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MASTER THESIS

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Grid Representations of Graphs and the Chromatic Number

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I would like to thank my supervisor Pavel Valtr for all the provided advice on the thesis and for supporting me both in my work and my studies.

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In Prague, August 2, 2012 Martin Balko

Název práce: Mřížková nakreslení grafů a chromatické číslo

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Abstrakt: V předložené práci se zabýváme mřížkovými nakresleními grafů a jejich souvislostmi s grafovými obarveními. Mřížkové nakreslení grafu zobrazuje vrcholy na body mřížky $\mathbb{Z}^d$ a hrany na úsečky, které se vyhýbají bodům odpovídajícím nekoncovým vrcholům. Nejdříve dokážeme, že graf je q^d-obarvitelný, d, $q \geq 2$, právě tehdy, když má mřížkové nakreslení, ve kterém každá úsečka protíná nanejvýš q mřížkových bodů. Poté se věnujeme mřížkovým nakreslením s omezeným počtem sloupců, kde představíme nové NP-úplné úlohy a rozšíříme některé známé výsledky. Také ukážeme ostrý dolní odhad na plochu mřížkového nakreslení pro úplné vyvážené k-partitní grafy, čímž dokážeme domněnku D. R. Wooda. Nakonec pro libovolný rovinný graf nalezneme rovinné mřížkové nakreslení, kde každá úsečka obsahuje pouze dva mřížkové body. Tím potvrdíme domněnky od autorů D. Flores Peñalozy a F. J. Zaragoza Martineze.

Klíčová slova: mřížková nakreslení, mřížka, chromatické číslo, rovina

Title: Grid representations of graphs and the chromatic number

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Abstract: In the thesis we study grid drawings of graphs and their connections with graph colorings. A grid drawing of a graph maps vertices to distinct points of the grid $\mathbb{Z}^d$ and edges to line segments that avoid grid points representing other vertices. We show that a graph G is q^d-colorable, d, $q \geq 2$, if and only if there is a grid drawing of G in $\mathbb{Z}^d$ in which no line segment intersects more than q grid points. Second, we study grid drawings with bounded number of columns, introducing some new NP-complete problems. We also show a sharp lower bound on the area of plane grid drawings of balanced complete k-partite graphs, proving a conjecture of David R. Wood. Finally, we show that any planar graph has a planar grid drawing where every line segment contains exactly two grid points. This result proves conjectures of D. Flores Peñaloza and F. J. Zaragoza Martinez.

Keywords: graph representations, grid, chromatic number, plane

Contents

Introduction

Graph drawing is an area of mathematics and computer science combining methods from graph theory, geometry and topology to derive suitable representations of graphs. There are many applications of this field, as graph drawings arising in other areas of application include, for example, Hasse diagrams, flow charts, Dessin d'enfants, computer network diagrams and so on.

In this thesis we study particular graph drawings in which vertices of a given graph are represented by grid points (points having integer coordinates) and each edge corresponds to a line segment between its endpoints. In addition, the only vertices a line segment intersects are its own endpoints. Recently, the grid drawings of graphs have attracted the attention of many researchers (see, for example [10, 13, 16, 21, 27, 30, 36]).

Although the main area of our interest are connections between grid drawings of graphs (in the grids of an arbitrary finite dimension) and graph colorings, we consider even other problems related to the grid drawings of graphs. Specifically, we study such drawings from three points of view – robustness, compactness and planarity.

By robustness we mean grid drawings with bounded number of grid points lying on a line segment. The compactness signifies somehow bounded space of the drawings and the last point of view, the planarity, involves studying grid drawings where two line segments meet only in extremal points. Exploring this topic, we use methods from graph theory, geometry, number theory and computational complexity theory.

Outline

Chapter 1 should give the reader a brief introduction to graph theory, complexity theory and number theory. Except of the basic terms and classical results of these fields, we also introduce a formal definition of grid drawings and some other related new terms. An experienced reader can skip most of this chapter, as he might find the majority of the mentioned definitions and statements familiar.

In **Chapter 2**, we study the grid drawings of graphs with bounded number of grid points per line segment. The main result of this chapter is Theorem 2.2 whose proof is shown in Section 2.2. Afterwards, we consider some of the immediate corollaries of this theorem. The second chapter ends with some other concluding remarks.

The second point of view, the compactness of the grid drawings, is studied in **Chapter 3**. We consider the grid drawings with bounded number of columns on which a given graph can be drawn or located (a graph can be located in $\mathbb{Z}^d$ if it has a grid drawing where every line segment contains exactly two grid points). We show that this problem is closely connected to a special form of a defective coloring in which some colors induce independent sets and some disjoint unions of paths. See Observation 3.2 and Theorem 3.4. We examine this particular coloring more deeply in Section 3.3 and we use it later to prove a more general result, Theorem 3.14, which implies NP-completeness of a problem of deciding whether a graph can be drawn/located on a l columns where $l \geq 2$ is a given integer.

This answers an open question of Cáceres et. al. [10]. Some other statements which strengthen known results are also shown in this chapter. For example Theorem 3.8.

We examine another type of compactness in **Chapter 4**. That is, the grid drawings in $\mathbb{Z}^2$ with small area. We mention some interesting already known results and then, in Section 4.2, we show a sharp lower bound on area of grid drawings of balanced complete k-partite graphs proving a conjecture asked by David R. Wood [36].

The main result of **Chapter 5** is Theorem 5.1, which says that any planar graph has a planar grid drawing in $\mathbb{Z}^2$ where every line segment intersects exactly two grid points. We use the Four Color Theorem to prove this statement and we show that these two results are actually equivalent. In the rest of this chapter we discuss the sizes of the obtained grid drawings.

At the end of the thesis we mention some related open problems and possible ways for further research together with a brief summary of our results.

An extended abstract containing some results from this thesis appeared in the proceedings[1] of the 28th European Workshop on Computational Geometry EuroCG '12 [5]. Afterwards, the paper was amongst selected papers invited to a special issue of the journal "Computational Geometry: Theory and Applications". I also presented a weaker version of Theorem 2.2 together with some older related results (not mentioned in the thesis) in the competition SVOČ 2011[4]'.

[1] The booklet of abstracts is available here: http://www.diei.unipg.it/eurocg2012/booklet.pdf

1. Preliminaries

This chapter should give the reader a brief introduction to graph theory, complexity theory and number theory. We mention some basic definitions and classical results which we refer to throughout the thesis. We also introduce the concept of grid drawings and we attach some related definitions.

A reader familiar with the semi-standard notation can skip most of this chapter and move on to Chapter 2 and alternatively return later for unfamiliar definitions and concepts.

1.1 Graph Theory

All graphs considered are simple, undirected, finite and loopless. It means that a *graph* G is given as a pair (V, E) where V is finite set of *vertices* and $E \subseteq \binom{V}{2}$ is a set of unordered pairs which we call *edges*.

Two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are said to be *isomorphic* if there is a bijection $\varphi\colon V_G \to V_H$ such that $uv \in E_G \Leftrightarrow \varphi(u)\varphi(v) \in E_H$. To simplify the notation, we consider all isomorphic graphs to be equal.

A *subgraph* $H = (V_H, E_H)$ of a graph $G = (V_G, E_G)$ is a graph that satisfies $V_H \subseteq V_G$ and $E_H \subseteq \binom{V_H}{2} \cap E_G$. If $E_H = \binom{V_H}{2} \cap E_G$, then we say that H is a *subgraph induced by the set* V_H, or simply it is an *induced subgraph* of G. In such case we write $H = G[V_H]$. The *degree of the vertex* $v \in V_G$ is a number of edges incident to v and we denote it as $d(v)$. The maximum degree of a graph G is denoted as $\Delta(G)$.

Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs with disjoint vertex sets. A *disjoint union of graphs* G *and* H is the graph with vertex set $V = V_G \cup V_H$ and edges $E = E_G \cup E_H$.

A *path* in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. A *cycle* is a path such that the start vertex and end vertex are the same. A path with no repeated vertices is called a *simple path* and a cycle with no repeated vertices or edges aside from the repetition of the start and end vertex is a *simple cycle*.

A *tree* is a graph in which any two vertices are connected by exactly one simple path. A *path graph* (or *path*, for short) is a tree which has a simple path which passes through all its vertices. The disjoint union of paths is called a *linear forest*. A *cycle graph* (or *cycle*, for short) is a graph that consists of a single simple cycle. A graph in which every pair of distinct vertices is connected by an edge is called *complete*. We also use the term *clique* for such graph and we denote the complete graph on n vertices as K_n.

Planar graphs are graphs that can be embedded in the Euclidean plane in such a way that vertices are represented by points and edges by curves connecting adjacent vertices that intersect only at their endpoints. A planar graph already drawn in the plane without edge crossings is called a *planar embedding of the graph*.

A *k-coloring* of a graph $G = (V, E)$ is a function $f\colon V \to C$ for some set C of k colors such that $f(u) \neq f(v)$ for every edge $uv \in E$. If such k-coloring of

G exists, then G is *k-colorable.* The *chromatic number* $\chi(G)$ of G is the least k such that G is k-colorable.

For $k \geq 2$, a *k-partite graph* is a graph whose vertices can be partitioned into k disjoint sets such that no two vertices within the same set are adjacent. Let $K(k,t)$ denote the balanced complete k-partite graph with t vertices in each color class. These graphs are very important, as every k-colorable graph on t vertices is a subgraph of $K(k,t)$.

Before introducing the concept of grid drawings, we conclude the part dedicated to graph theory by mentioning two classical results about planar graphs. The first one is called the Four Color Theorem and it shows the minimum number of colors that we need to color an arbitrary planar graph.

Theorem 1.1 (The Four Color Theorem, [2]). *Any planar graph is 4-colorable.*

The second statement, Fáry's theorem, concerns with drawings of planar graphs and it states that any planar graph has a planar embedding where edges are straight line segments.

Theorem 1.2 (Fáry's theorem, [20, 34, 35]). *Any planar graph can be drawn without crossings so that its edges are straight line segments.*

For more throughout introduction to graph theory, we recommend the books "Graph Theory" by R. Diestel [18] and "Modern Graph Theory" by B. Bollobás [8].

1.2 Grid Drawings

For an integer $d \geq 2$, a *column* in the grid $\mathbb{Z}^d$ with the *rank* $(x_1,\ldots,x_{d-1}) \in \mathbb{Z}^{d-1}$ is the set $\{(x_1,\ldots,x_{d-1},x) \mid x \in \mathbb{Z}\}$. Let $\overline{xy}$ denote the closed line segment joining two grid points $x,y \in \mathbb{Z}^d$. The line segment $\overline{xy}$ is *primitive* if $\overline{xy} \cap \mathbb{Z}^d = \{x,y\}$.

Definition A *grid drawing* $\phi(G)$ of G in $\mathbb{Z}^d$ is an injective mapping $\phi\colon V \to \mathbb{Z}^d$ such that, for every edge $uv \in E$ and vertex $w \in V$, $\phi(w) \in \overline{\phi(u)\,\phi(v)}$ implies that $w = u$ or $w = v$.

That is, a grid drawing represents vertices of G by a distinct grid points in $\mathbb{Z}^d$, and each edge by a line segment between its endpoints such that the only vertices an edge intersects are its own endpoints. Note that the condition on the grid points representing vertices is essential. Without it every graph could be drawn on a single arbitrary column. Also note that we allow edges to cross in a grid drawing.

The points $x,y \in P$, where $P \subseteq \mathbb{Z}^d$, are *(mutually) visible with respect to* P if $P \cap \overline{xy} = \{x,y\}$. If $P = \mathbb{Z}^d$, then we just say that u,v are *visible*. The *visibility graph* $v(P)$ of P has a vertex set P where two distinct vertices are adjacent if and only if they are visible with respect to P.

Now we can use the new terms of visibility for a description of the grid drawings. A graph G has a grid drawing in $\mathbb{Z}^d$ if and only if there is a set $P \subseteq \mathbb{Z}^d$ such that G is a subgraph of $v(P)$. Also, there is a primitive grid drawing of G in $\mathbb{Z}^d$ if and only if G is a subgraph of the visibility graph $v\left(\mathbb{Z}^d\right)$.

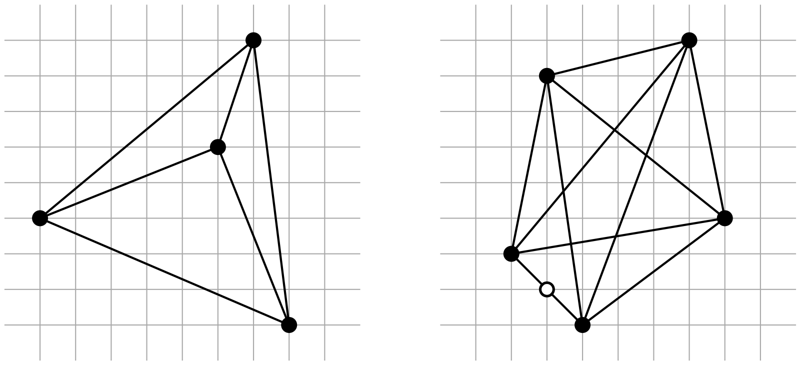

Figure 1.1: Some grid drawings

In the recent years a lot of interesting results concerning the grid drawings of graphs appeared. For example, suppose that we want to find a planar two-dimensional grid drawing of a given planar graph G such that the size of the drawing is as small as possible. The following three results proved by De Fraysseix, Pach, and Pollack [16], Schnyder [33], and Chrobak and Nakano [13], respectively, say that we can find such drawings in the grid with size $O(n) \times O(n)$ where n denotes the number of vertices of G. This grid size is asymptotically optimal.

Theorem 1.3 ([16])**.** *Any planar graph with n vertices has a planar grid drawing on the $2n-4 \times n-2$ grid and there is $O(n \log n)$ algorithm to effect this drawing.*

Theorem 1.4 ([33])**.** *Each planar graph with $n \geq 3$ vertices has a planar grid drawing on the $n-2 \times n-2$ grid and this drawing is computable in $O(n)$ time.*

Theorem 1.5 ([13])**.** *There is an algorithm which for a given a triangulated planar graph G with n vertices constructs a planar grid drawing of G into a $\omega \times (4\omega - 1)$ grid, $\omega \leq 2\left\lfloor \frac{2(n-1)}{3} \right\rfloor$, in linear time with respect to n.*

János Pach, Torsten Thiele and Géza Tóth [30] considered a similar problem in three-dimensional grid. Suppose we have an integer $r \geq 2$ and a given r-colorable graph G with n vertices. We want to map the vertices of G into distinct points of $\mathbb{Z}^3$ such that the straight-line segments representing the edges of G are pairwise non-crossing. In other words – we want to find a three-dimensional planar grid drawing of G. The authors showed that there exists such grid drawing of G which fits into a box of volume $O(n^2)$ and that this bound is optimal.

Theorem 1.6 ([30])**.** *For every $r \geq 2$ fixed, any r-colorable graph of n vertices has a planar three-dimensional grid drawing that fits into a rectangular box of volume $O(n^2)$. The order of magnitude of this bound cannot be improved.*

Although we consider a similar problem of drawing r-colorable graphs into two-dimensional grids with small area in Chapter 4, we mainly focus on the problems concerning the connections between grid drawings and graph colorings in this thesis.

1.3 Complexity

Complexity theory is a branch in theoretical computer science and mathematics that focuses on classifying computational problems according to their difficulty, and relating those classes to each other.

The *asymptotic notation* (or so called Bachmann-Landau notation) is used to describe the limiting behavior of a function when the argument tends towards a particular value or infinity. We also use it to classify algorithms by how they respond to changes in input size. Let $f(n)$ and $g(n)$ be two functions defined on some subset of the real numbers. We write $f(n) \in O(g(n))$ if there is a positive constant c such that for all sufficiently large values of n the inequality $0 \leq f(n) \leq cg(n)$ holds. Analogously, $f(n) \in \Omega(g(n))$ if $0 \leq cg(n) \leq f(n)$.

A *decision problem* is any arbitrary yes-or-no question on an infinite set of inputs. We can classify the complexity of a decision problem using the asymptotic notation, or we can assign the problem to a *complexity class*. For the exact definitions, please, see the book "Computational complexity" by C. M. Papadimitriou [31].

There is an infinite hierarchy of complexity classes, but we introduce only the most basic ones which we mention in the thesis. The complexity class $\mathsf{DTIME}(f(n))$ is the set of decision problems that can be solved by a deterministic Turing machine which runs in time $O(f(n))$ where n is the size of the input. The complexity class P can be defined as $\mathsf{P} = \bigcup_{k\in\mathbb{N}} \mathsf{DTIME}\left(n^k\right)$. That is, P is a class of of all the problems solvable in a time polynomial in the size of input on a deterministic Turing machine.

Similarly, the well-known class NP denotes the complexity class defined as $\bigcup_{k\in\mathbb{N}} \mathsf{NTIME}\left(n^k\right)$ where $\mathsf{NTIME}(f(n))$ is the set of decision problems that can be solved by a non-deterministic Turing machine in time $O(f(n))$. Thus NP is the class of all the problems solvable in time polynomial in the size of input on a non-deterministic Turing machine. We have the inclusion $\mathsf{P} \subseteq \mathsf{NP}$, but the question of equality remains open and it is one of the most famous problems in contemporary mathematics and theoretical computer science.

Let P_1 and P_2 be two decision problems with inputs I_1 and I_2, respectively. We say that P_1 is *polynomial time reducible* to P_2 if there is is an algorithm transforming I_1 in time polynomial in $|I_1|$ to I_2 such that the answer to $P(I_1)$ is "YES" if and only if the answer to $P(I_2)$ is "YES". In other words, P_1 can be solved in polynomial time by an oracle machine with an oracle for P_2.

A problem P_2 is NP-*hard* if there is an NP hard problem P_1 which is polynomial time reducible to P_2. Note that P_2 is at least as hard as P_1, because we can use P_2 to solve P_1. If P_2 is NP-hard an it is in the class NP, then we say that P_2 is NP-*complete problem*. These are the hardest problems in NP. There are decision problems that are NP-hard but not NP-complete, for example the Halting Problem which asks "Given a program and its input, will it run forever?".

Let us mention some known NP-complete problems which we use later in the thesis.

GRAPH k-COLORABILITY
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Is G k-colorable?

This particular problem is solvable for $k = 2$ in polynomial time, as a graph is two-colorable if and only if it is bipartite. For every fixed $k \geq 3$, the problem becomes NP-complete. For $k = 3$, the Graph k-Colorability problem remains NP-complete even if we restrict our attention to only planar graphs of degree four as shown by David P. Dailey [15]. Especially, it is NP-complete to compute the chromatic number of G.

3-SATISFIABILITY (3SAT)
Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$ (see [22] for definitions).
Question: Is there a satysfying truth assignment for C?

This is a classical NP-complete problem which is very useful for showing NP-completeness. As one of the most important NP-complete problems, 3SAT has many versions and two of them will be particularly helpful for us in the subsequent proofs. The following one, One-in-three 3SAT, is used in the proof of Lemma 3.12 in Chapter 3.

ONE-IN-THREE 3SAT
Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$.
Question: Is there a truth assignment for U such that each clause in C has exactly one true literal?

The following second variant of 3SAT helps us in the proof of Lemma 3.13.

NOT-ALL-EQUAL 3SAT
Instance: Set U of variables, collection C of clauses over U such that each clause $c \in C$ has $|c| = 3$.
Question: Is there a truth assignment for U such that each clause in C has at least one true and at least one false literal?

A list of some other NP-complete problems is available, for example, in the book "Computers and Intractability; A Guide to the Theory of NP-Completeness" by M. R. Garey and D. S. Johnson [22]. A nice introduction to complexity theory is also the book "Computational Complexity: A Modern Approach" by Sanjeev Arora and Boaz Barak [3].

1.4 Number Theory

Generally speaking, we can define number theory as the study of the properties of integers. It is one of the oldest and most fascinating fields of pure mathematics and it involves studying elementary objects made out of integers, properties of primes, congruences and so on. In this section we mention a few standard definitions and results which are key in the proofs of some statements in the thesis. An interested reader may find good introductions to number theory in the books "Recreations in the Theory of Numbers: The Queen of Mathematics Entertains" by A. H. Beiler [7] and "Elementary Number Theory with Applications" by T. Koshy [28].

For a positive integer n, the set $\mathbb{Z}_n$ is the finite set of integers modulo n. Two integers a and b are said to be *congruent modulo* n, written $a \equiv b(\bmod n)$, if the difference $a - b$ is an integer multiple of n. Congruence modulo n is equivalence relation and the equivalence class $[a]_n$ of the integer a is called the *residue class of the integer a modulo n.* Thus the set $\mathbb{Z}_n$ can be written as $\{[x]_n \mid x \in \mathbb{Z}\}$. If we define addition, subtraction, and multiplication on $\mathbb{Z}_n$ by the following rules $[a]_n + [b]_n = [a+b]_n$, $[a]_n - [b]_n = [a-b]_n$ and $[a]_n \cdot [b]_n = [ab]_n$, then $\mathbb{Z}_n$ becomes a commutative ring.

For $k \geq 2$, the greatest common divisor of integers $x_1, x_2, \ldots, x_k$ is denoted as $\gcd(x_1, x_2, \ldots, x_k)$. The *modular multiplicative inverse of an integer a modulo n* is an integer b such that $ab \equiv 1 \,(\bmod n)$. That is, it is the multiplicative inverse in the ring of integers modulo n and it exists if and only if $\gcd(a, n) = 1$. Thus $\mathbb{Z}_n$ is a field when n is prime and subdivision is defined as multiplication by inverse. The modular multiplicative inverse of a modulo n can be found with the extended Euclidean algorithm in time $O\left(\log^2 n\right)$, assuming $|a| < n$.

The following theorem shows that every linear system in the same single variable with pairwise relatively prime moduli has a unique solution. It is called the Chinese Remainder Theorem in honor of early contributions by Chinese mathematicians to the theory of congruences.

Theorem 1.7 (The Chinese Remainder Theorem)**.** *Let $m_1, m_2, \ldots, m_k$ be pairwise coprime integers and let $M = \prod_{i=1}^{k} m_i$. Then, for any given sequence of integers $r_1, r_2, \ldots, r_k$, there is a unique solution $n \in \mathbb{Z}_M$ such that $n \equiv r_i \,(\bmod m_i)$ for every $i \in \{1, 2, \ldots, k\}$.*

For completeness, we include a proof of this well-known theorem from the book by T. Koshy [28], as we need to know the details of the proof in Chapter 2, Corollary 2.9. The important feature of the proof is that it is constructive and we can find such solution in polynomial time. In addition, if we consider the numbers $n + iM$, where $i \in \mathbb{Z}$, then we see that there are infinitely many integer solutions.

Proof. Let $M_i = M/m_i$ for every $i \in \{1, 2, \ldots, k\}$. Since $\gcd(m_i, m_j) = 1$ for every distinct i and j, we get $\gcd(M_i, m_i) = 1$ for every i. Also we know that $M_i \equiv 0 \,(\bmod m_j)$ whenever $i \neq j$.

Since M_i and m_i are coprime, there is a unique solution y_i of $M_i y_i \equiv 1 \,(\bmod m_i)$. It is the modular multiplicative inverse of M_i modulo m_i. If we set $n = r_1 M_1 y_1 + r_2 M_2 y_2 + \cdots + r_k M_k y_k$, then it is not difficult to verify that n is the solution of our linear system, as, for every $j \in \{1, 2, \ldots, k\}$, we have

$$n = r_j M_j y_j + \sum_{i=1, i \neq j}^{k} r_i M_i y_i \equiv r_j \cdot 1 + \sum_{i=1, i \neq j}^{k} r_i \cdot 0 \cdot y_i \equiv r_j \,(\bmod m_j)\,.$$

We have thus proved the existence. Now it remains to show that this solution is unique. Assume to the contrary that n_1, $n_2 \in \mathbb{Z}_M$ are two distinct solutions. Since $n_1 \equiv r_i \,(\bmod m_i)$ and $n_2 \equiv r_i \,(\bmod m_i)$ for every $i \in \{1, 2, \ldots, k\}$, we get that every m_i divides $n_1 - n_2$ and thus M divides this difference as well. But then $n_1 \equiv n_2 \,(\bmod M)$ which is a contradiction. $\square$

For $x \in \mathbb{R}$, let $\pi(x)$ denote the number of primes less than or equal to x. It is called the *prime-counting function.* The following famous result, the Prime Number Theorem, gives a general description of how the primes are distributed amongst the positive integers.

Theorem 1.8 (The Prime Number Theorem, [17, 25])**.**

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln x} = 1$$

That is, as x gets larger, $\pi(x)$ approaches $x/\ln x$. This result was proved independently by Jacques Hadamard [25] and Charles Jean de la Vallée-Poussin [17] in 1896. Before those proofs, a slightly weaker form of the theorem was proved by Pafnuty Lvovich Chebyshev. Although he did not prove the Prime Number Theorem, his estimates were strong enough for him to prove Bertrand's postulate.

Theorem 1.9 (Bertrand's postulate, [12])**.** *If $n > 3$ is an integer, then there always exists at least one prime number p with $n < p < 2n - 2$.*

We use both these results in Chapter 4. The last statement, I would like to mention, provides a simple formula for calculating the area of a simple polygon constructed on a grid of equal-distanced points. We use this formula for estimating the sizes of certain planar grid drawings in Chapter 5.

Theorem 1.10 (Pick's theorem, [32])**.** *The area of a given simple closed lattice polygon equals $I + \frac{B}{2} - 1$, where I is the number of points in the interior of the polygon and B the number of lattice points on the polygon edges.*

2. Robustness of Grid Drawings

In this chapter, we are concerned with grid drawings in which the number of grid points that can appear on a single line segment is bounded.

Theorem 2.2, the main result of the second chapter, shows a connection between the chromatic number of a given graph and between the number of grid points that appear on line segments of its grid drawings. In the last section, we conclude with some additional remarks to the proof and we also show some interesting corollaries.

2.1 Bounded Number of Grid Points

The *robustness of a grid drawing* is understood as the maximum number of grid points a line segment of such drawing contains. For a given graph G we would like to find a grid drawing of G with the lowest robustness.

A graph G is said to be *(grid) locatable in* $\mathbb{Z}^d$ if there exists a grid drawing of G in $\mathbb{Z}^d$ where every edge is represented by a primitive line segment (such drawing is also called *primitive*). Finding a primitive grid drawing of G is called *locating the graph* G. D. Flores Peñaloza and F. J. Zaragoza Martinez [21] showed the following characterization:

Theorem 2.1 ([21, 27])**.** *A graph G is locatable in $\mathbb{Z}^2$ if and only if G is 4-colorable.*

Therefore not all graphs are locatable in $\mathbb{Z}^2$ and every (two-dimensional) grid drawing of any k-colorable graph, where $k > 4$, contains a line segment which contains at least three grid points. This leads us to a generalization of the concept of locatability. Let the number $gp\,(\phi\,(G))$ denote the maximum number of grid points any line segment of the grid drawing $\phi\,(G)$ contains.

Definition A graph G is *(grid) q-locatable* in $\mathbb{Z}^d$, for some integer $q \geq 2$, if there exists a grid drawing $\phi\,(G)$ in $\mathbb{Z}^d$ such that $gp\,(\phi\,(G)) \leq q$.

The *grid robustness of a graph* G is the minimum of $gp\,(\phi\,(G))$ among all grid drawings $\phi\,(G)$. For example, the graph K_5 has chromatic number five, thus it is not (two-)locatable in $\mathbb{Z}^2$. However the grid drawing in Figure 1.1 shows that K_5 is three-locatable in $\mathbb{Z}^2$ (the third grid point on one line segment is denoted by an empty circle). The main result of this chapter is a stronger version of Theorem 2.1.

Theorem 2.2. *For integers d, $q \geq 2$, a graph G is q^d-colorable if and only if G is q-locatable in $\mathbb{Z}^d$.*

The proof is constructive and for a given coloring of G with at most q^d colors we get a linear time algorithm which gives us a grid drawing $\phi\,(G)$ of G in $\mathbb{Z}^d$ where every line segment contains at most q grid points. Conversely, we also show that if we have such grid drawing of G, then it is quite trivial to get coloring of G with at most q^d colors.

2.2 Proof of Theorem 2.2

Let us show the following trivial but useful proposition before the actual proof.

Proposition 2.3 ([1])**.** *Let $d \in \mathbb{N}$ and $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ be two distinct grid points. Then a, b are mutually visible if and only if*

$$\gcd\left(|a_1 - b_1|, \ldots, |a_d - b_d|\right) = 1.$$

Proof. Clearly it suffices to prove the proposition when $b = (0, \ldots, 0)$. Assume a is visible from the origin and let $\alpha = \gcd(a_1, \ldots, a_d)$. If $\alpha > 1$, then $a_j = \alpha \tilde{a}_j$ for every $j \in \{1, \ldots, d\}$ and the grid point $(\tilde{a}_1, \ldots, \tilde{a}_d)$ lies on the line segment $\overline{ab}$. Conversely, assume $\gcd(a_1, \ldots, a_d) = 1$. If a grid point $(\tilde{a}_1, \ldots, \tilde{a}_d)$ lies on the line segment $\overline{ab}$, we have $\tilde{a}_1 = a_1 t, \ldots, \tilde{a}_d = a_d t$ where $0 < t < 1$. Hence $t \in \mathbb{Q}$, so $t = \frac{r}{s}$ where r, s are positive integers with $\gcd(r, s) = 1$. Thus $s\tilde{a}_1 = a_1 r, \ldots, s\tilde{a}_d = a_d r$, so $s | a_1 r, \ldots, s | a_d r$. But $\gcd(r, s) = 1$, so $s | a_1, \ldots, s | a_d$. Hence $s = 1$ since $\gcd(a_1, \ldots, a_d) = 1$. This contradicts the inequality $0 < t < 1$. Therefore the grid point a is visible from the origin. □

From the equation of a line it is not difficult to characterize the points lying on the line segment $\overline{ab}$.

Corollary 2.4. *Let $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d) \in \mathbb{Z}^d$ be distinct grid points and let*

$$\alpha = \gcd\left(|a_1 - b_1|, \ldots, |a_d - b_d|\right).$$

Then the line segment $\overline{ab}$ contains exactly the grid points of the form

$$\left(a_1 + i\frac{b_1 - a_1}{\alpha}, \ldots, a_d + i\frac{b_d - a_d}{\alpha}\right)$$

where $0 \le i \le \alpha$.

Now, we split the proof of Theorem 2.2 into two parts. First, we show the easier implication and then, after some auxiliary constructions, we give a proof of the reverse implication.

Lemma 2.5. *For integers d, $q \ge 2$, if the graph G is q-locatable in $\mathbb{Z}^d$, then it is q^d-colorable.*

Proof. Using Corollary 2.4 we can prove this lemma immediately by choosing an appropriate coloring of $\mathbb{Z}^d$. Let $\phi(G)$ be a grid drawing of the graph $G = (V, E)$ in $\mathbb{Z}^d$ having $gp(\phi(G)) \le q$. Consider the function $f\colon \mathbb{Z}^d \to \mathbb{Z}_q^d$ denoted as

$$f(x_1, \ldots, x_d) = \left(x_1 \,(\mathrm{mod} q), \ldots, x_d \,(\mathrm{mod} q)\right).$$

We use f as coloring of the grid with q^d colors and we show that it is also a proper vertex coloring of G. Assume to the contrary that $f(\phi(u)) = f(\phi(v))$ for some $uv \in E$. Then $u_1 \equiv v_1, \ldots, u_d \equiv v_d \,(\mathrm{mod} q)$ which implies

$$\gcd\left(|u_1 - v_1|, \ldots, |u_d - v_d|\right) \ge q.$$

According to our corollary, there are at least $q + 1$ grid points lying on the line segment $\overline{\phi(u)\,\phi(v)}$. This contradicts the fact that G is q-locatable via the drawing $\phi(G)$. □

Thus it remains to show the implication in the opposite direction. The main idea is to find a subset of $\mathbb{Z}^d$ which we can use for a convenient grid drawing of every q^d-colorable graph.

Assume that the dimension d is fixed and let p be a prime number. We define $V_{p,1}$ as the sequence $\{x_i\}_{i=0}^{p^d-1}$ such that each x_i is from the set $\mathbb{Z}_p^d$ and no two terms are equal. This definition is correct, as we can always find p distinct residues modulo p and, naturally, there are p^d distinct d-tuples of these residues. Now we define $V_{p,e}$ for $e \geq 2$ inductively. Assume as induction hypothesis that we have already set $V_{p,e-1}$. Now we place $V_{p,e}$ as a chain of p^d copies of $V_{p,e-1}$. Then we change the terms on the positions

$$i + p^{d(e-1)}, \ldots, i + \left(p^d - 1\right) p^{d(e-1)}$$

for every $i \in \left\{0, 1, \ldots, p^{d(e-1)} - 1\right\}$ in such way that the new terms are numbers from $\mathbb{Z}_{p^e}$ congruent to their predecessors modulo p^e and no two terms in $V_{p,e}$ are equal. For each element of $Z^d_{p^{e-1}}$ there are p^d congruent elements from $\mathbb{Z}^d_{p^e}$ modulo p^{e-1} and one of them is on the i-th position of $V_{p,e}$. Thus the definition of $V_{p,e}$ is, again, correct.

On Figure 2.1 there is an example of this construction with fixed parameters. The following sequence $V_{2,3}$ would have 64 terms and the first new one would be on position 16, assuming that the ranking starts at zero.

$V_{2,1}$: $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$

$\downarrow$

$V_{2,2}$: $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(0,2)$, $(2,1)$, $(1,2)$, $(1,3)$
$(2,0)$, $(2,1)$, $(3,0)$, $(3,1)$, $(2,2)$, $(2,3)$, $(3,2)$, $(3,3)$,

$\downarrow$

$\vdots$

Figure 2.1: Construction of $V_{2,e}$, $e = 1, 2, \ldots$, for $d = 2$

Continual repeating of the copies of $V_{p,e}$ gives us the infinite sequence $S_{p,e}$. We denote the i-th term of $S_{p,e}$ as $S_{p,e}[i]$ and the distance of two terms $S_{p,e}[i]$ and $S_{p,e}[j]$ is given by $|i - j|$. The following lemma shows an important feature of these sequences.

Lemma 2.6. *Let p be prime number and e positive integer. Then two terms of $S_{p,e}$ are equal if and only if p^{de} divides their distance.*

Proof. Suppose that our terms are on positions i and j. The case $i = j$ is apparent, thus we can assume $i \neq j$. From the definition two distinct terms equal if and only if both are in different copies of $V_{p,e}$, but on the same position in $V_{p,e}$. The length of $V_{p,e}$ is exactly p^{de}, so the distance between $S_{p,e}[i]$ and $S_{p,e}[j]$ is a multiple of p^{de}. □

Given an integer s, we set $f(p)$ as $\min\left\{e \in \mathbb{N} \mid p^{de} \geq s\right\}$ for every prime number $p < s$. Now, for every i, where $0 \leq i \leq s - 1$, we choose a distinct column of $\mathbb{Z}^d$ such that for every prime number $p < s$ the rank of this column is congruent to the first $d-1$ elements of the d-tuple $S_{p,e}[i]$ modulo $p^{f(p)}$. We label the chosen columns as $W_{0,s}, W_{1,s}, \ldots, W_{s-1,s}$. In every column $W_{i,s}$ we keep only the points with their last coordinate congruent to the last element of $S_{p,e}[i]$ modulo $p^{f(p)}$, again for every $p < s$. Finally we set $W_s = \bigcup_{i=0}^{s-1} W_{i,s}$.

Let us mention the last technical remark. If there is a prime $p \geq s$ such that ranks of two or more columns from W_s are congruent modulo p, then we assign distinct residues modulo p to these columns. Subsequently, we keep only the points with their last coordinate congruent to the assigned residue modulo p in each one of these columns. This method is correct, because the number of possible residues is at least s, thus every column can get a unique residue. According to the Chinese Remainder Theorem, every column of W_s still contains infinitely many points.

Example Assume we want to build W_9 in two-dimensional case. For $s = 9$, we have to define the sequences $S_{2,2}$, $S_{3,1}$, $S_{5,1}$ and $S_{7,1}$, as $2^4, 3^2, 5^2, 7^2 \geq 9$. No other sequences are required, because $9 \leq p$ for every other prime number p.

$$
\begin{aligned}
S_{2,2} &= (0,0)\,,(0,1)\,,(1,0)\,,(1,1)\,,(0,2)\,,(0,3)\,,(1,2)\,,(1,3)\,,(2,0)\,,\ldots \\
S_{3,1} &= (0,0)\,,(0,1)\,,(0,2)\,,(1,0)\,,(1,1)\,,(1,2)\,,(2,0)\,,(2,1)\,,(2,2)\,,\ldots \\
S_{5,1} &= (0,0)\,,(0,1)\,,(0,2)\,,(0,3)\,,(0,4)\,,(1,0)\,,(1,1)\,,(1,2)\,,(1,3)\,,\ldots \\
S_{7,1} &= (0,0)\,,(0,1)\,,(0,2)\,,(0,3)\,,(0,4)\,,(0,5)\,,(0,6)\,,(1,0)\,,(1,1)\,,\ldots
\end{aligned}
$$

Then we can get the set W_9 as a union of the following columns:

$$
\begin{aligned}
W_{0,9} &= \left\{(0,x) \in \mathbb{Z}^2 \mid x \equiv 0\,(\mathrm{mod}4)\,,0\,(\mathrm{mod}3)\,,0\,(\mathrm{mod}5)\,,0\,(\mathrm{mod}7)\right\} \\
W_{1,9} &= \left\{(420,x) \in \mathbb{Z}^2 \mid x \equiv 1\,(\mathrm{mod}4)\,,1\,(\mathrm{mod}3)\,,1\,(\mathrm{mod}5)\,,1\,(\mathrm{mod}7)\right\} \\
W_{2,9} &= \left\{(105,x) \in \mathbb{Z}^2 \mid x \equiv 0\,(\mathrm{mod}4)\,,2\,(\mathrm{mod}3)\,,2\,(\mathrm{mod}5)\,,2\,(\mathrm{mod}7)\right\} \\
W_{3,9} &= \left\{(385,x) \in \mathbb{Z}^2 \mid x \equiv 1\,(\mathrm{mod}4)\,,0\,(\mathrm{mod}3)\,,3\,(\mathrm{mod}5)\,,3\,(\mathrm{mod}7)\right\} \\
W_{4,9} &= \left\{(280,x) \in \mathbb{Z}^2 \mid x \equiv 2\,(\mathrm{mod}4)\,,1\,(\mathrm{mod}3)\,,4\,(\mathrm{mod}5)\,,4\,(\mathrm{mod}7)\right\} \\
W_{5,9} &= \left\{(196,x) \in \mathbb{Z}^2 \mid x \equiv 3\,(\mathrm{mod}4)\,,2\,(\mathrm{mod}3)\,,0\,(\mathrm{mod}5)\,,5\,(\mathrm{mod}7)\right\} \\
W_{6,9} &= \left\{(161,x) \in \mathbb{Z}^2 \mid x \equiv 2\,(\mathrm{mod}4)\,,0\,(\mathrm{mod}3)\,,1\,(\mathrm{mod}5)\,,6\,(\mathrm{mod}7)\right\} \\
W_{7,9} &= \left\{(281,x) \in \mathbb{Z}^2 \mid x \equiv 3\,(\mathrm{mod}4)\,,1\,(\mathrm{mod}3)\,,2\,(\mathrm{mod}5)\,,0\,(\mathrm{mod}7)\right\} \\
W_{8,9} &= \left\{(386,x) \in \mathbb{Z}^2 \mid x \equiv 0\,(\mathrm{mod}4)\,,2\,(\mathrm{mod}3)\,,3\,(\mathrm{mod}5)\,,1\,(\mathrm{mod}7)\right\}
\end{aligned}
$$

In the last step we ensure possible occurrences of prime numbers $p \geq s$ in decompositions of differences of ranks. For example, prime number 23 divides the difference of ranks 0 and 161. But, if we can keep only the points $(0,x) \in W_{0,9}$ and the points $(0,y) \in W_{6,9}$ such that x and y are not congruent modulo 23, then 23 does not divide $\gcd(|a_1 - b_1|, |a_2 - b_2|)$ for any $a \in W_{0,9}$ and $b \in W_{6,9}$.

The construction of the set W_s is not easy to describe, but it has nice properties that allow us to prove the crucial lemma in the proof of Theorem 2.2.

Lemma 2.7. *Let s be d-th power of integer $q \geq 2$. Let $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)$ be grid points located in distinct columns of the set W_s. Then*

$$\gcd(|a_1 - b_1|\,, \ldots, |a_d - b_d|) \leq q - 1.$$

Proof. Let α denote the greatest common divisor in the statement. Assume that the grid point a is in the column $W_{x,s}$ and the grid point b in the column $W_{y,s}$, $0 \leq x, y \leq s-1$ and $x \neq y$. The last remark in the construction of W_s guarantee that no prime number larger than $s-1$ divides α. Also, for every $e \in \mathbb{N}$ and prime number p, the power p^e divides α if and only if $S_{p,e}[x] = S_{p,e}[y]$. Since $p^e \mid \alpha$ implies that each coordinate of a is congruent to each coordinate of b modulo p^e and these coordinates are congruent to the d-tuples $S_{p,e}[x]$ and $S_{p,e}[y]$ modulo p^e. Thus p^e does not divide α for $e \geq f(p)$. Otherwise $S_{p,f(p)}[x] = S_{p,f(p)}[y]$ and, according to Lemma 2.6, the distance $|x-y|$ between them is at least $p^{df(p)}$, which is at least s. But this contradicts the inequality $0 \leq x, y \leq s-1$.

So we can assume that $\alpha = \prod_{i=1}^{k} p_i^{e_i}$ where p_i are prime numbers and $1 \leq e_i \leq f(p_i) - 1$. Then

$$\alpha^d = \prod_{i=1}^{k} p_i^{de_i} \leq s-1$$

holds. Since the expression of α implies that $S_{p_i,e_i}[x] = S_{p_i,e_i}[y]$ and, again, we get $p_i^{de_i} \mid |x-y|$ for every $i \in \{1, 2, \ldots, k\}$. Thus $\alpha^d \leq |x-y| \leq s-1$.

We know that s is d-th power of some integer $q \geq 2$ and we just showed that α^d is smaller d-th power than s. Thus $\alpha^d \leq (q-1)^d$. But this gives us the required inequality, as $(q-1) \leq \sqrt[d]{q^d} - 1$ holds trivially. $\square$

Now we can finally prove Theorem 2.2.

Proof of Theorem 2.2. The first implication is proved in Lemma 2.5, so assume that G is a q^d-colorable graph, $q \geq 2$. We need to find a grid drawing of G such that at most q grid points lie on any of its line segments. It suffices to show how to find such drawing for the graph $K\left(q^d, n\right)$ and arbitrary $n \in \mathbb{N}$, because every q^d-colorable graph on n vertices is its subgraph. We consider the set W_s for $s = q^d$ and we keep only the first n vertices of its first two columns. Then for every i, $2 \leq i \leq s-1$, we keep the first n points in the column $W_{i,s}$ such that all points in previous columns are visible from any of these points (with respect to other columns). These points in $W_{i,s}$ exist, because, unlike $W_{i,s}$, the previous columns are finite sets.

Afterwards we obtain the set $W_s(n) \subset W_s$ such that $K\left(q^d, n\right)$ is isomorphic to the visibility graph $\upsilon(W_s(n))$ and, according to Lemma 2.7, no line segment contains more then q grid points. Therefore we get suitable grid drawing of G and the second implication is proved. $\square$

2.3 Concluding Remarks

The construction of W_s used in the proof of Theorem 2.2 is not difficult to carry out for a given graph $G = (V, E)$ and its coloring with k colors. The position of its points can be computed in time $O(|V|)$ and we can use the procedure in the proof of the Chinese Remainder Theorem. The volume of the obtained grid drawing is constant with respect to number of vertices in the first $d-1$ coordinates and it depends on k.

One of the most interesting corollaries of this theorem are the following.

Corollary 2.8. *For an integer $d \geq 2$, a graph is 2^d-colorable if and only if it is locatable in $\mathbb{Z}^d$.*

Also note that a graph G has a primitive grid drawing in $\mathbb{Z}^{2d}$, $d \in \mathbb{N}$, if and only if there is a grid drawing of G in $\mathbb{Z}^2$ where no line segment contains more than 2^d grid points.

Corollary 2.9. *For given d, $q \geq 2$, it is* NP*-complete to decide whether or not the graph G is q-locatable in $\mathbb{Z}^d$.*

Proof. Clearly, this problem P belongs to NP. Theorem 2.2 shows a reduction of the Graph k-Colorability Problem to P. Revisiting the proof of the Chinese Remainder Theorem we see that this reduction is polynomial. □

3. Compactness

Our main concern in this chapter is how to draw a graph on the bounded number of columns in a grid. We give a complete characterization of graphs which can be drawn/located in a grid with fixed number of columns by proving Observation 3.2 and Theorem 3.4. We also discuss the relation between those two cases.

These characterizations seem to be useful, as we use them later to prove some other statements. For example, we show how to derive an upper bound on the number of columns, that we need to locate a given graph G, from the maximum degree of G. That is Theorem 3.8.

We show that such problems are closely connected to a special form of a defective coloring in which some colors induce independent sets and some disjoint unions of paths (linear forests). More statements about these colorings are proved in the last section where the main result, Theorem 3.14, implies that, for $l \geq 2$, it is NP-complete to decide whether or not a given graph can be drawn/located on l columns of the grid.

3.1 Minimum Number of Columns

For nonprimitive grid drawings, there is no loss of generality in assuming that the grid is two-dimensional. Since if we can find a grid drawing $\phi(G)$ in $\mathbb{Z}^d$, $d > 2$, on l columns, then we can transfer $\phi(G)$ on l columns in $\mathbb{Z}^2$. We just take each column of the original grid drawing and transfer its points to an arbitrary free column in the plane. Then we might have to shift some columns higher so that no point representing vertex lies on nonadjacent line segment. This is always possible, as the number of vertices in G is finite. By the same trick, we can also assume that there is no unused column between two columns in our drawing. If l is the minimum number of columns on which G can be drawn, then we say that this grid drawing of G is *compact*.

It is easy to see that if there is a grid drawing on $l \geq 2$ columns for a graph G, then G is l-locatable in $\mathbb{Z}^2$, because the differences of column ranks from such grid drawing are always lower then l and we can move the adjacent points of the same column such that the line segment between them is primitive. The implication in the reverse direction does not hold, as the graph K_7 is, according to Theorem 2.2, three-locatable, but it cannot be drawn on three columns, because the last vertex with any other two vertices induces three-cycle. Thus compactness is not the locatability in disguise. Theorem 2.2 gives us the following upper bound:

Corollary 3.1. *Every compact grid drawing of a graph G has at most $\chi(G)$ columns.*

However none of the shown bounds is tight, because there is, for example, a locatable graph with a compact grid drawing on three columns. See Figure 3.1.

Let us mention the following useful definition before we start to explore this problem more deeply.

Definition If there is a coloring of a graph G such that a colors induce independent sets (so called *normal colors*) and b colors induce linear forests (*path colors*), then we say that G is *(a, b)-colorable* and we call such coloring *mixed*.

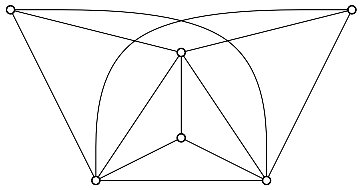

Figure 3.1: A locatable graph with a compact grid drawing on three columns

The following simple observation characterizes which graphs are embeddable on l columns in the terms of the graph theory.

Observation 3.2. *A graph G is embeddable on l columns if and only if G is $(0, l)$-colorable.*

Proof. We denote the path color classes as $V_1, V_2, \ldots, V_l$. In the grid drawing of G on l columns, the vertices represented by points of a single column define the set V_i. On the other hand, each induced subgraph $G[V_i]$ can be drawn on a single column and we can always shift the vertices in such way that the visibility of representing points is guaranteed. □

Thus embedding of a graph on few columns is equivalent with a special version of the mixed coloring. That is, an improper vertex coloring where every color class induces a linear forest. We call these color classes *path colors* for short. Naturally, the mixed coloring with colors inducing only independent sets is called *normal*. Also note that deciding whether G can be drawn on a single column is not difficult and it can be determined in linear time.

3.2 Compact Locating of Graphs

If we restrict our attention to only primitive grid drawings, then the situation changes radically. According to Theorem 2.2, only four-colorable graphs have primitive grid drawings in the plane. Thus we have to proceed to grid drawings in higher dimensions if we want to obtain primitive grid drawings of graphs with larger chromatic number. The minimum dimension of a grid on which a graph G can be located is $\lceil \log_2 \chi(G) \rceil$. Despite the fact that the situation with primitive grid drawings is quite different, Theorem 2.2 gives us the same upper bound on the minimum number of columns.

Corollary 3.3. *Every compact primitive grid drawing of a graph G has at most $\chi(G)$ columns in $\mathbb{Z}^d$ for $d = \lceil \log_2 \chi(G) \rceil$.*

However this bound is not tight even in the current case. For example, the graph K_5 cannot be located in $\mathbb{Z}^2$, as its chromatic number is five, but it can be located on three columns in $\mathbb{Z}^3$. Note that this number of columns is minimum, because three vertices on a single column induce a three-cycle. Thus compact primitive grid drawing of K_5 is on three columns in $\mathbb{Z}^3$.

In the previous case we assume that the set of columns in a compact grid drawing does not contain any holes. That is, there are no unused columns between two columns of this grid drawing. But now we cannot modify a primitive grid drawing by the same trick as before, because shifted line segments could intersect more grid points and the drawing would not be primitive. Thus it could happen that some primitive grid drawings on minimum number of columns are necessarily vast and sparse. Luckily, the following theorem shows that there are compact primitive grid drawings which take up little space. It also gives us a characterization of locating similar to Observation 3.2.

Theorem 3.4. *For a graph G, integers $d \geq 2$ and l, $2^{d-1} < l \leq 2^d$, the following statements are equivalent:*

1. *G can be located on l columns in $\mathbb{Z}^d$,*

2. *G is $\big(2l - 2^d, 2^d - l\big)$-colorable.*

The second statement says that G can be colored with at most l colors and at most $2^d - l$ of them are path colors. Note that if $l > 2^d$, then G cannot be located in $\mathbb{Z}^d$ at all, because, according to Corollary 2.8, only 2^d-colorable graphs can. We also describe the case $l \leq 2^{d-1}$ later by showing a similar characterization for nonprimitive grid drawings. See Observations 3.2 and 3.6.

Proof. Suppose that G is located on l columns in $\mathbb{Z}^d$. We construct a *congruence graph* C on the set of column ranks of such primitive grid drawing. Every vertex of this graph corresponds to a unique column rank and two vertices are adjacent if the corresponding ranks are congruent modulo two. The graph C is a disjoint union of complete graphs, because congruence is equivalence relation. The congruence graph always contains at most $2^{d-1} < l$ isolated vertices, because the maximum number of possible values of ranks modulo two is 2^{d-1}.

All points in the columns with ranks which lie in the same connected component of C can be colored with two normal colors. Since if we color the points with the odd last coordinate white and the points with the even last coordinate black, then no two monochromatic points can share an edge. If such ranks are congruent modulo two, then the line segment joining two adjacent monochromatic points would not be primitive. But this would be a contradiction, as the whole grid drawing is primitive. Thus we can use two colors in each connected component of C.

Now we show by induction on l that l colors is enough and that there is always at most $2^d - l$ path colors among them. This is sufficient, as every normal color is also a path color. Consider the case when $l = 2^{d-1} + 1$. Then we color the points of every column, whose rank corresponds to an isolated vertex in C, with a single path color, obtaining at most $l - 1$ path colors. Afterwards we color the points in all columns with ranks congruent modulo two using only two normal colors (by the same way as we showed before). Then the number of path colors decreases for every isolated vertex which becomes a larger component of C and we know it happens at least once, because $2^{d-1} < l$. Thus we have at most

$$l - 2 = 2^{d-1} + 1 - 2 = 2^{d-1} - 1 = 2^d - l$$

path colors. We also see that the total number of colors is at most l, because we add at most one new color for every vertex. Thus the case for $l = 2^{d-1} + 1$ is done.

Let us assume that this initial congruence graph contains all isolated vertices of the final congruence graph. Now suppose that our C contains l vertices and we know from the induction hypothesis that the condition holds for congruence graphs on $l-1$ vertices. We get the graph C by joining one vertex u to such smaller congruence graph. Due to the choice of initial graph, we know that u is not isolated in C. If we join u to some component of C which is colored with two normal colors, then we also color the points of a corresponding column with these colors. One color for points in even height, the other one for points in odd height. If $2^d - l$ drops bellow the number of used path colors, then we choose an isolated vertex whose column is colored with a single path color and color its points using two normal colors. One color is for the original one, the other is new for u.

If we join the new vertex u to a component colored with a path color, then it is an isolated vertex v and we can color points in the columns with ranks u and v using two normal colors. One is new for u, the other is original. The number of path colors is always at most $2^d - l$ and we added at most one new normal color in each step. Thus the conditions on the number of path and normal colors hold.

Now we prove the reverse implication. Let $V_1, V_2, \ldots, V_l$ be the colors (both normal and path) of our mixed coloring. Consider the set

$$\left\{(r_1, r_2, \ldots, r_{d-1}) \in \mathbb{Z}^{d-1} \mid r_1 \in \mathbb{Z}_4, r_i \in \mathbb{Z}_2\right\}.$$

The last $d-2$ coordinates $r_2, r_3, \ldots, r_{d-1}$ determine the set

$$\left\{(r_1, r_2, \ldots, r_{d-1}) \in \mathbb{Z}^{d-1} \mid r_1 \in \mathbb{Z}_4\right\}.$$

We mark it as $G_{r_2,\ldots,r_{d-1}}$ and its elements as $g_{i,r_2,\ldots,r_{d-1}} = (i, r_2, \ldots, r_{d-1})$, for $i = 0, 1, 2, 3$. For $d = 2$, there is only one such $G = \{0, 1, 2, 3\}$. Now we show a simple algorithm how to locate G on columns with ranks from this set. We repeat the following steps until there is no set of vertices left in our partition.

1. Take $G_{r_2,\ldots,r_{d-1}}$ that has not been chosen yet.

2. If there are two path colors V_i, V_j and there is no normal color, then map the vertices of V_i to points of column with rank $g_{0,r_2,\ldots,r_{d-1}}$ and the vertices of V_j to points of column with rank $g_{1,r_2,\ldots,r_{d-1}}$.

3. If there is a path color V_i and two normal colors V_j, V_k, then map the vertices of V_i to the column with rank $g_{1,r_2,\ldots,r_{d-1}}$. Also, map the vertices of V_j to points of column with rank $g_{0,r_2,\ldots,r_{d-1}}$ that have even d-th coordinate and the vertices of V_k to points of column with rank $g_{2,r_2,\ldots,r_{d-1}}$ that have odd d-th coordinate.

4. If there is no path color, then take four (or two, if there are not that many) normal colors V_i, V_j, V_k and V_m. Map V_i to points of column with rank $g_{0,r_2,\ldots,r_{d-1}}$ that have even d-th coordinate divisible by three and V_j to points of column with rank $g_{1,r_2,\ldots,r_{d-1}}$ that have even d-th coordinate too. Then, map V_k to points of column with rank $g_{2,r_2,\ldots,r_{d-1}}$ that have odd last coordinate and V_m to points of column with rank $g_{3,r_2,\ldots,r_{d-1}}$ that have odd last coordinate which is not divisible by three.

5. Remove the chosen colors.

Note that the total number of normal colors is even, because it equals $l - \left(2^d - l\right) = 2l - 2^d$. Thus if there is at least one such color in any step of the algorithm, then there is also another one, because we remove them by two or four.

The maximum number of possible steps is 2^{d-2}, since it is also the number of sets $G_{r_2,\ldots,r_{d-1}}$. We show that this number is sufficient. First, notice that for each path color V_i we lower l by one (if we start with empty partition and $l = 2^d$). Thus we can pair such V_i with a unique empty color (it does not contain any vertices) and we obtain 2^d colors such that some of them are path, some normal and some are empty. Each step of the algorithm takes four of these colors and locates their vertices. Thus we can locate vertices of all these $2^d = 4 \cdot 2^{d-2}$ colors within 2^{d-2} steps.

It is not difficult to see that the obtained grid drawing is primitive, as the only possible occurrence of nonprimitive line segment is between columns from the same set $G_{r_2,\ldots,r_{d-1}}$. But we mapped the vertices such that no line can intersect more than two grid points. □

Note that the dimension of a grid in the statement of this theorem is minimum for such choice of l, according to Corollary 2.8. The proof of this theorem also shows how to relocate a primitive grid drawing of G on a bounded number of columns, such that the new grid drawing is still primitive and it also requires small part of the grid (the first $d-1$ coordinates are constant). We also obtained relation between compact and primitive compact grid drawings.

Corollary 3.5. *Let G be a graph with a grid drawing on l columns. Then G can be located on k columns, $l \leq k \leq 2l$ in $\mathbb{Z}^{\lceil \log_2 k \rceil}$.*

Proof. The lower bound on k is apparent. For the upper bound, we just combine Observation 3.2 and Theorem 3.4 and we split every path color into two normal ones. □

We can also characterize graphs which can be located on at most 2^{d-1} columns.

Observation 3.6. *For a graph G and integers $d \geq 2$ and l, $1 \leq l \leq 2^{d-1}$, the following statements are equivalent:*

1. *G can be located on l columns in $\mathbb{Z}^d$,*
2. *G is embeddable on l columns (in $\mathbb{Z}^2$).*

Proof. Let G be located on 2^{d-1} columns in $\mathbb{Z}^d$. Then we can take all columns of this primitive grid drawing and arrange them in a consecutive order in the plane. Then we might have to shift some columns higher to satisfy the condition on mutual visibility with respect to points representing vertices. On the other hand, if there is a grid drawing of G on 2^{d-1} columns in the plane, then we take each column of this drawing and copy it on a unique point from the set

$$\{(r_1, \ldots, r_{d-1}) \mid r_i \in \{0, 1\}\} \subset \mathbb{Z}^{d-1}.$$

□

This observation is somehow intuitive, as every grid drawing on two columns is primitive. However, we know, according to Theorem 3.4, that for a larger number of columns this does not hold and locating becomes more restrictive than drawing.

Although we show that locating the graph on bounded number of columns is NP-complete in the following section, there are special classes of graphs for which we can find suitable estimations. The following theorem gives bounds that depend on the maximum degree of a graph. In order to show this, we need an auxiliary lemma proved by László Lovász.

Lemma 3.7 ([29])**.** *Let $G = (V, E)$ be a graph and let $k_1, k_2, \ldots, k_m$ be nonnegative integers with $k_1 + k_2 + \ldots + k_m \geq \Delta(G) - m + 1$. Then V can be partitioned into $V_1, V_2, \ldots, V_m$ so that $\Delta(G[V_i]) \leq k_i$, for all $i \in [m]$.*

Theorem 3.8. *Let $G = (V, E)$ be a graph with $\Delta(G) \leq 2^{d+1} - 1$, for $d \in \mathbb{N}$. Then G can be located on 2^d columns in $\mathbb{Z}^{d+1}$.*

Proof. According to Observation 3.6, it suffices to prove that G is embeddable on 2^d columns in the plane. To prove this we apply Observation 3.2. So eventually, we show by induction on d that the assumption in our theorem implies that V can be partitioned into $V_1, V_2, \ldots, V_{2^d}$ such that every induced subgraph $G[V_i]$ is isomorphic to a linear forest. As the basis of the induction we use the proof of a weaker theorem proved in [10].

For $d = 1$, the graph G is either a complete graph on four vertices or, according to Brooks' theorem, G can be colored with three colors. We know that the graph K_4 can be drawn on two columns, so the statement holds in the first case. In the second case, the vertices of G can be partitioned into three color classes C_1, C_2 and C_3 (we label the colors as c_1, c_2 and c_3). Consider the induced subgraph $G[C_1 \cup C_2]$. If there is a vertex of degree three, then we color it with the color c_3. Thus we ensured that $\Delta(G[C_1 \cup C_2]) \leq 2$. If there is a cycle left, then we choose its arbitrary vertex and color it with the new color c_4. Afterwards the graph $G[C_1 \cup C_2]$ is isomorphic to a linear forest, but there might be a vertex of degree three in the graph $G[C_3 \cup C_4]$. If there is such vertex, then we color it to c_1. After that, the graphs $G[C_1 \cup C_2]$ and $G[C_3 \cup C_4]$ are both linear forests.

Now we do the inductive step. Let the maximum degree of G be at most $2^{d+1} - 1$. Then, according to Lemma 3.7, V can be partitioned into V_1 and V_2 such that $\Delta(G[V_1]) \leq 2^d - 1$ and $\Delta(G[V_2]) \leq 2^d - 1$, if we set $m = 2$ and $k_1 = k_2 = 2^d - 1$. It follows from the inductive step that the vertices of each of the graphs $G[V_1]$, $G[V_2]$ can be partitioned into 2^{d-1} required sets. Together these partitions give the partition of V into $2^{d-1} + 2^{d-1} = 2^d$ sets. □

This bound is sharp in certain cases. For example for every complete graph on $2^{d+1}-1$ vertices, because the minimum number of columns on which we can draw a clique K_n is $\lceil \frac{n}{2} \rceil$ (the rest follows from the previous characterizations). Note that the reverse implication does not hold, as every star graph can be located on two columns in the plane and its maximum degree does not have to be bounded.

3.3 Mixed Colorings

We saw that drawing/locating of a graph with bounded number of columns is related to the mixed colorings. We use such colorings later in this section to prove NP-completeness of a problem of deciding whether a graph can or cannot be drawn/located on $l \geq 2$ columns. To do this we prove a more general result, Theorem 3.14, in this section. But first we show other related results.

We denote the class of all (a, b)-colorable graphs as $\mathcal{G}_{a,b}$ and call it a *mixed coloring type*. Then the following holds:

Observation 3.9. *$\mathcal{G}_{a,b} \subseteq \mathcal{G}_{c,d}$ if and only if there is a sequence $\{\mathcal{G}_{ai,bi}\}_{i=1}^{n}$ such that $a_1 = a$, $b_1 = b$, $a_n = c$, $b_n = d$ and $a_{i+1} = a_i+2$, $b_{i+1} = b_i-1$ or $a_{i+1} = a_i-1$, $b_{i+1} = b_i + 1$ for every $i \in \{1, \ldots, n-1\}$.*

That is, there is a sequence of steps where every step corresponds to a substitution of one path color by two normal colors or one normal by one path color.

Proof. It suffices to show only the second implication, as the other one is apparent. Let $K(a, b, t)$ denote the balanced complete $(a + b)$-partite graph with t vertices in every color class such that b color classes induce paths and the rest induce independents sets. Then $K(a, b, t) \in \mathcal{G}_{a,b}$ for every $t \in \mathbb{N}$ and for every $G \in \mathcal{G}_{a,b}$ there exists t such that G is a subgraph of $K(a, b, t)$.

Let $\mathcal{G}_{a,b} \subseteq \mathcal{G}_{c,d}$. Then we can color $K(a, b, t)$ with c path and d normal colors for every t. Assume a fixed coloring of $K(a, b, 3)$. Then we can modify it such that every path color contains only vertices from a single color class of $K(a, b, t)$. Since it suffices to swap colors between these color classes slightly, because every path color can contain at most three vertices (and those vetices are in at most two color classes of $K(a, b, 3)$, as otherwise there is a 3-cycle). Similarly, we can ensure that there is one path color or at most two normal colors in every color class of $K(a, b, 3)$. Then it is straightforward to expand such coloring for every $K(a, b, t)$ and thus obtain the sequence $\{\mathcal{G}_{ai,bi}\}_{i=1}^{n}$. $\square$

Now we can examine the partially ordered set of the set of all mixed coloring types ordered by inclusion. Figure 3.2 shows the modified Hasse diagram of this POSET where the inclusion corresponds to an oriented path between two types. The inclusion is not total order in this case, as there are incomparable elements.

According to Observation 3.2, the mixed coloring types which are drawn in the common grey site are classes of graphs that can be drawn on the same number of columns. The mixed coloring types denoted as black vertices correspond to the graph classes from Theorem 3.4. According to Corollary 2.8, we can split the POSET along these types into parts P_i, $i \geq 2$, such that any graph from a graph class lying in the part P_i can be located in the grid of dimension i.

The Four Color Theorem implies that every planar graph is $(4, 0)$-colorable and Wayne Goddard [23] showed that it is also $(0, 3)$-colorable. Thus we get the following corollary.

Corollary 3.10. *Every planar graph can be drawn on three columns.*

Cáceres et. al. [10] showed that every outerplanar graph can be drawn (and located) on two columns. In the same paper there is an example of a planar graph which is not $(2, 1)$-colorable. Thus we need four columns to locate an

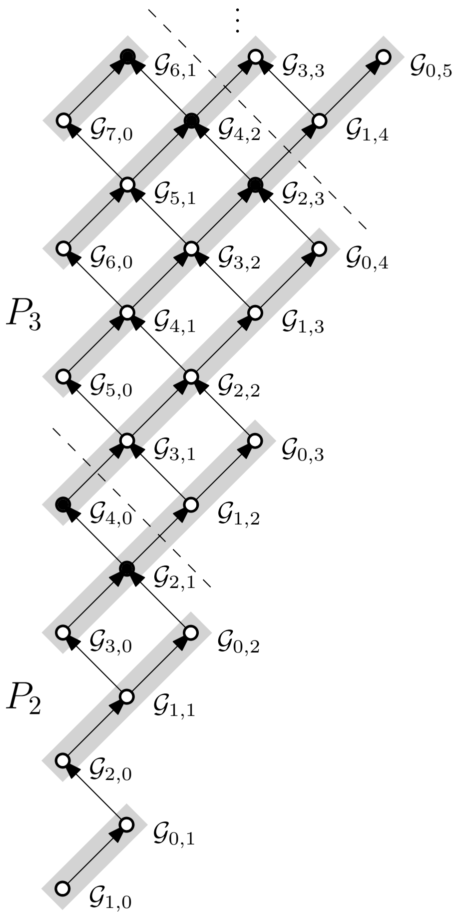

Figure 3.2: Mixed coloring types ordered by inclusion

arbitrary planar graph. The natural question is whether every planar graph is $(1, 2)$-colorable. The following proposition shows that using one normal and two path colors is insufficient too.

Proposition 3.11. *There is a planar graph which is not* $(1, 2)$*-colorable.*

Proof. Let α be the normal color and β and γ be the path colors we can use. Consider the gadget H depicted in part a) of Figure 3.3. This gadget is isomorphic to a complete graph on four vertices with a path on ten vertices inside each inner face. The path colors β and γ cannot both appear on the vertices of the outer face otherwise it is not possible to color the path adjacent to them. We could color at most four vertices of this path with β and γ in such case, but there would still be an edge with both vertices of color α. But this is not possible, since α is normal color. Thus the vertices of the outer face are colored α and one path color, say β.

Now we join three copies H_1, H_2 and H_3 of H as shown in Figure 3.3, part b), and we obtain the graph G. We see that G is not $(2, 1)$-colorable, because the only way how to color it with α, β and γ is to color K_4 with α and β and this is clearly not possible. $\square$

It is not difficult to prove that there is an outerplanar graph which is not $(1, 1)$-colorable, hence we know the tight estimations on mixed colorability of both planar and outerplanar graphs.

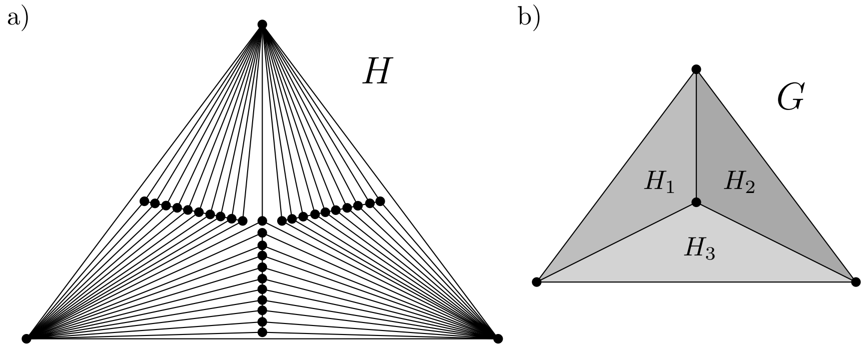

Figure 3.3: Construction of a planar graph which is not $(1,2)$-colorable

Now our main goal is to prove NP-completeness of problem of deciding whether a graph G is (a,b)-colorable for sufficiently large a and b. As a consequence we obtain that drawing/locating of graphs on bounded number of columns is a difficult task answering the open question in [10].

This problem is already partially solved, as Glenn G. Chappell, John Gimbel and Chris Hartman [11] proved that determining whether G can be colored with $l \geq 2$ path colors is NP-complete. Although this does not answer the question for locating of graphs (we need to prove the statement for general mixed colorings, not only for path colorings), we later apply a similar technique to prove NP-completeness of (a,b)-colorability for sufficiently large a and b.

In the following lemma we prove the initial case by using a reduction to the One-in-three 3SAT problem (see [22]).

Lemma 3.12. *It is* NP*-complete to decide whether or not a graph* $G=(V,E)$ *is* $(1,1)$*-colorable.*

Proof. Let F be a collection of m clauses $C_1, C_2, \ldots, C_m$ over n Boolean variables $v_1, v_2, \ldots, v_n$ such that each clause C_i contains exactly three literals $c_{i,1}$, $c_{i,2}$ and $c_{i,3}$. Each literal $c_{i,j}$, $i \in [m]$ and $j \in \{1,2,3\}$, is either v_k or $\overline{v_k}$ for some suitable $k \in [n]$. Let us remind that One-in-three 3SAT is a problem of determining whether there is a truth assignment e satisfying F such that each clause in F has exactly one true literal (and thus exactly two false literals).

We construct a graph $G\,(k)$ shown in Figure 3.4 for each variable v_k. Then, for each clause C_i, we construct a graph $G\,(C_i)$ which is isomorphic to K_3 and each one of its vertices represents a different literal of the clause C_i. Let $G\,(F)$ be a graph consisting of all the graphs $G\,(k)$ and $G\,(C_i)$ where the vertex $c_{i,j}$ is adjacent to $v \in V\,(G\,(k))$ if and only if the literal $c_{i,j}$ is $v \in \{v_k, \overline{v_k}\}$.

Suppose that G is colored with one path and one normal color, say black and white. Then the vertices v_k and $\overline{v}_k$ of $G\,(k)$ are colored differently. Otherwise they are black and the vertex u must be white. But then, since white is a normal color, w and t are black and induce a black 4-cycle together with v_k and $\overline{v}_k$. Also, if the vertices $x \in \{v_k, \overline{v}_k\}$ and $c_{i,j}$ are adjacent, then their colors are different too. Assume to the contrary that x (say $x = v_k$) and $c_{i,j}$ are both black and adjacent. Then we know that $\overline{v}_k$ is white and thus u and v are black. Hence v_k has three black neighbors which is a contradiction.

We define the truth assignment e for F as follows: if v_k is black, then $e\,(v_k)$ is true else $e\,(v_k)$ is false. The assignment e is correct, as the vertices v_k and $\overline{v}_k$ are

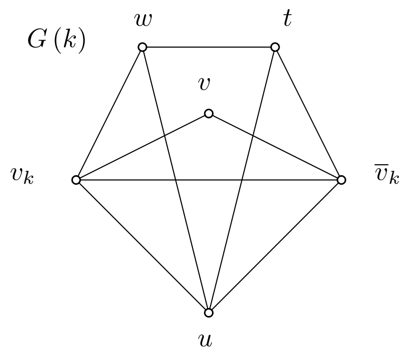

Figure 3.4: The graph $G(k)$

not monochromatic. In addition, there is exactly one true literal in every clause. Otherwise there would be a black 3-cycle or an edge with both vertices white in some $G(C_i)$.

Suppose that e satisfies F such that every clause has exactly one true and two false literals. Then we color the labeled vertices of each $G(C_i)$ white, if the corresponding literal is true; otherwise black. By the assumption, there is no monochromatic graph $G(C_i)$. After that, we color the vertex $v \in \{v_k, \overline{v}_k\}$ adjacent to $c_{i,j}$ black (white, respectively) if $c_{i,j}$ is white (black, respectively). Note that the vertices v_k and $\overline{v}_k$ are, again, differently colored. It remains to color the rest of graph $G(k)$ for each $k \in [n]$. One way how to do that is shown in Figure 3.5. □

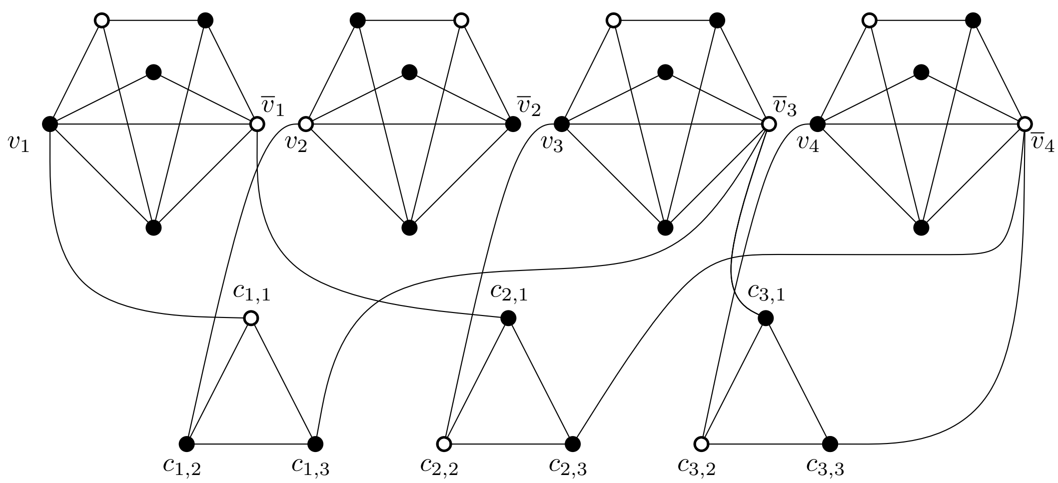

Figure 3.5: Coloring of $G(F)$ for $F = (v_1 \vee v_2 \vee \overline{v_3}) \wedge (\overline{v_1} \vee v_3 \vee \overline{v_4}) \wedge (\overline{v_3} \vee v_4 \vee \overline{v_4})$ and the truth assignment $v_1, v_3, v_4 \mapsto$ true and $v_2 \mapsto$ false

We use a reduction to the Graph k-Colorability Problem in the final statement, but this problem is NP-complete for at least three colors, thus we need to consider one more special case. That is $(0, 2)$-colorability. Although the following lemma is already known to be true [11], the known proof is based on the result with so called one-defective colorings. For completeness we include a short proof which uses a similar idea as the previous one (a variation of a technique used by Hoòng-Oanh Le [26]).

Lemma 3.13. *It is* NP*-complete to decide whether or not a graph* $G = (V, E)$ *is* $(0, 2)$*-colorable.*

Proof. The main idea is the same as before. We use a reduction to a variation of 3SAT problem, only this time we use Not-All-Equal 3SAT – a problem of determining whether there is a truth assignment satisfying a formula such that each clause has at least one true literal. So, let the notation be the same as in Lemma 3.12 with the only difference that instead of $G(k)$ we use the graph depicted in Figure 3.6.

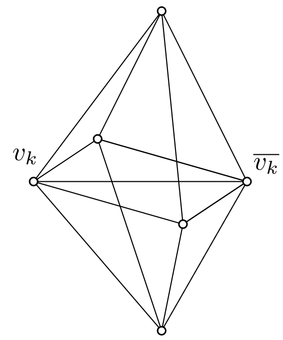

Figure 3.6: The new graph $G(k)$

Let G be colored with two path colors black and white. We can easily show that, again, it holds that the vertices v_k and $\overline{v}_k$ have distinct colors. Otherwise the remaining vertices of $G(k)$ induce a monochromatic 4-cycle. The adjacent vertices $x \in \{v_k, \overline{v_k}\}$ and $c_{i,j}$ are also heterochromatic. Otherwise x would have three neighbors of the same color.

Now, we can define the truth assignment as follows: if v_k is white, then $e(v_k)$ is true else $e(v_k)$ is false. The previous facts imply correctness of this assignment and there is at least one true literal in every clause, otherwise $G(C_i)$ would be monochromatic 3-cycle. The proof of the reverse implication is analogous too. $\square$

Theorem 3.14. *Let a and b be given nonnegative integers such that $a + b \geq 2$ and $(a, b) \neq (2, 0)$. Then it is* NP*-complete to decide whether or not a graph $G = (V, E)$ is (a, b)-colorable.*

Proof. We apply a reduction to the Graph k-Colorability Problem. That is, a problem of determining whether or not it is possible, for a given graph G and integer $k \geq 3$, to color G with k normal colors. If we set $k = a + b$, then we can assume, according to the previous lemmas, that $k \geq 3$. The Graph k-Colorability Problem is NP-complete in such case, thus we can consider the reduction. Suppose that G is a given graph. Let us create the graph H by joining two disjoint copies of the complete graph K_{a+2b-1} to every vertex v of G.

Suppose that G is colored with k normal colors. Then we color the cliques for every vertex v with all colors. Two vertices per path color and one per normal color.

On the other hand, if H is colored with a normal and b path colors, then G is colored with at most $k = a + b$ normal colors. Assume to the contrary that there is an edge uv with both vertices colored with the same path color (say black) in G. Then u has at least three black neighbors, because the sizes of the adjacent cliques imply that there is at least one other black vertex in every one of them. This is a contradiction, since the coloring of H is correct. $\square$

Applying Observation 3.2 and Theorem 3.4 we immediately get the following.

Corollary 3.15. *For a given integer $l \geq 2$ and a given graph G, it is* NP*-complete to decide whether or not it is possible to draw G on l columns.*

Corollary 3.16. *For a given integer $l \geq 2$ and a given graph G, it is* NP*-complete to decide whether or not it is possible to locate G on l columns (in a grid of sufficiently large dimension).*

4. Minimum Area

In the previous chapter, we showed some results concerning with grid drawings having bounded number of columns. The natural question is – what happens, if we fix the number of rows as well? This leads to another type of compactness – grid drawings with a small area and this is the main field of our interest in this chapter.

The first section is dedicated to already known results. We show a construction of a grid drawing of the balanced complete k-partite graphs $K(k,t)$ with asymptotically optimal area. After that, we consider so called aspect ratio of a grid drawing and its relation to the area.

The main result of the second section is a sharp lower bound on the area of the grid drawings of $K(k,t)$. This settles a problem of David R. Wood.

In the end of the chapter we focus on a multidimensional grid and we consider the volume of primitive grid drawings in $\mathbb{Z}^d$.

4.1 Grid Drawings with Small Area

Let w, h be positive integers. Suppose that $\phi(G)$ is a grid drawing of a graph $G = (V,E)$ in $\mathbb{Z}^2$ such that $\phi(v) = (X(v), Y(v))$ for every vertex $v \in V$. Then $\phi(G)$ is said to be $w \times h$ *grid drawing of* G, if $|X(u) - X(v)| < w$ and $|Y(u) - Y(v)| < h$ for all vertices $u, v \in V$. The number wh denotes the *area of a grid drawing* $\phi(G)$.

The grid drawings with small area were studied by David R. Wood [36] and they are closely connected to the famous “No-three-in-line” problem [24]. This problem was introduced by Henry Dudeney in 1917 and it asks for the maximum number of points that can be placed in the $n \times n$ grid so that no three points are collinear. Clearly $\phi(K_n)$ is a grid drawing of the complete graph $K_n = (V,E)$ if and only if $\{\phi(v) \mid v \in V\}$ is a set of grid points with no three collinear. Thus our problem of producing a grid drawing of a given graph with a small area is a generalization of this problem.

We show some interesting results of David R. Wood in this section. The first one gives us a construction of grid drawings of the graphs $K(k,t)$ for k, $t \geq 1$, with asymptotically optimal area (as we show later).

Theorem 4.1 ([36]). *For every k, $t \geq 1$, there is a $k \times pt$ grid drawing of the graph $K(k,t)$, where p is the least prime number that is greater than k.*

Proof. Let $V_0, V_1 \dots, V_{k-1}$ be the colors of a k-coloring of the graph $K(k,t)$ such that $V_i = \{v_{i,0}, v_{i,1}, \dots, v_{i,t-1}\}$ for every $i \in \{0, 1, \dots, k-1\}$. We define the grid drawing $\phi(K(k,t))$ by mapping the vertex $v_{i,j}$ to the grid point $(i, pj + (i^2 \bmod p))$ for every $0 \leq i \leq k-1$ and $0 \leq j \leq t-1$.

It suffices to prove that no three vertices from distinct color classes are collinear, as all vertices colored V_i are in the same vertical line. We know that three points (a_1, a_2), (b_1, b_2) and (c_1, c_2) are collinear if and only if the determinant

$$\begin{vmatrix} 1 & a_1 & a_2 \\ 1 & b_1 & b_2 \\ 1 & c_1 & c_2 \end{vmatrix} = 0.$$

Therefore, if we consider the points v_{i_1,j_1}, v_{i_2,j_2} and v_{i_3,j_3} from distinct color classes, then we get

$$\begin{vmatrix} 1 & i_1 & pj_1 + (i_1^2 \bmod p) \\ 1 & i_2 & pj_2 + (i_2^2 \bmod p) \\ 1 & i_3 & pj_3 + (i_3^2 \bmod p) \end{vmatrix} \equiv \begin{vmatrix} 1 & i_1 & i_1^2 \\ 1 & i_2 & i_2^2 \\ 1 & i_3 & i_3^2 \end{vmatrix} \equiv (i_1 - i_2)(i_1 - i_3)(i_2 - i_3) \, (\bmod p)$$

which is nonzero, because p is a prime and $1 \leq i_r - i_s \leq k - 1 \leq p - 1$ for every $1 \leq r < s \leq 3$.

Thus no three vertices from distinct color classes are mapped to collinear points. Therefore $\phi(K(k,t))$ is a correctly defined grid drawing, because the only points, which represent vertices, on a line segment are its own endpoints.

It remains to estimate the area of $\phi(K(k,t))$. For every vertex v, the inequalities $0 \leq X(v) \leq k - 1$ and $0 \leq Y(v) \leq p(t-1) + p - 1$ hold, hence we immediately get that $\phi(K(k,t))$ is a $k \times pt$ grid drawing. □

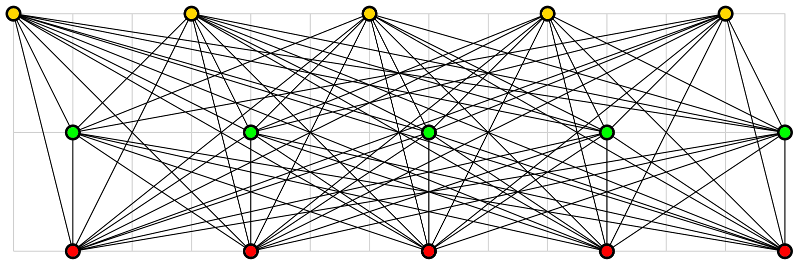

Figure 4.1: A rotated and scaled grid drawing from Theorem 4.1

Using the Bertrand's postulate we see that the area of obtained grid drawings is always less than $2k^2t$. An example of such drawing is shown in Figure 4.1. By Bertrand's Postulate and the Prime Number Theorem we can also derive the following corollary.

Corollary 4.2 ([36])**.** *For every $\epsilon > 0$, there exists k_ϵ such that for every $k \geq k_\epsilon$ and $t \geq 1$, $K(k,t)$ has a $k \times (1+\epsilon)kt$ grid drawing.*

Other result of David R. Wood, which I would like to mention, concerns with relation between the area of a grid drawing and ratio of its width to its height. The *aspect ratio of a $w \times h$ grid drawing* is the number $\max\{w,h\} / \min\{w,h\}$. The following theorem is a generalization of Theorem 4.1 for every k-colorable graphs using the aspect ratio as a parameter.

Theorem 4.3 ([36])**.** *Let G be a k-colorable graph on n vertices and let r be integer such that $1 \leq r \leq \frac{n}{k}$. Then G has a $\frac{2n}{r} \times 4n$ grid drawing.*

Note that the aspect ratio of the grid drawing is $2r$ and the area equals $\frac{8n^2}{r}$.

Proof. Suppose we have a k-coloring of G. Then we split each color class into smaller classes such that each one has exactly r vertices except the last one with at most r vertices. Thus we obtain at most $\frac{n}{r}$ sets with r vertices and at most k smaller sets. Since $r \leq \frac{n}{k}$, the total number of sets is at most $\frac{n}{r} + \frac{n}{r} = \frac{2n}{r}$.

Thus we have $\lfloor \frac{2n}{r} \rfloor$-coloring of G where every color contains at most r vertices. Hence G is a subgraph of $K\left(\lfloor \frac{2n}{r} \rfloor, r\right)$ and, according to Theorem 4.1, we know that G has a $\frac{2n}{r} \times 4n$ grid drawing. □

Note that for the choice $r = \lfloor \frac{n}{k} \rfloor$ we get a $O(k) \times O(n)$ grid drawing. The ideas used in the proofs are similar to methods used by J. Pach et al. [30] and V. Dujmović et al. [19]

4.2 Lower Bound on Area

Studying the grid drawings of the balanced complete k-partite graphs $K(k,t)$, $k,\ t \geq 1$, David R. Wood [36] showed that the area of such graphs is at least $\frac{k^2t}{4}$. He also asks whether it is possible to improve this bound to $\frac{k^2t}{2}$. For $t = 1$ the improvement is not possible, as we can find a grid drawing of $K(4,1)$ with area four. However we show that the conjecture is true for $t \geq 2$ by proving the following theorem.

Theorem 4.4. *For all $k \geq 1$ and $t \geq 2$ every grid drawing of the graph $K(k,t)$ has area at least $\frac{k^2t}{2}$.*

Note that this theorem implies that the area of the grid drawings obtained in Theorem 4.1 is asymptotically optimal.

Proof. We apply Wood's proof for the bound $\frac{k^2t}{4}$ in the beginning. Instead of estimating the area of a grid drawing we consider an easier problem. Assume that we forbid three collinear points of at least two colors only in vertical and horizontal direction. We call such grid drawings *incomplete*. Note that if we find a lower bound on the area of all incomplete grid drawings of $K(k,t)$, then the same bound holds for all grid drawings of $K(k,t)$, as the conditions on grid drawings are more restrictive.

Let w be a width and h a height of a given incomplete grid drawing $\psi(K(k,t))$. Without loss of generality let us assume that $w \geq h$. For $i \in \{1,\ldots,k\}$ let c_i denote the number of columns containing a vertex colored i and let r_i denote the same number for rows. Then we can use the arithmetic-harmonic means inequality (see [9] for example) to obtain the following

$$\left(\frac{1}{k}\sum_{i=1}^{k} c_i\right)\left(\frac{1}{k}\sum_{i=1}^{k}\frac{1}{c_i}\right) \geq 1.$$

As, clearly, $c_i r_i \geq t$ for every $i \in \{1,\ldots,k\}$, we get $\frac{1}{c_i} \leq \frac{r_i}{t}$ and thus

$$\left(\sum_{i=1}^{k} c_i\right)\left(\sum_{i=1}^{k} r_i\right) \geq k^2 t.$$

Note that we include a column (or a row) in the sum once if and only if it contains the points of only one color and twice if and only if it contains (two) points of two distinct colors. Hence $\sum_{i=1}^{k} c_i \leq (1+\epsilon)\,w$ and $\sum_{i=1}^{k} r_i \leq (1+\delta)\,h$ where $\epsilon, \delta \in [0,1]$ are numbers such that ϵw denotes the number of bichromatic columns and δh number of bichromatic rows. So we have

$$(1+\epsilon)(1+\delta)\,wh \geq \left(\sum_{i=1}^{k} c_i\right)\left(\sum_{i=1}^{k} r_i\right) \geq k^2 t.$$

This is the end of Wood's proof, as ϵ, $\delta \leq 1$ implies $wh \geq \frac{k^2t}{(1+\epsilon)(1+\delta)} \geq \frac{k^2t}{4}$. To obtain the better lower bound we modify the incomplete grid drawing such that the parameters w and h do not change too much. But first, we solve the trivial cases.

Note that $w \geq h \geq k$, since $t \geq 2$. If ϵ or δ is zero, then we get $wh \geq \frac{k^2t}{(1+\epsilon)(1+\delta)} \geq \frac{k^2t}{2}$ immediately. We also get the same bound if ϵ or δ equals one. This is because every column (or row) is bichromatic in such case and thus $w \geq \frac{kt}{2}$ or $h \geq \frac{kt}{2}$. The rest follows from $w \geq h \geq k$. The last trivial case is for $t = 2$, as we get $wh \geq k^2 = \frac{k^2t}{2}$. Therefore we can assume that $t \geq 3$ and $\delta, \epsilon \in (0,1)$.

Now we describe the modification of the incomplete grid drawing. We add k new rows above the original drawing, each one for one color. Then we replace each bichromatic row from original drawing to a new column such that the the points lie in an appropriate new row according to their colors. Thus every bichromatic row becomes a bichromatic column. We denote the region with shifted rows as R and the region with the rest of the drawing as S. One step of the modification is depicted in Figure 4.2.

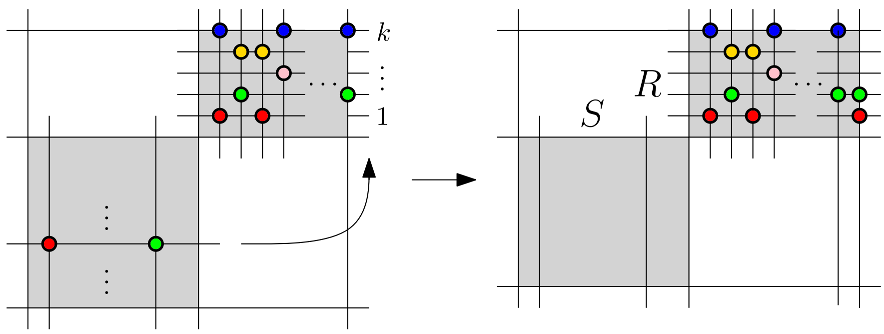

Figure 4.2: The modification of $\psi\left(K\left(k,t\right)\right)$

Let $\overline{w}$ and $\overline{h}$ denote the width and height of the modified incomplete grid drawing $\overline{\psi}\left(K\left(k,t\right)\right)$. Then we see that $\overline{w} \leq w + \delta h$ and $\overline{h} \leq h - \delta h + k$, because all δh bichromatic rows become new bichromatic columns and we added k new rows. We can decrease $\overline{h}$ by a simple trick. If there is a row in $\psi\left(K\left(k,t\right)\right)$ that contains only points of color i then we can shift the i-th new row to such level after the modification and decrease the number of the new rows by one.

Thus $\overline{h} \geq h - \delta h + b$ where $b = |B|$ and B is the set of all colors whose points are only in bichromatic rows. This is a better estimation, since $0 \leq b \leq k$. We also know that $\overline{w}\overline{h} \geq \frac{k^2t}{2}$, because there are no bichromatic rows in $\overline{\psi}\left(K\left(k,t\right)\right)$ and thus $\overline{\delta} = 0$. If $b = 0$, then we are done, as the following holds

$$\frac{k^2t}{2} \leq \overline{w}\overline{h} \leq (w + \delta h)(h - \delta h) \leq (1+\delta)(1-\delta)\,wh = \left(1 - \delta^2\right) wh.$$

Therefore it remains to show that the bound holds even for $b > 0$. To do this, we rearrange the points in $\psi\left(K\left(k,t\right)\right)$ such that its proportions remain the same, but we get the smaller area of $\overline{\psi}\left(K\left(k,t\right)\right)$. We also have to slightly revise the modification itself.

Let α be a color from the set B. We show that if we cannot save the new row for α, then we can save some column for this color. Suppose that α has points in at least two columns and that the points lying in the same rows as points colored α are not all in one column. Then we can perform adjustment shown in Figure 4.3. After this procedure the color α has its own row and thus we can save the new row for α after the modification. Note that δ decreases in this case.

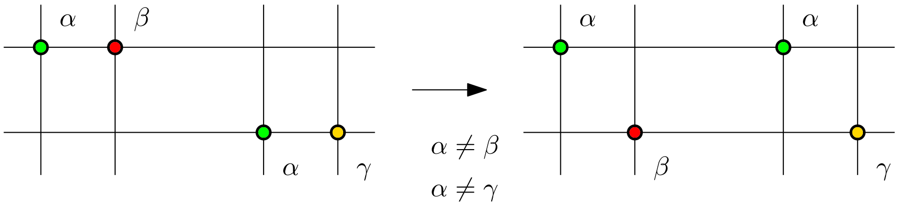

Figure 4.3: The adjustment of points colored α

If all points colored α are in the same column, then we cannot perform the adjustment. However we see that after the modification this column is empty and thus we can save it. The other unsuitable case is when α has at least two columns, but its points lie in the rows with points of a single column. Since $t \geq 3$ we see that all such points have the same color $\beta \in B$. We cannot save column for α in such case if and only if the color α does not have its own column. That is, every column containing a point colored α is bichromatic. If all these other points in those columns do not lie in the same row, then we can perform the same adjustment as before, only for rows. After that α has its own column which we can save after the modification.

Thus it remains to solve the case when all points and rows containing a point colored α are bichromatic and the points of the other two colors β and γ are in a single column and row. The way how to do this is drawn in Figure 4.4. We revise the modification itself. We replace all rows with points colored α and β as we did before, but instead of replacing the last row with a single column we use two columns (each one for one point). Then we take two arbitrary points colored γ and shift them into these columns in the region R. By this procedure we save all δh rows and also three columns – one for β and two for the points colored γ. On the other hand we added one extra new column, but it does not matter, as we can identify it with one the saved columns. At the end we can also say that the last saved column is for α.

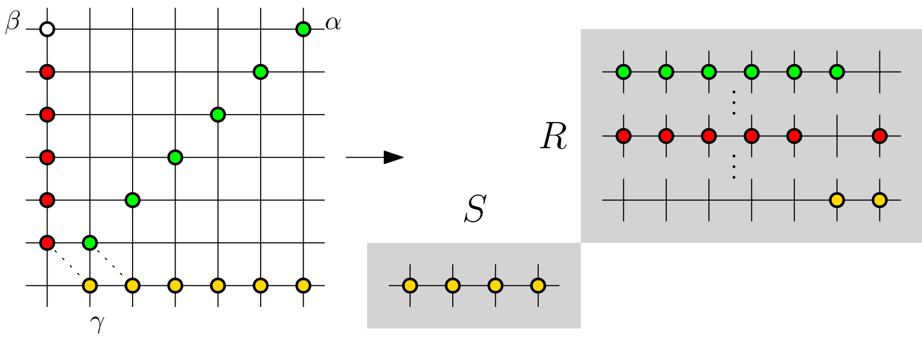

Figure 4.4: Solving the last case

Thus we showed that for each color we can either save a column or a row. Let c denote the number of colors for which we cannot save a row. Then we get that $\overline{w} \leq w + \delta h - c$ and $\overline{h} \leq h - \delta h + c$ where $\delta \in [0, 1]$ might differ from the original one, because of the adjustments. If we set $x = \delta h - c$, then we get

$$\frac{k^2 t}{2} \leq \overline{w}\overline{h} \leq (w + x)(h - x) = wh - wx + xh - x^2 \leq wh$$

as $w \geq h$ and $x \geq 0$. This holds, as the number of bichromatic rows δh is at least $\frac{ct}{2} \geq c$ (the colors for which we cannot save a row are in B and $t \geq 2$). $\square$

Although this bound is sharp in general, as it is the minimum area of a grid drawing of $K(2, t)$ for every $t \geq 1$, we can show that we can obtain even better bounds in some nontrivial special cases.

Corollary 4.5. *For all $k \geq 1$ and $t \geq 2$ every $w \times h$ grid drawing of the graph $K(k, t)$, $w \geq h$, has area at least $\frac{k^2 t^2}{4}$ if $\delta = 1$ and at least $\frac{k^2 t}{2(1-\delta^2)}$ if $\delta < 1$ and every color has its own row. The parameter δ denotes the ratio of the number of bichromatic rows to h.*

Proof. It suffices to revisit the proof of Theorem 4.4. $\square$

4.3 Volume of Primitive Grid Drawings

Using the grid drawings obtained in the proof of Theorem 3.4 we can show some estimations on the volume of primitive grid drawings (even in higher dimensions). First, we generalize the definition of the area of the grid drawing. Let $d \geq 2$ and $w_1, w_2, \ldots, w_d$ be positive integers. Suppose that $\phi(G)$ is a grid drawing of a graph $G = (V, E)$ in $\mathbb{Z}^d$ such that $\phi(v) = (X_1(v), X_2(v), \ldots, X_d(v))$ for every vertex $v \in V$. Then $\phi(G)$ is said to be $w_1 \times w_2 \times \ldots \times w_d$ *grid drawing of* G, if $|X_i(u) - X_i(v)| < w_i$ for all vertices $u, v \in V$ and every $i \in \{1, 2, \ldots, d\}$. The number $\prod_{i=1}^{d} w_i$ denotes the *volume of the grid drawing* $\phi(G)$.

Proposition 4.6. *For positive integers $d \geq 2$, $k \leq 2^d$ and t, there is a primitive grid drawing of $K(k, t)$ in $\mathbb{Z}^d$ with volume at most $24t \cdot \left\lceil \frac{k}{4} \right\rceil$.*

Proof. We map the color classes of $K(k, t)$ to the columns of $\mathbb{Z}^d$ with ranks from the set

$$\left\{(r_1, r_2, \ldots, r_{d-1}) \in \mathbb{Z}^{d-1} \mid r_1 \in \mathbb{Z}_4, r_i \in \mathbb{Z}_2\right\}$$

such that the following conditions on the last coordinates r_d hold:

$$\begin{aligned}
r_1 = 0 &\Rightarrow r_d \equiv 0 \,(\mathrm{mod} 2)\,, 0 \,(\mathrm{mod} 3)\\
r_1 = 1 &\Rightarrow r_d \equiv 0 \,(\mathrm{mod} 2)\\
r_1 = 2 &\Rightarrow r_d \equiv 1 \,(\mathrm{mod} 2)\\
r_1 = 3 &\Rightarrow r_d \equiv 1 \,(\mathrm{mod} 2)\,, 1, 2 \,(\mathrm{mod} 3)\,.
\end{aligned}$$

Then we know that the obtained grid drawing is primitive and it has a volume at most $24t \cdot \left\lceil \frac{k}{4} \right\rceil$, as the last coordinates of the points in the same column differ by at most six and each color class contains exactly t vertices. $\square$

According to the previous bounds, we see that this bound is asymptotically tight for primitive grid drawings of balanced complete k-partite graphs in plane and that the additional condition on primitivness does not violate our estimations too much.

5. Planar Grid Drawings

The last point of view on the grid drawings, the planarity, is studied in this chapter. We show that the Four Color Theorem together with Fáry's theorem imply the existence of a proper grid drawing for every planar graph, that is Theorem 5.1. Then we discuss the sizes of such drawings.

At the end of this chapter we show some known results about proper grid drawings of certain families of planar graphs.

5.1 Proper Drawings of General Planar Graphs

Although Theorem 2.2 and the Four Color Theorem imply that every planar graph is locatable in $\mathbb{Z}^2$, the drawings obtained by this approach do not have to be planar. On the other hand, De Fraysseix, Pach, and Pollack [16], Schnyder [33], and Chrobak and Nakano [13] proved that any planar graph on n vertices has a planar grid drawing which can be realized in grids of sizes $(2n-4) \times (n-2)$, $(n-2) \times (n-2)$ and $\lfloor 2(n-1)/3 \rfloor \times (4 \lfloor 2(n-1)/3 \rfloor - 1)$, respectively. Unfortunately, these drawings are not primitive.

Theorem 5.1. *There exists a proper grid drawing for every planar graph G.*

Proof. The main idea is to map a planar drawing of a graph, where line segments correspond to edges, to a grid such that no line segment contains more than two grid points. To find convenient coordinates we use the Four Color Theorem.

Let $G = (V, E)$ be a planar graph and let $\phi(G)$ be its initial planar embedding whose existence is ensured by, for example, Fáry's theorem. The mapping ϕ maps vertices of G to points with real coordinates in the plane. The edge $uv \in E$ corresponds to the line segment $\overline{\phi(u)\phi(v)}$ in the embedding $\phi(G)$. Let $f \colon V \to C$ be a vertex coloring of G with four colors and let $C = \{(0,0), (0,1), (1,0), (1,1)\}$. The first coordinate of color $c \in C$ is denoted as c_1, the second one as c_2. The existence of f is ensured by the Four Color Theorem.

Let $r \in \mathbb{R}$ denote the smallest distance such that every vertex can be shifted by r in any direction so that the condition on planarity still holds. We can set r as one half of the minimum distance between two points x, $y \in \mathbb{R}^2$ such that x and y belong to line segments which represent two vertex disjoint edges of G. The distance r is positive, otherwise we get a contradiction with planarity of $\phi(G)$. Thus, for every vertex $v \in V$, there is an open neighborhood $\Omega(v, r)$ of the point $\phi(v)$ such that any point $x \in \Omega(v, r)$ can represent the vertex v without violating the condition on planarity. Let us assume that no vertical line segment intersects two different neighborhoods $\Omega(u, r)$, $\Omega(v, r)$. Otherwise we can lower the distance r, as no two points $\phi(u)$, $\phi(v)$ lie on the same vertical line.

Now we put vertical lines across the whole plane such that the distance between two consecutive lines is $\epsilon > 0$. We choose the number ϵ such that every neighborhood is crossed by at least six lines (we can assume that $\epsilon = 1$). Then we choose one line and declare it as the initial line. Each line gets number according to its order, the initial line has number zero. Now for every vertex $v \in V$, we set $\phi(v) = x$, where x is a point from $\Omega(v, r)$ such that it lies on some vertical line with number l and $l \equiv f(v)_1 \,(\mathrm{mod} 2)$, $l \equiv f(v)_1 \,(\mathrm{mod} 3)$. We can always choose such line, because there are six consecutive lines crossing the neighborhood $\Omega(v,r)$. Thus numbers of these lines get through all values modulo two and three. In the rest of the proof, we assume that the first coordinates of points representing the vertices of G are integers. The point x is in $\Omega(v,r)$, so the modified embedding is still planar. By choosing appropriate lines we can also ensure that no two adjacent vertices lie on the same vertical line (but we might have to cross the neighborhoods by twelve lines).

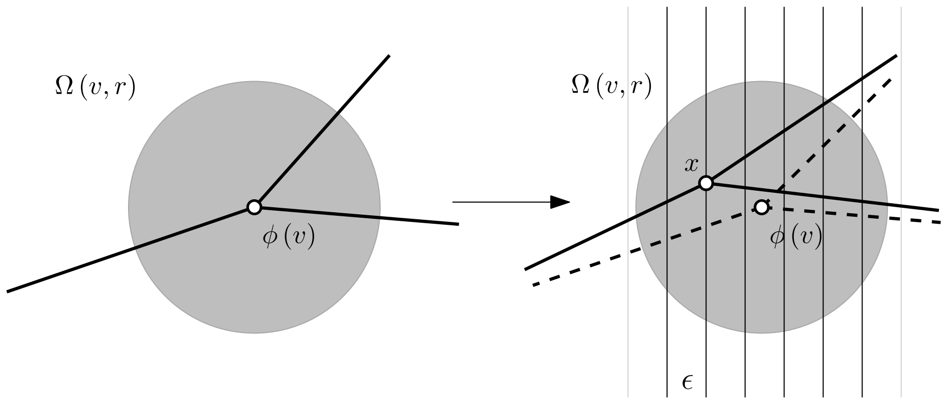

Figure 5.1: Placing the vertical lines

Let P denote the set of all prime numbers which appear in the decomposition of the difference $|\phi(u)_1 - \phi(v)_1|$ where $\phi(u)_1$, $\phi(v)_1$ are the first coordinates of points $\phi(u)$, $\phi(v)$ and $uv \in E$. The set P is finite, because no two points representing vertices lie on the same vertical line and thus the difference is always positive. Now we analogously put horizontal lines across the whole plane such that the distance between two consecutive lines is $\delta > 0$. This time we choose δ such that every vertical line is crossed by at least $\prod_{p \in P} p$ lines in every neighborhood.

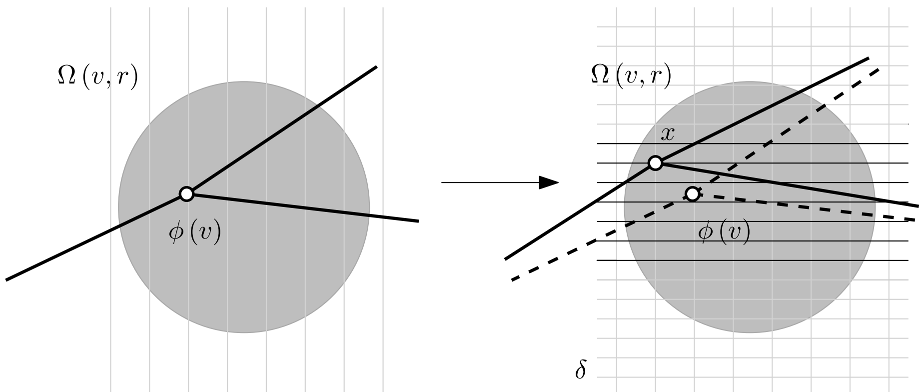

Figure 5.2: Placing the horizontal lines

Again, we declare one of these lines as initial and number them according to their order. Then, for every vertex $v \in V$, we set $\phi(v) = x$ such that $x \in \Omega(v,r)$, the first coordinate of $\phi(v)$ remains the same and x lies on the horizontal line with number l, where $l \equiv f(v)_2 \pmod{2}$, $l \equiv f(v)_2 \pmod{3}$. In addition, if there is another prime number p which divides the difference $|\phi(u)_1 - \phi(v)_1|$, $uv \in E$, then we set such horizontal lines for u and v that their numbers are not congruent modulo p. The different residues modulo p can be chosen according to the coloring

f. Each color corresponds to a unique residue modulo p ($p > 4$, so there is enough residues). We chose δ such that there is enough horizontal lines from which we can always choose the right ones.

Eventually the horizontal and vertical lines form an elongated grid which we can modify into a regular grid. It suffices to contract the grid such that the size of columns equals the size of rows, that is $\epsilon = \delta$. The contraction does not violate planarity, because the whole grid is regularly contracted, thus no positive distance can lower to zero. The coordinates of points are chosen such that every line segment is primitive, thus the embedding is planar and primitive. $\square$

This result gives an affirmative answer to the conjecture asked by D. Flores Peñaloza and F. J. Zaragoza Martinez [21]. The authors point out that proof of this statement would yield an alternate proof of the Four Color Theorem. However we use it as one of the assumptions. In fact, this theorem is equivalent to the Four Color Theorem, as the proof of the reverse implication is apparent if we use the function $f(x_1, x_2) = (x_1 \pmod 2, x_2 \pmod 2)$ as a coloring of $\mathbb{Z}^2$. If we use the Five Color Theorem in the proof instead, then we obtain three-locatable planar grid drawings of planar graphs.

Note that the choice of coordinates also gives us a coloring of G with at most four colors. In fact, we also proved a stronger conjecture from [21].

Corollary 5.2. *Any planar graph G is isomorphic to a plane subgraph H of the visibility graph of the integer lattice, in such a way that the function $g(a_1, a_2) = (a_1 \pmod 2, a_2 \pmod 2)$ is a coloring of H that uses exactly $\chi(G)$ colors.*

The grid drawings obtained by the proof can require large area with no reasonable bounds. However if we start with a nicer initial drawing, then we can estimate the upper bounds quite easily.

Suppose that the initial embedding is already a grid drawing of size $O(n) \times O(n)$ where n denotes the number of vertices of a given graph. The results of Chrobak, De Fraysseix, Pach, and Pollack, and Nakano [13, 16, 33] ensure the existence of such embedding. Then the following lemma gives us a lower bound on r. For completeness we include a short well-known proof using Pick's theorem [32].

Lemma 5.3. *Given an $n \times n$ integer grid, $n > 1$, the minimum nonzero distance from any grid point to any line segment is in $\Omega\left(\frac{1}{n}\right)$.*

Proof. Let a and b be the extremes of a line segment with grid vertices and c be any grid point (not on $\overline{ab}$). If we consider the triangle abc, then, according to Pick's theorem (see Section 1.4), the area of abc is at least $\frac{1}{2}$.

On the other hand the area of abc is one half of the length L of $\overline{ab}$ times the distance H from c to $\overline{ab}$. Since $L \leq \sqrt{2}n$ (consider the diagonal of the grid), we get

$$\frac{1}{2} \leq \frac{LH}{2} \leq \frac{\sqrt{2}nH}{2}$$

and therefore $H \geq \frac{1}{\sqrt{2}n}$. $\square$

Thus if the size of the initial grid drawing is $cn \times cn$, where $c > 0$ is some constant, then the minimum nonzero distance r from any point representing a vertex to any point representing an edge is in $\Omega\left(\frac{1}{n}\right)$. In the first part of the proof we refine the coordinates such that the neighborhood of every vertex is intersected by a constant number of vertical lines. The diameter of the neighborhoods is exactly $r \in \Omega\left(\frac{1}{n}\right)$, therefore the width of the new grid drawing is in $O\left(n^2\right)$.

All that is left is to estimate the height of the drawing. Following the proof we refine the vertical coordinates such that every neighborhood is intersected by at least $\prod_{p\in P} p$ horizontal lines. The diameter of the neighborhoods is now in $O(1)$, so if we find a function $f\colon \mathbb{N} \to \mathbb{N}$ such that the product $\prod_{p\in P} p$ is in $O(f(n))$, then we know that the height is in $O\left(n^2 f(n)\right)$ too.

We can focus on every vertex separately. Let v be a vertex of G and let P_v denote the set of prime numbers which divide the nonzero horizontal distance between the points $\phi(u)$ and $\phi(v)$ where $uv \in E$. Then the product of primes which divide the distance between u and v is in $O\left(n^2\right)$, as it is the width of the whole drawing. Therefore we get that

$$\prod_{p\in P_v} p \in O\left(n^{2d(u)}\right)$$

where $d(u)$ denotes the degree of u. According to the Chinese Remainder Theorem, we see that we can consider only the vertex with maximum degree Δ.

Hence we can find a proper grid drawing of any planar graph G with given coloring in the grid of size $O\left(n^2\right) \times O\left(n^{2\Delta+2}\right)$ where n denotes the number of vertices of G. Thus the rough estimation of the size of the drawing is polynomial for $\Delta \in O(1)$, quasi-polynomial for $\Delta \in O(\mathsf{poly}(\log n))$ and exponential for linear maximum degree.

Unfortunately we do not know how to embed the general planar graphs in a grid of polynomial size and the following question remains open.

Conjecture 5.4. *Every planar graph has a proper grid drawing of polynomial size.*

5.2 Specific Families of Planar Graphs

We show some already known results concerning the proper grid drawings of certain families of planar graphs in this section. As the first example we mention the planar bipartite graphs. Although the following proposition is an immediate corollary of Theorem 5.1, we can use its proof to derive a more reasonable upper bound (but still not really good) for the size of proper grid drawings of planar bipartite graphs.

Proposition 5.5 ([10])**.** *For every planar bipartite graph G, there exists a proper grid drawing of G.*

Proof. In a *2-book representation of a graph G* the vertices of G are mapped to a line (called *spine*) and edges are represented by curves that cross nor any other edge nor the spine. Lali Barrièrei and Clemens Huemer [6] showed that any planar quadrangulation Q admits a pair of 2-book embeddings of Q. The authors use such representations to embed the quadrangulation Q in the grid. They assign to each vertex the coordinates as its relative positions in the spines of both 2-book representations.

Note that a similar representation can be performed by forcing the differences of the X-coordinates to be relatively prime to each other. Since quadrangulations can be seen as maximal bipartite planar graphs, we get the required result using Proposition 2.3. $\square$

Corollary 5.6 ([10]). *For every planar bipartite graph G with n vertices, there exists a proper grid drawing of G which fits into a rectangle of area $O(n^2 \log n)$.*

Since we know the bounds on the number of columns of general grid drawings, we might consider the compactness of the proper grid drawings. However it is hopeless to try to obtain a constant bound on the number of columns for general planar graphs, as it is well known that some graphs with n vertices cannot be represented in any rectangular portion of the grid with one of its sides smaller than $\frac{n}{3}$. An example of such graph is shown in Figure 5.3.

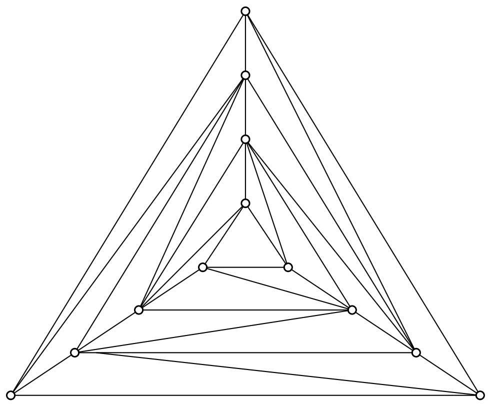

Figure 5.3: The nested triangle

We conclude this section by mentioning (without a proof) a complete characterization of graphs which have a proper grid drawing with two columns.

Proposition 5.7 ([14]). *A graph G has a proper grid drawing with two columns if and only if it can be extended to a maximal outerplanar graph such that its dual (excluding the vertex representing the outer face) is a path.*

An interesting problem, which remains open, is to improve the bounds of the area needed to properly embed a graph for certain families of planar graphs.

Conclusions

In this chapter, we briefly sum up author's results and mention some open problems and possible directions for further research.

We introduce the grid drawings of graphs in $\mathbb{Z}^d$, $d \geq 2$, and we study those drawings out of several points of view. First, we consider the situation in which the number of grid points a line segment of a grid drawing intersects is bounded. This case leads to the new definition of so called q-locatable graphs for an integer $q \geq 2$. The problem of characterizing such graphs is closely connected with graph coloring, as we show in Theorem 2.2. This result is a stronger version of a theorem proved by D. Flores Peñaloza and F. J. Zaragoza Martinez [21] and it has nice immediate corollaries, such as it is NP-complete to decide whether or not a given graph is q-locatable in $\mathbb{Z}^d$ for a given integer $q \geq 2$.

Second, we focus on the grid drawings with bounded number of columns showing a complete characterizations of graphs which have a (primitive) grid drawing on l columns for fixed $l \in \mathbb{N}$. This helps us understand the relationship between drawing and locating of graphs. These characterizations have, again, very much in common with graph coloring, only this time, we consider so called mixed coloring whose definition is introduced in Chapter 3. We use the mixed colorings to prove a more general result which implies NP-completeness of a problem of deciding whether it is possible to draw/locate a graph on l columns answering an open question of Cáceres et. al. [10]. We also prove some other statements about mixed colorings which concern planar and outerplanar graphs. It might be interesting to consider some other known graph classes and their relations to mixed coloring, as we might obtain some stronger bounds on the number of columns on which we can draw/locate such graphs.

In Chapter 4, we prove one of the conjectures asked by David R. Wood [36] which concerns with grid drawings having small area. We known that the problem of finding a grid drawing of a graph with minimum area is a generalization of the famous "No-three-in-line" problem. The reader might be interested in the other conjectures of David R. Wood which remained open, so let us mention these. The first one is a slightly modified version of Theorem 4.4. It also asks for the lower bound on the area of the grid drawings of complete k-partite graphs. However this time, we do not require the graphs to be balanced.

Conjecture ([36]) Has every grid drawing of any (non-balanced) complete k-partite graph with n vertices $\Omega\left(kn\right)$ area?

The second question would establish a trade-off between small area and small aspect ratio of a grid drawing.

Conjecture ([36]) Has every $w \times h$ grid drawing of $K\left(k,t\right)$ with aspect ratio $r = \frac{\max\{w,h\}}{\min\{w,h\}}$ $\Omega\left(\frac{k^2t^2}{r}\right)$ area?

We also consider planar graphs and their grid drawings. Although it has been known that there is a planar grid drawing of any planar graph on n verices in the grid of size $O\left(n\right) \times O\left(n\right)$, the question "Is there a planar and primitive grid drawing of an arbitrary planar graph?" was open (see [21]). We give an affirmative answer to this question by proving Theorem 5.1. However the size of the obtained grid drawings can be exponential and thus we ask whether there is, for an arbitrary planar graph, a planar and primitive grid drawing with polynomial size. Another disadvantage of the proof is that it uses the Four Color Theorem as one of the assumptions, but this is comprehensible, since these two results are actually equivalent. Perhaps the most intriguing question left open is whether there is a proof of Theorem 5.1 without using the Four Color Theorem, as it would yield an alternate proof of this classical result in graph theory.

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