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MASTER THESIS

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Markovské semigrupy

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I declare that I carried out this master thesis independently, and only with the cited sources, literature and other professional sources.

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In .... date ....

Název práce: Markovské semigrupy

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Abstrakt: V předložené práci studujeme existenci periodického řešení nekonečně rozměrné stochastické rovnice s periodickými koeficienty řízené cylindrickým Wienerovým procesem. Užitá teorie nekonečně rozměrných stochastických rovnic v Hilbertových prostorech a Markovských procesů je shrnuta v prvních dvou kapitolách. Ve třetí a závěrečné kapitole je prezentován samotný výsledek. Potřebné technické zázemí zejména z operátorové teorie je shrnuto v Dodatku. Náš důkaz existence periodického řešení příslušné rovnice je kombinací argumentů Chasminského, který zaručuje za jistých podmínek existenci periodického Markovského procesu, a výsledků Da Prata, Gątarka a Zabczyka pro existenci invariantní míry pro homomogenní stochastické rovnice v Hilbertových prostorech. Na závěr odvodíme postačující podmínky existence periodického řešení v řeči koeficientů užitím práce Ichikawovy a ilustrujeme výsledky na příkladě stochastické PDR. Práce je psaná v angličtině.

Klíčová slova: Stochastické diferenciální rovnice, Markovské procesy, Semigrupy

Title: Markov Semigroups

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Abstract: In the presented work we study the existence of periodic solution to infinite dimensional stochastic equation with periodic coefficients driven by Cylindrical Wiener process. Used theory of infinite dimensional stochastic equations in Hilbert spaces and Markov processes is summarized in the first two chapters. In the third and last chapter we present the result itself. Necessary technical background mostly from operator theory is encapsulated in the Appendix. The proof of existence of periodic solution of corresponding equation is a combination of arguments by Khasminskii, which ensure under suitable conditions the existence of periodic Markov process, and the results of Da Prato, Gątatrek and Zabczyk for the existence of invariant measure for homogeneous stochastic equation in Hilbert spaces. At the end we derive sufficient condition for the existence of periodic solution in the language of coefficients using the work of Ichikawa and illustrate the results by the example of Stochastic PDE. The work is written in English.

Keywords: Stochastic differential equations, Markov processes, Semigroups

Contents

Preface

While I am not sure, that preface is a necessary part of work such as diploma thesis, I could not resist to write one. It is here, where I can write casually, about things not completely related to mathematics and it was the part, on which the work was definitely most enjoyable for me.
The aim of the thesis is to present the thorough and detailed proof of the existence of a periodic solution to Stochastic evolution equation with periodic coefficients driven by Cylindrical Wiener process. I try to make the work as self-contained as possible. Due to the relatively heavy technical material surrounding the field, it is of course only moderately possible without going too far (and I definitely think excessively long thesis is an unwelcome solution for all involved, especially for the advisor and external examiner) or including some difficult proofs and concepts. However I do not like texts, where no proofs are offered, so we do include some proofs in the first two chapters, hopefully to offer balanced presentation in easily digestible style (a critique could yet argue that we include only the very easy pieces from the overall picture and such claim would not be completely unfair to be honest). So by this notion I mean that all the results ordinarily taught in the masters courses in our faculty are assumed and used without any references (especially results from key lectures such as Probability theory, Stochastic analysis, Stochastic differential equations and Markov processes, as well as basic facts from Functional analysis) and basically anything beyond should be said in at least informative way. After all, since I had to get acquainted with all the necessary material myself too, I simply include everything that is new or surprising to me. Since this is neither research article nor an official textbook, I dare to adopt sometimes a somewhat leisure style, because I do no longer think mathematical texts must be an all-time utter rigorous presentation in say Jarník's style (I had a great pleasure reading parts of splendid Kröner's book about elementary analysis [12], enjoying especially the less formal parts).
The method of obtaining an existence of a periodic solution is still just the employment of a famous method discovered by Krylov-Bogolyubov back in 1937 (see [2]). The technicalities grow though, fortunately practically all of the raising questions are solved by connection of finite dimensional Khasminski's results and the results presented in G. Da Prato and J. Zabczyk (see [4], [11]). When I began to write down the used material in the proof of periodic solution I discovered after a short trial that reasonable division of the thesis will be three parts. In the first one Cylindrical Wiener process is defined and we indicate how can one define the reasonable integration theory with respect to it. The second part gives the definition of so called mild solution of Stochastic evolution equation and we derive the basic existence and uniqueness result for Lipschitz nonlinearities together with the discussion of Markov property of solution. The last chapter presents the proof of a periodic solution itself. The used facts from operator theory we summarize in the Appendix.
Most of the necessary background material for infinite dimensional stochastic

equation we naturally draw from [6], sometimes we refer to [7] or [4]. In the case the material can be find in Seidler's excellent study material [22], we prefer to give reference to this perhaps only Czech literature on the related subjects. From the many existing reliable sources for functional analysis written by first class mathematicians I shall refer mostly to Lax with his formidable style [15], complementary material we sometimes capitalize from Lukeš's entertaining volume [18]. I like the idea expressed in Stroock [23], where he states in the preface that he wrote his book primarily for himself and to his own satisfaction. While I admit that my driving force for writing the thesis was the promise of obtaining a master's degree, I tried to write the thesis that would be comprehensible to me and would be written in a style that I alone enjoy and appreciate when reading mathematics from other sources.

1. Wiener process and stochastic integration in infinite dimension

1.1 Preliminaries

To simplify the expression in the whole work by a Hilbert space we will automatically understand a separable complete space with the inner product. Note that same approach is followed for example by such eminent analyst as Stein in [24], so we are not in a bad company. Unless otherwise specified, we suppose spaces to be real.
All the deterministic integral in the text with values in Banach spaces can be understood in Bochner sense. We use only the truly basic properties of integral in the text and since it is neither surprising nor very mathematically deep, we won't make a special covering of the topic in our text (an interested reader can see f. e. [20]). Same applies for random variables with values in Banach spaces, we need only elementary facts, for which one can look in [6]. The definition are however practically identical as in the classical case, so we omit a further discussion of the topic too.

Gaussian measure

Definition. *Let E be a separable Banach space. A probability measure μ is said to be a Gaussian measure, provided that the law of arbitrary linear functional E^* considered as a random variable on $(E, \mathcal{B}(E), \mu)$, is a Gaussian measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. If in addition the law of each $h \in E^*$ has mean zero, we call μ a symmetric Gaussian distribution.*

In the case μ is a Gaussian measure on a Hilbert space H, we can say more. In particular, there exists an element $m \in H$ and a symmetric nonnegative trace class operator $Q \in L(H)$ such that :

$$\int_H < h, x > \mu(dx) =< m, h > \ \ \forall h \in H,$$

$$\int_H < h_1, x >< h_2, x > \mu(dx)- < m, h_1 >< m, h_2 >=< Qh_1, h_2 > \ \ \forall h_1, h_2 \in H.$$

m is then called a mean and Q the covariance operator of μ. Gaussian measure with mean m and covariance operator Q we denote as $\mathcal{N}(m, Q)$. Naturally by a Gaussian process in H we understand a process whose finite dimensional distribution are Gaussian.

For an extensive treatment of Gaussian measures on infinite dimensional spaces one can look in [13] (although this book now is a bit outdated), sufficient covering of the topic is also in [6], where can be found the proofs of the above mentioned facts.

1.2 Q - Wiener process and Cylindrical Wiener process

Definition. *Let H be a Hilbert space and $Q \in L(H)$ a symmetric nonnegative trace class operator. A H valued stochastic process $W(t), t \geq 0$ is called a Q-Wiener process if*
(i) $W(0) = 0$
(ii) W has continous trajectories
(iii) W has independent increments
(iv) $\mathcal{L}(W(t) - W(s)) = \mathcal{N}(0, (t-s)Q), t \geq s \geq 0$.

It follows easily from definition that Q-Wiener process is a Gaussian process with mean zero and covariance operator tQ. Since Q is a trace class and symmetric operator, there exists an orthonormal basis $\{e_k\}$ and nonnegative real numbers $\{\lambda_k\}$, $\sum \lambda_k < \infty$ such that $Qe_k = \lambda_k e_k$, $\forall k$ is valid (this is the famous Hilbert-Schmidt spectral decompostion of compact normal operators, see Appendix for details). Let us note that straightforwardly from the Kolmogorov extension theorem one obtains for arbitrary trace class symmetric nonnegative operator Q on a Hilbert space H a corresponding Q-Wiener process (see [6], page 88). Our goal in this section is to define a Cylindrical Wiener process, the following easy proposition gives us formally a different description of Q- Wiener process, which however illuminates the correspondence (and also the difference) between these two notions.

Proposition 1.2.1. *Assume that W is a Q-Wiener process, with tr $Q < +\infty$. For arbitrary t, W has the expansion*

$$W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j} \beta_j(t) e_j \tag{1.1}$$

where

$$\beta_j(t) = \frac{1}{\sqrt{\lambda_j}} < W(t), e_j >, \; j = 1, 2, \ldots, \tag{1.2}$$

are real valued Brownian motions mutually independent on $(\Omega, \mathcal{F}, P)$ and the series 1.1 is convergent in $L^2(\Omega, \mathcal{F}, P)$.

Proof. Because $W(t)$ has a gaussian distribution on H, we easily deduce, that $\beta_j, j = 1, 2, \ldots$ is indeed a Wiener process on real line. To obtain the claimed independence of β_j, let $t > s > 0$, it follows from the (1.2)

$$E(\beta_i(t), \beta(j)) = \frac{1}{\sqrt{\lambda_i \lambda_j}} E(< W(t), e_i >< W(s), e_j >) =$$

$$= \frac{1}{\sqrt{\lambda_i \lambda_j}} [E(< W(t) - W(s), e_i >< W(s), e_j >)$$

$$+E(< W(s), e_i >< W(s), e_j >)]$$
$$= \frac{1}{\sqrt{\lambda_i \lambda_j}} s < Qe_i, e_j >= s\delta_{ij},$$

therefore thanks to the properties of normal distribution we have the desired independence of $\beta_i, i = 1, 2, \ldots$. The representation (1.1) itself is trivial, however one must check the convergence of the sum 1.2 in $L^2(\Omega, \mathcal{F}, P)$. To this purpose notice that for arbitrary $m \geq n \geq 1$,

$$E||\sum_{j=n}^{m} \sqrt{\lambda_j}\beta_j(t)e_j||^2 = E||\sum_{j=n}^{m} < W(t), e_j > e_j||^2$$

$$= E < \sum_{j=n}^{m} < W(t), e_j > e_j, \sum_{j=n}^{m} < W(t), e_j > e_j >$$

$$= \sum_{j=n}^{m} E << W(t), e_j > e_j, < W(t), e_j > e_j >= \sum_{j=n}^{m} E < W(t), e_j >^2$$

$$= t\sum_{j=n}^{m} < Qe_j, e_j >= t\sum_{j=n} m < \lambda_j e_j, e_j >= t\sum_{j=n}^{m} \lambda_j. \tag{1.3}$$

The last sum in (1.3) is convergent since Q is a trace class operator, hence the proposition follows. □

With a little more work the sum in (1.1) can be shown to be uniformly convergent even P-a.s. on $[0, T]$, for all $T > 0$. We arrive at the Cylindrical Wiener process by an analogy of (1.1), where we could formally define an expression $W(t) = \sum_{j=1}^{\infty} e_j\beta_j(t)$, however this sum fails to converge. To see that it is not even an element of $L^2(\Omega, \mathcal{F}, P)$ it suffices to repeat the calculation in the above proof. Relatively easy approach to Cylindrical Wiener process is offered by the following proposition taken from [6], which avoids the more demanding view of Cylindrical Wiener process as a process having values in certain dual space (see [7], [13]).

Proposition 1.2.2. *Let Q be a bounded, symmetric, strictly positive operator on a Hilbert space H. Let $H_0 = Q^{1/2}(H)$ with the induced norm $||h||_0 = ||Q^{-1/2}(h)||_H$, $h \in H_0$. Further let H_1 be an arbitrary Hilbert space such that H is embedded continuously into H_1 and moreover the embedding of H_0 into H_1 is Hilbert-Schmidt. Let $\{e_j\}$ be an orthonormal and complete basis in H_0 and $\{\beta_j\}$ a family of independent real valued Wiener processes. Then the formula*

$$W(t) = \sum_{j=1}^{\infty} e_j\beta_j(t), \tag{1.4}$$

defines a Q_1- Wiener process on H_1 with $tr\ Q_1 < +\infty$. Moreover we have $Im\, Q_1^{1/2} = H_0$ and $||h||_0 = ||Q_1^{-1/2}h||_1$.

Proof. The series defining $W(t)$ is convergent in $L^2(\Omega, \mathcal{F}, P, H_1)$ because

$$E(||\sum_{j=n}^{m} e_j\beta_j(t)||_1^2) = E < \sum_{j=n}^{m} e_j\beta_j(t), \sum_{j=n}^{m} e_j\beta_j(t) >_1=$$

$$= \sum_{j=n}^{m} < e_j, e_j >_1 E\beta_j^2(t) = \sum_{j=n}^{m} t||e_j||_1^2, \; m \geq n \geq 1 \tag{1.5}$$

and the embedding $J : H_0 \to H_1$ is Hilbert-Schmidt, hence the sum in (1.5) is convergent. Seeing that we obtain by (1.4) indeed a Q_1 - Wiener process on H_1 is easy from the definition and already said facts. By the definition of gaussian measure $\mathcal{L}(W(t)-W(s))$ is normal is equivalent to $< \sum\limits_{j=1}^{\infty} e_j\beta_j(t) - \sum\limits_{j=1}^{\infty} e_j\beta_j(s), h >_1$ being one dimensional normal, which reduces the question to a well known fact that the limit of one dimensional gaussian distributions in again gaussian. Item (i) is obvious and (iii) follows from the same property of β_j. Because the distribution is gaussian, there exists a corresponding trace class operator $Q_1 \in L(H_1)$. Hence we can find a version of (1.4) such that $W(t) = \sum_{j=1}^{\infty} e_j\beta_j(t)$ defines a Q_1-Wiener process on H_1. To prove the later part, calculate from the definition that

$$< Q_1 a, b >_1 = E < a, W(1) >_1 < b, W(1) >_1 = \sum_{j=1}^{\infty} < a, e_j >_1 < b, e_j >_1$$

$$= \sum_{1}^{\infty} < a, Je_j >_1 < b, Je_j >_1 = \sum_{j=1}^{\infty} < J^*a, e_j >_0 < J^*b, e_j >_0$$

$$=< J^*a, J^*b >_0 =< JJ^*a, b >_1,$$

where we used a Parseval equality in the penultimate equality. Hence from the well known property of operators on Hilbert space we get $Q_1 = JJ^*$. In particular

$$||Q_1^{1/2}a||_1^2 =< JJ^*a, a >_1 = ||J^*a||_0^2, \; a \in H_1. \tag{1.6}$$

Thus $Im\, Q_1^{1/2} = Im\, J = H_0$ and the operator $G = Q_1^{-1/2}J$ is bounded from H_0 onto H_1. From 1.6 it follows that $G^* = J^*Q_1^{-1/2}$ is an isometry, hence G itself is an isometry. So $||Q_1^{-1/2}u||_1 = ||Q_1^{-1/2}Ju||_1 = ||u||_0$ as desired. □

In the case the operator Q is trace class, then $Q^{1/2}$ is Hilbert-Schmidt operator (this is obvious from the trace class formula, see Theorem A.2.4) and if we simply put $H_1 = H$ we arrive at the previously defined concept of Q-Wiener process. If $tr\, Q = +\infty$ the constructed process is called a *Cylindrical Wiener process* on H. It is not uniquely determined for a sum (1.4), however last Proposition ensures that the spaces $Q_1^{1/2}(H_1)$ are identical for all possible extensions H_1.

1.3 Stochastic integration on Hilbert spaces

We will not formally describe the stochastic integration with respect to Cylindrical Wiener process, it is technically quite complicated and would lead us to far from our main topic, since our concern is not a stochastic integration itself. Instead let us just describe the definition domain of the integral and its useful properties for what follows.

Q-Wiener process case

For a Q-Wiener process in $(\Omega, \mathcal{F}, P)$ with values in Hilbert space H equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ in $\mathcal{F}$ (i. e. $W(t)$ is adapted to it and $W(t+h)-W(t)$ is independent of $\mathcal{F}_t$) the construction of stochastic integration can be carried in essentially the same way as in the one dimensional case (of course the technicalities grow). Let U be another a Hilbert space, denote as in the last proposition $H_0 = Q^{1/2}(H)$ the Hilbert subspace of H equipped with the same norm as above. Denote further $L_2^0 = L_2(H_0, U)$ the class of Hilbert-Schmidt operators between H_0 and U, which is again a Hilbert space with the inner product $< F, G >_0 = Tr(GQF^*)$. The integral is first defined in a natural way for a simple processes taking values in L_2^0 on $[0,T]$. The simple processes are dense in the space of predictable square integrable processes with values in L_2^0, i. e. predictable processes Φ on $[0,T]$, such that $||\Phi||_T = (E\int_0^T ||\Phi(s)||^2_{L_2^0} ds)^{1/2} < +\infty$, denote this space as $\mathcal{P}_W^2(0,T)$. Again similarly as in the finite dimension, by the localization method the space of integrands $\mathcal{P}_W^2(0,T)$ can be expanded to L_0^2 predictable processes satisfying a weaker condition $P(\int_0^T ||\Phi(s)||_{L_0^2} ds < +\infty)$. We denote this space as $\mathcal{P}_W(0,T)$. However one loses the martingale property of the stochastic integral with these processes and integrands. Sufficient treatment of integration in Hilbert spaces with respect to Q-Wiener process can be found for example in [6] and [7]. I think Chow's book is little more intelligible on this topic, on the other hand it is less complete than Da Prato and Zabczyk.

Cylindrical Wiener process

The above indicated construction can not be directly applied in the case of Cylindrical Wiener process W. One possible way of defining integral with respect to a process constructed in Proposition 1.2.2 is suggested in [6] (see page 99 there for details). The method is based on the approximative procedure and is relatively simple. The Wiener process defined in the Proposition 1.2.2 has the representation by the infinite sum $W(t) = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\beta_j(t)e_j$, where $\{\lambda_j, e_j\}$ is an eigensequence defined by Q_1 (see Proposition 1.2.1). The integral for $\Phi \in \mathcal{P}_W^2$ with respect to Wiener process of the form $W_n(t) = \sum\limits_{j=1}^{n} \sqrt{\lambda_j}\beta_j(t)e_j$ can be defined classically since W_n is a finite dimensional Wiener process. Nice ingenious argument based on the theory of integration with respect to $Q-$Wiener process gives that the sequence of stochastic integrals $X_\cdot = (\Phi \cdot W_N)_\cdot = \inf_0^\cdot \Phi(s)dW_s$ contains

a P-a.s. converging subsequence (the convergence being uniform on $[0,T]$) and the limit is independent of the choice of subsequence. The integral with respect to cylindrical Wiener process can thus be defined as such limit and the properties of stochastic integral then follows from properties of finite dimensional stochastic integration.
For the sake of clarity let us now summarize the basic properties of stochastic integral in a theorem, whose proof can be find in [6] or in [7] for the case of Q-Wiener process.

Theorem 1.3.1. *Assume that $\Phi \in \mathcal{P}_W^2(0,T)$, then the stochastic integral $X_t = \int_0^t \Phi(s)dW_s = (\Phi \cdot W)_t, 0 \le t \le T$ is a continuous L^2-martingale in H with mean zero and covariance operator $\int\limits_0^t \Phi(s)Q\Phi(s)^* ds$, i. e.*

$$E(X_t, g) = 0, \;\; E(X_t, g)(X_s, h) = E\int_0^{t\wedge s} (\Phi_r Q \Phi_r^* g, h) dr \;\forall g, h, \in H.$$

For $\Phi_1, \Phi_2 \in \mathcal{P}_W^2(0,T)$ we have $E(\Phi_i \cdot W(t)) = 0$, $i = 1, 2$ and

$$E < \Phi_1 \cdot W(t) >< \Phi_2 \cdot W(t) >= E\int\limits_0^{t\wedge s} Tr[(\Phi_2(r)Q^{1/2})(\Phi_1(r)Q^{1/2})^*]dr.$$

An essential tool for handling the nonlinear stochastic infinite dimensional equations is a factorization formula, which we will introduce later in the chapter two. Its proof is in fact an easy corollary of the stochastic Fubini theorem, which is of course a result in the integration theory, therefore we place it here. The proof can be find again in [6]. We denote the predictable σ-algebra on $[0,T] \times \Omega$ as $\mathcal{P}_T$.

Theorem 1.3.2 (Stochastic Fubini theorem)**.** *Let $(E, \mathcal{E})$ be a measurable space and let $\Phi : (t, \omega, x) \to \Phi(t, \omega, x)$ be a measurable mapping from $(\Omega_T \times E, \mathcal{P}_T \times \mathcal{B}(E))$ into $(L_2^0, \mathcal{B}(L_2^0))$, further let μ be a finite positive measure on $\mathcal{E}$. Assume that $\int_E ||\Phi(\cdot,\cdot,x)||_T \mu(dx) < +\infty$, then P-a.s.*

$$\int_E \left[\int\limits_0^T \Phi(t,\omega,x)dW(t)\right]\mu(dx) = \int\limits_0^T \left[\int_E \Phi(t,x)\mu(dx)\right] dW(t).$$

2. Infinite dimensional stochastic equations

In this chapter we will study mild solutions to nonlinear infinite dimensional equations with Cylindrical Wiener process as driving noise. The main purpose of this chapter is nevertheless to prepare all the necessary tools for third chapter, so we omit most complicated proofs and technical details, the emphasis is rather on the general overview, hence we also do not present the results in utmost generality available.

2.1 Formulation of the problem

Assume we have a fixed probability space $(\Omega, \mathcal{F}, P)$ with normal filtration (i. e. complete right-continuous). We will study the equation of the symbolic form

$$dX = (AX + F(t, X))dt + B(t, X)dW(t), \; X(0) = \xi. \tag{2.1}$$

on a Hilbert space U. W is in this chapter for us a Cylindrical Wiener process on H (possibly different space from U), i. e. (see previous chapter for details) Q_1 Wiener process on $H_1 \supset H$. To simplify the expression, we shall assume that $Q \in L(H)$ is an identity operator, thus we have $H_0 = H$ and the notation will be simplified accordingly. Denote $L(H, U)$ and $L_2(H, U)$ the classes of operators from H into U respectively Hilbert Schmidt operators from H into U, the Hilbert-Schmidt norm on $L_2(H, U)$ we denote simply $|| \cdot ||_{L_2}$. We will impose the following conditions, which roughly speaking ensure, that the integrals are well-defined and gives us an unique solution to problem (2.1) :

Hypothesis 2.1
(i) A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$, on U.
(ii)F is a measurable mapping from $[0, T] \times U$ into U and there exists a constant C_0 such that

$$|F(t, x)| \leq C_0(1 + |x|), \; x \in U,$$

$$|F(t, x) - F(t, y)| \leq C_0|x - y|, \; x, y \in U,$$

is valid for all $t \in [0, T]$.
(iii) For fixed t, B is a strongly continuous mapping from U into $L(H, U)$, i. e. for all $h \in H$ the mapping $u \to B(t, u)h$ from U into U is continuous. Moreover for any $t, s \geq 0$ and $x \in U$, $S(t)B(s, x)$ belongs to $L_2(H, U)$, and there exists a locally square integrable mapping

$$K : [0, +\infty) \to [0, +\infty), t \to K(t),$$

such that for any t there is a nontrivial interval J_t containing t, satisfying $\inf_{t\in[0,T]} J_t > 0$ so that for arbitrary $s \in J_t$ we have

$$||S(t)B(s,x)||_{L_2} \leq K(t)(1+|x|),\ x \in U$$
$$||S(t)B(s,x) - S(t)B(s,y)||_{L_2} \leq K(t)|x-y|,\ x,y \in U.$$

Comment. We do not need to specify the interval J in Hypothesis 2.1 to prove the existence and uniqueness of the solution, all what we need is to suppose the estimates hold on some interval J. We could replace the condition (iii) (as is done in [6], see their paragraph 7.1.1) for the purposes of this section by the following formally weaker one :
There exists $C_1 > 0$ such that

$$||B(t,x)|| \leq C_1(1+|x|);\ ||B(t,x) - B(t,y)|| \leq C_1|x-y|,$$

holds for all $t \in [0,T]$, $x \in U$ and for some $s > 0$

$$\int_0^s ||S(r)||_{L_2} dr < +\infty.$$

Obviously our conditions are implied by these (see Proposition A.3.2). Our somewhat artificial conditions (iii) will be of much use in chapter three though, hence we used this formulation.

Definition. *An $\mathcal{F}_t$-adapted process $X(t), t \geq 0$, is called a mild solution of the problem (2.1) if*

$$P\left(\int_0^t |X(s)|^2 ds < +\infty\right) = 1,\ P - a.s.$$

and for any $t \in [0,T]$ it satisfies an integral equation

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s))dW(s). \quad (2.2)$$

2.2 Basic existence and uniqueness result

The existence of the solution to the equation (2.1) can be proved using classical Picard iteration method via Banach contraction mapping principle (therefore also immediately establishing uniqueness), we will follow here mostly the exposition contained in [4]. While the method itself is unsophisticated, the technical details are quite demanding. To be able to execute the proof one needs primarily a maximal inequality for so-called stochastic convolution integral, which is the third term in equation (2.2). For the proof see [6] (or [7] for a sly Jan Seidler's proof), page 194.

Lemma 2.2.1. *For all $p \geq 2$, $t \in [0,T]$ and for arbitrary $L_2(H,U)$-valued predictable process $\Phi \in \mathcal{P}^2_W$, we have*

$$\sup_{s\in[0,t]} E\left|\int_0^s \Phi(r)dW(r)\right|^p \leq c_p \left(\int_0^t (E||\Phi(r)||^p_{L_2})^{2/p}dr\right)^{p/2},$$

where $c_p = (p(p-1)/2)^{p/2}$ is a constant dependent only on p.

Other useful inequality is similarly as in the finite dimensional case a Burkholder-Davis-Gundy inequality, whose proof based on the Hilbert spaces Ito's formula can be find in both [6] and [7].

Lemma 2.2.2 (Burkholder-Davis-Gundy inequality)**.** *Assume that $\Phi \in \mathcal{P}^2_W$. For any $p \geq 1$ there is a constant $c_p = (p(p-1)/2)^{p/2}$ such that*

$$E\left(\sup_{0\leq t\leq T}\left(\int_0^t \Phi(s)dW(s)\right)^p\right) \leq c_p E\left|\int_0^T ||\Phi(s)||^2_{L_2}ds\right|^{p/2}.$$

Both results stated above are fairly nontrivial and their proof requires a considerable effort. However, with these inequalities in hand we are now in a position to attack the equation (2.1) by the Banach fixed point theorem. Let us denote by $\mathcal{H}_{p,T}$ the Banach space of predictable U-valued processes $Y(t)$, $t \geq 0$, such that

$$||Y||_{p,T} = \sup_{t\in[0,Y]} (E|Y(t)|^p)^{1/p} < +\infty.$$

Theorem 2.2.3. *Assume Hypothesis 2.1 and let $p \geq 2$. Then for arbitrary $\mathcal{F}_0$-measurable initial condition ξ such that $E|\xi|^p < +\infty$ there exists a unique mild solution X of (2.1) in $\mathcal{H}_{p,T}$ and constant C_T, independent of ξ, such that*

$$\sup_{t\in[0,T]} E|X(t)|^p \leq C_T(1 + E|\xi|^p). \tag{2.3}$$

Proof. For a given $\mathcal{F}_0$-measurable $\xi \in L^p(\Omega, U)$ and $X \in \mathcal{H}_{p,T}$ we define a process $Y = D(\xi, X)$ by the formula

$$Y(t) = S(t)\xi + \int_0^t S(t-s)F(s,X(s))ds + \int_0^t S(t-s)B(s,X(s))dW(s), t \in [0,T].$$

We will establish an upper bound on the p-th moment of Y. To this end, set $M_T = \sup_{t\in[0,T]} ||S(t)||$. We obtain by a fairly standard inequality (the easiest way to see its validity is perhaps through application of Hölder inequality to counting measure, see [22], page 195)

$$E|Y(t)|^p \leq 3^{p-1}\left\{||S(t)||^p E|\xi|^p + E\left[\left|\int_0^t S(t-s)F(s,X(s))ds\right|^p\right]\right. \tag{2.4}$$
$$\left. + E\left[\left|\int_0^t S(t-s)B(s,X(s))dW(s)\right|^p\right]\right\}.$$

Further application of Hölder inequality and Fubini theorem together with the assumptions on middle term yields

$$E\left(\int_0^t |S(t-s)F(s,X(s))|ds\right)^p \le E\left(\int_0^t |S(t-s)^F(s,X(s))|^p ds\; t^{p-1}\right)$$

$$\le T^{p-1}M_T^p \int_0^t E|F(s,X(s))|^p ds \le 2^{p-1}T^p M_T^p C_0^p \sup_{s\in[0,t]} (1+E|X(s)|^p). \qquad (2.5)$$

To estimate the stochastic convolution term in (2.4) we employ lemma 2.2.1 to calculate

$$E\left|\int_0^t S(t-s)B(s,X(s))dW(s)\right|^p \le c_p \left[\int_0^t (E||S(t-s)B(s,X(s))||_{L_2}^p)^{2/p} ds\right]^{p/2}$$

$$\le 2^{p-1}\left(\int_0^t K^p(t-s)(1+E|X(s)|^p)^{2/p} ds\right)^{p/2}$$

$$\le 2^{p-1}\left(\int_0^t K^p(t-s)ds\right)^{p/2} \sup_{s\in[0,t]} (1+E|X(s)|^p).$$

Returning to the (2.4) we see now, that there exists a constant c_1, c_2 and c_3 such that

$$\sup_{t\in[0,T]} E|Y(t)|^p \le c_1 + c_2 E|\xi|^p + c_3 \sup_{t\in[0,T]} E|X(t)|^p. \qquad (2.6)$$

Hence we have proved $Y \in \mathcal{H}_{p,T}$. The proof of contraction now requires exactly the same procedure, only instead of at most linear growth supposition, one must employ the Lipschitz assumptions. We see nevertheless that for $X_1, X_2 \in \mathcal{H}_{p,T}$ and corresponding $Y_1 = D(\xi, X_1)$, $Y_2 = D(\xi, X_2)$ there exists constant c_{3T} such that

$$\sup_{t\in[0,T]} E|Y_1(t) - Y_2(t)|^p \le c_3 \sup_{t\in[0,T]} E|X_1(t) - X_2(t)|^p.$$

If one takes a proper look at the way we obtained these estimate, we can conclude that for T sufficiently small we have $c_{3T} < 1$, so on such interval we establish a unique solution of (2.1) in $\mathcal{H}_{p,T}$. The case of general T can be treated by iteration by considering the equation in intervals $[0,\tilde{T}], [\tilde{T}, 2\tilde{T}] \ldots$ so that $c_3(\tilde{T}) < 1$. Furthermore with such a $\tilde{T}$ we have from (2.6)

$$\sup_{t\in[0,\tilde{T}]} E|X(t)|^p \le \frac{1}{1-c_3(\tilde{T})}\left[c_1 + c_2 E|\xi|^p\right],$$

from which easily follows inequality (2.3). The case of general $T > 0$ is again solved by the same consideration. This finishes the proof. □

Immensely useful tool when dealing with stochastic evolution equations appears to be a factorization formula. We will not use it to prove a regularity results of solution, since it is not an easy thing, we do however make use of it in chapter.

Nevertheless since it is the result about decomposition of stochastic convolution, we do present it here. The formula itself is an immediate corollary of Stochastic Fubini theorem and famous Euler‘s reflection formula. However there appears to be no particularly simple proof of reflection formula, one can find either a complex analytic proof or a long real analytic Dedekind's proof [8], so we omit it. Instead we just show how to derive from it the exact result we shall use in the proof of factorization.

Lemma 2.2.4. *For any $0 < \alpha < 1$, we have the formula*

$$\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha} ds = \frac{\pi}{\sin \pi\alpha} \tag{2.7}$$

.

Proof. The Euler reflection formula reads for example (see [3], chapter 7)

$$0 < x < 1, \ \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

Let us further recall well-known formula for Beta function

$$B(x,\alpha) = \int_0^1 u^{x-1}(1-u)^{\alpha-1} du = \frac{\Gamma(x)\Gamma(\alpha)}{\Gamma(x+\alpha)}. \tag{2.8}$$

Because $\Gamma(1) = 1$, if we substitute $x = 1-\alpha$ in (2.8) we obtain together with Euler reflection formula

$$\frac{\pi}{\sin \pi\alpha} = \int_0^1 u^{-\alpha}(1-u)^{\alpha-1} du. \tag{2.9}$$

Now it suffices to put in (2.9) $s = r + u(t-r)$, so that $u = \frac{s-r}{t-r}$ and we arrive at the desired

$$\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha} ds = \frac{\pi}{\sin \pi\alpha},$$

which we wanted to prove. □

Lemma 2.2.5 (Factorization formula)**.** *Let $\Phi(t)$, $t \in [0,T]$ be a $L(H,U)$-valued process such that stochastic convolution integral*

$$W_A^\Phi(t) = \int_0^t S(t-s)\Phi(s)dW(s), \ t \in [0,T]$$

is well defined. Denote $Y_\alpha^\Phi(t) = \int_0^t (t-s)^{-\alpha} S(t-s)\Phi(s)dW(s), t \in [0,T]$ and suppose that for all $t \in [0,T]$

$$\int_0^t (t-s)^{\alpha-1} \left[\int_0^s (s-r)^{-2\alpha} E(||S(t-r)\Phi(r)||_{L_2}^2) dr \right]^{1/2} ds < +\infty.$$

Then the following formula holds

$$\int_0^t S(t-s)\Phi(s)dW(s) = \frac{\sin \alpha\pi}{\pi} \int_0^t (t-s)^{\alpha-1} S(t-s) Y_\alpha^\Phi(s) ds, \ t \in [0,T]. \tag{2.10}$$

Proof. From the (2.7) we have

$$W_A^{\Phi}(t) = \frac{\sin \alpha\pi}{\pi} \int_0^t S(t-r)\Phi(r) \left[\int_r^t (t-s)^{\alpha-1}(s-r)^{-\alpha} ds \right] dW(r).$$

Now we use Stochastic Fubini theorem (the lemma assumptions are make just to enable us this operation) to reverse integration order and also employ the semigroup properties of $S(t)$ to obtain

$$W_A^{\Phi}(t) = \frac{\sin \alpha\pi}{\pi} \int_0^t \int_0^s S(t-r)\Phi(r)(t-s)^{\alpha-1}(s-r)^{-\alpha} dW(r)dr$$

$$= \frac{\sin \alpha\pi}{\pi} \int_0^t (t-s)^{\alpha-1} S(t-s) \int_0^s S(s-r)\Phi(r)(r-s)^{-\alpha} dW(r)ds,$$

which is the formula (2.10) as claimed. □

This formula is surprisingly powerful and in fact as indicated above, the proof of existence of continuous solution to the (2.1) can be based on it (see [6], chapter 7 for further details). The proofs of regularity properties requires yet another nontrivial estimates and results, so we satisfy ourselves here with this statement.

2.3 Markov property of solution

In this section we list the important Markov property of solution to the (2.1), which will be an essential matter in the third chapter. We will not go through a somewhat lengthy proof procedure, since the proof itself is the same as is the finite dimensional case and is thus included in the standard lectures on Markov processes on our faculty (see [22] for a proof). In addition the whole issue in the infinite dimensional case is well discussed in [6]. For the proof machinery one needs to proof first a regular dependence of solution on initial data. From this point on, the proof is the same as in the finite dimensional case.
Let us first make some obvious remarks and considerations. In the same way as we considered solution on interval $[0,T]$, we can consider equation on interval $[s,T]$, the σ-field $\mathcal{F}_s$ plays the role of σ-field $\mathcal{F}_0$ and $W(t) - W(s), t \geq s$ is the Wiener process (the Markov property for infinite dimensional Wiener process is obvious from the definition, after all the situation in finite dimension is the same - nontrivial is so-called strong Markov property, i. e. same quality but for more general class of times - stopping times). From previous section we can conclude that for any $s \in [0,T]$ and for arbitrary U-valued $\mathcal{F}_s$- measurable random variable ξ, there exists an unique solution $X(t)$, $t \in [s,T]$ for the equation

$$X(t) = S(t-s)\xi + \int_s^t S(t-r)F(r,X(r))dr + \int_s^t S(t-r)B(r,X(r))dW(r),\ t \in [s,T]. \tag{2.11}$$

This solution will be denoted as $X(\cdot, s; \xi)$. For a deterministic starting points $\xi = x$ P-a.s., $x \in U$, the solution to (2.11) is denoted as $X(\cdot, s; x)$.

For a set $B \in \mathcal{B}(U)$ and $0 \leq s \leq T, x \in U$ we define naturally a transition function corresponding to equation (2.11) as

$$P(s, x; t, B) = P(X(t, s; x)) \in B).$$

Recall (see [**?**] for a thorough discussion) that sometimes we denote transition function as $P_{s,t}(x, B)$ and we change the indication freely according to our purposes. Based on our Hypothesis 2.1 one can now ([6], page 249) show that the family of processes $X(t, u; \xi)$, $t \in [u, T]$ is indeed a Markov process with the transition function $P_{s,t}, 0 \leq s \leq t \leq T$. In the case $P(s, x; t, B) = P(0, x; t - s, B)$, i. e. the transition function does not depend on time, the corresponding Markov process is called homogeneous Markov process, then simpler indication $P_t(x, B)$ is used for a transition functions. We have the following natural result, however the second part is what we need in the third chapter and the proof (while easy) for it is hard to find, so we include it.

Lemma 2.3.1. *(i) If the coefficients F and B do not depend on time, then the transition function itself meets $P_{s,t} = P_{0,t-s}$, $0 \leq s \leq t$.*
(ii) Suppose there exists a $p > 0$ such that $F(t, x) = F(t + p, x)$ and $B(t, x) = B(t + p, x)$ is valid for any $t \geq 0$ and $x \in U$. Then $P_{s,t} = P_{s+p,t+p}$ for $0 \leq s \leq t$.

Proof. (i) See [6], Corollary 9.10, the proof technique is yet analogous to the one employed below.
(ii) We need to show that the distribution of process $X(t+p, s+p; x)$ is the same as distribution of $X(t, s; x)$. From the definition we have

$$\begin{aligned} X(t + p, s + p; x) = S(t - s)x &+ \int_{s+p}^{t+p} S(t + p - u)F(u, X(u, s + p; x))du \\ &+ \int_{s+p}^{t+p} S(t + p - u)B(u, X(u, s + p; x))dW(u), \end{aligned}$$

hence using the assumptions we obtain

$$\begin{aligned} X(t + p, s + p; x) = S(t - s)x &+ \int_{s}^{t} S(t - u)F(u, X(u + p, s + p; x))du \\ &+ \int_{s}^{t} B(u, X(u + p, s + p; x))dW^p(u), \end{aligned} \tag{2.12}$$

where $W^p(u) = W(u + p) - W(p), u \geq 0$ is again a Cylindrical Wiener process. On the other hand we have

$$\begin{aligned} X(t, s; x) = S(t - s)x &+ \int_{s}^{t} S(t - s)F(u, X(u, s; x))ds \\ &+ \int_{s}^{t} B(u, X(u, s; x))dW(u), \end{aligned} \tag{2.13}$$

thus comparing expression (2.13) and (2.12) we find out that the distribution of processes must be the same, because they solve the same equation and we have the uniqueness of solution from Theorem 2.2.3. ☐

In the second case we shall say that the transition semigroup is p-periodic.

3. Existence of periodic solution for periodic equation

Now with all the necessary background reminded or mentioned we can begin focus on our original task - the proof of existence of periodic solution to the equation (2.1). The theorems in this section are original, but it has to be said that their proofs closely follow the pattern established in [6], only that we work in a different setting.

3.1 Periodic Markov process

By the Krylov-Bogolyubov method we prove that to a periodic transition function we can construct a periodic Markov process.

Definition. *A family of transition functions $P_{s,t}$, $0 \leq s \leq t$ on a separable complete metric space E is called Feller semigroup provided the space $C_b(E)$ is an invariant subspace for the operators $P_{s,t}$, that is $P_{s,t}f(\cdot) = \int_E f(y)P(s,\cdot,t,dy)$ is a bounded continuous function for arbitrary $f \in C_b(E)$.*

Note, that under our Hypothesis 2.1 we have that transition function $P_{s,t}$ corresponding to (2.1) by relationship $P_{s,t}(x,A) = P(X(t,s;x) \in A)$ is indeed a Feller semigroup of Markov operators (see chapter 9 in [6] for proof).
Main result of this section (and in fact only) is the following theorem, which appears in Khasminskii's book [11]. However Khasminskii doesn't actually present one, so we include it here.
To simplify the proof (I admit here that notation used in Khasminskii's original work [11] is particularly confusing for me, although reading my proof of theorem 3.1.1 does not give me much hope to be more intelligible) let us insert the following often used notation. Given a measure ν, we denote $P^*_{s,t}\nu$ the dual action of semigroup $P_{s,t}$ on ν, i. e. $P^*_{s,t}\nu(A) = \int P(s,y,t,A)\nu(dy)$. We also remind that for a Markov process X the dual semigroup $\left\{P^*_{s,t}, 0 \leq s \leq t\right\}$ also describes the evolution of a process, that is

$$P^*_{s,t}\mu(A) = \int P(s,y,t,A)\mu(dy)$$

is a probability that Markov process starting from time s with initial distribution μ hits set A in time t. For a thorough discussion of these matters see chapter 5 in [**?**].

Theorem 3.1.1. *Let E be a complete separable metric space and $P_{s,t}$, $0 \leq s \leq t$ a family of transition probabilities in E such that*
(i) $P_{s,t}$, $0 \leq s \leq t$ is Feller semigroup
(ii) $P_{s,t}$ is p-periodic

(iii) There exists $s_0 \in [0, +\infty)$ and measure ν on $\mathcal{B}(E)$ such that the set of probability measures $\left\{\frac{1}{n}\sum_{k=1}^{n} P^*_{s_0,s_0+kp}\nu, n \in \mathbb{N}\right\}$ is tight.
Then there exists a p-periodic Markov process with transition functions $P_{s,t}, 0 \le s \le t$.

Proof. From assumption the set $\left\{\frac{1}{n}\sum_{k=1}^{n} P^*_{s_0,s_0+kp}\nu, n \in \mathbb{N}\right\}$ is tight, therefore by Prokhorov theorem (there are many proofs of this fundamental result, for a thorough discussion of the theorem see Billingsley's classic [1], however perhaps simplest proof known to me is Varadarajan's, see [26] for details) there exists a subsequence $\{k_n\}$, $k_n \to +\infty$ and measure μ such that $\frac{1}{k_n}\sum_{j=1}^{k_n} P^*_{s_0,s_0+jp}\nu \xrightarrow{w} \mu$. We claim that process starting at s_0 with initial distribution μ is p-periodic. So we want to prove $P^*_{s_0,t}\mu = P^*_{s_0,t+p}\mu \ \forall t \in (s_0, +\infty)$. Because the measure is completely determined by continuous bounded functions, it suffices to prove $(P^*_{s,t}\mu)f = (P^*_{s,t+p}\mu)f$ for any $f \in C_b(E)$. We easily compute using the definition of weak convergence and Feller property

$$(P^*_{s_0,t}\mu)f = \int_E P_{s_0,t}f(y)d\mu(y) = \lim_{n\to\infty}\frac{1}{k_n}\sum_{j=1}^{k_n}\int_E P_{s_0,t}f(y)d(P^*_{s_0,s_0+jp}\nu)(y),$$

what we adjust thanks to the Chapman-Kolmogorov equality and periodicity of transition function to

$$= \lim_{n\to\infty}\frac{1}{k_n}\sum_{j=1}^{k_n}\int_E P_{s_0,s_0+jp}(P_{s_0,t}f)(y)d\nu(y) = \lim_{n\to\infty}\frac{1}{k_n}\sum_{j=1}^{k_n}\int_E P_{s_0,t+jp}f(y)d\nu(y). \tag{3.1}$$

Next we consider the same calculation applying to $(P^*_{s_0,t+p}\mu)f$, we obtain

$$(P^*_{s,t+p}\mu)f = \lim_{n\to\infty}\frac{1}{k_n}\sum_{j=1}^{k_n}\int_E P_{s_0,t+(j+1)p}f(y)d\nu(y). \tag{3.2}$$

Now if we compare (3.2) with (3.1) we arrive at

$$(P^*_{s_0,t}\mu)f - (P^*_{s,t+p}\mu)f = \lim_{n\to\infty}\frac{1}{k_n}\left[\int_E \left(P_{s_0,t+p}f(y) - P_{s_0,t+(k_n+1)p}f(y)\right)d\nu\right], \tag{3.3}$$

so that

$$|(P^*_{s_0,t}\mu)f - (P^*_{s,t+p}\mu)f| \le \limsup \frac{1}{k_n}2||f||_{C_b}2\nu(E) \to 0,$$

what we aimed to prove. □

As indicated above, the proof itself is fairly easy once we know what method to use, but the record is quite confusing and one must be focused when reading it.

3.2 Periodic solution

In this section we are going to prove our main theorem - the periodic solution to an equation with periodic coefficients. Remind that we examine the equation

$$dX(t) = (AX(t) + F(t, X(t)))dt + B(t, X(t))dW(t), \ X(0) = x \in U. \tag{3.4}$$

We will now prescribe for the convenience of reader the condition on (3.4) with which we will work in this section. We require the same conditions as in Hypothesis 2.1, however we need to strengthen the assumptions in order to make the proofs work.

Hypothesis 3.1
(i) A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, $t \geq 0$, on U.
(ii) The semigroup $S(t), t \geq 0$ is compact, i.e. $S(t)$ is compact operator for arbitrary $t > 0$.
(iii)F is a measurable mapping from $[0, T] \times U$ into U and there exists a constant C_0 such that

$$|F(t, x)| \leq C_0(1 + |x|), x \in U$$

$$|F(t, x) - F(t, y)| \leq C_0|x - y|, \ x, y \in U,$$

is valid for all $t \in [0, T]$.
(iv) The coeficients F and B are p-periodic, i.e. there exists $p > 0$ such that $F(t + p, x) = F(t, x)$ and $B(t + p, x) = B(t, x)$ for any $t \geq 0$ and $x \in U$.
(v) For fixed t B is a strongly continuous mapping from U into $L(H, U)$, i. e. for all $h \in H$ the mapping $u \to B(t, u)h$ from U into U is continuous. Moreover for any $t, s \geq 0$ and $x \in U$, $S(t)B(s, x)$ belongs to $L_2(H, U)$. There exists a locally square integrable mapping

$$K : [0, +\infty) \to [0, +\infty), t \to K(t),$$

such that for any $t, s \in [kp, (k + 1)p]$, $k \in \mathbb{N}_0$ we have

$$||S(t)B(s, x)||_{L_2} \leq K(t)(1 + |x|), x \in U$$

$$||S(t)B(s, x) - S(t)B(s, y)||_{L_2} \leq K(t)|x - y|, \ x, y \in U.$$

(vi) There exists $\alpha \in (0, \frac{1}{2})$ such that

$$\int_0^p t^{-2\alpha} K^2(s)ds < +\infty,$$

where K is the function from point (v).

For the simplicity in what follows we will now denote $X(t, 0, x)$ the solution to (3.4) in time t (only a minor correction of notation). Note first that in the finite dimensional case, the theorem 3.1.1 would practically give us all what we need. However the problem in the infinite dimensional case is with verifying the tightness of measures $\left\{\frac{1}{n}\sum_{k=1}^{n} P^*_{s_0,s_0+kp}\nu, n\in\mathbb{N}\right\}$. All this is caused simply by the fact, that a closed ball is never compact in infinite dimensional space. So while one has reasonable condition which pushes the solution of (3.4) to bounded balls, to extract a compact from it requires further assumptions and proof ingenuity. We will use a method found by Da Prato, Gątarek and Zabczyk in [5], which uses the notion of compact semigroup. We could apparently employ a similar method discovered by Maslowski in [10], where slightly stronger assumptions on the semigroup generated by A are made, namely he requires an analytical semigroup instead of compact one. Nevertheless the broadness of application does not differ by much, for example the important case of Laplace operator satisfies both conditions. We draw here from a more lucid presentation of their results in [4]. We first state the theorem itself and then prove it alongside couple of technical lemmas.

Theorem 3.2.1. *Assume that*
(i) Hypothesis 3.1 holds
(ii) There exists $x_0 \in U$ so that

$$\lim_{u\to\infty}\inf_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n} P(|(X(kp,0,x_0)| \le u)) = 1.$$

Then there exists a periodic Markov process corresponding to solution for problem (3.4).

To prove the theorem we shall use two more technical lemmas.

Lemma 3.2.2. *For any $\alpha \in (0,1]$ define operator $G_\alpha : L^r(0,p;U) \to U$ by the formula*

$$G_\alpha f = \int_0^p (p-s)^{\alpha-1} S(p-s) f(s) ds, \;\; f \in L^r(0,p;U).$$

Let $S(t), t > 0$ be compact operators, $r \ge 2$ and $\alpha > \frac{1}{r}$. Then G_α is compact operator.

Proof. We show that G_α is a limit of compact operators. For $\epsilon \in (0,p)$, $f \in L^r(0,p;U)$ define approximation

$$G_\alpha^\epsilon f = \int_0^{p-\epsilon} (p-s)^{\alpha-1} S(p-\epsilon-s) f(s) ds = S(\epsilon)\int_0^{p-\epsilon} (p-s)^{\alpha-1} S(p-\epsilon-s) f(s) ds.$$

From the compactness of $S(\epsilon)$ we obtain that G_α^ϵ is a compact operator as a composition of compact and bounded operator. For conjugate exponent $q = \frac{r}{r-1}$ one has

$$(\alpha-1)q + 1 = \frac{\alpha r - 1}{r-1} > 0.$$

Now with the aid of Hölder inequality we easily estimate

$$|G_\alpha f - G_\alpha^\epsilon f| = \left|\int_{p-\epsilon}^{p} (p-s)^{\alpha-1} S(p-s) f(s) ds\right|$$

$$\leq \left(\int_{p-\epsilon}^{p}(p-s)^{(\alpha-1)q}||S(p-s)||^{q}ds\right)^{1/q}\left(\int_{p-\epsilon}^{p}|f(s)|^{p}ds\right)^{1/p}$$

$$\leq M\left(\frac{\epsilon^{(\alpha-1)q+1}}{(\alpha-1)q+1}\right)^{1/q}||f||_{r},$$

where $M=\sup_{s\in[0,p]}||S(s)||$. This proves that $G_{\alpha}^{\epsilon}\xrightarrow{\epsilon\to 0_{+}}G_{\alpha}$ in the operator norm, so that G_{α} is compact as claimed. □

The second lemma is more complicated and it is a crucial result, because it gives a tightness of X in fixed time. The required tightness of weighted averages will follow from it relatively straightforwardly thanks to Chapman-Kolmogorov equality.

Lemma 3.2.3. *Assume that $r>2$, $\alpha\in(\frac{1}{r},\frac{1}{2})$ and that Hypotheses of Theorem 3.2.1 hold. Then for arbitrary $u>0$ and all $x\in U$ such that $|x|\leq u$, there exists a relatively compact set $C(u)\in U$ and $c>0$ such that*

$$P(X(p,0,x)\in C(u))\geq 1-cu^{-r}(1+|x|^{r}),\ u>0.$$

Proof. We employ the factorization formula introduced in the form (2.10). Set

$$Z(t,x)=\int_{0}^{t}(t-s)^{-\alpha}S(t-s)B(s,X(s,0,x))dW(s).$$

We have the identity

$$X(p,0,x)=S(p)x+G_{1}F(\cdot,X(\cdot,0,x))+\frac{sin\pi\alpha}{\pi}G_{\alpha}Z(\cdot,x).$$

From the previous lemma we have that the mapping

$$N:U\times L^{r}(0,p;U)\times L^{r}(0,p;U)\to U,\ N(y,g,h)=S(p)y+G_{1}g+G_{\alpha}h$$

is compact. Hence for arbitrary $u>0$ the set

$$C(u)=\{x\in U:x=S(1)y+G_{1}g+G_{\alpha}h,\ |y|\leq u,|g|_{r}\leq u,|h|_{r}\leq u\}$$

has compact closure in U.
Now let $|x|\leq u$. From the definition of $C(u)$ it follows that $X(p,0,x)\notin C(u)$ if and only if either $|F(\cdot,X(\cdot,0,x))|_{r}>u$ or $|Z(\cdot,x)|_{r}>\frac{\pi u}{sin\pi\alpha}$. Therefore

$$P(X(p,0,x)\notin C(u))\leq P(|F(\cdot,X(\cdot,0,x))|_{r}>u)+P\left(|Z(\cdot,x)|_{r}>\frac{\pi r}{\sin\pi\alpha}\right),$$

and by the Markov inequality we get estimate

$$P(X(p,0,x)\notin C(u))\leq u^{-r}E(|F(\cdot,X(\cdot,0,x))|_{r}^{r})+u^{-r}\frac{\sin^{r}\pi\alpha}{\pi^{r}}E(|Z(\cdot,x)|_{r}^{r}).\quad(3.5)$$

We now give estimates on the right hand terms to finish the proof of lemma.

$$E\int_0^p |Z(t,x)|^r dt = E\int_0^p \left|\int_0^t (t-s)^{-\alpha} S(t-s)B(s,X(s,0,x))dW(s)\right|^r dt$$

and the use of Burkholder-Davis-Gundy inequality (see lemma 2.2.2) gives the existence of $k>0$ that

$$E\int_0^p |Z(t,x)|^r dt \le kE\int_0^p \left(\int_0^t (t-s)^{-2\alpha}||S(t-s)B(s,X(s,0,x))||_{L_2}^2 ds\right)^{r/2} dt.$$

$$\le k2^{\frac{r}{2}}E\int_0^p \left(\int_0^p (t-s)^{-2\alpha}K^2(t-s)(1+|X(s,0,x)|^2)ds\right)^{r/2} dt$$

$$= k2^{\frac{r}{2}}E|u^{-2\alpha}K^2 * (1+|X(\cdot,0,x)|^2)|_{\frac{r}{2}}^{\frac{r}{2}},$$

where we denoted by u the real identity function. By the Young inequality (see Appendix for details) for convolution we infer now (laying $z=\frac{r}{2}$, $v=1$ and $q=\frac{r}{2}$ in Young inequality)

$$E\int_0^p |Z(t,x)|^r dt \le k2^{r/2}E|u^{-2\alpha}K^2|_1^{\frac{r}{2}}|1+|X(\cdot,0,x)|^2|_{\frac{r}{2}}^{\frac{r}{2}}$$

$$\le k2^{r/2}\left(\int_0^p t^{-2\alpha}K^2(t)dt\right)E\int_0^p (1+|X(s,0,x)|^2)^{r/2}ds. \tag{3.6}$$

Incorporating the estimate (2.3) we can deduce from (3.6) the existence of $k_1>0$ such that

$$E\int_0^p |Z(t,x)|^r dt \le k_1(1+|x|^r), x\in U. \tag{3.7}$$

Similarly thanks to (2.3) and Hypothesis 3.1 imposed on F (see estimate (2.5) for detailed derivation) we get the existence of $k_2>0$ such that

$$E\int_0^p |F(s,X(s,0,x))|^r ds \le k_2(1+|x|^r), x\in U. \tag{3.8}$$

Combining (3.8) and (3.7) we get after installing to (3.5)

$$P(X(p,0,x)\notin C(u)) \le u^{-r}(\pi^{-r}k_1+k_2)(1+|x|^r), \tag{3.9}$$

which proves the lemma. □

Proof of Theorem 3.2.1. As discussed previously, the Hypotheses 2.1 ensure that transition function $P_{s,t}, 0\le s\le t$ is a Feller Markovian semigroup and the solution to (3.4) is a continuous Markov process. In addition the periodicity of coefficients implies the periodicity of the transition function. On that account to use the Theorem 3.1.1 we only have to show that condition (iii) of this Theorem is fulfilled. We show it is true with $s_0 = 0$ and ν being the Dirac measure concentrated in point x_0. From the lemma 3.2.3 we know that there exists a compact set $C(u)$ such that

$$P(X(p,0,x) \in C(u)) \geq 1 - cu^{-r}(1 + |x|^r).$$

By the Chapman - Kolmogorov equality we compute

$$P(X(2p,0,x_0) \in C(u)) = P(0,x_0,2p,C(u))$$

$$= \int_U P(p,y,2p,C(u))P(0,x_0,p,dy),$$

and using periodicity of transition function we continue (we use a well established notation $\nu \circ T^{-1}$ to denote an image of measure ν under mapping T)

$$P(X(2p,0,x_0) \in C(u)) = \int_U P(0,y,p,C(u))P(0,x_0,p,dy)$$

$$= \int_U P(X(p,0,y) \in C(u))P \circ X(p,0,x_0)^{-1}$$

$$= \int_\Omega P(X(p,0,X(p,0,x_0)) \in C(u))dP$$

$$\geq \int_{\{|X(p,0,x_0)| \leq u_1\}} P(X(p,0,X(p,0,x_0)) \in C(u))dP$$

$$\geq \int_{\{|X(p,0,x_0)| \leq u_1\}} (1-cu^{-r}(1+|u_1|^r))dP = (1-cu^{-r}(1+|u_1|^r))P(|X(p,0,x_0)| \leq u_1).$$

Analogical computation for $P(X(3p,0,x_0) \in C(u))$ yields the estimate

$$P(X(3p,0,x_0) \in C(u)) \geq (1 - cu^{-r}(1 + |u_1|^r))P(|X(2p,0,x_0)| \leq u_1).$$

By obvious induction argument we are now able to deduce for arbitrary $u > u_1 > 0$ the existence of compact $C(u)$ such that

$$\frac{1}{n}\sum_{k=1}^{n} P(0,x_0,kp,C(u)) \geq (1-cu^{-r}(1+|u_1|^r))\frac{1}{n}\sum_{k=1}^{n} P(|X(kp,0,x)| \leq u_1) \quad (3.10)$$

is valid for any $n \in \mathbb{N}$. The equation (3.10) now implies that the condition (iii) of Theorem 3.1.1 is satisfied. □

While Theorem 3.2.1 gives a sufficient conditions for periodic solution to exist, it is not completely satisfying, since it does not give condition only in the language of coefficients F and B, but rather in terms of transition function associated with the equation (3.4). Therefore we now present a Theorem, which guarantees that condition (ii) of the Theorem 3.2.1 is accomplished. Notice first, that by Markov inequality to guarantee the fulfilment of condition (ii) in Theorem 3.2.1, it is enough to find x_0 such that for some $t_0 > 0$ and $r \geq 1$

$$\sup_{t \geq t_0} E|X(t,0,x)|^r. \tag{3.11}$$

We employ here the results obtained by Ichikawa in [14] and then apply them to obtain the validity of condition (3.11). Minding our purposes, we present them in simplified version, so as to avoid unnecessary complications and making the presentation as simple as possible.
Let $C^2(U)$ be the class of twice Frechét differentiable real valued continuous functions v on U, such that $v_u(u)$ and $v_{uu}(u)u_1$ for any $u_1 \in U$ are continuous on U. For such a function we define the operator

$$\mathcal{L}v(u,t) =< v_u(u), Au + F(t,u) > + \frac{1}{2} tr\ (B(t,u)^* v_{uu}(u) B(t,u)),\ u \in D(A). \tag{3.12}$$

Note that for such an operator to be reasonably defined we must stronger the assumptions we lay on B to suppose B is a Hilbert - Schmidt operator, then the trace in (3.12) is well defined.

Comment The usage of derivatives in Hilbert spaces above may be at first sight a little misleading to a novice in the field. We do follow here the common practice in identifying the first derivative with the element of Hilbert space that determines the corresponding linear functional. Similarly we understand the second derivative, which is formally a map from U into $L(U, L(U, \mathbb{R}))$, after identifying $L(U, \mathbb{R})$ with U, as a map with values in $L(U)$. Thus in this sense it is necessary to understand the equation (3.12). Proper lesson in differential calculus in Banach spaces (certainly sufficient or our purposes) can be drawn from [28], which is a thoroughly written introduction with author's characteristic elegant and precise style. With this having said, one then easily computes that for a function $x \to |x|^2$ from Hilbert space to $\mathbb{R}$, the first derivative is represented by element $2x$, while the second derivative is equated with the mapping $2I$.

We have the following Ichikawa's result (see [14] for a proof). It should be noted that he works with Q-Wiener process in difference with us, but since his proof is based only on the use of Itō's formula, it carries to our case as well.

Lemma 3.2.4. *Let $w(t,u) \in C^{1,2}([0,T] \times U)$ (i. e. the space of all functions having continuous time derivative and twice continuous space Frechét derivative) with properties*
(i) $|w(t,u)| + |w_u(t,u)| + |w_{uu}(t,u)| \leq k(1 + |u|^r)$ for some $k > 0$ and $r > 0$
(ii) $[w_t + \mathcal{L}w](t,u) \leq y(t,u)$, $u \in D(A)$ for some continuous function y on $[0,T] \times U$ with $|y(t,u)| \leq k_1(1 + |u|^{r_1})$, $k_1 > 0$, $r_1 > 0$. Then

$$Ew(t, X(t,0,x)) \leq Ew(0,x) + E \int_0^t y(s, X(s,0,x)) ds. \tag{3.13}$$

With this lemma in hand, it is now an easy task to arrive at the next theorem.

Theorem 3.2.5. *Suppose our Hypothesis 3.1 is satisfied. In addition for any $s \geq 0$ and $x \in U$ let $B(s,x)$ be a Hilbert-Schmidt operator. Suppose further that there exists numbers $c > 0, d > 0$ such that*

$$< 2x, Ax + F(t,x) > + \frac{1}{2} tr\ B^*(t,x)B(t,x) \leq -c|x|^2 + d \tag{3.14}$$

holds for $x \in D(A)$ and $t > 0$. Then for any $x \in U$

$$\sup_{t \geq 0} E|X(t,0,x)|^2 < +\infty. \tag{3.15}$$

Proof. We will use the lemma 3.2.4. Put $w(t,u) = e^{ct}|u|^2$ and observe $w_u(t,u) = e^{ct}2u$, $w_{uu}(t,u) = e^{ct}2I$, $w_t(t,u) = ce^{ct}|u|^2$. Thanks to our assumptions (3.14), the hypotheses of lemma 3.2.4 are now fulfilled with $y(t,u) = e^{ct}d$. If we now substitute into (3.13) and use Fubini theorem, we arrive at

$$Ee^{ct}|X(t,0,x)|^2 \leq |x|^2 + \int_0^t e^{cs} d ds. \tag{3.16}$$

Elementary computation of integral in (3.16) and division by e^{ct} yields

$$E|X(t,0,x)|^2 \leq e^{-ct}|x|^2 + e^{-ct}\frac{d}{c}(e^c t - 1),$$

from which (3.15) obviously follows. □

3.3 Example

We conclude the work by giving an example of stochastic PDE to which our Theorem 3.2.1 and 3.2.5 can be applied. We borrow the example from [5], but adjust it to illustrate our generalization of their results.
Let f, b be a measurable functions from $\mathbb{R} \times [0,T]$ to $\mathbb{R}$ such that for fixed t $f(t,\cdot)$ and $b(t,\cdot)$ are continuous functions and there exists $p > 0$ so that f and b are p-periodic. Further let f, b satisfy Lipschitz and at most linear growth condition, i.e. there exists $C > 0$ such that

$$|f(t,x) - f(t,y)| \leq C|x-y|; \ |b(t,x) - b(t,y)| \leq C|x-y|$$

$$|f(t,x)| \leq C(1 + |x|); \ |b(t,x)| \leq C(1 + |x|)$$

is valid for any $x, y \in \mathbb{R}$ and $t \in [0,T]$. Let $D \subset \mathbb{R}^d$ be a bounded domain (open set) with smooth boundary ∂D, Δ - Laplace operator on D and let W be a one dimensional standard Wiener process. Consider the following stochastic differential equation

$$dX(t,x) = [\Delta X(t,x) + f(t, X(t,x))]dt + b(t, X(t,x))dW(t), \tag{3.17}$$

$$X|_{\partial D} = 0,\ X(0, x) = X_0(x),$$

where $X_0 \in L^2(D)$ is given function. The question is now of course how to interpret the equation (3.17) as the equation of the form (2.1). It is a standard machinery, but a little demanding at first sight, since one must be very careful in which space precisely work.
To this end we denote $U = L^2(D)$, $Au = \Delta u$. Natural domain for A is then the space of twice differentiable functions that vanishes on the boundary, so that formally one put $D(A) = H^2(D) \cap H_0^1(D)$ (for a precise definition of these Sobolev spaces as well as discussion of their basic properties look in [9] or any other good PDE textbook). Moreover we put $H = \mathbb{R}$ and define mappings F and B as follows

$$F : [0, T] \times U \to U,\ F(t, u) = f(t, u(\cdot))$$

$$B : [0, T] \times U \to L_2(\mathbb{R}, U),\ B(t, u) = b(t, u(\cdot)),$$

where we regard $b(t, u(\cdot))$ as a multiplication operator form $\mathbb{R}$ into U, so that for $m \in \mathbb{R}$ we have $b(t, u(\cdot))(m) = m\ b(t, u(\cdot))$.
It is known that the semigroup generated by the Laplace operator is not only a C_0 class, but compact as well in this case (see [27], Theorem 7.2.7 for a proof). Thanks to the conditions we lay on f and b we see that F and B are well defined and fulfil the Hypothesis 3.1. We would like to verify the conditions (3.14) in order to employ the Theorem 3.2.5 to obtain the periodic solution to the equation (3.17). The easy calculation of the adjoint operator to $B(t, u)$ reveals that the operator B^*B mapping $\mathbb{R}$ to $\mathbb{R}$ is just $m \to m\ \int_D b^2(t, u(x))dx$. Since the trace for a mapping from $\mathbb{R}$ into $\mathbb{R}$ is a trivial notion we see that the (3.14) would be fulfilled if there exist a numbers $\tilde{c}, \tilde{d} > 0$ such that

$$\int_D \Delta u(x) 2u(x)dx + \int_D f(t, u(x))u(x)dx + \frac{1}{2}\int_D b^2(t, u(x))dx \qquad (3.18)$$

$$\leq -\tilde{c}\int_D u^2(x)dx + \tilde{d}$$

would hold for any $u \in D(A)$. We now make some considerations to simplify the form (3.18). The Laplace operator is a negative operator in our case (it is obvious by using the Green formula), so that if we leave out the first term, we can only increase the left side. As integral is a monotone operator, we get the validity of (3.18), provided that for any $t \in [0, T]$, $x \in \mathbb{R}$

$$f(t, x)x + b^2(t, x) \leq -cx^2 + d \qquad (3.19)$$

holds for some constants $c, d > 0$.
In other words, we have proved that an equation (3.17) with f, b described above and satisfying (3.19) has a periodic solution using our theory built in third chapter.

Conclusion

A lot of interesting questions connecting to the topic was left unanswered in the thesis and might be worth revisiting in the future. First of all our Hypothesis 2.1 respectively 3.1 allowed us to work comfortably, however our assumptions are overly restrictive in some cases. It would be interesting to drop the assumptions of global Lipschitzianity in our coefficients. Such consideration would though lead us to consider equations (see sections 7.2 and 7.3 in [6]) defined only on a Banach space, which is the subset of Hilbert space U. The whole issue would complicate significantly, so we have chosen to preserve current presentation to stay in our Hilbert space framework.
Another fruitful generalization could be to consider also the operator A dependent on time (lesson in such equation can be drawn for instance from Seidler's article [21]), so that we would have not only periodic coefficients, but also the operators themselves. We would then have to abandon the theory of one-parameter semigroups for two parameter systems to handle this situation.
The other important questions related to the study of periodic solution are uniqueness and convergence rate to the periodic solution. By the second notion we mean the question when and in what sense the transition measures $P(s, t, x, \cdot)$ converge to the periodic solution.
All these issues raised in the Conclusion could be a starting point to the further research on the topic, although the results would probably bear a little surprise, but rather served as a technical exercise.
Finally I would like to once again devote a special thank to my advisor professor Bohdan Maslowski, who guided me through all the possible pitfalls and devoted me a lot of his precious time, which enabled me to finish the work in a coveted time. It was a great experience for me to express it simply.

A. Appendix

In the Appendix we summarize the basic properties of operators and other supportive results used in the text. The purpose of this section is not an encyclopaedic overview. Instead we aim to provide enough background on the topic, so that main part of the text can be read without necessity of looking elsewhere.

A.1 Semigroup of linear operators

We follow here mostly [15], occasionally [19], where more comprehensive treatment of the topic can be found.

Definition. *A one-parameter semigroup of operators acting on complex Banach space X is a family of bounded linear operators $S(t), t \geq 0$, each mapping $X \to X$, satisfying*

$$S(t+s) = S(t)S(s) \text{ for all } t, s \geq 0; \;\; S(0) = I. \tag{A.1}$$

The semigroup is said to be strongly continuous (C_0 semigroup) at $t = 0$ if

$$\lim_{t \to 0_+} S(t)x = x, \;\; \forall x \in X. \tag{A.2}$$

Theorem A.1.1. *Let $S(t), t \geq 0$ be a C_0 semigroup of linear operators.*
(i) There exist constants b and k such that $S(t)$ is bounded in norm by

$$||S(t)|| \leq be^{kt}.$$

(ii) S(t)x is a strongly continuous function of t for every $x \in X$.

Proof. We claim that $||S(t)||$ is uniformly bounded in some neighbourhood of $t = 0$. If it was not the case, there would be a sequence $t_j \to 0_+$ such that $||S(t_j)|| \to \infty$. Then by the principle of uniform boundedness $S(t_j)x$ could not converge to x for all $x \in X$. This violates strong continuity at $t = 0$, therefore there exists $a > 0$, $b > 0$ such that $||S(t)|| \leq b$ for $t \leq a$.
Any t can be decomposed as $t = na + r$, $0 \leq r < a$, $n \in \mathbb{N}_0$. By the semigroup property (A.1) $S(t) = S^n(a)S(r)$. Therefore

$$||S(t)|| \leq ||S(a)||^n ||S(r)|| \leq b^{n+1} \leq be^{kt},$$

where $k = \frac{1}{a} \log b$. This proves (i).
To prove the second assertion, note that for any pair of positive numbers $s < t$ we can write by the semigroup property

$$S(t)x - S(s)x = S(s)\left[S(t-s)x - x\right].$$

Using the part (i) of the theorem and strong continuity at $t = 0$ (A.2) the theorem follows. □

However when encountering the applications of semigroup theory one usually does not encounter the semigroup itself, but its so called generator.

Definition. *Let $S(t), t \geq 0$ be a C_0 semigroup. We define it infinitesimal generator A as*

$$Ax = \lim_{h \to 0_+} \frac{S(h)x - x}{h}, \tag{A.3}$$

where the limit is understood in the norm (strong) sense. The domain of A, denoted $D(A)$, consists of all x for which the strong limit (A.3) exists.

We list now a basic properties of infinitesimal generator, for a proof one can look in [15], page 421.

Theorem A.1.2. *Let $S(t)$ be a C_0 semigroup and A its infinitesimal generator.*
(i) A commutes with $S(t)$, in the sense that if x belongs to $D(A)$, so does $S(t)x$ and $AS(t)x = S(t)Ax$.
(ii) The domain of A^n, n any natural number, is dense
(iii) A is closed operator

If the generator A would be defined everywhere in X, then by the Closed graph Theorem we would have that A is a bounded operator and in such cases the semigroup $S(t)$ is just the exponential function (see [15], section 34.1). But in most cases the operator A is unbounded and defined only on dense subset of X.

Theorem A.1.3. *A strongly continuous semigroup is uniquely determined by its infinitesimal generator.*

Proof. See [15], page 424. □

To prove that the C_0 semigroup is characterized by its infinitesimal generator is rather straightforward. Much deeper and more important question is though how to recognize the generator of C_0 semigroup and reconstruct the semigroup from the generator. This problem has been solved independently by Einar Hille and Kosaku Yosida in the late fourties and the corresponding famous theorem now bears their names. Recall that to every linear operator L we associate the resolvent set $\rho(L)$, which is the complement of spectrum of L, and the resolvent function $R(\lambda; L) = (\lambda I - L)^{-1}$.

Theorem A.1.4 (Hille - Yosida)**.** *Let A be a linear operator defined on a linear subspace $D(A)$ of Banach space X, $b, k > 0$ given numbers. A is the infinitesimal generator of a C_0 semigroup $S(t)$ satisfying $||S(t)|| \leq be^{kt}$, if and only if*
(i) $D(A)$ is dense in X.
(ii) The resolvent set $\rho(A)$ of A includes the set $(k, +\infty)$ and for any positive integer n

$$||R^n(\lambda; A)|| \leq \frac{b}{(\lambda - k)^n} \quad \forall \lambda > k.$$

For the proof of above theorem in this formulation see [19]. Let us note that usually with Hille and Yosida is associated the slightly less general formulation (Theorem 7, page 424 in [15]) dealing with contraction semigroups (i. e. semigroups satisfying $||S(t)|| \leq 1$ for all $t \geq 0$).
In the main part of the thesis in order to obtain our results we needed to suppose that the semigroup is even compact. It is therefore an important issue to know under which conditions A generates a compact semigroup, i. e. $S(t)$ is compact for every $t > 0$ (notice that since $S(0)$ is an identity, we can't have compactness for all $t \geq 0$ unless X is finite dimensional). Unfortunately such condition appears not to exist in general and to determine the compact semigroups is quite delicate issue. We have at least the following theorem though (see [19] for the proof).

Theorem A.1.5. *Let $S(t)$ be a C_0 semigroup and A its infinitesimal generator. $S(t)$ is a compact semigroup if and only if $S(t)$ is continuous in the uniform operator topology for $t > 0$ and $R(\lambda; A)$ is compact for $\lambda \in \rho(A)$.*

If the semigroup would be even uniformly continuous, i. e. $\lim_{t\to 0_+} ||S(t) - I||$, then we would have that compactness is equivalent to $R(\lambda; A)$ is compact for $\lambda \in \rho(A)$ ([19], page 50), but uniformly continuous semigroups are rarely useful concept, since it is equivalent to A being a bounded operator ([19], page 2), which is very restrictive requirement in applications.

A.2 Trace class operators

Trace class operators form a certain class of compact operators to which can be nicely extended some concepts of linear algebra. Throughout the section we work with mapping from Hilbert spaces H to H, the space can be considered over $\mathbb{C}$. Recall that by a positive operator we understand such operator T that $< Tx, x > \geq 0$ for all x in H.

Theorem A.2.1. *Every compact operator T can be factored as*

$$T = UA, \tag{A.4}$$

where A is a positive self-adjoint operator, and $U^*U = I$ on the range of A.

Proof. See [15], section 30.1. □

The operator A is called the absolute value of T and is compact. The relation (A.4) is called the polar decomposition of T. When T is compact, so is its absolute value A. The nonzero positive eigenvalues of A , which we denote as $\{s_j\}$, are called the singular values of T.

Definition. *A compact map T in $L(H)$ is in trace class when (the singular values are calculated with their multiplicity)*

$$\sum_1^\infty s_j(T) < +\infty.$$

The sum is called the trace norm of $T : ||T||_{L_1} = \sum s_j(T)$.
We denote the linear space of trace class operators as $L_1(H)$. There exists another equivalent description of nuclear operators (see [18], page 99 for the proof), which may be slightly more convenient for the proofs of their basic properties. We say that $T \in L(H)$ is a trace class operator provided there exists sequences $\{x_n\}$ and $\{y_n\}$ in H such that $\sum_n |x_n||y_n| < +\infty$ and

$$Tx = \sum_n (x, x_n) y_n \ \forall x \in H.$$

The trace class norm can be computed also as

$$||T||_{L_1} = \inf \left\{ \sum_n |x_n||y_n| : Tx = \sum_n < x, x_n > y_n \text{ for } x \in H \text{ and } \sum_n |x_n||y_n| < \infty \right\}.$$

The basic properties of the trace class operators may be summarized as follows.

Theorem A.2.2. *The space $L_1(H)$ endowed with the $|| \cdot ||_{L_1}$ norm is a Banach space and two sided ideal in $L(H)$. In addition for $T \in L_1(H)$ we have $T^* \in L_1(H)$ and $||T||_{L_1} = ||T^*||_{L_1}$.*

Proof. [18], page 100. □

Key notion for trace class operators is that of a trace. To see it is well defined characteristic, let us first prove a simple lemma.

Lemma A.2.3. *Let $\{u_j\}$ and $\{e_j\}$ be orthonormal bases of Hilbert space H. If T is a nuclear operator mapping H into H, then*

$$\sum_{j=1}^{\infty} < Tu_j, u_j > = \sum_{k=1}^{\infty} < Te_k, e_k > .$$

Moreover $\{< Te_k, e_k >\} \in \ell^1$.

Proof. We have $Tx = \sum_n < x, x_n > y_n$ from the definition of trace class operator. We calculate then

$$\sum_{k=1}^{\infty} | < Te_k, e_k > | = \sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} < e_k, x_n >< y_n, e_k > \right|$$

$$\leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} | < e_k, x_n >< y_n, e_k > | \leq \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} < e_n, x_n >^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} < y_n, e_k >^2 \right)^{\frac{1}{2}}$$

$$= \sum_{n=1}^{\infty} |x_n||y_n| < +\infty,$$

where we used Schwarz inequality and subsequently Parseval equality. Hence we have proved $\{< Te_k, e_k >\} \in \ell^1$. So with the aid of Fubini-Tonelli theorem and again Parseval equality we conclude

$$\sum_{k=1}^{\infty} < Te_k, e_k >= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} < e_k, x_n >< y_n, e_k >$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} < y_n, e_k >< e_k, x_n >= \sum_{n=1}^{\infty} < y_n, x_n >,$$

which finishes the proof. □

Definition. *For a trace class operator $T \in L_1(H)$ and arbitrary orthonormal basis $\{e_n\}$ in H we define its trace as $tr\ T = \sum_n (Te_n, e_n)$.*

The most important and adequately deep result about trace is so called trace class formula, which gives another expression of the trace. The Theorem is usually credited to Victor Borisovich Lidskii, who who published the result in 1959 in [16] (for a clear presentation of the proof see [15], section 30.3).

Theorem A.2.4 (Trace class formula)**.** *The trace of trace class operator is the sum of its eigenvalues :*

$$tr\ T = \sum \lambda_j(T) \tag{A.5}$$

It is worth mentioning that if T is in addition normal operator, then the trace class formula (A.5) immediately follows from Hilbert-Schmidt spectral theorem, since then we have the orthonormal basis consisting of eigenvectors of T, so that

$$tr\ T = \sum_n (Te_n, e_n) = \sum_n \lambda_n.$$

For the sake of completeness let us state the Hilbert-Schmidt spectral theorem, for a proof see again [15] or [18].

Theorem A.2.5 (Hilbert-Schmidt spectral theorem)**.** *Let $T \in L(H)$ be a compact and normal operator. Then there exists orthonormal basis $\{e_n\}$ of H consisting of eigenvectors, i. e. $Te_n = \lambda_n e_n$. The eigenvalues λ_n are real and their only point of accumulation is* 0.

A.3 Hilbert-Schmidt operators

Another important class of compact operators with which we worked a lot in the text forms Hilbert-Schmidt operators.

Definition. *Let H and U be Hilbert spaces. A linear operator $T \in L(H, U)$ is Hilbert-Schmidt class if for some orthonormal basis $\{e_j\}$ of H*

$$\sum_{k=1}^{\infty} |Te_k|^2 < \infty. \tag{A.6}$$

We denote the space of Hilbert-Schmidt operators as $L_2(H,U)$. The definition would not be of much use of course, if the (A.6) depended on the choice of basis. It is nevertheless a trivial exercise for Parseval equality and Fubini theorem (of course one can argue more elementary in this case) to show that condition (A.6) is fulfilled for any orthonormal basis $\{u_j\}$ and independent of it, since

$$\sum_{k=1}^{\infty}|Te_k|^2 = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}| < Te_k, u_j > |^2\right)$$

$$= \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}| < Te_k, u_j > |^2\right) = \sum_{j=1}^{\infty}|Tu_j|^2.$$

The same calculation also shows that if T is Hilbert-Schmidt operator, so is T^* and that $||T||_{L_2} = ||T^*||_{L_2}$.
We define the Hilbert-Schmidt norm on $L_2(H,U)$ as the square root of sum (A.6), so that $||T||_{L_2} = \left(\sum_{j=1}^{\infty}|Te_k|^2\right)^{1/2}$. We summarize some of the basic useful properties (none of them being particularly deep) in the next theorem.

Theorem A.3.1. *Let $T \in L(H,U)$ be a Hilbert-Schmidt operator. The following is valid :*
(i) $||T|| \leq ||T||_{L_2}$
(ii) T is a limit of finite dimensional operators, hence T is a compact operator
(iii) The space $L_2(H,U)$ is a Hilbert space
(iv) The space $L_2(H,U)$ is closed ideal in $L(H,U)$.

Proof. See [18] for most of the Propositions. □

The following proposition relates the concept of Hilbert - Schmidt operator to other types of operators. Since Lax in [15] serves it as an exercise, we try to make a self contained proof based on the already mentioned facts about the operators.

Proposition A.3.2. *Let H be a Hilbert space, then*
(i) Every Trace class operator acting on H is also of Hilbert-Schmidt class
(ii) Product of two Hilbert Schmidt operator T and V in $L(H)$ is in trace class and $||TV||_{L_1} \leq ||T||_{L_2}||V||_{L_2}$
(iii) Every trace class operator can be written as a product of two Hilbert-Schmidt operators
(iv) Let T be a Hilbert - Schmidt operator and $N \in L(H)$. Then both compositions BN and NB are Hilbert - Schmidt operators and for both norms the estimate $|| \cdot ||_{L_2} \leq ||N||||B||_{L_2}$ holds.

Proof. (i) Let T be a trace class operator. Then by the Theorem A.2.2 we see that T^* is also trace class operator and hence T^*T is a trace class operator too as the product of two trace class operators. So if we denote $\{\alpha_n\}$ the eigenvalues of T^*T, we calculate for an orthonormal basis $\{e_n\}$ by the trace class formula (A.5)

$$\sum_k < Te_k, Te_k >= \sum_k < T^*Te_k, e_k >= \sum_k \alpha_k < \infty,$$

which proves the assertion (i).
(ii) For T a trace class operator we have (see [15], page 332 for a proof) also the following description of trace norm :

$$||T||_{L_1} = \sup \sum_n | < Tu_n, e_n > |,$$

where the supremum ranges over all orthonormal bases $\{e_n\}$ and $\{u_n\}$. If B and V are Hilbert-Schmidt operators, then by the Schwarz inequality

$$\sum_n | < BVu_n, e_n > | = \sum_n | < Vu_n, B^*e_n > | \leq \sum_n |Vu_n||B^*e_n|$$

$$\leq \left(\sum_n |Vu_n|^2\right)^{\frac{1}{2}} \left(\sum_n |B^*e_n|^2\right)^{\frac{1}{2}} \leq ||V||_{L_2}||B^*||_{L_2}.$$

If we pass to supremum and realize that $||B^*||_{L_2} = ||B||_{L_2}$, we obtain the proposition together with the estimate $||BV||_{L_1} \leq ||B||_{L_2}||V||_{L_2}$.
(iii) Recall that by the polar decomposition (A.4) we can factorize trace class operator T as $T = UA$ with A being compact self-adjoint operator and U an unitary operator on Range A. We set $B = A^{\frac{1}{2}}$ and $V = UA^{\frac{1}{2}}$. Obviously $T = VB$, thus it remains to prove that B and V are Hilbert-Schmidt operators. However it follows easily, since we have (well-known property of functional calculus for self-adjoint operators ensures that $A^{\frac{1}{2}}$ is also self-adjoint)

$$\sum_k < A^{\frac{1}{2}}e_k, A^{\frac{1}{2}}e_k > = \sum_k < Ae_k, e_k >= \sum_k \lambda_k < \infty,$$

proving that B is Hilbert-Schmidt operator. For V it suffices to make the same calculation. U is the unitary operator on the Range of A, therefore

$$\sum_k < UA^{\frac{1}{2}}e_k, UA^{\frac{1}{2}} > = \sum_k < A^{\frac{1}{2}}e_k, A^{\frac{1}{2}}e_k > < \infty,$$

which finishes the proof.
(iv) Standard calculation (using $||N^*|| = ||N||$) gives

$$\sum_k < NTe_k, NTe_k >= \sum_k < Te_k, N^*NTe_k >$$

$$\leq \sum_k |Te_k||N^*NTe_k| \leq \left(\sum_k < Te_k, Te_k >\right)^{1/2} ||N||^2 \left(\sum_k < Te_k, Te_k\right)^{1/2}$$

$$\leq ||T||_{L_2}^2 ||N||^2,$$

taking the square root now we obtain part of the assertion. On the other hand, if we use the fact that Hilbert - Schmidt norm of the adjoint operator does not change, we can write

$$\sum_k < TNe_k, TNe_k > = \sum_k < N^*T^*e_k, N^*T^*e_k >$$

$$= \sum_k < T^*e_k, NN^*Te_k > \leq \sum_k ||N||^2 ||T||_{L_2}^2.$$

For the last inequality argue as above. □

While it is not necessary for the text itself, it is nonetheless quite interesting and not obvious from our definitions, why one denotes the space of Hilbert-Schmidt operators and Trace class operator L_2 and L_1 respectively. To shed light on the used notation, let us mention few facts from the Hilbert space operator theory, for their proof or further references see [18], paragraph 10.5.
The polar decomposition of operators introduced in (A.4) can be obtained for a wider class of operators than compact ones in fact. However it turns out, that compact operators are precisely those, for which the sequence of singular values $\{s_j\}$ converges to zero, in other words belongs to the space c_0. Trace class operators are operators, for which the sum of singular values lies in ℓ^1. Hilbert - Schmidt operators are described as the class of operators for which the sum of singular values $\{s_j\}$ belongs into ℓ^2.
In addition we have got an analogy to the classical situation in dual spaces. While for the sequence spaces we have that ℓ^1 is isometrically isomorphic to the dual of c_0 and ℓ^∞ is the dual to ℓ^1, for the operators, we have certain parallel. The space $L_1(H)$ is isometrically isomorphic to the dual of space of compact operators, on the contrary the space of all bounded operators $L(H)$ is the dual to the Trace class operator space.

A.4 Miscellaneous

In this section we place everything what I think deserves a special mention, but does not account by its extent to separate section. In the proof of lemma 3.2.3 we made use of Young's inequality. Since it probably does not belong to common equipment of probability student, we make reference of it here.

Theorem A.4.1 (Young inequality). *If $1 \leq v, q, z \leq \infty$, $\frac{1}{v} + \frac{1}{q} = 1 + \frac{1}{z}$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ then*

$$f * g(x) = \int f(x-y)g(y)dy$$

exists for almost all x and defines a function $f * g \in L^z(\mathbb{R}^n)$. Moreover

$$|f * g|_z \leq |f|_v |g|_q.$$

Proof. See [17], Theorem 26.20. $\square$

In the main part of the thesis we also often used terms symmetric operator and self-adjoint operator without making differences between the notions. There is a difference, but in our cases the concepts coincides, because the operators discussed are bounded. Let us clear up now the issue.

Definition. *A densely defined operator $T : H \to H$ on a Hilbert space is said to be symmetric provided T^* is an extension of T (i. e. $D(T^*) \supset D(T)$ and $T = T^*$ on $D(T)$). Equivalently $D(T)$ is dense and*

$$< Tu, v >=< u, Tv >, \text{ for } u, v \in H.$$

While we say that operator T is self-adjoint, if $T^* = T$. For a bounded everywhere defined operator is then being symmetric equivalent to being self - adjoint. For a more detailed discussion, look for instance in [25], page 511.

Used notations

A^*	The adjoint of an operator
$\mathcal{B}(X)$	Borel σ-algebra on space X
$f * g$	The convolution of functions f and g
$D(A)$	The domain of operator A
I	The identity operator
Δ	The Laplace operator, i. e. $\Delta u = \sum \frac{\partial^2 u}{\partial^2 x_i}$
$(L^p, \lvert \cdot \rvert_p)$	The Lebesgue space of all p-integrable functions
$\lvert \cdot \rvert$	The norm on Normed linear space
$< \cdot, \cdot >$	The inner product on Hilbert space
$P \circ X^{-1}$	The image of measure P at mapping X
$\mathbb{R}$, $\mathbb{C}$	The real respectively complex field
$C(X, Y)$	The space of all continuous functions from X into Y
$C_b(X)$	The space of all bounded continuous function with domain X
$C^n(X)$	The space of all n times Frechét differentiable function defined on X
c_0	The space of sequences converging to zero
ℓ_p	The space of sequences satisfying $\sum_n \lvert a_n \rvert^p < +\infty$
$(L(X, Y), \lvert\lvert \cdot \rvert\rvert)$	The space of linear operators mapping X into Y
$(L_1(H), \lvert\lvert \cdot \rvert\rvert_{L_1})$	The space of Trace class operators with its natural norm
$(L_2(H, U), \lvert\lvert \cdot \rvert\rvert_{L_2})$	The space of Hilbert-Schmidt operators with its natural norm
$tr\ T$	The trace of an operator T
$\xrightarrow{w}$	Weak convergence of measures

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