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KATEGORIE FUZZY MNOŽIN

CATEGORIES OF FUZZY SETS

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DECLARATION OF UNAIDED WORK

Hereby I declare that this thesis is my own unaided work and that I used only the sources listed in references exclusively.

Prague, May 2009. Signature:

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ABSTRACT

Category theory provides very useful tools for studying mathematical structures and phenomena. One of the structures that is studied in a category-theoretical manner are fuzzy sets. If we consider fuzzy sets as objects and set up certain kind of structure preserving mappings as morphisms, we can obtain a suitable category for our purposes. Goal of this work is to give an overview of preferably all important category-theoretical approaches to fuzzy sets that were done throughout relatively short history of category-theoretical modelling of fuzzy sets.

KEYWORDS

Fuzzy sets, Fuzzy relations, Categories of fuzzy sets.

CONTENTS

1. INTRODUCTION

It is evident from the title of this work that the main area of our interest in the next chapters lies at the intersection of category theory and fuzzy set theory. We attempt to give an overview of preferably all important category-theoretical approaches to fuzzy sets.
Let us say (informally) a little more about the two fields and their interconnection now.

1.1 CATEGORY THEORY

Category theory was introduced by **Samuel Eilenberg** and **Saunders Mac Lane** in the early 1940´s [ref. 3, 4] in connection with algebraic topology. Briefly speaking, it is a general theory of mathematical structures and relationships between them and it provides a unifying notion and theory for these structures. Fundamental conceps of the theory are objects, morphisms, categories, functors, natural transformations, diagrams, limits, etc. Eminency of category theory consits in its generality, high degree of abstraction and mainly in its simple and synoptical language.
Categories occur in every part of mathematics and also in related fields such as computer science, theoretical physics and logic. Such entities as sets, groups, vector spaces, ordered sets, topological spaces, automata, formal languages, etc., all naturally give rise to categories. Category theory allows us to uncover general properties of mathematical structures and to understand how (apparently) different structures are interrelated and what they have in common.
Category theory provides a conceptual framework for mathematics and thus can be seen as an alternative to set theory as a foundational theory for mathematics.

1.2 FUZZY SET THEORY

The concept of fuzzy set was introduced by **Lotfi A. Zadeh** in his pioneering work from 1965 [ref. 30]. Since then it has become an important concept in applied mathematics and engineering. The idea is quite simple and straightforward. For classical sets (often called crisp sets in fuzzy context) characteristic functions take values from $\{0, 1\}$. Zadeh´s generalization of classical sets consists in the fact that characteristic functions of fuzzy sets (then called membership functions) take values in the real unit interval $[0, 1]$. Fuzzy sets are uniquely characterized by their membership functions. In fact, they are directly identified with their membership functions generally.
Another important generalization was done by Zadeh´s student **Joseph A. Goguen** in his 1967 paper [ref. 6]. Goguen generalized Zadeh´s idea in the sense that membership functions could take values not only in the real unit interval but in some kind of ordered structure generally. Usually it is required that this structure be at least a poset or a lattice. Examples of the most commonly used target structures are (beside the real unit interval) completely distributive lattices, complete Heyting algebras, various kinds of ordered monoids, various kinds of quantales, etc.
One can see that just the membership function´s target structure presents itself to be one of the main criteria for classification of fuzzy sets. For purely fuzzy-set-theoretical purposes it actually is, but later we will see that for our category-theoretical purposes there are some more suitable criteria which arise directly out of category-theoretical concepts.

1.3 CATEGORY-THEORETICAL APPROACHES TO FUZZY SETS

Goguen´s contribution does not consist only in the fact of generalization of target structures, but it also (we would like to say "above all", for our purposes) consists in the fact that he introduced category-theoretical approach to fuzzy sets. In his work from 1967, which was mentioned above, he introduced the first category of fuzzy sets ever, category S(L). It is a category of L-fuzzy sets and L-fuzzy relations, where L signifies the generalization of target structure mentioned above. In case of category S(L), L stands for a complete lattice ordered semigroup. In his 1967 paper, Goguen did not give any particular results about S(L) (except the main one that S(L) constitutes a category). Later, in his 1974 article [ref. 8], he considered another category of L-fuzzy sets, category Set(L). Here L stands for some completely distributive lattice. Some interesting results about category Set(L) were given by Goguen himself and also by other authors, mainly by L. N. Stout [ref. 25, 26, 27].
Goguen´s early works have become a starting point for other authors. Various kinds of target structures have been considered, objects, morphisms and composition of morphisms have been defined in many different ways, thus many different categories of fuzzy sets (or fuzzy relations) have been obtained throughout the years.
As our aim is to give an overview of these categories, we need to choose some criteria for their classification in an effort to keep this work well-arranged. As mentioned above, the type of target structure of membership functions seems to be a good criterion for this classification. But there are more suitable criteria, for our category-theoretical purposes, which put similar categories to the same class. Thus we have decided to classify categories in this work by the facts whether objects are fuzzy sets or objects are fuzzy relations and whether morphisms are (fuzzy) functions or morphisms are (fuzzy) relations. Thereby we have the following classes of categories now:

- Categories of fuzzy sets (as objects) and (fuzzy) functions (as morphisms).
- Categories of fuzzy sets (as objects) and (fuzzy) relations (as morphisms).
- Categories of fuzzy relations (as objects) and (fuzzy) functions (as morphisms).
- Categories of fuzzy relations (as objects) and (fuzzy) relations (as morphisms).

Categories of the same type (in the same class) will be ordered by resemblance. Very similar categories will be put together.
We believe that this classification is suitable for category-theoretical purposes and that it will keep the work well-arranged from categorical point of view at least.

2. PRELIMINARIES

In this chapter we present some necessary category-theoretic, (fuzzy) set-theoretic and algebraic concepts, which we will be dealing with in the next chapter. These definitions and concepts are primarily taken over from [ref. 1, 9, 18] for category-theoretic section and from [ref. 2, 21, 29] for (fuzzy) set-theoretic and algebraic section. We also recommend these works for more detailed study of the fields.

2.1 CATEGORY-THEORETIC BASICS

Category theory was introduced by **Samuel Eilenberg** and **Saunders Mac Lane** in the 1940´s [ref. 3, 4]. It deals with mathematical structures and relationships between them and provides a unifying theory and terminology for these structures.

Definition 2.1.1
A **category** C is a quadruple $C = (\mathrm{Ob}(C), \mathrm{Hom}_C, \circ, 1)$ consisting of:
1) A class $\mathrm{Ob}(C)$, whose members are called (C-)**objects**.
2) For each pair (A, B) of objects, a set $\mathrm{Hom}_C(A, B)$, whose members are called (C-) **morphisms** from A to B. The sets $\mathrm{Hom}_C(A, B)$ are pairwise disjoint.
3) A binary operation from $\mathrm{Hom}_C(A, B) \times \mathrm{Hom}_C(B, C)$ to $\mathrm{Hom}_C(A, C)$ called **composition** of morphisms. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms. Then their composition is denoted by $f \circ g$. Composition of morphisms is **associative**, i.e. for all morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$ $(f \circ g) \circ h = f \circ (g \circ h)$ holds.
4) For every object B, there exist a morphism $1_B : B \longrightarrow B$ called the **identity morphism** such that for all $f : A \longrightarrow B$ and $g : B \longrightarrow C$, $f \circ 1_B = f$ and $1_B \circ g = g$.

Remark:
If $f \in \mathrm{Hom}_C(A, B)$ (denoted by $f : A \longrightarrow B$), then A is called a **domain** of f (denoted by $A = \mathrm{dom}(f)$) and B is called a **codomain** of f (denoted by $B = \mathrm{cod}(f)$). The class of all C-morphisms (denoted by $\mathrm{Hom}(C)$) is defined to be the union of all the sets $\mathrm{Hom}_C(A, B)$.

We can also take a more global viewpoint and consider categories themselves as objects. Then the "morphisms" between categories are called **functors**. More precisely:

Definition 2.1.2
Let C and D be categories. Then a **functor** F from the category C to the category D is a function that assigns to each C-object A an D-object F(A) and to each C-morphism $f : A \longrightarrow B$ a D-morphism $F(f) : F(A) \longrightarrow F(B)$ in such a way that:
1) F preserves composition of morphisms, i.e. $F(f \circ g) = F(f) \circ F(g)$ for all C-morphisms $f \in \mathrm{Hom}_C(A, B)$ and $g \in \mathrm{Hom}_C(B, C)$
2) F preserves the identity morphisms, i.e. $F(1_A) = 1_{F(A)}$ for all C-objects $A \in \mathrm{Ob}(C)$.

Examples of categories commonly used in mathematics:
The category **Set** of all sets (as objects) and all functions (as morphisms).
The category **Rel** of all sets and binary relations.
The category **Ord** of all preordered sets and monotonic functions.
The category **Grp** of all groups and group homomorphisms.
The category **Top** of all topological spaces and continuous functions.
The category **Met** of all metric spaces and metric functions.
The category **Vect** of all vector spaces and linear functions.

Let us consider the following situation. Let category C be given and let us consider category D which has the same objects as category C and there are also morphisms between the same objects as in C, but these morphisms are of the opposite direction. Category D is called the **dual** (or **opposite**) **category** of C. More precisely:

Definition 2.1.3
Let $C = (\mathrm{Ob}(C), \mathrm{Hom}(C), \circ, 1)$ be a category. Then the **dual** (or **opposite**) **category** of C is the category $C^{op} = (\mathrm{Ob}(C^{op}), \mathrm{Hom}(C^{op}), \circ^{op}, 1^{op})$ such that:
1) $\mathrm{Ob}(C^{op}) = \mathrm{Ob}(C)$
2) $\mathrm{Hom}_{C}{}^{op}(A, B) = \mathrm{Hom}_C(B, A)$, for all $A, B \in \mathrm{Ob}(C^{op})$
3) $f \circ^{op} g = g \circ f$, for all $f, g \in \mathrm{Hom}(C^{op})$
4) $1_A{}^{op} = 1_A$, for all $A \in \mathrm{Ob}(C^{op})$.

Thanks to the definition 2.1.3 each concept P gives rise to its dual concept P^{op}, which is often denoted by coP (co-P), for example equalizers and coequalizers, products and coproducts, wellpowered and co-wellpowered, etc.

Definition 2.1.4
Let C be a category. We say that C is a **small category** iff $\mathrm{Ob}(C)$ is actually set, not proper class. A category which is not small is called **large category**.

Definition 2.1.5
Let C be a category. Then D is a **subcategory** of C iff:
1) $\mathrm{Ob}(D) \subseteq \mathrm{Ob}(C)$.
2) $\mathrm{Hom}_D(A, B) \subseteq \mathrm{Hom}_C(A, B)$, for all $A, B \in \mathrm{Ob}(C)$.
If $\mathrm{Hom}_D(A, B) = \mathrm{Hom}_C(A, B)$, for all $A, B \in \mathrm{Ob}(C)$ we say that D is a **full subcategory** of C.

Definition 2.1.6
Let $f : B \longrightarrow C$ be a morphism. Then f is a **monomorphism** iff for all morphisms g_1 and $g_2 : A \longrightarrow B$ the following holds: if $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.

Definition 2.1.7
Let $f : A \longrightarrow B$ be a morphism. Then f is an **epimorphism** iff for all morphisms g_1 and $g_2 : B \longrightarrow C$ the following holds: if $f \circ g_1 = f \circ g_2$, then $g_1 = g_2$.

Definition 2.1.8
Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be morphisms. Then g is called an **inverse** morphism of f iff $f \circ g = 1_A$ and $g \circ f = 1_B$. If there exists an inverse morphism of f, then f is called an **isomorphism**.

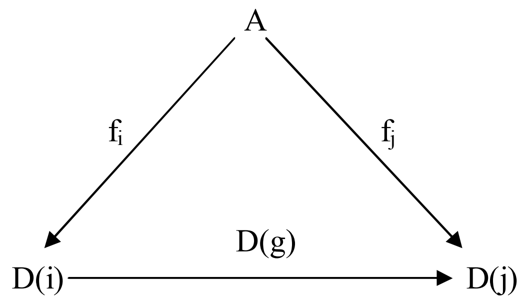
Definition 2.1.9
Let A be an object of a given category C. We say that the pair (B, f) is a **subobject** of an object A iff $B \in \mathrm{Ob}(C)$ and $f : B \longrightarrow A$ is a monomorphism. Dually, we say that the pair (B, f) is a **quotient object** of an object A iff $B \in \mathrm{Ob}(C)$ and $f : A \longrightarrow B$ is an epimorphism.
Remark:
An inclusion relation between subobjects of a given object A is defined by the following way. Let (B, f) and (C, g) be subobjects of A. We put $(B, f) \subseteq (C, g)$ iff there exists a morphism $h : B \longrightarrow C$ such that $f = h \circ g$. (Such morphism h will always be a monomorphism).

Definition 2.1.10
Let *C* be a category. We say that *C* is a **balanced category** iff every morphism in *C* which is both monomorphism and epimorphism has an inverse morphism in *C*.

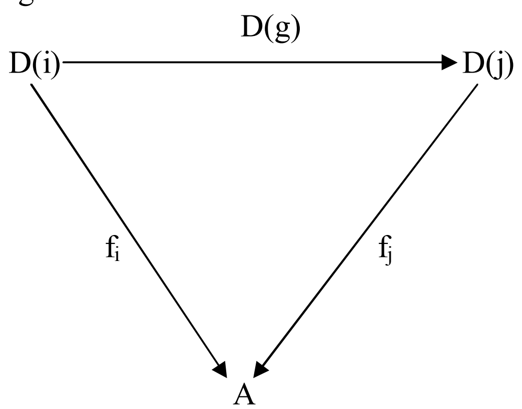
At this point, it is suitable to introduce a very important and useful concept of category theory. It is one of the most fundamental concepts of the theory and it describes many basic constructions in mathematics. It is the concept of a **limit** (and dually a **colimit**) of a **diagram**.

Definition 2.1.11
Let *C* be a category. Then a **diagram** in the category *C* is a functor $D : I \longrightarrow C$. The domain of D, category *I*, is called the **index category** or the **scheme** of the diagram D.
Remark:
Let *I* be a **preorder**, i.e. a category in which there is at most one morphism between two objects and in which a reflexive and transitive binary relation $\leq$ on objects is defined as follows: $A \leq B$ iff there is a morphism between A and B. If every pair of objects of *I* has a lower bound, then *I* is called **down-directed**. Dually, if every pair of elements of *I* has an upper bound, then *I* is called **up-directed**. A diagram D is called **finite** iff its scheme *I* is finite, i.e. *I* has a finite number of objects and a finite number of morphisms between them.

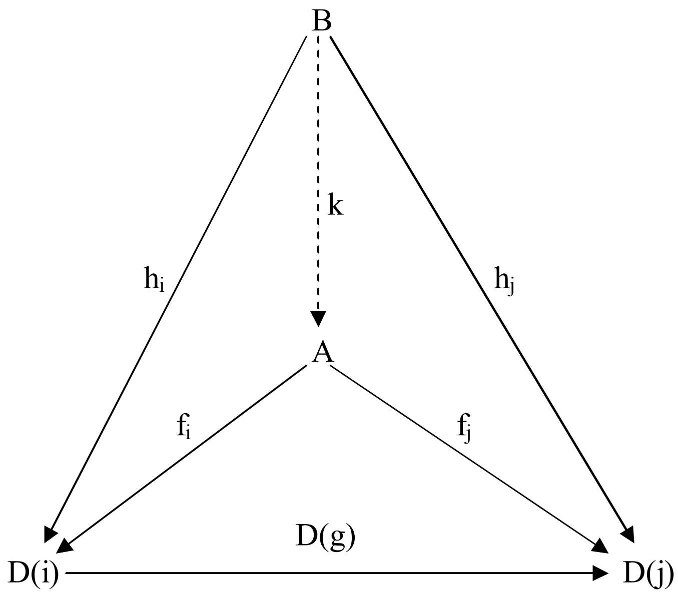
Definition 2.1.12
Let $D : I \longrightarrow C$ be a diagram in a category *C*. Then a **cone** over the diagram D is an object $A \in Ob(C)$, called a vertex, together with a family of morphisms $f_i : A \longrightarrow D(i) \in Hom(C)$ for each $i \in Ob(I)$ such that $f_i \circ D(g) = f_j$ for each morphism $g : i \longrightarrow j \in Hom(I)$, i.e. such that the following triangle commutes:



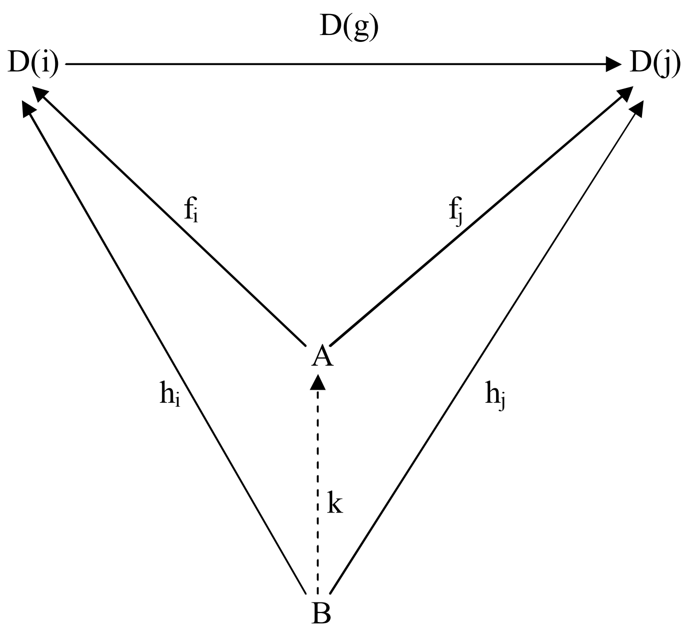
Definition 2.1.13
Let $D : I \longrightarrow C$ be a diagram in a category *C*. Then a **co-cone** under the diagram D is an object $A \in Ob(C)$ together with a family of morphisms $f_i : D(i) \longrightarrow A \in Hom(C)$ for each $i \in Ob(I)$ such that $D(g) \circ f_j = f_i$ for each morphism $g : i \longrightarrow j \in Hom(I)$, i.e. such that the following triangle commutes:



Definition 2.1.14
Let D : $I \longrightarrow C$ be a diagram in a category C and let A ∈ Ob(C) together with a family of morphisms f_i : A ⟶ D(i) ∈ Hom(C) for each i ∈ Ob(I) be a cone over the diagram D. Then this cone is a **limit** of the diagram D iff for any other cone (B ∈ Ob(C), {h_i : B ⟶ D(i) ∈ Hom(C) | for each i ∈ Ob(I)}) over the digram D there exists a unique morphism k : B ⟶ A such that k ∘ f_i = h_i for all i ∈ I, i.e. such that the following picture commutes:

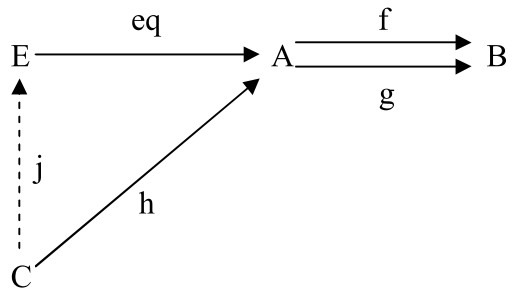


Definition 2.1.15
Let D : $I \longrightarrow C$ be a diagram in a category C and let A ∈ Ob(C) together with a family of morphisms f_i : A ⟶ D(i) ∈ Hom(C) for each i ∈ Ob(I) be a co-cone under the diagram D. Then this co-cone is a **colimit** of the diagram D iff for any other co-cone (B ∈ Ob(C), {h_i : B ⟶ D(i) ∈ Hom(C) | for each i ∈ Ob(I)}) under the digram D there exists a unique morphism k : B ⟶ A such that k ∘ f_i = h_i for all i ∈ I, i.e. such that the following picture commutes:



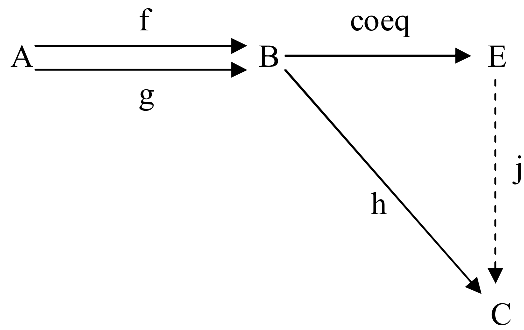
Definition 2.1.16
Let D : $I \longrightarrow C$ be a diagram in a category C. If I is a down-directed poset (considered as a category), then (co)limits of diagrams with scheme I are called **inverse** (or **projective**) (co)**limits**. Dually, if I is an up-directed poset (considered as a category), then (co)limits of diagrams with scheme I are called **direct** (co)**limits**.

Definition 2.1.17
Let f, g : A ⟶ B be (parallel) morphisms. Then an object E together with a morphism eq : E ⟶ A is an **equalizer** of f and g iff eq ∘ f = eq ∘ g and for any other object C together with a morphism h : C ⟶ A such that h ∘ f = h ∘ g there exists a unique morphism j : C ⟶ E such that j ∘ eq = h, i.e. such that the following picture commutes:



Remark: equalizers are limits of diagrams with scheme • ⇉ •. If the two arrows are replaced by an arbitrary set of arrows, then limits of diagrams with such schemes are called **multiple equalizers**.

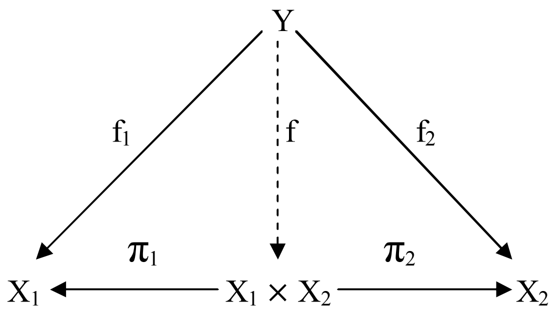
Definition 2.1.18
Let f, g : A ⟶ B be (parallel) morphisms. Then an object E together with a morphism coeq : B ⟶ E is an **coequalizer** of f and g iff f ∘ coeq = g ∘ coeq and for any other object C together with a morphism h : B ⟶ C such that f ∘ h = g ∘ h there exists a unique morphism j : E ⟶ C such that coeq ∘ j = h, i.e. such that the following picture commutes:



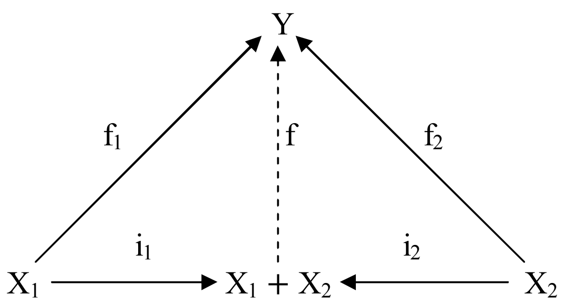
Remark: coequalizers are colimits of diagrams with scheme • ⇉ •. If the two arrows are replaced by an arbitrary set of arrows, then colimits of diagrams with such schemes are called **multiple coequalizers**.

Definition 2.1.19
Let I ∈ Ob(C). We say that I is an **initial object** of a category C iff for every object X ∈ Ob(C), there exists precisely one morphism from I to X. Dually, we say that T ∈ Ob(C) is a **terminal object** of a category C iff for all X ∈ Ob(C), there exists precisely one morphism from X to T.

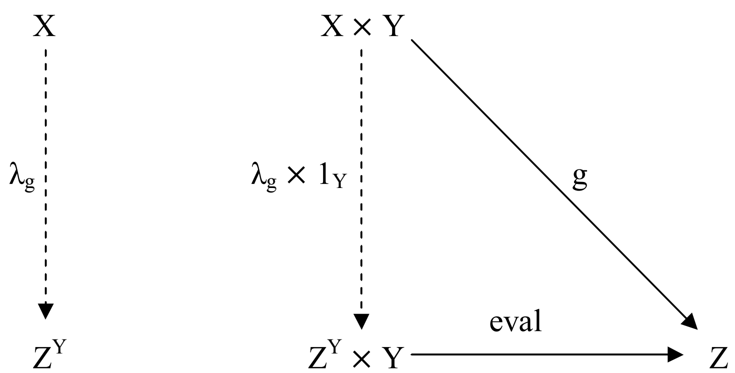
Definition 2.1.20
Let $X_1 \in Ob(C)$ and $X_2 \in Ob(C)$. Then a **product** of X_1 and X_2 is an object $X_1 \times X_2 \in Ob(C)$ together with a pair of morphisms $\pi_1 : X_1 \times X_2 \longrightarrow X_1$ and $\pi_2 : X_1 \times X_2 \longrightarrow X_2$ such that for any $Y \in Ob(C)$ together with a pair of morphisms $f_1 : Y \longrightarrow X_1$ and $f_2 : Y \longrightarrow X_2$ there exist a unique morphism $f : Y \longrightarrow X_1 \times X_2$ such that $f_1 = f \circ \pi_1$ and $f_2 = f \circ \pi_2$, i.e. such that the following picture commutes:



Definition 2.1.21
Let $X_1 \in Ob(C)$ and $X_2 \in Ob(C)$. Then a **coproduct** of X_1 and X_2 is an object $X_1 + X_2 \in Ob(C)$ together with a pair of morphisms $i_1 : X_1 \longrightarrow X_1 + X_2$ and $i_2 : X_2 \longrightarrow X_1 + X_2$ such that for any $Y \in Ob(C)$ together with a pair of morphisms $f_1 : X_1 \longrightarrow Y$ and $f_2 : X_2 \longrightarrow Y$ there exist a unique morphism $f : X_1 + X_2 \longrightarrow Y$ such that $f_1 = i_1 \circ f$ and $f_2 = i_2 \circ f$,
i.e. such that the following picture commutes:



Definition 2.1.22
Let C be a category with products for every pair of objects and let $Y \in Ob(C)$ and $Z \in Ob(C)$. Then an object Z^Y together with a morphism $eval : Z^Y \times Y \longrightarrow Z$ is an **exponential** of Y and Z iff for any object X and a morphism $g : X \times Y \longrightarrow Z$ there is a unique morphism $\lambda_g : X \longrightarrow Z^Y$ such that the following picture commutes:

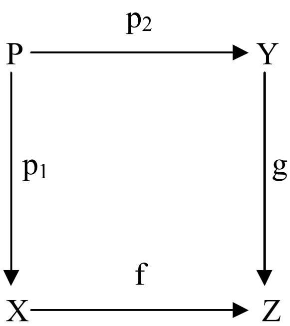


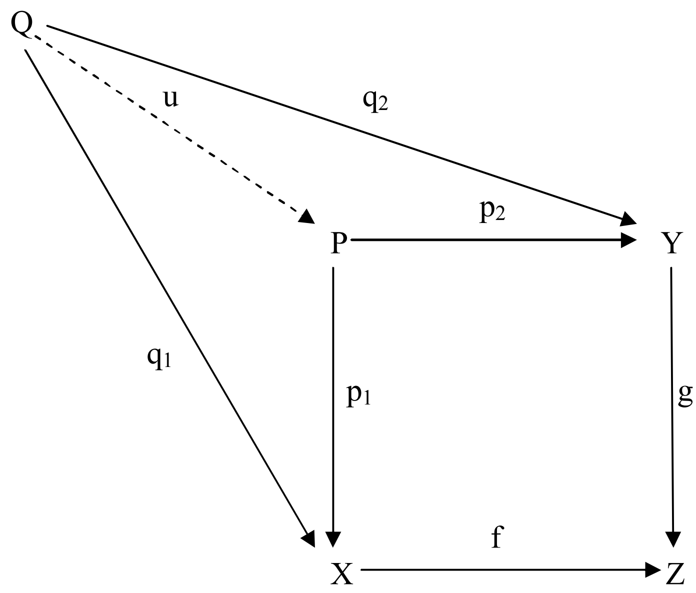
Definition 2.1.23
Let C be a category. Then C is called a **Cartesian closed** category iff the following properties hold:
1) C has a terminal object.
2) For any two objects X and Y of C, there exist a product $X \times Y$ in C.
3) For any two objects X and Y of C, there exist an exponential Y^X in C.

Definition 2.1.24
Let C be a category and let $A \in Ob(C)$ be given. Then the **slice category** $C{\downarrow}A$ of C-objects **over** A has as objects all C-morphisms with codomain A. Let $f : B \longrightarrow A$ and $g : C \longrightarrow A$ be an objects of the category $C{\downarrow}A$. Then a $C{\downarrow}A$-morphism from f to g is a C-morphism $h : B \longrightarrow C$ such that $f = h \circ g$.

Definition 2.1.25
A category C is called **locally Cartesian closed** iff all of its slice categories are Cartesian closed.

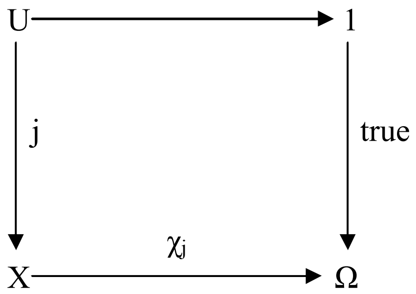
Definition 2.1.26
Let $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ be morphisms. Then a **pullback** of f and g is an object P together with two morphisms $p_1 : P \longrightarrow X$ and $p_2 : P \longrightarrow Y$ such that $p_1 \circ f = p_2 \circ g$, i.e. such that the following picture commutes, and which moreover has the universal property, i.e. for any other such triple (Q, q_1, q_2) there exists a unique morphism $u : Q \longrightarrow P$ such that $q_1 = u \circ p_1$ and $q_2 = u \circ p_2$, i.e. such that the following picture commutes:





Definition 2.1.27
A morphism $m : A \longrightarrow B$ is called a **regular monomorphism** iff it is an equalizer of some pair of morphisms.

Definition 2.1.28
Let *C* be a category with a terminal object 1. Then we say that a *C*-object Ω together with a *C*-morphism true : 1 ⟶ Ω is a (**regular**) **subobject classifier** for *C* if the following property holds: for each (regular) monomorphism j : U ⟶ X there is a unique morphism χ_j : X ⟶ Ω such that the following commutative picture represents a pullback:



Definition 2.1.29
Let *C* be a category. Then category *C* is a **complete category** iff every diagram D : *I* ⟶ *C*, where *I* is a small category has a limit in *C* (we also say that all small limits exist in *C*). Dually, *C* is a **cocomplete category** iff every diagram D : *I* ⟶ *C*, where *I* is a small category has a colimit in *C* (we also say that all small colimits exist in *C*). Category *C* is called **finitely** (co)**complete** iff every finite diagram has a (co)limit in *C*.

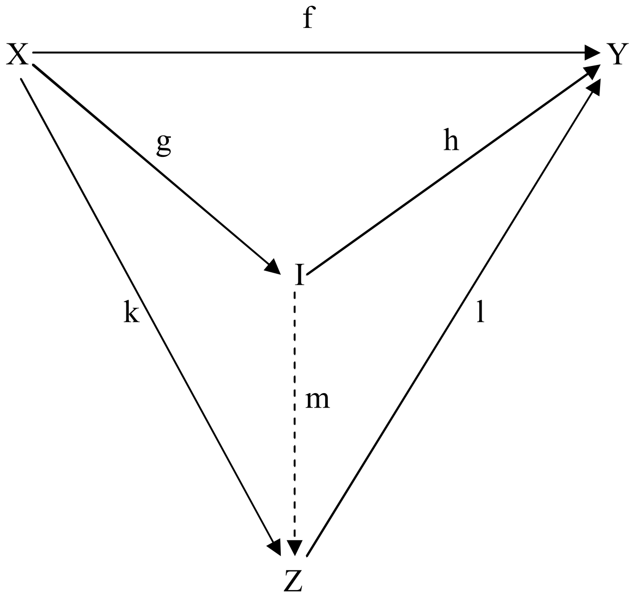
Definition 2.1.30
A category *C* is an (elementary) **topos** iff the following properties hold:
1) *C* is finitely complete category.
2) *C* is finitely cocomplete category.
3) For any two objects X and Y of *C*, there exist an exponential Y^X in *C*.
4) *C* has a subobject classifier.

Definition 2.1.31
A category *C* is a **quasitopos** iff the following properties hold:
1) *C* is finitely complete category.
2) *C* is finitely cocomplete category.
3) *C* is locally Cartesian closed category.
4) *C* has a regular subobject classifier.

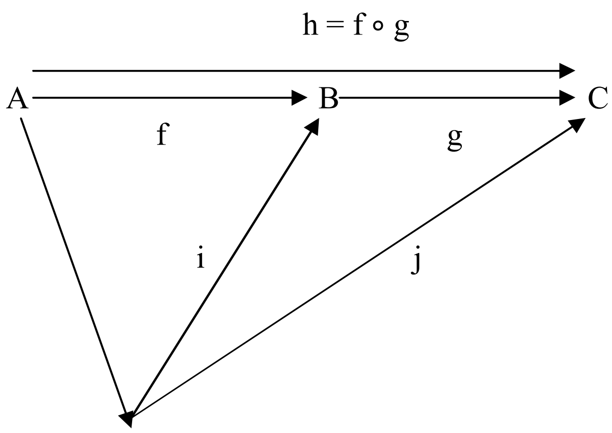
Definition 2.1.32
Let *C* be a category. Then category *C* is **cdl-ordered** iff for each object A in *C* its class P(A) of subobjects is a complete distributive lattice with respect to subobject inclusion. Category *C* is **disjointedly cdl-ordered** iff it is cdl-ordered and each cdl P(A) is pseudo-complemented and has for any two (different) atoms the universal upper bound as lattice join of their pseudo-complements.

Definition 2.1.33
Let *C* be a category and let P be an object of *C*. Then P is an **atomic monic** iff each morphism f : P → A is a monomorphism, for each non-initial A there exist a morphism g : P → A and each subobject f : P → A is atomic in the lattice P(A). An object P of a category *C* is a **projective generator** iff each morphism h : P → B factors through every epimorphism f : A → B and for every two different morphisms f , g : A → B there is a morphism h : P → A such that h ∘ f ≠ h ∘ g.

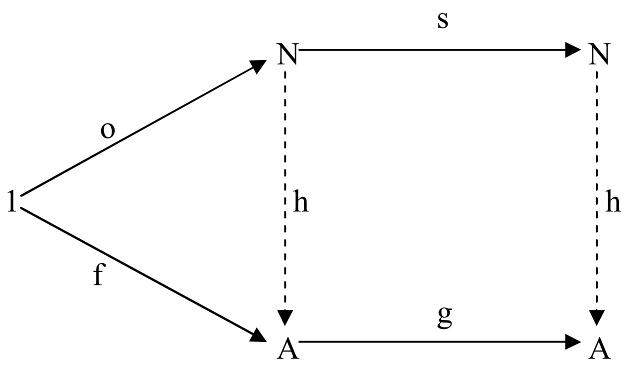
Definition 2.1.34
Let f : X ⟶ Y be a morphism. A monomorphism h : I ⟶ Y is the **image** of f iff the following properties are satisfied:
1) There exist a morphism g : X ⟶ I such that f = g ∘ h.
2) For any object Z with a morphism k : X ⟶ Z and a monomorphism l : Z ⟶ Y such that f = k ∘ l there exist a unique morphism m : I ⟶ Z such that k = g ∘ m and h = m ∘ l, i.e. such that the following diagram commutes:



Definition 2.1.35
Let *C* be a category. Then category *C* **has images** iff each *C*-morphism has an image in *C*.
Let f : A ⟶ B and g : B ⟶ C be *C*-morphisms and let h = f ∘ g be their composition.
Let us denote by i the image of f and by j the image of h = f ∘ g. Then category *C* **has associative images** iff for each such morphisms i ∘ g = j holds. That is iff the following diagram commutes:

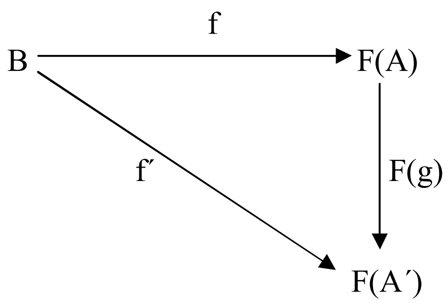


Definition 2.1.36
Let *C* be a category with a terminal object 1. An object N together with morphisms $o : 1 \longrightarrow N$ and $s : N \longrightarrow N$ is called a **natural number object** (NNO) iff for any other object A and morphisms $f : 1 \longrightarrow A$ and $g : A \longrightarrow A$ there exist a unique morphism $h : N \longrightarrow A$ such that $o \circ h = f$ and $s \circ h = h \circ g$, i.e. such that the following diagram commutes:



Definition 2.1.37
Let *C* and *D* be categories. Then the **product category** $C \times D$ of categories *C* and *D* is defined as follows:
1) Objects are pairs (A, B), where $A \in Ob(C)$ and $B \in Ob(D)$.
2) Morphisms from (A_1, B_1) to (A_2, B_2) are pairs (f, g), where $f : A_1 \longrightarrow A_2 \in Hom(C)$ and $g : B_1 \longrightarrow B_2 \in Hom(D)$.
3) Composition of two (composable) morphisms (f_1, g_1) and (f_2, g_2) is defined as $(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2)$.
4) Identity morphism on an object (A, B) is defined as $1_{(A, B)} = (1_A, 1_B)$.

Definition 2.1.38
Let *C* and *D* be categories, let F be a functor from *C* to *D* and let $B \in Ob(D)$. Then an F-**structured arrow with domain** B is a pair (f, A), where $A \in Ob(C)$ and $f : B \longrightarrow F(A) \in Hom(D)$. An F-structured arrow (f, A) with domain B is called F-**universal for** B iff for each F-structured arrow (f', A') with domain B there exists a unique *C*-morphism $g : A \longrightarrow A'$ such that $f' = f \circ F(g)$, i.e. such that the following diagram commutes:



Definition 2.1.39
Let *C* and *D* be categories and let F be a functor from *C* to *D*. We say that F is an **adjoint functor** iff for every *D*-object B there exist an F-universal arrow with domain B.
Remark:
Dual notion of adjoint functor is that of **co-adjoint functor**. A functor F : *C* ⟶ *D* is **co-adjoint** iff for every *D*-object B there exist an F-co-universal arrow with codomain B. Every adjoint functor has a unique co-adjoint functor. Conversely, if a functor F has a co-adjoint functor, then F is an adjoint functor. Dually, a functor is co-adjoint iff it has an (unique) adjoint. Thus adjoint functors and co-adjoint functors naturally come in pairs. Let F : *C* ⟶ *D* and G : *D* ⟶ *C* are functors such that F is adjoint functor of G and G is co-adjoint functor of F. Then F is (also) called a **right adjoint** of G and G is (also) called a **left adjoint** of F.

Definition 2.1.40
Led *C* and *D* be categories and let F : *C* ⟶ *D* be a functor from *C* to *D*. The functor F induces a function $F_{A, B}$: Hom_{C}(A, B) ⟶ Hom_{D}(F(A), F(B)) by $F_{A, B}$ (f) = F(f) for all morphisms f ∈ Hom_{C}(A, B). Then F is called a **faithful functor** iff $F_{A, B}$ is an injective function.

Definition 2.1.41
Let *X* be a category. Then a **concrete category over *X*** is a pair (*C*, U), where *C* is a category and U : *C* ⟶ *X* is a faithful functor. A concrete category over **Set** is called simply a **concrete category** (also a **construct**).

Definition 2.1.42
Let (*C*, U) be a concrete category over a category *X*. Then a **fibre** of an *X*-object X is the preordered class of all *C*-objects A such that U(A) = X ordered by:
A ≤ B iff 1_X : U(A) ⟶ U(B) is a *C*-morphism. A fibre is called **small** if it is a (preordered) set.

Definition 2.1.43
Let F : *A* ⟶ *B* be a functor. We say that F is an **amnestic functor** iff each *A*-isomorphism f is an identity whenever F(f) is an identity.

Definition 2.1.44
Let (*C*, U) be a concrete category over a category *X*. We say that (*C*, U) is an **amnestic category** iff U is an amnestic functor.

Definition 2.1.45
Let F be a functor from a category *C* to a category *D*. We say that F is **topological** iff for each *D*-cone (A ∈ Ob(*D*), {f_i : A ⟶ F(B(i)) ∈ Hom(*D*) | B(i) ∈ Ob(*C*), for each i ∈ Ob(*I*)}) there exists a unique *C*-cone (A′ ∈ Ob(*C*), {f_i': A′ ⟶ B(i) ∈ Hom(*C*) | for each i ∈ Ob(*I*)}) such that for any other *C*-cone (D ∈ Ob(*C*), {g_i : D ⟶ B(i) ∈ Hom(*C*) | for each i ∈ Ob(*I*)}) and each *D*-morphism h : F(D) ⟶ F(A′) such that F(g_i) = h ∘ F(f_i') holds for all i there exists a unique *C*-morphism h′ : D ⟶ A′ such that g_i = h′ ∘ f_i' holds for all i and h = F(h′), and moreover whenever the *D*-cone (A ∈ Ob(*D*), {f_i : A ⟶ F(B(i)) ∈ Hom(*D*) | B(i) ∈ Ob(*C*), for each i ∈ Ob(*I*)}) is a limit of D ∘ F for an arbitrary diagram D : *I* ⟶ *C* over the category *C*, then the *C*-cone (A′ ∈ Ob(*C*), {f_i': A′ ⟶ B(i) ∈ Hom(*C*) | for each i ∈ Ob(*I*)}) is a limit of D such that F[(A′ ∈ Ob(*C*), {f_i': A′ ⟶ B(i) ∈ Hom(*C*) | for each i ∈ Ob(*I*)})] = (A ∈ Ob(*D*), {f_i : A ⟶ F(B(i)) ∈ Hom(*D*) | B(i) ∈ Ob(*C*), for each i ∈ Ob(*I*)}).

Definition 2.1.46
Let (C, U) be a concrete category over X. We say that (C, U) is a **topological category over** X iff U is topological. If (C, U) is a concrete category (over **Set**) then (C, U) is called simply a **topological category**.

Definition 2.1.47
Let C be a category. We say that category C is **wellpowered** iff for each A ∈ Ob(C) the class of its (pairwise non-isomorphic) subobjects is a set. Dually, we say that category C is **co-wellpowered** iff for each A ∈ Ob(C) the class of it′s (pairwise non-isomorphic) quotient objects is a set.

Definition 2.1.48
Let C be a category with objects (called 0-cells) A, B, C,... and morphisms (called 1-cells) f, g, h,... A **2-category** 2-C on the category C has, additionally, a class of 2-cells $\alpha : f \longrightarrow g$ with domain f and codomain g, where $f, g : A \longrightarrow B$ are parallel morphisms in the category C. These 2-cells have two different kinds of composition. Let $\alpha : f \longrightarrow g$ and $\alpha' : f' \longrightarrow g'$ be 2-cells, where $f, g : A \longrightarrow B$ and $f', g' : B \longrightarrow C$ are 1-cells. Then there is a horizontal composition of 2-cells $\alpha \circ \alpha' : f \circ f' \longrightarrow g \circ g'$. We require that 2-cells form a category under this horizontal composition. This means that for each object B there is an identity 2-cell $1_{1B} : 1_B \longrightarrow 1_B$ such that for each 2-cells $\alpha : f \longrightarrow g$, where $f, g : A \longrightarrow B$, and $\alpha' : f' \longrightarrow g'$, where $f', g' : B \longrightarrow C$, $\alpha \circ 1_{1B} = \alpha$ and $1_{1B} \circ \alpha' = \alpha'$ hold. Horizontal composition of 2-cells is associative due to associativity of composition of morphisms (1-cells) in C. Second kind of composition of 2-cells is called a vertical composition, denoted by •. Let $\alpha : f \longrightarrow g$ and $\beta : g \longrightarrow h$ be 2-cells, where $f, g, h : A \longrightarrow B$ are parallel 1-cells. Then a vertical composition of α and β is given by $\alpha \bullet \beta : f \longrightarrow h$ and is associative. There are also vertical identity 2-cells $1_f : f \longrightarrow f$, for each 1-cell $f : A \longrightarrow B$, for this composition. Moreover, there are two additional axioms relating the horizontal compositon to the vertical one. First, we require that the horizontal composition of two vertical identities is itself a vertical identity. Let $1_f : f \longrightarrow f$, where $f : A \longrightarrow B$, and $1_g : g \longrightarrow g$, where $g : B \longrightarrow C$, be (horizontally composible) vertical identities. Then $1_f \circ 1_g = 1_{fg}$, where 1_{fg} is the vertical identity on $f \circ g : A \longrightarrow C$. Second, let $\alpha : f \longrightarrow g$, $\beta : g \longrightarrow h$, where $f, g, h : A \longrightarrow B$ are parallel 1-cells, and $\alpha' : f' \longrightarrow g'$, $\beta' : g' \longrightarrow h'$, where $f', g', h' : B \longrightarrow C$ are parallel 1-cells, be 2-cells. Then the following equation must be satisfied: $(\alpha \circ \alpha') \bullet (\beta \circ \beta') = (\alpha \bullet \beta) \circ (\alpha' \bullet \beta') : f \circ f' \longrightarrow h \circ h'$, where $f \circ f' : A \longrightarrow C$ and $h \circ h' : A \longrightarrow C$ are the compositions of 1-cells f, f′ and h, h′ respectively.

2.2 (FUZZY) SET-THEORETIC AND ALGEBRAIC BASICS

The concept of **fuzzy set** was introduced by **Lotfi A. Zadeh** in 1965 in [ref. 30] as a generalization (extension) of the classical concept of a set. Zadeh defined fuzzy sets as follows:

Definition 2.2.1
Let X be an ordinary (crisp) set. Then a fuzzy set A on the set X is defined as a (membership) function $A : X \longrightarrow [0, 1]$.

An important generalization of the concept was done by **Joseph A. Goguen**, who was a student of Zadeh, in 1967 in [ref. 6]. Goguen´s generalization consists in the fact that he considered ordered sturctures beyond the unit interval, in which the membership functions take values. His definition of fuzzy sets (L-fuzzy sets) is as follows:

Definition 2.2.2
An **L-fuzzy set** A on a set X is a function $A : X \longrightarrow L$, where X is an ordinary (crisp) set and L is an ordered structure (set).

Usually it is required that L be at least a poset [def. 2.2.3] or a lattice [def. 2.2.8].
The structure of L in the previous definition is actually one of the main criteria for classification of fuzzy sets. So now we present some of the algebraic structures which we will be working with.

Definition 2.2.3
Let P be a set and let $\leq$ be a binary relation over P such that the following conditions hold for all a, b and $c \in P$:
1) $a \leq a$ (reflexivity)
2) if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)
3) if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity).
Then $\leq$ is called a **partial order** and the pair $(P, \leq)$ is called a **partially ordered set** (or a **poset**).

Definition 2.2.4
Let S be a set and let $* : S \times S \longrightarrow S$ be an associative binary operation on S. Then the pair $(S, *)$ is called a **semigroup**.

Definition 2.2.5
Let $(M, *)$ be a semigroup. We say that $(M, *)$ is a **monoid** iff there exist $e \in M$ such that $a * e = e * a = a$ holds for all $a \in M$ (e is called the **unit** element of the monoid).
A monoid $(M, *)$ is called **commutative** (or Abelian) iff $x * y = y * x$ holds for all $x, y \in M$.
A structure $(M, \leq, *)$, where $\leq$ is a partail order is called an **ordered monoid** iff the following conditions hold:
1) if $x \leq y$ then $z * x \leq z * y$
2) if $x \leq y$ then $x * z \leq y * z$, for all $x, y, z \in M$.

Definition 2.2.6
Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. Then $p \in P$ is called:
1) An **upper bound** of A iff $a \leq p$ holds for all $a \in A$.
2) A **lower bound** of A iff $p \leq a$ holds for all $a \in A$.
3) The **greatest element** of A iff $a \leq p$ holds for all $a \in A$ and $p \in A$.
4) The **least element** of A iff $p \leq a$ holds for all $a \in A$ and $p \in A$.

Definition 2.2.7
Let $(P, \leq)$ be a partially ordered set and let $A \subseteq P$. Then $p \in P$ is called:
1) The **supremum** of A iff p is the least element of the set of all upper bounds of A (shortly: p is the least upper bound of A).
2) The **infimum** of A iff p is the greatest element of the set of all lower bounds of A (shortly: p is the greatest lower bound of A).

Definition 2.2.8 (algebraic)
Let L be a set and let $\wedge$ and $\vee$ be binary operations on L satisfying the following axioms for all $x, y, z \in L$:
1) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, $x \vee (y \vee z) = (x \vee y) \vee z$ (associativity)
2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$ (commutativity)
3) $x \wedge (x \vee y) = x$, $x \vee (x \wedge y) = x$ (absorption).
Let us define the **canonical ordering** of L as follows:
$x \leq y$ iff $x \wedge y = x$ iff $x \vee y = y$. Then the relation $\leq$ is a **partial order**.
Then the quadruple $(L, \wedge, \vee, \leq)$ is called a **lattice**.

Definition 2.2.9 (set-theoretic)
Let $(L, \leq)$ be a poset. We say that $(L, \leq)$ is a **lattice** iff the following conditions hold for all $x, y \in L$:
1) The set $\{x, y\}$ has a supremum in L, denoted by $x \vee y$
2) The set $\{x, y\}$ has an infimum in L, denoted by $x \wedge y$.

Remark: definitions 2.2.8 and 2.2.9 are equivalent.

Definition 2.2.10
Let L be a lattice. We say that L is a **complete lattice** if for every subset $A \subseteq L$ the following conditions hold :
1) A has a supremum in L, denoted by $\bigvee A$
2) A has an infimum in L, denoted by $\bigwedge A$.

Definition 2.2.11
Let $(M, \leq, *)$ be an ordered monoid. We say that $(M, \leq, *)$ is a **lattice-ordered monoid** iff the following properties hold:
1) $(M, \leq)$ is a lattice
2) $x * (y \vee z) = (x * y) \vee (x * z)$ and $(x \vee y) * z = (x * z) \vee (y * z)$ hold for all $x, y, z \in M$.
A lattice-ordered monoid $(M, \leq, *)$ is called a **completely lattice-ordered monoid** iff the following properties hold:
3) $(M, \leq)$ is a complete lattice
4) $x * (\bigvee_i y_i) = \bigvee_i (x * y_i)$ holds for all $x, y \in M$, $i \in I$ some index set.
An ordered monoid $(M, \leq, *)$ is called **integral** iff the unit element of $(M, *)$ is also the universal upper bound of $(M, \leq)$.

Definition 2.2.12
Let L be a lattice. We say that L is a **distributive lattice** if the following conditions of distributivity hold for all $x, y, z \in L$:
1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Definition 2.2.13
Let L be a complete lattice. Then L is called a **completely distributive lattice** iff for any doubly indexed family $\{x_{j,k} \mid j \in J, k \in K_j\}$ of elements of L the following equation holds:

$$\bigwedge_j \bigvee_k x_{j,k} = \bigvee_f \bigwedge_j x_{j,f(j)}$$

for all $j \in J$, $k \in K_j$, $f \in F$, where F is a set of choice functions f choosing for each index $j \in J$ some index $f(j) \in K_j$.

Definition 2.2.14
Let L be a lattice. We say that L is a **bounded lattice** if there exist the greatest and the least element (with respect to $\leq$) in L (denoted by 1 and 0, or $\top$ and $\bot$, respectively).

Definition 2.2.15
Consider an algebra $\mathbf{L} = (L, \wedge, \vee, *, \longrightarrow, 0, 1)$. We say that **L** is a **residuated lattice** iff the following conditions hold:
1) $(L, \wedge, \vee, 0, 1)$ is a lattice with the greatest element 1 and the least element 0 (with respect to the lattice ordering $\leq$).
2) $(L, *, 1)$ is a commutative monoid.
3) $*$ and $\longrightarrow$ form an adjoint pair, i.e. $z \leq (x \longrightarrow y)$ iff $x * z \leq y$ holds for all $x, y, z \in L$.

Definition 2.2.16
Let H be a bounded lattice with the gratest element 1 and the least element 0 and let $\longrightarrow$ be a binary operation on H satisfying the following conditions for all $x, y, z \in H$:
1) $x \longrightarrow x = 1$
2) $x \wedge (x \longrightarrow y) = x \wedge y$
3) $y \wedge (x \longrightarrow y) = y$
4) $x \longrightarrow (y \wedge z) = (x \longrightarrow y) \wedge (x \longrightarrow z)$.
Then we say that H is a **Heyting algebra**. An element $x \longrightarrow y$ of H is called the **relative pseudo-complement** of x with respect to y.
A Heyting algebra is **complete** iff H is a complete lattice.

Definition 2.2.17
Let $\mathbf{L} = (L, \wedge, \vee, *, \longrightarrow, 0, 1)$ be a residuated lattice. Then **L** is called an **MV-algebra** iff $(x \longrightarrow y) \longrightarrow y = x \vee y$ holds for all $x, y \in L$.
An MV-algebra is **complete** iff L is a complete lattice.

Definition 2.2.18
Consider the triple $M = (L, \leq, *)$. We say that $M = (L, \leq, *)$ is a **GL-monoid** if the following conditions hold:
1) M is a lattice-ordered commutative monoid.
2) $(L, \leq)$ is a complete and bounded lattice.
3) The greatest element in L (with respect to $\leq$) is the unit element of the semigroup $(L, *)$.
4) The least element in L (with respect to $\leq$) is the zero element of the semigroup $(L, *)$.
5) For all $a, b \in L$ there is a $c \in L$ such that $a = b * c$ (divisibility).
6) The semigroup operation $*$ has an adjoint, i.e. a residuation $\longrightarrow$ such that $a * b \leq c$ iff $a \leq b \longrightarrow c$ holds for all $a, b, c \in L$.

Definition 2.2.19
Consider a structure $\mathbf{Q} = (Q, \leq, *)$. We say that **Q** is a **quantale** if the following conditions hold:
1) $(Q, \leq)$ is a complete lattice
2) $(Q, *)$ is a semigroup satisfying the following additional properties:
2.1) $(\bigvee_i x_i) * y = \bigvee_i (x_i * y)$
2.2) $y * (\bigvee_i x_i) = \bigvee_i (y * x_i)$ for all $x_i, y \in Q$, $i \in I$ any index set.
Remark:
1) Each quantale has, because of the completness of its lattice structure, a universal lower bound $\perp$ which is the zero element of a semigroup $(Q, *)$. A quantale also has a universal upper bound $\top$, but this need not be the unit element of the semigroup.
2) Every quantale is **left-** and **right-residuated**, i.e. there exist binary operations $\longrightarrow_l$ and $\longrightarrow_r$ satisfying the adjointness conditions:
$x * y \leq z$ iff $x \leq y \longrightarrow_l z$
$x * y \leq z$ iff $y \leq x \longrightarrow_r z$

Definition 2.2.20
A quantale is called **commutative** iff $*$ is commutative operation.
A quantale is called **unital** iff it has an identity element for the operation $*$.
An element e of a quantale Q is called a **quasi-unit** of Q iff $x * e * y = x * y$ holds for all $x, y \in Q$.
A quantale Q is called **involutive** iff it is endowed with an order preserving involution $^{\bullet}$ provided with the property $(a * b)^{\bullet} = b^{\bullet} * a^{\bullet}$ for all $a, b \in Q$.
We say that $\top$ is **right-extensive** resp. **left-extensive** iff $x \leq x * \top$ resp. $x \leq \top * x$ hold for all $x \in Q$. If $\top$ is right-extensive and left-extensive, then $\top$ is called **extensive**.
A quantale Q is called (strictly) **right-sided** resp. (strictly) **left-sided** iff $x * \top \leq (=)\ x$ resp. $\top * x \leq (=)\ x$ hold for all $x \in Q$. Quantales which are (strictly) right-sided and (strictly) left-sided are called (strictly) **two-sided**.
A quantale Q is called **right-symmetric** resp. **left-symmetric** iff $x * (y * z) = x * (z * y)$ resp. $(x * y) * z = (y * x) * z$ hold for all $x, y, z \in Q$.
A quantale Q is called **bi-symmetric** iff $(a * b) * (c * d) = (a * c) * (b * d)$ holds for all $a, b, c, d \in Q$.
A right-sided quantale Q is called a **right Gelfand quantale** iff $(Q, *)$ is an idempotent semigroup, i.e. $x * x = x$ holds for all $x \in Q$.
A right-sided and right-symmetric quantale Q is called a **right GL-quantale** iff the following conditions hold:
i) $(x \longrightarrow_l x) * x = x$ for all $x \in Q$.
ii) the subquantale of all two-sided elements of Q is a GL-monoid.

Definition 2.2.21
Let L be a lattice. Then L is called a **projectorial lattice** iff each element $a \in L$ is equipped with a functor $a \sqcap (-) : L \longrightarrow L$ called a-projection, such that the following properties hold, for all $x \in L$:
1) $a \sqcap x \leq a$.
2) if $x \leq a$ then $a \sqcap x = x$.
3) $x \sqcap a = \bot$ iff $a \sqcap x = \bot$.
If in addition we are given a right adjoint functor $a \Downarrow (-)$ to $a \sqcap (-)$, then L is called a **projectale**.

Definition 2.2.22
Let A and B be sets. Then their **Cartesian product** $A \times B$ is defined as the set of all pairs (a, b) with $a \in A$ and $b \in B$, i.e. $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$.

Definition 2.2.23
Let A and B be nonempty sets. Then an **L-fuzzy relation** between A and B is defined as a function $A \times B \longrightarrow L$, where L is an ordered structure (set).

Definition 2.2.24
Let $R: A \times B \longrightarrow L$ and $S: B \times C \longrightarrow L$ be L-fuzzy relations.
Then their (**sup-min-**) **composition** $R \circ S$ is defined as follows:
$(R \circ S)(a, c) = \bigvee_b \{R(a, b) \wedge S(b, c)\}$, where $a \in A$, $b \in B$ and $c \in C$.

3. CATEGORIES OF FUZZY SETS

3.1 CATEGORIES OF FUZZY SETS (AS OBJECTS) AND (FUZZY) FUNCTIONS (AS MORPHISMS)

Category S(V) (J. A. Goguen, 1969) [ref. 7]

Objects:
Objects are V-sets, i.e. the pairs (X, A), where X is a crisp set and $A : X \longrightarrow V$ is a function. Here $V = (P, \leq)$ is fixed partially ordered set [def. 2.2.3].

Morphisms:
Let (X, A) and (Y, B) be objects. Morphisms $f : (X, A) \longrightarrow (Y, B)$ are defined as the functions $f : X \longrightarrow Y$ such that $A(x) \leq B(f(x))$ holds for all $x \in X$.

Composition of morphisms:
Let $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ be morphisms. Then their composition $f \circ g : (X, A) \longrightarrow (Z, C)$ is defined as the composition of functions. Let $h = f \circ g$, then $h \in Hom((X, A), (Z, C))$ due to transitivity of $\leq$.

Associativity of composition:
Let $f : (X, A) \longrightarrow (Y, B)$, $g : (Y, B) \longrightarrow (Z, C)$ and $h : (Z, C) \longrightarrow (W, D)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to associativity of composition of functions.

Identity morphisms:
Let (X, A) be an object. Then the identity morphism on (X, A) is defined as the identity function $1_X : X \longrightarrow X$ and is an element of Hom((X, A), (X, A)) due to reflexivity of $\leq$.

Main results (properties):

• If V has just one point, then **S(V)** ≅ **Set**.

• If V = {0,1}, where $0 < 1$, then **S(V)** is the category of pairs (X, A) of sets, where $A \subseteq X$.

• If V is complete, then **S(V)** is a complete category [def. 2.1.29].

• If V is complete, then **S(V)** is a cocomplete category [def. 2.1.29].

• If V is complete, then **S(V)** has an exponential object [def. 2.1.22].

• If V is complete, then **S(V)** has a natural number object [def. 2.1.36].

Remark:
There are (at least) another four categories which are based on the category **S(V)**. Objects, morphisms, composition of morphisms and identity morphisms in these categories are defined in the same manner as in **S(V)**. Differences between the particular categories are given by different target structures of membership functions of their objects (i.e. fuzzy sets). The categories are as follows:

Category Set(L) (J. A. Goguen, 1974) [ref. 8, cited according to ref. 10, 25, 26]

Target structure of membership functions is a completely distributive lattice [def. 2.2.13].

Main results (properties):
• Category (categories of the form **Set(L)**) **Set(L)** is characterized by these axioms:

1. **Set(L)** has an initial object 0 and a terminal object 1 [def. 2.1.19].

2. **Set(L)** has associative images [def. 2.1.35].

3. **Set(L)** is disjointedly cdl-ordered [def. 2.1.32].

4. **Set(L)** has coproducts [def. 2.1.21] which are disjoint unions.

5. **Set(L)** has an atomic monic projective generator P [def. 2.1.33].

6. P + P is not isomorphic to P.

• Category **Set(L)** is a quasitopos [def. 2.1.31].

• Category **Set(L)** is a topos [def. 2.1.30] iff L is {0}, in which case **Set(L)** ≅ **Set**.

• Category **Set(L)** is a Cartesian closed category [def. 2.1.23] iff L is a complete Heyting algebra [def. 2.2.16].

• Category **Set(L)** is balanced [def. 2.1.10] iff L is a one point lattice [def. 2.2.8] with 0 = 1.

Category Set(Q) (L. N. Stout, 1995) [ref. 27]

Target structure of membership functions is a quantale [def. 2.2.19].
Stout studies previous Goguen´s category **Set(L)**. He shows what happens when we add in other operations on the lattice L. For details see [ref. 27].

Category Set(L) (L. N. Stout, 1995) [ref. 27]

Target structure of membership functions is a projectale [def. 2.2.21].
In [ref. 27, p. 230] Stout gives the following upshot:
"Category **Set(L)** has a propositional logic which can be thought of as being based on quantum logic. When we move up to internal first order logic we get either a fragment because of the absence of existential quantifiers or a much more complicated structure because the existential quantifier is the left adjoint to a different substitution functor than the universal quantification is right adjoint to. The category does have the higher order internal logic given by weak representation of unbalanced subobjects, as do all categories of lattice valued fuzzy sets."

Category $\mathbf{Set_{dc}(H)}$ (O. Wyler, 1995) [ref. 28]

Index **dc** in $\mathbf{Set_{dc}}$ refers to the fact that objects are **d**iscrete fuzzy sets and morphisms are **c**risp mappings.
Target structure of membership functions is a complete Heyting algebra [def. 2.2.16].

Main results (properties):

- Category $\mathbf{Set_{dc}(H)}$ is a topological category over [def. 2.1.46] **Set** with small fibres [def. 2.1.42].

- Category $\mathbf{Set_{dc}(H)}$ is a complete category [def. 2.1.29].

- Category $\mathbf{Set_{dc}(H)}$ is a cocomplete category [def. 2.1.29].

- Category $\mathbf{Set_{dc}(H)}$ is a quasitopos [def. 2.1.31].

Category Set(JCPos) (S. A. Solovyov, 2006) [ref. 22]

Category **Set(JCPos)** is a concrete category over [def. 2.1.41] the product category [def. 2.1.37] **Set** × **JCPos**. It is the category of lattice-valued subsets of sets. Category **JCPos** is the category of complete lattices [def. 2.2.10] and join-preserving maps.

Objects:
Objects are lattice-valued subsets of sets, i.e. maps $X : X' \longrightarrow L_X$, where $(X', L_X) \in \mathrm{Ob}(\mathbf{Set} \times \mathbf{JCPos})$, i.e. X' is a set and L_X is a complete lattice [def. 2.2.10].

Morphisms:
Let $X : X' \longrightarrow L_X$ and $Y : Y' \longrightarrow L_Y$ be objects. Then morphisms from X to Y are pairs $f = (f_S, f_J)$ such that $(X \circ f_J)(x) \leq (f_S \circ Y)(x)$ for all $x \in X'$, where $f_S \in \mathrm{Hom}_{\mathbf{Set}}(X', Y')$ and $f_J \in \mathrm{Hom}_{\mathbf{JCPos}}(L_X, L_Y)$.

Composition of morphisms:
Let $f = (f_S , f_J)$ and $g = (g_S , g_J)$ be morphisms such that $\mathrm{cod}(f_S) = \mathrm{dom}(g_S)$ and $\mathrm{cod}(f_J) = \mathrm{dom}(g_J)$. Then their composition is defined as usual product category composition, i.e. $f \circ g = (f_S \circ g_S , f_J \circ g_J)$.

Associativity of composition:
Let $f = (f_S , f_J)$, $g = (g_S , g_J)$ and $h = (h_S , h_J)$ be morphisms such that $\mathrm{cod}(f_S) = \mathrm{dom}(g_S)$, $\mathrm{cod}(g_S) = \mathrm{dom}(h_S)$, $\mathrm{cod}(f_J) = \mathrm{dom}(g_J)$ and $\mathrm{cod}(g_J) = \mathrm{dom}(h_J)$. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to associativity of composition of morphisms in **Set** and in **JCPos**.

Identity morphisms:
Let $X : X' \longrightarrow L_X$ be an object and let $1_{X'}$ and 1_{L_X} be the identity morphisms of X' in **Set** and of L_X in **JCPos** respectively. Identity morphism 1_X is defined as the pair $(1_{X'} , 1_{L_X})$, i.e. the usual product category identity morphism.

Main results (properties):

• Category **Set(JCPos)** is an amnestic category [def. 2.1.44].

• Category **Set(JCPos)** is a topological category [def. 2.1.46].

• Category **Set(JCPos)** is a complete category [def. 2.1.29].

• Category **Set(JCPos)** is a cocomplete category [def. 2.1.29].

• Category **Set(JCPos)** is a wellpowered category [def. 2.1.47].

• Category **Set(JCPos)** is a co-wellpowered category [def. 2.1.47].

• Category **Set(JCPos)** is not a cartesian closed category [def. 2.1.23].

Category X(A) (S. A. Solovyov, 2006) [ref. 23, 24]

Category **X(A)** is a generalization of the category **Set(JCPos)** [page 28]. Suppose (**A**, U) is a concrete category over **X** [def. 2.1.41] such that the following conditions hold:
1) **A** is a 2-category [def. 2.1.48]
2) U is an adjoint functor [def. 2.1.39].

Objects:
Objects are triples (X, α, A), where $X \in Ob(\mathbf{X})$, $A \in Ob(\mathbf{A})$ and $\alpha : X \longrightarrow U(A) \in Mor(\mathbf{X})$.
Convention: Objects will be identified with $\alpha : X \longrightarrow U(A)$.

Morphisms:
Let $\alpha : X \longrightarrow U(A)$ and $\beta : Y \longrightarrow U(B)$ be objects. Let $f : X \longrightarrow Y \in Hom(\mathbf{X})$ and $g : A \longrightarrow B \in Hom(\mathbf{A})$. Then morphisms from $\alpha : X \longrightarrow U(A)$ to $\beta : Y \longrightarrow U(B)$ are the $\mathbf{X} \times \mathbf{A}$ morphisms (f, g) such that $\alpha \circ U(g) \sqsubseteq f \circ \beta$, where $\sqsubseteq$ is defined as follows:
Take any two objects $\alpha_1 : X \longrightarrow U(A)$ and $\alpha_2 : X \longrightarrow U(A)$. Due to adjointness of U we have $\alpha_1' : F(X) \longrightarrow A$ and $\alpha_2' : F(X) \longrightarrow A \in Hom(\mathbf{A})$, where F is an adjoint functor of U.
Define $\alpha_1 \sqsubseteq \alpha_2$ iff there exist a 2-cell $\tau : \alpha_1' \longrightarrow \alpha_2'$.

Composition of morphisms:
The fact that **X**(**A**)-morphisms are closed under composition can be easily checked with the help of the following condition:
Suppose $f : X \longrightarrow Y$, $\alpha : Y \longrightarrow U(A)$, $\beta : Y \longrightarrow U(A)$ and $U(\varphi) : U(A) \longrightarrow U(B)$ are **X**-morphisms. If $\alpha \sqsubseteq \beta$, then $f \circ \alpha \sqsubseteq f \circ \beta$ and $\alpha \circ U(\varphi) \sqsubseteq \beta \circ U(\varphi)$.

Associativity of composition:
Composition of morphisms is associative due to associativity of composition of 2-cells in 2-category **A**.

Identity morphisms:
Let $\alpha : X \longrightarrow U(A)$ be an object and let 1_X be the identity morphism of the object X in the category **X** and 1_A be the identity morphism of the object A in the category **A**. Then the identity morphism of $\alpha : X \longrightarrow U(A)$ is defined as the $\mathbf{X} \times \mathbf{A}$ identity morphism $(1_X, 1_A)$ such that $\alpha \circ U(1_A) \sqsubseteq 1_X \circ \alpha$, which holds evidently due to the fact that $\alpha \sqsubseteq \alpha$.

Main results (properties):
- Category **X**(**A**) is an amnestic category [def. 2.1.44].

- Category **X**(**A**) is a topological category over [def. 2.1.46] the product category [def. 2.1.37] $\mathbf{X} \times \mathbf{A}$.

- In [ref. 23, p. 853] Solovyov gives the following upshot:
  "By analogy with the notion of a lattice-valued set one can introduce the notion of an **A**-valued object in a category as follows: Given a concrete category (**A**, U) over **X**, define an A-valued object in the category **X** to be an U-structured arrow $f : X \longrightarrow U(A)$. In such a way **X(A)** becomes the category of **A**-valued objects. The definition of **X(A)** allows one to consider different realizations of the category **A**, for example, such categories as the category **Top** or the category **Grp**."

3.2 CATEGORIES OF FUZZY SETS (AS OBJECTS) AND (FUZZY) RELATIONS (AS MORPHISMS)

Category S(L) (J. A. Goguen, 1967) [ref. 6]

Objects:
Objects are L-fuzzy sets, i.e. the pairs (X, A), where X is a crisp set and $A : X \rightarrow L$ is an L-fuzzy subset of X. Here L is a complete lattice-ordered semigroup [def. 2.2.4, 2.2.5, 2.2.11] $(L, \wedge, \vee, *)$ satisfying the complete distributive laws:
$a * (\bigvee_i b_i) = \bigvee_i (a * b_i)$ and $(\bigvee_i b_i) * a = \bigvee_i (b_i * a)$, for all $a, b \in L$, $i \in I$.

Morphisms:
Let $\mathbf{A} = (X, A)$ and $\mathbf{B} = (Y, B)$ be objects. Let p be an arbitrary fixed element (of some crisp set). Then we let A also denote the L-relation $\{p\} \times X \longrightarrow L$ defined by $A(p, x) = A(x)$ for all $x \in X$. Let R be an L-fuzzy relation [def. 2.2.23] from X to Y. Then the composition $A \circ R$ is given by $(A \circ R)(p, y) = \bigvee_x \{A(p, x) * R(x, y)\}$ (so called sup-$*$-composition), for all $x \in X$ and $y \in Y$. Finally, morphisms from **A** to **B** are triples (A, B, R), where R is an L-fuzzy relation such that $(A \circ R)(p, y) \le B(y)$ holds for all $y \in Y$.

Composition of morphisms:
Let $(A, B, R) \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ and $(B, C, S) \in \mathrm{Hom}(\mathbf{B}, \mathbf{C})$. Then their composition $(A, B, R) \circ (B, C, S)$ is defined to be $(A, C, R \circ S)$ where $R \circ S$ is a sup-$*$-composition of L-fuzzy relations R and S, i.e. $(R \circ S)(x, z) = \bigvee_y \{R(x, y) * S(y, z)\}$ holds for all $x \in X$, $y \in Y$ and $z \in Z$. $(A, C, R \circ S) \in \mathrm{Hom}(\mathbf{A}, \mathbf{C})$ iff $(A \circ (R \circ S))(p, z) \le C(z)$ holds for all $z \in Z$. This holds due to the following property [ref. 6, p. 168] (and due to the transitivity of $\le$):
$A \circ (R \circ S) = (A \circ R) \circ S$.

Associativity of composition:
Let $(A, B, R) \in \mathrm{Hom}(\mathbf{A}, \mathbf{B})$, $(B, C, S) \in \mathrm{Hom}(\mathbf{B}, \mathbf{C})$ and $(C, D, T) \in \mathrm{Hom}(\mathbf{C}, \mathbf{D})$. Then $(A, C, R \circ S) \circ (C, D, T) = (A, B, R) \circ (B, D, S \circ T)$ holds due to the associativity of composition of L-fuzzy relations:

$$\begin{aligned}((R \circ S) \circ T)(x, w) &= \textstyle\bigvee_z \{\bigvee_y \{R(x, y) * S(y, z)\} * T(z, w)\} \\ &= \textstyle\bigvee_{y, z} \{R(x, y) * S(y, z) * T(z, w)\} \\ &= \textstyle\bigvee_y \{R(x, y) * \bigvee_z \{S(y, z) * T(z, w)\}\} \\ &= (R \circ (S \circ T))(x, w)\end{aligned}$$

holds for all L-fuzzy relations R from X to Y, S from Y to Z, T from Z to W and for all $x \in X$, $y \in Y$, $z \in Z$ and $w \in W$.

Identity morphisms:
Let $\mathbf{A} = (X, A)$ be an object. Then the identity morphism on $\mathbf{A} = (X, A)$ is defined as an L-fuzzy relation $(A, A, 1_X) \in \mathrm{Hom}(\mathbf{A}, \mathbf{A})$, where 1_X is the identity function from X to X.

Main results (properties):
- Category **S(L)** is thus the category of L-fuzzy sets (as objects) and L-fuzzy relations (as morphisms). Goguen also considered (also in [ref. 6]) the subcategory [def. 2.1.5] **F(L)** whose morphisms involve only functional relations. Category **F(L)** is thus the category of L-fuzzy sets (as objects) and L-fuzzy functions (as morphisms).

Category Fuz(H) (M. Eytan, 1981) [ref. 5]

Objects:
Objects are H-valued fuzzy sets, i.e. the pairs (X, A), where X is a crisp set and $A : X \to H$ is a map from X to a complete Heyting algebra H [def. 2.2.16].

Morphisms:
Let (X, A) and (Y, B) be H-valued fuzzy sets. Morphisms $f : (X, A) \longrightarrow (Y, B)$ between these fuzzy sets are defined as maps $f : X \times Y \longrightarrow H$ satisfying:
1) $f(x, y) \leq A(x)$, $f(x, y) \leq B(y)$ for all $x \in X$, $y \in Y$.
2) $A(x) \leq \bigvee_{y \in Y} f(x, y)$ for all $x \in X$.
3) $f(x, y) \wedge f(x, y') \leq e(y, y')$, where $e(y, y') = \top$ if $y = y'$ and $e(y, y') = \bot$ if $y \neq y'$ for all $x \in X$ and $y, y' \in Y$.

Composition of morphisms:
Let $f : (X, A) \longrightarrow (Y, B)$ and $g : (Y, B) \longrightarrow (Z, C)$ be morphisms. Then their composition is defined to be a map $f \circ g : X \times Z \longrightarrow H$ with $(f \circ g)(x, z) = \bigvee_{y \in Y} (f(x, y) \wedge g(y, z))$ for all $x \in X$, $z \in Z$.

Associativity of composition:
Let $f : (X, A) \longrightarrow (Y, B)$, $g : (Y, B) \longrightarrow (Z, C)$ and $h : (Z, C) \longrightarrow (W, D)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the following identities:

$$
\begin{aligned}
((f \circ g) \circ h)(x, w) &= \textstyle\bigvee_{z \in Z} ((f \circ g)(x, z) \wedge h(z, w)) \\
&= \textstyle\bigvee_{z \in Z} (\bigvee_{y \in Y} (f(x, y) \wedge g(y, z)) \wedge h(z, w)) \\
&= \textstyle\bigvee_{z \in Z} \bigvee_{y \in Y} (f(x, y) \wedge g(y, z) \wedge h(z, w)) \\
&= \textstyle\bigvee_{y \in Y} \bigvee_{z \in Z} (f(x, y) \wedge g(y, z) \wedge h(z, w)) \\
&= \textstyle\bigvee_{y \in Y} (f(x, y) \wedge \bigvee_{z \in Z} (g(y, z) \wedge h(z, w))) \\
&= \textstyle\bigvee_{y \in Y} (f(x, y) \wedge (g \circ h)(y, w)) \\
&= (f \circ (g \circ h))(x, w), \text{ for all } x \in X,\ w \in W.
\end{aligned}
$$

Identity morphisms:
Let (X, A) be an object. Then the identity morphism on (X, A) is defined to be the map $i : X \times X \longrightarrow H$ with $i(x, x') = A(x) \wedge A(x') \wedge e(x, x')$ for all $x, x' \in X$.

Main results (properties):
- Eytan claimed in his 1981 paper [ref. 5] that the category **Fuz(H)** is a topos [def. 2.1.30], but it is not. According to Stout´s 1991 paper [ref. 26], the crucial point seems to be that in the category **Fuz(H)** there is no graded equality relation available.

- According to Stout´s 1984 paper [ref. 25], the category **Fuz(H)** is a logos, i.e. a category in which first order theories have natural models.

3.3 CATEGORIES OF FUZZY RELATIONS (AS OBJECTS) AND (FUZZY) FUNCTIONS (AS MORPHISMS)

L. N. Stout, 1991 [ref. 26]

Objects:
Objects are pairs (A, α), where A is a set and $\alpha : A \times A \longrightarrow [0, 1]$ is a map satisfying $\alpha(a, a') = \alpha(a', a)$, for all $a, a' \in A$.

Morphisms:
Let (A, α) and (B, β) be objects. Morphisms between (A, α) and (B, β) are defined as functions $f : A \longrightarrow B$ satisfying $\alpha(a, a') \leq \beta(f(a), f(a'))$, for all $a, a' \in A$.

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms. Then their composition $f \circ g : (A, \alpha) \longrightarrow (C, \gamma)$ is defined as the composition of functions. Let $h = f \circ g$, then $h \in \mathrm{Hom}((A, \alpha), (C, \gamma))$ due to transitivity of $\leq$.

Associativity of composition:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$, $g : (B, \beta) \longrightarrow (C, \gamma)$ and $h : (C, \gamma) \longrightarrow (D, \delta)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to associativity of composition of functions.

Identity morphisms:
Let (A, α) be an object. Then the identity morphism on (A, α) is defined as the identity function $1_A : A \longrightarrow A$ and is an element of $\mathrm{Hom}((A, \alpha), (A, \alpha))$ due to reflexivity of $\leq$.

Main results (properties):
- This category is a cartesian closed category [def. 2.1.23].

- Products [def. 2.1.20] are given by $(A, \alpha) \times (B, \beta) = (A \times B, \alpha \wedge \beta)$.

- The exponential [def. 2.1.22] is given by $(B, \beta)^{(A, \alpha)} = (\{f : A \longrightarrow B\}, \zeta)$, where $\zeta(f, g) = \inf\{\alpha(a, a') \Longrightarrow \beta(f(a), g(a'))\}$, where $a \Longrightarrow b = \max\{h \mid h \wedge a \leq b\}$.

Category Ω-Set (U. Höhle, L. N. Stout, 1991) [ref. 17]

Objects:
Objects are Ω-valued sets, i.e. the pairs (X, E), where X is a set and $E : X \times X \longrightarrow \Omega$ is a map to a complete Heyting algebra Ω [def. 2.2.16] satisfying the following conditions:
1) $E(x, y) = E(y, x)$ (symmetry)
2) $E(x, y) \wedge E(y, z) \leq E(x, z)$ (transitivity),
for all $x, y, z \in X$.

Morphisms:
Let (X, E) and (Y, F) be objects. Then morphisms between (X, E) and (Y, F) are defined as the mappings $f : X \longrightarrow Y$ satisfying these conditions:
1) $F(f(x), f(x)) \leq E(x, x)$ (strictness)
2) $E(x_1, x_2) \leq F(f(x_1), f(x_2))$ (preservation of equality),
for all $x, x_1, x_2 \in X$.

Composition of morphisms:
Let $f : (X, E) \longrightarrow (Y, F)$ and $g : (Y, F) \longrightarrow (Z, G)$ be morphisms. Then their composition is the usual composition of functions and $f \circ g \in \mathrm{Hom}((X, E), (Z, G))$ due to transitivity of ≤.

Associativity of composition:
Let $f : (X, E) \longrightarrow (Y, F)$, $g : (Y, F) \longrightarrow (Z, G)$ and $h : (Z, G) \longrightarrow (W, H)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the associativity of composition of functions.

Identity morphisms:
Let (X, E) be an object. Then the identity morphism on (X, E) is defined as the usual identity function $1_{(X, E)} : X \longrightarrow X$ and $1_{(X, E)} \in \mathrm{Hom}_{\text{M-Set}}((X, E), (X, E))$ due to the reflexivity of ≤.

Main results (properties):
- The authors give particular results about the Kleisli category **$(\Omega\text{-Set})_T$** and the Eilenberg-Moore category **$(\Omega\text{-Set})^T$**. These results, as well as the concepts of Kleisli category and Eilenberg-Moore category, are out of the scope of this work already. For details see [ref. 17].

Category SM-Set (U. Höhle, L. N. Stout, 1991) [ref. 17]

Objects:
Let $M = (L, \leq, *)$ be an integral, commutative, completely lattice-ordered monoid with zero [def. 2.2.11] satisfying the following additional property:
If $a \in L$ is idempotent with respect to $*$ and if $A \subseteq L$ such that $a \leq \bigvee A$, then $a \leq \bigvee\{b * b \mid b \in A\}$.
Objects are strong M-valued sets, i.e. the pairs (X, E) where X is a set and $E : X \times X \longrightarrow L$ is a map satisfying the following conditions:
1) $E(x, x) * E(x, y) * E(y, y) = E(x, y)$ (strong strictness)
2) $E(x, y) = E(y, x)$ (symmetry)
3) $E(x, y) * E(y, z) \leq E(x, z)$ (strong transitivity),
for all $x, y, z \in X$.

Morphisms:
Let (X, E) and (Y, F) be objects. Then morphisms from (X, E) to (Y, F) are defined as the maps $f : X \longrightarrow Y$ satisfying the following conditions:
1) $F(f(x), f(x)) \leq E(x, x)$ (strictness)
2) $E(x_1, x_2) \leq F(f(x_1), f(x_2))$ (preservation of equality),
for all $x, x_1, x_2 \in X$.

Composition of morphisms:
Let $f : (X, E) \longrightarrow (Y, F)$ and $g : (Y, F) \longrightarrow (Z, G)$ be morphisms. Then their composition $f \circ g$ is defined as the usual composition of functions and $f \circ g \in \mathrm{Hom}_{\text{SM-Set}}((X, E), (Z, G))$ due to the transitivity of $\leq$.

Associativity of composition:
Let $f : (X, E) \longrightarrow (Y, F)$, $g : (Y, F) \longrightarrow (Z, G)$ and $h : (Z, G) \longrightarrow (W, H)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the associativity of composition of functions.

Identity morphisms:
Let (X, E) be an object. Then the identity morphism on (X, E) is defined as the usual identity function $1_{(X, E)} : X \longrightarrow X$ and $1_{(X, E)} \in \mathrm{Hom}_{\text{M-Set}}((X, E), (X, E))$ due to the reflexivity of $\leq$.

Main results (properties):
- Category **SM-Set** is a strong form of the preceding category **Ω-Set** [page 33].

- The authors give particular results about the Kleisli category **(SM-Set)$_T$**. These results, as well as the concept of Kleisli category, are out of the scope of this work already. For details see [ref. 17].

Category M-Set (U. Höhle, L. N. Stout, 1991) [ref. 17] and (U. Höhle, 1992) [ref. 14]

Objects:
Let $M = (L, \leq, *)$ be a GL-monoid [def. 2.2.18].
Objects are separated M-valued sets, i.e. the pairs (X, E), where X is a set and $E : X \times X \longrightarrow L$ is a map satisfying the following axioms:
1) $E(x, y) \leq E(x, x) \wedge E(y, y)$ (strictness)
2) $E(x, y) = E(y, x)$ (symmetry)
3) $E(x, y) * (E(y, y) \Longrightarrow E(y, z)) \leq E(x, z)$ (transitivity)
4) $E(x, x) \vee E(y, y) \leq E(x, y) \Longrightarrow x = y$ (separation), for all $x, y, z \in X$.

Morphisms:
Let (X, E) and (Y, F) be separated M-valued sets. Then morphisms between (X, E) and (Y, F) are mappings $f : X \longrightarrow Y$ satisfying the following axioms:
1) $F(f(x), f(x)) \leq E(x, x)$ (strictness)
2) $E(x, y) \leq F(f(x), f(y))$ (preservation of equality), for all $x, y \in X$.

Composition of morphisms:
Let $f : (X, E) \longrightarrow (Y, F)$ and $g : (Y, F) \longrightarrow (Z, G)$ be morphisms. Then their composition $f \circ g$ is defined as the usual composition of functions and $f \circ g \in Hom_{M\text{-}Set}((X, E), (Z, G))$ due to the transitivity of $\leq$.

Associativity of composition:
Let $f : (X, E) \longrightarrow (Y, F)$, $g : (Y, F) \longrightarrow (Z, G)$ and $h : (Z, G) \longrightarrow (W, H)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the associativity of composition of functions.

Identity morphisms:
Let (X, E) be an object. Then the identity morphism on (X, E) is defined as the usual identity function $1_{(X, E)} : X \longrightarrow X$ and $1_{(X, E)} \in Hom_{M\text{-}Set}((X, E), (X, E))$ due to the reflexivity of $\leq$.

Main results (properties):
- Category **M-Set** has an initial object [def. 2.1.19] $0 = (\emptyset, \emptyset)$ and a terminal object [def. 2.1.19] $1 = (L, \wedge)$, where L is an underlying lattice [def. 2.2.8] of the GL-monoid [2.2.18] and $\wedge$ is the meet operation in L.

- Category **M-Set** has set-indexed products [def. 2.1.20].

- Category **M-Set** has set-indexed coproducts [def. 2.1.21].

- Category **M-Set** has multiple equalizers [def. 2.1.17].

- Category **M-Set** has multiple coequalizers [def. 2.1.18].

- Category **M-Set** has inverse limits [def. 2.1.16].

- Category **M-Set** has direct limits [def. 2.1.16].

- Category **M-Set** is a complete category [def. 2.1.29].

- Category **M-Set** is a cocomplete category [def. 2.1.29].

Category $\mathbf{Set_{tc}(H)}$ (O. Wyler, 1995) [ref. 28]

Index **tc** in $\mathbf{Set_{tc}}$ refers to the fact that objects are **t**otally fuzzy sets (relations) and morphisms are **c**risp mappings.

Objects:
Objects are H-valued totally fuzzy sets, i.e. the pairs (A, α), where A is a crisp set and α is a fuzzy equality mapping $A \times A \longrightarrow H$, where H is a complete Heyting algebra [def. 2.2.16], satisfying these two conditions of symmetry and transitivity:
1) $\alpha(x, y) = \alpha(y, x)$
2) $\alpha(x, y) \wedge \alpha(y, z) \leq \alpha(x, z)$,
for all $x, y, z \in A$.

Morphisms:
Let (A, α) and (B, β) be H-valued totally fuzzy sets. Morphisms between (A, α) and (B, β) are defined as the crisp mappings $f : A \longrightarrow B$, satisfying $\alpha(x, y) \leq \beta(f(x), f(y))$, for all $x, y \in A$.

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms. Then their composition $f \circ g : (A, \alpha) \longrightarrow (C, \gamma)$ is a crisp mapping $f \circ g : A \longrightarrow C$, satisfying
$\alpha(x, y) \leq \gamma(g(f(x)), g(f(y)))$, for all $x, y \in A$.

Associativity of composition:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$, $g : (B, \beta) \longrightarrow (C, \gamma)$ and $h : (C, \gamma) \longrightarrow (D, \delta)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to associativity of composition of functions.

Identity morphisms:
Let (A, α) be an object. Then the identity morphism on (A, α) is defined as the identity function $1_A : A \longrightarrow A$ and is an element of Hom((A, α), (A, α)) due to reflexivity of ≤.

Main results (properties):
- Category $\mathbf{Set_{tc}(H)}$ is a concrete category over **Set** [def. 2.1.41].

- Category $\mathbf{Set_{tc}(H)}$ is a topological category over **Set** [def. 2.1.46].

- Category $\mathbf{Set_{tc}(H)}$ has the category $\mathbf{Set_{dc}(H)}$ [page 27] as a full concrete subcategory [def. 2.1.5, 2.1.41].

- Category $\mathbf{Set_{tc}(H)}$ is a cartesian closed category [def. 2.1.23], but not a quasitopos [def. 2.1.31].

Category $\mathbf{Set_{te}(H)}$ (O. Wyler, 1995) [ref. 28]

Index **te** in $\mathbf{Set_{te}}$ refers to the fact that objects are **t**otally fuzzy sets (relations) and morphisms are **e**xtensional maps.

Objects:
Objects are H-valued totally fuzzy sets, i.e. the pairs (A, α), where A is a crisp set and α is a fuzzy equality mapping $A \times A \longrightarrow H$, where H is a complete Heyting algebra [def. 2.2.16], satisfying the two conditions of symmetry and transitivity:
1) $\alpha(x, y) = \alpha(y, x)$
2) $\alpha(x, y) \wedge \alpha(y, z) \leq \alpha(x, z)$,
for all $x, y, z \in A$.

Morphisms:
Let (A, α) and (B, β) be H-valued totally fuzzy sets. Let f and g be morphisms between (A, α) and (B, β) as defined in the category $\mathbf{Set_{tc}(H)}$ [page 36], i.e. the crisp mappings $f : A \longrightarrow B$ and $g : A \longrightarrow B$, satisfying $\alpha(x, y) \leq \beta(f(x), f(y))$ and $\alpha(x, y) \leq \beta(g(x), g(y))$ respectively, for all $x, y \in A$. We say that the maps f and g are extensionally equal if the following condition holds: $\alpha(x, x) \leq \beta(f(x), g(x))$, for all $x \in A$.
Extensional equality is an equivalence relation. We define an extensional map $[f] : A \longrightarrow B$ as an equivalence class of a map $f : A \longrightarrow B$ for this relation. Finally, the extensional morphisms from (A, α) to (B, β) are defined as these extensional maps, i.e. the equivalence classes of extensionally equal morphisms from (A, α) to (B, β) in the sense of category $\mathbf{Set_{tc}(H)}$ [page 36].

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms in the sense of category $\mathbf{Set_{tc}(H)}$ [page 36]. Then their composition in category $\mathbf{Set_{te}(H)}$ is given by $[f] \circ [g] = [f \circ g]$.

Associativity of composition:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$, $g : (B, \beta) \longrightarrow (C, \gamma)$ and $h : (C, \gamma) \longrightarrow (D, \delta)$ be morphisms in the sense of the category $\mathbf{Set_{tc}(H)}$ [page 36]. Then the composition of morphisms in $\mathbf{Set_{te}(H)}$ is associative due to the equivalence of the following identities:
$([f] \circ [g]) \circ [h] = [f] \circ ([g] \circ [h])$
$([f \circ g]) \circ [h] = [f] \circ ([g \circ h])$
$[f \circ g \circ h] = [f \circ g \circ h]$.

Identity morphisms:
Let (A, α) be an object and let 1_A be the identity morphism on (A, α) in the sense of the category $\mathbf{Set_{tc}(H)}$ [page 36]. Then the identity morphism on (A,α) is defined as $[1_A]$, i.e. the equivalence class of the map 1_A for the extensional equality relation.

Main results (properties):
• Category $\mathbf{Set_{te}(H)}$ is a quasitopos [def. 2.1.31] with H-valued internal logic.

Category Q-Set (U. Höhle, 1998) [ref. 15]

Objects:
Q = (L, $\leq$, $*$) is a right GL-quantale [def. 2.2.20].
Objects are Q-valued sets, i.e. the pairs (A, α), where A is a crisp set and $\alpha : A \times A \longrightarrow L$ is a map satisfying these conditions:
1) $\alpha(x, x) * (\alpha(x, x) \longrightarrow_r \alpha(x, y)) = \alpha(x, y)) \leq \top * \alpha(y, y)$ (strictness)
2) $(\alpha(y, y) \longrightarrow_l \alpha(y, y)) * \alpha(x, y) \leq \alpha(y, x)$ (right symmetry)
3) $\alpha(x, y) * (\alpha(y, y) \longrightarrow_r \alpha(y, z)) \leq \alpha(x, z)$ (transitivity),
for all $x, y, z \in A$.

Morphisms:
Let (A, α) and (B, β) be objects. Then morphisms between (A, α) and (B, β) are mappings $f : A \longrightarrow B$ satisfying the following conditions:
1) $\alpha(x, y) \leq \beta(f(x), f(y))$ (preservation of equality)
2) $\alpha(x, x) \leq \beta(f(x), f(x))$ (strictness),
for all $x, y \in A$.

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms. Then their composition $f \circ g$ is defined as the usual composition of functions and $f \circ g \in Hom_{Q\text{-}Set}((A, \alpha), (C, \gamma))$ due to the transitivity of $\leq$.

Associativity of composition:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$, $g : (B, \beta) \longrightarrow (C, \gamma)$ and $h : (C, \gamma) \longrightarrow (D, \delta)$ be morphisms.
Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the associativity of composition of functions.

Identity morphisms:
Let **A** = (A, α) be an object. Then the identity morphism on **A** = (A, α) is defined as the usual identity function $1_{\mathbf{A}} : A \longrightarrow A$ and is an element of $Hom_{Q\text{-}Set}((A, \alpha), (A, \alpha))$ due to the reflexivity of $\leq$.

Main results (properties):
• Category **Q-Set** is a generalization of the category **M-Set** mentioned above [page 35].

Category CΩ-Set (V. Novák, I. Perfilieva, J. Močkoř, 1999) [ref. 20]

Objects:
Let $H = (\Omega, \vee, \wedge, \rightarrow)$ be a complete Heyting algebra [def. 2.2.16].
Let $\mathbf{A} = (A, \alpha)$, where A is a set and $\alpha : A \times A \longrightarrow \Omega$ is a function satisfying these conditions:
1) $\alpha(x, y) = \alpha(y, x)$ (symmetry)
2) $\alpha(x, y) \wedge \alpha(y, z) \leq \alpha(x, z)$ (transitivity),
for all $x, y, z \in A$, be an Ω-set, i.e. an object of the category **Ω-Set** defined below [page 45].
We define a singleton in **A** as a function $s : A \longrightarrow \Omega$ such that the following conditions hold:
1) $s(x) \wedge \alpha(x, y) \leq s(y)$
2) $s(x) \wedge s(y) \leq \alpha(x, y)$,
for all $x, y \in A$. We say that $\mathbf{A} = (A, \alpha)$ is a complete Ω-set if for any singleton s there exists the unique $a \in A$ such that $s(x) = \alpha(x, a)$, for all $x \in A$. Finally, the objects are these complete Ω-sets.

Morphisms:
Let $\mathbf{A} = (A, \alpha)$ and $\mathbf{B} = (B, \beta)$ be complete Ω-sets. Then morphisms between $\mathbf{A} = (A, \alpha)$ and $\mathbf{B} = (B, \beta)$ are defined as the maps $f : A \longrightarrow B$ satisfying the following conditions:
1) $\alpha(x, y) \leq \beta(f(x), f(y))$
2) $\beta(f(x), f(x)) \leq \alpha(x, x)$,
for all $x, y \in A$. Morphisms with these properties are called the strong morphisms.

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be strong morphisms. Then their composition $f \circ g$ is defined as the usual composition of functions and $f \circ g \in \mathrm{Hom}_{C\Omega\text{-Set}}((A, \alpha), (C, \gamma))$ due to the transitivity of ≤.

Associativity of composition:
Composition of morphisms is associative due to the associativity of composition of functions.

Identity morphisms:
Let $\mathbf{A} = (A, \alpha)$ be an object. Then the identity morphism on **A** is defined as the usual identity function $1_{\mathbf{A}} : A \longrightarrow A$ and $1_{\mathbf{A}} \in \mathrm{Hom}_{C\Omega\text{-Set}}((A, \alpha), (A, \alpha))$ due to the reflexivity of ≤.

Main results (properties):
• Category **CΩ-Set** is a topos [def. 2.1.30].

• Category **CΩ-Set** is equivalent to the category **Ω-Set** mentioned below [page 45].

Category Ω - FSet (V. Novák, I. Perfilieva, J. Močkoř, 1999) [ref. 20]

Objects:
Let $\Omega = (L, \vee, \wedge, *, \longrightarrow)$ be a complete MV-algebra [def. 2.2.17].
Objects are Ω-fuzzy sets, i.e. the pairs (A, α), where A is a set and $\alpha : A \times A \longrightarrow L$ is a map satisfying these conditions:
1) $\alpha(x, y) \leq \alpha(x, x) \wedge \alpha(y, y)$
2) $\alpha(x, y) = \alpha(y, x)$
3) $\alpha(x, y) * (\alpha(y, y) \longrightarrow \alpha(y, z)) \leq \alpha(x, z)$,
for all $x, y, z \in A$.

Morphisms:
Let (A, α) and (B, β) be Ω-fuzzy sets. Then morphisms between (A, α) and (B, β) are defined as the functions $f : A \longrightarrow B$ satisfying these conditions:
1) $\alpha(x, y) \leq \beta(f(x), f(y))$
2) $\alpha(x, x) = \beta(f(x), f(x))$,
for all $x, y \in A$.

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms. Then their composition $f \circ g$ is defined as the usual composition of functions and $f \circ g \in Hom_{\Omega\text{-FSet}}((A, \alpha), (C, \gamma))$ due to the transitivity of ≤.

Associativity of composition:
Composition of morphisms is associative due to the associativity of composition of functions.

Identity morphisms:
Let **A** = (A, α) be an object. Then the identity morphism on **A** is defined as the usual identity function $1_{\mathbf{A}} : A \longrightarrow A$ and $1_{\mathbf{A}} \in Hom_{\Omega\text{-FSet}}((A, \alpha), (A, \alpha))$ due to the reflexivity of ≤.

Main results (properties):
- Category **Ω - FSet** is a complete category [def. 2.1.29].

- Category **Ω - FSet** is not a topos [def. 2.1.30] since there does not exist a subobject classifier [def. 2.1.28] in the category **Ω - FSet**.

Category Q-Set (U. Höhle, T. Kubiak, 2008) [ref. 16]

Objects:
Let $Q = (Q, \leq, *, ^{\bullet})$ be a bi-symmetric and involutive quantale in which the universal upper bound is extensive and a quasi-unit [def. 2.2.20].
Objects are Q-valued sets, i.e. the pairs (X, E), where X is a set and E is a Q-valued equality on X, i.e. E is a map $X \times X \longrightarrow Q$ satisfying the following conditions:
1) $E(x, y) \leq E(x, x) \wedge E(y, y)$ (strictness)
2) $E(x, x) * (E(x, x) \longrightarrow_r E(x, y)) = E(x, y)$ (divisibility)
3) $E(x, y) = E(y, x)^{\bullet}$ (symmetry)
4) $E(x, y) * (E(y, y) \longrightarrow_r E(y, z)) \leq E(x, z)$ (transitivity),
for all $x, y, z \in X$.

Morphisms:
Let (X, E) and (Y, F) be Q-valued sets. Then morphisms between (X, E) and (Y, F) are defined as structure preserving maps $f : X \longrightarrow Y$ satisfying these conditions:
1) $E(x, x) = F(f(x), f(x))$ (invariance of extent)
2) $E(x_1, x_2) \leq F(f(x_1), f(x_2))$ (preservation of equality),
for all $x, x_1, x_2 \in X$.

Composition of morphisms:
Let $f : (X, E) \longrightarrow (Y, F)$ and $g : (Y, F) \longrightarrow (Z, G)$ be morphisms. Then their composition $f \circ g$ is defined as the usual composition of functions and $f \circ g \in \mathrm{Hom}_{Q\text{-}Set}((X, E), (Z, G))$ due to the transitivity of $\leq$.

Associativity of composition:
Let $f : (X, E) \longrightarrow (Y, F)$, $g : (Y, F) \longrightarrow (Z, G)$ and $h : (Z, G) \longrightarrow (W, H)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the associativity of composition of functions.

Identity morphisms:
Let (X, E) be an object. Then the identity morphism on (X, E) is defined as the usual identity function $1_{(X, E)} : X \longrightarrow X$ and is an element of $\mathrm{Hom}_{Q\text{-}Set}((X, E), (X, E))$ due to the reflexivity of $\leq$.

Main results (properties):

- Category **Q-Set** is a complete category [def. 2.1.29].

- Category **Q-Set** is a cocomplete category [def. 2.1.29].

- The authors give particular results about the Kleisli category $\mathbf{(Q\text{-}Set)_T}$ and the Eilenberg-Moore category $\mathbf{(Q\text{-}Set)^T}$. It is also shown that the category **Q-Set** is biclosed. These results, as well as the concepts of Kleisli category and Eilenberg-Moore category, are out of the scope of this work already. For details see [ref. 16].

3.4 CATEGORIES OF FUZZY RELATIONS (AS OBJECTS) AND (FUZZY) RELATIONS (AS MORPHISMS)

Category Set(H) (D. Higgs, 1973) [ref. 12, 13, cited according to ref. 5, 26]

Objects:
Objects are H-valued fuzzy sets, i.e. the pairs (A, α), where A is a set and $\alpha : A \times A \longrightarrow H$ is a mapping to a complete Heyting algebra H [def. 2.2.16] satisfying the following conditions:
1) $\alpha(a, a') = \alpha(a', a)$ (symmetry)
2) $\alpha(a, a') \wedge \alpha(a', a'') \leq \alpha(a, a'')$ (transitivity),
for all $a, a', a'' \in A$.

Morphisms:
Let (A, α) and (B, β) be objects. Then morphisms between (A, α) and (B, β) are defined as the mappings $f : A \times B \longrightarrow H$ satisfying the following conditions:
1) $\alpha(a, a') \wedge f(a, b) \leq f(a', b)$ (f respects equality on A)
2) $\beta(b, b') \wedge f(a, b) \leq f(a, b')$ (f respects equality on B)
3) $\bigvee_{b \in B} (f(a, b)) = \alpha(a, a)$ (f is total)
4) $f(a, b) \wedge f(a, b') \leq \beta(b, b')$ (f is single-valued),
for all $a, a' \in A$ and $b, b' \in B$.

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms. Then their composition $f \circ g : (A, \alpha) \longrightarrow (C, \gamma)$ is given by the following equality:
$(f \circ g)(a, c) = \bigvee_{b \in B} (f(a, b) \wedge g(b, c))$,
for all $a \in A$ and $c \in C$.

Associativity of composition:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$, $g : (B, \beta) \longrightarrow (C, \gamma)$ and $h : (C, \gamma) \longrightarrow (D, \delta)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the following identities:

$$\begin{aligned}
((f \circ g) \circ h)(a, d) &= \bigvee_{c \in C} ((f \circ g)(a, c) \wedge h(c, d)) \\
&= \bigvee_{c \in C} (\bigvee_{b \in B} (f(a, b) \wedge g(b, c)) \wedge h(c, d)) \\
&= \bigvee_{c \in C} \bigvee_{b \in B} (f(a, b) \wedge g(b, c) \wedge h(c, d)) \\
&= \bigvee_{b \in B} \bigvee_{c \in C} (f(a, b) \wedge g(b, c) \wedge h(c, d)) \\
&= \bigvee_{b \in B} (f(a, b) \wedge \bigvee_{c \in C} (g(b, c) \wedge h(c, d))) \\
&= \bigvee_{b \in B} (f(a, b) \wedge (g \circ h)(b, d)) \\
&= (f \circ (g \circ h))(a, d), \text{ for all } a \in A, d \in D.
\end{aligned}$$

Identity morphisms:
Let **A** = (A, α) be an object. Then the identity morphism on (A, α) is defined by taking α itself as the identity morphism on (A, α).

Main results (properties):
- Category **Set(H)** is a topos [def. 2.1.30]. Due to this fact the category **Set(H)** is widely known as the **Higgs topos** and it has become a paradigm for other authors.

Category $\mathbf{Set_{tf}(H)}$ (O. Wyler, 1995) [ref. 28]

Index **tf** in $\mathbf{Set_{tf}}$ refers to the fact that objects are **t**otally fuzzy sets (relations) and morphisms are **f**uzzy functions (relations).

Objects:
Objects are H-valued totally fuzzy sets, i.e. the pairs (A, α), where A is a crisp set and α is a fuzzy equality mapping $A \times A \longrightarrow H$, where H is a complete Heyting algebra [def. 2.2.16], satisfying these two conditions of symmetry and transitivity:
1) $\alpha(x, y) = \alpha(y, x)$
2) $\alpha(x, y) \wedge \alpha(y, z) \leq \alpha(x, z)$,
for all $x, y, z \in A$.

Morphisms:
Let (A, α) and (B, β) be H-valued totally fuzzy sets. Binary fuzzy relation on a set $(A, \alpha) \times (B, \beta)$ is defined as a mapping $R : A \times B \longrightarrow H$, satisfying these conditions:
1) $R(x, y) \leq \alpha(x, x) \wedge \beta(y, y)$
2) $R(x, y) \wedge \alpha(x, x') \wedge \beta(y, y') \leq R(x', y')$,
for all $x, x' \in A$ and $y, y' \in B$.
If R also satisfies:
3) $R(x, y) \wedge R(x, y') \leq \beta(y, y')$
4) $\alpha(x, x) = \bigvee_y R(x, y)$,
for all $x \in A$ and $y, y' \in B$, then R is called a fuzzy function from (A, α) to (B, β).
Morphisms are defined as fuzzy functions.

Composition of morphisms:
Let $R : (A, \alpha) \longrightarrow (B, \beta)$ and $S : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms. Then their composition $R \circ S : (A, \alpha) \longrightarrow (C, \gamma)$ is a fuzzy function $R \circ S : A \times C \longrightarrow H$, satisfying
$(R \circ S)(x, z) = \bigvee_{y \in B} (R(x, y) \wedge S(y, z))$,
for all $x \in A$, $z \in C$.

Associativity of composition:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$, $g : (B, \beta) \longrightarrow (C, \gamma)$ and $h : (C, \gamma) \longrightarrow (D, \delta)$ be morphisms.
Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the following identities:

$$
\begin{aligned}
((f \circ g) \circ h)(x, w) &= \bigvee_{z \in Z} ((f \circ g)(x, z) \wedge h(z, w)) \\
&= \bigvee_{z \in Z} (\bigvee_{y \in Y} (f(x, y) \wedge g(y, z)) \wedge h(z, w)) \\
&= \bigvee_{z \in Z} \bigvee_{y \in Y} (f(x, y) \wedge g(y, z) \wedge h(z, w)) \\
&= \bigvee_{y \in Y} \bigvee_{z \in Z} (f(x, y) \wedge g(y, z) \wedge h(z, w)) \\
&= \bigvee_{y \in Y} (f(x, y) \wedge \bigvee_{z \in Z} (g(y, z) \wedge h(z, w))) \\
&= \bigvee_{y \in Y} (f(x, y) \wedge (g \circ h)(y, w)) \\
&= (f \circ (g \circ h))(x, w), \text{ for all } x \in X, w \in W.
\end{aligned}
$$

Identity morphisms:
Let (A, α) be an object. Then the identity morphism on (A, α) is defined as an identity fuzzy function, i.e. $1_A : A \times A \longrightarrow H$ satisfying the conditions 1) - 4) in the definiton of morphisms.

Main results (properties):
• Category $\mathbf{Set_{tf}(H)}$ is a topos [def. 2.1.30] with H-valued internal logic.

Category Q-Sets (C. J. Mulvey, M. Nawaz, 1995) [ref. 19]

Objects:
Objects are quantal sets over Q, i.e. the triples **A** = (A, E, [. = .]), where A is a set, E : A ⟶ Q and [. = .] : A × A ⟶ Q are mappings to a right Gelfand quantale Q [def. 2.2.20] satisfying the following conditions:
1) $E(a) * E(a) \leq [\, a = a \,]$
2) $[\, a = b \,] \leq E(a) * E(b)$
3) $E(a) * [\, b = a \,] \leq [\, a = b \,]$
4) $[\, a = b \,] * [\, b = c \,] \leq [\, a = c \,]$,
for all a, b, c ∈ A.
The objects E(a) and [a = b] of the quantale Q will be called respectively the extent and the equality of the quantal set **A**.

Morphisms:
Let **A** and **B** be objects. Then morphisms from **A** to **B** are defined as the maps f : A × B ⟶ Q satisfying the following conditions:
1) $f(a, b) \leq E(a) * E(b)$
2) $[\, a = a' \,] * f(a', b) \leq f(a, b)$
3) $f(a, b) * [\, b = b' \,] \leq f(a, b')$
4) $E(b) * E(b') * f(a, b) * f(a, b') \leq [\, b = b' \,]$
5) $E(a) \leq \bigvee_b f(a, b)$,
for all a, a′ ∈ A and b, b′ ∈ B.

Composition of morphisms:
Let f : **A** ⟶ **B** and g : **B** ⟶ **C** be morphisms. Then their composition f ∘ g : **A** ⟶ **C** is defined by setting $(f \circ g)(a, c) = \bigvee_{b \in B} (f(a, b) * g(b, c))$, for all a ∈ A and c ∈ C.

Associativity of composition:
Let f : **A** ⟶ **B,** g : **B** ⟶ **C** and h : **C** ⟶ **D** be morphisms. Then (f ∘ g) ∘ h = f ∘ (g ∘ h) holds due to the following identities:

$$\begin{aligned}((f \circ g) \circ h)(a, d) &= \bigvee_{c \in C} ((f \circ g)(a, c) * h(c, d))\\ &= \bigvee_{c \in C} (\bigvee_{b \in B} (f(a, b) * g(b, c)) * h(c, d))\\ &= \bigvee_{c \in C} \bigvee_{b \in B} (f(a, b) * g(b, c) * h(c, d))\\ &= \bigvee_{b \in B} \bigvee_{c \in C} (f(a, b) * g(b, c) * h(c, d))\\ &= \bigvee_{b \in B} (f(a, b) * \bigvee_{c \in C} (g(b, c) * h(c, d)))\\ &= \bigvee_{b \in B} (f(a, b) * (g \circ h)(b, d))\\ &= (f \circ (g \circ h))(a, d), \text{ for all } a \in A, d \in D.\end{aligned}$$

Identity morphisms:
Let **A** be an object. Then the identity morphism $1_{\mathbf{A}}$: **A** ⟶ **A** is given by the equality relation [. = .] : A × A ⟶ Q.

Main results (properties):
- The authors also consider the categories **Complete Q-Sets** (with so called complete quantal sets as objects), **Presheaves/Q** and **Sheaves/Q** and show their mutual relationships.
  For details see [ref. 19].

Category Ω-Set (V. Novák, I. Perfilieva, J. Močkoř, 1999) [ref. 20]

Objects:
Let $H = (\Omega, \vee, \wedge, \rightarrow)$ be a complete Heyting algebra [def. 2.2.16].
Objects are Ω-sets, i.e. the pairs $\mathbf{A} = (A, \alpha)$, where A is a set and $\alpha : A \times A \longrightarrow \Omega$ is a function satisfying the following conditions:
1) $\alpha(x, y) = \alpha(y, x)$ (symmetry)
2) $\alpha(x, y) \wedge \alpha(y, z) \leq \alpha(x, z)$ (transitivity),
for all $x, y, z \in A$.

Morphisms:
Let $\mathbf{A} = (A, \alpha)$ and $\mathbf{B} = (B, \beta)$ be Ω-sets. Then morphisms between $\mathbf{A} = (A, \alpha)$ and $\mathbf{B} = (B, \beta)$ are defined as the maps $f : A \times B \longrightarrow \Omega$ satisfying the following conditions:
1) $\alpha(x, x') \wedge f(x, y) \leq f(x', y)$
2) $\beta(y, y') \wedge f(x, y) \leq f(x, y')$
3) $f(x, y) \wedge f(x, y') \leq \beta(y, y')$
4) $\alpha(x, x) = \bigvee\{f(x, y) : y \in B\}$,
for all $x, x' \in A$ and $y, y' \in B$.

Composition of morphisms:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$ and $g : (B, \beta) \longrightarrow (C, \gamma)$ be morphisms. Then their composition (so called sup-max-composition) is defined as the function $f \circ g : A \times C \longrightarrow \Omega$ such that $(f \circ g)(x, z) = \bigvee_{y\in B} (f(x, y) \vee g(y, z))$ holds, for all $x \in A$ and $z \in C$.

Associativity of composition:
Let $f : (A, \alpha) \longrightarrow (B, \beta)$, $g : (B, \beta) \longrightarrow (C, \gamma)$ and $h : (C, \gamma) \longrightarrow (D, \delta)$ be morphisms. Then $(f \circ g) \circ h = f \circ (g \circ h)$ holds due to the following identities:

$$\begin{aligned}((f \circ g) \circ h)(x, w) &= \textstyle\bigvee_{z\in C} ((f \circ g)(x, z) \vee h(z, w)) \\ &= \textstyle\bigvee_{z\in C} (\bigvee_{y\in B} (f(x, y) \vee g(y, z)) \vee h(z, w)) \\ &= \textstyle\bigvee_{z\in C} \bigvee_{y\in B} (f(x, y) \vee g(y, z) \vee h(z, w)) \\ &= \textstyle\bigvee_{y\in B} \bigvee_{z\in C} (f(x, y) \vee g(y, z) \vee h(z, w)) \\ &= \textstyle\bigvee_{y\in B} (f(x, y) \vee \bigvee_{z\in C} (g(y, z) \vee h(z, w))) \\ &= \textstyle\bigvee_{y\in B} (f(x, y) \vee (g \circ h)(y, w)) \\ &= (f \circ (g \circ h))(x, w), \text{ for all } x \in A, w \in D.\end{aligned}$$

Identity morphisms:
Let $\mathbf{A} = (A, \alpha)$ be an object. Then the identity morphism on $\mathbf{A} = (A, \alpha)$ is defined as a map $1_A : A \times A \longrightarrow \Omega$ satisfying the conditions in the definition of morphisms.

Main results (properties):
• Category **Ω-Set** is a topos [def. 2.1.30].

• Category **Ω-Set** is equivalent to the category **CΩ-Set** mentioned above [page 39].

4. CONCLUSION

Goal of this work was to give an overwiev of preferably all important category-theoretical approaches to fuzzy sets that were done throughout relatively short history of category-theoretical modelling of fuzzy sets. This intention predetermines the character of this work to be a survey. But it is not a historical survey, it is a thematic one. Categories are sorted by their "shape" and properties. One of the main criteria of selection of relevant papers was an affinity of there modeled fuzzy sets with these occurring in formal fuzzy logic. The most important and interesting categories for our purposes were these equipped with some kind of monoidal operation on a membership functions structure.
One of the main problems which we were dealing with was an extent of this work. Which categories and which of their properties are “elementary enough“ to be mentioned here and which are outside of the scope of the work already? We have decided to present here every category we have met with in our sources and list its the most fundamental and important category-theoretical properties like completness (cocompletness), cartesian closedness, if it is a topos (quasitopos), existence of basic category-theoretical constructions, limits (colimits), etc. Thus reader can find here categories from Goguen´s early works to Solovyov´s and Höhle´s works from recent years.
It is not possible to unambiguously say in which way the development of category-theoretical approaches to fuzzy sets will proceed. Author´s opinion is that especially Solovyov´s recent works [ref. 22, 23, 24] provide very interesting themes for subsequent research.

5. REFERENCES

[1] ADÁMEK, J., HERRLICH, H., STRECKER, G. E., Abstract and Concrete Categories: The Joy of Cats., Wiley, New York, 1990.

[2] BALCAR, B., ŠTĚPÁNEK, P., Teorie množin., 2nd edition, Academia, Praha, 2000.

[3] EILENBERG, S., MAC LANE, S., Group Extensions and Homology., Annals of Mathematics 43, p. 757-831, 1942.

[4] EILENBERG, S., MAC LANE, S., General Theory of Natural Equivalences., Transactions of the American Mathematical Society 58, p. 231-294, 1945.

[5] EYTAN, M., Fuzzy Sets: A Topos-Logical Point of View., Fuzzy Sets and Systems 5, p. 47-67, 1981.

[6] GOGUEN, J. A., L-Fuzzy Sets., Journal of Mathematical Analysis and Applications 18, p. 145-174, 1967.

[7] GOGUEN, J. A., Categories of V-Sets., Bulletin of the American Mathematical Society 75, p. 622-624, 1969.

[8] GOGUEN, J. A., Concept Representation in Natural and Artifical Languages: Axioms, Extensions, and Applications for Fuzzy Sets., International Journal of Man-Machine Studies 6, p. 513-516, 1974.

[9] GOLDBLATT, R., Topoi: The Categorial Analysis of Logic., Dover Publications, NewYork, 2006.

[10] GOTTWALD, S., Universes of Fuzzy Sets and Axiomatizations of Fuzzy Set Theory. Part II: Category Theoretic Approaches., Studia Logica 84, p. 23-50, 2006.

[11] GRAHAM, I., Fuzzy Sets and Toposes-Towards Higher Order Fuzzy Logic., Fuzzy Sets and Systems 23, p. 19-32, 1987.

[12] HIGGS, D., A Category Approach to Boolean-Valued Set Theory., Preprint, University of Waterloo, 1973.

[13] HIGGS, D., Injectivity in the Topos of Complete Heyting Algebra Valued Sets., Canadian Journal of Mathematics 36, p. 550-568, 1984.

[14] HÖHLE, U., M-Valued Sets and Sheaves Over Integral Commutative CL-Monoids., S. E. Rodabaugh et al. (eds.), Applications of Category Theory to Fuzzy Subsets, p. 33-72, Kluwer Academic Publishers, 1992.

[15] HÖHLE, U., GL-Quantales: Q-Valued Sets and Their Singletons., Studia Logica 61, p. 123-148, 1998.

[16] HÖHLE, U., KUBIAK, T., Quantale Sets and Their Singleton Monad., Preprint, 2008.

[17] HÖHLE, U., STOUT, L. N., Foundations of Fuzzy Sets., Fuzzy Sets and Systems 40, p. 257-296, 1991.

[18] MAC LANE, S., Categories for the Working Mathematician., 2nd edition, Springer-Verlag, New York, 1998.

[19] MULVEY, C. J., NAWAZ, M., Quantales: Quantal Sets., U. Höhle and E. P. Klement (eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, p. 159-217, Kluwer Academic Publishers, 1995.

[20] NOVÁK, V., PERFILIEVA, I., MOČKOŘ, J., Mathematical Principles of Fuzzy Logic., Kluwer Academic Publishers, 1999.

[21] ONO, H., Substructural Logics and Residuated Lattices - an Introduction., V. F. Hendricks and J. Malinowski (eds.), Trends in Logic 20, p. 177-212, Kluwer Academic Publishers, 2003.

[22] SOLOVYOV, S. A., On the Category Set(JCPos)., Fuzzy Sets and Systems 157, p. 459-465, 2006.

[23] SOLOVYOV, S. A., Categories of Lattice-Valued Sets as Categories of Arrows., Fuzzy Sets and Systems 157, p. 843-854, 2006.

[24] SOLOVYOV, S. A., On a Generalization of Goguen´s Category Set(L)., Fuzzy Sets and Systems 158, p. 367-385, 2007.

[25] STOUT, L. N., Topoi and Categories of Fuzzy Sets., Fuzzy Sets and Systems 12, p. 169-184, 1984.

[26] STOUT, L. N., A Survey of Fuzzy Sets and Topos Theory., Fuzzy Sets and Systems 42, p. 3-14, 1991.

[27] STOUT, L. N., Categories of Fuzzy Sets with Values in a Quantale or Projectale., U. Höhle and E. P. Klement (eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, p. 219-234, Kluwer Academic Publishers, 1995.

[28] WYLER, O., Fuzzy Logic and Categories of Fuzzy Sets., U. Höhle and E. P. Klement (eds.), Non-Classical Logics and Their Applications to Fuzzy Subsets, p. 235-268, Kluwer Academic Publishers, 1995.

[29] WINTER, M., Goguen Categories: A Categorical Approach to L-Fuzzy Relations., Springer, 2007.

[30] ZADEH, L. A., Fuzzy Sets., Information and Control 8, p. 338-353, 1965.