Univerzita Karlova v Praze
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DIPLOMOVÁ PRÁCE

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Ramseyovské otázky v euklidovském prostoru

Ramsey-type questions in Euclidean space

Katedra aplikované matematiky

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Abstrakt: Jedním ze základních problémů euklidovské Ramseyovy teorie je určení barevnosti euklidovského prostoru. Barevnost prostoru je nejmenší počet barev, se kterými lze celý prostor obarvit tak, aby žádné dva stejnobarevné body nebyly v jednotkové vzdálenosti.

V práci je ukázáno, že barevnost šestirozměrného reálného prostoru je alespoň 11 a že barevnost sedmirozměrného racionálního prostoru je alespoň 15. Dále je předveden nový důkaz dolního odhadu devět pro barevnost pětirozměrného reálného prostoru a zjednodušen důkaz dolního odhadu sedm pro čtyřrozměrný reálný prostor.

Je známo, že barevnost n-rozměrného reálného prostoru roste exponenciálně v n. Ukážeme některé podprostory reálného prostoru, pro které barevnost roste pomaleji než exponenciálně.

Dále shrneme předchozí výsledky pro obecné normované prostory a některé konkrétní neeuklidovské prostory.

Klíčová slova: barevnost prostoru, jedna-vzdálenostní graf, euklidovský prostor

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Abstract: One of the problems in Euclidean Ramsey theory is to determine the chromatic number of the Euclidean space. The chromatic number of a space is the minimum number of colors with which the whole space can be colored so that no two points of the same color are at unit distance.

We prove that the chromatic number of the six-dimensional real space is at least 11 and that the chromatic number of the seven-dimensional rational space is at least 15. In addition we give a new proof of the lower bound 9 for the chromatic number of the five-dimensional real space. We also simplify the proof of the lower bound 7 for the four-dimensional real space.

It is known that the chromatic number of the n-dimensional real space grows exponentially in n. We show some of its subspaces, in which the growth is slower than exponential.

We also summarize previous results for normed spaces in general and for some interesting non-Euclidean spaces.

Keywords: chromatic number of a space, unit-distance graph, Euclidean space

Chapter 1

Introduction

A typical problem in Euclidean Ramsey theory is to determine the largest number p, such that each coloring of the n-dimensional Euclidean space with p colors contains a monochromatic congruent copy of a fixed finite configuration of points. The simplest configuration about which this question can be asked consists of two points unit distance apart. This thesis is focused merely on this configuration.

The *chromatic number of a space* $\mathbb{S}$ is the minimum number of colors needed to color all the points of $\mathbb{S}$ so that no two points of the same color are at unit distance. We will consider only spaces such that all their points have real coordinates. In Chapter 2 we will focus on n-dimensional real spaces $\mathbb{R}^n$, that is, spaces containing all the points with real coordinates. In Chapter 3, we will study n-dimensional rational spaces $\mathbb{Q}^n$, that is, spaces containing all the points with rational coordinates.

Except for Chapter 4, the distance between points $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ will be the Euclidean distance, which is defined as

$$\operatorname{dist}(x, y) = \|x - y\|_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

A *unit-distance graph* in a space $\mathbb{S}$ is a graph that can be embedded in $\mathbb{S}$ so that the distance between each two adjacent vertices is 1.

The chromatic number $\chi(\mathbb{S})$ of the space $\mathbb{S}$ is equal to the chromatic number of the graph whose vertices are all the points of the space and two vertices are joined by an edge if and only if the distance of their corresponding points is one. This graph is an infinite unit-distance graph. However, a theorem of de Bruijn and Erdős [6] says, that if an infinite graph has a finite chromatic number, then it has a finite subgraph with the same chromatic number. Because the chromatic number of the n-dimensional real space is finite, its chromatic number is the maximum of the chromatic numbers of finite unit-distance graphs in the n-dimensional space.

For any space $\mathbb{S}$ and $r \in \mathbb{R}^+$, let $\chi(\mathbb{S}, r)$ denote the chromatic number of the graph whose vertices are all the points in $\mathbb{S}$ and edges connect the pairs of vertices distance r apart. The distance r is then called the *forbidden distance*. In this notation, $\chi(\mathbb{S}, 1) = \chi(\mathbb{S})$.

The structure of the n-dimensional real space does not change after scaling. Therefore, instead of forbidding two monochromatic points to be distance 1 apart, we can equivalently use any $r \in \mathbb{R}^+$ as the forbidden distance. This is also true in rational spaces if $r \in \mathbb{Q}^+$. Therefore, we can conclude with the following

Observation 1.

$$\forall r \in \mathbb{R}^+ : \chi(\mathbb{R}^n, r) = \chi(\mathbb{R}^n)$$
$$\forall q \in \mathbb{Q}^+ : \chi(\mathbb{Q}^n, q) = \chi(\mathbb{Q}^n)$$

The situation in the line is simple; the chromatic number is 2. However, the problem of finding the chromatic number of the plane is still open. The problem is also known as the *Hadwiger-Nelson problem* and was originated around 1950 by Edward Nelson and first published in [10].

The following table summarizes the best known bounds for some real spaces.

	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$	$\mathbb{R}^6$	$\mathbb{R}^7$	$\mathbb{R}^8$
Lower bound	4[16]	6[17]	7[2]	9[2]	**11**	15[19]	16[13]
Upper bound	7[11]	15[5, 18]	54[18]				

In general,

$$(1.239 + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n \quad [19], [7].$$

These bounds will be discussed in more detail in Chapter 2.

The most important new result of this thesis is the lower bound 11 for the six-dimensional real space presented in Section 2.2. It improves the previous bound with value 10 by Larman and Rogers [13].

Another new result is a simpler proof for the lower bound 9 in the five-dimensional real space. This lower bound originates from [2], where the author did not aim to optimize the size of the unit-distance graph and it has more than 2000 vertices. In Section 2.3, we construct a unit-distance graph in $\mathbb{R}^5$ with 21 vertices which needs 9 colors.

We will also simplify the proof of the lower bound $\chi(\mathbb{R}^4) \geq 7$ in Subsection 2.1.2.

In rational spaces, the chromatic numbers are known up to dimension 4, the values are

$$\chi(\mathbb{Q}^2) = 2 \ [22],$$
$$\chi(\mathbb{Q}^3) = 2 \ [12],$$
$$\chi(\mathbb{Q}^4) = 4 \ [1].$$

For higher-dimensional rational spaces, the best upper bounds are equal to the upper bounds for the real spaces and the lower bounds are in the following table.

	$\mathbb{Q}^5$	$\mathbb{Q}^6$	$\mathbb{Q}^7$	$\mathbb{Q}^8$	$\mathbb{Q}^n$
Lower bound	7[14]	10[15]	**15**	16[15]	$(1.173 + o(1))^n$[19]

See Chapter 3 for more information.

The new lower bound 15 for $\mathbb{Q}^7$ is presented in Subsection 3.2.2. It improves previous unpublished lower bound 14, which was found by Matthias Mann and is published only on the website `http://www.unit-distance-graphs.com`. Both the previous and the new lower bound were proven by a computer program that determined a lower bound on the chromatic number of some unit-distance graph.

In Chapter 4, we present several previous and new results in some other spaces. This includes, for example, results in vector spaces with non-Euclidean distance. In Subsection 4.5.1 we show tight upper bounds on the chromatic numbers of rational spaces with irrational forbidden distance in dimensions 2,3 and 4. In Subsection 4.5.2, we present some subspaces of $\mathbb{R}^n$ whose chromatic number grows slower than exponentially in n.

Chapter 2

Real Euclidean spaces

2.1 Previous results

2.1.1 Plane

The lower bound $\chi(\mathbb{R}^2) \geq 4$ follows from the construction of the graph in Figure 2.1, called the *Moser spindle*. The Moser spindle was constructed by Leo Moser and William Moser [16] and is a unit-distance graph in the plane whose chromatic number is 4.

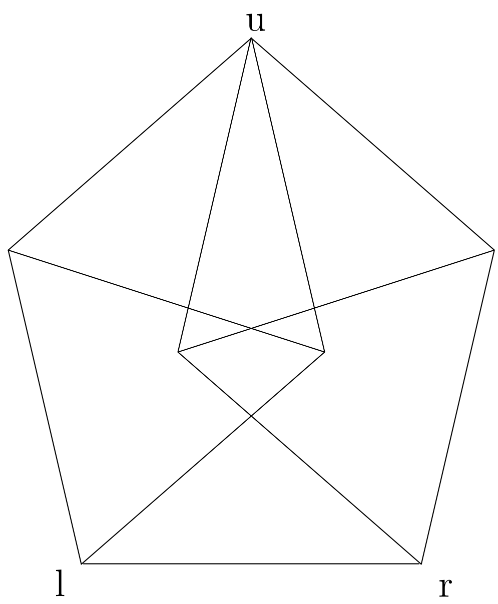

Figure 2.1: The Moser spindle

If the spindle could be properly colored with three colors, vertices u and l would have the same color, because their common neighbors require two colors. Similarly vertices u and r would have the same color, which leads to a contradiction, because l and r must have different colors. The chromatic number of the spindle is therefore at least four and it is easy to find a proper 4-coloring.

The spindle construction can be generalized to higher dimensions.

Lemma 1. *(Raiskii 1970 [20])*

Take some unit-distance graph G in $\mathbb{R}^{n-1}$ whose vertices lie on a sphere $\mathcal{C}$ with radius $r \leq \sqrt{15}/4$. Then we can construct a unit-distance graph in $\mathbb{R}^n$ whose chromatic number is at least $\chi(G)+2$.

Proof. Take points u, l and r in $\mathbb{R}^n$ such that

$$\operatorname{dist}(u,l) = \operatorname{dist}(u,r) = 2 \cdot \sqrt{1-r^2},$$
$$\operatorname{dist}(l,r) = 1.$$

Then place one copy of G on the hyperplane orthogonal to the segment ul so that the center of the sphere $\mathcal{C}$ is in the center of the segment ul. Then all the vertices of G are at distance 1 from both u and l. If we have no more than $\chi(G)+1$ colors, both u and l must receive the same color. Similarly we place another copy of G orthogonal to ur. All in all, l and r received the same color and $\chi(G)+1$ colors are not enough to color the graph. □

In arbitrary dimension $n \geq 2$, the $(n-1)$-dimensional simplex with edge length 1 lies on a sphere with radius at most $1/2$ and its chromatic number is n. This gives a unit-distance graph in $\mathbb{R}^n$ with chromatic number $n+2$, which is called the *Moser-Raiskii spindle* [20]. The Moser spindle is obtained if G is the one-dimensional simplex, which consists of two points distance 1 apart.

2.1.2 Dimensions 3 and 4

The best lower bound in $\mathbb{R}^3$ has value 6 and was found by Nechushtan [17] in 2002. This was the first improvement since the Moser-Raiskii spindle from 1970. The construction is quite complicated, so we will not reproduce it here.

In dimension four, the best lower bound is 7. Like in dimension 3, this was the first improvement since the Moser-Raiskii spindle.

Theorem 1. *(Cantwell 1996 [2])*

$$\chi\left(\mathbb{R}^4\right) \geq 7.$$

The original proof uses another Ramsey-type result from the same paper.

Theorem 2. *(Cantwell 1996 [2])*

For any b, if $\mathbb{R}^4$ is 2-colored, then a monochromatic square with side b is formed.

However, the use of Theorem 2 can be easily avoided as we will show in the remainder of this subsection. Although the new proof is different from the original one, the main ideas remain the same.

Proof. We will prove $\chi(\mathbb{R}^4, \sqrt{2}) > 6$.

Let the *standard configuration* be the set of points of $\mathbb{R}^5$ with exactly two coordinates equal to 1 and the rest equal to 0. It will be denoted by $\mathcal{S}_5$. We can embed $\mathcal{S}_5$ in $\mathbb{R}^4$, because all its points lie on the hyperplane that contains the points $x = (x_1, x_2, x_3, x_4, x_5)$, such that $\sum_{i=1}^{5} x_i = 2$.

The $\sqrt{2}$-distance graph formed by the points of $\mathcal{S}_5$ is the line-graph of K_5, the complete graph on 5 vertices. To see this, we will look at the characteristic vectors of each edge of K_5. Each such vector has five elements two of which are 1 and the rest is 0. This defines the bijection between the points of $\mathcal{S}_5$ and the edges of K_5. Two edges of K_5 are adjacent in the line-graph of K_5 if and only if they share one vertex; two points of $\mathcal{S}_5$ are at distance $\sqrt{2}$ if there is exactly one coordinate in which both points have value 1.

Further observe, that for any four points in $\mathbb{R}^4$ that form a square, there is a standard configuration such, that the four points correspond to the edges of a four-cycle of the underlying K_5.

The *octahedron* is the polytope with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ and the *four-dimensional cross polytope* is the polytope with vertices $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$, $(0, 0, 0, \pm 1)$. The only pairs of vertices of the cross-polytope that are not $\sqrt{2}$ apart, are the four pairs of vertices that have the same nonzero coordinate.

If the cross polytope is colored with six colors, then at least two pairs of vertices, which have the same nonzero coordinate, are monochromatic. These two pairs form a square colored with two colors, let them be blue and green. We embed the square in a standard configuration, so that the vertices of the square form edges of a four-cycle in the corresponding K_5. Let the vertices of the four-cycle be s, t, u and v and let the fifth vertex be w. If the edges of K_5 are colored with six colors, then the four edges going from w must use all the four colors different from blue and green. See Figure 2.2 for one such coloring.

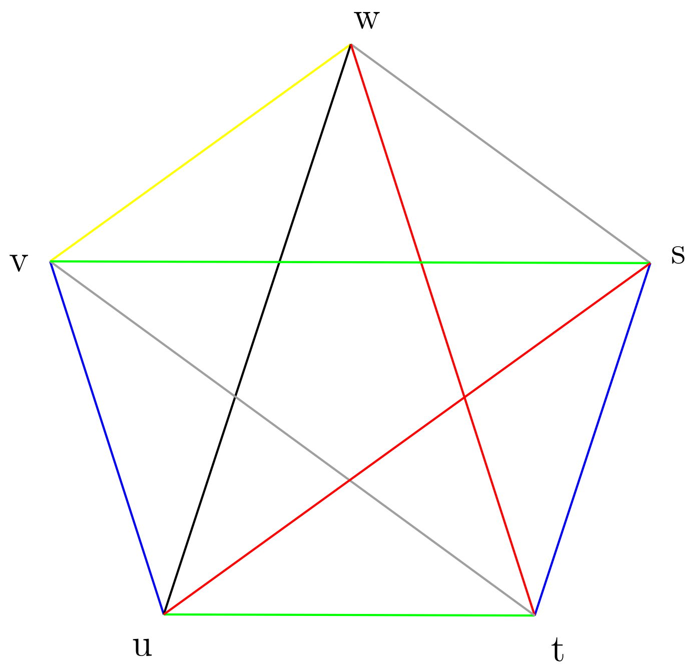

Figure 2.2: Coloring of the standard configuration

Edge $\{v, t\}$ has the color of $\{w, x\}$, where x is either s or u. The edges of the graph induced by vertices $\{s, t, u, v, w\} \setminus \{x\}$ correspond to an octahedron whose vertices are colored with six different colors. But there exist points at distance $\sqrt{2}$ from all the vertices of the octahedron, which can not be colored. One such point is $(0, 0, 0, 1)$, which is at distance $\sqrt{2}$ from each of the points $(\pm 1, 0, 0, 0)$, $(0, \pm 1, 0, 0)$, $(0, 0, \pm 1, 0)$.

$\square$

This allows us to construct a unit-distance graph with 32 vertices which needs seven colors. The cross-polytope has 8 vertices, some three of his four octahedra are completed to form the standard configuration (12 more vertices). Each octahedron in some of these standard configurations, that is not one of the three original octahedra, gets one point at distance $\sqrt{2}$ from all its vertices (12 more vertices).

2.1.3 Dimensions 5 and 6

Larman and Rogers [13] proved that the $\sqrt{2}$-distance graph on the vertices of the five-dimensional half-cube requires eight colors. The *five-dimensional half-cube* $\mathcal{H}_5$ is a polytope whose vertices are the 16 points in $\mathbb{R}^5$ with all coordinates equal to 0 or 1, such that the sum of the coordinates is even. We connect two vertices of the half-cube by an edge if they are at distance $\sqrt{2}$, which occurs if the vertices differ in exactly two coordinates.

Cantwell [2] used a construction based on the half-cube to show $\chi(\mathbb{R}^5) \geq 9$.

If we scale the space down by $\sqrt{2}$, the vertices of the half-cube will lie on a sphere with radius $\sqrt{5}/\sqrt{8}$ and the forbidden distance will be 1. Therefore we can use the spindle construction from Lemma 1 and obtain $\chi(\mathbb{R}^6) \geq 10$.

2.1.4 Dimensions 7 and 8

The first lower bound for these dimensions came from Larman and Rogers [13]. They used the vertices of the seven-dimensional Gosset polytope, which can be defined in several ways, here we use the definition from [21]. The *seven-dimensional Gosset polytope* is the polytope whose vertices are all the vectors that can be obtained by putting signs in all the eight ways to the characteristic vectors of the lines of the Fano plane. The 2-distance graph on the vertices of the polytope has 56 vertices and independence number 4 and thus needs at least 14 colors.

To show that each independent set has no more than four vertices, we first observe which vertices are adjacent, that is, are at distance 2. Two vertices that correspond to the same line of the Fano plane are adjacent if and only if exactly one of their nonzero coordinates differs in sign. Two vertices that correspond to two different lines of the Fano plane have exactly one nonzero coordinate in common and are adjacent if and only if this coordinate has the same sign in both vertices.

We distinct several cases:

- Each vertex of the independent set corresponds to a different line of the Fano plane.

  No triple of the vertices can have one nonzero coordinate in common. Otherwise, two vertices would have the same sign of the common coordinate and would be adjacent.

  For contradiction, let A be an independent set with five vertices and let r be the number of pairs (coordinate c, vertex in A with nonzero in c). There are seven coordinates and each is nonzero in at most two vertices, thus $r \leq 7 * 2$ There are five vertices and each has three nonzero coordinates, thus $r = 5 * 3 > 7 * 2$, a contradiction.

- One or more pairs of vertices of the independent set correspond to the same line, but no three vertices correspond to the same line

Let u and v be two vertices corresponding to the same line. The signs in u and v differ for at least two coordinates. Each other vertex must have zero in the two coordinates; otherwise the sign of the common nonzero coordinate would be the same as its sign in either u or v. All other vertices correspond to a single line; otherwise there would be vertices corresponding to three different lines with one nonzero coordinate in common and two of the vertices would be adjacent. Therefore all the vertices of the independent set correspond to at most two different lines and thus there are at most four vertices.

- At least three vertices of the independent set correspond to the same line p

  Without loss of generality, one of the vertices is $(1, 1, 1, 0, 0, 0, 0)$.

  - Vertex $(-1, -1, -1, 0, 0, 0, 0)$ is in the independent set

    Each other vertex that corresponds to the line p differs in exactly one sign from one of the two above mentioned vertices that are in the independent set and thus no three vertices of the independent set correspond to the same line.

  - Vertex $(-1, -1, -1, 0, 0, 0, 0)$ is not in the independent set

    Without loss of generality $(-1, -1, 1, 0, 0, 0, 0)$ and $(-1, 1, -1, 0, 0, 0, 0)$ are in the independent set. Since each of the three nonzero coordinates of p is already used with both signs, each vertex corresponding to a different line is adjacent to one of the three vertices. The only other vertex corresponding to the line p that can be in the independent set is $(1, -1, -1, 0, 0, 0, 0)$, so the independent set has at most four vertices.

After scaling the space down by 2, the graph becomes a unit-distance graph whose vertices lie on a sphere with radius $\sqrt{3}/2$. We can thus use the general spindle construction from Lemma 1 and $\chi(\mathbb{R}^8) \geq 16$

As an interesting fact, notice that the five-dimensional half-cube, which was used to show the best lower bounds in dimensions 5 and 6, is another Gosset polytope.

In another definition of the seven-dimensional Gosset polytope, the vertices lie on a hyperplane in $\mathbb{R}^8$. Let

$$\begin{aligned}
\mathcal{F}_1 &= \{F = (f_1, \dots, f_8) : \forall i : f_i \in \{0, 1\}, f_1 + \cdots + f_8 = 4\}, \\
\mathcal{F}_2 &= \{F = (f_1, \dots, f_8) : \forall i : f_i \in \{0, 2\}, f_1 + \cdots + f_8 = 4\}, \\
\mathcal{F}_{1,-1} &= \{F = (f_1, \dots, f_8) : \forall i : f_i \in \{-1, 1\}, f_1 + \cdots + f_8 = 4\}.
\end{aligned}$$

The points of $\mathcal{F}_2 \cup \mathcal{F}_{1,-1}$ form the vertices of the Gosset polytope. This time, the forbidden distance is $\sqrt{8}$.

Raigorodskii [19] added the points of $\mathcal{F}_1$ and showed that this graph, which has 126 vertices, needs at least 15 colors.

2.1.5 Higher dimensions

A trivial lower bound for any dimension follows from the fact that the unit-distance graph of the n-dimensional simplex is a complete graph. Thus $\chi(\mathbb{R}^n) \geq n+1$. The Moser-Raiskii spindle mentioned in Subsection 2.1.1 improves this to $\chi(\mathbb{R}^n) \geq n + 2$.

Larman and Rogers in an appendix of their paper [13] cite an unpublished lemma of Paul Erdős and Vera Sós. Their configuration is the following. Consider in $\mathbb{R}^{n+1}$ all $0-1$ vectors with exactly three 1-elements. This configuration lies on a hyperplane and therefore can be embedded in $\mathbb{R}^n$. Let G_{n+1} be the 2-distance graph on these points.

$$\alpha(G_{n+1}) = \begin{cases} n+1 & \text{if } n+1 \equiv 0 \mod 4 \\ n & \text{if } n+1 \equiv 1 \mod 4 \\ n-1 & \text{if } n+1 \equiv 2 \text{ or } 3 \mod 4 \end{cases} \tag{2.1}$$

Because G_{n+1} has $\Theta(n^3)$ vertices, its chromatic number grows quadratically in the dimension n.

Let us prove Equation 2.1. We will consider the vertices as three-element subsets of $\{1, 2, \dots, n+1\}$. The independent set is a set of vertices, each pair of which has either two or no elements in common. The equation holds for $n \leq 4$. Let A be some independent set of G_{n+1}. We analyse all possible cases:

- Each two vertices from A have no element in common.

  For any vertex, the three elements it contains are not used in any other vertex. Therefore the size of such independent set is at most $n/3$.

- Vertices $u, v \in A$ have two elements in common.

  - For some vertex $w \in A$, the intersection of u, v and w has two elements.

    If some other vertex of A has non-empty intersection with u, it must contain the two elements of $u \cap v \cap w$, since otherwise, its intersection with v or w would have only one element. Therefore there are vertices $v_1 = v, v_2 = u, v_3 = w, v_4, \dots, v_j$, such that $|v_1 \cap v_2 \cap \dots \cap v_j| = 2$ and no other vertex has non-empty intersection with any of them. Because $|v_1 \cup v_2 \cup \dots \cup v_j| = j+2$, we obtain

    $$|A| \leq j + \alpha(G_{n-j-1}).$$

  - Each other vertex contains at most one element from $u \cap v$.

    Each other vertex with non-empty intersection with u or v contains one element from $u \cap v$, the only one element of $u \setminus v$ and the only element of $v \setminus u$. Therefore there can be only two such vertices w and z and

    $$|A| \leq 4 + \alpha(G_{n-3}).$$

  If $n \geq 4$, we can always find the four independent vertices u, v, w and z, whose union has only four elements, Therefore, for any n, there is an independent set A with $4 + \alpha(G_{n-3})$ elements and the equality in Equation 2.1 holds.

The biggest step was made by the construction of Frankl and Wilson [7] which gives an exponential lower bound. This followed from their intersection theorem for set systems. This intersection theorem is also the basis of two other breakthrough results. One is the best constructive lower bound for the Ramsey number $R(k,k)$ in the same paper and the other one is the counterexample to the Borsuk conjecture by J. Kahn and G. Kalai.

The Frankl-Wilson construction takes some prime number $p < n/2$. The vertices of the graph are all the points in $\mathbb{R}^{n+1}$ whose $2p-1$ coordinates are 1 and the remaining ones are 0. The configuration lies on a hyperplane and can be embedded in $\mathbb{R}^n$. The $\sqrt{2p}$-distance graph on these points then has $\binom{n+1}{2p-1}$ vertices and Frankl and Wilson showed that its independence number is at most $\binom{n+1}{p-1}$. For a good choice of p, the chromatic number of such graph is at least $(1.2 + o(1))^n$. Notice that the choice $p = 2$ gives exactly the Erdős-Sós construction.

The exponential lower bound was later improved by Raigorodskii [19] to $(1.239+o(1))^n$. As the vertices of his $\sqrt{2p}$-distance graph, he uses points from the set

$$\{x = (x_1, x_2, \ldots, x_n) : x_i \in \{0, 1, -1\}, |\{i : x_i \neq 0\}| = a, |\{i : x_i = -1\}| = b\},$$

for an appropriate choice of a and b. An upper bound on the independence number then gives the above mentioned lower bound on the chromatic number.

2.1.6 Upper bounds

The best upper bound in the plane has value 7 and was found by Hadwiger [11]. The bound is obtained by tiling with copies of a regular hexagon whose diameter is slightly less than 1. The tiling is depicted on Figure 2.3. Boundary points can be colored with the color of any of the neighboring regions.

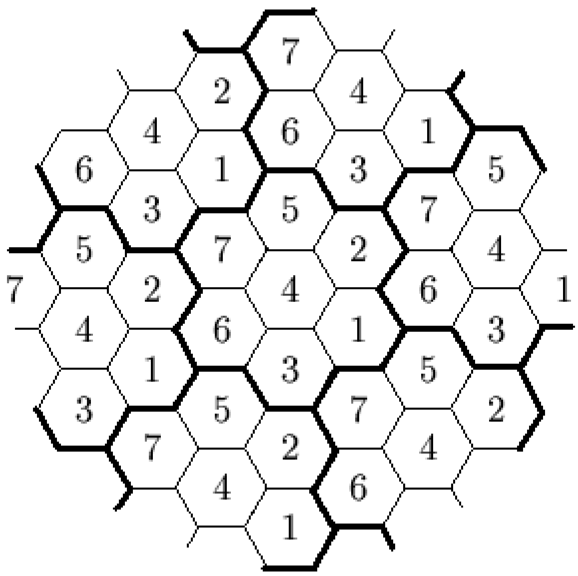

Figure 2.3: Coloring of the plane with 7 colors

In the three-dimensional space, the currently best upper bound was found independently by Coulson [5] and Radoicic and Toth [18]. Their tiling of $\mathbb{R}^3$ with copies of the permutahedron of order 4 (see Figure 2.4) allows coloring with 15 colors. The *permutahedron* of order n is the convex hull of all vectors that can be obtained by permuting the coordinates of the vector $(1, 2, \ldots, n)$. All points of the permutahedron satisfy $\sum_{i=1}^{n} x_i = n * (n + 1)/2$, therefore they lie on a hyperplane and thus the permutahedron of order n can be embedded in $\mathbb{R}^{n-1}$.

Notice that the regular hexagon, which was used to tile the plane, is the permutahedron of order 3.

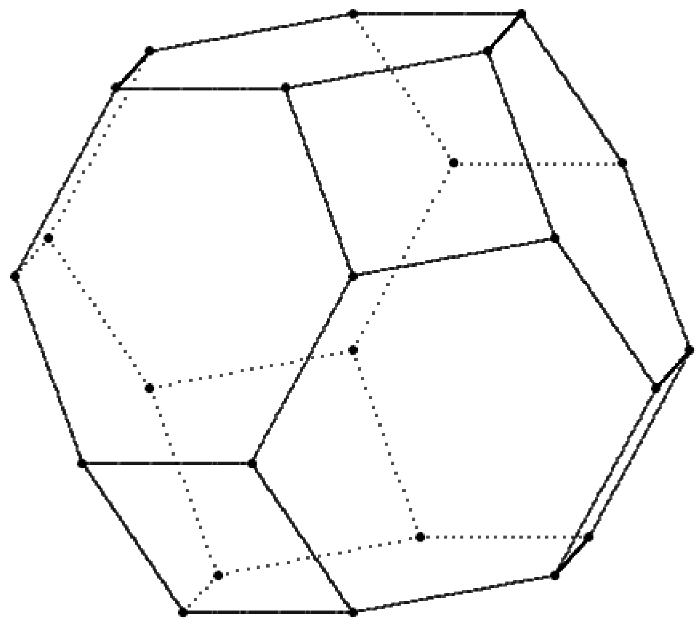

Figure 2.4: Permutahedron of order 4

Radoicic and Toth [18] mention that using the same method, they found a coloring of $\mathbb{R}^4$ with 54 colors.

For higher dimensions, the smallest known upper bound is $(3 + o(1))^n$ and is due to Larman and Rogers [13]. They also showed that the conjectured sphere packing in $\mathbb{R}^n$ with density $(2^{-0.5} + o(1))^n$ would imply $\chi(\mathbb{R}^n) \leq (2^{1.5} + o(1))^n$.

2.2 New lower bound for $\mathbb{R}^6$

Theorem 3. *Chromatic number of the Euclidean space $\mathbb{R}^6$ is at least 11.*

We will use $\sqrt{2}$ as the forbidden distance. The proof uses some properties of the half-cube $\mathcal{H}_5$, which was defined in Subsection 2.1.3. All the vertices of the half-cube lie on a sphere with radius $\sqrt{5}/2$ centered at the point $(1/2, 1/2, 1/2, 1/2, 1/2)$ called the *center of the half-cube.*

By a *facet of the half-cube* we will mean the set of vertices of the half-cube all of which have the same value of one coordinate. There are the following ten facets:

$$\mathcal{F}_{2j-1} = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathcal{H}_5, x_j = 0\right\}, j = 1, \ldots, 5$$
$$\mathcal{F}_{2j} = \left\{(x_1, x_2, x_3, x_4, x_5) \in \mathcal{H}_5, x_j = 1\right\}, j = 1, \ldots, 5$$

Each facet lies on a three-dimensional sphere with radius 1; the center of the sphere is called the *center of the facet.* The pair of facets $\{\mathcal{F}_{2j-1}, \mathcal{F}_{2j}\}$ is called a *pair of opposite facets.* A facet with colored vertices is a *rainbow facet* if each of its 8 vertices has a different color.

Lemma 2. *The $\sqrt{2}$-distance graph on the vertices of the half-cube $\mathcal{H}_5$ has chromatic number at least 8. Moreover, in each 8-coloring, there exists a pair of opposite facets both of which are rainbow.*

Proof. We will determine which vertices can have the same color as the vertex $(0,0,0,0,0)$. All vertices with two coordinates equal to 1 are its neighbors, so we can use only the vertices with four coordinates equal to 1. But the distance between each two of them is $\sqrt{2}$ and therefore only one of them can have the same color as $(0,0,0,0,0)$. It follows from the symmetries of the half-cube, that each color class has at most two vertices and therefore the sixteen vertices of the half-cube require at least eight colors.

Moreover, each color class is composed of two vertices that differ in exactly four coordinates and therefore lie in exactly one common facet.

If a facet is not rainbow, it must contain two vertices with the same color. Therefore at most eight of the ten facets are not rainbow. A rainbow facet contains exactly one vertex of each color class. Therefore the other vertex of that color class lies on the opposite facet which is then rainbow. □

Lemma 3. *If the space $\mathbb{R}^6$ is colored with 10 colors, then there is a five-dimensional half-cube whose vertices are colored with 8 colors.*

Proof. For contradiction, each half-cube has vertices colored with 9 colors. Because the vertices of the half-cube lie on a sphere with radius $\sqrt{5}/2$, we can use the general spindle construction from Lemma 1 and the six-dimensional space needs at least 11 colors.

□

Lemma 4. *Let S and T be two points in the plane distance 1 apart and let $\mathbb{C}_1$ and $\mathbb{C}_2$ be the circles of radius 1 centered at S and T respectively. Then the set $\mathbb{C}_1 \cup \mathbb{C}_2$ contains the vertices of an equilateral triangle with sides of length $\sqrt{2}$.*

Proof. Take a point $A \in \mathbb{C}_2$ at distance $(\sqrt{6}+\sqrt{2})/2$ from S. From the circle $\mathbb{C}_1$, take the points B and C at distance $\sqrt{2}$ from A. A simple computation reveals that the distance between B and C is $\sqrt{2}$ and therefore A, B and C are the vertices of the required equilateral triangle. □

Lemma 5. *If the space $\mathbb{R}^6$ is colored with 10 colors, then no five-dimensional half-cube has vertices colored with 8 colors.*

Proof. Suppose that there is a half-cube with vertices colored with 8 colors. From Lemma 2, it has a pair $(\mathcal{F},\mathcal{G})$ of opposite rainbow facets. Let S and T be the centers of $\mathcal{F}$ and $\mathcal{G}$, respectively. The two facets lie on two parallel four-dimensional subspaces, so we can take the plane $\mathbb{P}$ orthogonal to both of them. Because the line joining the centers S and T is orthogonal to the facets, we can select such $\mathbb{P}$, that contains both S and T. Each point that lies in the plane $\mathbb{P}$ and is at distance 1 from S, is $\sqrt{2}$ far from each vertex of facet $\mathcal{F}$ and therefore is colored with one of the two colors different from the eight colors used on the vertices of the half-cube. Similarly, each point of $\mathbb{P}$ at distance 1 from T is colored with one of the two colors. But from Lemma 4, the two circles of radius 1 centered at S and T require 3 colors. □

If the space $\mathbb{R}^6$ is colored with at most 10 colors, then Lemmas 3 and 5 lead to a contradiction. The proof leads to a unit-distance graph in $\mathbb{R}^6$ with 65 vertices and chromatic number at least 11.

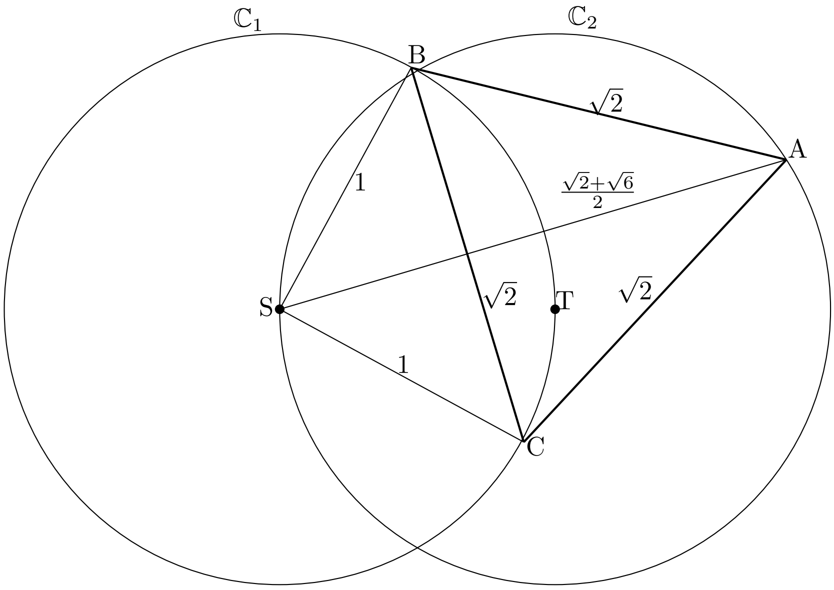

Figure 2.5: Situation in Lemma 4

2.3 New proof of the lower bound in dimension 5

We will construct a $\sqrt{2}$-distance graph in $\mathbb{R}^5$ with chromatic number 9. Another graph with these properties was already found by Kent Cantwell [2], but our construction is simpler and leads to a graph with fewer vertices.

Theorem 4. *(Cantwell 1996 [2])*
Chromatic number of the Euclidean space $\mathbb{R}^5$ *is at least 9.*

Proof. Suppose that $\mathbb{R}^5$ is colored with 8 colors. By Lemma 2, the vertices of the half-cube are colored with 8 colors and the half-cube has a pair of opposite rainbow facets. The center of arbitrary one of the two facets is at distance $\sqrt{2}$ from all the vertices of the opposite facet and thus there is no color left to color it. □

The $\sqrt{2}$-distance graph with chromatic number 9 is the graph whose vertices are the vertices of the half-cube together with the center of one facet from each pair of opposite facets. The graph thus has 21 vertices.

Chapter 3

Rational spaces

3.1 Known exact values (dimensions 2, 3 and 4)

The first result in rational spaces came from Woodall [22], who proved that the chromatic number of the rational plane is equal to 2. Later Johnson [12] showed that $\chi(\mathbb{Q}^3) = 2$. The value $\chi(\mathbb{Q}^4) = 4$ is due to Benda and Perles [1]; although the paper was published in 2000, it existed as a preprint since 1976.

A simpler proof for these dimensions was found by Chilakamarri [4] and will be presented in this section.

The following lemma allows us to move from rational spaces to integer grids.

Lemma 6. *(Chilakamarri 1993 [4])*
For any $n \in \mathbb{N}$

$$\chi(\mathbb{Q}^n) = \max_{r\in\mathbb{N}}\{\chi(\mathbb{Z}^n, r)\}.$$

Proof. We only need to prove $\chi(\mathbb{Q}^n) \leq \max_{r\in\mathbb{N}}\{\chi(\mathbb{Z}^n, r)\}$, because the fact that $\mathbb{Z}^n \subset \mathbb{Q}^n$ implies $\chi(\mathbb{Z}^n, r) \leq \chi(\mathbb{Q}^n, r) = \chi(\mathbb{Q}^n)$.

The theorem of de Bruijn and Erdős mentioned in Chapter 1 implies the existence of a finite unit-distance graph G in $\mathbb{Q}^n$ whose chromatic number is equal to the chromatic number of $\mathbb{Q}^n$. Because G has finitely many vertices, we can multiply their coordinates by some $r \in \mathbb{N}$ to obtain vertices whose all coordinates are integers. This is then an r-distance graph in $\mathbb{Z}^n$. □

Lemma 7. *(Chilakamarri 1993 [4])*
For any odd $r \in \mathbb{N}$ and any $n \in \mathbb{N}$

$$\chi(\mathbb{Z}^n, r) = 2.$$

Proof. Color all the points whose sum of coordinates is even with one color and the remaining ones with the other color. We need to show that two points x, y with the same parity of the sum of coordinates are never adjacent in the r-distance graph.

Take $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$, such that $\sum x_i \equiv \sum y_i \mod 2$. For contradiction, suppose that x and y are adjacent, which means that they are at distance r. The difference $x_i - y_i$ is odd for an even number of coordinates i. Therefore $(x_i - y_i)^2$ is odd for an even number of coordinates i. This implies

$$r^2 = \sum (x_i - y_i)^2 \equiv 0 \mod 2,$$

which is a contradiction because r is odd. □

3.1.1 Dimensions 2 and 3

Lemma 8. *(Chilakamarri 1993 [4])*
For any $r \in \mathbb{N}$ and $n \in \{2,3\}$

$$\chi(\mathbb{Z}^n, 2r) \leq \chi(\mathbb{Z}^n, r).$$

Proof. We will show the proof only for $\mathbb{Z}^3$, proof for the smaller dimension is the same.
Take any $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ at distance $2r$ and let $\delta_i = x_i - y_i$.

$\delta_i \mod 2$	$\delta_i^2 \mod 4$
0	0
1	1

From the table, we can see that all δ_i are even, because $\delta_1^2 + \delta_2^2 + \delta_3^2 = r^2 \equiv 0 \mod 4$. Therefore there exists a $2r$-distance graph with all coordinates of all its vertices even and whose chromatic number is equal to $\chi(\mathbb{Z}^3, 2r)$. Dividing all coordinates by 2 transforms the graph to an r-distance graph in $\mathbb{Z}^3$.

□

If we put Lemmas 6, 7 and 8 together, we obtain

Theorem 5. *(Woodall 1973 [22], Johnson 1982 [12])*

$$\chi(\mathbb{Q}^2) = \chi(\mathbb{Q}^3) = 2$$

3.1.2 Dimension 4

Lemma 9. *(Chilakamarri 1993 [4])*
For any integer r

$$\chi(\mathbb{Z}^4, r) \leq 4.$$

Proof. Take any $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ at distance r and let $\delta_i = x_i - y_i$.

$\delta_i \mod 4$	$\delta_i^2 \mod 8$
0	0
1	1
2	4
3	1

- r is odd

  From Lemma 7, $\chi(\mathbb{Z}^4, r) \leq 2$.

- $r \equiv 2 \mod 4$, thus $r^2 \equiv 4 \mod 8$

  Each point x gets the color defined by the pair (parity of x_1, parity of $\lfloor (\sum_{i=1}^4 x_i)/2 \rfloor$). If x and y are adjacent, then either all δ_i are odd, or for one i, $\delta_i \equiv 2 \bmod 4$ while the others are multiples of 4. In the first case, the first element of the color is different in x and y, while in the second, the second element differs.

- $r \equiv 0 \mod 4$, thus $r^2 \equiv 0 \mod 8$

  From the table it follows that all δ_i are even and like in Lemma 8, $\chi(\mathbb{Z}^4, r) \leq \chi(\mathbb{Z}^4, r/2)$.

$\square$

Theorem 6. *(Benda and Perles 2000 [1])*

$$\chi(\mathbb{Q}^4) = 4$$

Proof. Inequality $\chi(\mathbb{Q}^4) \geq 4$ follows from the fact that the unit-distance graph whose vertices are the four points $(\pm 1/2, 0, 0, 0), (0, \pm 1/2, 1/2, 1/2)$. is the complete graph K_4.

The other inequality follows from Lemmas 6 and 9.

$\square$

3.2 Lower bounds in higher dimensions

3.2.1 Previous results

In dimension 5, the best lower bound has value 7 and was found by Matthias Mann [14]. He constructed a unit-distance graph whose chromatic number was first determined by a computer program and later by a theoretical proof as well.

With his computer program, Mann also found a unit-distance graph in $\mathbb{Q}^6$ which needs 10 colors [15].

In the same paper [15], Mann also showed that $\mathbb{Q}^7$ needs at least 13 colors. He later improved this to 14, but it was not published yet; the description of the graph, that needs 14 colors can be downloaded at website `http://www.unit-distance-graphs.com`. However, if we arrange the Gosset polytope as in Subsection 2.1.4, it forms another seven-dimensional 2-distance graph with rational coordinates and with chromatic number 14.

In $\mathbb{Q}^8$, the best lower bound 16 is from [15] as well. However, the Gosset polytope can also be used to show $\chi(\mathbb{Q}^8) \geq 16$. We cannot use the general spindle construction, because it would lead to irrational coordinates, but in this special case we can do a simpler construction. We place the polytope in the eight-dimensional space so that the eighth coordinate of all its vertices is zero. All the vertices lie on a sphere with radius $\sqrt{3}$ centered at the origin of coordinates. Now we add points $(0, 0, 0, 0, 0, 0, 0, \pm 1)$ which are at distance 2 from all the vertices of the polytope. Their mutual distance is 2 and therefore they need two additional colors.

In higher dimensions, the construction of Paul Erdős and Vera Sós for n-dimensional real space (see Subsection 2.1.5) gives a 2-distance graph in $\mathbb{Q}^{n+1}$ and provides the best lower bounds $\chi(\mathbb{Q}^{11}) \geq 19$, $\chi(\mathbb{Q}^{12}) \geq 19$, $\chi(\mathbb{Q}^{13}) \geq 24 \ldots$.

For dimensions even higher, the exponential lower bound $\chi(\mathbb{Q}^n) \geq (1.173 + o(1))^n$ was given by Raigorodskii [19].

3.2.2 New lower bound for $\mathbb{Q}^7$

Theorem 7.

$$\chi\left(\mathbb{Q}^7\right) \geq 15$$

Proof. The main part of the proof was done by a computer program, which can be found in Appendix A and at `http://kam.mff.cuni.cz/~cibulka/diplomka/q7_15col.c` .

We add several points to the vertices of the Gosset polytope to increase the chromatic number to 15.

Let

$$\mathcal{U}_3 = \{(x_1, x_2, \ldots, x_7) : x_i \in \{0, 1, -1\}, |\{i : x_i \neq 0\}| = 3\},$$
$$\mathcal{U}_7 = \{(x_1, x_2, \ldots, x_7) : x_i \in \{1, -1\}\}.$$

Then $|\mathcal{U}_3| = 280$ and $|\mathcal{U}_7| = 128$. The forbidden distance is 2. With the above mentioned computer program, we can determine that the 2-distance graph whose vertices are points from $\mathcal{U}_3$ has independence number 20. Therefore, if $\mathcal{U}_3 \cup \mathcal{U}_7$ is colored with 14 colors, then each color class has exactly 20 vertices from $\mathcal{U}_3$. Again, using the computer program, we can show that if an independent set in $\mathcal{U}_3 \cup \mathcal{U}_7$ has 20 vertices from $\mathcal{U}_3$, then it has at most 7 vertices from $\mathcal{U}_7$. Therefore 14 colors are not enough. □

Notice, that this proof can be restated as an example of a more general approach. In this approach we first assign weights to the vertices of a graph. A lower bound on the chromatic number of the graph is then obtained as the ratio of the sum of weights of all the vertices and the weight of the maximum weighted independent set of the graph. In our case, we could for example set weight of each vertex of $\mathcal{U}_3$ to 1000 and weight of each vertex of $\mathcal{U}_7$ to 1.

Chapter 4

Other interesting spaces

4.1 Introduction

In this chapter we will look at some other metric spaces. We will focus on real and rational spaces and integer grids with metrics $l_1 \dots l_\infty$.

The distance between $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in the metric l_k is defined as

$$\mathrm{dist}_k(x, y) = \|x - y\|_k = \sqrt[k]{\sum_{i=1}^{n} |x_i - y_i|^k}.$$

The distance defined by the metric l_2 is the Euclidean distance which was used in the previous chapters.

In the metric l_∞, the distance between x and y is

$$\mathrm{dist}_\infty(x, y) = \|x - y\|_\infty = \lim_{k \to \infty} \sqrt[k]{\sum_{i=1}^{n} |x_i - y_i|^k} = \max_{i \in \{1,2,\dots,n\}} |x_i - y_i|.$$

We will use $\mathbb{R}_k^n$, $\mathbb{Q}_k^n$ and $\mathbb{Z}_k^n$ to denote the n-dimensional spaces with the l_k metric, where $k \in \{1, \dots \infty\}$.

4.2 Normed spaces in general

Definition 1. *Let $\mathbb{V}$ be a vector space over a subfield F of $\mathbb{R}$.* Norm $\|\cdot\|$ *on $\mathbb{V}$ is a function from $\mathbb{V}$ to nonnegative real numbers that satisfies*

1. $\forall a \in F\ \forall v \in \mathbb{V} : \|a \cdot v\| = |a| \cdot \|v\|$
2. $\forall u, v \in \mathbb{V} : \|u + v\| \leq \|u\| + \|v\|$

The distance between points $x, y \in \mathbb{V}$ is then defined as $\|x - y\|$.

All spaces with the $l_1 \dots l_\infty$ metrics are normed spaces.

Chilakamarri studied the chromatic number of the plane with an arbitrary norm.

Theorem 8. *(Chilakamarri 1991 [3])*
Let $\mathbb{V} = (\mathbb{R}^2, \|\cdot\|)$ be a normed vector space. Then the chromatic number of $\mathbb{V}$ satisfies

$$4 \leq \chi(\mathbb{V}) \leq 7.$$

In addition, if the unit circle is a parallelogram or a hexagon, the chromatic number is exactly 4.

Another interesting result is the exponential upper bound on the chromatic number of any n-dimensional normed space.

Theorem 9. *(Füredi, Kang [9])*
Let n be large enough and $\mathbb{V} = (\mathbb{R}^n, \|\cdot\|)$ be a normed vector space. Then the chromatic number of the space $\mathbb{V}$ satisfies

$$\chi(\mathbb{V}) \leq c(n \ln n)5^n.$$

However, it is not known whether there is any exponential lower bound, that would work for every normed space.

4.3 Spaces with the l_∞ metric

The situation with the l_∞ metric is simple and the values of the chromatic number are known in all dimensions.

Proposition 1. *For any dimension n*

$$\chi(\mathbb{R}^n_\infty) = \chi(\mathbb{Q}^n_\infty) = \chi(\mathbb{Z}^n_\infty) = 2^n.$$

Proof. Because $\chi(\mathbb{R}^n_\infty) \geq \chi(\mathbb{Q}^n_\infty) \geq \chi(\mathbb{Z}^n_\infty)$, it is enough to show that $\chi(\mathbb{Z}^n_\infty) \geq 2^n$ and $\chi(\mathbb{R}^n_\infty) \leq 2^n$.

To color the space $\mathbb{R}^n_\infty$, we assign to each point x the color defined by the n-element vector whose i-th element is equal to the parity of $\lfloor x_i \rfloor$. If $\|x - y\|_\infty = 1$ then one of the coordinates differs by 1 and x and y have different colors.

To show the lower bound, we take all the points whose all coordinates are either 1 or 0. For each two of the points, there is at least one coordinate in which they differ by 1 and in each other coordinate they differ by at most 1. Thus each two points are distance 1 apart and the points form a complete graph on 2^n vertices.

□

4.4 Spaces with the $l_1, l_2, \ldots$ metrics

4.4.1 Real spaces

Like with the l_2 metric, in any l_i metric, both best upper and lower bounds are exponential. The exponential upper bound follows from Theorem 9 and the exponential lower bound from the following

Theorem 10. *(Füredi, Kang 2004 [8])*
Let q be an odd prime and let $n = 4q - 1$. For every natural number k,

$$\chi(\mathbb{R}^n_k) \geq 1.139^n.$$

4.4.2 Rational spaces of small dimensions

We can use a lemma analogous to Lemmas 6 and 7:

Lemma 10.
For any k and n

$$\chi(\mathbb{Q}_k^n) = \max_{r\in\mathbb{N}}\{\chi(\mathbb{Z}_k^n, r)\}.$$

For any odd $r \in \mathbb{N}$ and any $k, n \in \mathbb{N}$

$$\chi(\mathbb{Z}_k^n, r) = 2.$$

The proof of the lemma is the same as in the case of the l_2 metric.

Theorem 11. *For any k and any dimension $n < 2^k$*

$$\chi(\mathbb{Q}_k^n) = 2.$$

Proof. Thanks to Lemma 10, we just need to prove $\chi(\mathbb{Z}_k^n, r) = 2$ if r is even. Take any $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ at distance r and let $\delta_i = x_i - y_i$.

$\delta_i \mod 2$	$\delta_i^k \mod 2^k$
0	0
1	1

Because the number of δ_i is smaller than 2^k and their sum is a multiple of 2^k, they all need to be even. Therefore we can shrink the graph and $\chi(\mathbb{Z}_k^n, r) \le \chi(\mathbb{Z}_k^n, r/2) \le 2$.
□

4.4.3 The l_1 metric

The l_1 metric was studied by Zoltán Füredi and Jeong-Hyun Kang [8]. In this subsection we will present some of their results.

Theorem 12. *(Füredi, Kang 2004 [8])*
For any natural numbers r and n

$$\chi(\mathbb{Z}_1^n, 2r) \ge 2n.$$

In addition,

$$\chi(\mathbb{Z}_1^n, 2) = 2n.$$

Proof. We take all the points with one coordinate equal to r or $-r$ and the others equal to 0. Each two of the points are at distance $2r$ from each other. Therefore their $2r$-distance graph is a complete graph on $2n$ vertices.

If the forbidden distance is 2, we define the color of $x = (x_1, x_2, \ldots, x_n)$ as

$$f(x) = \left\lfloor \sum_{i=1}^{n} \left(i - \frac{1}{2}\right) x_i \right\rfloor \pmod{2n}.$$

The coloring uses $2n$ colors and a simple case analysis shows that it is a proper coloring.
□

Proposition 2. *(Füredi, Kang 2004 [8])*

$$\chi(\mathbb{Z}_1^2, 2) = \chi(\mathbb{Q}_1^2) = \chi(\mathbb{R}_1^2) = 4$$

Proof. Theorem 12 provides the inequality $\chi(\mathbb{Z}_1^2, 2) \geq 4$.

The coloring of the plane depicted in Figure 4.1 uses 4 colors and thus $\chi(\mathbb{R}_1^2) \leq 4$. $\square$

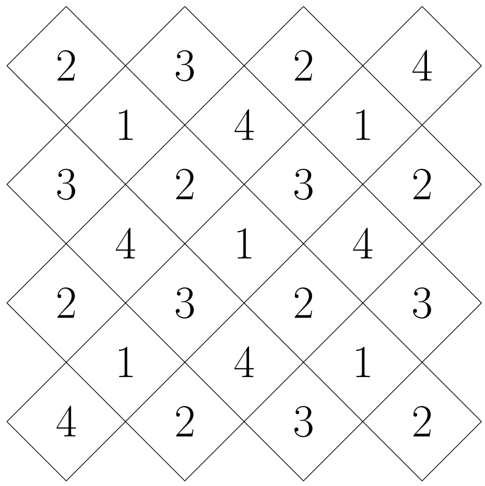

Figure 4.1: The 4-coloring of the plane with the l_1 metric. The diagonals of the squares have length 1. Boundary points get the color of the rightmost of the neighboring squares.

Notice that $\chi(\mathbb{R}_1^2) = 4$ can also be obtained as a corollary of Theorem 8.

4.5 Spaces $(d\mathbb{Q})^n$

In this section, we will be using only the Euclidean distance. We will study the chromatic number of spaces $(d\mathbb{Q})^n$, which is a vector space over rational multiples of some irrational number d. This is equivalent to determining $\chi(\mathbb{Q}^n, 1/d)$.

4.5.1 Dimensions 2, 3 and 4

Lemma 11. *For any $n \in \mathbb{N}$ and $d \in \mathbb{R}^+$*

$$\chi(\mathbb{Q}^n, d) = \max_{r \in \mathbb{N}} \{\chi(\mathbb{Z}^n, d \cdot r)\}.$$

The proof is similar to the proof of Lemma 6.

Theorem 13. *For any positive real number d:*

$$\begin{aligned} \chi(\mathbb{Q}^2, d) &\leq 2, \\ \chi(\mathbb{Q}^3, d) &\leq 4, \\ \chi(\mathbb{Q}^4, d) &\leq 4. \end{aligned}$$

Proof. It is enough to prove $\chi(\mathbb{Z}^2, d) \leq 2$, $\chi(\mathbb{Z}^3, d) \leq 4$ and $\chi(\mathbb{Z}^4, d) \leq 4$ for any positive real number d.

Let $\delta_i = y_i - x_i$.

- $d^2 \notin \mathbb{N}$

  The unit-distance graph has no edges and its chromatic number is 1.

- d^2 is odd

  The unit-distance graph can be colored with 2 colors.

- $d^2 \equiv 0 \mod 4$

  - $n \leq 3$

    All δ_i are even. We can thus shrink any unit-distance graph with the highest chromatic number and $\chi(\mathbb{Z}^n, d) \leq \chi(\mathbb{Z}^n, d/2)$.

  - $n = 4$

    If $d^2 \equiv 0 \mod 8$, then all δ_i are even and $\chi(\mathbb{Z}^n, d) \leq \chi(\mathbb{Z}^n, d/2)$. If $d^2 \equiv 4 \mod 8$, then either all δ_i are odd or for one i, $\delta_i \equiv 2 \mod 4$ while for all the other i, $\delta_i \equiv 0 \mod 4$. The colors then can be defined by the pair (parity of x_1, parity of $\lfloor \sum_{i=1}^{4} x_i/2 \rfloor$).

- $d^2 \equiv 2 \mod 4$

  Exactly 2 of δ_i are odd and the rest is even.

  - $n = 2$

    Parity of the first coordinate differs in each two adjacent vertices. Therefore the graph is bipartite.

  - $n = 3$

    Parity of the first or the second coordinate differs in each two adjacent vertices. Therefore we can define the color of x to be the pair (parity of x_1, parity of x_2) and 4 colors are enough.

  - $n = 4$

    We color x by the the pair (parity of $(x_1 + x_2)$, parity of $(x_1 + x_3)$) and a simple case analysis shows that it is a proper coloring.

□

For some choice of d, these lower bounds can be achieved. For example, if $d = \sqrt{2}$, the four points $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(0, 1, 1)$ form a clique, thus $\chi(\mathbb{Z}^4, \sqrt{2}) = \chi(\mathbb{Z}^3, \sqrt{2}) = 4$.

4.5.2 Higher dimensions

The new construction for the lower bound on the chromatic number of the five-dimensional real space presented in Section 2.3 gives the lower bound

$$\chi(\mathbb{Q}^5, \sqrt{2}) \geq \chi(\mathbb{Z}^5, \sqrt{8}) \geq 9.$$

The construction of Raigorodskii (see Subsection 2.1.5), which was used to show the best lower bound on the chromatic number of high-dimensional real spaces, gives

$$\max_{r\in\mathbb{N}} \left\{\chi\left(\mathbb{Z}^n, \sqrt{r}\right)\right\} \geq (1.239 + o(1))^n.$$

In the construction, this is achieved with large values of r, particularly $r \approx n/2$. As we will now show, this would not be possible for small r.

Theorem 14. *For any natural numbers n and r*

$$\chi(\mathbb{Z}^n, \sqrt{r}) \leq 2^r \binom{n+r-1}{r} + 1 \leq \left(\frac{2e(n+r)}{r}\right)^r$$

Proof. We will bound the number of points with integer coordinates at distance $\sqrt{r}$ from a given point $u \in \mathbb{Z}^n$. Let v be at distance $\sqrt{r}$ from u and let $\delta = u - v$. Then $\sum_{i=1}^n \delta_i^2 = r$. Vector δ is thus in the set

$$\mathcal{D}_n = \{\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{Z}^n : \sum_{i=1}^n \delta_i^2 = r\}.$$

Size of the set $\mathcal{D}_n$ equals the number of points with integer coordinates at distance $\sqrt{r}$ from u and thus $\chi(\mathbb{Z}^n, r) \leq |\mathcal{D}_n| + 1$.

Each nonzero δ_i can be positive or negative, which gives at most 2^r possibilities to assign signs. Let

$$\mathcal{D}'_n = \{\delta' = (\delta'_1, \delta'_2, \ldots, \delta'_n) \in \mathbb{Z}^n : \ \forall i : \delta'_i \geq 0 \ \& \ \sum_{i=1}^n \delta'_i = r\}.$$

Number of $\delta \in \mathcal{D}_n$ with all coordinates nonnegative is equal to the number of $\delta' \in \mathcal{D}'_n$ such, that each δ'_i is a square of an integer. Thus $|\mathcal{D}_n| \leq 2^r|\mathcal{D}'_n|$. Elements of $\mathcal{D}'_n$ are in a bijection with the set of vectors with r 1-elements and $n-1$ 0-elements. The bijection is defined as

$$(\delta'_1, \delta'_2, \ldots, \delta'_n) \leftrightarrow (1^{\delta'_1}, 0, 1^{\delta'_2}, 0, \ldots, 0, 1^{\delta'_n}),$$

where 1^t means t 1-elements in a row. Thus $\chi(\mathbb{Z}^n, r) \leq 2^r|\mathcal{D}'_n| + 1 = 2^r\binom{n+r-1}{r} + 1$.

The second inequality of the claim needs just a little bit of counting:

$$2^r\binom{n+r-1}{r} + 1 = 2^r\left(\binom{n+r}{r} - \binom{n+r-1}{r-1}\right) + 1 \leq 2^r\binom{n+r}{r} \leq \left(\frac{2e(n+r)}{r}\right)^r,$$

where the last inequality follows from the well known formula

$$\binom{m}{k} \leq \left(\frac{em}{k}\right)^k.$$

□

Corollary 1.

1. *If $r(n) = o(n)$, then $\chi\left(\mathbb{Z}^n, \sqrt{r(n)}\right)$ grows slower than exponentially, that is*
$$\log\left(\chi\left(\mathbb{Z}^n, \sqrt{r(n)}\right)\right) = o(n).$$

2. *If $r \leq n/23$, then*
$$\chi(\mathbb{Z}^n, \sqrt{r}) \leq 1.236^n.$$

Proof.

1. The presumption $\lim_{n\to\infty} r(n)/n = 0$ implies
$$\lim_{n\to\infty} \frac{r(n)}{4en} \log\left(\frac{4en}{r(n)}\right) = 0.$$
Therefore
$$\lim_{n\to\infty} \frac{\log(\chi(\mathbb{Z}^n, r(n)))}{n} \leq \lim_{n\to\infty} \frac{r(n)}{n} \log\left(\frac{2e(n + r(n))}{r(n)}\right) \leq \lim_{n\to\infty} \frac{r(n)}{n} \log\left(\frac{4en}{r(n)}\right) = 0.$$

2. A simple calculation shows that for a fixed positive value of n, the function
$$f(r) = \left(\frac{2e(n + r)}{r}\right)^r$$
is nondecreasing on $(0, \infty)$.
Therefore, if $r \leq n/23$, then
$$\chi(\mathbb{Z}^n, r) \leq \left(\frac{2e(\frac{24}{23}n)}{\frac{1}{23}n}\right)^{\frac{1}{23}n} = \left(\sqrt[23]{48e}\right)^n \leq 1.236^n.$$

□

For $r = 2$, Theorem 14 gives a quadratic upper bound, but we can find a linear upper bound. A simple modification to the coloring from the proof of Theorem 12 provides the upper bound $\chi(\mathbb{Z}^n, \sqrt{2}) \leq 2n - 2$. But we can obtain even stronger result.

Theorem 15. *For any dimension n, let t be the smallest power of 2 at least as large as n. Then*
$$\chi(\mathbb{Z}^n, \sqrt{2}) \leq t.$$

Proof. We will show that if n is a power of two, then $\chi(\mathbb{Z}^n, \sqrt{2}) \leq n$.

For $i \in \{0, 1, \dots, \log_2 n - 1\}$, let I_i be the set of indices of coordinates, whose binary digit on position i is 1, that is
$$I_i = \{j \in \{0, 1, \dots, n - 1\} : \left\lfloor \frac{j}{2^i} \right\rfloor \equiv 1 \mod 2\}.$$

Let the color of a point $x = (x_0, x_1, \ldots, x_{n-1})$ be the vector $(c_0, c_1, \ldots, c_{log_2 n-1})$, where

$$c_i = \left(\sum_{j \in I_i} x_j \right) \mod 2.$$

There are $2^{log_2 n} = n$ possible colors. If two points are at distance $\sqrt{2}$ then they differ in exactly two coordinates by 1. The indices of the two coordinates differ in some of their binary digits, let i denote the digit. Then the i-th element of the vector of the color differs in the two points and the points have different colors.

□

The upper bound is tight for $n \leq 8$. This is trivial for $n \leq 2$. If $n > 2$, it follows from the results already presented, in particular from $\chi(\mathbb{Z}^3, \sqrt{2}) \geq 4$ (see Subsection 4.5.1) and $\chi(\mathbb{Z}^5, \sqrt{2}) \geq 8$ (see the construction of the five-dimensional half-cube in Subsection 2.1.3).

Chapter 5

Conclusions and future directions

In real and rational spaces, the gaps between the best lower and upper bounds on the chromatic numbers are still huge and offer lots of opportunities for new results.

The small-dimensional rational spaces seem to be more promising. All best lower bounds in dimensions from 5 to 8 were determined by a computer program, although in dimensions 5 and 8, the lower bound was also proven theoretically without help of a computer. Most of these lower bounds were found by Matthias Mann, who constructed a unit-distance graph and then exactly determined its chromatic number. In this thesis, we presented a new approach based on determining only a lower bound on the chromatic number. We applied it in dimension 7, where we used a computer program to show that a graph has no independent set with given properties, which lead to a new lower bound.

The approach using the independent sets has the advantage, that it is usually much faster to find a maximum independent set than to determine the chromatic number. As a consequnce, all the graphs used by Matthias Mann have less than a hundred of vertices while our graph for $\mathbb{Q}^7$ has 408 vertices. On the other hand, the lower bound calculated using the maximum independent set is often smaller than the exact chromatic number of the graph.

Particularly, it would be interesting to improve the lower bound in the five-dimensional rational space, which is the smallest dimension in which the exact value is not known. Chilakamarri [4] conjectures that $\chi(\mathbb{Q}^5) = 8$ and thus if this conjecture is true, the best known lower bound, which has value 7, can be increased only by 1.

Another interesting result would be to improve the lower bound in $\mathbb{Q}^8$, because this would also increase the lower bound in $\mathbb{R}^8$.

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Appendix A

Program used in the proof of Theorem 7

```
/* The purpose of this program is to prove that the chromatic number of the
 * 7-dimensional rational space is at least 15. This is done by showing that
 * the chromatic number of the graph, whose vertices are all the points with
 * three or seven coordinates from {1,-1} and the rest of coordinates equal
 * to 0, has chromatic number at least 15. There are 280 vertices with 3
 * nonzero coordinates and 128 vertices with 7 nonzero coordinates.
 * An independent set will be called 'bad' if it has more than 20 vertices
 * with 3 nonzero coordinates or 20 vertices with 3 nonzero coordinates and
 * more than 7 vertices with 7 nonzero coordinates. The program shows that
 * there is no bad independent set which implies that the chromatic number
 * of the graph is at least 15.
 */

#include <stdio.h>
#include <stdlib.h>
#include <assert.h>

#define DIM 7 // dimension of the space
#define ADJ_DIST 4 // square of the unit-distance
#define MAXVER 1000 // maximum number of vertices for which we will have space


typedef struct Gr
{
  int vernum; // number of vertices
  int edgelist[MAXVER+2][MAXVER+2]; // edgelist[v] is list of neighbors of v
  int edgenum[MAXVER+2]; // number of neighbors of the vertex v
} Gr;

struct Gr g;

int vervec[MAXVER+2][DIM+2]; // vervec[v] is vector of coordinates of v
int avail[MAXVER+2]; // vertex is available if it is in or can be added to the
                     // independent set
```

```
int numav3, numav7; // number of available vertices with 3 (resp. 7) nonzero
                    // coordinates
int unavailed[MAXVER+2]; // from when the vertex is not available
int minbr = 1000000; // used for output of progress


void addedge(int from, int to)
{
  g.edgelist[from][g.edgenum[from]++]=to;
  g.edgelist[to][g.edgenum[to]++]=from;
}


/* backtracking function for searching for a bad independent set
 * v is the vertex on which we will backtrack */
void isbt(int v)
{
  int i, tmp;

  // stop if there cannot be a bad independent set
  if(numav3<20 || (numav3==20 && numav7<8)) return;
  // check whether we have not found the bad independent set
  if(v==g.vernum) { printf("\nThe assumption is not true.\n"); exit(0); }

  // try not adding v
  if(v>0) // we can permute coordinates to have vertex 0 in independent set
  {
    // update the array avail, numav3 and numav7
    if(avail[v])
    {
      if(v<280) numav3--; else numav7--;
      avail[v] = 0;
      unavailed[v] = v;
    }
    //backtrack
    isbt(v+1);
    // restore the array avail, numav3 and numav7
    if(unavailed[v]==v)
    {
      avail[v] = 1;
      if(v<280) numav3++; else numav7++;
    }
    if(v<minbr) { minbr=v; printf("Branched at %d\n", v); } // output progress
  }
  if(!avail[v]) return; // do not try adding v which is not available

  // try adding v
  // update the array avail, numav3 and numav7
  for(i=0;i<g.edgenum[v];i++)
  {
```

```
    tmp = g.edgelist[v][i];
    if(tmp>v && avail[tmp])
    {
      avail[tmp]=0;
      unavailed[tmp]=v;
      if(tmp<280) numav3--; else numav7--;
    }
  }
  // backtrack
  isbt(v+1);
  // restore the array avail, numav3 and numav7
  for(i=0;i<g.edgenum[v];i++)
  {
    tmp = g.edgelist[v][i];
    if(unavailed[tmp]==v)
    {
      avail[tmp]=1;
      if(tmp<280) numav3++; else numav7++;
    }
  }
}


int tmpvec[DIM+2]; // vector of the newly generated vertex

/* backtracking function for generating vertices
 * el is the currently generated coordinate
 * nonz is the number of nonzero coordinates that remain */
void gen_vec(int el, int nonz)
{
  int i;

  if(nonz<0) return;
  if(el==DIM)
  {
    if(nonz==0)
    { // add the new vertex
      for(i=0;i<DIM;i++) vervec[g.vernum][i] = tmpvec[i];
      g.vernum++;
    }
    return;
  }
  // try setting all possible values to coordinate el and backtrack
  tmpvec[el] = 0;
  gen_vec(el+1, nonz);
  tmpvec[el] = 1;
  gen_vec(el+1, nonz-1);
  tmpvec[el] = -1;
  gen_vec(el+1, nonz-1);
}
```

```
int main(int argc, char **argv)
{
  int i,j,k;
  int dist;

  // generate vertices
  g.vernum=0;
  gen_vec(0,3);
  assert(g.vernum == 280);
  gen_vec(0,7);
  assert(g.vernum == 408);

  // generate edges
  for(i=0;i<g.vernum;i++) for(j=i+1;j<g.vernum;j++)
  {
    dist=0;
    for(k=0;k<DIM;k++) if(vervec[i][k]!=vervec[j][k])
    {
      if(vervec[i][k]==0 || vervec[j][k]==0) dist++; else dist+=4;
    }
    if(dist==ADJ_DIST) addedge(i,j);
  }

  // prepare for searching the ind. set
  for(i=0;i<g.vernum;i++) { avail[i]=1; unavailed[i]=-1; }
  numav3 = 280;
  numav7 = 128;
  // search for a bad independent set
  isbt(0);
  printf("The chromatic number of Q^7 is at least 15.\n");
  return 0;
}
```