} \subseteq U$ and $\bigcap_{\alpha<\xi} -X_\alpha \in U$ $= -\bigcup_{\alpha<\xi} X_\alpha \in U$. Using ultrafilter property again we get $\bigcup_{\alpha<\xi} X_\alpha \notin U$.

Now let $\{X_\alpha \mid \alpha < \xi\} \subseteq U$. From the definition of ultrafilter $\{-X_\alpha \mid \alpha < \xi\}$ is a family of subsets not in U but then $\bigcup_{\alpha<\xi} -X_\alpha \notin U = -\bigcap_{\alpha<\xi} X_\alpha \notin U$ and using ultrafilter property again we get $\bigcap_{\alpha<\xi} X_\alpha \in U$.

Now from left to right let $\{X_\alpha \mid \alpha < \xi\}$ be a family of subsets not in U. Then $\sum m(X_\alpha) = 0$ and using κ-additivity $m(\bigcup_\alpha X_\alpha) = 0$. We get $\bigcup_{\alpha<\xi} X_\alpha \notin U$.

From right to left let $\sum m(X_\alpha) = 0$. Then for every α, $m(X_\alpha) = 0$ and $\{X_\alpha \mid \alpha < \xi\}$ is a family of subsets not in U. We get $\bigcup_{\alpha<\xi} X_\alpha \notin U$. This results in $m(\bigcup_\alpha X_\alpha) = 0$. $\square$

Remark: Let $\kappa > \omega$ be a cardinal number, We say κ is measurable if and only if there is a κ-complete non-principal ultrafilter over κ.

1.2 Inaccessible cardinals

We start with a quote about inaccessible cardinals from [2]:"...the least among them has such an exorbitant magnitude that it will hardly ever come into consideration for the usual purposes of set theory." Those were words of German mathematician Hausdorff in regards of the possible existence of inaccessible cardinals.

Definition 1.8 A cardinal number κ is:

(i) *regular* if $cf(\kappa) = \kappa$, where $cf(\kappa)$ denotes cofinality of κ,

(ii) a *weak limit* if κ is neither 0 nor a successor cardinal,

(iii) a *strong limit* if κ is a weak limit and for every $\lambda < \kappa$, $2^\lambda < \kappa$.

Let $\kappa > \omega$ be a cardinal number. We say κ is *inaccessible* if and only if κ is regular and a strong limit.

In other words an inaccessible cardinal is a cardinal κ above ω, which cannot be reached by repeated powerset operation (strong limit) and cannot be broken into smaller collections of smaller parts (regularity).

The rest of this section illustrates some of the consequences of existence of inacessible cardinals.

Definition 1.9 Let ZFC be an abbreviation for Zermelo-Fraenkel set theory with AC, i.e., finite set of axioms $\{\varphi_1, \ldots, \varphi_n\}+$ schema of replacement. We say $\ulcorner\varphi\urcorner$ is a formal (arithmetic) version of metamathematical formula φ and denote $\ulcorner\mathrm{ZFC}\urcorner$ the set $\{\ulcorner\varphi_1\urcorner, \ldots, \ulcorner\varphi_n\urcorner\} \cup$ instances of formal schema of replacement.

Lemma 1.10 *If κ is an inaccessible cardinal, then $V_\kappa \models \ulcorner\mathrm{ZFC}\urcorner$.*

Proof. For this lemma to hold we need to verify if the following is true:

$$\mathrm{ZFC} \vdash V_\kappa \models \ulcorner\mathrm{ZFC}\urcorner.$$

If φ is a metamathematical formula, V_κ a set, it holds that

$$\mathrm{ZFC} \vdash \varphi^{V_\kappa} \leftrightarrow \mathrm{ZFC} \vdash V_\kappa \models \ulcorner\varphi\urcorner.$$

We can immediately see that for every $\varphi \in \mathrm{ZFC}\backslash\mathrm{Replacement}$ $\mathrm{ZFC} \vdash \varphi^{V_\kappa} \leftrightarrow \mathrm{ZFC} \vdash V_\kappa \models \ulcorner\varphi\urcorner$. To show Replacement we will need to check every instance of this axiomatic schema. The following holds for Replacement :

If $\mathrm{ZFC} \vdash$ for every $F : V_\kappa \to V_\kappa$, for every $a \in V_\kappa$, $F[a] \in V_\kappa$ then $\mathrm{ZFC} \vdash V_\kappa \models \ulcorner\mathrm{Replacement}\urcorner$.

It remains to show that the condition above holds to verify schema of Replacement and that all other axioms of ZFC hold in V_κ.

For every $\alpha > 0$, axioms of extensionality, foundation, subsets, empty set, union and choice are true in V_α. If α is a limit ordinal, axioms of pairs and power set are true in V_α. If $\alpha > \omega$, then axiom of infinity is true in V_α as well. We state this as fact. Since κ is inaccessible, it is both bigger than ω, and is a limit, thus verifying the above.

We still need to show the condition of formal Replacement in V_κ. That means to see whether for every $F : V_\kappa \to V_\kappa$, for every $a \in V_\kappa$, $F[a] \in V_\kappa$.

Since κ is inaccessible we have $|V_\kappa|=\kappa$ and $|a|<\kappa$ for every $a \in V_\kappa$. If F is a function from V_κ, $a \in V_\kappa$ into V_κ, then $|F[a]| \leq |a| < \kappa$ and since κ is regular, $F[a] \subseteq V_\alpha$ for some $\alpha<\kappa$. From this $F[a] \in V_\kappa$.
This concludes the proof. □

Let I be the statement: "There is an inaccessible cardinal."

Corollary 1.11 $\mathrm{ZFC} + \mathrm{I} \vdash \mathrm{Con}(\ulcorner \mathrm{ZFC} \urcorner)$, *where* Con *denotes consistence of a theory.*

Theorem 1.12 *The existence of inaccessible cardinals is not provable in* ZFC. *Moreover, it cannot be shown by finite means that the existence of inaccessible cardinals is consistent with* ZFC.

Proof. If I holds and κ is the least inaccessible cardinal, then $V_\kappa \models \mathrm{ZFC} +$ "There is no inaccessible cardinal". But if there is no inaccessible cardinal, we take the universe as the model and conclude the existence of inaccessible cardinals cannot be proven in ZFC.

To show the second part, let's assume for contradiction that there is a finite means to show $\mathrm{Con}(\mathrm{ZFC}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{I})$.

Because the argument is finite there is a way to formulate it formally in ZFC and thus also in ZFC + I:

$$\mathrm{ZFC} + \mathrm{I} \vdash \mathrm{Con}(\ulcorner \mathrm{ZFC} \urcorner) \to \mathrm{Con}(\ulcorner \mathrm{ZFC} \urcorner + \mathrm{I}).$$

We already have

$$\mathrm{ZFC} + \mathrm{I} \vdash \mathrm{Con}(\ulcorner \mathrm{ZFC} \urcorner),$$

thus using modus ponens, we obtain

$$\mathrm{ZFC} + \mathrm{I} \vdash \mathrm{Con}(\ulcorner \mathrm{ZFC} \urcorner + \mathrm{I}),$$

which is a contradiction to Gödel's Second Incompleteness Theorem. □

For $\kappa > \omega$ we can now formulate the following theorem:

Theorem 1.13 (Ulam-Tarski) *Let $\kappa > \omega$ be a cardinal number. If κ is measurable, then κ is inaccessible.*

2 Constructible universe L

In this chapter we take a little detour to look at some of the contributions to the set theory by the foremost mathematical logician and his speculations about large cardinals. It would not be without his contributions that the axiomatic set theory and large cardinals got as much attention from the mathematical community.

Theorems throughout this chapter are stated without proofs, that can be found in [3].

2.1 Axiom of constructibility V=L

The person behind the principles of constructibility is Kurt Gödel. His article *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis* (see [7]) was the first showing the results Gödel made.

We state that set y is definable over a structure $\mathcal{M}$ if and only if there is a first-order formula $\varphi(v_0, \ldots, v_n)$ in the language of $\mathcal{M}$ and parameters $a_1, \ldots, a_n$ in the domain of $\mathcal{M}$ such that: $z \in y$ if and only if $\mathcal{M} \vDash \varphi[z, a_1, \ldots, a_n]$. For any set x,

$$\mathrm{def}(x) = \{y \subseteq x \mid y \text{ is definable over } \langle x, \in \rangle\}.$$

Clearly, $x \in \mathrm{def}(x)$ and $x \subseteq \mathrm{def}(x) \subseteq P(x)$.

Definition 2.1 We say class T is *transitive* if every element of T is a subset of T, i.e., when $x \in T$, $x \subseteq T$.

Definition 2.2 For L a transitive set we define by transfinite recursion

(i) $L(0) = 0$, $L(\alpha + 1) = \mathrm{def}(L(\alpha))$,
(ii) $L(\alpha) = \bigcup_{\xi<\alpha} L(\xi)$ when α is a limit ordinal and
(iii) $\mathbf{L}$=$\bigcup_{\alpha\in\mathbf{ON}} L(\alpha)$.

Note that each $L(\alpha)$ is a transitive set, thus $\mathbf{L}$ is a transitive class.

Gödel named $\mathbf{L}$ the class of all constructible sets and the following axiom of constructibility: $\mathrm{V} = \mathbf{L}$ $(\forall x(x \in \mathbf{L}))$ meaning 'Every set is constructible.' Moreover Gödel proved that

$\mathrm{ZF} + \mathrm{V} = \mathbf{L} \vdash \mathrm{AC}$ and
$\mathrm{ZF} + \mathrm{V} = \mathbf{L} \vdash \mathrm{GCH}$.

Theorem 2.3 $\mathbf{L}$ *is a model of* ZF*.*

Corollary 2.4 $\mathbf{L}$ *is a model of* $\mathrm{ZF} + \mathrm{V} = \mathbf{L}$*.*

Corollary 2.5 $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZF} + \mathrm{V} = \mathbf{L})$.

Corollary 2.6 $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZFC} + \mathrm{GCH})$.

2.2 Inner Models

Definition 2.7 An *inner model* M of ZF is a transitive class that contains all ordinals and satisfies the axioms of ZF, i.e., M is a model of ZF.

From what we have estabilished so far, we can immediately state that $\mathbf{L}$ is an inner model of ZF since it is a transitive class containing all ordinals and it satisfies all axioms of ZF.

To show $\mathbf{L}$ is the least inner model, we first extend the notion of relativization to classes:

Theorem 2.8 *If* M *is an inner model, then* $\mathbf{L} \subseteq M$*. In other words* $\mathbf{L}$ *is the least inner model of* ZF*.*

3 Ultrapower construction

3.1 Defining the Ultrapower

Let S be a nonempty set and let $\{\mathfrak{A}_x \mid x \in S\}$ be a system of models (for an uncountable language $\mathcal{L}$) with universe A_x for every $x \in S$. Let U be a non-principal filter over S. Let

$$\prod_{x \in S} A_x$$

be the set of functions $f : S \to \bigcup_{x \in S} A_x$ such that $f(x) \in A_x$ for every $x \in S$.

Definition 3.1 For $f, g \in \prod_{x \in S} A_x$ we define

$$f =_U g \text{ if and only if } \{x \in S \mid f(x) = g(x)\} \in U.$$

Lemma 3.2 *The relation $=_U$ on $\prod_{x \in S} A_x$ is an equivalence relation.*

Proof. Let $f, g, h \in \prod_{x \in S} A_x$. Reflexivity of $=_U$ holds because $f =_U f$ if and only if $\{x \in S \mid f(x) = f(x)\} \in U$ translates into $S \in U$ which holds for every filter. Symmetry for $=_U$ holds because relation $=$ is symmetric. For transitivity we assume $f =_U g =_U h$. Then by Definition 3.1

$$\{x \in S \mid f(x) = g(x)\} \in U \quad \text{and} \quad \{x \in S \mid g(x) = h(x)\} \in U$$

and also

$$\{x \in S \mid f(x) = g(x)\} \cap \{x \in S \mid g(x) = h(x)\} \in U$$

since U is closed under intersection. Thus

$$\{x \in S \mid f(x) = g(x) = h(x)\} \in U.$$

Because

$$\{x \in S \mid f(x) = g(x) = h(x)\} \subseteq \{x \in S \mid f(x) = h(x)\}$$

is true, from the definition of a filter we have

$$\{x \in S \mid f(x) = h(x)\} \in U.$$

Then by Definition 3.1 $f =_U h$. $\square$

Definition 3.3 Let $[f] = \{g \in \prod_{x \in S} A_x \mid f =_U g\}$ be the equivalence class of $f \in \prod_{x \in S} A_x$.

We say $\mathfrak{A}$ is called a *reduced product* of $\{\mathfrak{A}_x \mid x \in S\}$ by U if

(i) $A = \{[f] \mid f \in \prod_{x \in S} A_x\}$ is the universe of $\mathfrak{A}$,

(ii) for every constant c we have
$c^{\mathfrak{A}} = [f]$, where $f(x) = c^{\mathfrak{A}_x}$ for every $x \in S$,

(iii) for every function symbol F, if $f_1, \ldots, f_n \in \prod_{x \in S} A_x$, then
$F^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f]$, where $f(x) = F^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))$ for every $x \in S$,

(iv) for every predicate P, if $f_1, \ldots, f_n \in \prod_{x \in S} A_x$, then
$P^{\mathfrak{A}}([f_1], \ldots, [f_n])$ if and only if $\{x \in S \mid P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in U$.

Lemma 3.4 *Conditions* (iii) *and* (iv) *in Definition 3.3 do not depend on the choice of representatives from the equivalence classes* $[f_1], \ldots, [f_n]$.

Proof. For (iv) let $[f_1], \ldots, [f_n] \in A$ and let $f_1, \ldots, f_n$ be the representatives of $[f_1], \ldots, [f_n]$ such that
$P^{\mathfrak{A}}([f_1], \ldots, [f_n])$ if and only if $\{x \in S \mid P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in U$. Let $g_1 =_U f_1, \ldots, g_n =_U f_n$.

From the definition of $=_U$ we get

$$\{x \in S \mid f_1(x) = g_1(x)\} \in U, \ldots, \{x \in S | f_n(x) = g_n(x)\} \in U$$
$$\text{and } \{x \in S \mid P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in U.$$

Since U is closed under intersection we obtain

$$\{x \in S \mid f_1(x) = g_1(x)\} \cap, \ldots, \cap \{x \in S \mid f_n(x) = g_n(x)\} \cap$$
$$\cap \{x \in S \mid P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \in U.$$

Thus

$$\{x \in S \mid f_1(x) = g_1(x)\} \cap, \ldots, \cap \{x \in S \mid f_n(x) = g_n(x)\} \cap$$
$$\cap \{x \in S \mid P^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))\} \subseteq \{x \in S \mid P^{\mathfrak{A}_x}(g_1(x), \ldots, g_n(x))\}$$

and since U is upward-closed we get

$$\{x \in S \mid P^{\mathfrak{A}_x}(g_1(x), \ldots, g_n(x))\} \in U.$$

Condition (iii) is proven analogously. $\square$

Reduced products are particularly important when the filter is an ultrafilter. If U is an ultrafilter over S, then the reduced product $\mathfrak{A}$ defined in Definition 3.3 is called the *ultraproduct* of $\{\mathfrak{A}_x \mid x \in S\}$ by U and we use the notation

$$Ult_U\{\mathfrak{A}_x \mid x \in S\}$$

instead of $\mathfrak{A}$.

Theorem 3.5 (Łoś) *Let U be an ultrafilter over S and let $\mathfrak{A}$ be the ultraproduct of $\{\mathfrak{A}_x \mid x \in S\}$ by U. If φ is a formula, then for every $f_1, \ldots, f_n \in \prod_{x\in S} A_x$*

$$\mathfrak{A} \models \varphi([f_1], \ldots, [f_n]) \textit{ if and only if } \{x \in S \mid \mathfrak{A}_x \models \varphi(f_1(x), \ldots, f_n(x))\} \in U.$$

Proof. We prove this by induction on the complexity of formulas. We start with the evaluation of terms.

Terms: For every term t and $f_1, \ldots, f_n \in \prod_{x\in S} A_x$,

(1) $t^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f],$ where $f(x) = t^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x))$ for every $x \in S$

We prove this by induction on the complexity of terms.
For a constant c, where $t = c$, we need to show

$$c^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f], \text{ where } f(x) = c^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x)) \text{ for every } x \in S.$$

But this is equivalent to

$$c^{\mathfrak{A}} = [f] \text{ where } f(x) = c^{\mathfrak{A}_x} \text{ for all } x \in S$$

which is exactly Definition 3.3.
For a variable v, where $t = v$, we need to show

$$v^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f], \text{ where } f(x) = v^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x)) \text{ for every } x \in S.$$

But

$$v^{\mathfrak{A}}([f_1], \ldots, [f_n]) = ([f_1], \ldots, [f_n])(v) = [f_v]$$

and for every $x \in S$ we have

$$v^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x)) = (f_1(x), \ldots, f_n(x))(v) = f_v(x)$$

thus

$$v^{\mathfrak{A}}([f_1], \ldots, [f_n]) = [f_v], \text{ where } f_v(x) = v^{\mathfrak{A}_x}(f_1(x), \ldots, f_n(x)) \text{ for every } x \in S.$$

Now suppose $t = F(t_1, \dots, t_n)$, where F is a function and $t_1, \dots, t_n$ are terms. We assume (1) is true for $t_1, \dots, t_n$. Then

$$(F(t_1, \dots, t_n))^{\mathfrak{A}}([f_1], \dots, [f_n]) = F^{\mathfrak{A}}(t_1^{\mathfrak{A}}([f_1], \dots, [f_n]), \dots, t_n^{\mathfrak{A}}([f_1], \dots, [f_n]))$$

and from (1) for every i, $t_i^{\mathfrak{A}}([f_1], \dots, [f_n]) = [g_i]$,
where $g_i(x) = t_i^{\mathfrak{A}_x}(f_1(x), \dots, f_n(x))$ for every $x \in S$.

Thus we write $F^{\mathfrak{A}}([g_1], \dots, [g_n])$ and from Definition 3.3 we get

$$F^{\mathfrak{A}}([g_1], \dots, [g_n]) = [f], \text{ where } f(x) = F^{\mathfrak{A}_x}(g_1(x), \dots, g_n(x)) \text{ for every } x \in S.$$

But then

$$f(x) = F^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}(f_1(x), \dots, f_n(x)), \dots, t_n^{\mathfrak{A}_x}(f_1(x), \dots, f_n(x))) \text{ for every } x \in S.$$

This is equivalent to

$$f(x) = (F(t_1, \dots, t_n))^{\mathfrak{A}_x}(f_1(x), \dots, f_n(x)) \text{ for every } x \in S$$

proving the theorem for terms.

Atomic formulas: Consider an n-ary predicate P and an expression $\mathfrak{A} \vDash P(t_1, \dots, t_n)([f_1], \dots, [f_m])$, where $[f_1], \dots, [f_m]$ is the evaluation of m-many free variables in terms $(t_1, \dots, t_n)$. Then we have

$$\begin{aligned}
(2)\quad \mathfrak{A} \vDash & P(t_1, \dots, t_n)([f_1], \dots, [f_m]) \\
& \leftrightarrow P^{\mathfrak{A}}(t_1^{\mathfrak{A}}([f_1], \dots, [f_m]), \dots, t_n^{\mathfrak{A}}([f_1], \dots, [f_m])) \\
& \leftrightarrow \{x \in S \mid P^{\mathfrak{A}_x}(t_1^{\mathfrak{A}_x}(f_1(x), \dots, f_m(x)), \dots, t_n^{\mathfrak{A}_x}(f_1(x), \dots, f_m(x))\} \in U \\
& \leftrightarrow \{x \in S \mid \mathfrak{A}_x \vDash P(t_1, \dots, t_n)(f_1(x), \dots, f_m(x))\} \in U
\end{aligned}$$

Logical connectives: We assume the theorem holds for atomic formula φ and show it holds for $\neg\varphi$ as well (we use the ultrafilter property $X \in U$ if and only if $-X \notin U$).

$$\begin{aligned}
\mathfrak{A} \vDash \neg\varphi([f_1], \dots, [f_n]) \quad & \leftrightarrow \quad \text{not } \mathfrak{A} \vDash \varphi([f_1], \dots, [f_n]) \\
& \leftrightarrow \quad \{x \in S \mid \mathfrak{A}_x \vDash \varphi(f_1(x), \dots, f_n(x))\} \notin U \\
& \leftrightarrow \quad \{x \in S \mid \mathfrak{A}_x \nvDash \varphi(f_1(x), \dots, f_n(x)\} \in U \\
& \leftrightarrow \quad \{x \in S \mid \mathfrak{A}_x \vDash \neg\varphi(f_1(x), \dots, f_n(x)\} \in U.
\end{aligned}$$

Assuming the theorem holds for atomic formulas φ and ψ,

$$\begin{aligned}\mathfrak{A} \models \varphi \wedge \psi &\leftrightarrow \mathfrak{A} \models \varphi \text{ and } \mathfrak{A} \models \psi \\ &\leftrightarrow \{x \in S \mid \mathfrak{A}_x \models \varphi\} \in U \text{ and } \{x \in S \mid \mathfrak{A}_x \models \psi\} \in U \\ &\leftrightarrow \{x \in S \mid \mathfrak{A}_x \models \varphi \wedge \psi\} \in U.\end{aligned}$$

Last equivalence holds due to filter property $X \in U$ and $Y \in U$ if and only if $X \cap Y \in U$.

Existential quantifier: We assume the theorem holds for atomic formula φ and show it holds for the formula $\exists u \varphi$. Let

$$\mathfrak{A} \models \exists u \varphi([f_1], \dots, [f_n]). \tag{3}$$

Then there is some $g \in \prod_{x \in S} A_x$ such that $\mathfrak{A} \models \varphi([g], [f_1], \dots, [f_n])$ and since we assumed the theorem holds for φ, we have

$$\{x \in S \mid \mathfrak{A}_x \models \varphi(g(x), f_1(x), \dots, f_n(x))\} \in U. \tag{4}$$

But then there exists some u (in particular $u = g(x)$) such that

$$\{x \in S \mid \mathfrak{A}_x \models \exists u \varphi(f_1(x), \dots, f_n(x))\} \in U. \tag{5}$$

To show the other implication, assume (5). For each $x \in S$ let $u_x \in A_x$ be such that $\mathfrak{A}_x \models \varphi(u_x, f_1(x), \dots, f_n(x))$ if such u_x exists, and arbitrary otherwise. If we define $g \in \prod_{x \in S} A_x$ by $g(x) = u_x$, then we have (4) and from our assumption also

$$\mathfrak{A} \models \varphi([g], [f_1], \dots, [f_n]),$$

which implies (3). $\square$

Corollary 3.6 *Let U be an ultrafilter over S and let $\mathfrak{A}$ be the ultraproduct of $\{\mathfrak{A}_x \mid x \in S\}$ by U. If σ is a sentence, then*

$$\mathfrak{A} \models \sigma \textit{ if and only if } \{x \in S \mid \mathfrak{A}_x \models \sigma\} \in U.$$

When

$$\{x \in S \mid \mathfrak{A}_x \models \varphi(f_1(x), \dots, f_n(x))\} \in U$$

holds, we say that $\mathfrak{A}_x$ satisfies $\varphi(f_1(x), \dots, f_n(x))$ for *allmost all* x, or that $\mathfrak{A}_x \models \varphi(f_1(x), \dots, f_n(x))$ holds *almost everywhere*. Using this terminology, Łoś's theorem states that $\varphi([f_1],\dots,[f_n])$ holds in the ultraproduct if and only if for almost all x, $\varphi(f_1(x),\dots,f_n(x))$ holds in $\mathfrak{A}_x$.

Let us consider now a special case of ultraproducts. When each $\mathfrak{A}_x$ is the same model $\mathfrak{A}$, we call the ultraproduct an *ultrapower* of $\mathfrak{A}$; denoted $Ult_U\mathfrak{A}$.

Corollary 3.7 *An ultrapower $Ult_U\mathfrak{A}$ is elementary equivalent to $\mathfrak{A}$.*

Proof. Note that for $Ult_U\mathfrak{A}$ to be elementary equivalent to $\mathfrak{A}$ only requires that we show both of them satisfy the same sentences. By Corollary 3.6 we get $Ult_U\mathfrak{A} \vDash \sigma$ if and only if $\{x \mid \mathfrak{A} \vDash \sigma\} \in U$. For this to hold the set $\{x \mid \mathfrak{A} \vDash \sigma\}$ is either whole S when $\mathfrak{A} \vDash \sigma$ or empty when $\mathfrak{A} \nvDash \sigma$. $\square$

We further show that there exists an elementary embedding between $\mathfrak{A}$ and its ultrapower $Ult_U\mathfrak{A}$. If U is an ultrafilter over S, we define $j : \mathfrak{A} \to Ult_U\mathfrak{A}$ a *canonical embedding* as follows:

Definition 3.8 For each $a \in A$, let c_a be the constant function where $\forall x \in S$

$$c_a(x) = a \quad \text{and} \quad j(a) = [c_a].$$

Corollary 3.9 *The canonical embedding $j : \mathfrak{A} \to Ult_U\mathfrak{A}$ is an elementary embedding.*

Proof. Let $a_1,\dots,a_n \in A$, then, using Łoś's theorem, we obtain that $Ult_U\mathfrak{A} \vDash \varphi(j(a_1),\dots,j(a_n))$ if and only if $Ult_U\mathfrak{A} \vDash \varphi([c_{a_1}],\dots,[c_{a_n}])$ iff $\mathfrak{A} \vDash \varphi(a_1,\dots,a_n)$ for almost all x (this holds because we previously defined c_a a constant function with value a for every $x \in S$) if and only if $\mathfrak{A} \vDash \varphi(a_1,\dots,a_n)$. $\square$

3.2 Extending to proper classes

The last section described the ultrapower construction on sets. Our next goal should be to extend this construction to models that are proper classes.

The language of set theory has only one binary predicate $\in$ and so models of set theory consist of the universe M(so far we have only considered M to be a set) and a binary relation R, which interprets $\in$.

We want to extend the concept to consider models of set theory that are proper classes. We have to be careful how we make the generalization, in order not to obtain a theory proving its own consistence, because that is a contradiction with Gödel's Second Incompleteness Theorem.

Definition 3.10 Let M be a class, let R be a binary relation on M and let $\varphi(x_1, \dots, x_n)$ be a formula of the language of set theory. The *relativization* of φ to M, R is the formula

$$\varphi^{M,R}(x_1, \dots, x_n) \tag{6}$$

with the folowing inductive definition:

$$\begin{aligned} (x \in y)^{M,R} &\leftrightarrow x \; R \; y \\ (x = y)^{M,R} &\leftrightarrow x = y \\ (\neg\varphi)^{M,R} &\leftrightarrow \neg\varphi^{M,R} \\ (\varphi \wedge \psi)^{M,R} &\leftrightarrow \varphi^{M,R} \wedge \psi^{M,R} \\ (\exists x \varphi)^{M,R} &\leftrightarrow (\exists x \in M)\varphi^{M,R} \end{aligned}$$

Other connectives and $\forall$ quantifier are defined similarly.

If R is $\in$, we write φ^M instead of $\varphi^{M,\in}$.

When using $\varphi^{M,R}(x_1, \dots, x_n)$, it is usually assumed variables $x_1, \dots, x_n \in M$ to make sense. Instead of (6),

$$(M, R) \vDash \varphi(x_1, \dots, x_n)$$

is used to describe that the model (M, R) satisfies φ. It should be noted though, that while it's clearly without a problem in every particular case of φ, the general satisfaction relation is formally undefinable in ZF.

Now we use the generalized technique to construct ultrapowers of the universe. Let U be an ultrafilter over a set S. Consider the class of all functions with domain S. Using Theorem 3.5 and Corollary 3.6 let

$$f =^* g \quad \text{if and only if} \quad \{x \in S \mid f(x) = g(x)\} \in U, \tag{7}$$

$$f \in^* g \quad \text{if and only if} \quad \{x \in S \mid f(x) \in g(x)\} \in U. \tag{8}$$

By Lemma 3.4, $\in^*$ again does not depend on the choice of representatives.

The *rank* function is then defined as follows:

$$\text{rank}(x) = \text{the least } \alpha \text{ such that } x \in V_{\alpha+1}.$$

Each V_α is the collection of all sets of rank less than α and also

(i) If $x \in y$ then rank$(x)<$ rank(y).

(ii) rank(α)=α.

For each f we denote $[f]$ the equivalence class of f in $=^*$:

(9) $$[f] = \{g \mid f =^* g \text{ and } \forall h(h =^* f \to \text{rank}(g) \leq \text{rank}(h))\}.$$

Note that $[f]$ in (9) is not the actual equivalence class of f in $=^*$. It is just a nonempty subset of the equivalence class f. The reason for making this trick is that the real equivalence class $[f] = \{g \mid g =^* f\}$ is a proper class and we cannot consider the class Ult(U,V) $= \{[f] \mid f : \kappa \to \mathrm{V}\}$ for formal reasons. (9) is a way to get around this problem.

Let Ult(U,V) = Ult$_U$(V) be the class of all $[f]$, where $f : S \to \mathrm{V}$ is a function. Now consider the model (Ult(U,V),$\in^*$). Łoś's Theorem holds in this context as well: Let $\varphi(x_1, \ldots, x_n)$ be a formula of set theory, then

$$\text{Ult(U,V)} \vDash \varphi([f_1], \ldots, [f_n]) \quad \text{iff} \quad \{x \in S \mid \mathrm{V} \vDash \varphi(f_1(x), \ldots, f_n(x))\} \in U$$

is true. If σ is a sentence, then Ult(U,V) $\vDash \sigma$ if and only if σ holds in V. From Corollary 3.7 Ult(U,V) is elementary equivalent to the universe (V,$\in$). The canonical embedding j is defined exactly according to Definition 3.8. Thus by Corrolary 3.9, j is an elementary embedding such that j :V$\to$ Ult(U,V) which gives us:

$$\begin{aligned} \text{Ult}(U,\mathrm{V}) \vDash \varphi([c_{a_1}], \ldots, [c_{a_n}]) &\quad \text{if and only if} \quad \mathrm{V} \vDash \varphi(c_{a_1}(x), \ldots, c_{a_n}(x)) \\ \text{Ult}(U,\mathrm{V}) \vDash \varphi(j(a_1), \ldots, j(a_n)) &\quad \text{if and only if} \quad \mathrm{V} \vDash \varphi(a_1, \ldots, a_n) \end{aligned}$$

whenever $\varphi(x_1, \ldots, x_n)$ is a formula of set theory. As we have shown, there is an elementary embedding from V to Ult(U,V).

Note that since $\in^*$ does not depend on the choice of representatives, we may write $f \in^* g$ when $[f] \in^* [g]$. We will further use just the notation $[f] \in^* [g]$.

The most important application of ultrapowers in set theory are those in which (Ult(U,V),$\in^*$) is well-founded. As we show below, well-founded ultrapowers are closely related to measurable cardinals.

Definition 3.11 The model (Ult(U,V),$\in^*$) is *well-founded*, if every nonempty set $X \subseteq$ Ult(U,V) has $\in^*$-minimal element.

Since the above definition is quite impractical to use, we introduce the following characterization.

Proposition 3.12: The following statements are equivalent:

(i) Every nonempty set $X \subseteq \mathrm{Ult}(U,\mathrm{V})$ has $\in^*$-minimal element.
(ii) There exists no infinite descending $\in^*$-sequence

$$[f_0] \ni^* [f_1] \ni^* \ldots \ni^* [f_k] \ni^* \ldots \ (k \in \omega).$$

Proof. From left to right we assume (i) holds, but that there is an infinite descending $\in^*$-sequence in $\mathrm{Ult}(U,\mathrm{V})$. However, the infinite descending $\in^*$-sequence is a nonempty subset of $\mathrm{Ult}(U,\mathrm{V})$, thus has no minimal element. That contradicts (i).

From right to left we assume (ii) holds, but that there exists a nonempty $X \subseteq \mathrm{Ult}(U,\mathrm{V})$ without $\in^*$-minimal element. However, X must then contain an infinite descending $\in^*$-sequence contradicting (ii). □

By the following Lemma, a σ-complete ultrafilter is sufficient for $(\mathrm{Ult}(U,\mathrm{V}),\in^*)$ to be a well-founded model.

Lemma 3.13 *If U is a σ-complete ultrafilter, then* $(\mathrm{Ult}(U,\mathrm{V}), \in^*)$ *is a well-founded model.*

Proof. We shall prove exactly what we already put forth; that for U a σ-complete ultrafilter over S, there is no infinite descending $\in^*$-sequence.

Let us assume that $[f_0], [f_1], \ldots, [f_n], \ldots,$ is such a descending sequence. Thus each set

$$X_n = \{x \in S \mid f_{n+1}(x) \in f_n(x)\}$$

is in the ultrafilter. From σ-completeness of U the intersection $X = \bigcap_n X_n$ is also in U and therefore nonempty. Let x be an arbitrary element of X. Then we get

$$f_0(x) \ni f_1(x) \ni \ldots \ni f_n(x) \ni \ldots$$

an infinite descending $\in$-sequence, which is a contradiction with axiom of foundation, which implies there is no infinite descending $\in^*$-sequence. □

3.3 Mostowski's Collapsing Theorem

In this section we prove Mostowski's Collapsing Theorem, which is essential for the proof of Dana Scott's theorem. It shows that the transitive collapse of an extensional set-like well-founded relation is one-to-one and that every extensional class is $\in$-isomorphic to a transitive class.

But first we define

Definition 3.14 Let a, b be sets and let R be a binary relation. Then

$$\mathrm{ext}_R(a) = \{b \mid b \; R \; a\}$$

is a *R-extension* of a.

Definition 3.15 Let X be a class and let R be a binary relation on X. We say that R is *set-like* if for every $a \in X$, $\mathrm{ext}_R(a)$ is a set.

Proposition 3.16: The relation $\in^*$ is set-like.

Proof. Let $[f], [g] \in A$. The relation $\in^*$ is then set-like if

$$\mathrm{ext}_{\in^*}([f]) = \{[g] \mid [g] \; \in^* \; [f]\}$$

is a set.

For every equivalence class $[g]$ define a function g' such that $g(x) = g'(x)$ whenever $g(x) \in f(x)$ and $g'(x) = 0$ otherwise.

Since $[g] \in^* [f]$ holds, we know that $g =^* g'$. It follows that every $[g] \in^* [f]$ can be represented by a function g' from S to $\bigcup \mathrm{rng}(f) \cup \{0\}$. Since f is a set, we have $\bigcup \mathrm{rng}(f) \cup \{0\}$ is a set and since there are only set-many functions from a set S to another set, there are only set-many such g'. It follows that there are only set-many elements $[g]$ such that $[g] \in^* [f]$. □

Definition 3.17 A relation R on a class X is *well-founded* if every nonempty subset Y of X has an R-minimal element.

By Proposition 3.12, $\in^*$ is well-founded.

Definition 3.18 A well-founded relation R on a class X is *extensional* if

$$\mathrm{ext}_R(Y) \neq \mathrm{ext}_R(Z) \tag{10}$$

whenever Y and Z are distinct elements of X.

A class M is *extensional* if the relation $\in$ on M is extensional, i.e., if for any distinct X and $Y \in M$, $X \cap M \neq Y \cap M$.

Proposition 3.19: The relation $\in^*$ is extensional.

Proof. Let $[f], [g] \in A$. Assume for contradiction that

$$ext_{\in^*}([g]) \neq ext_{\in^*}([f]) \text{ and } f =^* g.$$

Therefore

$$\{[h_1] \mid [h_1] \in^* [g]\} \neq \{[h_2] \mid [h_2] \in^* [f]\} \text{ and } f =^* g$$

holds. Using Łoś's theorem we get

$$\{[h_1] \mid \{x \in S \mid h_1(x) \in g(x)\} \in U\} \neq \{[h_2] \mid \{x \in S \mid h_2(x) \in f(x)\} \in U\}$$
$$\text{and } \{x \in S \mid g(x) = f(x)\} \in U.$$

But then

$$\begin{aligned}\{[h_1] \mid \{x \in S \mid h_1(x) \in g(x)\} \in U\} &= \{[h_1] \mid \{x \in S \mid h_1(x) \in f(x)\} \in U\} \\ &= \{[h_2] \mid \{x \in S \mid h_2(x) \in f(x)\} \in U\}\end{aligned}$$

and that is a contradiction. □

Following theorems come from [3].

Theorem 3.20 (∈-Induction) *Let T be a transitive class and let ϕ be a property. Assume that:*

(i) $\phi(\emptyset)$,
(ii) *if $x \in T$ and $\phi(z)$ holds for every $z \in x$, then $\phi(x)$.*

Then every $x \in T$ has the property ϕ.

Proof. Let C be the class of all $x \in T$ that do not have the property ϕ. If C is nonempty, then it has some ∈-minimal element x. Now, using (i), we have $x \neq \emptyset$ and from minimality of x we see for every $z \in x$, $\phi(z)$ holds. Thus, using (ii), we obtain $\phi(x)$ holds, but that is a contradiction with $x \in C$. □

Theorem 3.21 (Well-Founded Induction) *Let R be a well-founded relation on a class X. Let ϕ be a property. Assume that:*

(i) *Every R-minimal element x of X has property ϕ,*
(ii) *if $x \in X$ and $\phi(z)$ holds for every z such that $z \; R \; x$, then $\phi(x)$.*

Then every $x \in X$ has the property ϕ.

Proof. Similarly as in $\in$-induction. Let C be the class of all $x \in X$ that do not have the property ϕ. If C is nonempty, then it has some R-minimal element x. Now, using (i), we get x is not minimal in X. Thus from minimality of x in C we obtain that for every z satisfying $z \; R \; x$, $\phi(z)$ holds. Hence, using (ii), $\phi(x)$ holds as well and that is a contradiction with $x \in C$. $\square$

Theorem 3.22 (Well-Founded Recursion) *Let R be a well-founded relation on a class X. Let $G : X \times V \to V$ be a function. Then there is a unique function $F : X \to V$ such that*

$$F(x) = G(x, F \restriction \mathrm{ext}_R(x)) \tag{11}$$

for every $x \in X$.

Proof. We first show the uniqueness of F. Let F, F' both satisfy (11) and assume they are not the same. Because R is a well-founded relation, there is some $x \in X$, which is R-minimal, with the property that $F(x) \neq F'(x)$. But now by (11),

$$F(x) = G(x, F \restriction \mathrm{ext}_R(x)) = G(x, F' \restriction \mathrm{ext}_R(x)) = F'(x).$$

That is a contradiction.

Let $F(x) = y$, if there exists a function f such that $\mathrm{dom}(f) \subseteq X$ and

$$\begin{aligned} \forall z \in \mathrm{dom}(f), \quad & f(z) = G(z, f \restriction \mathrm{ext}_R(z)) \text{ and} \\ & f(x) = y. \end{aligned}$$

The existence of F satisfying (11) comes from Theorem 3.21. For more details see [4], p.103. $\square$

Example 3.23 (The Rank Function) We define by induction for all $x \in X$ and R a well-founded relation:

$$\rho(x) = \sup\{\rho(z) + 1 \mid z \; R \; x\}.$$

The range of ρ is either an ordinal or the class Ord. For all $x, y \in X$,

$$x \; R \; y \to \rho(x) < \rho(y).$$

The last example is just a conversion of the former rank function definition.

Theorem 3.24 *Let M_1, M_2 be transitive classes and let π be an $\in$-isomorphism of M_1 onto M_2, i.e., π is one-to-one and*

$$x \in v \leftrightarrow \pi(x) \in \pi(v).$$

Then $M_1 = M_2$ and $\pi(x) = x$ for every $x \in M_1$.

Proof. Assume that $\pi(z) = z$ for each $z \in x$ and let $y = \pi(x)$.

We have $x \subseteq y$ because if $z \in x$, then $z = \pi(z) \in \pi(x) = y$ and $z \in y$.

We also have $y \subseteq x$: Let $t \in y$. Since $y \in M_2$, there is $z \in M_1$ such that $\pi(z) = t$. Since $\pi(z) \in y$, we have $z \in x$, and so $t = \pi(z) = z$. Thus $t \in x$.

Therefore $\pi(x) = x$ for all $x \in M_1$, and $M_2 = M_1$. □

Now we finally have all we need to prove the following Theorem.

Theorem 3.25 (Mostowski's Collapsing Theorem)

(i) *If R is a well-founded set-like extensional relation on a class X, then there is a transitive class M and an isomorphism π between (X, R) and $(M, \in)$. The transitive class M and the isomorphism π are unique.*
(ii) *In particular, every extensional class X is isomorphic to a transitive class M. The transitive class M and the isomorphism π are unique.*
(iii) *In case* (ii), *if $T \subseteq X$ is transitive, then $\pi x = x$ for every $x \in T$.*

Proof. Because (ii) is just a special case of (i), we will aim to prove the existence of an isomorphism in the general case and get (ii) in the process.

Since R must be a well-founded set-like relation, we define π in the following way: Every $\pi(x)$ will be defined throught $\pi(z)$'s, where $z \; R \; x$. We let for each $x \in X$

$$\pi(x) = \{\pi(z) \mid z \; R \; x\}. \tag{12}$$

This definition is correct because of Theorem 3.22. The function π maps X onto a class $M = \pi(X)$. To see M is transitive, take $x, y \in X$. If $x \; R \; y$, then $\pi(x) \in \pi(y)$ from (12). From Example 3.23 we get $\mathrm{rank}(\pi(x)) < \mathrm{rank}(\pi(y))$, also $\pi(x) \in V_\alpha$ for some α and $\pi(y) \in V_\beta$ for some $\beta > \alpha$. Since $\beta > \alpha$, $V_\alpha \subseteq V_\beta$ and thus $\pi(x) \subseteq \pi(y)$.

To show π is also one-to-one, we use the extensionality of R. Let $z \in M$ be of least rank such that $z = \pi(x) = \pi(y)$ for some $x \neq y$. Then from Definition 3.18, $\mathrm{ext}_R(x) \neq \mathrm{ext}_R(y)$ hence there is $u \in \mathrm{ext}_R(x)$ such that $u \notin \mathrm{ext}_R(y)$. Let

$t = \pi(u)$. Since $t \in z$, where $z = \pi(y)$, there is $v \in \mathrm{ext}_R(y)$ such that $t = \pi(v)$. Thus we have $t = \pi(u) = \pi(v)$ and $u \neq v$. But since $t \in z$, t is of lesser rank than z and thus z isn't of least rank; a contradiction.

Now it's easy to prove

$$x \; R \; y \leftrightarrow \pi(x) \in \pi(y). \tag{13}$$

If $x \; R \; y$ holds, then from (12) $\pi(x) \in \pi(y)$. For the second implication, if $\pi(x) \in \pi(y)$, then again by (12) for some $z \; R \; y$, $\pi(x) = \pi(z)$ and since we proved π is one-to-one, we have $x = z$ and so $x \; R \; y$ as well.

To see π and the transitive class $M = \pi(X)$ are unique:
If π_1 is an isomorphism of X onto M_1, π_2 is an isomorphism of X onto M_2, then $\pi_2\pi_1^{-1}$ is an isomorphism between M_1 and M_2. From Theorem 3.24 we get that $M_1 = M_2$ and $\pi_2\pi_1^{-1}$ is an identity mapping. Thus $\pi_1 = \pi_2$.

It only remains to prove (iii). If $T \subseteq X$ is transitive, then we first observe that $x \subseteq X$ for every $x \in T$ and then $x \cap X = x$ and also

$$\pi(x) = \{\pi(z) \mid z \in x\}$$

for all $x \in T$. Obviously $\pi(\emptyset) = \emptyset$ and also if $x \in T$ and $\pi(z) = z$ holds for every $z \in x$, then $\pi(x) = \pi(z) = \{z \mid z \in x\} = x$. Now, using Theorem 3.20, we get $\pi(x) = x$ for all $x \in T$. □

3.4 Critical point

Now we recall all the previous results to prove the theorem of Danna Scott published in [1], which distinguishes universe of sets from Gödel's constructible universe under the assumption that there is a measurable cardinal.

In order to use Mostowski's Collapsing Theorem, we need a well-founded set-like extensional relation R on class X, that assures the existence of an isomorphism π between X and a transitive class M, both being unique.

Let $(\mathrm{Ult}(U,\mathrm{V}),\in^*) = (X, R)$. We have already proven $\in^*$ is well-founded, set-like and extensional relation if U is a σ-complete ultrafilter. By Mostowski's Collapsing Theorem there exists a one-to-one mapping π of $\mathrm{Ult}(U,\mathrm{V})$ onto a transitive class M, such that $[f] \in^* [g]$ if and only if $\pi([f]) \in \pi([g])$. To simplify the notation we will identify each $[f]$ with its image $\pi([f])$.

Therefore if U is a σ-complete ultrafilter, then $\mathrm{Ult}(U,\mathrm{V})$ denotes the transitive collapse of the ultrapower and for each function f on S, $[f]$ is an element of the transitive class $\mathrm{Ult}(U,\mathrm{V})$. We say that function f represents $[f] \in \mathrm{Ult}(U,\mathrm{V})$. Again if U is a σ-complete ultrafilter, then $M = \text{Ult}(U,\text{V})$ is an inner model and j is an elementary embedding $j : \text{V} \to M$ (see Corollary 3.7).

If α is an ordinal and since j is an elementary embedding, then $j(\alpha)$ is an ordinal as well. Moreover if $\alpha < \beta$, then $j(\alpha) < j(\beta)$. This implies $\alpha \leq j(\alpha)$ for every ordinal number α. Since $\alpha + 1$ is a successor ordinal of α, we get $j(\alpha + 1) = j(\alpha) + 1$ and $j(n) = n$ from the definition of j.

In order to prove $j(\omega) = \omega$, assume that $[f] < \omega$, then $f(x) < \omega$ for almost all $x \in S$ and by σ-completeness, for almost all x there is some $n < \omega$ such that $f(x) = n$. Using the same argument for λ-complete ultrafilter U, we obtain $j(\gamma) = \gamma$ for all $\gamma < \lambda$.

Definition 3.26 Let κ be a measurable cardinal and let U be a non-principal κ-complete ultrafilter over κ. Then d is the *diagonal function* on κ defined as follows:

$$\text{For every } \alpha < \kappa,\ d(\alpha) = \alpha.$$

Lemma 3.27 *Let κ be a measurable cardinal, let U be a non-principal κ-complete ultrafilter over κ and let d be the diagonal function on κ. Let $j : \text{V} \to M$ be an elementary embedding. Then for every $\gamma < \kappa$, $\gamma = j(\gamma) < [d]$; in particular $\kappa \leq [d]$.*

Proof. Since U is κ-complete, it holds for every $\gamma < \kappa$, that $\gamma \notin U$. Thus we have $\gamma < d(\alpha)$ for almost all α. Hence $\gamma = j(\gamma) < [d]$ for every $\gamma < \kappa$ and also $\kappa \leq [d]$. □

From Lemma 3.27 we have $[d] \geq \kappa$ for every $\gamma < \kappa$. We also have $[d] < j(\kappa)$ because $\alpha = d(\alpha) < \kappa$ for every $\alpha < \kappa$ and then $j([d]) < j(\kappa)$ from the definition of j. We also get $[d] = j[d]$ thus $[d] < j(\kappa)$ from the definition. We obtain $j(\kappa) > \kappa$.

Now we can finally prove Scott's theorem mentioned at the start of this section.

Theorem 3.28 (Scott) *Let $\mathbf{L}$ be the class of all constructible sets. If there is a measurable cardinal, then $\text{V} \neq \mathbf{L}$.*

Proof. Assume that $\text{V} = \mathbf{L}$ and let κ be the least measurable cardinal. Let U be a non-principal κ-complete ultrafilter over κ and let $j : \text{V} \to M$ be an elementary embedding. From the argument above, $j(\kappa) > \kappa$.

Since we assumed $V = \mathbf{L}$ and we know $\mathbf{L}$ is the least inner model, the only transitive model containing all ordinals is the universe itself. We have $V = M = \mathbf{L}$ and the mapping $j : V \to V$. Because κ is the least measurable cardinal, we get

$$V \vDash j(\kappa) \text{ is the least measurable cardinal.}$$

But then $j(\kappa)$ is the least measurable cardinal. This is a contradiction, since $j(\kappa) > \kappa$ and κ is the least measurable cardinal. $\square$

Remark: It is now clear, that the existence of a measurable cardinal has a very interesting consequence, i.e., existence of a nontrivial elementary embedding of the universe into a transitive model. It can be shown conversely, that if $j : V \to M$ is a nontrivial elementary embedding, then there exists a measurable cardinal.

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