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DIPLOMOVÁ PRÁCE

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Vlastnosti grafů velkého obvodu

(Properties of graphs with large girth)

Katedra aplikované matematiky

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Název práce: Vlastnosti grafů velkého obvodu

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Abstrakt: V práci zkoumáme dva náhodné procesy pro kubické grafy velkého obvodu. První proces nalezne pravděpodobnostní distribuci na hranových řezech takovou, že každá hrana je v náhodně vybraném řezu s pravděpodobností alespoň 0.88672. Jako důsledek odvodíme dolní odhad na velikost největšího řezu pro kubické grafy velkého obvodu a pro náhodné kubické grafy, a dále též horní odhad na váhu nejmenšího zlomkového pokrytí hranovými řezy pro kubické grafy velkého obvodu.
Druhý proces nalezne pravděpodobnostní distribuci na nezavislých množinách takovou, že každý vrchol je v nezávislé množině s pravděpodobností alespoň 0.4352. Z toho plyne dolní odhad na velikost největší nezavíslé množiny pro kubické grafy velkého obvodu a pro náhodné kubické grafy, a dále též horní odhad na zlomkovou barevnost pro kubické grafy velkého obvodu.

Klíčová slova: obvod, kubický graf, náhodné grafy, maximální řez, nezávislá množina, zlomkové obarvení

Title: Properties of graphs with large girth

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Abstract: In this work we study two random procedures in cubic graphs with large girth. The first procedure finds a probability distribution on edge-cuts such that each edge is in a randomly chosen cut with probability at least 0.88672. As corollaries, we derive lower bounds for the size of maximum cut in cubic graphs with large girth and in random cubic graphs, and also an upper bound for the fractional cut covering number in cubic graphs with large girth.
The second procedure finds a probability distribution on independent sets such that each vertex is in an independent set with probability at least 0.4352. This implies lower bounds for the size of maximum independent set in cubic graphs with large girth and in random cubic graphs, as well as an upper bound for the fractional chromatic number in cubic graphs with large girth.

Keywords: girth, cubic graph, random graphs, maximum cut, independent set, fractional coloring

Contents

Chapter 1

Introduction

In this work we give lower bounds for the size of the maximum independent set and the maximum edge-cut in cubic graphs with large girth. We prove that there exists an integer g such that every n-vertex cubic graph with girth at least g contains an edge-cut of size at least $1.33008n$ and an independent set of size at least $0.4352n$. These bounds also translates to asymptotic lower bounds for the size of the maximum edge-cut and the maximum independent set in random cubic graphs. We also show that every cubic graph with girth at least g has the fractional chromatic number at most 2.2978 and the fractional cut covering number at most 1.12776. The methods we present here can also be generalized to r-regular graphs for $r \geq 4$ by considering more complicated arguments.

This work extends my bachelor thesis defended in 2010 at Charles University in Prague and the results presented here are also a part of my joint papers with František Kardoš and Daniel Kráľ [13, 14].

1.1 Preliminaries

The reader is referred to the book of Bondy and Murty [2] for the basic notation from graph theory and to the book of Alon and Spencer [1] for that from probability theory. All graphs considered in this work are without loops and multiple edges.

An *independent set* is a subset of vertices such that no two of them are adjacent. The *independence number* $\alpha(G)$ of a graph G is the size of the largest independent set in G. An *edge-cut* or simply a *cut* in a graph $G = (V, E)$ defined by $X \subseteq V$ is the set of edges with exactly one end vertex in X (and exactly one end vertex in $V \setminus X$) and *maximum (edge-)cut* is simply the cut with the maximum number of edges. The *girth* of a graph G is the length of the shortest cycle of G.

1.2 Fractional colorings and cut coverings

A *fractional coloring* of a graph G is an assignment of non-negative weights to independent sets in G such that for each vertex v of G the sum of weights of the sets containing v is at least one. The *fractional chromatic number* $c_f(G)$ of G is the minimum sum of weights of independent sets forming a fractional coloring.

A parameter analogous to a fractional coloring is a *fractional cut covering* of a graph G. It was first introduced by Šámal [20] (under the name cubical colorings). He also related this parameter to graph homomorphisms. The aim now is to assign non-negative weights to edge-cuts in G such that for each edge e of G the sum of weights of the cuts containing e is at least one. The *fractional cut covering number* is the minimum sum of weights of cuts forming a fractional cut covering.

1.3 Random cubic graphs

A random n-vertex cubic graph is a graph on the vertex set $[n] = \{1, 2, \ldots, n\}$ uniformly chosen among all cubic graphs on $[n]$. A detailed description and basic properties of this model can be found for example in [12].

Many properties of random cubic graphs (and more generally random regular graphs) and cubic (regular) graphs with large girth are closely related. On one hand, Wormald showed in [21] that a random cubic graph asymptotically almost surely (a.a.s.) contains only $o(n)$ cycles shorter than a fixed integer g. Therefore, a.a.s. we can remove a small number (this means $o(n)$) of vertices to obtain a subgraph with large girth and only $o(n)$ vertices of degree less than three. This idea could be applied for translating lower bounds on the size of maximum independent set or maximum cut as well as for translating bounds on many other graph parameters.

On the other hand, Hoppen and Wormald [11] have recently developed a technique for translating many results for random r-regular graphs to r-regular graphs with sufficiently large girth. In particular, they are able to translate bounds obtained by analyzing the performance of so called locally greedy algorithms for a random regular graphs. These algorithms and their analysis provide the currently best known asymptotic bounds to many parameters of random regular graphs, for example an upper bound on the size of the smallest dominating set [4]. The main tool for the analysis of such algorithms as well as for analysis of many other random processes is the *differential equation method* developed by Wormald. An overview of the method and some of its applications can be found in [23].

Chapter 2

Edge-cuts

In this chapter, we present our results on edge-cuts and fractional cut coverings in cubic graphs with large girth and edge-cuts in random cubic graphs. We improve the best known lower bound on the size of the maximum cut in a cubic graph with sufficiently large girth as well as the best known asymptotic lower bound on the size of the maximum cut in a random cubic graph.

Previously the best known lower bound for the size of the maximum cut in cubic graphs with large girth is $9n/7 - o(n) = 1.28571n - o(n)$ given by Zýka [24]. The lower bound for that in random cubic graphs is $1.32595n$ given by Díaz, Do, Serna and Wormald [3]. On the other hand, the best known upper bound is $0.9351m = 1.4026n$ which applies to both cases. The upper bound was first announced by McKay [16]; the proof can be found in [9].

The main result of this chapter is the following.

Theorem 2.1. *If G is a cubic graph with girth at least $16\,353\,933$, then there exists a probability distribution such that each edge of G is contained in an edge-cut drawn according to this distribution with probability at least 0.88672.*

Before we start with the proof of Theorem 2.1, we state three corollaries of this theorem.

By considering the expected size of a cut drawn according to the distribution from Theorem 2.1, we get the following.

Corollary 2.2. *Every n-vertex cubic graph with girth at least $16\,353\,933$ contains an edge-cut of size at least $1.33008n$.*

Since a random cubic graph asymptotically almost surely contains only $o(n)$ cycles shorter than a fixed integer g [21], the lower bound on the size of an edge-cut also translates to random cubic graphs.

Corollary 2.3. *A random n-vertex cubic graph asymptotically almost surely contains an edge-cut of size at least $1.33008n$.*

Proof. Let G be a randomly chosen n-vertex cubic graph. The results of [21] imply then we can a.a.s. remove $o(n)$ vertices and obtain a subcubic subgraph G' with girth at least $16\,353\,933$. Let n_1 and n_2 be the numbers of vertices of G' with degree one and two, respectively. Observe that $n_1 + n_2 = o(n)$.

Let R be a $(2n_1 + n_2)$-regular graph with girth at least $16\,353\,933$. Replace each vertex of R with a copy of G' in such a way that the edges of R are incident with vertices of degree one and two in the copies of G' and the resulting graph is cubic. Observe that the obtained graph H has girth at least 16 353 933. Applying Corollary 2.2 to H yields an edge-cut C of size at least $1.33008N$ where N is the number of vertices of H. Observe that the number of the edges corresponding to those of R among all the edges of H is $o(N)$. Therefore, at least one copy of G' in H contains at least $1.33008n - o(n)$ edges of C. $\square$

The last corollary relates Theorem 2.1 to the problem of fractional coverings the edges with edge-cuts. We show how to construct from the probability distribution given by Theorem 2.1 a fractional cut covering.

Corollary 2.4. *Every n-vertex cubic graph G with girth at least* 16 353 933 *has the fractional cut covering number at most* 1.12776.

Proof. Consider the probability distribution given by Theorem 2.1 for G. If the probability of a cut C to be drawn in this distribution is $p(C)$, assign C weight $p(C)/0.88672$. It is straightforward to verify that we have obtained a fractional cut covering of weight $1/0.88672 = 1.12776$. $\square$

2.1 Structure of the proof

Our proof is inspired by the method which was developed by Hoppen in [10] for obtaining lower bounds on independent sets and induced forests. In order to prove Theorem 2.1, we design a randomized procedure for obtaining an edge-cut (of large size). The key tool for our analysis is the independence lemma (Lemma 2.6) which is given in Section 2.3. This lemma is used for to simplify the recurrence relations appearing in the analysis. The recurrences describing the behavior of the randomized procedure are derived in Section 2.4. The actual performance of the procedure is based on setting up the parameters of the procedure and solving the recurrences numerically. This is discussed in Section 2.5.

The sought probabilistic distribution is obtained by processing a cubic graph $G = (V, E)$ by the procedure which produces an edge-cut of it. The graph is processed in a fixed number of rounds K. The required assumption on the girth of G will depend only on the number K. We will iteratively construct two disjoint subsets $R \subseteq V$ and $B \subseteq V$; the vertices contained in R are referred to as red vertices and those in B as blue ones. The aim of the procedure is to maximize the number of red-blue edges. The vertices that are not red nor blue will be called white.

All vertices are initially white. In every round, each white vertex is recolored to red or blue with a certain probability depending on the number of its red and blue neighbors, as well as on the number of current round. Once a vertex is colored red or blue, its color stays the same in all the remaining rounds of the procedure.

2.2 Detailed description

We now describe the randomized procedure in more detail. We first introduce some notation. Let $I_j := \{(r, b) : r \in \mathbb{N}_0, b \in \mathbb{N}_0, r + b \leq j\}$, i.e., the set I_j contains all pairs r and b of non-negative integers such that $r + b \leq j$. For example, $I_2 = \{(0,0), (0,1), (1,0), (1,1), (2,0), (0,2)\}$. Note that $|I_j| = \binom{j+2}{2}$. Let

$G = (V, E)$ be a cubic graph and v a vertex of G. Throughout the analysis, $r(v)$ will refer to the number of red neighbors of v and $b(v)$ to the number of its blue neighbors. Therefore, $3 - r(v) - b(v)$ is the number of the white neighbors of v. If the vertex v is clear from the context, we just use r and b instead of $r(v)$ and $b(v)$.

Our randomized procedure is parametrized by the following parameters:

- an integer K,
- probabilities $P_k^{r,b}(W)$ for all $k \in [K]$ and $(r, b) \in I_3$,
- probabilities $P_k^{r,b}(R)$ for all $k \in [K]$ and $(r, b) \in I_3$ and
- probabilities $P_k^{r,b}(B)$ for all $k \in [K]$ and $(r, b) \in I_3$.

We require that it holds that $P_k^{r,b}(W) + P_k^{r,b}(R) + P_k^{r,b}(B) = 1$ for all $k \in [K]$ and $(r, b) \in I_3$.

The integer $K \in \mathbb{N}_0$ denotes the number of rounds that are performed. Throughout the procedure, vertices of the input graph G have one of the three colors: white (W), red (R) and blue (B). Let $W_k \subseteq V(G)$ denotes the set of white vertices after the k-th round. Analogously, we define R_k and B_k as the sets of red vertices and blue vertices, respectively. As we have already mentioned, at the beginning of the process $W_0 := V, R_0 := \emptyset$ and $B_0 := \emptyset$. For $(r, b) \in I_3$ we define $W_k^{r,b} \subseteq W_k$ to be the set of white vertices with exactly r red neighbors and b blue neighbors. Hence the sets $W_k^{r,b}$ forms a partition of W_k for every k. Note that $W_0^{0,0} = V$ and $W_0^{r,b} = \emptyset$ for all $(0, 0) \neq (r, b) \in I_3$.

Consider the coloring of G obtained after the k-th round. The $(k+1)$-th round of the procedure is performed as follows. Let v be a vertex from $W_k^{r,b}$. With the probability $P_{k+1}^{r,b}(R)$ we change the color of v to red, with the probability $P_{k+1}^{r,b}(B)$ we recolor it to blue and with the probability $P_{k+1}^{r,b}(W)$ it remains white. If v is after the k-th round colored red or blue, it will not change its color during the $(k + 1)$-th round.

Before we can proceed further, we have to introduce some additional notation. For a vertex $v \in V(G)$ let T_v^d denote the subgraph of G induced by vertices at the distance from v at most d. Observe that if the girth of G is larger than $2d+1$, then the subgraph T_v^d is a tree.

Now we show that if the girth of G is sufficiently large, then the probabilities that after the k-th round a vertex v has white, red or blue color, respectively, do not depend on the choice of v. We start with the following proposition.

Proposition 2.5. *Let G be a cubic graph with girth at least $2K$ and v a vertex of G. For every $k \in [K]$ the probability that the subgraph T_v^{K-k} has after the k-th round a certain coloring is determined by the coloring of T_v^{K-k+1} after the $(k-1)$-th round.*

Proof. The color of a vertex $u \in T_v^{K-k}$ after the k-th round depends only on the colors of u and its neighbors after the $(k-1)$-th round. Since all the neighbors of u are contained in T_v^{K-k+1}, the proposition follows. □

Suppose that the girth of G is at least $2K$. Since for any $k \in [K]$ the structure of a subgraph T_v^{K-k} does not depend on the choice of v, i.e., it is always a tree with all inner vertices of degree three, using a simple inductive argument together with Proposition 2.5 we conclude that the following probabilities do not depend on the choice of v:

$$w_k := \mathbf{P}\big[v \in W_k\big]\ , \quad r_k := \mathbf{P}\big[v \in R_k\big]\ , \quad b_k := \mathbf{P}\big[v \in B_k\big]\ .$$

Analogously, for any $k \in [K-1]$ and $(r, b) \in I_3$ the probability that after the k-th round a vertex v is white and has r red neighbors and b blue neighbors does not depend on the choice of v as well. Therefore, we can define

$$w_k^{r,b} := \mathbf{P}\big[v \in W_k^{r,b} \ \big|\ v \in W_k\big]\ .$$

Also observe that if the girth of G is larger than $2d+2$, then for every edge $uv \in E(G)$ the subgraph of G induced by vertices x satisfying $\min\{d(x, u), d(x, v)\} \leq d$ is a tree. If the girth of G is at least $2K + 1$, the same reasoning as before yields the following. The probability that for an edge $uv \in E(G)$ either u is red and v is blue after the k-th round, or v is red and u is blue after the k-th round does not depend on the choice of uv. This probability will be denoted by

$$p_k := \mathbf{P}\big[(u \in R_k \wedge v \in B_k) \vee (u \in B_k \wedge v \in R_k)\ \big|\ uv \in E(G)\big]\ .$$

2.3 Independence lemma

In this section we present a key tool which we use in the analysis of the randomized procedure. Our analysis follows the approach used in [10].

We start with a definition. If G is a cubic graph with girth at least $2K+1$, uv is an edge of G and d is an integer between 0 and $K-1$, $T_{v,u}^d$ denotes the component of $T_v^d - u$ containing the vertex v. We refer to v as to the root of $T_{v,u}^d$. From the assumption on the girth it follows that all the subgraphs $T_{v,u}^d$ are isomorphic to the same rooted binary tree $\mathcal{T}^d$ of depth d.

Let $k \in [K]$. For a set $V' \subseteq V(G)$ let $c_k(V')$ denote the coloring of vertices V' after the k-th round. The set of all colorings of $\mathcal{T}^{K-k}$ such that the root of the tree is white is denoted by $\mathcal{C}_k$. Observe that by the girth assumption for any $\gamma \in \mathcal{C}_k$ the probability $\mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right) = \gamma\big]$ does not depend on the edge uv.

We are ready to prove the main lemma of this section.

Lemma 2.6 (Independence lemma)**.** *Consider the randomized procedure with parameters K and $P_i^{r,b}(C)$, where $i \in [K], (r, b) \in I_3$ and $C \in \{W, R, B\}$. Let G be a cubic graph with girth at least $2K+1$, uv an edge of G, k an integer smaller than K and γ_u and γ_v two colorings from $\mathcal{C}_k$. Conditioned by the event $uv \in W_k$, the events $c_k\left(T_{v,u}^{K-k}\right) = \gamma_v$ and $c_k\left(T_{v,u}^{K-k}\right) = \gamma_u$ are independent. In other words, the probabilities*

$$\mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \ \big|\ uv \subseteq W_k\big] \tag{2.1}$$

and

$$\mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \ \big|\ v \in W_k \wedge c_k\left(T_{u,v}^{K-k}\right) = \gamma_u\big] \tag{2.2}$$

are equal.

Proof. The proof proceeds by induction on k. After the first round each vertex has a color C with the probability $P_1^{0,0}(C)$ independently on other vertices. Hence, the claim holds for $k = 1$.

Assume now that $k > 1$. By the definition of the conditional probability and the fact that the event $uv \subseteq W_k$ immediately implies that the event $uv \subseteq W_{k-1}$ occurs, (2.1) is equal to

$$\frac{\mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \wedge u \in W_k \,\middle|\, uv \subseteq W_{k-1}\right]}{\mathbf{P}\left[uv \subseteq W_k \,\middle|\, uv \subseteq W_{k-1}\right]} . \tag{2.3}$$

Analogously, (2.2) is equal to

$$\frac{\mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \wedge c_k\left(T_{u,v}^{K-k}\right) = \gamma_u \,\middle|\, uv \subseteq W_{k-1}\right]}{\mathbf{P}\left[v \in W_k \wedge c_k\left(T_{u,v}^{K-k}\right) = \gamma_u \,\middle|\, uv \subseteq W_{k-1}\right]} . \tag{2.4}$$

Now we expand the numerator of (2.3).

$$\begin{aligned}
&\sum_{\gamma'_u \in \mathcal{C}_{k-1}} \sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}\left[c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \mid uv \subseteq W_{k-1}\right] \\
&\times \mathbf{P}\left[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \mid v \in W_{k-1} \wedge c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u\right] \\
&\times \mathbf{P}\left[u \in W_k \mid c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v\right] \\
&\times \mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \mid c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_k\right] .
\end{aligned}$$

By the induction hypothesis, for any two colorings $\gamma'_u, \gamma'_v \in \mathcal{C}_{k-1}$ the probabilities

$$\mathbf{P}\left[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \,\middle|\, v \in W_{k-1} \wedge c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u\right]$$

and

$$\mathbf{P}\left[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \,\middle|\, uv \subseteq W_{k-1}\right]$$

are equal.

Since the new color of u is determined only by the colors of the neighbors of u, it follows that the probabilities

$$\mathbf{P}\left[u \in W_k \,\middle|\, c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v\right]$$

and

$$\mathbf{P}\left[u \in W_k \,\middle|\, c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge v \in W_{k-1}\right]$$

are also equal.

Analogously, for any vertex $w \in T_{v,u}^{K-k} \setminus \{v\}$ the new color of w does not depend on γ'_u at all. Applying the same reasoning for v yields that the probabilities

$$\mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \,\middle|\, c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_k\right]$$

and

$$\mathbf{P}\left[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \,\middle|\, c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_{k-1}\right]$$

are equal as well. Note that in the last equality we have also used that the random choices of new colors for arbitrary two vertices in the $(k+1)$-th round are independent.

By changing the order of summation, we conclude that the nominator of (2.3) is equal to

$$\begin{aligned}&\left(\sum_{\gamma'_u \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \bigm| uv \subseteq W_{k-1}\big]\right.\\&\qquad\left.\times\, \mathbf{P}\big[u \in W_k \bigm| c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge v \in W_{k-1}\big]\right)\\\times&\left(\sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \bigm| uv \subseteq W_{k-1}\big]\right.\\&\qquad\left.\times\, \mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \bigm| c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_{k-1}\big]\right).\end{aligned}$$

Along the same lines, the denominator of (2.3) is equal to

$$\begin{aligned}&\left(\sum_{\gamma'_u \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \bigm| uv \subseteq W_{k-1}\big]\right.\\&\qquad\left.\times\, \mathbf{P}\big[u \in W_k \bigm| c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge v \in W_{k-1}\big]\right)\\\times&\left(\sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \bigm| uv \subseteq W_{k-1}\big]\right.\\&\qquad\left.\times\, \mathbf{P}\big[v \in W_k \bigm| c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_{k-1}\big]\right).\end{aligned}$$

Canceling out the sum over γ'_u which is the same in both numerator and denominator of (2.3), we derive that (2.1) is equal to

$$\frac{\sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \mid uv \subseteq W_{k-1}] \times \mathbf{P}[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_{k-1}]}{\sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \mid uv \subseteq W_{k-1}] \times \mathbf{P}[v \in W_k \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_{k-1}]}. \tag{2.5}$$

We apply the same trimming to the numerator and denominator of (2.4). The numerator is first expanded to

$$\begin{aligned}&\left(\sum_{\gamma'_u \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \bigm| uv \subseteq W_{k-1}\big]\right.\\&\qquad\left.\times\, \mathbf{P}\big[c_k\left(T_{u,v}^{K-k}\right) = \gamma_u \bigm| c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge v \in W_{k-1}\big]\right)\\\times&\left(\sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \bigm| uv \subseteq W_{k-1}\big]\right.\\&\qquad\left.\times\, \mathbf{P}\big[c_k\left(T_{v,u}^{K-k}\right) = \gamma_v \bigm| c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_{k-1}\big]\right)\end{aligned}$$

and the denominator is then expanded to

$$\left(\sum_{\gamma'_u \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \mid uv \subseteq W_{k-1}\big] \right.$$
$$\left. \times \mathbf{P}\big[c_k\left(T_{u,v}^{K-k}\right) = \gamma_u \mid c_{k-1}\left(T_{u,v}^{K-k+1}\right) = \gamma'_u \wedge v \in W_{k-1}\big] \right)$$
$$\times \left(\sum_{\gamma'_v \in \mathcal{C}_{k-1}} \mathbf{P}\big[c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \mid uv \subseteq W_{k-1}\big] \right.$$
$$\left. \times \mathbf{P}\big[v \in W_k \mid c_{k-1}\left(T_{v,u}^{K-k+1}\right) = \gamma'_v \wedge u \in W_{k-1}\big] \right) .$$

By canceling out the sum over γ'_u, we obtain (2.5). Therefore the expressions (2.1) and (2.2) are equal. □

2.4 Recurrence relations

In this section we derive recurrence relations for the probabilities describing the behavior of the randomized procedure.

Fix parameters K and $P_k^{r,b}(C)$, $k \in [K], (r,b) \in I_3$ and $C \in \{W, R, B\}$. We will inductively show that the probabilities describing the state of the procedure after the $(k+1)$-th round can be computed using only the probabilities describing the state after the k-th round. This yields the recurrence relations for the probabilities which is the main goal of this section.

We start with determining the probabilities after the initialization round. It is easy to see that the probabilities r_1, b_1, w_1, p_1 and $w_1^{r,b}$ are

$$\begin{aligned}
&r_1 = P_1^{0,0}(R) ,\\
&b_1 = P_1^{0,0}(B) ,\\
&w_1 = 1 - r_1 - b_1 ,\\
&p_1 = 2 \cdot P_1^{0,0}(R) \cdot P_1^{0,0}(B) \quad \text{and}\\
&w_1^{r,k} = \binom{3}{r}\binom{3-r}{b} \cdot \left(P_1^{0,0}(R)\right)^r \cdot \left(P_1^{0,0}(B)\right)^b \cdot \left(1 - P_1^{0,0}(R) - P_1^{0,0}(B)\right)^{3-r-b}\\
&\text{for } (r,b) \in I_3 .
\end{aligned}$$

Now we show how to compute the probabilities r_{k+1}, b_{k+1} and w_{k+1} from r_k, b_k, w_k and $w_k^{r,b}$. We start with the formula for r_{k+1}. If a vertex v is colored red after the $(k+1)$-th round, then it was after the k-th round either already colored red, or it was white, had r red neighbors, b blue neighbors and it was recolored to red. The latter happened with probability $P_{k+1}^{r,b}(R)$. The probability of the first event is r_k and that of the second event is $w_k \cdot w_k^{r,b} \cdot P_{k+1}^{r,b}(R)$. This yields that

$$r_{k+1} = r_k + w_k \cdot \sum_{(r,b) \in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(R) .$$

Analogously, we can compute

$$b_{k+1} = b_k + w_k \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(B) \ ,$$

and finally w_{k+1} is given by

$$w_{k+1} = 1 - r_{k+1} - b_{k+1} \ .$$

Before we proceed with the recurrences for p_{k+1} and $w_{k+1}^{r,b}$, let us introduce some auxiliary notation. All of the following quantities are fully determined by $w_k^{r,b}$, but this notation will help to make the formulas simpler. We start with the probability that a vertex v has white color after the $(k+1)$-th round conditioned by the event it had white color after the k-th round. This quantity will be denoted by $w_{\to k+1}$. It is straightforward to check that

$$w_{\to k+1} := \mathbf{P}\big[v \in W_{k+1} \bigm| v \in W_k\big] = \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(W) \ .$$

Next, we consider the probability that the vertex u is white after the k-th round conditioned by the event that a fixed neighbor v of u is white. This will be denoted by q_k^{W-W}. Again it is easy to check that

$$q_k^{W-W} := \mathbf{P}\big[uv \subseteq W_k \bigm| v \in W_k\big] = \sum_{(r,b)\in I_2} \frac{3-r-b}{3} \cdot w_k^{r,b} \ .$$

Finally, for a color $C \in \{W, R, B\}$ and an edge $e = uv$, $q_{\to k+1}^{(C)}$ denotes the probability that u has the color C after the $(k+1)$-th round conditioned by the event that both u and v were white after the k-th round. We infer from the definition of the conditional probability that

$$q_{\to k+1}^{(R)} := \mathbf{P}\big[u \in R_{k+1} \bigm| uv \subseteq W_k\big] = \sum_{(r,b)\in I_2} \frac{w_k^{r,b} \cdot (3-r-b) \cdot P_{k+1}^{r,b}(R)}{3 \cdot q_k^{W-W}} \ ,$$

$$q_{\to k+1}^{(B)} := \mathbf{P}\big[u \in B_{k+1} \bigm| uv \subseteq W_k\big] = \sum_{(r,b)\in I_2} \frac{w_k^{r,b} \cdot (3-r-b) \cdot P_{k+1}^{r,b}(B)}{3 \cdot q_k^{W-W}} \ ,$$

$$q_{\to k+1}^{(W)} := \mathbf{P}\big[u \in W_{k+1} \bigm| uv \subseteq W_k\big] = \sum_{(r,b)\in I_2} \frac{w_k^{r,b} \cdot (3-r-b) \cdot P_{k+1}^{r,b}(W)}{3 \cdot q_k^{W-W}} \ .$$

Now we are ready to present the remaining recurrences. We start with p_{k+1}, i.e., the probability than an edge $e = uv$ is red-blue after the $(k+1)$-th round. Note that once we color a vertex x with either red or blue color, the color of x in the future rounds will stay the same. Therefore, we can split the contribution to p_{k+1} to the following four types.

1. $e \cap W_k = \emptyset$: This event happens with the probability p_k and the colors stay the same.

2. $e \cap W_k = \{v\}$: Suppose first that u is blue. The probability that we have such configuration after k-th round is $w_k \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot b/3$. In this case, the edge e become red-blue after the $(k+1)$-th round with the probability $P_{k+1}^{r,b}(R)$. Analogously, if u is red, the contribution of this case is $w_k \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(B) \cdot r/3$.

3. $e \cap W_k = \{u\}$: This case is symmetric to the previous one.

4. $e \subseteq W_k$: The probability that v has white color is w_k. With the probability $w_k^{r,b} \cdot (3-r-b)/3$, v has r red neighbors, b blue neighbors and u is white. The probability that v becomes red is $P_{k+1}^{r,b}(R)$, and using the independence lemma (Lemma 2.6) the neighborhood of u does not depend on the colors of the other neighbors of v. Therefore, the probability that u becomes blue is $q_{\to k+1}^{(B)}$. On the other hand, the probability that v becomes red and u becomes blue is $P_{k+1}^{r,b}(B) \cdot q_{\to k+1}^{(R)}$.

The just presented analysis yields that

$$p_{k+1} = p_k + \frac{w_k}{3} \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(R) \cdot \left(2b + (3-r-b) \cdot q_{\to k+1}^{(B)}\right)$$
$$+ \frac{w_k}{3} \cdot \sum_{(r,b)\in I_3} w_k^{r,b} \cdot P_{k+1}^{r,b}(B) \cdot \left(2r + (3-r-b) \cdot q_{\to k+1}^{(R)}\right) .$$

We finish this section with the recurrence relations for the probabilities $w_{k+1}^{r,b}$. Observe that

$$w_{k+1}^{r,b} = \frac{\mathbf{P}\left[v \in W_{k+1}^{r,b}\right]}{\mathbf{P}\left[v \in W_{k+1}\right]} = \frac{\mathbf{P}\left[v \in W_{k+1}^{r,b} \mid v \in W_k\right]}{\mathbf{P}\left[v \in W_{k+1} \mid v \in W_k\right]} . \tag{2.6}$$

The second equality holds because each of the events $v \in W_{k+1}$ and $v \in W_{k+1}^{r,b}$ immediately implies that the event $v \in W_k$ occurs. The denominator of (2.6) is equal to $w_{\to k+1}$, so it remains to derive the formula for the nominator.

Let $N_k^W(v)$ denote the set of white neighbors of v after the k-th round. Using the same argument as for deriving the formula for p_{k+1}, the color after the $(k+1)$-th round of a white neighbor $u \in N_k^W(v)$ will be red with the probability $q_{\to k+1}^{(R)}$. Analogously, it will be blue with the probability $q_{\to k+1}^{(B)}$ and white with the probability $q_{\to k+1}^{(W)}$. Finally, by Lemma 2.6 and the fact that in all rounds we recolor each white vertex independently of the others, the new color of a neighbor $u_1 \in N_k^W(v)$ does not depend on the new color of another neighbor $u_2 \in N_k^W(v)$. Therefore, it holds for each$(r,b) \in I_3$ that

$$w_{k+1}^{r,b} = \sum_{\substack{\overline{r}\le r\\ \overline{b}\le b}} \frac{w_k^{\overline{r},\overline{b}} \cdot P_{k+1}^{\overline{r},\overline{b}}(W) \cdot \binom{3-\overline{r}-\overline{b}}{r-\overline{r}}\binom{3-r-\overline{b}}{b-\overline{b}} \cdot \left(q_{\to k+1}^{(R)}\right)^{r-\overline{r}} \cdot \left(q_{\to k+1}^{(B)}\right)^{b-\overline{b}} \cdot \left(q_{\to k+1}^{(W)}\right)^{3-r-b}}{w_{\to k+1}} .$$

2.5 Setting up the parameters

In this section we set up the parameters in the randomized procedure. In the first round, we pick a vertex with a small probability p_0 and color it either red or blue. The next rounds of the procedure are split into two phases, which consist of K_1 and K_2 rounds, respectively. Therefore, the total number of rounds K is equal to $K_1 + K_2 + 1$.

In the rounds of the first phase, with probability p_B (p_R), where $p_R \ll p_B$, we color a vertex with exactly one red (blue) neighbor by blue (red). If a vertex has at least two neighbors of the same color, we color it with the other color with probability one. In all the other cases we do nothing.

With one exception, the rounds of the second phase are performed identically to the rounds of the first phase. The exception is that a white vertex with one red, one blue and one white neighbor is colored red with probability $p_{RB}/2$ or blue with probability $p_{RB}/2$. The choice of p_{RB} is such that $p_{RB} \ll p_R$.

Specifically, we set:

- $K := K_1 + K_2 + 1$,
- $P_1^{0,0}(R) := p_0/2$, $P_1^{0,0}(B) := p_0/2$,
- $P_k^{r,b}(R) := 1$ for $(r,b) \in I_3 \cap \{(r,b) : b \geq 2\}$ for $k \in [2,\ldots,K]$,
- $P_k^{r,b}(W) := 1$ for $(r,b) \in I_3 \cap \{(r,b) : r \geq 2\}$ for $k \in [2,\ldots,K]$,
- $P_k^{0,1}(R) := p_R$, $P_k^{1,0}(B) := p_B$ for $k \in [2,\ldots,K]$,
- $P_k^{1,1}(R) := p_{RB}/2$, $P_k^{1,1}(B) := p_{RB}/2$ for $k \in [K_1+2,\ldots,K]$,
- $P_k^{r,b}(R) := 0$ for all the other choices of r and b,
- $P_k^{r,b}(B) := 0$ for all the other choices of r and b and
- $P_k^{r,b}(W) := 1 - P_k^{r,b}(R) - P_k^{r,b}(B)$ for $(r,b) \in I_3$.

The recurrences presented in this chapter were solved numerically using the C program provided in the Appendix A. The particular choice of parameters used in the program was $p_0 = 10^{-5}, p_B = 1/10, p_R = 10^{-5}, p_{RB} = 10^{-6}$, $K_1 = 1\,672\,413$ and $K_2 = 6\,504\,552$.

The choice of K_1 was made in such a way that at the end of the first phase, i.e. after the (K_1+1)-th round, the probability that a vertex is white and has exactly one non-white neighbor is less than 10^{-7}. Analogously, the choice of K_2 was made in a way that at the end of the process, i.e. after the K-th round, the probability that a vertex is white is less than 10^{-7}. We also estimated the precision of our calculations based on the representation of float numbers to avoid rounding errors effecting the presented bound on significant digits. Solving the recurrences for the above choice of parameters we have obtained that $p_K > 0.88672$.

Chapter 3

Independent sets

In this chapter, we present our results about independent sets and fractional colorings in cubic graphs with large girth and about independent sets in random cubic graphs. Analogously to the case of edge-cuts, our results improve the previously best known lower bound on the size of maximum independent set and the upper bound on the fractional chromatic number in a cubic graph with sufficiently large girth as well as the best known asymptotic lower bound on the size of maximum independent set in a random cubic graph. The structure of the proof is very similar to the structure presented in Chapter 2, however some details of the analysis are more complicated.

3.1 Motivation and results

Let us start with surveying previous results. Inspired by Nešetřil's Pentagon Conjecture [19] on circular colorings of cubic graphs with large girth, Hatami and Zhu [7] showed that every subcubic graph with sufficiently large girth has fractional chromatic number at most $8/3 + \varepsilon$ for any $\varepsilon > 0$. We provide a simple proof of this statement in Section 3.9 (under a weaker assumption of having large odd girth). They also showed that a triangle-free subcubic graph has fractional chromatic number at most $3 - 3/64$. Lu and Peng [15] improved the bound to $3 - 3/43$. On the other hand, there are triangle-free cubic graphs with fractional chromatic number $14/5$; Heckman and Thomas [8] conjectured that this value is also the upper bound on the fractional chromatic number for any triangle-free subcubic graph.

For the independence number, Hoppen [10] showed that every n-vertex cubic graph with sufficiently large girth has the independence number at least $0.4328n$; this bound matches an earlier bound of Wormald [22], independently proven by Frieze and Suen [6], for random cubic graphs. The lower bound for random cubic graphs was further improved by Duckworth and Zito [5] to $0.4347n$. The best known asymptotic upper bound for the size of maximum independent set in a random cubic graph is $0.45537n$ derived by McKay [17] and this bound also applies to the independence number of cubic graphs with large girth. In [17], McKay mentions that experimental evidence suggests that almost all n-vertex cubic graphs contain independent sets of size $0.439n$; his more recent experiments [18] suggest that the best possible lower bound might be even as large as $0.447n$.

The main result of this chapter is the following.

Theorem 3.1. *There exists $g > 0$ such that for every cubic graph G with girth at least g, there is a probability distribution such that each vertex is contained in an independent set drawn according to this distribution with probability at least* 0.4352*.*

Analogously to Theorem 2.1, Theorem 3.1 has the following three corollaries.

Corollary 3.2. *There exists $g > 0$ such that every n-vertex subcubic graph with girth at least g contains an independent set of size at least* $0.4352n$*.*

Corollary 3.3. *There exists $g > 0$ such that every cubic graph with girth at least g has the fractional chromatic number at most* 2.2978*.*

Corollary 3.4. *A random n-vertex cubic graph asymptotically almost surely contains an independent set of size at least* $0.4352n$*.*

3.2 Structure of the proof

Analogously as in Chapter 2, we design a randomized procedure for obtaining an independent set (of large size). First we describe a procedure RAND-INDEP for obtaining a random independent set in infinite cubic trees and analyze its behavior in Section 3.5. The core of our analysis is again the independence lemma (Lemma 3.6) which is analogous to the independence lemma from Chapter 2 (Lemma 2.6) but it becomes more complicated because of a more complex nature of the randomized procedure. We then show that the procedure RAND-INDEP can be modified for cubic graphs with sufficiently large girth while keeping its performance. The modified procedure will be called RAND-INDEP-CUBIC. The actual performance of the procedure is based on solving the derived recurrences numerically.

3.3 Randomized procedure RAND-INDEP

The procedure RAND-INDEP is parametrized by three numbers: the number K of its rounds and probabilities p_1 and p_2.

Throughout the procedure, vertices of the input graph have one of the three colors: white, blue and red. At the beginning, all vertices are white. In each round, some of the white vertices are recolored in such a way that *red* vertices always form an independent set and all vertices adjacent to red vertices as well as some of other vertices (see details below) are *blue*. All other vertices of the input graph are *white*. Red and blue vertices are never again recolored during the procedure. When we talk of a *degree* of a vertex, we mean the number of its white neighbors. Observe that the neighbors of a white vertex that are not white must be blue.

The first round of the procedure is special and differs from the other rounds. As we have already said, at the very beginning, all vertices of the input graph are white. In the first round, we randomly and independently with probability p_1 mark some vertices as active. Active vertices with no active neighbor become red and vertices with at least one active neighbor become blue. In particular, if two adjacent vertices are active, they both become blue (as well as their neighbors). At this point, we should note that the probability p_1 will be very small.

In the second and the remaining rounds, we first color all white vertices of degree zero by red. We then consider all paths formed by white vertices with end vertices of degree one or three and with all inner vertices of degree two. Based on the degrees of their end vertices, we refer to these paths as paths of type 1↔1, 1↔3 or 3↔3. Edges joining two vertices of degree one or three are considered as such paths of length one. Finally, each vertex of degree two is now activated with probability p_2 independently of all other vertices.

For each path of type 1↔3, we color the end vertex of degree one with red and we then color all the inner vertices with red and blue in the alternating way. If the neighbor of the end vertex of degree three becomes red, we also color the end vertex of degree three with blue. In other words, we color the end vertex of degree three by blue if the path has an odd length. For a path of type 1↔1, we choose randomly one of its end vertices, color it red and color all the remaining vertices of the path with red and blue in the alternating way. We refer to the chosen end vertex as the beginning of this path. Note that whether and which vertices of degree two on the paths of type 1↔1 or 1↔3 are activated do not influence the above procedure.

Paths of type 3↔3 are processed as follows. A path of type 3↔3 becomes active if at least one of its inner vertices is activated, i.e., a path of length ℓ is active with probability $1-(1-p_2)^{\ell-1}$. Note that paths with no inner vertices, i.e., edges between two vertices of degree three, are never active. For each active path, flip the fair coin to select one of its end vertices of degree three, color this vertex by blue and its neighbor on the path by red. The remaining inner vertices of the path are colored with red and blue in the alternating way. Again we refer to the choosen end vertex as the beginning of this path. The other end vertex of degree three is colored blue if its neighbor on the path becomes red, i.e., if the path has an even length.

Note that a vertex that has degree three at the beginning of the round cannot become red during the round but it can become blue because of several different paths ending at it.

3.4 Analysis of RAND-INDEP

Let us start with introducing some notation used in the analysis. For any edge uv of a cubic graph G, $T_{u,v}$ is the component of $G-v$ containing the vertex u; we sometimes refer to u as to the root of $T_{u,v}$. If G is the infinite cubic tree, then it is a union of $T_{u,v}, T_{v,u}$ and the edge uv. The subgraph of $T_{u,v}$ induced by vertices at distance at most d from u is denoted by $T^d_{u,v}$. Observe that if the girth of G is larger than $2d+1$, all the subgraphs $T^d_{u,v}$ are isomorphic to the same rooted tree $\mathcal{T}^d$ of depth d. The infinite rooted tree with the root of degree two and all inner vertices of degree three will be denoted as T^∞.

Since any automorphism of the infinite cubic tree yields an automorphism of the probability space of the vertex colorings constructed by our procedure, the probability that a vertex u has a fixed color after the k-th round does not depend on the choice of u. Hence, we can use w_k, b_k and r_k to denote a probability that a fixed vertex of the infinite cubic tree is white, blue and red, respectively, after the k-th round. Similarly, $w_k^{(i)}$ denotes the probability that a fixed vertex has degree i after the k-th round conditioned by the event that it is white after the k-th round, i.e., the probability that a fixed vertex is white and has degree two after the k-th round is $w_k \cdot w_k^{(2)}$.

The sets of white, blue and red vertices after the k-th round of the randomized procedure described in Section 3.3 will be denoted by W_k, B_k and R_k, respectively. Similarly, $W_k^{(i)}$ denotes the set of white vertices with degree i after the k-th round, i.e., $W_k = W_k^{(0)} \cup W_k^{(1)} \cup W_k^{(2)} \cup W_k^{(3)}$. Finally, $c_k(T_{u,v})$ denotes the coloring of $T_{u,v}$ and $c_k\left(T_{u,v}^d\right)$ the coloring of $T_{u,v}^d$ after the k-th round. The set $\mathcal{C}_k^d$ will consist of all possible colorings γ of $\mathcal{T}^d$ such that the probability of $c_k\left(T_{u,v}^d\right) = \gamma$ is non-zero in the infinite cubic tree (note that this probability does not depend on the edge uv) and such that the root of $\mathcal{T}^d$ is colored white. We extend this notation and use $\mathcal{C}_k^\infty$ to denote all such colorings of $\mathcal{T}^\infty$.

Since the infinite cubic tree is strongly edge-transitive, we conclude that the probability that $T_{u,v}$ has a given coloring from $\mathcal{C}_k^\infty$ after the k-th round does not depend on the choice of uv. Similarly, the probability that both u and v are white after the k-th round does not depend on the choice of the edge uv. To simplify our notation, this event is denoted by $uv \subseteq W_k$. Finally, the probability that u has degree $i \in \{1, 2, 3\}$ after the k-th round conditioned by $uv \subseteq W_k$ does also not depend on the choice of the edge uv. This probability is denoted by $q_k^{(i)}$.

We now show that the probability that both end-vertices of an edge uv are white after the k-th round can be computed as a product of two probabilities. This will be crucial in the proof of the Independence Lemma. To be able to state the next lemma, we need to introduce additional notation. For $\gamma \in \mathcal{C}_{k-1}^\infty$, define

$$P_k(u, v, \gamma) := \mathbf{P}\left[u \text{ stays white regarding } T_{u,v} \;\middle|\; c_{k-1}(T_{u,v}) = \gamma \wedge v \in W_{k-1}\right]$$

where the phrase "u stays white regarding $T_{u,v}$" means

- if $k = 1$, that neither u nor a neighbor of it in $T_{u,v}$ is active, and
- if $k > 1$, that neither
  - u has degree one after the $(k-1)$ round,
  - u has degree two after the $(k-1)$ round and the path from u formed by vertices of degree two in $T_{u,v}$ ends at a vertex of degree one,
  - u has degree two after the $(k-1)$ round, the path from u formed by vertices of degree two in $T_{u,v}$ ends at a vertex of degree three and at least one of the vertices of degree two on this path is activated, nor
  - u has degree three after the $(k-1)$-th round and is colored blue because of a path of type $1\leftrightarrow 3$ or $3\leftrightarrow 3$ fully contained in $T_{u,v}$.

Informally, this phrase represents that there is no reason for u not to stay white based on the coloring of $T_{u,v}$ and vertices in $T_{u,v}$ activated in the k-th round.

Lemma 3.5. *Consider the randomized procedure for the infinite cubic tree. Let k be an integer, uv an edge of the tree and γ_u and γ_v two colorings from $\mathcal{C}_{k-1}^\infty$. The probability*

$$\mathbf{P}\left[uv \subseteq W_k \;\middle|\; c_{k-1}(T_{u,v}) = \gamma_u \wedge c_{k-1}(T_{v,u}) = \gamma_v\right], \tag{3.1}$$

i.e., the probability that both u and v are white after the k-th round conditioned by $c_{k-1}(T_{u,v}) = \gamma_u$ and $c_{k-1}(T_{v,u}) = \gamma_v$, is equal to

$$P_k(u, v, \gamma_u) \cdot P_k(v, u, \gamma_v) .$$

Proof. We distinguish the cases $k = 1$ and $k > 1$. If $k = 1$, $\mathcal{C}_0^\infty$ contains a single coloring γ_0 where all vertices are white. Hence, the probability (3.1) is $\mathbf{P}\big[uv \subseteq W_1 \bigm| c_{k-1}(T_{u,v}) = \gamma_0 \wedge c_{k-1}(T_{v,u}) = \gamma_0\big]$ and it is equal to $\mathbf{P}\big[uv \subseteq W_1\big] = (1-p_1)^6$. On the other hand, $P_1(u, v, \gamma_0) = P_1(v, u, \gamma_0) = (1-p_1)^3$. The assertion of the lemma follows.

Suppose that $k > 1$. Note that there is a unique coloring of the infinite cubic tree that coincides with γ_u on $T_{u,v}$ and with γ_v on $T_{v,u}$. If u has degree one after the $(k-1)$-th round, then (3.1) is zero as well as $P_k(u, v, \gamma_u) = 0$. If u has degree two and lie on a path of type $1 \leftrightarrow 1$ or $1 \leftrightarrow 3$, then (3.1) is zero and $P_k(u, v, \gamma_u) = 0$ or $P_k(v, u, \gamma_v) = 0$ depending which of the trees $T_{u,v}$ and $T_{v,u}$ contains the vertex of degree one. If u has degree two and lie on a path of type $3 \leftrightarrow 3$ of length ℓ, then (3.1) is equal to $(1-p_2)^{\ell-1}$. Let ℓ_1 and ℓ_2 be the number of vertices of degree two on this path in $T_{u,v}$ and $T_{v,u}$, respectively. Observe that $\ell_1 + \ell_2 = \ell - 1$. Since $P_k(u, v, \gamma_u) = (1 - p_2)^{\ell_1}$ and $P_k(v, u, \gamma_v) = (1 - p_2)^{\ell_2}$, the claimed equality holds.

Hence, we can now assume that the degree of u is three. By symmetry, the degree of v is also three. Note that u can only become blue and only because of an active path of type $3 \leftrightarrow 3$ ending at u. This happens with probability $1 - P_k(u, v, \gamma_u)$. Similarly, v becomes blue with probability $1 - P_k(v, u, \gamma_v)$. Since the event that u becomes blue and v becomes blue conditioned by $c_{k-1}(T_{u,v}) = \gamma_u$ and $c_{k-1}(T_{v,u}) = \gamma_v$ are independent, it follows that (3.1) is also equal to $P_k(u, v, \gamma_u) \cdot P_k(v, u, \gamma_v)$ in this case. ☐

Lemma 3.5 plays a crucial role in the Independence Lemma, which we now prove. Its proof enlights how we designed our randomized procedure.

Lemma 3.6 (Independence Lemma). *Consider the randomized procedure for the infinite cubic tree. Let k be an integer, uv an edge of the tree and Γ_u and Γ_v two measurable subsets of $\mathcal{C}_{k-1}^\infty$. Conditioned by the event $uv \subseteq W_k$, the events that $c_k(T_{u,v}) \in \Gamma_u$ and $c_k(T_{v,u}) \in \Gamma_v$ are independent. In other words,*

$$\mathbf{P}\big[c_k(T_{u,v}) \in \Gamma_u \bigm| uv \subseteq W_k\big] = \mathbf{P}\big[c_k(T_{u,v}) \in \Gamma_u \bigm| uv \subseteq W_k \wedge c_k(T_{v,u}) \in \Gamma_v\big] .$$

Proof. The proof proceeds by induction on k. If $k = 1$, the event $uv \subseteq W_1$ implies that neither u, v nor their neighbors is active during the first round. Conditioned by this, the other vertices of the infinite cubic tree are marked active with probability p_1 randomly and independently. The result of the marking in $T_{u,v}$ fully determine the coloring of the vertices of $T_{u,v}$ and is independent of the marking and coloring of $T_{v,u}$. Hence, the claim follows.

Assume that $k > 1$. Fix subsets Γ_u and Γ_v. We aim at showing that the probabilities

$$\mathbf{P}\big[c_k(T_{u,v}) \in \Gamma_u \bigm| uv \subseteq W_k\big] \tag{3.2}$$

and

$$\mathbf{P}\big[c_k(T_{u,v}) \in \Gamma_u \bigm| u \in W_k \wedge c_k(T_{v,u}) \in \Gamma_v\big] \tag{3.3}$$

are equal.

The definition of the conditional probability and the fact that the probability $\mathbf{P}\big[uv \subseteq W_{k-1} | uv \subseteq W_k\big]$ is one yield that (3.2) is equal to

$$\frac{\mathbf{P}\big[c_k\left(T_{u,v}\right) \in \Gamma_u \wedge v \in W_k \;\big|\; uv \subseteq W_{k-1}\big]}{\mathbf{P}\big[uv \subseteq W_k \;\big|\; uv \subseteq W_{k-1}\big]} \; . \tag{3.4}$$

By the induction hypothesis, for any two subsets Γ'_u and Γ'_v of C^{∞}_{k-1}, the probabilities

$$\mathbf{P}\big[c_{k-1}\left(T_{v,u}\right) \in \Gamma'_v \;\big|\; u \in W_{k-1} \wedge c_{k-1}\left(T_{u,v}\right) \in \Gamma'_u\big]$$

and

$$\mathbf{P}\big[c_{k-1}\left(T_{v,u}\right) \in \Gamma'_v \;\big|\; uv \subseteq W_{k-1}\big]$$

are equal. Hence, the numerator of (3.4) is equal to the following:

$$\begin{aligned}\int_{\gamma'_u,\gamma'_v \in C^{\infty}_{k-1}} & \mathbf{P}\big[uv \subseteq W_k \;\big|\; c_{k-1}\left(T_{u,v}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}\right) = \gamma'_v\big] \times \\ & \mathbf{P}\big[c_k\left(T_{u,v}\right) \in \Gamma_u \;\big|\; uv \subseteq W_k \wedge c_{k-1}\left(T_{u,v}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}\right) = \gamma'_v\big] \\ & \mathbf{dP}\big[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \;\big|\; uv \subseteq W_{k-1}\big] \mathbf{dP}\big[c_{k-1}\left(T_{v,u}\right) = \gamma'_v \;\big|\; uv \subseteq W_{k-1}\big].\end{aligned}$$

Observe that when conditioning by $uv \subseteq W_k$, the event $c_k\left(T_{u,v}\right) \in \Gamma_u$ is independent of $c_{k-1}\left(T_{v,u}\right) = \gamma'_v$. Hence, the double integral can be rewritten to

$$\begin{aligned}\int_{\gamma'_u,\gamma'_v \in C^{\infty}_{k-1}} & \mathbf{P}\big[uv \subseteq W_k \;\big|\; c_{k-1}\left(T_{u,v}\right) = \gamma'_u \wedge c_{k-1}\left(T_{v,u}\right) = \gamma'_v\big] \times \\ & \mathbf{P}\big[c_k\left(T_{u,v}\right) \in \Gamma_u \;\big|\; uv \subseteq W_k \wedge c_{k-1}\left(T_{u,v}\right) = \gamma'_u\big] \\ & \mathbf{dP}\big[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \;\big|\; uv \subseteq W_{k-1}\big] \mathbf{dP}\big[c_{k-1}\left(T_{v,u}\right) = \gamma'_v \;\big|\; uv \subseteq W_{k-1}\big] \; .\end{aligned}$$

An application of Lemma 3.5 then yields that the double integral is equal to

$$\begin{aligned}\int_{\gamma'_u,\gamma'_v \in C^{\infty}_{k-1}} & P_k(u,v,\gamma'_u) \cdot P_k(v,u,\gamma'_v) \cdot \mathbf{P}\big[c_k\left(T_{u,v}\right) \in \Gamma_u \;\big|\; uv \subseteq W_k \wedge c_{k-1}\left(T_{u,v}\right) = \gamma'_u\big] \\ & \mathbf{dP}\big[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \;\big|\; uv \subseteq W_{k-1}\big] \mathbf{dP}\big[c_{k-1}\left(T_{v,u}\right) = \gamma'_v \;\big|\; uv \subseteq W_{k-1}\big] \; .\end{aligned}$$

Regrouping the terms containing γ'_u only and γ'_v only, we obtain that the numerator of (3.4) is equal to

$$\begin{aligned}&\Bigg(\int_{\gamma'_u \in C^{\infty}_{k-1}} P_k(u,v,\gamma'_u) \times \mathbf{P}\big[c_k\left(T_{u,v}\right) \in \Gamma_u \;\big|\; uv \subseteq W_k \wedge c_{k-1}\left(T_{u,v}\right) = \gamma'_u\big] \\ &\qquad \mathbf{dP}\big[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \;\big|\; uv \subseteq W_{k-1}\big]\Bigg) \times \\ &\Bigg(\int_{\gamma'_v \in C^{\infty}_{k-1}} P_k(v,u,\gamma'_v) \mathbf{dP}\big[c_{k-1}\left(T_{v,u}\right) = \gamma'_v \;\big|\; uv \subseteq W_{k-1}\big]\Bigg) \; .\end{aligned}$$

Along the same lines, the denominator of (3.4) can be expressed as

$$\begin{aligned}&\Bigg(\int_{\gamma'_u \in C^{\infty}_{k-1}} P_k(u,v,\gamma'_u) \mathbf{dP}\big[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \;\big|\; uv \subseteq W_{k-1}\big]\Bigg) \times \\ &\Bigg(\int_{\gamma'_v \in C^{\infty}_{k-1}} P_k(v,u,\gamma'_v) \mathbf{dP}\big[c_{k-1}\left(T_{v,u}\right) = \gamma'_v \;\big|\; uv \subseteq W_{k-1}\big]\Bigg) \; .\end{aligned}$$

Cancelling out the integral over γ'_v which is the same in the numerator and the denominator of (3.4), we obtain that (3.2) is equal to

$$\frac{\int_{\gamma'_u \in C^\infty_{k-1}} P_k(u,v,\gamma'_u)\mathbf{P}\left[c_k\left(T_{u,v}\right) \in \Gamma_u \mid uv \subseteq W_k \wedge c_{k-1}\left(T_{u,v}\right) = \gamma'_u\right]\mathbf{dP}\left[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \mid uv \subseteq W_{k-1}\right]}{\int_{\gamma'_u \in C^\infty_{k-1}} P_k(u,v,\gamma'_u)\mathbf{dP}\left[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \mid uv \subseteq W_{k-1}\right]} . \tag{3.5}$$

The same trimming is applied to (3.3). First, the probability (3.3) is expressed as

$$\frac{\mathbf{P}\left[c_k\left(T_{u,v}\right) \in \Gamma_u \wedge c_k\left(T_{v,u}\right) \in \Gamma_v \mid uv \subseteq W_{k-1}\right]}{\mathbf{P}\left[u \in W_k \wedge c_k\left(T_{v,u}\right) \in \Gamma_v \mid uv \subseteq W_{k-1}\right]} . \tag{3.6}$$

The numerator of (3.6) is then expanded to

$$\begin{aligned}&\left(\int_{\gamma'_u \in C^\infty_{k-1}} P_k(u,v,\gamma'_u)\mathbf{P}\left[c_k\left(T_{u,v}\right) \in \Gamma_u \mid uv \subseteq W_k \wedge c_{k-1}\left(T_{u,v}\right) = \gamma'_u\right]\right.\\ &\qquad\left.\mathbf{dP}\left[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \mid uv \subseteq W_{k-1}\right]\right)\times\\ &\left(\int_{\gamma'_v \in C^\infty_{k-1}} \mathbf{P}\left[c_k\left(T_{v,u}\right) \in \Gamma_v \mid uv \subseteq W_k \wedge c_{k-1}\left(T_{v,u}\right) = \gamma'_v \cdot P_k(v,u,\gamma'_v)\right]\right.\\ &\qquad\left.\mathbf{dP}\left[c_{k-1}\left(T_{v,u}\right) = \gamma'_v \mid uv \subseteq W_{k-1}\right]\right)\end{aligned}$$

and the denominator of (3.3) is expanded to

$$\begin{aligned}&\left(\int_{\gamma'_u \in C^\infty_{k-1}} P_k(u,v,\gamma'_u)\mathbf{dP}\left[c_{k-1}\left(T_{u,v}\right) = \gamma'_u \mid uv \subseteq W_{k-1}\right]\right)\times\\ &\left(\int_{\gamma'_v \in C^\infty_{k-1}} \mathbf{P}\left[c_k\left(T_{v,u}\right) \in \Gamma_v \mid uv \subseteq W_k \wedge c_{k-1}\left(T_{v,u}\right) = \gamma'_v \cdot P_k(v,u,\gamma'_v)\right]\right.\\ &\qquad\left.\mathbf{dP}\left[c_{k-1}\left(T_{v,u}\right) = \gamma'_v \mid uv \subseteq W_{k-1}\right]\right) .\end{aligned}$$

We obtain (3.5) by cancelling out the integrals over γ'_v. The proof is now finished. $\square$

3.5 Recurrence relations

We now derive recurrence relations for the probabilities describing the behavior of the randomized procedure. Analogously as in Section 2.4, we will inductively show that the probabilities describing the state of the procedure after the $(k+1)$-th round can be computed using only the probabilities describing the state after the k-th round.

Recall that $w_k^{(i)}$ is the probability that a fixed vertex u has degree i after the k-th round conditioned by the event that u is white after the k-th round. Also recall that $q_k^{(i)}$ is the probability that a fixed vertex u with a fixed neighbor v has degree i after the k-th round conditioned by the event that both u and v are white after the k-th round, i.e., $uv \subseteq W_k$. Finally, w_k, r_k and b_k are the probabilities that a fixed vertex is white, red and blue, respectively, after the k-th round.

If u is white, the *white subtree* of $T_{u,v}$ is the maximal subtree containing u and white vertices only. We claim that the probability that the white subtree of $T_{u,v}$ is isomorphic to a tree in a given subset $\mathcal{T}_0$ after the k-th round, conditioned by the event that both u and v are white after the k-th round, can be computed from the values of $q_k^{(i)}$. Indeed, if $T_0 \in \mathcal{T}_0$, the probability that u has degree i as in T_0 is $q_k^{(i)}$. Now, if the degree of u is i as in T_0 and z is a neighbor of u, the values of $q_k^{(i)}$ again determine the probability that the degree of z is as in T_0. By Lemma 3.6, the probabilities that u and z have certain degrees, conditioned by the event that they are both white, are independent. Inductively, we can proceed with other vertices of T_0. Applying standard probability arguments, we see that the values of $q_k^{(i)}$ fully determine the probability that the white subtree of $T_{u,v}$ after the k-th round is isomorphic to a tree in $\mathcal{T}_0$.

After the first round, the probabilities w_k, r_k, b_k and $q_k^{(i)}$, $i \in \{0,1,2,3\}$, are the following.

$$
\begin{aligned}
b_1 &= 1-(1-p_1)^3 & q_1^{(1)} &= \left(1-(1-p_1)^2\right)^2\\
r_1 &= p_1(1-b_1) = p_1(1-p_1)^3 & w_1^{(3)} &= (1-p_1)^6\\
w_1 &= 1-b_1-r_1 & w_1^{(2)} &= 3\cdot(1-p_1)^4\left(1-(1-p_1)^2\right)\\
q_1^{(3)} &= (1-p_1)^4 & w_1^{(1)} &= 3\cdot(1-p_1)^2\left(1-(1-p_1)^2\right)^2\\
q_1^{(2)} &= 2\cdot(1-p_1)^2\left(1-(1-p_1)^2\right) & w_1^{(0)} &= \left(1-(1-p_1)^2\right)^3
\end{aligned}
$$

A vertex becomes blue if at least one of its neighbors is active and it becomes red if it is active and none of its neighbors is also active. Otherwise, a vertex stays white. This leads to the formulas above.

To derive formulas for the probabilities w_k, r_k, b_k and $q_k^{(i)}$ for $k \geq 2$, we introduce additional notation. The recurrence relations can be expressed using $q_k^{(i)}$ only, but additional notation will help us to simplify expressions appearing in our analysis. For a given edge uv of the infinite tree, let $P_k^{\to 1}$ be the probability that the white subtree of $T_{u,v}$ contains a path from u to a vertex of degree one with all inner vertices of degree two after the k-th round conditioned by the event $uv \subseteq W_k$. Note that such a path may have length zero ending at u (this happens if the degree of u is one). Quantities $P_k^{E\to 1}$ and $P_k^{O\to 1}$ are probabilities that the length of such a path is even or odd, respectively. Analogously, $P_k^{\to 3}$, $P_k^{E\to 3}$ and $P_k^{O\to 3}$ are probabilities that the white subtree of $T_{u,v}$ contains a path, an even path and an odd path, respectively, from u to a vertex of degree three with all inner vertices of degree two after the k-th round conditioned by the event $uv \subseteq W_k$.

Using Lemma 3.6, we conclude that the values of the just defined probabilities can be computed as follows.

$$
\begin{aligned}
P_k^{\to 1} &= q_k^{(1)} \cdot \textstyle\sum_{\ell \geq 0} \left(q_k^{(2)}\right)^{\ell} = \frac{q_k^{(1)}}{1-q_k^{(2)}} & P_k^{\to 3} &= q_k^{(3)} \cdot \textstyle\sum_{\ell \geq 0} \left(q_k^{(2)}\right)^{\ell} = \frac{q_k^{(3)}}{1-q_k^{(2)}}\\
P_k^{O\to 1} &= q_k^{(1)} \cdot \textstyle\sum_{\ell \geq 0} \left(q_k^{(2)}\right)^{2\ell} = \frac{q_k^{(1)}}{1-\left(q_k^{(2)}\right)^2} & P_k^{O\to 3} &= q_k^{(3)} \cdot \textstyle\sum_{\ell \geq 0} \left(q_k^{(2)}\right)^{2\ell} = \frac{q_k^{(3)}}{1-\left(q_k^{(2)}\right)^2}\\
P_k^{E\to 1} &= P_k^{\to 1} - P_k^{O\to 1} = q_k^{(2)} \cdot P_k^{O\to 1} & P_k^{E\to 3} &= P_k^{\to 3} - P_k^{O\to 3} = q_k^{(2)} \cdot P_k^{O\to 3}
\end{aligned}
$$

Observe that $P_k^{\to 1} + P_k^{\to 3} = 1$.

The formulas for the above probabilities can be easily altered to express the probabilities that a path exists and one of its inner vertices is active; simply, instead of multiplying by $q_k^{(2)}$, we multiply by $p_2 q_k^{(2)}$. Now, $\widehat{P}_k^{\to 3}$ is the probability that the white subtree of T_{uv} contains a path from u to a vertex of degree three with all inner vertices of degree two after the k-th round and none of them become active, conditioned by $uv \subseteq W_k$. Analogously to the previous paragraph, we use $\widehat{P}^{O\to 3}$ and $\widehat{P}^{E\to 3}$. The probabilities $\widehat{P}_k^{\to 3}$, $\widehat{P}^{O\to 3}$ and $\widehat{P}^{E\to 3}$ can be computed in the following way.

$$\begin{aligned}
\widehat{P}_k^{\to 3} =& q_k^{(3)} \cdot \textstyle\sum_{\ell\geq 0}\left(q_k^{(2)} \cdot (1-p_2)\right)^{\ell} = \frac{q_k^{(3)}}{1-q_k^{(2)}\cdot(1-p_2)} \\
\widehat{P}_k^{O\to 3} =& q_k^{(3)} \cdot \textstyle\sum_{\ell\geq 1}\left(q_k^{(2)} \cdot (1-p_2)\right)^{2\ell} = \frac{q_k^{(3)}}{1-\left(q_k^{(2)}\right)^2\cdot(1-p_2)^2} \\
\widehat{P}_k^{E\to 3} =& \widehat{P}_k^{\to 1} - \widehat{P}_k^{O\to 3} = q_k^{(2)} \cdot (1-p_2) \cdot \widehat{P}_k^{O\to 3}
\end{aligned}$$

The probabilities $\widetilde{P}_k^{O\to 3}$ and $\widetilde{P}^{E\to 3}$ are the probabilities that such an odd/even path exists and at least one of its inner vertices become active. Note that $P_k^{O\to 3} = \widehat{P}_k^{O\to 3} + \widetilde{P}_k^{O\to 3}$. The value of $\widetilde{P}_k^{O\to 3}$ is given by the equation

$$\widetilde{P}_k^{O\to 3} = \left(q_k^{(2)}\right)^2 \cdot \left((1-p_2)^2 \cdot \widetilde{P}_k^{O\to 3} + \left(1-(1-p_2)^2\right) \cdot P_k^{O\to 3}\right) ,$$

which can be manipulated to

$$\widetilde{P}_k^{O\to 3} = \frac{\left(q_k^{(2)}\right)^2 \cdot \left(1-(1-p_2)^2\right) \cdot P_k^{O\to 3}}{1-\left(q_k^{(2)}\right)^2 \cdot (1-p_2)^2} .$$

Using the expression for $\widetilde{P}_k^{O\to 3}$, we derive that $\widetilde{P}_k^{E\to 3}$ is equal to the following.

$$\widetilde{P}_k^{E\to 3} = q_k^{(2)} \cdot \left(p_2 \cdot P_k^{O\to 3} + (1-p_2) \cdot \widetilde{P}_k^{O\to 3}\right)$$

We now show how to compute the probabilities w_{k+1}, b_{k+1} and r_{k+1}. Since blue and red vertices keep their colors once assigned, we have to focus on the probability that a white vertex changes a color. We distinguish vertices based on their degrees.

A vertex of degree zero. Such a vertex is always recolored to red.

A vertex of degree one. Such a vertex is always recolored. Its new color is blue only if lies on an odd path to another vertex of degree one and the other end is chosen to be the beginning of the path. This leads to the following equalities.

$$\mathbf{P}\left[u \in R_{k+1} \mid u \in W_k^{(1)}\right] = \frac{1}{2}P_k^{O\to 1} + P_k^{E\to 1} + P_k^{\to 3} ,$$

and

$$\mathbf{P}\left[u \in B_{k+1} \mid u \in W_k^{(1)}\right] = \frac{1}{2}P_k^{O\to 1} .$$

A vertex of degree two. Since we have already computed the probabilities that the paths of white vertices with degree two leading in the two directions from the vertex end at a vertex of degree one/three, have odd/even length and contain an active vertex, we can easily determine the probability that the vertex stays white or becomes red or blue. Note that for odd paths with type 1↔1 and 3↔3, in addition, the random choice of the start of the path comes in the play. It is then straightforward to derive the following.

$$\begin{aligned}
&\mathbf{P}\left[u \in R_{k+1} \mid u \in W_k^{(2)}\right] = \left(P_k^{E\to 1}\right)^2 + \frac{2}{2} P_k^{O\to 1} P_k^{E\to 1} + 2 P_k^{E\to 1} P_k^{\to 3} \\
&+ (1-p_2)\cdot\left(\left(\widetilde{P}_k^{O\to 3}\right)^2 + 2\widetilde{P}_k^{O\to 3}\widehat{P}_k^{O\to 3} + \frac{2}{2}\widetilde{P}_k^{O\to 3}\widetilde{P}_k^{E\to 3} + \frac{2}{2}\widehat{P}_k^{O\to 3}\widetilde{P}_k^{E\to 3} + \frac{2}{2}\widetilde{P}_k^{O\to 3}\widehat{P}_k^{E\to 3}\right) \\
&+ p_2\cdot\left(\left(P_k^{O\to 3}\right)^2 + \frac{2}{2} P_k^{E\to 3} P_k^{O\to 3}\right) \\
&\mathbf{P}\left[u \in B_{k+1} \mid u \in W_k^{(2)}\right] = \left(P_k^{O\to 1}\right)^2 + \frac{2}{2} P_k^{O\to 1} P_k^{E\to 1} + 2 P_k^{O\to 1} P_k^{\to 3} \\
&+ (1-p_2)\cdot\left(\left(\widetilde{P}_k^{E\to 3}\right)^2 + 2\widetilde{P}_k^{E\to 3}\widehat{P}_k^{E\to 3} + \frac{2}{2}\widetilde{P}_k^{O\to 3}\widetilde{P}_k^{E\to 3} + \frac{2}{2}\widehat{P}_k^{O\to 3}\widetilde{P}_k^{E\to 3} + \frac{2}{2}\widetilde{P}_k^{O\to 3}\widehat{P}_k^{E\to 3}\right) \\
&+ p_2\cdot\left(\left(P_k^{E\to 3}\right)^2 + \frac{2}{2} P_k^{E\to 3} P_k^{O\to 3}\right)
\end{aligned}$$

A vertex of degree three. The vertex either stays white or is recolored to blue. It stays white if (and only if) all the three paths with inner vertices being white and with degree two are among the following: even paths to a vertex of degree one, non-activated paths to a vertex of degree three, or activated odd paths to a vertex of degree three that was chosen as the beginning (and thus recolored with blue). Hence we obtain that

$$\mathbf{P}\left[u \in B_{k+1} \mid u \in W_k^{(3)}\right] = 1 - \left(P_k^{E\to 1} + \widehat{P}_k^{\to 3} + \frac{1}{2}\widetilde{P}_k^{O\to 3}\right)^3 . \tag{3.7}$$

Plugging all the probabilities from the above analysis together yields that

$$\begin{aligned}
r_{k+1} =& r_k + w_k \cdot \left(\sum_{i=0}^{2} w_k^{(i)} \cdot \mathbf{P}\left[u \in R_{k+1} \mid u \in W_k^{(i)}\right]\right) \\
b_{k+1} =& b_k + w_k \cdot \left(\sum_{i=1}^{3} w_k^{(i)} \cdot \mathbf{P}\left[u \in B_{k+1} \mid u \in W_k^{(i)}\right]\right) \\
w_{k+1} =& 1 - r_{k+1} - b_{k+1} .
\end{aligned}$$

The crucial for the whole analysis is computing the values of $w_{k+1}^{(i)}$. Suppose that u is a white vertex after the k-th round. The values of $w_k^{(i)}$ determine the probability that u has degree i and the values of $q_k^{(i)}$ determine the probabilities that white neighbors of u have certain degrees. In particular, the probability that u has degree i and its neighbors have degrees $j_1, \ldots, j_i$ after the k-th round conditioned by u being white after the k-th round is equal to

$$w_k^{(i)} \cdot \prod_{j \in \{j_1, \ldots, j_i\}} q_k^{(j)} .$$

In what follows, the vector of degrees $j_1, \ldots, j_i$ will be denoted by $\overrightarrow{J}$.

Let $R^{(i)}_{k+1}(\overrightarrow{J})$ be the probability that u is white after the $(k+1)$-th round conditioned by the event that u is white, has degree i and its white neighbors have degrees $\overrightarrow{J}$ after the k-th round. Note that the value of $R^{(i)}_{k+1}(\overrightarrow{J})$ is the same for all permutation of entries/degrees of the vector $\overrightarrow{J}$.

If u has degree three and all its neighbors also have degree three after the k-th round, the probability $R^{(3)}_{k+1}(3,3,3)$ is equal to one: no vertex of degree three can be colored by red and thus the color of u stays white. On the other hand, if u or any of its neighbor has degree one, u does definitely not stay white and the corresponding probability $R^{(i)}_{k+1}(\overrightarrow{J})$ is equal to zero.

We now analyze the value of $R^{(i)}_{k+1}(\overrightarrow{J})$ for the remaining combinations of i and $\overrightarrow{J}$. If $i=2$, then u stays white only if it lies on a non-active path of degree-two vertices between two vertices of degree three. Consequently, it holds that

$$\begin{aligned}R^{(2)}_{k+1}(2,2) &= (1-p_2)^3 \cdot \left(\widehat{P}^{\to 3}_k\right)^2\\ R^{(2)}_{k+1}(3,2) &= (1-p_2)^2 \cdot \widehat{P}^{\to 3}_k\\ R^{(2)}_{k+1}(3,3) &= (1-p_2)\end{aligned}$$

If $i=3$, then u stays white if and only if for every neighbor v of degree two of u, the path of degree-two vertices from v to u is

- an odd path to a vertex of degree one,
- a non-activated path to a vertex of degree three,
- an activated even path to vertex of degree three and u is not chosen as the beginning.

Based on this, we obtain that the values of $R^{(3)}_{k+1}(\overrightarrow{J})$ for the remaining choices of $\overrightarrow{J}$ are the as follows.

$$\begin{aligned}R^{(3)}_{k+1}(2,2,2) &= \left(P^{O\to 1}_k + (1-p_2)\cdot\left(\widehat{P}^{\to 3}_k + \frac{1}{2}\widetilde{P}^{E\to 3}_k\right) + p_2 \cdot \frac{1}{2}P^{E\to 3}_k\right)^3\\ R^{(3)}_{k+1}(2,2,3) &= \left(P^{O\to 1}_k + (1-p_2)\cdot\left(\widehat{P}^{\to 3}_k + \frac{1}{2}\widetilde{P}^{E\to 3}_k\right) + p_2 \cdot \frac{1}{2}P^{E\to 3}_k\right)^2\\ R^{(3)}_{k+1}(2,3,3) &= P^{O\to 1}_k + (1-p_2)\cdot\left(\widehat{P}^{\to 3}_k + \frac{1}{2}\widetilde{P}^{E\to 3}_k\right) + p_2 \cdot \frac{1}{2}P^{E\to 3}_k\end{aligned}$$

We now focus on computing the probabilities $R^{(i\to i')}_{k+1}(\overrightarrow{J})$ that a vertex u is a white vertex of degree i' after the $(k+1)$-th round conditioned by the event that u is a white vertex with degree i with neighbors of degrees in $\overrightarrow{J}$ after the k-th round and u is also white after the $(k+1)$-th round. For example, $R^{(2\to i')}_{k+1}(2,2)$ is equal to one for $i'=2$ and to zero for $i'\neq 2$. To derive formulas for the probabilities $R^{(i\to i')}_{k+1}$, we have to introduce some additional notation: $S^{(i,j)}_{k+1}$ for $(i,j)\in\{(2,2),(2,3),(3,2),(3,3)\}$ will denote the probability that a vertex v is white after the $(k+1)$-th round conditioned by the event that v is a white vertex of degree j after the k-th round and a fixed (white) neighbor u of v has degree i after the k-th round and u is white after the $(k+1)$-th round. It is easy to see that $S_{k+1}^{(2,2)} = 1$. If $j = 3$, the event we condition by guarantees that one of the neighbors of v is white after the $(k+1)$-th round. Hence, we derive that

$$S_{k+1}^{(2,3)} = S_{k+1}^{(3,3)} = \left(P_k^{E\to 1} + \widehat{P}_k^{\to 3} + \frac{1}{2}\widetilde{P}_k^{O\to 3} \right)^2 .$$

Using the probabilities $S_{k+1}^{(i,j)}$, we can easily express some of the probabilities $R_{k+1}^{(i\to i')}(\overrightarrow{J})$.

$$\begin{aligned}
R_{k+1}^{(2\to 2)}(2,3) &= S_{k+1}^{(2,3)} & R_{k+1}^{(2\to 2)}(3,3) &= \left(S_{k+1}^{(2,3)}\right)^2 \\
R_{k+1}^{(2\to 1)}(2,3) &= 1 - S_{k+1}^{(2,3)} & R_{k+1}^{(2\to 1)}(3,3) &= 2 \cdot S_{k+1}^{(2,3)} \left(1 - S_{k+1}^{(2,3)}\right) \\
R_{k+1}^{(2\to 0)}(2,3) &= 0 & R_{k+1}^{(2\to 0)}(3,3) &= \left(1 - S_{k+1}^{(2,3)}\right)^2 \\
R_{k+1}^{(3\to 3)}(3,3,3) &= \left(S_{k+1}^{(3,3)}\right)^3 & R_{k+1}^{(3\to 1)}(3,3,3) &= 3 \cdot S_{k+1}^{(3,3)} \left(1 - S_{k+1}^{(3,3)}\right)^2 \\
R_{k+1}^{(3\to 2)}(3,3,3) &= 3 \cdot \left(S_{k+1}^{(3,3)}\right)^2 \left(1 - S_{k+1}^{(3,3)}\right) & R_{k+1}^{(3\to 0)}(3,3,3) &= \left(1 - S_{k+1}^{(3,3)}\right)^3
\end{aligned}$$

We now determine $S_{k+1}^{(3,2)}$, i.e., the probability that a vertex v is white after the $(k+1)$-th round conditioned by the event that v has degree two after the k-th round and a fixed white neighbor u of u that has degree three after the k-th round is white after the $(k+1)$-th round. Observe that v is white after the $(k+1)$-th round only if v is contained in a non-active 3$\leftrightarrow$3 path. Since we condition by the event that u is white after the $(k+1)$-th round, v cannot be contained in an active 3$\leftrightarrow$3 path of even length or an active 3$\leftrightarrow$3 odd path with u being chosen as the beginning of this path. Hence, the value of $S_{k+1}^{(3,2)}$ is the following.

$$S_{k+1}^{(3,2)} = \frac{(1-p_2)\cdot \widehat{P}_k^{\to 3}}{P_k^{O\to 1} + (1-p_2)\cdot \left(\widehat{P}_k^{\to 3} + \frac{1}{2}\widetilde{P}_k^{E\to 3}\right) + p_2 \cdot \frac{1}{2} P_k^{E\to 3}} .$$

Using $S^{(3,2)}_{k+1}$, the remaining values of $R^{(i\to i')}_{k+1}(\overrightarrow{J})$ can be expressed as follows.

$$
\begin{aligned}
R^{(3\to 3)}_{k+1}(3,3,2) &= \left(S^{(3,3)}_{k+1}\right)^2 \cdot S^{(3,2)}_{k+1}\\
R^{(3\to 2)}_{k+1}(3,3,2) &= \left(S^{(3,3)}_{k+1}\right)^2 \left(1-S^{(3,2)}_{k+1}\right) + 2\cdot\left(1-S^{(3,3)}_{k+1}\right) S^{(3,3)}_{k+1}\cdot S^{(3,2)}_{k+1}\\
R^{(3\to 1)}_{k+1}(3,3,2) &= \left(1-S^{(3,3)}_{k+1}\right)^2 S^{(3,2)}_{k+1} + 2\cdot S^{(3,3)}_{k+1}\left(1-S^{(3,3)}_{k+1}\right)\left(1-S^{(3,2)}_{k+1}\right)\\
R^{(3\to 0)}_{k+1}(3,3,2) &= \left(1-S^{(3,3)}_{k+1}\right)^2\left(1-S^{(3,2)}_{k+1}\right)\\
R^{(3\to 3)}_{k+1}(2,2,3) &= \left(S^{(3,2)}_{k+1}\right)^2 \cdot S^{(3,3)}_{k+1}\\
R^{(3\to 2)}_{k+1}(2,2,3) &= \left(S^{(3,2)}_{k+1}\right)^2 \left(1-S^{(3,3)}_{k+1}\right) + 2\cdot\left(1-S^{(3,2)}_{k+1}\right) S^{(3,2)}_{k+1}\cdot S^{(3,3)}_{k+1}\\
R^{(3\to 1)}_{k+1}(2,2,3) &= \left(1-S^{(3,2)}_{k+1}\right)^2 S^{(3,3)}_{k+1} + 2\cdot S^{(3,2)}_{k+1}\left(1-S^{(3,2)}_{k+1}\right)\left(1-S^{(3,3)}_{k+1}\right)\\
R^{(3\to 0)}_{k+1}(2,2,3) &= \left(1-S^{(3,2)}_{k+1}\right)^2\left(1-S^{(3,3)}_{k+1}\right)\\
R^{(3\to 3)}_{k+1}(2,2,2) &= \left(S^{(3,2)}_{k+1}\right)^3\\
R^{(3\to 2)}_{k+1}(2,2,2) &= 3\cdot\left(S^{(3,2)}_{k+1}\right)^2\left(1-S^{(3,2)}_{k+1}\right)\\
R^{(3\to 1)}_{k+1}(2,2,2) &= 3\cdot S^{(3,2)}_{k+1}\left(1-S^{(3,2)}_{k+1}\right)^2\\
R^{(3\to 0)}_{k+1}(2,2,2) &= \left(1-S^{(3,2)}_{k+1}\right)^3
\end{aligned}
$$

Using $R^{(i\to i')}_{k+1}(\overrightarrow{J})$, we can compute $w^{(i')}_{k+1}$, $i'\in\{0,1,2,3\}$. In the formula below, $\mathcal{J}_i$ denotes the set of all possible vectors $\overrightarrow{J}$ with i entries and all entries either two or three. The denominator of the formula is the probability that the vertex u is white after the $(k+1)$-th round conditioned by the event that u is white after the k-th round; the nominator is the probability that u is white and has degree i' after the $(k+1)$-th round conditioned by the event that u is white after the k-th round.

$$
w^{(i')}_{k+1} = \frac{\displaystyle\sum_{\substack{i\geq 2\\ i\geq i'}}\sum_{\overrightarrow{J}\in\mathcal{J}_i} w^{(i)}_k\cdot\prod_{j\in\overrightarrow{J}} q^{(j)}_k\cdot R^{(i)}_{k+1}\left(\overrightarrow{J}\right)\cdot R^{(i\to i')}_{k+1}\left(\overrightarrow{J}\right)}{\displaystyle\sum_{i\geq 2}\sum_{\overrightarrow{J}\in\mathcal{J}_i} w^{(i)}_k\cdot\prod_{j\in\overrightarrow{J}} q^{(j)}_k\cdot R^{(i)}_{k+1}\left(\overrightarrow{J}\right)}
$$

It remains to exhibit the recurrence relations for the values of $q^{(i)}_{k+1}$. Let uv be an edge of the tree, $i\geq 1$ and $\overrightarrow{J}\in\mathcal{J}_i$. Observe that the probability that u has degree i and its neighbors have degrees $\overrightarrow{J}$ after the k-th round conditioned by the event $uv\subseteq W_k$ is exactly

$$
q^{(i)}_k\cdot\prod_{j\in\overrightarrow{J}} q^{(j)}_k .
$$

In what follows, we will assume that the first coordinate j_1 of $\overrightarrow{J}$ corresponds to the vector v.

The probabilities $Q^{(i)}_{k+1}$ are defined analogically to $R^{(i)}_{k+1}$ with an additional requirement that v is also white after the $(k+1)$-th round. Formally, $Q^{(i)}_{k+1}\left(\overrightarrow{J}\right)$

is the probability that a vertex u and its fixed neighbor v are both white after the $(k+1)$-th round conditioned by the event that $uv \subseteq W_k$, u has degree i and its white neighbors have degrees $\overrightarrow{J}$ after the k-th round. Observe that the following holds.

$$\begin{aligned}
Q^{(2)}_{k+1}(2,2) &= R^{(2)}_{k+1}(2,2) & Q^{(2)}_{k+1}(3,2) &= R^{(2)}_{k+1}(3,2) \cdot S^{(2,3)}_{k+1}\\
Q^{(2)}_{k+1}(2,3) &= R^{(2)}_{k+1}(2,3) = R^{(2)}_{k+1}(3,2) & Q^{(2)}_{k+1}(3,3) &= R^{(2)}_{k+1}(3,3) \cdot S^{(2,3)}_{k+1}\\
Q^{(3)}_{k+1}(j_1,j_2,j_3) &= R^{(3)}_{k+1}(j_1,j_2,j_3) \cdot S^{(3,j_1)}_{k+1}
\end{aligned}$$

Similarly, $Q^{(i\to i')}_{k+1}$ is the probability that a vertex u is a white vertex of degree $i' \geq 1$ and its fixed neighbor v is white after the $(k+1)$-th round conditioned by the event that u is a white vertex with degree i with neighbors of degrees in $\overrightarrow{J}$ after the k-th round and $uv \subseteq W_{k+1}$. Using the arguments analogous to those to derive the formulas for $R^{(i\to i')}_{k+1}(\overrightarrow{J})$, we obtain the following formulas for $Q^{(i\to i')}_{k+1}$. We provide the list of recurrences to compute the values of $Q^{(i\to i')}_{k+1}$ and leave the actual derivation to the reader.

$$\begin{aligned}
Q^{(2\to2)}_{k+1}(2,2) &= Q^{(2\to2)}_{k+1}(3,2) = 1 & Q^{(2\to2)}_{k+1}(2,3) &= Q^{(2\to2)}_{k+1}(3,3) = S^{(2,3)}_{k+1}\\
Q^{(2\to1)}_{k+1}(2,2) &= Q^{(2\to1)}_{k+1}(3,2) = 0 & Q^{(2\to1)}_{k+1}(2,3) &= Q^{(2\to1)}_{k+1}(3,3) = 1 - S^{(2,3)}_{k+1}
\end{aligned}$$

$$\begin{aligned}
Q^{(3\to3)}_{k+1}(3,3,3) &= \left(S^{(3,3)}_{k+1}\right)^2 & Q^{(3\to3)}_{k+1}(3,2,2) &= \left(S^{(3,2)}_{k+1}\right)^2\\
Q^{(3\to2)}_{k+1}(3,3,3) &= 2\cdot S^{(3,3)}_{k+1}\left(1-S^{(3,3)}_{k+1}\right) & Q^{(3\to2)}_{k+1}(3,2,2) &= 2\cdot\left(1-S^{(3,2)}_{k+1}\right)S^{(3,2)}_{k+1}\\
Q^{(3\to1)}_{k+1}(3,3,3) &= \left(1-S^{(3,3)}_{k+1}\right)^2 & Q^{(3\to1)}_{k+1}(3,2,2) &= \left(1-S^{(3,2)}_{k+1}\right)^2\\
Q^{(3\to3)}_{k+1}(3,3,2) &= S^{(3,3)}_{k+1}\cdot S^{(3,2)}_{k+1} & Q^{(3\to3)}_{k+1}(2,3,3) &= \left(S^{(3,3)}_{k+1}\right)^2\\
Q^{(3\to2)}_{k+1}(3,3,2) &= S^{(3,3)}_{k+1}\left(1-S^{(3,2)}_{k+1}\right)+\left(1-S^{(3,3)}_{k+1}\right)S^{(3,2)}_{k+1} & Q^{(3\to2)}_{k+1}(2,3,3) &= 2\cdot\left(1-S^{(3,3)}_{k+1}\right)S^{(3,3)}_{k+1}\\
Q^{(3\to1)}_{k+1}(3,3,2) &= \left(1-S^{(3,3)}_{k+1}\right)\left(1-S^{(3,2)}_{k+1}\right) & Q^{(3\to1)}_{k+1}(2,3,3) &= \left(1-S^{(3,3)}_{k+1}\right)^2\\
Q^{(3\to3)}_{k+1}(2,2,3) &= S^{(3,2)}_{k+1}\cdot S^{(3,3)}_{k+1} & Q^{(3\to3)}_{k+1}(2,2,2) &= \left(S^{(3,2)}_{k+1}\right)^2\\
Q^{(3\to2)}_{k+1}(2,2,3) &= \left(1-S^{(3,2)}_{k+1}\right)S^{(3,3)}_{k+1}+S^{(3,2)}_{k+1}\left(1-S^{(3,3)}_{k+1}\right) & Q^{(3\to2)}_{k+1}(2,2,2) &= 2\cdot\left(1-S^{(3,2)}_{k+1}\right)S^{(3,2)}_{k+1}\\
Q^{(3\to1)}_{k+1}(2,2,3) &= \left(1-S^{(3,2)}_{k+1}\right)\left(1-S^{(3,3)}_{k+1}\right) & Q^{(3\to1)}_{k+1}(2,2,2) &= \left(1-S^{(3,2)}_{k+1}\right)^2
\end{aligned}$$

Using the values of $Q^{(i)}_{k+1}$ and $Q^{(i\to i')}_{k+1}$, we can compute the values of $q^{(i')}_{k+1}$ for $i' \in \{1,2,3\}$. The denominator of the formula is the probability that the vertices u and v are white after the $(k+1)$-th round conditioned by the event that u and v are white after the k-th round; the nominator is the probability that u are v are white and u has degree i' after the $(k+1)$-th round conditioned by the event that u and v are white after the k-th round.

$$q^{(i')}_{k+1} = \frac{\sum\limits_{\substack{i\geq 2\\ i\geq i'}} \sum\limits_{\overrightarrow{J}\in\mathcal{J}_i} q^{(i)}_k \cdot \prod\limits_{j\in\overrightarrow{J}} q^{(j)}_k \cdot Q^{(i)}_{k+1}\left(\overrightarrow{J}\right)\cdot Q^{(i\to i')}_{k+1}\left(\overrightarrow{J}\right)}{\sum\limits_{i\geq 2} \sum\limits_{\overrightarrow{J}\in\mathcal{J}_i} q^{(i)}_k \cdot \prod\limits_{j\in\overrightarrow{J}} q^{(j)}_k \cdot Q^{(i)}_{k+1}\left(\overrightarrow{J}\right)}$$

3.6 Setting up the parameters

The recurrences presented in the previous section were solved numerically using the Python program provided in the Appendix B. The particular choice of parameters used in the program was $p_1 = p_2 = 10^{-5}$ and $K = 307449$. The choice of K was made in such a way that $w_K \leq 10^{-6}$. We also estimated the precision of our calculations based on the representation of float numbers to avoid rounding errors effecting the presented bound on significant digits. Solving the recurrences for the above choice of parameters we obtain that $r_K > 0.4352$.

3.7 Procedure RAND-INDEP-CUBIC

In this section, we show how to modify the procedure RAND-INDEP so that it can be applied to cubic graphs with large girth. In order to use the analysis presented in the previous sections, we have to cope with the dependence of some of the events caused by the presence of cycles in the graph. To do so, we introduce an additional parameter L which will control the length of paths causing the dependencies. Then, we will be able to guarantee that the probability that a fixed vertex of a given cubic graph is red is at least $r_K - o(1)$ assuming the girth of the graph is at least $8KL + 2$. In particular, if L tends to infinity, we approach the same probability as for the infinite cubic tree.

We now describe how the randomized procedure is altered. Let G be the given cubic graph. We produce a sequence $G_0, \overline{G}_1, G_1, \ldots, \overline{G}_K, G_K$ of vertex-colored subcubic graphs; the only vertices in these graphs that have less than three neighbors will always be assigned a new color—black. The graphs can also contain some additional vertices which do not correspond to the vertices of G; such vertices will be called *virtual*. The graph G_0 is the cubic graph G with all vertices colored white. Assume that G_{k-1} is already defined. Let $\overline{G}_k$ be the graph and its vertex coloring obtained from G_{k-1} using the randomized procedure for the k-th round exactly as described for the infinite tree.

Before we describe how the graph G_k is obtained from $\overline{G}_k$, we need to introduce additional notation. Let $y_0, y_1, \ldots, y_{2L}$ be a fixed path in the infinite cubic tree. Now consider restrictions to T_{y_1,y_0} of colorings of the tree after the k-th round that satisfy that

1. the vertices $y_0, y_1, \ldots, y_{2L}$ are white after the k-th round and

2. the vertices $y_1, \ldots, y_{2L-1}$ have degree two.

Let $\mathcal{D}_k$ be the probability distribution on these restrictions such that the probability of each restriction is proportional to the probability of the original coloring after the k-th round. In other words, we discard the colorings that do not satisfy the two constraints, restrict them to T_{y_1,y_0} and normalize the probabilities.

All graphs G_k will satisfy that all paths joining non-virtual vertices contain non-virtual vertices only. The graph G_k is obtained from $\overline{G}_k$ by performing the following operation on every path P of type $1\leftrightarrow 1$, $1\leftrightarrow 3$ or $3\leftrightarrow 3$ between vertices a and b that has length at least $2L$ and contains at least one non-virtual inner vertex. Let x_u be the non-virtual inner vertex of P that is the closest to a and x_v the one closest to b. Let P_x be the subpath between x_u and x_v (inclusively) in $\overline{G}_k$. Let u be the neighbor of x_u on P towards a, and v the neighbor of x_v on P towards b. Observe that, since no path between non-virtual vertices contains a virtual vertex, if a is non-virtual, then x_u is the neighbor of a on P and $u = a$. Similarly, if b is non-virtual, then v is b.

We now modify the graph $\overline{G}_k$ as follows. Color the vertices of P_x black and remove the edges $x_u u$ and $x_v v$ from the graph. Then attach to u and v rooted trees T_u and T_v, all of them fully comprised of virtual vertices, such that the colorings of T_u and T_v are randomly sampled according to the distribution $\mathcal{D}_k$. The roots of the trees will become adjacent to u or v, respectively. These trees are later referred to as *virtual trees*. Observe that we have created no path between two non-virtual vertices containing a virtual vertex.

After the K-th round, the vertices of G receive colors of their counterparts in G_K. In this way, the vertices of G are colored white, blue, red and black and the red vertices form an independent set.

3.8 Refining the analysis

We argue that the analysis for the infinite cubic trees presented in Section 3.5 also applies to cubic graphs with large girth. We start with some additional definitions. Let u and v be two vertices of G_k. The vertex v is *reachable* from u if both u and v are white and there exists a path in G_k between u and v comprised of white vertices with all inner vertices having degree two. Clearly, the relation of being reachable is symmetric. The vertex v is *near* to u if both u and v are white and either v is reachable from u or v is a neighbor of a white vertex reachable from u. Note that the relation of being near is not symmetric in general. For a subset $X \subseteq V(G_k)$ of white vertices, $N_k(X) \subseteq V(G_k)$ is the set of white vertices that are near to a vertex of X in G_k.

We now prove the following theorem.

Theorem 3.7. *Let K be an integer, p_1 and p_2 positive reals, G a cubic graph and v a vertex of G. For every $\varepsilon > 0$ there exists an integer L such that if G has girth at least $8KL+2$, then the probability that the vertex v will be red in G at the end of the randomized procedure is at least $r_K - \varepsilon$ where r_K is the probability that a fixed vertex of the infinite cubic tree is red after the K-th round of the randomized procedure RAND-INDEP with parameters p_1 and p_2.*

Proof. We keep the notation introduced in the description of the randomized procedure. As the first step in the proof, we establish the following two claims.

Claim 1. *Let k be a non-negative integer, u a vertex of G_k, c one of the colors, γ_1 and γ_2 two colorings of vertices of G_k such that u is white in both γ_1 and γ_2. If the set of vertices near to u in γ_1 and γ_2 induce isomorphic trees rooted at u, then the probability that u has the color c in $\overline{G}_{k+1}$ conditioned by the event that G_k is colored as in γ_1, and the probability that u has the color c in $\overline{G}_{k+1}$ conditioned by the event that G_k is colored as in γ_2, are the same.*

Indeed, the color of u in $\overline{G}_{k+1}$ is influenced only by the length and the types of the white paths in G_k containing u. All the vertices of these paths are near to u as well as the white neighbors of the other end-vertices of these paths (which are necessary to determine the types of the paths). By the assumption on the colorings γ_1 and γ_2, the two probabilities from the statement of the claim are the same.

We now introduce yet another definition. For a subcubic graph H with vertices colored white, red, blue or black, a vertex $v \in H$ is said to be *d-close* to a white vertex $u \in H$ if v is white and there exists a path P from u to v comprised of white vertices such that

- the length of P is at most d, or
- P contains a vertex w such that w is at distance at most d from u on P and each of the first $(2L-1)$ vertices following w (if they exist) has degree two.

Finally, a d-close tree of u is the subgraph comprised of all vertices d-close to u (for our choice of d, this subgraph will always be tree) that is rooted at u. By the definition, the d-close tree of u contains white vertices only.

Observe that if v is d-close to u, then it is also d'-close to u for every $d' > d$. Also observe that if a vertex v of a virtual subtree is d-close to a non-virtual vertex u, then all the white vertices lying in the same white component of the virtual subtree are also d-close to u: indeed, consider a path P witnessing that v is d-close to u and let P_0 be its subpath from u to the root of the virtual tree. The path P_0 witnesses that the root is d-close to u and P_0 can be prolonged by the path comprised of $2L-1$ degree-two virtual vertices to a path to any white vertex v' of the same white component as v. The new path now witnesses that v' is also d-close to u.

Let us look at d-close sets in the infinite cubic trees.

Claim 2. *Let d be a non-negative integer, T an infinite cubic tree with vertices colored red, blue and white, and u a white vertex of T. If a vertex v is d-close to u in T, then every vertex v' that is near to v is $(d+2L)$-close to u.*

Let P be a path from u to v that witnesses that v is d-close to u. Assume first that the length of P is at most d. If v' lies on P or is a neighbor of a vertex of P, then v' is $(d+1)$-close to u. Otherwise, consider the path P' from u to v'; observe that P is a subpath of P' and all the vertices following v on P with a possible exception of v' and the vertex immediately preceding it have degree two. If the length of P' is at most $d+2L$, then v' is $(d+2L)$-close to u. Otherwise, v is followed by at least $2L-1$ vertices of degree two and v' is again $(d+2L)$-close to u.

Assume now that the length of P is larger than d. Then, P contains a vertex w at distance at most d from u such that the first $2L-1$ vertices following w (if they exist) have degree two. If v' lies on P or is adjacent to a vertex of P, then v' lies on P after w or is adjacent to w. In both cases, v' is $(d+1)$-close to u. In the remaining case, we again consider the path P' from u to v' which must be an extension of P (otherwise, v' would lie on P or it would be adjacent to a vertex on P). If there are at least $2L-1$ vertices following w on P, then v' is d-close to u. Otherwise, either P' contains $2L-1$ vertices of degree two following w or the length of P' is at most $d+2L$. In both cases, v' is $(d+2L)$-close to u.

We are ready to prove our main claim.

Claim 3. *Let $k \leq K-1$ be a positive integer, T a rooted subcubic tree such that its root is not contained in a path of degree-two vertices of length at least $2L$, u a vertex of the infinite cubic tree and v a non-virtual vertex of G_k. The probability that u is white and the $4(K-k)L$-close tree of u in the infinite tree is isomorphic to T after the k-th round is the same as the probability that v is white and the $4(K-k)L$-close tree of v in G_k is isomorphic to T after the k-th round.*

The proof proceeds by induction on k. For $k=0$, both in the infinite tree and in G_0, the probability is equal to one if T is the full rooted cubic tree of depth $4KL$ and it is zero, otherwise. Here, we use the girth assumption to derive that the subgraph of G induced by $4KL$-close vertices to u is a tree (otherwise, G would contain a cycle of length at most $8KL+1$).

Suppose $k>0$. Let $\widetilde{T}$ be the $4(K-k+1)L$-close tree of v in G_{k-1} and W the set of the vertices that are $(4(K-k)L+2L)$-close to v in G_{k-1}. The degree and the color of each vertex in $\overline{G}_k$ is determined by vertices that are near to it in G_{k-1} by Claim 1. In particular, the degrees and the colors of the vertices of W are determined by $\widetilde{T}$ by Claim 2. The induction assumption implies that every vertex of W is white and has a given degree i in $\overline{G}_k$ with the same probability as its counterpart in the infinite cubic tree (assuming that the vertex u of the infinite tree does not lie on a path of degree-two vertices of length at least $2L$ after the $(k-1)$-th round). If v lies on a path with at least $2L-1$ degree-two vertices in $\overline{G}_k$, it becomes black.

The set W contains all vertices that are $4(K-k)L$-close to v in G_k with the exception of the new virtual vertices that are $4(K-k)L$-close to v. Since the colorings of newly added virtual trees have been sampled according to the distribution $\mathcal{D}_k$, Lemma 3.6 implies that the probability that v is white and the $4(K-k)L$-close tree of v is equal to T is the same as the corresponding probability for u in the infinite cubic tree (conditioned by not being contained in a path of degree-two vertices of length at least $2L$).

Claim 4. *Let k be a non-negative integer and u a vertex of the infinite cubic tree. The probability that u is white and lies on a white path of length at least $2L$ after the k-th round is at most $2 \cdot \left(q_k^{(2)}\right)^{L-1}$.*

If u lies on such a path, its degree must be two and the length of the path from u in one of the two directions is at least $L-1$. The probability that this happens for each of the two possible directions from u is at most $\left(q_k^{(2)}\right)^{L-1}$. The claim now follows.

Let $p_0 = \sum_{k=1}^{K} 2\left(q_k^{(2)}\right)^{L-1}$. Observe that $q_k^{(2)} < 1$. Indeed, if a vertex u of the infinite tree and all the vertices at distance at most $2K$ from u are white after the first round, then u and its three neighbors must have degree three after the K-th round (all vertices at distance at most $2(K+1-k)$ from u are white after the k-th round). Since this happens with non-zero probability, $q_k^{(3)} > 0$ and thus $q_k^{(2)} < 1$. This implies that p_0 tends to 0 with L approaching the infinity.

Fix a vertex v of G. By Claim 1, the probability that v is colored red in the k-th round is fully determined by the vertices that are near to v in G_{k-1}. All such vertices are also $2L$-close to v by Claim 2. Consider a rooted subcubic tree T. If the root of T does not lie on a path with inner vertices of degree two with length at least $2L$, then the probability that v is white and the $2L$-close tree of v is isomorphic to T is the same as the analogous probability for a vertex of the infinite tree by Claim 3. Since the probability that v is white and it lies on a path of length at least $2L$ in its $2L$-close tree at some point during the randomized procedure is at most p_0 by Claim 4, the probability that v is colored red in G_K is at least $r_K - p_0$. Since $p_0 < \varepsilon$ for L sufficiently large, the statement of the theorem follows. $\square$

3.9 Graphs with large odd girth

We conclude with a remark related to Corollary 3.3 whether the fractional chromatic number of cubic graphs is bounded away from 3 under a weaker assumption that the odd girth (the length of the shortest odd cycle) is large. This is indeed the case as we now show.

Theorem 3.8. *Let $g \geq 5$ be an odd integer. The fractional chromatic number of every subcubic graph G with odd girth at least g is at most $\frac{8}{3-6/(g+1)}$.*

Proof. Clearly, we can assume that G is bridgeless. If G contains two or more vertices of degree two, then we include G in a large cubic bridgeless graph with the same odd girth. Hence, we can assume that G contains at most one vertex of degree two. Consequently, G has a 2-factor F.

We now construct a probability distribution on the independent sets such that each vertex is included in the independent set chosen according to this distribution with probability at least $3(1-2/(g+1))/8$. This implies the claim of the theorem.

Number the vertices of each cycle of F from 1 to ℓ where ℓ is the length of the cycle. Choose randomly a number k between 1 and $(g+1)/2$ and let W be the set of all vertices with indices equal to k modulo $(g+1)/2$. Hence, each vertex is not in W with probability $1 - 2/(g+1)$.

Let $V_1, \ldots, V_m$ be the sets formed by the paths of $F \setminus W$. Since each set V_i, $i = 1, \ldots, m$, contains at most $g - 1$ vertices, the subgraph $G[V_i]$ induced by V_i is bipartite. Choose randomly (and independently of the other subgraphs) one of its two color classes and color its vertices red. Observe that if an edge has both its end-points colored red, then it must be an edge of the matching M complementary to F. If this happens, choose randomly one vertex of this edge and uncolor it.

The resulting set of red vertices is independent. We estimate the probability that a vertex v is red conditioned by $v \notin W$. With probability $1/2$, v is initially colored red. However, with probability at most $1/2$ its neighbor through an edge of M is also colored red (it can happen that this neighbor is in W). If this is the case, then the vertex v is uncolored with probability $1/2$. Consequently, the probability that v is red is at least $1/2 \cdot (1 - 1/4) = 3/8$. Multiplying by the probability that $v \notin W$, which is $1 - 2/(g + 1)$, we obtain that the vertex v is included in the independent set with probability at least $3(1 - 2/(g + 1))/8$ as claimed earlier. $\square$

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Appendix A

Program for edge-cuts

```
#include<stdio.h>
#include<stdlib.h>
#include <math.h>

#define THRESHOLD 0.0000001
#define Pinit 0.00001/2 //p_0
#define Pred  0.00001   //p_r
#define Prb   0.000001  //p_rb
#define Pblue 0.1   //p_b

double prob_r[4][4];     //P_k^{r,b}(R)
double prob_b[4][4];     //P_k^{r,b}(B)
double prob_w[4][4];     //P_k^{r,b}(W)

int k;

double p_r,p_b,p_w; //r_k, b_k, w_k
double cut;         //p_k^c

double p[4][4];     //p_k^{r,b}
double qr,qb,qw;    //q_k->k+1^{R/B/W}


int binom(int n, int k) {
    int i;
    int r = 1;

    if (n < k) return 0;

    for (i=1; i<=k; i++) {
    r*=n;
    r/=i;
        n--;
    }

    return r;
}

void init() {
    int r,b,i;

    k=1;

    p_r = Pinit;
    p_b = Pinit;
    p_w = 1 - 2*Pinit;
    cut = 2*Pinit*Pinit;

    for (r=0; r<4; r++) for (b=0; b<4-r; b++) {
        p[r][b] = binom(3,r) * binom(3-r,b);
        for (i=0;i<r+b;i++) p[r][b]*= Pinit;
      for (i=0;i<3-r-b;i++) p[r][b]*=1-2*Pinit;
    }
}
```

```
void comp_trans() {
    int r,b;
    double pe = 0; //p_k^WW

    for (r=0; r<3; r++) for (b=0; b<3-r; b++)
        pe += p[r][b]*(3-r-b)/3;

    qr = qb = qw = 0;

    for (r=0; r<3; r++) for (b=0; b<3-r; b++) {
        qr += p[r][b]*prob_r[r][b]*(3-r-b)/3;
        qb += p[r][b]*prob_b[r][b]*(3-r-b)/3;
        qw += p[r][b]*prob_w[r][b]*(3-r-b)/3;
    }

    qr /= pe; qb /= pe; qw /= pe;
}

void step() {
    int r,b,r_,b_, i;
    double x;

    double pw; //p_k^W
    double p_new[4][4];

    k++;
    comp_trans();

    x = 0;
    for (r=0; r<4; r++) for (b=0; b<4-r; b++) {
        x+=p[r][b] * ( prob_r[r][b]*(b+.5*(3-r-b)*qb)
          + prob_b[r][b]*(r+.5*(3-r-b)*qr) );
    }
    cut+=2*p_w*x/3;

    x = 0;
    for (r=0; r<4; r++) for (b=0; b<4-r; b++)
        x+= p[r][b]*prob_r[r][b];
    p_r += p_w*x;

    x = 0;
    for (r=0; r<4; r++) for (b=0; b<4-r; b++)
        x+= p[r][b]*prob_b[r][b];
    p_b += p_w*x;

    p_w = 1-p_b-p_r;

    pw = 0;
    for (r=0; r<4; r++) for (b=0; b<4-r;b++)
        pw += p[r][b]*prob_w[r][b];


    for (r=0; r<4; r++) for (b=0; b<4-r; b++) {
        p_new[r][b]=0;

        for (r_=0; r_<=r; r_++) for (b_=0; b_<=b; b_++) {
            x = p[r_][b_] * prob_w[r_][b_] * binom(3-r_-b_,r-r_)
              * binom(3-r-b_,b-b_);

            for (i=0; i<r-r_; i++) x*=qr;
            for (i=0; i<b-b_; i++) x*=qb;
            for (i=0; i<3-r-b; i++) x*=qw;

            p_new[r][b] += x;
        }
        p_new[r][b] /= pw;
    }

    for (r=0; r<4; r++) for (b=0; b<4-r;b++)
        p[r][b] = p_new[r][b];
}
```

```
int main() {
    int r,b;

    for (r=0; r<4; r++) for (b=0; b<4-r; b++) {
        prob_r[r][b] = prob_b[r][b] = 0;
        if (r+b==0) { prob_w[r][b] = 1; continue; }
        if (r>=2) { prob_b[r][b] = 1; continue; }
        if (b>=2) { prob_r[r][b] = 1; continue; }
        if (r+b==2) { prob_w[r][b] = 1; continue; }

        if (r) {
            prob_b[r][b] = Pblue; prob_w[r][b] = 1-Pblue;
            continue;
        }
        if (b) {
            prob_r[r][b] = Pred; prob_w[r][b] = 1 - Pred;
            continue;
        }
    }

    init();

    while (p_w*(p[1][0] + p[0][1]) > THRESHOLD) {
        step();
    }

    prob_r[1][1]=Prb/2;
    prob_b[1][1]=Prb/2;
    prob_w[1][1]=1-Prb;

    while (p_w > THRESHOLD) {
        step();
    }

    printf("%d rounds (girth => %d), prob. %.7lf (r=%.7lf b=%.7lf)\n",
        k,2*k+1,cut,p_r,p_b);

    return 0;
}
```

Appendix B

Program for independent sets

```
W_THOLD = 1.0/1000000
p_1 = 0.00001
p_2 = 0.00001

def state():
   print("%d: r=%f b=%f w=%f q3=%f q2=%f q1=%f w3=%f w2=%f w1=%f w0=%f"
       % (k, p_r, p_b, p_w, q[3], q[2], q[1], w[3], w[2], w[1], w[0]))

def gen_C():
   S= {1: q[1], 2: q[2], 3: q[3]}
   for u in (1,2,3):
      A=(1,2,3)
      B=S if (u>1) else {"-":1}
      C=S if (u>2) else {"-":1}
      for a in A:
       for b in B:
        for c in C:
           C3[(u,a,b,c)] = w[u]*q[a]*B[b]*C[c]
           C2[(u,a,b,c)] = q[u]*q[a]*B[b]*C[c]


def init():
   global p_r, p_w, p_b

   t = (1-p_1)**2
   w[3] = t**3
   w[2] = 3*(t**2)*(1-t)
   w[1] = 3*t*(1-t)**2
   w[0] = (1-t)**3

   q[3] = t**2
   q[2] = 2*t*(1-t)
   q[1] = (1-t)**2

   p_b = 1-(1-p_1)**3
   p_r = p_1*(1-p_b)
   p_w = 1 - p_r - p_b

k=0
p_w = 1
p_b = p_r = 0

w = [0,0,0,3]
q = [0,0,0,1]
C3 = {}
C2 = {}

while (p_w > W_THOLD):
   k+=1

   if (k==1):
      init()
      state()
      continue
```

```
o1 = q[1]/(1-q[2]**2)
e1 = q[2]*o1
p1 = o1 + e1

o3 = q[3]/(1-q[2]**2)
e3 = q[2]*o3
p3 = o3 + e3

p3_n = q[3]/(1-q[2]*(1-p_2))
o3_n = q[3]/(1-q[2]**2 * (1-p_2)**2)
e3_n = q[2]*(1-p_2)*o3_n

o3_y = (q[2]**2 *(1-(1-p_2)**2)*o3)/(1-q[2]**2 *(1-p_2)**2)
e3_y = q[2]*(p_2*o3 + (1-p_2)*o3_y)

Pr = [1,0,0,0]
Pb = [0,0,0,0]

Pr[1] = (e1 + o1*.5) + p3
Pb[1] = o1*.5

Pr[2] = e1**2 + e1*o1 + 2*e1*p3
Pr[2]+= (1-p_2)*(o3_y**2 +2*o3_y*o3_n+o3_y*e3_y+o3_n*e3_y+o3_y*e3_n)
Pr[2]+= p_2*(o3**2 + o3*e3)

Pb[2] =  o1**2 + e1*o1 + 2*o1*p3
Pb[2]+= (1-p_2)*(e3_y**2 +2*e3_y*e3_n+o3_y*e3_y+o3_n*e3_y+o3_y*e3_n)
Pb[2]+= p_2*(e3**2+o3*e3)

Pb[3] = 1-(p3_n+e1+o3_y/2)**3

p_r+= p_w*(w[0]+w[1]*Pr[1]+w[2]*Pr[2])
p_b+= p_w*(w[1]*Pb[1]+w[2]*Pb[2]+w[3]*Pb[3])
p_w = 1-p_r-p_b

r32 = o1+(1-p_2)*(p3_n+.5*e3_y)+p_2*(.5*e3)
s33 = (p3_n+e1+o3_y/2)**2
s32 = (1-p_2)*p3_n / r32

gen_C()

T = [0,0,0,0]
sum = 0
for C in C3:
   deg1=False
   for x in C:
      if x==1:
         deg1=True
         break
   if deg1: continue

   N={}
   C_p = C3[C]



   if (C[0]==2):
       N[3] = {0:1}
      C_p*=1-p_2
      for i in (1,2):
         if (C[i]==3):
            N[i] = {0: s33 }
            N[i][1] = 1-N[i][0]
         elif (C[i]==2):
            C_p *= (1-p_2)*p3_n
            N[i] = {0: 1}

   elif (C[0]==3):
      for i in (1,2,3):
         if (C[i]==3):
            N[i] = {0: s33 }
         elif (C[i]==2):
```

```
            C_p *= r32
            N[i] = {0: s32 }

         N[i][1] = 1-N[i][0]

     sum += C_p
   for a in N[1]:
      for b in N[2]:
         for c in N[3]:
            T[C[0]-a-b-c] += C_p*N[1][a]*N[2][b]*N[3][c]

for i in (0,1,2,3): w[i] = T[i]/sum

T = [0,0,0,0]
sum = 0
for C in C2:
   deg1=False
   for x in C:
      if x==1:
         deg1=True
         break
   if deg1: continue

   N={}
   C_p = C2[C]

   if (C[1]==3):
      C_p *= s33
   elif (C[1]==2):
      C_p *= (1-p_2)*p3_n

   if (C[0]==2):
       N[3] = {0:1}
      C_p*=1-p_2

      if (C[2]==3):
         N[2] = {0: s33 }
         N[2][1] = 1-N[2][0]
      elif (C[2]==2):
         C_p *= (1-p_2)*p3_n
         N[2] = {0: 1}

   elif (C[0]==3):
      for i in (2,3):
         if (C[i]==3):
            N[i] = {0: s33 }
         elif (C[i]==2):
            C_p *= r32
            N[i] = {0: s32 }

         N[i][1] = 1-N[i][0]

     sum += C_p
   for b in N[2]:
      for c in N[3]:
         T[C[0]-b-c] += C_p*N[2][b]*N[3][c]

for i in (1,2,3): q[i] = T[i]/sum

state()
```