Essays in Market Design

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A dissertation

submitted to

the Department of Economics

in partial fulfillment of the requirements

for the degree of

Doctor of Philosophy

Boston College
Morrissey College of Arts and Sciences
Graduate School

March 2021

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This dissertation consists of two essays in market design.

In the first chapter, we study affirmative action policies in college admissions and hiring. A college or firm makes admissions or hiring decisions in which each candidate is characterized by priority ranking and type, which may depend on race, gender, or socioeconomic status. The admissions or hiring committee faces a trade-off between meritocracy and diversity: while a *merit-first choice rule* may admit candidates of the same type, a *diversity-first choice rule* may be unfair due to priority violations. To formalize this trade-off, we introduce a measure of meritocracy and a measure of diversity for choice rules. Then, we investigate how to resolve the tension between them. A choice rule that uses both *reserves* and *quotas* can be viewed as a compromise and is a generalization of the two extreme rules. The first result is comparative statics for this class of choice rules: we show that as parameters change and the choice rule becomes more meritorious, it also becomes less diverse. The second result is a characterization of the choice rule, which may help admissions or hiring committees to decide their policies.

In the second chapter, we introduce a method to measure manipulability of a matching mechanism and use theory and simulation to study *constrained mechanisms* in

school choice. First, we show that the implications from existing measures are strongly dependent on the full preference domain assumption. Our measure is more robust. The implications from existing measures can be carried over as well: while the recent school admissions reforms did not fully eliminate incentives to manipulate, they discouraged manipulation. Second, we use simulations for quantitative analysis. Our results support the recent school admissions reforms quantitatively, as well as qualitatively: they largely eliminated the incentives to manipulate. In addition, while the qualitative implications from theory are parallel to existing measures, the quantitative implications from simulations confirm a significant difference.

Acknowledgement

First, I would like to express my sincere gratitude to Professor M. Utku Ünver, Professor M. Bumin Yenmez, and Professor Tayfun Sönmez. Their research on market design is insightful, and their series of studies have heavily influenced my research. Professor Ünver kindly agreed to become my advisor. My research questions were not polished initially, but he has advised me patiently and led me to the accomplishment of this thesis. I have learned the essence of market design through invaluable discussions with him. Thank you, Professor Ünver. I hope to make a significant contribution to the field of market design.

Professor Yenmez was also my advisor and the one with whom I talked about my research most often. The most fortunate in my Ph.D. program was that he moved to Boston College during my second year. He carefully listened to any single progress of my research and always gave me very sharp but warm comments on it. The first chapter of this thesis is based on his excellent research. He also cheered me up during the job search period, especially when I was not confident of getting a good academic job. Thank you, Professor Yenmez. Your continuous encouragement helped me working on the projects, which led to completing this thesis. I hope to research with you in the future.

Professor Sönmez is very strict with others and with himself. The worst research presentation in my Ph.D. program was the first presentation at the dissertation workshop in front of him. I was sometimes depressed by his harsh comments, but my research and presentation have improved thanks to his help. His research is exciting and has a lot of real-life applications. The second chapter is based on his research. Thank you, Professor

Sönmez. I respect your research attitude and will try my best to be as close to it as possible.

I thank Professor Keisuke Bando, Professor Kentaro Tomoeda, and Professor Hideo Konishi. Professor Bando led me into the field of market design. Professor Tomoeda is my coauthor and friend. The second chapter was started when he was staying at Boston College. Professor Konishi supported me both in research and life.

My graduate school experience could not have been the same without my friends. Special thanks to my classmates, Norihiro Komura and Tomohide Mineyama. Although our research fields are different, I have learned a lot from them. I would also like to thank Kohei Iwasaki, Manshu Khanna, Marco Antonio Robles Sanchez, and Akina Yoshimura for discussing my research ideas during the graduate study at Boston College.

Finally, I am grateful to my parents Tsuyoshi and Fumiko Imamura, for caring about my life, health, and the progress of my thesis from Japan.

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Chapter 1

Meritocracy versus Diversity

1.1 Introduction

Affirmative action is a scheme to promote diversity based on different characteristics, such as gender and race. Furthermore, it is used to close socioeconomic gaps that exist between various groups in society. To this end, it involves policies designed to increase the participation of underrepresented groups in public areas such as employment, education, and business contracting, thereby giving these groups a higher chance of participation.

Affirmative action policies highlight the basic tension between meritocracy and diversity. Since they give a higher chance to underrepresented groups, they may not be meritorious: while a more meritorious candidate may be rejected, a less meritorious candidate from an underrepresented group may be admitted. On the other hand, a policy based on only merit does not promote diversity, as many candidates from the overrepresented group may be admitted, since they are likely to be more meritorious than candidates from the other groups.

The tension between meritocracy and diversity can be a very contentious issue. For example, Harvard's college admissions policies were brought to court by an organization representing a group of Asian-American students called "Students for Fair Admissions." The plaintiffs claimed that Harvard used race and ethnicity as a predominant factor in admissions decisions for diversity purposes, which intentionally discriminated against Asian-American applicants.[1] However, this claim was rejected by a federal judge who stated that the university met the strict constitutional standards for considering race in its admissions process. Moreover, the judge discussed the benefits of diversity, such as fostering tolerance, acceptance, and understanding that will ultimately make race-conscious admissions obsolete. She further stated that it was not yet time to look beyond race in college admissions, which suggests that the current race-conscious admissions process should be justified due to diversity concerns.[2]

While there is rich market-design literature on how to incorporate various diversity constraints into a choice rule, little attention has been paid to how the choice of these constraints affects meritocracy. In this paper, we propose a measure of meritocracy and a measure of diversity to formalize the tension and a potential resolution. We obtain a choice rule generated by reserves and quotas (*reserves-and-quotas rule*) as a resolution of the tension. Reserves provide soft lower bounds while quotas provide hard upper bounds on the number of candidates of different types. The first result is comparative statics in this class of choice rules: as parameters change and the choice rule becomes more meritorious, it also becomes less diverse. While we focus on the specific class of choice rules, it includes choice rules in market-design literature. We obtain two choice rules from the literature—the responsive rule and the ideal-distribution rule—as two extremes of the class of reserves-and-quotas rules. The responsive rule is the most meritorious and least diverse rule among this class, whereas the ideal-distribution rule is the least meritorious and most diverse rule.

The second result is a characterization of the reserves-and-quotas rule, which could help colleges or firms decide their admissions or hiring policies. Two out of the four axioms are particularly important. The first axiom is *across-types* $\succ$*-compatibility*. We face the trade-off between meritocracy and diversity. This axiom states when the college uses the priority

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[1] Students for Fair Admissions also accused the University of North Carolina and the University of Texas of discrimination. See https://candidatesforfairadmissions.org/about/.

[2] For more details, see the following NY Times article: https://www.nytimes.com/2019/10/01/us/harvard-admissions-lawsuit.html.

and when it concerns diversity. The other axiom is *substitutability*, which is necessary for the existence of stable (or fair) matchings. It also enables us to use the most popular mechanism in practice—the deferred acceptance algorithm—to find a stable matching.

The reserves-and-quotas rule is also closely related to the recently developed *reserve system* literature, which analyzes the allocation of resources by reserves. A key observation is that allocation depends on not only the size of reserve but also the *order* which seat is processed (Dur et al. [21]). In terms of the order, our choice rule processes all reserves before open seats. We provide another characterization to understand the relationship between our choice rule and alternative reserve systems. In particular, we introduce a new axiom *meritorious monotonicity in reserve and quota size*, which is motivated by our comparative statics. This axiom ensures transparency: when a diversity constraint by reserves and quotas is relaxed, a choice rule becomes more meritorious. The reserves-and-quotas rule is the only natural choice rule which satisfy the new axiom. In terms of reserves system, processing reserves first is the only transparent way.

We provide two additional results. First, we study the case of endogenous priorities. The priority is exogenous in our model. The first characterization is generalized to the case in which not only reserves and quotas, but also the priority, are endogenous. Second, we study a *separable* choice rule. This property states that the college never uses the priority to compare candidates of different types. While the ideal-distribution rule is separable, there are other separable choice rules. We study the class of separable choice rules with the new property.

The rest of Chapter 1 is organized as follows. Section 2 presents the model. Section 3 presents our main results. Section 4 provides a characterization. Section 5 provides additional results. Section 6 provides the concluding remarks. Appendix provides the proofs of the results and verifies the independence of the axioms.

1.1.1 Related literature

Our results are closely related to the results obtained by Echenique and Yenmez [24]. They study the tension between the existence of stable matchings and diversity concerns. Three choice rules are proposed to resolve this tension; specifically, they characterize the ideal-distribution rule, the reserves rule, and the quotas rule. Doğan [20] corrects the mistake in the characterization of the reserves rule. These choice rules are included in the class of our reserves-and-quotas rules. While our first characterization builds on their results, there are two major differences in our results. First, we identify the role of substitutability, which is the main axiom in Echenique and Yenmez's characterizations, by providing another characterization of our choice rule without substitutability. Second, and perhaps more importantly, we provide not only the characterizations but also the trade-off between meritocracy and diversity, which is the main topic in our work.

In two recent papers, Erdil and Kumano [26] and Kojima et al. [37] consider choice rules similar to the ideal-distribution rule. However, their models and motivations are different from those in this paper. First, their models are different. Erdil and Kumano [26] allow indifferences in the priority, whereas the preference is strict in our model. Also, while their choice rule assigns empty reserved seats for one type to other types of candidates, the ideal-distribution rule never assigns them due to diversity concerns. Kojima et al. [37] consider a matching market in which there are multiple colleges and multiple candidates. However, this paper mainly focuses on the case of one college and multiple candidates. Also, Kojima et al. [37] do not assume types of candidates. Second, the motivations in their papers differ. Erdil and Kumano [26] focus on inefficiency due to indifferences and study how to improve this. Kojima et al. [37] study how to incorporate distributional constraints into matching theory. To do so, Kojima et al. [37] change the definition of stability and show the existence of stable matchings using techniques in discrete convex analysis. We study how diversity constraints affect meritocracy and provide a formal trade-off.

Diversity concerns have been widely studied in school choice literature. Abdulkadiroğlu and Sönmez [2] note the problem of school choice with diversity concerns. They propose a choice rule generated by quotas, which have been analyzed by Abdulkadiroğlu [1] and Ergin and Sönmez [27]. Kojima [35] shows an impossibility result: affirmative action policies based on majority quotas may hurt minority candidates. To overcome this difficulty, Hafalir et al. [32] propose an affirmative action based on minority reserves. Doğan [19] proposes another solution which never hurts minority candidates. More generally, Ehlers et al. [25] study affirmative action policies when there are both upper and lower type-specific bounds. They propose solutions based on whether these bounds are hard or soft. The reserves-and-quotas rule in this paper can be regarded as a choice rule with hard upper and soft lower type-specific bounds. The hard lower type-specific bounds are further analyzed by Fragiadakis [28], Fragiadakis and Troyan [29], and Tomoeda [48]. Other papers consider specific choice rules similar to the choice rule generated by reserves (Aygun and Bó [4]; Dur et al. [21]; Dur et al. [22]; Kominers and Sönmez [38]; Westkamp [49]). These papers study how to incorporate diversity constraints into school choice. In this paper, we investigate the impact of diversity constraints on meritocracy.

Recently, affirmative action with complex constraints have been studied in various settings (Aygun and Turhan [6]; Baswana et al. [8]; Hamada et al. [39]; Sönmez and Yenmez [45] and [46]; Thakur [47]). For example, several papers study a model in which candidates possibly have multiple types. However, this paper assumes that each candidate has one type. These models are outside the scope of our analysis.

1.2 Preliminaries

This section presents the model and preliminary results. We describe the model in the first subsection and the choice rules in the literature in the second subsection. Then, in the third subsection, we study the ideal-distribution rule by introducing a new property.

1.2.1 Model

Let $\mathcal{S} = \{s_1, ..., s_n\}$ be a nonempty finite set of candidates. There is a college. The college's choice rule C is a function that maps for each nonempty set $S \subseteq \mathcal{S}$ to a subset $C(S) \subseteq S$. Let k denote the capacity of the college. It is such that $|C(S)| \leq k$ for each $S \subseteq \mathcal{S}$. A priority is a binary relation $\succ$ on S that is complete, transitive and antisymmetric. The candidates is partitioned into d different types. Let $T = \{t_1, ..., t_d\}$ be the set of types and $\tau : \mathcal{S} \to T$ be the type function. Let $\mathcal{S}_t$ be the set of type t candidates; i.e., $\mathcal{S}_t \equiv \{s \in \mathcal{S} : \tau(s) = t\}$. Similarly, for each $S \subseteq \mathcal{S}$, let S_t be the set of type t candidates in S; i.e., $S_t \equiv S \cap \mathcal{S}_t$. We use a function $\xi : 2^{\mathcal{S}} \to \mathbf{Z}_+^{\mathbf{d}}$ to describe the number of candidates of each type in each particular set. Thus, $\xi(S) \equiv (|S_{t_1}|, ..., |S_{t_d}|) \in \mathbf{Z}_+^{\mathbf{d}}$ consists of the number of candidates of each type in S. We term $\xi(S)$ the distribution of candidates in S. We shall assume that the college is not large enough to admit all candidates of a given type: $k < |\mathcal{S}_t|$ for every $t \in T$.

1.2.2 Choice rules

We introduce two choice rules. The first choice rule is a responsive rule as meritocracy first. The other choice rule is an ideal-distribution rule as diversity first.

First, we introduce a responsive rule. This rule is meritocracy first: the college always uses the priority to compare candidates of different types.

Definition 1. A choice rule C is **responsive** if for all $S \subseteq \mathcal{S}$

$$C(S) = \begin{cases} S & \text{if } |S| \leq k \\ \{s_1^*, ..., s_k^*\} & \text{otherwise} \end{cases}$$

where $s_1^* = \arg\max_{\succ} S$, and for all $i = 2, ..., k$, $s_i^* = \arg\max_{\succ} S \setminus \{s_1^*, ..., s_{i-1}^*\}$.

Next, we introduce an ideal-distribution rule proposed by Echenique and Yenmez [24].

In this rule, the college sets an ideal distribution of candidate types and minimizes the distance to it.

Definition 2 (Ideal-distribution rule). A choice rule C is generated by an **ideal distribution** for priority $\succ$ if there exists a vector $z^* \in \mathbf{Z}_+^{\mathbf{d}}$ with $\sum_{t \in T} z_t^* \leq k$ such that for all $S \subseteq \mathcal{S}$,

(i) $\xi(C(S))$ is the closest vector to z^* (in Euclidean distance) in $B(\xi(S))$, where $B(x) \equiv \{y \in Z_+^d : \sum_{i \in T} y_i \leq k$ and $\forall i \in T, y_i \leq x_i\}$ and

(ii) type-t candidates in $C(S)$ have a higher priority than all type-t candidate in $S \setminus C(S)$ for all $t \in T$.

In words, the ideal-distribution rule consists of two stages. First, the college chooses a distribution of candidates $\xi(C(S))$ that is as close to z^* as possible. Second, given the distribution $\xi(C(S))$, it admits the highest priority candidates up to $\xi(C(S))_t$ for each type t.

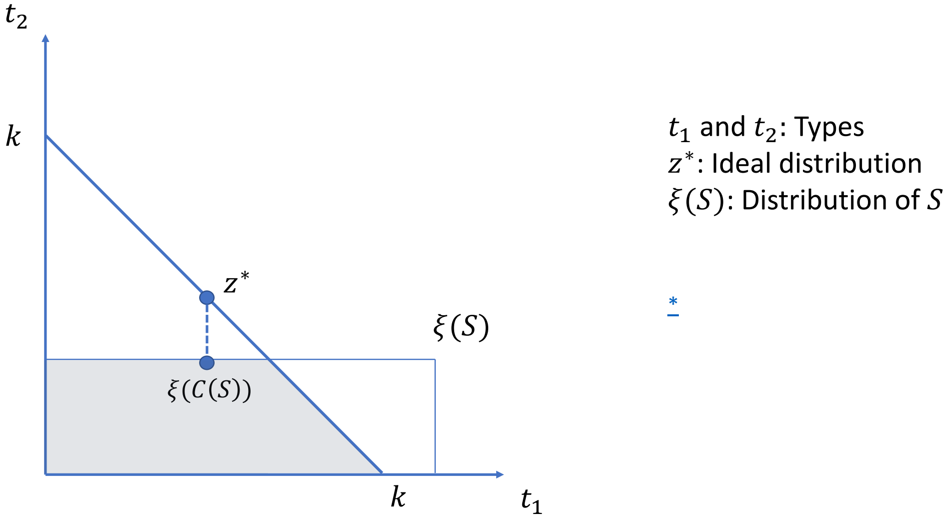

Figure 1.1: Ideal-distribution rule.

1.2.3 Separability: alternative characterization of ideal-distribution rule

We study the ideal-distribution rule by introducing a new property. While the ideal-distribution rule is intuitive and simple, it is extreme: the college never uses the priority to compare candidates of different types. Echenique and Yenmez [24] describe it as diversity first: the rule emphasizes diversity over individual candidates' priorities. We introduce a new property for choice rules to formalize the idea.

Definition 3. A choice rule C is **separable** if there exists a collection of choice rules $\{C_t\}_{t\in T}$ such that

(i) C_t is a choice rule defined on $\mathcal{S}_t$ for each type t and

(ii) $C(S) = \cup_{t\in T} C_t(S_t)$ for all $S \subseteq \mathcal{S}$.

We provide an alternative characterization of the ideal-distribution rule based on separability. The proof is contained in Appendix.

Proposition 1. *A choice rule C is generated by an ideal distribution if and only if there exists $z^* \in \mathbf{Z}^{\mathbf{d}}_{+}$ with $\sum_{t\in T} z^*_t \leq k$ such that C is separable and for each type t, C_t is a responsive rule whose capacity is z^*_t.*

The ideal-distribution rule is extreme in its view of how to resolve the tension between meritocracy and diversity. The college insists in achieving the diversity objective z^*. There are two disadvantages. First, it would create many priority violations: while low priority candidates are admitted, high priority candidates of other types are rejected. Second, it would create many empty seats: if no candidates from some type t apply to this college, z_t seats are unassigned, even if candidates of other types are rejected. We will discuss this issue in the next section.

1.3 Main results

In this section, we study a reserves-and-quotas rule as a potential flexible choice rule. The key property of the ideal-distribution rule is separability: the college *never* uses the priority to compare candidates of different types. The reserves-and-quota rule relaxes it: the college *sometimes* uses the priority to compare candidates of different types. We provide two results for this choice rule. The first result is the trade-off between meritocracy and diversity. The second result is the characterizations.

1.3.1 Reserves-and-quotas rule

We define our main rule, the reserves-and-quotas rule. Reserves provide soft lower bounds while quotas provide hard upper bounds on the number of candidates of different types. The college can use the priority to compare two candidates of different types when their types meet minimum and maximum requirements.

To define the choice rule, given $S \subseteq \mathcal{S}$, let $H(S, l, (l_t)_{t \in T})$ be the largest subset of S that includes the highest priority candidates in S according to $\succ$ such that there are no more than l candidates in total and l_t candidates of type t. Formally, for all $S' \subseteq S$ such that $|S'| \leq l$ and $|S'_t| \leq l_t$ for each $t \in T$, $H(S, l, (l_t)_{t \in T})$ satisfies

(i) $|S'| \leq |H(S, l, (l_t)_{t \in T})|$ and

(ii) if $|S'| = |H(S, l, (l_t)_{t \in T})|$ and $S' \neq H(S, l, (l_t)_{t \in T})$, then for all $s \in H(S, l, (l_t)_{t \in T}) \setminus S'$ and $s' \in S' \setminus H(S, l, (l_t)_{t \in T})$, we have $s \succ s'$.

Definition 4 (Reserves-and-quotas rule). A choice rule C is generated by reserves and quotas for priority $\succ$ if there exist parameters $r = (r_t)_{t \in T}$ and $q = (q_t)_{t \in T}$ such that $r_t \leq q_t$

for each type t, $\sum_{t \in T} r_t \leq k \leq \sum_{t \in T} q_t$, and for all $S \subseteq \mathcal{S}$, $C(S) = C^1(S) \cup C^2(S)$ where

$$C^1(S) \equiv H(S, k, (r_t)_{t \in T}) \text{ and}$$
$$C^2(S) \equiv H(S \setminus C^1(S), k - |C^1(S)|, (q_t - r_t)_{t \in T}).$$

In words, the reserves-and-quotas rule consists of two stages. First (reserves stage), the college considers each type t separately and admits $\min\{|S_t|, r_t\}$ type-t candidates with the highest priority. Second (open seat stage), the college considers the candidates from all types that are rejected at the first stage. Then, it admits these candidates with the highest priority up to its remaining capacity, while keeping its hard upper type-specific bound q_t for each type t.

It is important to mention four points related to the reserves-and-quotas rule. First, the class of reserves-and-quotas rules includes a lot of choice rules in market-design literature. The responsive rule and the ideal-distribution rule are obtained as special cases. The reserves-and-quotas rule is the responsive rule when for each type t, the number of reserves is equal to zero, and the number of quotas is greater than or equal to the capacity, that is, $r_t = 0$ and $q_t \geq k$ for all $t \in T$. The reserves-and-quotas rule is the ideal-distribution rule when for each type t, the numbers of reserves and quotas equal the diversity goal z_t^*, that is, $r_t = z_t^* = q_t$ for all $t \in T$.[3] Clearly, the class of the reserves-and-quotas rule also includes the reserves rule and the quotas rule proposed by Hafalir et al. [32] and Abdulkadiroğlu and Sönmez [2], respectively. Our contribution is to provide the unified framework, the reserves-and-quotas rule, and show the trade-off between meritocracy and diversity based on the parameters $(r_t)_{t \in T}$ and $(q_t)_{t \in T}$.

Second, the reserves-and-quotas rule is a variant of the choice rules proposed in the literature. Ehlers et al. [25] propose the choice rule with soft lower bounds $(r_t)_{t \in T}$ and soft upper bounds $(q_t)_{t \in T}$. The reserves-and-quotas rule is generated by soft lower bounds $(r_t)_{t \in T}$

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[3] The proofs are provided in Appendix.

and hard upper bounds $(q_t)_{t \in T}$. The difference between these rules is that the reserves-and-quotas rule does not admit more than q_t students of type t, whereas the choice rule by Ehlers et al. [25] sometimes admits more than q_t students of type t. Formally, we obtain their choice rule by adding the third stage C^3 to the reserves-and-quotas rule where for all $S \subseteq \mathcal{S}$,

$$C^3(S) \equiv H(S \setminus (C^1(S) \cup C^2(S)), k - |C^1(S) \cup C^2(S)|, (k - q_t)_{t \in T}).$$

Third, the reserves-and-quotas rule is not a unique choice rule to satisfy the constraint by soft lower bounds $(r_t)_{t \in T}$ and hard upper bounds $(q_t)_{t \in T}$; there are other choice rules to implement the same number of reserves and quotas. A real-life example is the choice rule used in elite public high schools in Chicago (Kominers and Sönmez [38]; Dur et al. [22]) or the vertical reserves in India (Sönmez and Yenmez [45] and [46]). In words, this choice rule also consists of two stages.[4] First, it considers candidates from all types and admits $k - \sum_{t \in T} r_t$ candidates with the highest priority. Second, it considers each type t separately and type-t candidates that are rejected at the first stage. Then, it admits r_t candidates with the highest priority for each type t. Intuitively, this rule assigns reserves at the second stage. Thus, this rule implements the reserves in the opposite order to our rule: the reserves-and-quotas rule assigns reserves at the first stage. It is worth noting that our results on the reserves-and-quotas rule cannot be carried out with the reserves rule used in Chicago. We discuss this issue in the following sections.

Last, the reserves-and-quotas rule sometimes creates empty seats even if it rejects some candidates. It is observed in practice to leave seats empty due to diversity concerns. An example is the kindergarten assignment in Louisville, KY. Parents who tried to enroll their children in kindergarten sued the Louisville school district. They argued that there was a racial quota since the schools did not accept them although there were plenty of empty seats. The school district contends that it's not discriminating against anyone, but instead is trying

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[4] This rule has no quotas.

to n

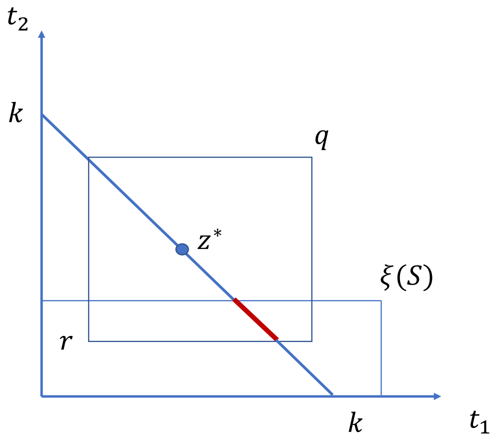

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Figure 1.2: Reserves-and-quotas rule.

1.3.2 Trade-off between meritocracy and diversity

In this section, we investigate the trade-off between meritocracy and diversity. The responsive rule cannot promote diversity, and the ideal-distribution rule is unfair due to many priority violations. The reserves-and-quotas rule lies between the two rules. We formally discuss the trade-off by introducing a measure of meritocracy and a measure of diversity for choice rules.

First, we introduce a measure of meritocracy. Upon enumerating $\mathcal{S}$ from highest to lowest according to $s_1^* \succ s_2^* \succ s_3^* \succ ... \succ s_n^*$, we define F_S for all $S \subseteq \mathcal{S}$ and $l \in \{1, ..., n\}$ as follows.

$$F_S(l) = |\{s \in S : s \succeq s_l^*\}|.$$

Definition 5. A choice rule C is **more meritorious** than C' if for all $S \subseteq \mathcal{S}$ and $l \in$

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[5] For more details, see the following ABC News article:
http://abcnews.go.com/Politics/SupremeCourt/story?id=2693451.

$\{1, ..., n\}$, we have $F_{C(S)}(l) \geq F_{C'(S)}(l)$.[6]

In words, for all priority ranking l, C always admits more candidates whose priority ranking is weakly higher than l. A more meritorious choice rule reduces the numbers of priority violations and empty seats. For example, $F_{\{s_1^*, s_2^*\}}(l) \geq F_{\{s_2^*, s_3^*\}}(l)$ and $F_{\{s_1^*, s_2^*\}}(l) \geq F_{\{s_1^*\}}(l)$ for all $l \in \{1, ..., n\}$.

Next, we introduce a measure of diversity. Suppose that the college has a diversity goal z^* with $\sum_{t \in T} z_t^* \leq k$ on the distribution of candidate types. The measure of diversity here is a distance from the diversity goal z^*, considering the Manhattan norm (or L^1 norm), which is defined as $\|x\| \equiv \sum_{i=1}^{d} |x_i|$.[7]

Definition 6. A choice rule C is **less diverse** than C' if for all $S \subseteq \mathcal{S}$,

$$\|z^* - \xi(C(S))\| \geq \|z^* - \xi(C'(S))\|.$$

Now we provide our first main result. It formalizes the trade-off between meritocracy and diversity in the class of reserves-and-quotas rules. The proof is contained in Appendix.

Theorem 1. *Let C be a choice rule generated by reserves r and quotas q and C' be a choice rule generated by reserves r' and quotas q'. Suppose $r_t \leq r'_t \leq z_t^* \leq q'_t \leq q_t$ for every type $t \in T$. Then C is more meritorious than C' and C is less diverse than C'.*

Theorem 1 yields the following result: the responsive rule and the ideal-distribution rule are two extremes in the class of reserves-and-quotas rules. The proof is contained in Appendix.

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[6] This is equivalent to the following definition. Let $R(S, l)$ be the lth highest priority candidate in S. Formally, $|\{s \in S : s \succeq R(S, l)\}| = l$ when $l \leq |S|$. For a case of $|S| < l$, let introduce a fictitious candidate $s_\emptyset$ whose priority is lower than any other candidate $s \in \mathcal{S}$ i.e., for all $s \in \mathcal{S}$, we have $s \succ s_\emptyset$. For $|S| < l$, define $R(S, l) = s_\emptyset$. We say that a function $f : \mathcal{S} \to \mathbb{R}$ is compatible with $\succ$ when for all $s, s' \in \mathcal{S}$, $s \succ s'$ if and only if $f(s) > f(s')$. A choice rule C is more meritorious than C' if and only if for all $S \subseteq \mathcal{S}$, $l \in \{1, ..., n\}$, and f compatible with $\succ$, we have $\sum_{i=1}^{l} f(R(C(S), i)) \geq \sum_{i=1}^{l} f(R(C'(S), i))$.

[7] Our results rely on this norm. For example, Theorem 1 cannot be extend to the measure of diversity based on L^2. On the other hand, the ideal-distribution rule works with any L^p norm for $p < \infty$.

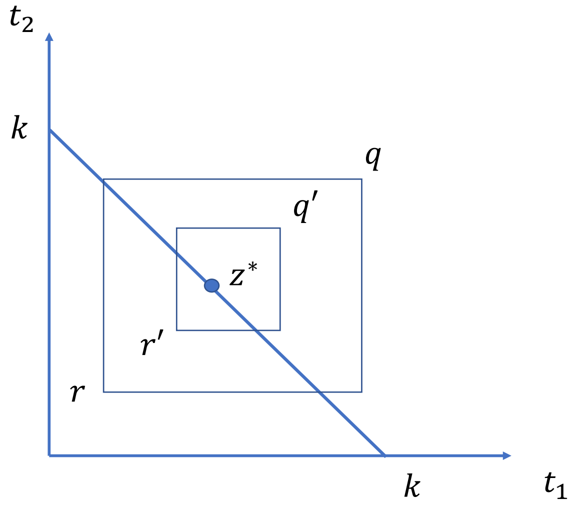

Figure 1.3: Trade-off between meritocracy and diversity.

Proposition 2. *The responsive rule is more meritorious and less diverse than any other reserves-and-quotas rules. Any reserves-and-quotas rules are more meritorious and less diverse than the ideal-distribution rule.*

While the class of reserves-and-quotas rules has the trade-off, it is unclear in other classes of choice rules for diversity. For example, the reserves rule in public high schools in Chicago does not have such a trade-off: reducing reserves does not mean that it is more meritorious. The following example illustrates the fact.

Example 1. There are three candidates s_1, s_2, and s_3 and two types t_1 and t_2. Suppose that $\tau(s_1) = \tau(s_3) = t_1$ and $\tau(s_2) = t_2$. A priority $\succ$ ranks them from first to last as $s_1 \succ s_2 \succ s_3$. A capacity k is equal to 2. Let C be the choice rule used in Chicago generated by $r_{t_1} = r_{t_2} = 1$. Also, let C' be the choice rule used in Chicago generated by $r'_{t_1} = 1$ and $r'_{t_2} = 0$. Notice that C' is obtained from C by reducing a reserve for t_2. The choice rule C admits s_1 and s_2. On the other hand, the choice rule C' admits s_1 and s_3. Thus, C' is not more meritorious than C since $F_{\{s_1,s_2\}}(2) > F_{\{s_1,s_3\}}(2)$.

The example leads to a natural question: what a class of choice rules does have the trade-off? In Section 4, we will answer this question: The reserves-and-quotas rule is the one and only natural choice rule which has the trade-off.

1.3.3 Characterizations of reserves-and-quotas rule

This section provides a characterization of the reserves-and-quotas rule using four axioms. Also, we provide another characterization in a different case in which reserves and quotas are exogenously specified to identify the role of substitutability—one of the main axioms.

Characterization

We introduce four axioms. The first axiom is substitutability, as introduced by Kelso and Crawford [33], which is necessary for the existence of stable or fair matchings. Substitutability, together with the irrelevance of rejected contracts introduced by Aygün and Sönmez [5], is sufficient for the existence of stable matchings.[8] This also enables us to use the most popular mechanism in practice, the deferred acceptance algorithm, to find a stable matching. This axiom states that if a candidate s is chosen in a set of candidates S, s is chosen in any subset of S including her.

Definition 7. A choice rule C satisfies **substitutability** if for all $S \subseteq S' \subseteq \mathcal{S}$ and $s \in S$, $s \in C(S')$ implies $s \in C(S)$.

The second axiom is within-type $\succ$-compatibility introduced by Echenique and Yenmez [24]. This is fairness within type: the college should always use the priority to compare candidates of the same type.[9] This axiom states that if a candidate s is chosen over another s' of the same type, s has a higher priority than s'.

Definition 8. A choice rule C satisfies **within-type $\succ$-compatibility** if for all $S \subseteq \mathcal{S}$ and $s, s' \in S$, whenever $s \in C(S)$, $s' \in S \setminus C(S)$ and $\tau(s) = \tau(s')$, we have $s \succ s'$.

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[8] Substitutability and rejection maximality, the third axiom, imply the irrelevance of rejected contracts. Thus, the reserves-and-quotas rule satisfies sufficiency for the existence of stable matchings.

[9] It is also referred to as *inter se merit* in the context of affirmative action in India; see Sönmez and Yenmez [45] and [46] for more detail.

The third axiom is rejection maximality introduced by Echenique and Yenmez [24]. This is a weak efficiency condition: acceptance implies rejection maximality.[10] In words, if a type-t candidate is rejected from a set when there is an empty seat, then the number of type-t candidates chosen from this set is weakly greater than the corresponding number for any set that does not have more type-t candidates than this set.

Definition 9. A choice rule C satisfies **rejection maximality** if for all $s \in S \subseteq \mathcal{S}$, $s \in S \backslash C(S)$ and $|C(S)| < k$ implies that for all S' with $|S'_{\tau(s)}| \leq |S_{\tau(s)}|$, we have $|C(S')_{\tau(s)}| \leq |C(S)_{\tau(s)}|$.

The last axiom is our new axiom, across-types $\succ$-compatibility. We face the trade-off between meritocracy and diversity. Thus, the key point is to determine when the college uses the priority and when it concerns diversity. This axiom states that the number of candidates of each type determines when the priority is used.

To introduce this axiom, we need two concepts introduced by Echenique and Yenmez [24] and Doğan [20]. A type t is **saturated** in a set of candidates S if there exists S' such that $|S'_t| = |C(S)_t|$ and $|C(S')_t| < |C(S)_t|$. Saturation says that the number of admitted type-t candidates in S is judged to be enough for diversity. This is because the college admits fewer type-t candidates in another situation S' though the number of available type-t candidates is the same for that of the admitted type-t candidates in S. A type t is **demanded** in a set of candidates S if there exists S' such that $|S'_t| = |S_t|$ and $|C(S')_t| > |C(S)_t|$. Demanded type t means the college can admit more type-t candidates without sacrificing diversity. This is because the college admits more type-t candidates in another situation S', though the number of available type-t candidates is the same in S.

Now, we can introduce our new axiom.

Definition 10. A choice rule C satisfies **across-types $\succ$-compatibility** if for all $S \subseteq \mathcal{S}$ and $s, s' \in S$, whenever $s \in C(S)$, $s' \in S \setminus C(S)$, $\tau(s)$ is saturated in S, and $\tau(s')$ is demanded in S, we have $s \succ s'$.

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[10] A choice rule C satisfies acceptance if for all $S \subseteq \mathcal{S}$, $|C(S)| = \min\{k, |S|\}$.

This axiom decides when to use the priority. If a saturated type candidate s is admitted and a demanded type candidate s' is rejected, then the admission of s is explained by the high priority of s and the rejection of s' is explained by the low priority of s'. From a different view, the axiom states that when the college focuses on diversity. Consider a situation S where the college rejects a candidate s and admits a candidate s', whereas s has the higher priority than s': i.e., $s \in S \setminus C(S)$, $s' \in C(S)$, and $s \succ s'$. According to the axiom, the college overrules an objection by s for either or both of two diversity reasons. First, $\tau(s)$ is the non-demanded type, meaning that there are many type-$\tau(s)$ candidates relative to some diversity objectives. Thus, the college cannot admit more type-$\tau(s)$ candidates to promote diversity. Second, for another admitted candidate s', $\tau(s')$ is non-saturated, meaning that there are few type-$\tau(s')$ candidates relative to some diversity objectives. Thus, the college must admit s' regardless of the priority.

We provide our second main result. The proof is contained in Appendix. We also verify the independence axioms in Appendix.

Theorem 2. *A choice rule is generated by reserves and quotas for priority $\succ$ if and only if it satisfies substitutability, within-type $\succ$-compatibility, rejection maximality, and across-types $\succ$-compatibility.*

Another characterization: role of substitutability

In this section, we discuss the role of substitutability in our characterization. One might think that it is difficult to label substitutability as an axiom. While substitutability is necessary for the existence of stable matchings, stability is not a concept of choice rules. Neither stability nor substitutability is necessary, especially in the case of one college. We identify the role of substitutability: it constructs reserves and quotas. Specifically, we show that in the case where reserves and quotas are exogenously specified, substitutability can be dropped from our characterization, with small modifications. In contrast, the reserves and quotas are not exogenously specified in the first characterization: our axioms do not involve the reserves and quotas. In practice, colleges or firms often specify the number of reserves and quotas. Thus, our characterization without substitutability is still helpful for admissions or hiring committees in deciding their policies.

Exogenously given the reserves r and the quotas q, we focus on a class of choice rules satisfying the constraints by r and q. Specifically, a choice rule C satisfies **the feasible constraints by the reserves r and quotas q** if for all $S \subseteq \mathcal{S}$ and $t \in T$, we have $\min\{r_t, |S_t|\} \leq |C(S)_t| \leq q_t$. Also, we slightly change the two axioms to suit the case of exogenous reserves r and quotas q.

Definition 11. A choice rule C satisfies **rejection maximality*** if for all $s \in S \subseteq \mathcal{S}$, whenever $s \in S \setminus C(S)$ and $|C(S)| < k$, we have $|C(S)_{\tau(s)}| = q_{\tau(s)}$.

Definition 12. A choice rule C satisfies **across-types* $\succ$-compatibility** if for all $S \subseteq \mathcal{S}$ and $s, s' \in S$, whenever $s \in C(S)$, $s' \in S \setminus C(S)$, $|C(S)_{\tau(s)}| > r_{\tau(s)}$, and $|C(S)_{\tau(s')}| < q_{\tau(s')}$, we have $s \succ s'$.

Recall that there are many ways to satisfy the feasible constraints: the reserves-and-quotas rule is not unique. An example is the reserves rule used in public high schools in Chicago or the vertical reserves rule in India.[11] While this rule satisfies the feasible constraints by the reserves and quotas, it violates across-types* $\succ$-compatibility. The following example illustrates the fact.

Example 2. There are four candidates s_1, s_2, s_3 and s_4 and two types t_1 and t_2. Suppose that $\tau(s_1) = \tau(s_2) = \tau(s_3) = t_1$ and $\tau(s_4) = t_2$. A priority $\succ$ ranks them from first to last as $s_1 \succ s_4 \succ s_2 \succ s_3$. A capacity k is equal to 2. There is one reserve for t_1 and no reserve for t_2, $r_{t_1} = 1$ and $r_{t_2} = 0$. There are non-binding quotas for t_1 and t_2, $q_{t_1} = q_{t_2} = 2$. Let C be the choice rule used in Chicago. Since $C(\{s_1, s_2, s_4\}) = \{s_1, s_2\}$, we have $|C(\{s_1, s_2, s_4\})_{t_1}| = 2 > 1$ and $|C(\{s_1, s_2, s_4\})_{t_2}| = 0 < 2$. However $s_2 \in C(\{s_1, s_2, s_4\})$, $s_4 \in S \setminus C(\{s_1, s_2, s_4\})$ and $s_4 \succ s_2$, a violation of across-types* $\succ$-compatibility.[12]

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[11] See Section 3 for how the reserves rule used in Chicago works.

[12] This rule also violates across-types $\succ$-compatibility. Notice that t_1 is saturated in $\{s_1, s_2, s_4\}$ since

Our third result states that the three axioms are enough to characterize the reserves-and-quotas rule in the class of choice rules satisfying feasible constraints by the exogenous reserves and quotas. The proof is contained in Appendix. We also verify the independence axioms in Appendix.

Theorem 3. *Let r and q be reserves and quotas, respectively. Suppose that a choice rule satisfies feasibility constraints by r and q. Then it is the reserves-and-quotas rule for priority $\succ$ if and only if it satisfies within-type $\succ$-compatibility, rejection maximality*, and across-types* $\succ$-compatibility.*

1.4 Comparative statics as axiom

In this section, we provide the third characterization of the class of the reserves-and-quotas rules. We introduce a new axiom, *meritorious monotonicity in reserve and quota size*. This axiom (i) is motivated by our comparative statics in Theorem 1 and (ii) is transparency for a practitioner and individuals in a market. Then, we discuss the relationship between our new axiom and the choice rules in the literature, *reserve system* and *soft reserves and soft quotas*.

1.4.1 New axiom

Our axiom states that as a constraint generated by reserves and quotas is relaxed, a choice rule should become more meritorious. To capture a change of a constraint, we focus on a class of choice rule parameterized by reserves r and quotas q. Specifically, the college's choice rule C is a function that maps for each nonempty set $S \subseteq \mathcal{S}$ and non-zero vectors $r, q \in \mathbf{Z}_+^{\mathbf{d}}$ to a subset $C(S : r, q) \subseteq S$. It is such that $C(S : r, q) \subseteq S$ and $|C(S : r, q)| \leq k$

$|\{s_2, s_3, s_4\}_{t_1}| = |C(\{s_1, s_2, s_4\})_{t_1}|$ and $|C(\{s_2, s_3, s_4\})_{t_1}| < |C(\{s_1, s_2, s_4\})_{t_1}|$. Also, note that t_2 is demanded in $\{s_1, s_2, s_4\}$ since $|\{s_1, s_2, s_4\}_{t_2}| = |\{s_4\}_{t_2}|$ and $|C(\{s_1, s_2, s_4\})_{t_1}| < |C(\{s_4\})_{t_1}|$. However $s_2 \in C(\{s_1, s_2, s_4\})$, $s_4 \in S \setminus C(\{s_1, s_2, s_4\})$ and $s_4 \succ s_2$, a violation of across-types $\succ$-compatibility.

for each $S \subseteq \mathcal{S}$ and $r, q \in \mathbf{Z}^{\mathbf{d}}_{+}$. $C(S : r, q)$ means the set of admitted candidates in S given the reserves r and quotas q. Now we introduce our new axiom.

Definition 13. A choice function C satisfies **meritorious monotonicity in reserve and quota size** if for all $S \subseteq \mathcal{S}$, $r, r' \in \mathbf{Z}^{\mathbf{d}}_{+}$ and $q, q' \in \mathbf{Z}^{\mathbf{d}}_{+}$, whenever $r_t \leq r'_t$ and $q'_t \leq q_t$ for all $t \in T$, $C(: r, q)$ is more meritorious than $C(: r', q')$.

It is important to mention two points related to our axiom. First, our axiom is motivated by Theorem 1. Thus, the class of the reserves-and-quotas rules clearly satisfies it. As we have seen in Example 1, the reserves rule used in Chicago public high school choice (Dur et al. [22]) violates it. In addition, as we will see the next section, the other rules proposed in the literature violate our axiom as well. Therefore, the class of reserves-and-quotas rule is the only natural class that has clear comparative statics.[13]

Second, this axiom ensures transparency of rule to a practitioner and participants in the market. Affirmative action is often implemented through a reserves system (Pathak et al. [43]). However, the functioning of the systems is counter intuitive, and thus a practitioner and participants in the market often misunderstand it (Dur et al. [21]; Pathak et al. [41]; Delacrétaz [18]). In addition, a reserve and quota size often changes in real life. For example, to promote diversity, each public school in Chicago increase the reserve size for students from specific neighborhoods in 2012 (Dur et al. [22]). The elimination of reserves for local neighborhood applicants in Boston's school choice system (Dur et al. [21]) serves as another example. Given the complexity of the system, it is not easy to understand what will happen with these changes. Our axiom ensures that the rule works the way the practitioner and participants intended it to.

We need additional concepts. Exogenously given the reserves r and the quotas q, we focus on a class of choice rules satisfying the constraints by r and q. Specifically, a choice rule C satisfies **the feasible constraints by the reserves** r **and quotas** q if for all $S \subseteq \mathcal{S}$

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[13] We can define a monotonicity axiom about diversity. However, our axiom is enough to characterize the class of reserves-and-quotas rules. We will discuss it more in-depth in the Appendix.

and $t \in T$, we have $\min\{r_t, |S_t|\} \leq |C(S)_t| \leq q_t$. Also, we slightly change the two axioms to suit this case.

Definition 14. A choice rule C satisfies **within-type $\succ$-compatibility** if for all $S \subseteq \mathcal{S}$, $r, q \in \mathbf{Z}^{\mathbf{d}}_{+}$ and $s, s' \in S$, whenever $s \in C(S : r, q)$, $s' \in S \setminus C(S : r, q)$ and $\tau(s) = \tau(s')$, we have $s \succ s'$.

Definition 15. A choice rule C satisfies **rejection maximality*** if for all $s \in S \subseteq \mathcal{S}$ and $r, q \in \mathbf{Z}^{\mathbf{d}}_{+}$, whenever $s \in S \setminus C(S : r, q)$ and $|C(S : r, q)| < k$, we have $|C(S : r, q)_{\tau(s)}| = q_{\tau(s)}$.

We provide our third characterization. The proof is contained in Appendix. We also verify the independence axioms in Appendix.

Theorem 4. *Let C be a feasible choice rule. Then it is the reserves-and-quotas rule for priority $\succ$ if and only if it satisfies within-type $\succ$-compatibility, rejection maximality*, and meritorious monotonicity in reserve and quota size.*

1.4.2 Choice rule in literature

Reserve system

Pathak et al. [43] propose a reserve system to allocate medical resources (e.g., ventilators, ICU beds, drugs, and vaccines) to reconcile various ethical values. Their concept generalizes a reserve system with sequential processing introduced by Kominers and Sönmez [38]. Recently, the importance of the precedence order is identified in the literature. Practical examples are affirmative action in India (Sönmez and Yenmez [45]; Sönmez and Yenmez [46]), school choice in the U.S. (Dur et al. [21]), and H1B visa allocation (Pathak et al. [40]).

The reserves-and-quotas rule is closely related to the reserve system literature. In terms of the order, our choice rule processes all reserves before open seats. While that approach is similar to our reserves-and-quotas rule, there are at least two important differences. First, the reserve system does not have quotas. Extending this rule to incorporate quotas would be a non-trivial and interesting question. Second, perhaps more importantly, the reserves-and-quotas rule is not included in the class of the reserve systems even without quotas. Specifically, a reserve system with any precedence order is not equivalent to the reserves-and-quotas rule.[14] On the other hand, any reserve system satisfies feasibility, within-type $\succ$-compatibility, and rejection maximality*. Therefore, Theorem 4 imply that it violates only meritorious monotonicity in reserve and quota size. In other words, this new axiom makes the difference.

Soft reserves and soft quotas

Ehlers et al. [25] propose the choice rule with soft lower bounds $(r_t)_{t\in T}$ and soft upper bounds $(q_t)_{t\in T}$.[15] The reserves-and-quotas rule is a variant of this choice rule. However, this class of choice rules has no comparative statics as we study in Theorem 1. The following example illustrates that the class of the choice rules violate meritorious monotonicity in reserve and quota size.

Example 3. There are five candidates s_1, s_2, s_3, s_4 and s_5 and tree types t_1, t_2 and t_3. Suppose that $\tau(s_1) = \tau(s_4) = t_1$, $\tau(s_2) = \tau(s_4) = t_2$ and $\tau(s_5) = t_3$. A priority $\succ$ ranks them from first to last as $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5$. A capacity k is equal to 3. Let C be the choice rule proposed by Ehlers et al. [25] generated by $r_{t_1} = r_{t_2} = r_{t_2} = 0$ and $q_{t_1} = 2, q_{t_1} = q_{t_1} = 1$. Also, let C' be the choice rule proposed by Ehlers et al. [25] generated by $r_{t_1} = r_{t_2} = r_{t_2} = 0$ and $q_{t_1} = q_{t_1} = q_{t_1} = 1$. Notice that C is obtained from C' by relaxing the constraint by soft reserves and soft quotas. The choice rule C admits s_1, s_2 and s_4. On the other hand, the choice rule C' admits s_1, s_2 and s_3. This is a violation of meritorious monotonicity in reserve and quota size: C is not more meritorious than C since

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[14] The cause of the difference is how to allocate empty reserves. While the reserve system with the order converts an empty reserve to an open seat immediately, the reserves-and-quotas moves an empty reserve last.

[15] The formal definition is provided in Section 3

$F_{\{s_1,s_2,s_3\}}(3) > F_{\{s_1,s_2,s_4\}}(3)$.

1.5 Additional results

1.5.1 Endogenous priority

In this section, we generalize Theorem 2 so that the class of reserves-and-quotas rules does not depend on a particular priority. We introduce a property based on the strong axiom of revealed preference. This axiom allows for the endogenous construction of a priority over candidates.

Definition 16. A choice rule C satisfies **across-types strong axiom of revealed preference** if there are no sequences $\{s_k\}_{k=1}^{K}$ and $\{S_k\}_{k=1}^{K}$, candidates and sets of candidates, respectively, such that, for all k

(i) $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;

(ii) $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_k)$ is demanded in S_{k+1} and $\tau(s_{k+1})$ is saturated in S_{k+1}.

(using addition mod K).

This axiom excludes the existence of certain cycles in the revealed preference. There are two cases to consider. First, if $\tau(s_{k+1}) = \tau(s_k)$, it is revealed that s_{k+1} has a higher priority than s_k. Second, if $\tau(s_{k+1}) \neq \tau(s_k)$, we require $\tau(s_k)$ is demanded in S_{k+1} and $\tau(s_{k+1})$ is saturated in S_{k+1}. In this case, we can say that s_{k+1} has a higher priority than s_k in the revealed preference, even if they have different types.

Theorem 5. *A choice rule is generated by reserves and quotas for some priority $\succ$ if and only if it satisfies substitutability, rejection maximality, and across-types strong axiom of revealed preference.*

Proof. Suppose that C satisfies the axioms. We show that C is generated by reserves and quotas for some $\succ$.

Step 1: Construction of a binary relation $\succ^*$ over $\mathcal{S}$.

Define the binary relation $\succ^*$ over $\mathcal{S}$ as follows. For each $s, s' \in \mathcal{S}$ such that $\tau(s) = \tau(s')$, $s \succ^* s'$ if and only if there exists $S \subseteq \mathcal{S}$ such that $s \in C(S)$ and $s' \in S \setminus C(S)$. For each $s, s' \in \mathcal{S}$ such that $\tau(s) \neq \tau(s')$, $s \succ^* s'$ if and only if there exists $S \subseteq \mathcal{S}$ such that $s \in C(S)$, $s' \in S \setminus C(S)$, $\tau(s)$ is saturated in S, and $\tau(s')$ is demanded in S. While $\succ^*$ is not a complete order, $\succ^*$ has a linear extension $\succ$ to $\mathcal{S}$ by across-types strong axiom of revealed preference. Specifically, there exists a linear order $\succ$ over $\mathcal{S}$ such that for every $s, s' \in \mathcal{S}$, $s \succ^* s'$ imply $s \succ s'$.

Step 2: C satisfies within-type $\succ$-compatibility and across-types $\succ$-compatibility for the constructed $\succ$.

For all $S \subseteq \mathcal{S}$ and $s, s' \in S$, if $s \in C(S)$, $s' \in S \setminus C(S)$ and $\tau(s) = \tau(s')$, then $s \succ^* s'$ by the definition of $\succ^*$. We also get $s \succ s'$ since $\succ$ is a linear extension of $\succ^*$. Thus, C satisfies within-type $\succ$-compatibility. For every $S \subseteq \mathcal{S}$ and every $s, s' \in S$, if $s \in C(S)$, $s' \in S \setminus C(S)$, $\tau(s)$ is saturated in S, and $\tau(s')$ is demanded in S, then $s \succ^* s'$ by the definition of $\succ^*$. Again, we also get $s \succ s'$ since $\succ$ is a linear extension of $\succ^*$. Thus, C satisfies across-types $\succ$-compatibility.

Note that C satisfies all axioms in Theorem 1. Thus, C is generated by reserves-and-quotas.

For the other direction, it is enough to show C generated by reserves and quotas for $\succ$ satisfies across-types strong axiom of revealed preference. Suppose toward a contradiction: there are sequences $\{s\}_{k=1}^{K}$ and $\{S\}_{k=1}^{K}$, of candidates and sets of candidates, respectively, with the properties in the definition. This means $\succ$ admits a cycle: $s_K \succ s_{K-1} \succ \ldots \succ s_1 \succ s_K$, a contradiction to $\succ$ being a linear order. □

1.5.2 Separable choice rules

In this section, we study the class of separable choice rules. While the ideal-distribution rule is separable, it is not a unique separable choice rule. We introduce a new monotonicity property and show that together with substitutability, the choice rule is still separable. Then, we study the class of choice rules.

New monotonicity

We introduce a new monotonicity property. Echenique and Yenmez [24] introduce monotonicity, which is the key axiom in their characterization of the ideal-distribution rule. The definition is as follows.

Definition 17. A choice rule C satisfies **monotonicity** if for all $S, S' \subseteq \mathcal{S}$ such that $|S_t| \leq |S'_t|$ for every type t, we have $|C(S)_t| \leq |C(S')_t|$ for every type t.

This new property relaxes monotonicity based on the law of aggregate demand.[16] A choice rule satisfying this property compares two sets of candidates with respect to a set inclusion of each type, instead of the number of each type.

Definition 18. A choice rule C satisfies **the type law of aggregate demand** if for all $S, S' \subseteq \mathcal{S}$ such that $S_t \subseteq S'_t$ for every $t \in T$, we have $|C(S)_t| \leq |C(S')_t|$ for every $t \in T$.

We show that a choice rule satisfying substitutability and the type law of aggregate demand is still separable.

Theorem 6. *If a choice rule satisfies substitutability and the type law of aggregate demand, then it is separable.*

Proof. We show that for all $t \in T$ and all $S, S' \subseteq \mathcal{S}$ with $S_t = S'_t$, we have $C(S)_t = C(S')_t$. For all $t \in T$ and all $S, S' \subseteq \mathcal{S}$ with $S_t = S'_t$, we have $|C(S)_t| \leq |C(S \cup S')_t|$. This is because for every $t' \in T$, we have $S_{t'} \subseteq (S \cup S')_{t'}$ and the type law of aggregate demand. By substitutability, we have $C(S \cup S')_t \cap S_t \subseteq C(S)_t$. Since $S_t = (S \cup S')_t$, we have $C(S \cup S')_t \cap S_t = C(S \cup S')_t$. Since $|C(S)_t| \leq |C(S \cup S')_t|$ and $C(S \cup S')_t \subseteq C(S)_t$, we have $C(S)_t = C(S \cup S')_t$. Similarly we have $C(S')_t = C(S \cup S')_t$ and thus $C(S)_t = C(S')_t$. $\square$

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[16] A choice rule C satisfies the law of aggregate demand if for all $S, S' \subseteq \mathcal{S}$ such that $S \subseteq S'$, we have $|C(S)| \leq |C(S')|$.

Class of separable choice rules

We study the class of choice rules that satisfy substitutability, the type law of aggregate demand, and within-type $\succ$-compatibility. This class is larger than the class of ideal-distribution rules: Echenique and Yenmez [24] characterize the ideal-distribution rule by substitutability, monotonicity, and within-type $\succ$-compatibility. First, we give an example in which a choice rule is in our class, but not the ideal-distribution rule. Second, we show that our rule also has a representation based on the reserves and quotas. Then, the ideal-distribution rule appears again, when reserves equal quotas. Third, however, the class of choice rules has no clear trade-off between meritocracy and diversity as in the class of reserves-and-quotas rules.

In the following example, a choice rule satisfies substitutability, the type law of aggregate demand, and within-type $\succ$-compatibility, but violates monotonicity. Thus it is not the ideal-distribution rule.

Example 4. There are three candidates s_1, s_2 and s_3. Suppose that $\tau(s_1) = \tau(s_2) = \tau(s_3)$. A priority ranks them from first to last as $s_1 \succ s_2 \succ s_3$. Consider the following choice rule.

$$C(S) = \begin{cases} \{s_1, s_2\} & \text{if } \{s_1, s_2\} \subseteq S \\ \arg\max_{\succ} S & \text{otherwise} \end{cases}$$

It is easy to check that C satisfies substitutability. Since all candidates are the same type, C satisfies the type law of aggregate demand and within-type $\succ$-compatibility. However, it violates monotonicity: $|\{s_1, s_2\}| = |\{s_1, s_3\}|$ and $|C(\{s_1, s_2\})| = |\{s_1, s_2\}| \neq |C(\{s_1, s_3\})| = |\{s_1\}|$. Thus, C is not the ideal-distribution rule.

In the example, while the college admits all candidates when the first and second highest priority candidates are available, it admits only one candidate otherwise. In other words, the college's capacity depends on the set of candidates.

The choice rule also has a representation based on reserves and quotas. The proof is contained in Appendix.

Proposition 3. *Suppose that a choice rule C satisfies substitutability, the type law of aggregate demand, and within-type $\succ$-compatibility. Then there exist reserves r and quotas q such that C satisfies the feasible constraints by r and q. Moreover, C is the ideal-distribution rule when $r_t = q_t$ for each type t.*

By Proposition 1, the ideal-distribution rule allocates the fixed capacity for each type and applies the responsive rule. Here, the choice rule allocates a flexible capacity for each type. As long as the choice rule satisfies the feasible constraint, it can change the capacity depending on the set of candidates. For example, suppose that $r_t = 50$ and $q_t = 100$ for some t. Then, the college admits 100 candidates of type t when the highest priority candidates apply. On the other hand, it admits only 50 candidates of type t when the lowest priority candidates apply.

While the choice rule also has a representation based on reserves and quotas, it has no clear trade-off between meritocracy and diversity as in our reserves-and-quota rule. Either reducing reserves or increasing quotas possibly leads to more priority violations. This is because we cannot use the priority to compare candidates of different types by separability.

1.6 Concluding remarks

Affirmative action is used in various settings to promote diversity and has caused a debate about meritocracy and diversity. In this paper, we study the tension between meritocracy and diversity, with the reserves-and-quotas rule provided as a compromise solution.

First, we formalize the trade-off between meritocracy and diversity by introducing the two measures of choice rules. We show the clear trade-off in the class of reserves-and-quotas rules: as the number of reserves and quotas changes so that the rule becomes more meritorious, it is less diverse. As a by-product of this result, we provide a unified view of the choice rules in market design literature. The responsive rule is most meritorious and least diverse, while the ideal-distribution rule is least meritorious and most diverse. Second, we provide characterizations of reserves-and-quotas rules. These results help organizations that want to promote diversity, to decide on their policies. Also, we identify the role of substitutability, one of the main axioms in our result and the characterizations by Echenique and Yenmez [24].

While we focus on the class of the reserves-and-quotas rules, some choice rules for diversity in the literature are not included in this class. We illustrate that one such rule, the reserves rule as used in public high schools in Chicago, has no clear trade-off based on our measures. Two possible directions for future research are as follows. The first direction is to show a trade-off in this class of choice rules by providing a new measure of choice rules. The other direction is to provide a characterization of the choice rule. These remain possible avenues for future research.

1.7 Appendix

1.7.1 Omitted Proofs

Proof of Theorem 1

The proof consists of two steps. We start with lemmas.

Lemma 1. *Suppose that* $C(S) \setminus C'(S) \neq \emptyset$ *and* $C'(S) \setminus C(S) \neq \emptyset$*. Then, for all* $s \in C(S) \setminus C'(S)$ *and* $s' \in C'(S) \setminus C(S)$*, we have* $s \succ s'$*.*

Proof. $s \in S \setminus C'(S)$ implies $|C'(S)_{\tau(s)}| \geq r'_{\tau(s)} \geq r_{\tau(s)}$. $s \in C(S)$ implies $|C(S)_{\tau(s)}| > |C'(S)_{\tau(s)}|$. Together with facts, we have $|C(S)_{\tau(s)}| > r_{\tau(s)}$. Since $s' \in C'(S) \setminus C(S)$, we have $q_{\tau(s')} \geq q'_{\tau(s')} \geq |C'(S)_{\tau(s')}| > |C(S)_{\tau(s')}|$. Since C satisfies across-types* $\succ$-compatibility, we have $s \succ s'$. □

Lemma 2. *Let* C^1 *be a choice rule generated by reserves* r^1 *and quotas* q^1 *and* C^2 *be it generated by reserves* r^2 *and quotas* q^2*. Suppose* $q^1_t \leq q^2_t$ *for every* $t \in T$*. Then, for all* $S \subseteq \mathcal{S}$*, we have* $|C^1(S)| \leq |C^2(S)|$*.*

Proof. Suppose not: there exists $S \subseteq \mathcal{S}$ such that $|C^2(S)| < |C^1(S)|$. Thus, there exists $t \in T$ such that $|C^2(S)_t| < |C^1(S)_t|$. This implies $S_t \setminus C^2(S)_t \neq \emptyset$. By the definition of the capacity, $|C^2(S)| < |C^1(S)| \leq k$. Since C^2 satisfies rejection maximality*, we have $|C^2(S)_t| = q^2_t$. However, this implies $q^1_t \leq q^2_t = |C^2(S)_t| < |C^1(S)_t|$, a contradiction to the definition of q^1_t. □

Step 1: C is more meritorious than C'.

By Lemma 2, for all $S \subseteq \mathcal{S}$, we have $|C'(S)| \leq |C(S)|$. There are two cases to consider.

Case 1: $C'(S) \subseteq C(S)$.

For all $l \in \{1, ..., n\}$, we have $\{s \in C'(S) : s \succeq s^*_l\} \subseteq \{s \in C(S) : s \succeq s^*_l\}$. Thus, we have $F_{C'(S)}(l) \leq F_{C(S)}(l)$.

Case 2: $C'(S) \not\subseteq C(S)$.

By Lemma 1 and $|C'(S)| \leq |C(S)|$, for all $l \in \{1, ..., n\}$, we have $|\{s \in C'(S) \setminus C(S) : s \succeq s_l^*\}| \leq |\{s \in C(S) \setminus C'(S) : s \succeq s_l^*\}|$. Thus, we have

$$\begin{aligned} F_{C'(S)}(l) &= |\{s \in C'(S) \setminus C(S) : s \succeq s_l^*\}| + |\{s \in C'(S) \cap C(S) : s \succeq s_l^*\}| \\ &\leq |\{s \in C(S) \setminus C'(S) : s \succeq s_l^*\}| + |\{s \in C'(S) \cap C(S) : s \succeq s_l^*\}| \\ &= F_{C(S)}(l). \end{aligned}$$

We complete Step 1.

Step 2: C is less diverse than C'.

Fix any type $t \in T$ with $r_t < z_t^*$. Construct another choice rule C^* generated by reserves r^* and quotas q^*. r^* and q^* are defined as follows. $r_t^* = r_t + 1$ for t and $r_{t'}^* = r_{t'}$ for $t' \neq t$ and $q_{t'}^* = q_{t'}$ for all $t' \in T$. In words, the choice rule C^* is obtained from C by increasing reserves for t by one and keeping other reserves and quotas same.

Step 2-1. C is less diverse than C^*.

Claim 1. *For all* $S \subseteq \mathcal{S}$, $|C^*(S) \setminus C(S)| \leq 1$.

Proof. Suppose not. There exists $S \subseteq \mathcal{S}$ such that $|C^*(S) \setminus C(S)| > 1$. Pick for any $s_1, s_2 \in C^*(S) \setminus C(S)$. Note that $C(S)_{\tau(s_i)} \subsetneq C^*(S)_{\tau(s_i)}$ for $i = 1, 2$ by within-type $\succ$-compatibility of C and C^*. Thus, $|C^*(S)_{\tau(s_i)}| > r_{\tau(s_i)}$. By the construction of r^*, without loss of generality, we can assume that $|C^*(S)_{\tau(s_1)}| > r^*_{\tau(s_1)}$. By Lemma 2, $|C(S)| = |C^*(S)|$. Thus, $|C(S) \setminus C^*(S)| > 1$. Pick $s' \in C(S) \setminus C^*(S)$. Note that $C^*(S)_{\tau(s')} \subsetneq C(S)_{\tau(s')}$ and thus $|C^*(S)_{\tau(s')}| < |C(S)_{\tau(s')}| \leq q_{\tau(s')} = q^*_{\tau(s')}$. By across-types* $\succ$-compatibility of C^*, we have $s_1 \succ s'$. However, by Lemma 1, $s' \succ s_1$, a contradiction. □

Claim 2. *For all* $S \subseteq \mathcal{S}$, *if* $|C^*(S) \setminus C(S)| = 1$, *then (1)* $\tau(s) = t$ *for* $s \in C^*(S) \setminus C(S)$ *and (2)* $|C(S)_t| = r_t$.

Proof. First part: suppose not. There exists $S \subseteq \mathcal{S}$ such that $|C^*(S) \setminus C(S)| = 1$, but $\tau(s) \neq t$ for $s \in C^*(S) \setminus C(S)$. Following the same argument in Claim 1, $r_{\tau(s)} <$

$|C^*(S)_{\tau(s)}| < q_{\tau(s)} = q^*_{\tau(s)}$. By the assumption $\tau(s) \neq t$, we have $r_{\tau(s)} = r^*_{\tau(s)}$ and thus $r^*_{\tau(s)} < |C^*(S)_{\tau(s)}|$. By Lemma 2, $|C(S)| = |C^*(S)|$. Thus, $|C(S) \setminus C^*(S)| = 1$. Pick $s' \in C(S) \setminus C^*(S)$. Following the same argument in Claim 1, $|C^*(S)_{\tau(s')}| < q^*_{\tau(s')}$. By across-types* $\succ$-compatibility of C^*, we have $s \succ s'$. However, by Lemma 1, $s' \succ s$, a contradiction.

Second part: suppose not. There exists $S \subseteq \mathcal{S}$ such that $|C^*(S) \setminus C(S)| = 1$, but $|C(S)_t| \neq r_t$. Since $s \in S \setminus C(S)$, $|C(S)_t| > r_t$. By the first part in this claim, $|C^*(S)_t| = |C(S)_t| + 1 > r_t + 1 = r^*_t$. By Lemma 2, $|C(S)| = |C^*(S)|$. Thus, $|C(S) \setminus C^*(S)| = 1$. Pick $s' \in C(S) \setminus C^*(S)$. Following the same argument in Claim 1, $|C^*(S)_{\tau(s')}| < q^*_{\tau(s')}$. By across-types* $\succ$-compatibility of C^*, $s \succ s'$. However, by Lemma 1, $s' \succ s$, a contradiction. $\square$

Now we show that C is less diverse than C^*: for all $S \subseteq \mathcal{S}$, $\|z^* - \xi(C(S))\| \geq \|z^* - \xi(C^*(S))\|$. If $C(S) \neq C^*(S)$, then by Lemma 2, $|C(S)| = |C^*(S)|$. By Claim 1, $|C^*(S) \setminus C(S)| = 1$ and $|C(S) \setminus C^*(S)| = 1$. By Claim 2, $\tau(s) = t$ for $s \in C^*(S) \setminus C(S)$ and $|C(S)_t| = r_t$. Thus, we have $|z^* - r_t| - |z^*_t - r^*_t| = 1$. Let t' be $\tau(s')$ for $s' \in C(S) \setminus C^*(S)$. Since $|C^*(S) \setminus C(S)| \leq 1$, we have $|z^*_{t'} - \xi(C(S))_{t'}| - |z^*_{t'} - \xi(C^*(S))_{t'}| \geq -1$. The two inequalities imply

$$\begin{aligned}
\|z^* - \xi(C(S))\| - \|z^* - \xi(C^*(S))\| &= |z^*_t - \xi(C(S))_t| - |z^*_t - \xi(C^*(S))_t| \\
&\quad + |z^*_{t'} - \xi(C(S))_{t'}| - |z^*_{t'} - \xi(C^*(S))_{t'}| \\
&= |z^*_t - r_t| - |z^*_t - r^*_t| \\
&\quad + |z^*_{t'} - \xi(C(S))_{t'}| - |z^*_{t'} - \xi(C^*(S))_{t'}| \\
&\geq 0.
\end{aligned}$$

We complete Step 2-1.

Fix any type $t \in T$ with $z^*_t < q_t$. Construct another choice rule $\hat{C}$ generated by reserves $\hat{r}$ and quotas $\hat{q}$. $\hat{r}$ and quotas $\hat{q}$ are defined as follows. $\hat{r}_{t'} = r_{t'}$ for all $t' \in T$ and $\hat{q}_t = \hat{q} - 1$

for t and $\hat{q}_{t'} = q_{t'}$ for $t' \neq t$. In words, $\hat{C}$ is obtained from C by decreasing quotas for t by one and keeping other reserves and quotas same.

Step 2-2. C is less diverse than $\hat{C}$.

Claim 3. *For all $S \subseteq \mathcal{S}$, $|C(S) \setminus \hat{C}(S)| \leq 1$.*

Proof. Suppose not. There exists $S \subseteq \mathcal{S}$ such that $|C(S) \setminus \hat{C}(S)| > 1$. Pick for any $s_1, s_2 \in C(S) \setminus \hat{C}(S)$. Note that $\hat{C}(S)_{\tau(s_i)} \subsetneq C(S)_{\tau(s_i)}$ for $i = 1, 2$. Thus, $|\hat{C}(S)_{\tau(s_i)}| < |C(S)_{\tau(s_i)}| \leq q_{\tau(s_i)}$ for $i = 1, 2$. By the construction of $\hat{q}$, without loss of generality, we can assume that $|\hat{C}(S)_{\tau(s_1)}| < \hat{q}_{\tau(s_1)}$. By $s_1 \in S \setminus \hat{C}(S)$, $|\hat{C}(S)_{\tau(s_1)}| < \hat{q}_{\tau(s_1)}$ and rejection maximality* of $\hat{C}$, we have $|\hat{C}(S)_{\tau(s_1)}| = k$. Thus, $|\hat{C}(S) \setminus C(S)| > 1$. Pick $s' \in \hat{C}(S) \setminus C(S)$. Note that $C(S)_{\tau(s')} \subsetneq \hat{C}(S)_{\tau(s')}$ and thus $\hat{r}_{\tau(s')} = r_{\tau(s')} \leq |C(S)_{\tau(s')}| < |\hat{C}(S)_{\tau(s')}|$. By across-types* $\succ$-compatibility of $\hat{C}$, $s' \succ s_1$. However, by Lemma 1, $s_1 \succ s'$, a contradiction. □

Claim 4. *For all $S \subseteq \mathcal{S}$, if $|C(S) \setminus \hat{C}(S)| = 1$, then (1) $\tau(s) = t$ for $s \in C(S) \setminus \hat{C}(S)$ and (2) $|C(S)_t| = q_t$.*

Proof. First part: suppose not. There exists $S \subseteq \mathcal{S}$ such that $|C(S) \setminus \hat{C}(S)| = 1$, but $\tau(s) \neq t$ for $s \in C(S) \setminus \hat{C}(S)$. Following the same argument in Claim 3, $|\hat{C}(S)_{\tau(s)}| < |C(S)_{\tau(s)}| \leq q_{\tau(s)}$. By the assumption $\tau(s) \neq t$, $q_{\tau(s)} = \hat{q}_{\tau(s)}$. Following the same argument in Claim 3, there exists $s' \in \hat{C}(S) \setminus C(S)$ such that $\hat{r}_{\tau(s')} < |\hat{C}(S)_{\tau(s')}|$. By across-types* $\succ$-compatibility of $\hat{C}$, $s' \succ s$. However, by Lemma 1, $s \succ s'$, a contradiction.

Second part: suppose not. There exists $S \subseteq \mathcal{S}$ such that $|C(S) \setminus \hat{C}(S)| = 1$, but $|C(S)_t| < q_t$. By the first part in this claim, $|\hat{C}(S)_t| = |C(S)_t| - 1 < q_t - 1 = \hat{q}_t$. Since $s \in S \setminus \hat{C}(S)$, $|\hat{C}(S)_t| < \hat{q}_t$, and rejection maximality* of $\hat{C}$, we have $|\hat{C}(S)| = k$. Thus, $|\hat{C}(S) \setminus C(S)| > 1$. Pick $s' \in \hat{C}(S) \setminus C(S)$. Following the same argument in Claim 3, $\hat{r}_{\tau(s')} < |\hat{C}(S)_{\tau(s')}|$. By across-types* $\succ$-compatibility of $\hat{C}$, $s' \succ s$. However, by Lemma 1, $s \succ s'$, a contradiction. □

Now we show that C is less diverse than $\hat{C}$: for all $S \subseteq \mathcal{S}$, $\|z^* - \xi(C(S))\| \geq \|z^* - \xi(\hat{C}(S))\|$. If $C(S) \neq \hat{C}(S)$, then by Lemma 2, $|C(S)| \geq |\hat{C}(S)|$. Thus, $|C(S) \setminus \hat{C}(S)| = 1$ and $|\hat{C}(S) \setminus C(S)| \leq 1$. By Claim 4, $\tau(s) = t$ for $s \in C(S) \setminus \hat{C}(S)$ and $|C(S)_t| = q_t$. Thus, $|z^*_t - q_t| - |z^*_t - \hat{q}_t| = 1$. If $\hat{C}(S) \setminus C(S) \neq \emptyset$, let t' be $\tau(s')$ for $s' \in \hat{C}(S) \setminus C(S)$. Since $|\hat{C}(S) \setminus C(S)| \leq 1$, we have $|z^*_{t'} - \xi(C(S))_{t'}| - |z^*_{t'} - \xi(\hat{C}(S))_{t'}| \geq -1$. The two inequalities imply

$$\begin{aligned}
\|z^* - \xi(C(S))\| - \|z^* - \xi(\hat{C}(S))\| &= |z^*_t - \xi(C(S))_t| - |z^*_t - \xi(\hat{C}(S))_t| \\
&\quad + |z^*_{t'} - \xi(C(S))_{t'}| - |z^*_{t'} - \xi(\hat{C}(S))_{t'}| \\
&= |z^*_t - q_t| - |z^*_t - \hat{q}_t| \\
&\quad + |z^*_{t'} - \xi(C(S))_{t'}| - |z^*_{t'} - \xi(\hat{C}(S))_{t'}| \\
&\geq 0.
\end{aligned}$$

We complete Step 2-2.

To complete Step 2, note that C' can be obtained from C by sequentially constructing C^* and $\hat{C}$ from C. By Step 2-1 and Step 2-2, C is less diverse than C'.

Proof of Theorem 2

We show the first part: when C satisfies the axioms, there exist $r = (r_t)_{t \in T} \in Z^d_+$ and $q = (q_t)_{t \in T} \in Z^d_+$ such that C can be regard as the reserves-and-quotas rule. Let $q_t \equiv |C(\mathcal{S}_t)|$. We construct the reserve of type t as follows. Let $X_t \equiv \{S \subset \mathcal{S} : \exists s \in C(S), \exists s' \in S \setminus C(S) \text{ such that } \tau(s) = t, \tau(s') \neq t, |C(S)_{\tau(s')}| < q_{\tau(s')} \text{ and } s' \succ s\}$ and let $r_t \equiv \max_{S \in X_t} |C(S)_t|$. If $X_t = \emptyset$, set $r_t = 0$. We also need the following property of choice rules.

Definition 19. A choice rule C satisfies the irrelevance of rejected candidates (IRS) if for all $S, S' \subseteq \mathcal{S}$, $C(S') \subseteq S \subseteq S'$ imply that $C(S) = C(S')$.

Substitutability and IRS are equivalent to the following property.

Definition 20. A choice rule C is path independent (PI) if for all $S, S' \subseteq \mathcal{S}$, $C(S \cup S') = C(S \cup C(S'))$.

To prove the first part, we will establish claims.

Claim 5. *If C satisfies substitutability and rejection maximality, it also satisfies PI.*

Proof. It is enough to show C satisfies IRC. First, we show that rejection maximality imply the law of aggregate demand. Consider any $S, S' \subseteq \mathcal{S}$ with $S \subseteq S'$. If $|C(S')| = k$, then $|C(S)| \leq |C(S')|$. Thus, suppose $|C(S')| < k$. There are two cases to consider. Case 1: $S' \setminus C(S') = \emptyset$. Since $S \subseteq S'$, we have $S \subseteq C(S')$ and $|C(S)| \leq |C(S')|$. Case 2: $S' \setminus C(S') \neq \emptyset$. Notice that for all $t \in T$, $|S_t| \leq |S'_t|$ since $S \subseteq S'$. For all $t \in T$ such that $|C(S')_t| < |S'_t|$, by rejection maximality, we have $|C(S)_t| \leq |C(S')_t|$. For all $t \in T$ such that $|C(S')_t| = |S'_t|$, we have $|C(S)_t| \leq |C(S')_t|$. Thus, we have $|C(S)| \leq |C(S')|$.

Second, we show that substitutability and the law of aggregate demand imply IRC. Consider any $S, S' \subseteq \mathcal{S}$ with $C(S') \subseteq S \subseteq S'$. Since $S \subseteq S'$ and the law of aggregate demand, we have $|C(S)| \leq |C(S')|$. Since $C(S') \subseteq S \subseteq S'$ and substitutability, we have $C(S') \cap S = C(S') \subseteq C(S)$. By $|C(S)| \leq |C(S')|$ and $C(S') \subseteq C(S)$, we have $C(S') = C(S)$.

□

Claim 6. *For all $S \subseteq \mathcal{S}$ and $t \in T$, if $r_t < |C(S)_t|$, then t is saturated in S.*

Proof. By the definition of r_t, there exists $S^* \in X_t$ such that $|C(S^*)_t| = r_t$. Let S' be the set obtained by adding $|C(S)_t| - r_t$ type t candidates to $C(S^*)$. Specifically, $S' = C(S^*) \cup \{s_1, ..., s_{|C(S)_t| - r_t}\}$ where $\{s_1, ..., s_{|C(S)_t| - r_t}\} \subseteq \mathcal{S}_t \setminus C(S^*)_t$. Notice that $|S'_t| = |C(S)_t|$. We show that $|C(S')_t| < |S'_t| = |C(S)_t|$, and t is saturated in S. Suppose not: $|C(S')_t| = |S'_t|$. Since $S^* \in X_t$, there exist $s \in C(S^*)$ and $s' \in S \setminus C(S^*)$ such that $\tau(s) = t$, $\tau(s') \neq t$, $|C(S^*)_{\tau(s')}| < q_{\tau(s')}$ and $s' \succ s$. Consider $S^* \cup \{s_1, ..., s_{|C(S)_t| - r_t}\}$. Since $C(S^*) \subseteq S'$ and

$|C(S')_t| = |S'_t|$, we have $s \in C(S')$. By path independence,

$$\begin{aligned} &C(S^* \cup \{s_1, ..., s_{|C(S)_t|-r_t}\}) \\ &= C(C(S^*) \cup \{s_1, ..., s_{|C(S)_t|-r_t}\}) \\ &= C(S') \end{aligned}$$

Since $C(S^* \cup \{s_1, ..., s_{|C(S)_t|-r_t}\}) = C(S')$, we have $s \in C(S^* \cup \{s_1, ..., s_{|C(S)_t|-r_t}\})$, $s' \in S \setminus C(S^* \cup \{s_1, ..., s_{|C(S)_t|-r_t}\})$ and $|C(S^* \cup \{s_1, ..., s_{|C(S)_t|-r_t}\})_{\tau(s')}| < q_{\tau(s')}$. However, $r_t = |C(S^*)_t| < |S'_t| = |C(S')_t| = |C(S^* \cup \{s_1, ..., s_{|C(S)_t|-r_t}\})_t|$, we get a contradiction to maximality of r_t. □

Claim 7. *For all $S \subseteq \mathcal{S}$ and $t \in T$, if $|C(S)_t| < \min\{|S_t|, q_t\}$, then t is demanded in S.*

Proof. There are two cases to consider. Case 1: $|S_t| \leq q_t$. By the definition of q_t, we have $q_t = |C(\mathcal{S}_t)|$. Let be the set S' obtained from $C(\mathcal{S}_t)$ by removing $q_t - |S_t|$ candidates. Specifically, $S' = C(\mathcal{S}_t) \setminus \{s_1, ..., s_{q_t - |S_t|}\}$ where $\{s_1, ..., s_{q_t - |S_t|}\} \subseteq C(\mathcal{S}_t)$. By substitutability, we have $C(S') = S'$. Since $|S_t| = |S'_t|$ and $|C(S)_t| < |C(S')|_t$, t is demanded in S. Case 2: $q_t < |S_t|$. Notice that $|S_t| \leq |\mathcal{S}_t|$. Let S' be the set obtained from $\mathcal{S}_t$ by removing $|\mathcal{S}_t| - |S_t|$ rejected candidates. Specifically, $S' = \mathcal{S}_t \setminus \{s_1, ..., s_{|\mathcal{S}_t| - |S_t|}\}$ where $\{s_1, ..., s_{|\mathcal{S}_t| - |S_t|}\} \subseteq (\mathcal{S}_t \setminus C(\mathcal{S}_t))$. By IRC, we have $C(S') = C(\mathcal{S}_t)$. Since $|S_t| = |S'_t|$ and $|C(S)_t| < |C(S')|_t$, t is demanded in S. □

Claim 8. *For all $S \subseteq \mathcal{S}$ and $t \in T$, if $|C(S)_t| < \min\{|S_t|, q_t\}$, then $|C(S)| = k$.*

Proof. Suppose not: there exist $S \in \mathcal{S}$ and $t \in T$ such that $|C(S)_t| < \min\{|S_t|, q_t\}$, and $|C(S)| < k$. By the definition of q_t, $|C(\mathcal{S}_t)| = q_t$. There are two cases to consider. Case 1: $q_t \leq |S_t|$. Since $|C(\mathcal{S}_t)_t| \leq |S_t|$ and $|C(S)_t| < |C(\mathcal{S}_t)_t|$, we get a contradiction to rejection maximality. Case 2: $q_t > |S_t|$. Let S' be the set obtained from $C(\mathcal{S}_t)$ by removing $q_t - |S_t|$ candidates. Specifically, $S' = C(\mathcal{S}_t) \setminus \{s_1, ..., s_{q_t - |S_t|}\}$ where $\{s_1, ..., s_{q_t - |S_t|}\} \subseteq C(\mathcal{S}_t)$. By substitutability, $C(S') = S'$. However, we have $|S'_t| = |S_t|$ and $|C(S)_t| < |C(S')_t|$, a contradiction to rejection maximality. □

Claim 9. *For all* $S \subseteq \mathcal{S}$ *and* $t \in T$, $\min\{r_t, |S_t|\} \leq |C(S)_t| \leq q_t$.

Proof. First we show that $|C(S)_t| \leq q_t$. By the definition of q_t, we have $|C(\mathcal{S}_t)| = q_t$. If $|C(\mathcal{S}_t)| = k$, then it is clear $|C(S)_t| \leq |C(\mathcal{S}_t)| = q_t$. Thus, suppose $|C(\mathcal{S}_t)| < k$. By the definition of $\mathcal{S}_t$, we have $|S_t| \leq |\mathcal{S}_t|$. Since we assume $k < |\mathcal{S}_t|$, we have $\mathcal{S}_t \setminus C(\mathcal{S}_t) \neq \emptyset$. By rejection maximality, $|C(S)_t| \leq |C(\mathcal{S}_t)| = q_t$.

Second, we show that $\min\{r_t, |S_t|\} \leq |C(S)_t|$. Suppose by way of contradiction that $|C(S)_t| < |S_t|$ and $|C(S)_t| < r_t$. By the definition of r_t, there is $S' \in X_t$ such that $|C(S')_t| = r_t$. Let $S^* \equiv S' \setminus (S'_t \setminus C(S')_t)$. In words, S^* is obtained from S' by removing rejected type-t candidates in S'. By IRS, we have $C(S') = C(S^*)$. Notice that all type t candidates in S^* are admitted and $|S^*_t| = |C(S^*)_t| = r_t$. There two cases to consider. Case 1: $|S_t| \leq r_t$. Let S'' be the set of candidate obtained from S by adding $r_t - |S_t|$ type t candidates so that $|S''_t| = r_t$. By substitutability, we have $|C(S'')_t| \leq |C(S)_t| + r_t - |S_t| < r_t = |C(S^*)_t|$. By Claim 6, type t is saturated in S^*. However, $S^* \in X_t$, a contradiction to across-types $\succ$-compatibility. Case 2: $|S_t| > r_t$. Let S'' be the set of candidates obtained from S by removing $|S_t| - r_t$ rejected type t candidates from S. By IRS, we have $C(S) = C(S'')$. Now type t is saturated in S^* because $|C(S'')_t| < |S''_t| = r_t = |C(S^*)_t|$ and Claim 6. However, $S^* \in X_t$, a contradiction to across-types $\succ$-compatibility. □

Given these claims, we prove the first part. The proof consists of three steps.

Step 1: For all $S \subseteq \mathcal{S}$, $C^1(S) \subseteq C(S)$.

Recall that $C^1(S) \equiv H(S, k, (r_t)_{t\in T})$. Suppose not: there exist $S \subseteq \mathcal{S}$ and $s \in S$ such that $s \in C^1(S)$ and $s \in S \setminus C(S)$. Since $s \in C^1(S)$, s is the at least r_tth highest priority candidate of type $\tau(s)$ in S that is, $|\{s' \in S : \tau(s') = \tau(s), s' \succ s\}| < r_t$. Since $s \in S \setminus C(S)$ and Claim 9, $r_{\tau(s)} \leq |C(S)_{\tau(s)}|$. Thus, there exists $\tilde{s}$ such that $\tilde{s} \in C(S)$, $\tau(\tilde{s}) = \tau(s)$, and $s \succ \tilde{s}$, a contradiction to within-type $\succ$-compatibility.

Step 2: For all $S \subseteq \mathcal{S}$, $C^2(S) \subseteq C(S)$.

Suppose not: there exist $S \subseteq \mathcal{S}$ and $s \in S$ such that $s \in C^2(S)$ and $s \in S \setminus C(S)$. By within-type $\succ$-compatibility of C and the definition of C^1 and C^2, we have $C(S)_{\tau(s)} \subsetneq (C^1(S) \cup C^2(S))_{\tau(s)}$. Thus, $|C(S)_{\tau(s)}| < q_{\tau(s)}$. By Claim 7, $\tau(s)$ is demanded in S. By Claim 8, $|C(S)| = k$, and thus there exists $s' \in S$ such that $s' \in C(S) \setminus (C^1(S) \cup C^2(S))$. By within-type $\succ$-compatibility, $\tau(s') \neq \tau(s)$ and $(C^1(S) \cup C^2(S))_{\tau(s')} \subsetneq C(S)_{\tau(s')}$. Thus, $r_{\tau(s')} < |C(S)_{\tau(s')}|$. By Claim 6, $\tau(s')$ is saturated in S. By across-types $\succ$-compatibility, $s' \succ s$. However, $|(C^1(S) \cup C^2(S))_{\tau(s)}| > r_{\tau(s)}$ since $s \in C^2(S)$. Also, $|(C^1(S) \cup C^2(S))_{\tau(s')}| < q_{\tau(s')}$ since $(C^1(S) \cup C^2(S))_{\tau(s')} \subsetneq C(S)_{\tau(s')}$. By the definition of C^1 and C^2, we have $s \succ s'$, a contradiction.

Step 3: C is generated by reserves r and quotas q for priority $\succ$.

We show that $C^1(S) \cup C^2(S) = C(S)$. Suppose not: $C^1(S) \cup C^2(S) \subsetneq C(S)$. Then, $|C^1(S) \cup C^2(S)| < |C(S)|$. By the definition of C^1 and C^2, either $|C(S)| > k$ or $|C(S)_t| > q_t$ for some $t \in T$, a contradiction. We complete the first part in Theorem 2.

To prove the second part in Theorem 2, suppose that C is generated by reserves $r = (r_t)_{t \in T} \in Z^d_+$ and quotas $q = (q_t)_{t \in T} \in Z^d_+$ for $\succ$. We show that C satisfies the four axioms. By the definition of C^1 and C^2 , C satisfies within-type $\succ$-compatibility. For all $S \subseteq \mathcal{S}$, if $s \in S \setminus C(S)$ and $|C(S)| < k$, then $|C(S)_{\tau(s)}| = q_{\tau(s)}$. Thus, C satisfies rejection maximality. Given $r = (r_t)_{t \in T} \in Z^d_+$ and $q = (q_t)_{t \in T} \in Z^d_+$, it is easy to see that for all $t \in T$ and $S \subseteq \mathcal{S}$, if t is saturated in S, then $|C(S)_t| > r_t$; if t is demanded in S, then $|C(S)_t| < q_t$. Thus, C satisfies across-types $\succ$-compatibility. We show that C also satisfies substitutability: for all $s \in S \subseteq S' \subseteq \mathcal{S}$, $s \in S \setminus C(S)$ implies $s \in S' \setminus C(S')$. To see that, let $l(s, S)$ be the priority ranking of s in S. There are two cases to consider. Case 1: $|C(S)_{\tau(s)}| = q_{\tau(s)}$. $l(s, S_{\tau(s)}) > q_{\tau(s)}$ since $s \in S \setminus C(S)$ and the definition of C. $S \subseteq S'$ implies $l(s, S'_{\tau(s)}) \geq l(s, S_{\tau(s)}) > q_{\tau(s)}$ and thus $s \in S' \setminus C(S')$. Case 2: $|C(S)| < q_{\tau(s)}$. $s \in S \setminus C(S)$ implies $s \in S \setminus C^1(S)$ and $r_{\tau(s)} < l(s, S_\tau(s))$. Since $S_{\tau(s)} \subseteq S'_{\tau(s)}$, we have $r_t < l(s, S_{\tau(s)}) \leq l(s, S'_{\tau(s)})$. Thus, $s \in S' \setminus C^1(S')$. Also $s \in S \setminus C(S)$ implies $s \in S \setminus C^2(S)$. Consider a student of the lowest priority in $C^2(S)$: i.e., $\arg\min_{\succ} H(S \setminus C^1(S), k-|C^1(S)|, (q_t-r_t)_{t\in T})$. Since $s \in S \setminus C^2(S)$, we have $\arg\min_{\succ} H(S \setminus C^1(S), k-|C^1(S)|, (q_t-r_t)_{t\in T}) \succ s$. Since $S \subseteq S'$, we have $|C^1(S)| \leq |C^1(S')|$ and $\arg\min_{\succ} H(S' \setminus C^1(S'), k-|C^1(S')|, (q_t-r_t)_{t\in T}) \succeq \arg\min_{\succ} H(S \setminus C^1(S), k-|C^1(S)|, (q_t-r_t)_{t\in T})$. Thus, $\arg\min_{\succ} H(S' \setminus C^1(S'), k-|C^1(S')|, (q_t-r_t)_{t\in T}) \succ s$ and $s \in S' \setminus C^2(S')$. Together with these facts, $s \in S' \setminus C(S')$.

Proof of Theorem 3

We show that the first part: C is the reserves-and-quotas rule if it satisfies within-type $\succ$-compatibility, rejection maximality*, and across-types* $\succ$-compatibility. The proof consists of three steps.

Step 1: For all $S \subseteq \mathcal{S}$, $C^1(S) \subseteq C(S)$.

Suppose not: there exist $S \subseteq \mathcal{S}$ and $s \in S$ such that $s \in C^1(S)$ and $s \in S \setminus C(S)$. Since $s \in C^1(S)$, s is the at least r_tth highest priority candidate of type $\tau(s)$ in S that is, $|\{s' \in S : \tau(s') = \tau(s), s' \succ s\}| < r_t$. Since $s \in S \setminus C(S)$ and C satisfies the feasible constraints by r and q, we have $r_{\tau(s)} \leq |C(S)_{\tau(s)}|$. Thus, there exists $\tilde{s}$ such that $\tilde{s} \in C(S)$, $\tau(\tilde{s}) = \tau(s)$, and $s \succ \tilde{s}$, a contradiction to within-type $\succ$-compatibility.

Step 2: For all $S \subseteq \mathcal{S}$, $C^2(S) \subseteq C(S)$.

Suppose not: there exist $S \subseteq \mathcal{S}$ and $s \in S$ such that $s \in C^2(S)$ and $s \in S \setminus C(S)$. By within-type $\succ$-compatibility of C and the definition of C^1 and C^2, $C(S)_{\tau(s)} \subsetneq (C^1(S) \cup C^2(S))_{\tau(s)}$. Thus, $|C(S)|_{\tau(s)} < q_{\tau(s)}$. By rejection maximality*, $|C(S)| = k$ and thus there exists $s' \in S$ such that $s' \in C(S) \setminus (C^1(S) \cup C^2(S))$. By within-type $\succ$-compatibility of C and the definition of C^1 and C^2, $\tau(s') \neq \tau(s)$ and $(C^1(S) \cup C^2(S))_{\tau(s')} \subsetneq C(S)_{\tau(s')}$. Thus, $r_{\tau(s')} < |C(S)_{\tau(s')}|$. By across-types* $\succ$-compatibility, we have $s' \succ s$. However, $|(C^1(S) \cup C^2(S))_{\tau(s)}| > r_{\tau(s)}$ since $s \in C^2(S)$. Also, $|(C^1(S) \cup C^2(S))_{\tau(s')}| < q_{\tau(s')}$

since $(C^1(S) \cup C^2(S))_{\tau(s')} \subsetneq C(S)_{\tau(s')}$. By the definition of C^1 and C^2, we have $s \succ s'$, a contradiction.

Step 3: C is generated by reserves r and quotas q for priority $\succ$.

We show that $C^1(S) \cup C^2(S) = C(S)$. Suppose not: $C^1(S) \cup C^2(S) \subsetneq C(S)$. Then $|C^1(S) \cup C^2(S)| < |C(S)|$. By the definition of C^1 and C^2, either $|C(S)| > k$ or $|C(S)_t| > q_t$ for some $t \in T$, a contradiction.

Next, we show the second part. Suppose that C is generated by reserves r and quotas q. By the definition of C^1 and C^2, C satisfies the feasible constraints by r and q. By Theorem 2, we haven shown C satisfies within-type $\succ$-compatibility. For rejection maximality*, suppose by way of contradiction that there exists $s \in S \subseteq \mathcal{S}$ such that $s \in S \setminus C(S)$, $|C(S)| < k$, and $|C(S)_{\tau(s)}| < q_{\tau(s)}$. Notice that $|C(S) \cup \{s\}| \leq k$ and $|C(S)_{\tau(s)} \cup \{s\}| \leq q_{\tau(s)}$, we get a contradiction to the maximality of C^1 and C^2. For across-types* $\succ$-compatibility, suppose by way of contradiction that there exist $S \subseteq \mathcal{S}$ and $s, s' \in S$ such that $s \in C(S)$, $s' \in S \setminus C(S)$, $|C(S)_{\tau(s)}| > r_{\tau(s)}$, $|C(S)_{\tau(s')}| < q_{\tau(s')}$, and $s' \succ s$. Since $s' \in S \setminus C(S)$, $|C(S)_{\tau(s')}| < q_{\tau(s')}$, and rejection* maximality of C, we have $|C(S)| = k$. Let $S^* \equiv (C(S) \setminus \{s\}) \cup \{s'\}$. Notice that $|S^*| = k$, $|S^*_{\tau(s)}| \geq r_{\tau(s)}$ and $|S^*_{\tau(s')}| \leq q_{\tau(s')}$. This contradicts that C^1 and C^2 chooses the highest priority candidates with respect to the constraints.

Proof of Proposition 1

Suppose that there exists z^* such that C is separable and for each type t, C_t is a responsive rule on $\mathcal{S}_t$ whose capacity is z^*_t. Echenique and Yenmez [24] show that the ideal-distribution rule is characterized by substitutability, monotonicity, and within-type $\succ$-compatibility. Thus, it is enough to check that C satisfies these three axioms. Since C_t is responsive on $\mathcal{S}_t$ for each $t \in T$, C satisfies substitutability and within-type $\succ$-compatibility. We show that for all $S, S' \subseteq \mathcal{S}$ and $t \in T$, if $|S_t| \leq |S'_t|$, then $|C(S)_t| \leq |C(S')_t|$. Suppose not:

there exist $S, S' \subseteq \mathcal{S}$ and $t \in T$ such that $|S_t| \leq |S'_t|$ and $|C(S)_t| > |C(S')_t|$. Notice that $|C(S')_t| < |C(S)_t| \leq |S_t| \leq |S'_t|$. Since C_t satisfies acceptance on $\mathcal{S}_t$, $|C(S')_t| < |S'_t|$ imply $|C(S')_t| = z^*_t$. This implies $z^*_t < |C(S)_t|$, a contradiction for the fact that C_t is a responsive rule whose capacity is z^*_t. This property is stronger than monotonicity, and thus C satisfies monotonicity.

Proof of Proposition 2

We show that the class of reserves and quotas rules includes the responsive rule and the ideal-distribution rule as special cases. Then, Proposition 2 follows from Theorem 1.

First, we show that C is the responsive rule when $r_t = 0$ and $q_t \geq k$ for all $t \in T$. Since $r_t = 0$ for all $t \in T$, we have $C^1(S) = \emptyset$ for all $S \subseteq \mathcal{S}$. Since $q_t \geq k$ for all $t \in T$, $C = C^2$ satisfies acceptance. Let C' be the responsive rule whose capacity is k. We show that for all $S \subseteq \mathcal{S}$, $C(S) = C'(S)$. Suppose not: there exists $S \subseteq \mathcal{S}$ such that $C(S) \neq C'(S)$. Since both C and C' satisfy acceptance, $|C(S)| = |C'(S)| = k$. Thus, there are $s \in C(S) \setminus C'(S)$ and $s' \in C'(S) \setminus C(S)$. Since $s \in C(S)$, $s' \in S \setminus C(S)$, $C^1(S) = \emptyset$, and the definition of C^2, we have $s \succ s'$. However, since $s' \in C'(S)$, $s \in S \setminus C'(S)$, and the definition of the responsive rule, we have $s' \succ s$, a contradiction.

Next, we show that C is the ideal-distribution rule when $r_t = z^*_t = q_t$ for all $t \in T$. Since $r_t = z^*_t = q_t$ for all $t \in T$, we have $C^2(S) = \emptyset$ for all $S \subseteq \mathcal{S}$. This means $C = C^1$, and thus C is separable. For each $t \in T$, C_t is the responsive rule whose capacity is z^*_t since $r_t = z^*_t$ and the definition of C^1. By Proposition 1, C is the ideal-distribution rule.

Proof of Proposition 3

For each type t, the reserves r_t and quotas q_t are defined as $r_t \equiv \min_{S \subseteq \mathcal{S}}\{|S_t| : |C(S)_t| = |S_t|\}$ and $q_t \equiv \max_{S \subseteq \mathcal{S}} |C(S)_t|$, respectively. By the definitions of r and q, the choice rule C satisfies the feasible constraints by r and q. By Theorem 5, C is separable. For each $t \in T$, if $r_t = q_t$, then C_t is the responsive rule on $\mathcal{S}_t$. By Proposition 1, C is the ideal distribution rule.

Proof of Theorem 4

Proof. Let C be the reserves-and-quotas rule. Let C' be a choice rule satisfying the four axioms. By a way of contradiction, suppose that for some $S \subseteq \mathcal{S}$ and $r, q \in \mathbf{Z}_+^{\mathbf{d}}$, we have $C(S, r, q) \neq C'(S, r, q)$.

Claim 10. $|C(S, r, q)| = |C'(S, r, q)| = k$.

Proof. First we show $|C(S, r, q)| = |C'(S, r, q)|$. Suppose not: $|C(S, r, q)| \neq |C'(S, r, q)|$. There are two cases to consider. Case 1: $|C(S, r, q)| > |C'(S, r, q)|$. There exists $t \in T$ such that $|C(S, r, q)_t| > |C'(S, r, q)_t|$. By the feasible constraint, we have $q_t > |C(S, r, q)_t| > |C'(S, r, q)_t|$. Note that $S_t \setminus C'(S, r, q)_t \neq \emptyset$ since $|C(S, r, q)| \neq |C'(S, r, q)|$. However, $|C(S, r, q)| > |C'(S, r, q)|$ imply $k > |C'(S, r, q)|$, a violation of rejection maximality. Case 2: $|C(S, r, q)| < |C'(S, r, q)|$. Since the reserves-and-quotas rule C satisfies rejection maximality, we can apply the same argument in Case 1.

Second we show $|C(S, r, q)| = k$. Suppose not: $|C(S, r, q)| < k$. Since $C(S, r, q) \neq C'(S, r, q)$ and $|C(S, r, q)| = |C'(S, r, q)|$, we have $C'(S, r, q) \backslash C(S, r, q) \neq \emptyset$. Pick any $s \in C'(S, r, q) \setminus C(S, r, q)$. Since C' satisfies the feasible constraint, we have $|C'(S, r, q)_{\tau(s)}| \leq q_{\tau(s)}$. By within-type $\succ$-compatibility of C' and $s \in C'(S, r, q) \setminus C(S, r, q)$, we have $C(S, r, q)_{\tau(s)} \subsetneq C'(S, r, q)_{\tau(s)}$ and thus $|C(S, r, q)_{\tau(s)}| < q_{\tau(s)}$. However, together with $|C(S, r, q)| < k$ and $s \in S \setminus C(S, r, q)$, it contradicts C satisfies rejection maximality. $\square$

Lemma 3. *$F_{C(S,r,q)}(l) \geq F_{C'(S,r,q)}(l)$ for all $l \in \{1, ..., n\}$.*

Proof. Since $C(S, r, q) \neq C'(S, r, q)$ and $|C(S, r, q)| = |C'(S, r, q)|$, we have $C(S, r, q) \setminus C'(S, r, q) \neq \emptyset$, $C'(S, r, q) \setminus C(S, r, q) \neq \emptyset$, and $|C(S, r, q) \setminus C'(S, r, q)| = |C'(S, r, q) \setminus C(S, r, q)|$. It is suffice to show that for all $s \in C(S, r, q) \setminus C'(S, r, q)$ and $s' \in C'(S, r, q) \setminus C(S, r, q)$, we have $s \succ s'$. First, we show that $|C(S, r, q)_{\tau(s)}| > r_{\tau(s)}$. $s \in S \setminus C'(S)$ imply $|C'(S, r)_{\tau(s)}| \geq r_{\tau(s)}$. By within-type $\succ$-compatibility of C and $s \in C(S, r, q) \setminus C'(S, r, q)$, we have $C'(S,r,q)_{\tau(s)} \subsetneq C(S,r,q)_{\tau(s)}$ and thus $|C(S,r,q)_{\tau(s)}| > r_{\tau(s)}$. Second, we show that $|C(S,r,q)_{\tau(s')}| < q_{\tau(s')}$. Since C' satisfies the feasible constraint, we have $|C'(S,r,q)_{\tau(s')}| \leq q_{\tau(s)}$. By within-type $\succ$-compatibility of C and $s' \in C'(S,r,q) \setminus C(S,r,q)$, we have $C(S,r,q)_{\tau(s')} \subsetneq C'(S,r,q)_{\tau(s')}$ and thus $|C(S,r,q)_{\tau(s')}| < q_{\tau(s')}$. By the definition of the reserves-and-quotas rule, we have $s \succ s'$. □

Now we prove Theorem 4. Define $r', q' \in \mathbf{Z}_{+}^{\mathbf{d}}$ by setting $r'_t = |C(S,r,q)_t| = q'_t$ for all $t \in T$. Notice that $\sum_{t\in T} r'_t = k = \sum_{t \in T} q'_t$ by Claim. Together with these facts, $\xi(C(S,r,q))$ is a unique solution for the feasible constraint by r' and q'. Thus, we have $\xi(C'(S,r',q')) = \xi(C(S,r,q))$. Since C' satisfy within-type $\succ$-compatibility, we have $C'(S,r',q') = C(S,r,q)$. By Lemma, we have $F_{C'(S,r',q')}(l) \geq F_{C'(S,r,q)}(l)$ for all $l \in \{1,...,n\}$. However, we have $r_t \leq |C(S,r,q)_t| = r'_t$ and $q'_t = |C(S,r,q)_t| \leq q_t$ for all $t \in T$, a contradiction.

□

1.7.2 Independence of Axioms

We show the independence of axioms in Theorem 2, 3, 4, and 5.

Axioms in Theorem 5

Example 1 (Violating only rejection maximality). Let $\mathcal{S} = \{s_1, s_2, s_3\}$, $k = 2$ and $\tau(s_1) = \tau(s_2) = \tau(s_3) = t$. Consider the following choice rule: $C(\{s_1,s_2,s_3\}) = C(\{s_1,s_2\}) = C(\{s_1,s_3\}) = C(\{s_1\}) = \{s_1\}$, $C(\{s_2,s_3\}) = \{s_2,s_3\}$, $C(\{s_2\}) = \{s_2\}$ and $C(\{s_3\}) = \{s_3\}$. C satisfies substitutability. C also satisfies across-types strong axiom of revealed preference since all candidates are the same type. It violates rejection maximality since $|\{s_1,s_2,s_3\}_t| > |\{s_2,s_3\}_t|$ and $|C(\{s_1,s_2,s_3\})_t| < |C(\{s_2,s_3\})_t| = k$.

Example 2 (Violating only substitutability). Let $\mathcal{S} = \{s_1,s_2,s_3,s_4\}$, $k = 2$ and $\tau(s_1) = \tau(s_2) = \tau(s_3) = t_1$ and $\tau(s_4) = t_2$. Consider the following choice rule: $C(\{s_1,s_2,s_3,s_4\}) =$

$C(\{s_1, s_2, s_3\}) = \{s_1, s_2\}, C(\{s_1, s_2, s_4\}) = C(\{s_1, s_3, s_4\}) = \{s_1, s_4\}, C(\{s_2, s_3, s_4\}) = \{s_2, s_4\}$, $C(\{s_1, s_2\}) = C(\{s_1, s_3\}) = \{s_1\}$, $C(\{s_2, s_3\}) = \{s_2\}$, and $C(S) = S$ for the remaining S.

Let $\succ$ be defined as follows: $s \succ s'$ if there exists $S \supseteq \{s.s'\}$ such that $s \in C(S)$ and $s' \notin C(S)$ and either $\tau(s) = \tau(s')$ or $\tau(s)$ is saturated and $\tau(s')$ is demanded in S. Notice that t_1 is neither demanded nor saturated in any $S \subseteq \mathcal{S}$. Thus, we focus on t_1 only and get $s_1 \succ s_2 \succ s_3$. Since there is no cycle, across-types strong axiom of revealed preference is satisfied. For rejection maximality, there are three cases where $|C(S)| < k$. Rejection maximality is satisfied since $|C(\{s_1, s_2\})| = |C(\{s_1, s_3\})| = |C(\{s_2, s_3\})|$. To see that substitutability is not satisfied, note $s_2 \in C(\{s_1, s_2, s_3, s_4\})$ and $s_2 \notin C(\{s_1, s_2, s_4\})$.

Example 3 (Violating only across-types strong axiom of revealed preference). Let $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$, $k = 2$ and $\tau(s_1) = \tau(s_2) = \tau(s_3) = \tau(s_4) = t$. Consider the following choice rule: $C(\{s_1, s_2, s_3, s_4\}) = C(\{s_1, s_2, s_3\}) = C(\{s_1, s_2, s_4\}) = \{s_1, s_2\}$, $C(\{s_1, s_3, s_4\}) = \{s_1, s_3\}$, $C(\{s_2, s_3, s_4\}) = \{s_2, s_4\}$ and $C(S) = S$ for the remaining S. C satisfies rejection maximality and substitutability. But it does not satisfy across-types strong axiom of revealed preference since s_3, s_4 and $\{\{s_2, s_3, s_4\}, \{s_1, s_3, s_4\}\}$ satisfies $s_4 \in C(\{s_2, s_3, s_4\})$, $s_3 \in \{s_2, s_3, s_4\} \setminus C(\{s_2, s_3, s_4\})$, $s_3 \in C(\{s_1, s_3, s_4\})$, $s_4 \in \{s_1, s_3, s_4\} \setminus C(\{s_1, s_3, s_4\})$, and $\tau(s_3) = \tau(s_4) = t$.

Axioms in Theorem 2

Example 4 (Violating only rejection maximality). Consider the choice rule in Example 1. Let $\succ$ be as follows: $s_1 \succ s_2 \succ s_3$. As argued in Example 1, C satisfies substitutability but not rejection maximality. Moreover, C satisfies within-type $\succ$-compatibility and across-types $\succ$-compatibility for $\succ$.

Example 5 (Violating only substitutability). Consider the choice rule in Example 2. Let $\succ$ be as follows: $s_1 \succ s_2 \succ s_3 \succ s_4$. As argued in Example 2, C satisfies rejection maximality but not substitutability. It is easy to see C satisfies within-type $\succ$-compatibility. t_1 is neither demanded nor saturated in any $S \subseteq \mathcal{S}$, as argued in Example 2. Thus, across-types $\succ$-compatibility is also satisfied.

Example 6 (Violating only within-type $\succ$-compatibility). Consider the choice rule in Example 3. As argued in Example 3, C satisfies rejection maximality and substitutability. Across-types $\succ$-compatibility is also satisfied since there is only one type. But it fails within-type $\succ$-compatibility for any $\succ$; $s_4 \in C(\{s_2, s_3, s_4\})$ and $s_3 \in \{s_2, s_3, s_4\} \setminus C(\{s_2, s_3, s_4\})$ imply $s_4 \succ s_3$; $s_3 \in C(\{s_1, s_3, s_4\})$ and $s_4 \in \{s_1, s_3, s_4\} \setminus C(\{s_1, s_3, s_4\})$ imply $s_3 \succ s_4$.

Example 7 (Violating only across-types $\succ$-compatibility). Consider the choice rule in Example 3 but suppose that all candidates have different types. As argued in Example 3, C satisfies rejection maximality and substitutability. Within-type $\succ$-compatibility is also satisfied since all candidates have different types. But it fails across-types $\succ$-compatibility for any $\succ$. Since $s_4 \in C(\{s_2, s_3, s_4\})$, $s_3 \in \{s_2, s_3, s_4\} \setminus C(\{s_2, s_3, s_4\})$, $\tau(s_4)$ is saturated in $\{s_2, s_3, s_4\}$, and $\tau(s_3)$ is demanded in $\{s_2, s_3, s_4\}$, we have $s_4 \succ s_3$. On the other hand, since $s_3 \in C(\{s_1, s_3, s_4\})$, $s_4 \in \{s_1, s_3, s_4\} \setminus C(\{s_1, s_3, s_4\})$, $\tau(s_3)$ is saturated in $\{s_1, s_3, s_4\}$, and $\tau(s_4)$ is demanded in $\{s_1, s_3, s_4\}$, we have $s_3 \succ s_4$.

Axioms in Theorem 3

Example 8 (Violating only rejection maximality*). Consider the choice rule in Example 1. Let $\succ$ be as follows: $s_1 \succ s_2 \succ s_3$. Let $r_t = 0$ and $q_t = 2$. C satisfies feasible constraints by r and q. C violates rejection maximality* since $|C(\{s_1, s_2, s_3\})_t| < q_t$ and

$|C(\{s_1, s_2, s_3\})| < k$. Moreover, C satisfies within-type $\succ$-compatibility and across-types* $\succ$-compatibility for $\succ$.

Example 9 (Violating only within-type $\succ$-compatibility). Consider the choice rule in Example 3. Let $r_t = 0$ and $q_t = 2$. C satisfies feasible constraints by r and q. Note that C satisfies acceptance which is stronger than rejection maximality*. Across-types* $\succ$-compatibility is also satisfied since there is only one type. But it fails within-type $\succ$-compatibility for any $\succ$; $s_4 \in C(\{s_2, s_3, s_4\})$ and $s_3 \in \{s_2, s_3, s_4\} \setminus C(\{s_2, s_3, s_4\})$ imply $s_4 \succ s_3$; $s_3 \in C(\{s_1, s_3, s_4\})$ and $s_4 \in \{s_1, s_3, s_4\} \setminus C(\{s_1, s_3, s_4\})$ imply $s_3 \succ s_4$.

Example 10 (Violating only across-types* $\succ$-compatibility). Consider the choice rule in Example 3 but suppose that all candidates have different types. Let $r_{\tau(s_i)} = 0$ and $q_{\tau(s_i)} = 2$ for $i = 1, ..., 4$. C satisfies feasible constraints by r and q. Note that C satisfies acceptance which is stronger than rejection maximality*. Within-type $\succ$-compatibility is also satisfied since all candidates have different types. But it fails across-types* $\succ$-compatibility for any $\succ$. Since $s_4 \in C(\{s_2, s_3, s_4\})$, $s_3 \in \{s_2, s_3, s_4\} \setminus C(\{s_2, s_3, s_4\})$, $|C(\{s_2, s_3, s_4\})_{\tau(s_4)}| > r_{\tau(s_4)}$, and $|C(\{s_2, s_3, s_4\})_{\tau(s_3)}| < q_{\tau(s_3)}$, we have $s_4 \succ s_3$. On the other hand, since $s_3 \in C(\{s_1, s_3, s_4\})$, $s_4 \in \{s_1, s_3, s_4\} \setminus C(\{s_1, s_3, s_4\})$, $|C(\{s_1, s_3, s_4\})_{\tau(s_3)}| > r_{\tau(s_3)}$, and $|C(\{s_1, s_3, s_4\})_{\tau(s_4)}| < q_{\tau(s_4)}$, we have $s_3 \succ s_4$.

Axioms in Theorem 4

Example 11 (Violating only within-type $\succ$-compatibility). Let $\mathcal{S} = \{s_1, s_2\}$, $k = 1$ $\tau(s_1) = \tau(s_2) = t$, and $s_1 \succ s_2$. Consider the following choice rule: $C(\{s_1, s_2\}) = \{s_2\}$ and $C(S) = S$ for the remaining S. C satisfies feasibility, rejection maximality* and meritorious monotonicity in reserve and quota size. But it does not satisfy within-type $\succ$-compatibility.

Example 12 (Violating only rejection maximality*). Fix any type t. Consider the following choice rule. For all r and q, $C(\colon r, q)$ is the reserves-and-quotas rule generated by r and q except that for all $S \subseteq \mathcal{S}$, $|C(S)_t| = \min\{r_t, |S_t|\}$. By the definition of the rule, C satisfies feasibility and within-type $\succ$-compatibility. It also satisfies meritorious monotonicity in reserve and quota size by Theorem 1. But it dose not satisfy rejection maximality*.

Example 12 (Violating only meritorious monotonicity in reserve and quota size). As we discussed in Section 3.2, the reserves rule in public high schools in Chicago does not satisfy the reserves rule in public high schools in Chicago. On the other hand, it satisfies feasibility, within-type $\succ$-compatibility and rejection maximality*.

Chapter 2

Measuring Manipulability of Matching Mechanisms

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2.1 Introduction

In the last decades, policymakers in several school districts have attempted to simplify the strategic aspects of school admissions in their school choice plans. A first change occurred to Boston's grade K-12 assignment system, known as the Boston mechanism, in place since 1999. Abdulkadiroğlu and Sönmez [2] show that this mechanism is vulnerable to strategic manipulation, and suggest two alternatives which are not. In June 2005, the Boston school committee voted to replace the Boston mechanism with the student proposing deferred acceptance (DA) mechanism (Gale and Shapley [30]), a mechanism where participants can do no better than report their preferences truthfully.

While the transition from the Boston mechanism to the DA mechanism is clear in terms of discouraging manipulation, many school admissions reforms include variants of the two mechanisms. These mechanisms are constrained in the sense that they consider a subset of choices on a student's rank order list. Due to the constraint, all these mechanisms are manipulable. For example, two admissions reforms took place in Chicago public high school choice. The first reform was the transition from the constrained Boston mechanism to the constrained DA mechanism. Both the mechanisms consider the first four choices on a student's rank order list. The second reform changed to the constrained DA that considers the first six choices. Besides, similar school admissions reforms took place in England and other places. A natural idea is that the new mechanisms are "less manipulable" than the abandoned mechanisms even that all of them are manipulable.

We introduce a method to measure manipulability of a matching mechanism. Our measure counts the number of students who can manipulate a mechanism for each preference profile. We use theory and simulation to apply our measure to the constrained mechanisms DA^k and BOS^k where these mechanisms consider the first k choices on a student's rank order list. First, to motivate to study our measure, we demonstrate a problem of the existing measure introduced by Pathak and Sönmez [42]. They show that (i) manipulability of DA^k decreases in k, and (ii) DA^k is less manipulable than BOS^k. These results support recent admissions reforms: while the recent school admissions reforms did not fully eliminate incentives to manipulate, they discouraged manipulation. We find that the implications from their measure are strongly dependent on the full preference domain assumption. Specifically, we show that the mechanisms in their results are equally manipulable under *tiered preference domain*.

Second, we show that our measure is more robust. The implications from existing measures can be carried over in both the full and the tiered preference domains. In addition, while these results are parallel to the results in the literature, the proof is different. We use the key property of DA, *individual rational monotonicity* proposed by Kojima and Manea [36], to prove them which are theoretically interesting for themselves. These results confirm parallel to results in literature and provide further justifications for admissions reforms in Chicago, England, and other places (Pathak and Sönmez [42]; Decerf and Van der Linden[17]; Bonkoungou and Nesterov [9]).

Third, we use simulation for quantitative analysis. The first simulation is on our manipulability measure and provides further quantitative analysis of our theoretical results. Two implications obtained from the simulations are particularly important; (i) switching from BOS^k to DA^k largely reduce manipulability; (ii) long preference list is not necessary if we can allow small manipulability under DA^k. These implications support the recent school admissions reforms quantitatively, as well as qualitatively: they largely eliminated the incentive to manipulate. The second simulation is on the measure by Pathak and Sönmez [42]. While our qualitative results from theory are parallel to their measure, the quantitative results from simulations confirm a significant difference.

The rest of Chapter 2 is organized as follows. Section 2 presents the model. Section 3 presents the mechanisms that we study in this paper. Section 4 reviews the measure in the literature. Section 5 provides our main results. Section 6 provides the concluding remarks. Appendix provides the omitted proofs.

2.1.1 Related literature

Our results are closely related to the results obtained by Pathak and Sönmez [42]. They introduce a method to compare mechanisms by their vulnerability to manipulation and justify recent school admissions reforms. While our qualitative results from theory are parallel to their measures, there are two major differences in our results. First, our measure is more robust to a restricted domain. Second, our paper also provides new insights by quantitative analysis via simulations. In two recent papers, Decerf and Van der Linden [17] and Bonkoungou and Nesterov [9] also study manipulability in school choice. Decerf and Van der Linden [17] focus on dominant strategies and rely on a measure introduced by Arribillaga and Massó [3] in the context of voting. While their implication is parallel to that by Pathak and Sönmez [42], they also study tie-breaking rule from a manipulability perspective. Bonkoungou and Nesterov [9] introduce a new concept, strategic accessibility, and correct mistakes one of the results by Pathak and Sönmez [42] while keeping their implications. As we discuss later, all comparison criteria have limitations, and our analysis should be viewed as complementary to these approaches. All approaches are complementary, as they provide different measures available for market designers to choose from depending on the purpose of the application at hand.

Our paper contributes to the literature on constrained school choice. Haeringer and Klijn [31] introduce the constrained school choice problem and study the efficiency and stability properties of the Nash equilibria of constrained school choice mechanisms. Chen and Kesten [13] study a variant of constrained mechanism which is used in Chinese college admissions. Calsamiglia et al. [10] and Chen and Kesten [14] study constrained mechanisms experimentally replicating the design of Chen and Sönmez [15]. Recently, Dur and Morrill [23] and Decerf and Van der Linden [16] show how constraints on the number of schools students can report can make the constrained DA more efficient than the unconstrained DA. The efficiency of constrained DA is also discussed by Che and Tercieux [12].

Our paper also contributes to the literature on a measure of incentives to mechanisms. In the market design literature, Azevedo and Budish [7] propose a relaxation of strategy-proofness based on the idea that vulnerability to manipulation disappears in large economies for some mechanisms, but not others. Focusing on voting applications, Peleg [44] introduce a profile-counting approach, which is similar to ours. Carroll [11] proposes another criterion to evaluate voting mechanisms based on the extent to which they encourage manipulation. Arribillaga and Massó [3] introduce a measure that focuses on the number of dominant strategies that a voting mechanism provides.

2.2 Model

There is a finite set of students $I = \{i_1, ..., i_n\}$ and a finite set of schools $S = \{s_1, ..., s_m\}$ where $m \geq 2$. A student can be unassigned to any school in S. $\emptyset$ represents not being assigned to any school in S. Each student i has a preference relation R_i over $S \cup \{\emptyset\}$, where P_i is the strict counter part of R_i. We denote $\mathcal{R}_i$ by the set of all preference relations. Let $R = (R_i)_{i \in I} \in \times_{i \in I} \mathcal{R}_i$ denote the preference profile. We denote $\mathcal{R} \equiv \times_{i \in I} \mathcal{R}_i$ by the set of all preference profiles. A **(preference) domain** is a set $\mathcal{D} \equiv \times \mathcal{D}_i \subseteq \mathcal{R}$. We say that $\mathcal{D}$ is the **full domain** if $\mathcal{D} = \mathcal{R}$. A school s is acceptable for student i if $sP_i\emptyset$. Each school s has a priority $\succ_s$ and capacity q_s. $\succ_s$ is a strict ranking over I, and q_s is the number of seats in s. Let $\succ = (\succ_s)_{s \in S}$ and $q = (q_s)_{s \in S}$ denote the priority profile and the capacity profile, respectively. Let $\mathcal{F}_s$ and $\mathcal{F}$ denote the set of all priorities and the set of all priority profiles, respectively.

A **matching** μ is a function from $S \cup I$ to $2^{I \cup S}$ such that (i) for all student $i \in I$, $|\mu(i)| \leq 1$ and $\mu(i) \subseteq S$; (ii) for all school $s \in S$, $|\mu(s)| \leq q_s$ and $\mu(s) \subseteq I$; (iii) for all pair of student and school $(i, s) \in I \times S$, $\mu(i) = \{s\}$ if and only if $i \in \mu(s)$. When $\mu(i) \neq \{\emptyset\}$, $\mu(i)$ denotes the school that student i is assigned to. $\mu(s)$ denotes the set of students who are assigned to school s. Under a matching, every student is assigned to a school or unassigned, and every school s can accept at most q_s students.

Given priority and capacity profiles $(\succ, q)$, the matching outcome is determined according to the mechanism based only on the submitted preferences of students. A **mechanism** φ is a function from the set of preference profile to the set of matching. For ease of notation, we will sometimes use $\varphi_i(R)$ to refer to $\varphi(R)(i)$.

In this paper, we focus on a constrained version of mechanism φ. Let φ^k be a **constrained mechanism** that considers the first k choices on a student's preference profile. To formally define φ^k, we need to introduce R^k, which is obtained from R as follows. If $|\{s : sP_i\emptyset\}| > k$, then R_i^k satisfies (1) for all $s, s' \in S$, sR_is' if and only if sR_i^ks', and (2) $|\{s \in S : sP_i^k\emptyset\}| = k - 1$. If $|\{s : sP_i\emptyset\}| \leq k$, then $R_i^k = R_i$. Given this concept, a constrained mechanism is defined by $\varphi^k(R) \equiv \varphi(R^k)$.

2.3 Mechanisms

In this section, we describe the two mechanism that we study in the context of school choice: the Boston mechanism and the student proposing deferred acceptance (DA) mechanism.

2.3.1 The Boston mechanism

Consider a preference profile R submitted by the students. The Boston mechanism finds a matching through the following procedure.

Step 1. Set $q_s^1 \equiv q_s$ for all $s \in S$. Each student i proposes to the school that is ranked first in R_i (if $\emptyset$ is ranked first in R_i, then i remains unassigned). Each school s assigns up to q_s^1 seats to its proposers one at a time following the priority order $\succ_s$. Remaining students are rejected. Let q_s^2 denote the number of available seats at school s.

Step l, $l \geq 2$. Each student i that is rejected in Step $l-1$ proposes to the school that is ranked lth in R_i (if $\emptyset$ is ranked lth in R_i, then i remains unassigned). School s assigns up to q_s^l seats to its (new) proposers one at a time following the priority order $\succ_s$. Remaining students are rejected. Let q_s^{l+1} denote the number of available seats at school s.

The procedure stops when no student is rejected. Any remaining student remains unassigned. Let $BOS(R)$ denote the matching. The mechanism BOS is the Boston mechanism, or BOS for short.

2.3.2 The student proposing deferred acceptance mechanism

The student proposing deferred acceptance (DA) mechanism was introduced by Gale and Shapley [30] and was studied in the context of school choice by Abdulkadiroğlu and Sönmez [2]. Let R be a preference profile submitted by the students. The DA mechanism finds a matching through the following procedure.

Step 1. Each student i proposes to the school that is ranked first in R_i (if there is no such school then i remains unassigned). Each school s tentatively assigns up to q_s seats to its proposers one at a time following the priority order $\succ_s$. Remaining students are rejected.

Step l, $l \geq 2$. Each student i that is rejected in Step $l-1$ proposes to the next school in the ordered list R_i (if there is no such school then i remains unassigned). Each school s considers the new proposers and the students that have a (tentative) seat at s. School s tentatively assigns up to q_s seats to these students one at a time following the priority order $\succ_s$. Remaining students are rejected.

The algorithm stops when no student is rejected. Each student is assigned to his final tentative school. Let $DA(R)$ denote the matching. The mechanism DA is the student proposing deferred acceptance mechanism, or DA for short.

2.4 Manipulability measure by Pathak and Sönmez (2013)

In this section, we study the manipulability measure introduced by Pathak and Sönmez [42]. First, we define their measure and introduce their results. Second, we illustrate that this measure sometimes leads to counter intuitive results. Third, based the observation from examples, we show that the measure does not work under some preference domain.

2.4.1 Definition

We need several concepts to define the manipulability measure. A mechanism φ is **manipulable** by student i at preference profile R if there exists a preference $\tilde{R}_i$ such that $\varphi_i(\tilde{R}_i, R_{-i}) P_i \varphi_i(R)$. We will say that preference profile R is **vulnerable** under mechanism φ if φ is manipulable by some student at R. A mechanism is **strategy-proof** if it is not manipulable by any player at any problem. It is known that DA is strategy-proof, but BOS is not. Moreover, neither constrained DA^k nor BOS^k is strategy-proof.

Now we describe the manipulability measure by Pathak and Sönmez [42].

Definition 21. A mechanism ψ is **at least as manipulable** as mechanism φ on domain $\mathcal{D}$

if for all $R \in \mathcal{D}$,

$$R \text{ is vulnerable under mechanism } \varphi \Rightarrow R \text{ is vulnerable under mechanism } \psi.$$

Two mechanisms can be equally manipulable if they are manipulable for exactly the same set of preference profiles. The next definition rules out this possibility.

Definition 22. A mechanism ψ is **more manipulable** than mechanism φ on domain $\mathcal{D}$ if

(i) ψ is at least as manipulable as φ on domain and $\mathcal{D}$

(ii) there exists $R \in \mathcal{D}$ such that R is vulnerable under mechanism ψ but not under φ.

Pathak and Sönmez [42] support recent admissions reforms in Chicago, England and other places. According to their measure, while the reforms did not fully eliminate incentives to manipulate, they discourage manipulation.

Theorem 7 (Pathak and Sönmez [42])**.** *For all k and l with $1 \leq k < l \leq |S|$,*

(i) BOS^l is more manipulable than DA^l on the full domain $\mathcal{R}$;

(ii) DA^k is more manipulable than DA^l on the full domain $\mathcal{R}$.

The measure introduced by Pathak and Sönmez [42] makes weak requirement that preference profile R is vulnerable under mechanism φ. If there is any one student can manipulate φ at R, R is vulnerable under mechanism φ. Due to the weak requirement, this measure sometimes leads to counter intuitive results: there are preference profile R and two mechanisms such that while R is vulnerable under both mechanisms, it is natural that R is ”less vulnerable” under one than the other in some way. The following examples illustrate this fact.

Example 5. Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}$, and $q_1 = q_2 = q_3 = 1$. Each school s has a common prioirty, $i_1 \succ_s i_2 \succ_s i_3 \succ_s i_4$. Let R be as given below, where vertical dots represent arbitrary rankings of remaining schools.

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}
s_1	s_1	s_1	s_2
$\vdots$	s_2	s_2	s_3
	s_3	s_3	$\vdots$

Consider DA^2. The matching produced by DA^2 at R is given by

$$DA^2(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_4 \end{pmatrix}.$$

Consider BOS^2. The matching produced by BOS^2 at R is given by

$$BOS^2(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_4 & \emptyset \end{pmatrix}.$$

DA^2 is manipulable by i_3 at R since i_3 can be better off by reporting s_3 as the first choice. BOS^2 is manipulable by i_2 at R since i_2 can be better off by reporting s_3 as the first choice. Similarly, BOS^2 is manipulable by i_3 at R. Note that the set of students who can manipulate DA^2 at R is strictly included in that of BOS^2 at R i.e., $\{i_3\} \subsetneq \{i_2, i_3\}$.

Example 6. Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, and $q_1 = q_2 = q_3 = 1$. Each school s has a common prioirty, $i_1 \succ_s i_2 \succ_s i_3$. Let R be as given below.

R_{i_1}	R_{i_2}	R_{i_3}
s_1	s_1	s_1
$\vdots$	s_2	s_2
	$\vdots$	s_3

Consider DA^2. The matching produced by DA^2 at R is given by

$$DA^2(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & \emptyset \end{pmatrix}.$$

Consider DA^1. The matching produced by DA^1 at R is given by

$$DA^1(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & \emptyset & \emptyset \end{pmatrix}.$$

DA^2 is manipulable by i_3 at R since i_3 can be better off by reporting s_3 as the first choice. DA^1 is manipulable by i_2 at R since i_2 can be better off by reporting s_3 as the first choice. Similarly, DA^1 is manipulable by i_3 at R. Note that the set of students who can manipulate DA^2 at R is strictly included in that of DA^1 at R i.e., $\{i_3\} \subseteq \{i_2, i_3\}$.

In the examples, one might think DA^2 is less manipulable than BOS^2 (DA^1) at R since the set of students who can manipulate DA^2 at R is strictly included. However, preference profile R is equally vulnerable under the both mechanisms in terms of the measure by Pathak and Sönmez [42]. This is because the requirement for preference to be vulnerable is weak: if any one student s can manipulate φ at R, then R is vulnerable under φ. It is natural that R is "more vulnerable" under BOS^2 (DA^1) in some sense.

The weak requirement for preference to be vulnerable causes at least two problems. First, two mechanisms are often equally manipulable since a preference profile is often vulnerable under both mechanisms. Second, even if one mechanism is more manipulable than another, the difference by this measure would be small. These are disadvantages since we want to compare mechanisms by their vulnerability to manipulation. We will discuss the first issue by theory and the second by simulation the following sections.

2.4.2 Measure on restricted domain

Based on the observation from the examples, we show that the measure by Pathak and Sönmez [42] does not work under some preference domain: the mechanisms in Theorem 1 are equally manipulable.

We consider a *tiered preference domain* studied by Kesten and Kurino [34]. For $T \subsetneq S$, the set of **T-tier preferences** is defined as

$$\mathcal{R}_i(T) \equiv \{R_i \in \mathcal{R}_i : \forall t \in T, \forall s \in S \setminus T, tP_i s \text{ and } tP_i\emptyset\}.$$

In words, under preference $R_i \in \mathcal{R}_i(T)$ agent i's top $|T|$ choices in T. Denote $\mathcal{R}(T) \equiv \times_{i\in I}\mathcal{R}_i(T)$. We say that a domain $\mathcal{D}$ is the **T-tiered preference domain** if $\mathcal{D} \equiv \mathcal{R}(T)$.

In addition, we impose two assumptions.

Assumption 1. The school capacities are in short supply: $|I| > |\sum_{s\in S} q_s|$.

Assumption 2. For each student i, all school s are acceptable: for all $i \in I$, $R_i \in \mathcal{R}_i(T)$ and $s \in S$, $sP_i\emptyset$.

Pathak and Sönmez [42] impose Assumption 1 and 2 to analyze the admissions reforms in Chicago public high school choice.

Proposition 4. *Suppose Assumption 1 and 2 and let $T \subsetneq S$. Then for all k and l with $1 \leq k < l \leq |T|$,*

(i) DA^l is at least as manipulable as BOS^l on the T-tiered preference domain $\mathcal{R}(T)$;

(ii) DA^l is at least as manipulable as DA^k on the T-tiered preference domain $\mathcal{R}(T)$.

Proof. It is suffice to show that for all $R \in \mathcal{R}(T)$, R is vulnerable under DA^l. There exists student i such that $DA^l_i(R) = \emptyset$ since $|I| > |\sum_{s\in T} q_s|$ by the first assumption. Notice that for all $s \in S \setminus T$, $DA^l_s(R) = \emptyset$. By the second assumption, we have $sP_i\emptyset$. Thus, i can manipulate DA^l at R: $s = DA^l(\tilde{R}_i, R_{-i})P_i DA^l_i(R)$ where s is the first choice at $\tilde{R}_i$. $\square$

This result states that the implication in Theorem 1 cannot be carried over the T-tired preference domain. By the definition of vulnerability, a preference profile is often vulnerable under any mechanism. Due to this problem, any preference profile from the natural domain is always vulnerable under many mechanisms. As a results, we cannot distinguish the mechanisms by their vulnerability of manipulation. This motivates to introduce our manipulability measure.

2.5 Main results

In this section, we provide our main results. First, we define our measure using a motivating example. Second, we show that our measure can carry out the implications by Pathak and Sönmez [42] in the general preference domain. Third, we show that our measure is more robust: it works under the tier preference domain as well. Fourth, we run simulations for quantitative analysis and support the recent admissions reforms quantitatively, as well as qualitatively.

2.5.1 New measure

We motivate to introduce our measure by using a example. Example 1 and Example 2 provide preferences that the set of students who can manipulate the mechanism is strictly included in the other. A natural conjecture is that if student i cannot manipulate DA^k at preference profile R, then i cannot also manipulate DA^{k+1} at R.[1] That is, opportunities of manipulation for each student i is decreasing in k for each preference profile R. The following example illustrates that this is not in case

Example 7. Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}$, and $q_1 = q_2 = q_3 = 1$. Each school s has a common prioirty, $i_1 \succ_s i_2 \succ_s i_3 \succ_s i_4$. Let R be as given below.

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_1	s_1	s_2	s_2
$\vdots$	s_2	s_1	s_3
	s_3	s_3	$\vdots$

Consider DA^1. The matching produced by DA^1 at R is given by

$$DA^1(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_3 & \emptyset \end{pmatrix}.$$

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[1] This measure is studied in Pathak and Sönmez [42] as the *strong manipulability*. In terms of the measure, Example 3 illustrates that DA^k is not strongly manipulable than DA^{k+1}.

Consider DA^2. The matching produced by DA^2 at R is given by

$$DA^2(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & i_4 \end{pmatrix}.$$

DA^1 is manipulable by i_2 at R since i_2 can be better off by reporting s_3 as the first choice. Similarly, DA^1 is manipulable by i_4. DA^2 is manipulable by i_3 at R since i_3 can be better off by reporting s_3 as the first choice. Note that the set of students who can manipulate DA^2 at R is not included in that of DA^1 at R i.e., $\{i_3\} \subseteq \{i_2, i_4\}$. However, the number of students who can manipulate DA^2 is smaller i.e., $|\{i_3\}| < |\{i_2, i_4\}|$.

Now we introduce our measure. Our notion is based on the following idea. For preference profile R, we count the number of students who can manipulate φ at R. According to our measure, R is more vulnerable *in number* under BOS^2 than that under DA^2 in Example 7. We formalize our measure as follows.

Definition 23. A mechanism ψ is **at least as manipulable in number** as mechanism φ on domain $\mathcal{D}$ if for all $R \in \mathcal{D}$,

$$|\{i \in I : \psi \text{ is manipulable by } i \text{ at } R\}| \geq |\{i \in I : \varphi \text{ is manipulable by } i \text{ at } R\}|.$$

Definition 24. A mechanism ψ is **more manipulable in number** than mechanism φ on domain $\mathcal{D}$ if

(i) ψ is at least as manipulable in number as φ on domain $\mathcal{D}$ and

(ii) there exists $R \in \mathcal{D}$ such that

$$|\{i \in I : \psi \text{ is manipulable by } i \text{ at } R\}| > |\{i \in I : \varphi \text{ is manipulable by } i \text{ at } R\}|.$$

2.5.2 Theoretical results

In this section, we provide our theoretical results. First we show that our measure can keep the implications in Theorem 1 by Pathak and Sönmez [42]. While these results are parallel to the results in literature, the proof is different. We use the key property of DA, *individual rational monotonicity* proposed by Kojima and Manea [36], to prove them which are theoretically interesting for themselves. Second, unlike Pathak and Sönmez [42], our measure keeps the implication under the tier preference domain as well.

New Measure on full domain

First, we show that how the implications in Theorem 1 can be carried out under our measure as well. While this result is parallel to the result by Pathak and Sönmez [42], its proof is different. Our measure makes analysis hard since we need to count the number of students who can manipulate. We use the property of the DA mechanism, individual rational (IR) monotonicity proposed by Kojima and Manea [36]. Intuitively, IR monotonicity states that all students be weakly better off when some students claim fewer schools. A key observation is that R^k can be seen that all students claim fewer schools from R^{k+1}. Thus, we can apply IR monotonicity.

Now we provide the first theoretical result.

Theorem 8. *For all k and l with $1 \leq k < l \leq |S|$,*

(i) BOS^l is more manipulable in number than DA^l on the full domain $\mathcal{R}$;

(ii) DA^k is more manipulable in number than DA^l on the full domain $\mathcal{R}$.

Proof. **First part.** The proof is contained in Appendix.

Second part. We use individual rational monotonicity, introduced by Kojima and Manea [36].[2] We say that R'_i is an **individually rational monotonic transformation** of R_i at

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[2] Kojima and Manea [36] provide a characterization of the DA mechanism based on individual rational monotonicity.

$s \in S \cup \{\emptyset\}$ (R'_i i.r.m.t. R_i at s) if for each $t \in S \cup \{\emptyset\}$, tP'_is and $tP'_i\emptyset$ implies tP_is. In words, any school ranked above both s and $\emptyset$ at R'_i is also ranked above s at R_i. We say that R' is an individually rational monotonic transformation of R at a matching μ (R' i.r.m.t. R at μ) if for each $i \in I$, R'_i i.r.m.t. R_i at μ_i. We write $\mu' R\mu$ if and only if for each $i \in N$, $\mu'_i R_i \mu_i$.

Definition 25. A mechanism φ satisfies **individual rational monotonicity** if for all $R, R' \in \mathcal{R}$,

$$R' \text{ i.r.m.t. } R \text{ at } \varphi(R) \Rightarrow \varphi(R')R'\varphi(R).$$

In words, individual rational monotonicity states that if students change their preference profile according to an individual rational monotonic transformation, then the matching is weakly preferable for all students to the original matching.

We start with lemmas.

Lemma 4. *Let k with $1 \leq k \leq |S|$. For all student $i \in I$ and preference profile $R \in \mathcal{R}$, if DA^k is manipulable by i at R, then $DA^k_i(R) = \emptyset$.*

Proof. For the sake of contradiction, suppose that $DA^k_i(R) \neq \emptyset$. By the definition of the constrained mechanism, we have $DA^k_i(R) = DA_i(R^k)$. Since $DA^k_i(R) \neq \emptyset$, we have $DA_i(R^k) = DA_i(R_i, R^k_{-i})$. Since DA^k is manipulable by i at R, there exists $\tilde{R}_i$ such that $DA^k_i(\tilde{R}_i, R_{-i})P_iDA^k_i(R)$. Again, by definition of the constrained mechanism, we have $DA^k_i(\tilde{R}_i, R_{-i}) = DA_i(\tilde{R}^k_i, R^k_{-i})$. Together with these facts, we have $DA_i(\tilde{R}^k_i, R^k_{-i})P_iDA_i(R_i, R^k_{-i})_i$, which is a contradiction to strategy-proofness of DA. □

Define $S_i(R_{-i}, \varphi) \equiv \{s \in S : s = \varphi_i(\tilde{R}_i, R_{-i}) \text{ for some } \tilde{R}_i\}$. In words, $S_i(R_{-i}, \varphi)$ is a set of schools that i can be potentially matched at R_{-i} under φ.

Lemma 5. *Let k with $1 \leq k < |S|$. For all preference profile $R \in \mathcal{R}$, student $i \in I$, and school $s \in S$,*

$$S_i(R_{-i}, DA^{k+1}) \subseteq S_i(R_{-i}, DA^k).$$

Proof. Pick any $s \in S_i(R_{-i}, DA^{k+1})$ with $s = DA_i^{k+1}(\tilde{R}_i, R_{-i})$. Construct $\hat{R}_i$ from $\tilde{R}_i$ by ranking s first, that is, $s\hat{R}_i s'$ for all $s' \in S \cup \{\emptyset\}$. Notice $\hat{R}_i^k$ i.r.m.t. $\tilde{R}_i^{k+1}$ at $s = DA_i^{k+1}(\tilde{R}_i, R_{-i})$. We also note R_j^k i.r.m.t. R_j^{k+1} at $DA_j^{k+1}(\tilde{R}_i, R_{-i})$ for every $j \in S \setminus \{i\}$. Thus, we have $(\hat{R}_i^k, R_{-i}^k)$ i.r.m.t. $(\tilde{R}_i^{k+1}, R_{-i}^{k+1})$ at $DA(\tilde{R}_i^{k+1}, R_{-i}^{k+1})$. By individual rational monotonicity, we have $DA(\hat{R}_i^k, R_{-i}^k)\hat{R}_i DA_i(\tilde{R}_i^{k+1}, R_{-i}^{k+1}) = s$. Since s is ranked fist at $\hat{R}_i$, we have $DA(\hat{R}_i^k, R_{-i}^k) = s$. Thus, we have $s \in S_i(R_{-i}, DA^k)$, which completes the proof. □

Let $I(\varphi, \psi, R)$ be the set of students who manipulate under mechanism φ but not under mechanism ψ at preference profile R.

Lemma 6. *Let k with $1 \leq k < |S|$. For all preference profile $R \in \mathcal{R}$ and student $i \in I$,*

$$i \in I(DA^{k+1}, DA^k, R) \Rightarrow DA_i^{k+1}(R) = \emptyset \text{ and } DA_i^k(R) \neq \emptyset.$$

Proof. $DA_i^{k+1}(R) = \emptyset$ follow from Lemma 1. Suppose $DA_i^k(R) = \emptyset$. Since $i \in I(DA^{k+1}, DA^k, R)$, i can manipulate DA^{k+1} at R. Thus, there exits $s \in S$ such that $s \in S_i(R_{-i}, DA^{k+1})$ and $sP_i\emptyset$. We have $s \in S_i(R_{-i}, DA^k)$ by Lemma 3 and thus $sP_iDA_i^k(R)$, which contradicts that i cannot manipulate DA^k at R. □

Lemma 7. *Let k with $1 \leq k < |S|$. For all preference profile $R \in \mathcal{R}$ and student $i \in I$,*

$$DA_i^k(R) = \emptyset \text{ and } DA_i^{k+1}(R) \neq \emptyset \Rightarrow i \in I(DA^k, DA^{k+1}, R).$$

Proof. By Lemma 1, i cannot manipulate DA^{k+1} at R. By Lemma 2, we have $DA_i^{k+1}(R) \in$

$S_i(R_{-i}, DA^k)$. Thus, i can manipulate DA^k at R. □

Lemma 8. *Let k with $1 \leq k < |S|$. For all preference profile $R \in \mathcal{R}$, we have*

$$|\{i \in I : DA_i^k(R) \neq \emptyset\}| \leq |\{i \in I : DA_i^{k+1}(R) \neq \emptyset\}|.$$

Proof. For the sake of contradiction, suppose that

$$|\{i \in I : DA_i^k(R) \neq \emptyset\}| > |\{i \in I : DA_i^{k+1}(R) \neq \emptyset\}|.$$

By the definition of a matching,

$$\begin{aligned}\sum_{s \in S} |\{i \in I : DA_i^k(R) = s\}| &= |\{i \in I : DA_i^k(R) \neq \emptyset\}| \\ &> |\{i \in I : DA_i^{k+1}(R) \neq \emptyset\}| \\ &= \sum_{s \in S} |\{i \in I : DA_i^{k+1}(R) = s\}|.\end{aligned}$$

Thus, there exists school $s \in S$ such that

$$|\{i \in I : DA_i^k(P, \succ) = s\}| > \{i \in I : DA_i^{k+1}(P, \succ) = s\}|.$$

This implies that there exists student $j \in I$ such that $DA_j^{k+1}(R) \neq s$ and $DA_j^k(R) = s$. Notice R^k i.r.m.t. R^{k+1} at $DA(R^{k+1})$. By individual rational monotonicity, we have $DA(R^k)R^kDA(R^{k+1})$. For student j, we have $s = DA_j(R^k)P_j^kDA_j(R^{k+1})$. Together with $sP_j^k\emptyset$, $sP_j^kDA_j(R^{k+1})$, and R_j^k i.r.m.t. R_j^{k+1} at $DA_j(R^{k+1})$, we have $sP_j^{k+1}DA_j(R^{k+1})$. However, $|\{i \in I : DA_i^{k+1}(R) = s\}| < q_s$, which is a contradiction to non-wastefulness of DA. □

Now we show Theorem 1. It is sufficient to show that

$$|I(DA^{k+1}, DA^{k+1}, R)| \leq |I(DA^k, DA^{k+1}, R)|.$$

By the lemmas, we have

$$\begin{aligned}
|I(DA^{k+1}, DA^k, R)| &\leq |\{i \in I : DA_i^{k+1}(R) \neq \emptyset, DA_i^k = \emptyset\}| \\
&\leq |\{i \in I : DA_i^k(R) \neq \emptyset, DA_i^{k+1} = \emptyset\}| \\
&\leq |I(DA^k, DA^{k+1}, R)|.
\end{aligned}$$

The first inequality follows from Lemma 3, the second inequality follows from Lemma 5, and the third inequality follows from Lemma 4.

□

Measure in restricted domain

We show that our measure is more robust. Our measure works under the tier preference domain as well. Specifically, implications in Theorem 1 can be carried over the tier preference domain based on our measure.

We provide the second theoretical result. Here, we impose same assumptions in Section 4.2.

Proposition 5. *Suppose Assumption 1 and 2 and let* $T \subsetneq S$. *Then for all* k *and* l *with* $1 \leq k < l \leq |T|$,

(i) BOS^l *is more manipulable in number than* DA^l *on the* T*-tiered preference domain* $\mathcal{R}(T)$;

(ii) DA^k *is more manipulable in number than* DA^l *on the* T*-tiered preference domain* $\mathcal{R}(T)$.

Proof. **First part**: By Theorem 8, BOS^l is at least as manipulable in number as DA^l on the T-tiered preference domain $\mathcal{R}(T)$. The following example illustrates that there exists $R \in \mathcal{R}(T)$ such that

$$|I(DA^2, BOS^2, R)| < |I(BOS^2, DA^2, R)|.$$

Thus, we show that BOS^2 is more manipulable in number than DA^2 on the T-tiered preference domain $\mathcal{R}(T)$.

Example 8. Let $I = \{i_1, i_2, i_3, i_4, i_5\}$, $S = \{s_1, s_2, s_3, s_4\}$, $T = \{s_1, s_2, s_3\}$, and $q_1 = q_2 = q_3 = q_4 = q_5 = 1$. Each school s has a common prioirty, $i_1 \succ_s i_2 \succ_s i_3 \succ_s i_4 \succ i_5$. Let R be as given below.

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}
s_1	s_1	s_1	s_2	s_1
$\vdots$	s_2	s_2	s_3	s_2
	s_3	s_3	$\vdots$	$\vdots$

Consider DA^2. The matching produced by DA^2 at R is given by

$$DA^2(R) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_1 & i_2 & i_4 & \emptyset \end{pmatrix}.$$

i_3 and i_5 have the incentive to manipulate to report s_4 as the first choice. Consider BOS^2. The matching produced by BOS^2 at R is given by

$$BOS^2(R) = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ i_1 & i_4 & \emptyset & \emptyset \end{pmatrix}.$$

i_2, i_3 and i_5 have the incentive to manipulate to report s_4 as the first choice. Thus, we have

$$|I(DA^2, BOS^2, R)| < |I(BOS^2, DA^2, R)|.$$

We can generalize this example: for all k with $2 < k \leq |T|$, BOS^k is more manipulable in number than DA^k on $\mathcal{R}(T)$. Pick any k with $2 < k \leq |T|$. Consider additional $k-1$ schools and $k-1$ students, $\{s_1^*, ..., s_{k-1}^*\}$ and $\{i_1^*, ..., i_{k-1}^*\}$. Let $T = \{s_1^*, ..., s_{k-1}^*\}$ and $q_l = 1$ for all $l \in \{1, ..., k-1\}$. We also consider that

- $s^* P_j s$ for all $j \in \{i_1, ..., i_5\}$, $s^* \in \{s_1^*, ..., s_{k-1}^*\}$, and $s \in \{s_1, s_2, s_3, s_4\}$;
- i_l^* ranks s_l^* first for all $l \in \{1, ..., k-1\}$;
- Each school s has a common prioirty, $i_1^* \succ i_2^* \succ ... \succ i_{k-1}^* \succ i_1 \succ i_2 \succ i_3 \succ i_4 \succ i_5$.

Notice that i_l^* has no incentive to manipulate since i_l^* is assigned her first choice s_l^* under both BOS^k and DA^k. Thus, as in the example above, we have

$$|I(DA^k, BOS^k, R)| < |I(BOS^k, DA^k, R)|.$$

Second part: By Theorem 8, DA^{k-1} is at least as manipulable in number as DA^k on the T-tiered preference domain $\mathcal{R}(T)$. The following example illustrates that there exists $R \in \mathcal{R}(T)$ such that

$$|I(DA^2, DA^1, R)| < |I(DA^1, DA^2, R)|.$$

Thus, we show that DA^1 is more manipulable in number than DA^2 on the T-tiered preference domain $\mathcal{R}(T)$.

Example 9. Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}$, $T = \{s_1, s_2\}$, and $q_1 = q_2 = q_3 = 1$. Each school s has a common prioirty, $i_1 \succ_s i_2 \succ_s i_3 \succ_s i_4$. Let R be as given below.

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}
s_1	s_1	s_1	s_1
$\vdots$	s_2	s_2	s_2
	s_3	s_3	s_3

Consider DA^1. The matching produced by DA^1 at R is given by

$$DA^1(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & \emptyset & \emptyset \end{pmatrix}.$$

i_2, i_3 and i_4 have the incentive to manipulate to report s_3 as the first choice. Consider DA^2. The matching produced by DA^2 at R is given by

$$DA^2(R) = \begin{pmatrix} s_1 & s_2 & s_3 \\ i_1 & i_2 & \emptyset \end{pmatrix}.$$

i_3 and i_4 has the incentive to manipulate to report s_3 as the first choice. Thus, we have

$$|I(DA^2, DA^1, R)| < |I(DA^2, DA^1, R)|.$$

We can generalize this example: for all k with $2 < k \leq |T|$, DA^{k-1} is more manipulable in number than DA^k on $\mathcal{R}(T)$. Pick any k with $2 < k \leq |T|$. Consider additional $k-1$ schools and $k-1$ students, $\{s^*_1, ..., s^*_{k-1}\}$ and $\{i^*_1, ..., i^*_{k-1}\}$. Let $T = \{s^*_1, ..., s^*_{k-1}\}$ and $q_l = 1$ for all $l \in \{1, ..., k-1\}$. We also consider that

- $s^* P_j s$ for all $j \in \{i_1, ..., i_5\}$, $s^* \in \{s^*_1, ..., s^*_{k-1}\}$, and $s \in \{s_1, s_2, s_3\}$;
- i^*_l ranks s^*_l first for all $l \in \{1, ..., k-1\}$, and
- Each school s has a common prioirty, $i^*_1 \succ i^*_2 \succ ... \succ i^*_{k-1} \succ i_1 \succ i_2 \succ i_3 \succ i_4 \succ i_5$.

Notice that i^*_l has no incentive to manipulate since i^*_l is assigned her first choice s^*_l under both DA^{k-1} and DA^k. Thus, as in the example above, we have

$$|I(DA^k, DA^{k-1}, R)| < |I(DA^{k-1}, DA^k, R)|.$$

□

The results in this section qualitatively support school admissions reforms in school districts: the admission reforms successfully discourage manipulation. These implications lead to a quantitative question: How much these reforms reduce manipulability? We will answer this question by using simulation the following section.

2.5.3 Simulation

In this section, we use simulations for quantitative analysis. The first simulation is on our manipulability measure and provides further quantitative analysis of our theoretical results. The second simulation is on the measure by Pathak and Sönmez [42]. While our qualitative results from theory are parallel to their measure, the quantitative results from simulations confirm a significant difference.

There are at least two advantages for qualitative analysis. First, quantitative analysis enables us to measure the magnitude of manipulability. We want to know how much the current school choice mechanisms are manipulable since they are still manipulable. The second advantage is a clear understanding of the differences between the two measures. While the two measures differ qualitatively in the restricted domain, they lead to the same implications in the full domain. If there is a quantitative difference in the general domain, we might understand each measure better. However, it is hard to get a quantitative result by theoretical analysis. Thus, we run simulations to get observations for quantitative analysis.

For quantitative analysis, we define a magnitude of manipulability of mechanism φ in terms of our measure,

$$M(\varphi) \equiv \frac{\sum_{R \in \mathcal{R}} |\{i \in I : \varphi \text{ is manipulable by } i \text{ at } R\}|}{|\mathcal{R}| \times |I|}.$$

Similarly, we define a magnitude of manipulability of mechanism φ in terms of the measure by Pathak and Sönmez [42],

$$M_{PS}(\varphi) = \frac{|\{R \in \mathcal{R} : R \text{ is vulnerable under mechanism } \varphi\}|}{|\mathcal{R}|}.$$

To run simulation, we impose several assumptions throughout this section.[3] First, we assume that every student accepts all schools, and every school accepts all students. Second, the number of students is equal to the number of total capacities of schools i.e., $|I| = \sum_{s \in S} q_s$. Third, we assume homogeneous capacities i.e., $q_s = q_{s'}$ for all $s, s' \in S$. The setting of our simulation is as follows. The number of students is 50. We run three types of simulations by setting that a pair of the number of schools and capacities $(|S|, q_s)$ to $(25, 2)$, $(10, 5)$, and $(5, 10)$. Each school s draws priority $\succ_s$ from a uniform distribution over $\mathcal{F}_s$. This priority profile $\succ$ is fixed. Then, for computational burden, we run the mechanisms 100 times where each agent i draws a preference relation R_i from a uniform distribution over $\mathcal{R}_i$.

The first set of simulations suggests two implications of DA^k on our measure: Figure 1, 2, and 3 show that (i) $M(DA^k)$ is quickly decreasing, and (ii) large k is not necessary if we can allow a small level of $M(DA^k)$. These implications possibly support the recent school admissions reforms quantitatively, as well as qualitatively. While the current matching mechanisms are still manipulable, their manipulability would be sufficiently low.

The second set of simulations suggests two implications of the measure by Pathak and Sönmez [42]. Figure 4, 5, and 6 provides a contrast with our measure: (i) $M_{PS}(DA^k)$ is not quickly decreasing, and (ii) large k is necessary even if we can allow a small level of

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[3] Our theoretical results do not rely on these assumptions.

$M_{PS}(DA^k)$. The definition of vulnerability would explain the difference between the two measures. It is rare that no one has an incentive to manipulate, especially for small k.

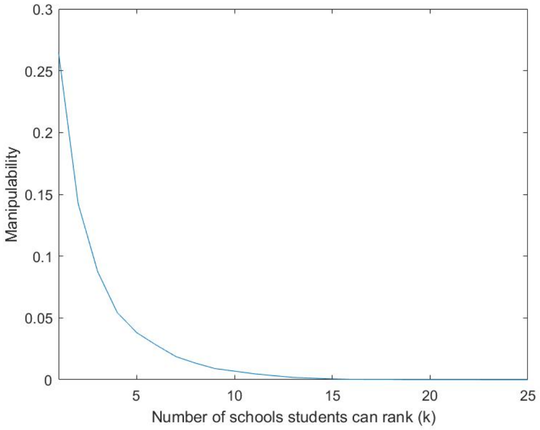

Figure 2.1: $M(DA^k)$ with $(|S|, q_s) = (25, 2)$.

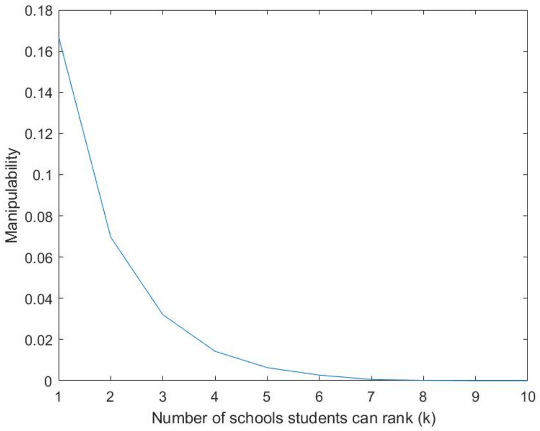

Figure 2.2: $M(DA^k)$ with $(|S|, q_s) = (10, 5)$.

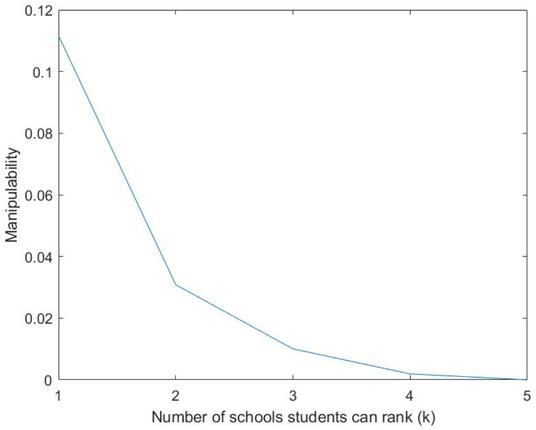

Figure 2.3: $M(DA^k)$ with $(|S|, q_s) = (5, 10)$.

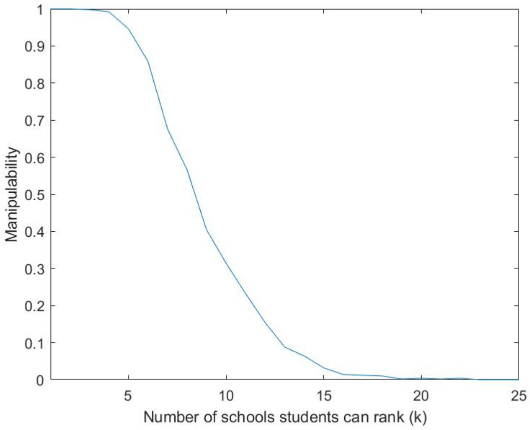

Figure 2.4: $M_{PS}(DA^k)$ with $(|S|, q_s) = (25, 2)$.

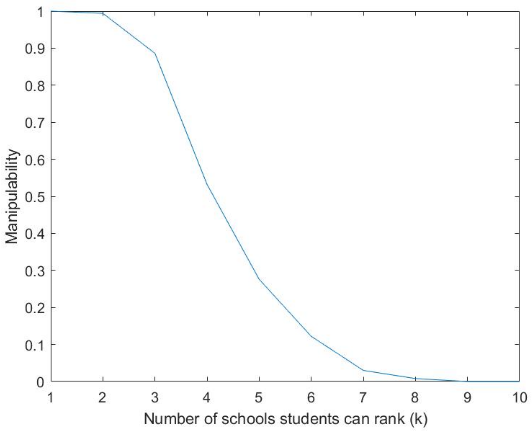

Figure 2.5: $M_{PS}(DA^k)$ with $(|S|, q_s) = (10, 5)$.

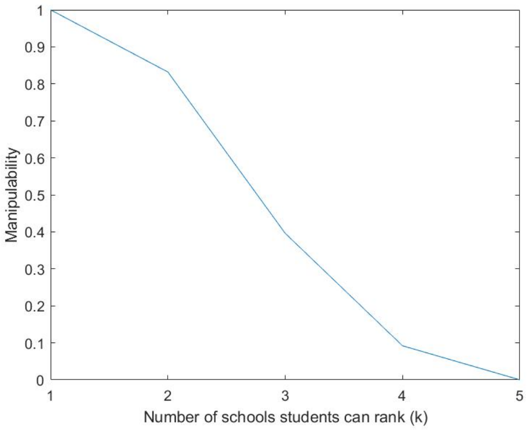

Figure 2.6: $M_{PS}(DA^k)$ with $(|S|, q_s) = (5, 10)$.

2.6 Appendix: Omitted proofs

Proof of the first part in Theorem 2

Proof of the first part in Theorem 2

Proof. Fix any preference profile R. Pick any student i such that i can manipulate DA^k at R. The proof consists of two steps.

Step 1: There exists student $j(i)$, possibly $j(i) = i$, such that $j(i)$ can manipulate BOS^k at R.

By the definition of DA^k, if $DA_i^k(R) \neq \emptyset$, then i cannot manipulate $DA^k(R)$. Thus, we have $DA_i^k(R) = \emptyset$. There are two cases to consider.

Case 1: $BOS_i^k(R) = \emptyset$. We show that i can manipulate $BOS^k(R)$. Suppose not. Since i can manipulate DA^k at R, there exists $\tilde{R}_i$ and $s \in S$ such that $s = DA_i^k(\tilde{R}_i, R_{-i}) P_i DA_i^k(R)$. By Lemma 1, there exists student j such that $BOS_j^k(R) = s$, $j \succ_s i$, and $r(s, R_j) = 1$. Since $s = DA_i^k(\tilde{R}_i, R_{-i})$, we have $DA_j^k(\tilde{R}_i, R_{-i}) \neq s$. However, this violates that $DA^k(\tilde{R}_i, R_{-i})$ is stable at $(\tilde{R}_i^k, R_{-i}^k)$.

Case 2: $BOS_i^k(R) \neq \emptyset$.

Construct a sequence of students $(i_0, ..., i_m)$ such that

(i) $i_0 = i$;

(ii) For all $l \in \{1, ..., m-1\}$, we have $BOS_{i_{l-1}}^k(R) = DA_{i_l}^k(R) \neq \emptyset$ and $BOS_{i_l}^k(R) = DA_{i_{l+1}}^k(R)$;

(iii) For m, $DA_{i_m}^k(R) = BOS_{i_{m-1}}^k(R) \neq \emptyset$ and either $BOS_{i_m}^k(R) = \emptyset$ or $BOS_{i_m}^k(R) = s$ where $DA_s^k(R) = \emptyset$.

For student i, we have $BOS_i^k(R) P_j DA_i^k(P) = \emptyset$. Also, we have $DA_{i_m}^k(R) P_{i_m} BOS_{i_m}^k(R)$ since $DA_{i_m}^k(R) \neq \emptyset$ and $DA_s^k(R) = \emptyset$. Thus, there exists $m' \leq m$ such that (i) for all $l < m'$, we have $BOS_{i_l}^k(R) P_{i_l} DA_{i_l}^k(R)$ and (ii) $DA_{i_{m'}}^k(R) P_{i_{m'}} BOS_{i_{m'}}^k(R)$. Notice that $i_{m'}$

can manipulate $BOS^k(R)$. This is because $DA^k_{i_{m'}}(R) = BOS^k_{i_{m'-1}}(R) P_{i_{m'-1}} DA^k_{i_{m'-1}}(R)$ and stability of DA^k at R^k imply $i_{m'} \succ_{BOS^k_{i_{m'-1}}(R)} i_{m'-1}$.

Step 2: If students i_1 and i_2 with $i_1 \neq i_2$ can manipulate DA^k at R, then $j(i_1) \neq j(i_2)$.

If either $i^1 = j(i^1)$ or $i^2 = j(i^2)$, then $j(i^1) \neq j(i^2)$. Thus, suppose $i^1 \neq j(i^1)$ and $i^2 \neq j(i^2)$. By the argument in Case 2, there exist sequences of students $(i^1_0, ..., i^1_m)$ and $(i^2_0, ..., i^2_p)$. We show that $\{i^1_0, ..., i^1_m\} \cap \{i^2_0, ..., i^2_p\} = \emptyset$. The proof proceeds by induction on $l \in \{0, 1, ..., m\}$.

Base step: Since $DA^k_{i^1_0}(R) = \emptyset$ and $DA^k_j(R) \neq \emptyset$ for all $j \in \{i^2_1, ..., i^2_p\}$, we have $i^1_0 \neq j$. By the assumption in Step 2, we have $i^1_0 \neq i^2_0$. Thus, $i^1_0 \notin \{i^2_0, ..., i^2_p\}$.

Inductive step: Suppose that $i^1_l \in \{i^2_0, ..., i^2_p\}$. We show that $i^1_{l+1}\{i^2_0, ..., i^2_p\}$. Suppose not. By the construction of the sequence of $(i^2_0, ..., i^2_p)$, there exists $j \in \{i^2_0, ..., i^2_p\}$ such that $BOS^k_j(R) = DA^k_{i^1_{l+1}}(R)$. By the construction of the sequence of $(i^1_0, ..., i^1_m)$, $BOS^k_{i^1_l}(R) = DA^k_{i^1_{l+1}}(R)$. However, we have $i^1_l \neq j$ by the inductive hypothesis, a contradiction for one-to-one matching. $\square$

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