Semi-empirical mass formula for Λ-hypernuclei

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Abstract: We have studied the A-dependence of the separation energy, S_Λ, of the Λ-hyperon in Λ-hypernuclei by fitting the parameters of a semi-empirical mass formula employed in previous works, which is based on the hypernuclear finite size. We extend this formula by considering two corrections: one takes into account the finite mass of the hypernucleus in terms of a Λ reduced mass and the other implements the effects of charge symmetry breaking (CSB). Both corrections are found non-negligible for finite hypernuclei. Moreover, our study reveals some inconsistencies in the S_Λ data of mirror hypernuclei, preventing us from obtaining a well-determined CSB parameter.

I. INTRODUCTION

Hypernuclei are bound systems of nucleons and one or more strange baryons named hyperons [1, 2]. They are created in hadronic or electromagnetic reactions on nuclei, in which a nucleon is replaced by a hyperon and, because of that, the final hypernucleus has different properties than the target nucleus. These reactions have disclosed several grounds and excited states of Λ-hypernuclei covering the whole periodic table.

The study of Λ-hypernuclei is interesting because it permits obtaining information on the ΛN interaction, which is a very difficult task from scattering experiments because the Λ lifetime is 10^{-10} s [3].

The main objective of this work is to build a semi-empirical hypernuclear mass formula which describes satisfactorily the ground-state masses of the observed Λ-hypernuclei. There exist several works on this subject in the literature [4–6]. We aim to go beyond these studies taking additional considerations. On the one hand, we will take into account the finite mass of the nucleus upon replacing the Λ-mass by the reduced mass of the Λ-nucleus system. On the other hand, we will introduce a new term that accounts for the charge-symmetry violation recently seen in light hypernuclei [7] and will analyze its consequences on other hypernuclear species.

II. FORMALISM

A. Semi-empirical approach to the Λ-separation energy

A Λ-hypernucleus is denoted by ${}^{A}_{\Lambda}\mathrm{Z}$ where Z is the number of protons and $A = N + Z + 1$ denotes the total number of baryons with N being the number of neutrons. The mass of a Λ-hypernucleus can be related to its binding energy by:

$$M(N, Z, \Lambda) = Nm_n + Zm_p + m_\Lambda - B(N, Z, \Lambda)$$

where m_n is the neutron mass, m_p is the proton mass and m_Λ is the Λ-baryon mass.

Expressing the hypernuclear mass in terms of the mass of the nuclear core, $M(N, Z, \Lambda) = M_c(N, Z) + m_\Lambda - S_\Lambda$, we can relate the binding energy of the hypernucleus to that of the nucleus and the separation energy of the Λ, S_Λ:

$$B(N, Z, \Lambda) = B(N, Z) + S_\Lambda.$$

In this work, we will obtain a semi-empirical parametrization of S_Λ. A few experimental values of S_Λ are shown in the next table.

Hypernuclei	S_Λ (MeV)
${}^{4}_{\Lambda}\mathrm{H}$	2.04 ± 0.08
${}^{4}_{\Lambda}\mathrm{He}$	2.39 ± 0.07
${}^{5}_{\Lambda}\mathrm{He}$	3.12 ± 0.06
${}^{6}_{\Lambda}\mathrm{He}$	4.18 ± 0.14
${}^{7}_{\Lambda}\mathrm{He}$	5.55 ± 0.21
${}^{7}_{\Lambda}\mathrm{Li}$	5.82 ± 0.16
${}^{7}_{\Lambda}\mathrm{Be}$	5.16 ± 0.12
${}^{8}_{\Lambda}\mathrm{He}$	7.16 ± 0.74
${}^{8}_{\Lambda}\mathrm{Li}$	6.80 ± 0.07

TABLE I: Experimental values of S_Λ for the lightest Λ-hypernuclei, taken from the compilation done in [7].

Analytical expectation of E_Λ

In this section, we obtain the analytic value of the Λ ground state energy E_Λ inside a potential well $V(r)$:

$$V(r) = \begin{cases} -V_0 & 0 < r \le R \\ 0 & r > R \end{cases}$$

of depth V_0 and size R, which will be identified with the size of the hypernucleus, to gain insight on the parametrization for S_Λ that will be introduced in the next section.

For a Λ in its ground state, having angular momentum L = 0, the solution of the 3D-Schrödinger equation gives

$$E_\Lambda = -V_0 + \frac{\pi^2\hbar^2}{2m_\Lambda R^2}. \tag{1}$$

Analytical expectation of S_Λ

To write a theoretical formula for S_Λ we simply make the identification $S_\Lambda = |E_\Lambda|$ and implement the A-dependence on the radius R. In most studies the hypernucleus radius R is taken as:

$$R = r_0 \cdot A^{1/3} \tag{2}$$

with $r_0 \approx 1.1 - 1.2$ fm.

However, Millener et al. pointed out in [8] that an A-dependent r_0, of the type

$$r_0 = a + b \cdot A^{-2/3}, \tag{3}$$

provided a better description of the Λ separation energies. To obtain the expression of S_Λ we first write the $1/R^2$ factor in Eq. (1) using the prescription of Eq. (2) corrected by Eq. (3):

$$\begin{aligned} \frac{1}{R^2} &= \frac{1}{(a^2 A^{2/3} + 2ab + b^2 A^{-2/3})} \\ &= \frac{1}{a^2 A^{2/3}\,(1 + \frac{2b}{a} A^{-2/3} + \frac{b^2}{a^2} A^{-4/3})} \\ &= \frac{1}{a^2 A^{2/3}} \left(1 - \frac{2b}{a} A^{-2/3} - \frac{b^2}{a^2} A^{-4/3} + ...\right) \\ &= \frac{1}{a^2}\, A^{-2/3} - \frac{2b}{a^3}\, A^{-4/3} - \frac{b^2}{a^4}\, A^{-6/3}, \end{aligned} \tag{4}$$

where a Taylor's expansion has been done up to $A^{-6/3}$ terms. With this, we obtain

$$S_\Lambda = V_0 - \frac{\hbar^2 \pi^2}{2 m_\Lambda a^2} \left(A^{-2/3} - \frac{2b}{a} A^{-4/3} - \frac{b^2}{a^2} A^{-6/3} \right), \tag{5}$$

which leads us to write the following analytical expectation for S_Λ:

$$S_\Lambda = a_0 + a_1 A^{-2/3} + a_2 A^{-4/3} + a_3 A^{-6/3}. \tag{6}$$

In fact, using the values of $a = 1.128$ fm and $b = 0.439$ fm given in [8] and $m_\Lambda c^2$ =1115.68 MeV, we can calculate theoretically the coefficients in this expansion. They are $a_0 = V_0 = 28$ MeV, $a_1 = -135.32$ MeV, $a_2 = 105.33$ MeV and $a_3 = 20.50$ MeV.

B. Reduced mass correction

In addition to the first correction tied to the hypernuclear radius, we can consider the correction due to the finite mass of the hypernuclear core. Instead of m_Λ we will use the Λ reduced mass:

$$\mu_\Lambda = \frac{m_\Lambda M_A}{m_\Lambda + M_A}. \tag{7}$$

Replacing m_Λ by μ_Λ in Eq. (5) we obtain:

$$\begin{aligned} S_\Lambda = V_0 &- \frac{\hbar^2 \pi^2}{2a^2 M_\Lambda} \left(A^{-2/3} - \frac{2b}{a} A^{-4/3} - \frac{b^2}{a^2} A^{-6/3} \right) \\ &- \frac{\hbar^2 \pi^2}{2a^2 u} \left(A^{-5/3} - \frac{2b}{a} A^{-7/3} - \frac{b^2}{a^2} A^{-9/3} \right) \end{aligned} \tag{8}$$

where we have employed the prescription $M_A = A \cdot u$, where $u = 931.49$ MeV$/c^2$. We observe that the first part of the Eq. (8) is the same as Eq. (5), being the second part the one that provides the reduced mass correction.

C. Charge symmetry breaking corrections

Up until this part of the study, we have only assumed the A-dependence in the separation energy of the Λ baryon. According to Eq. (6), hypernuclei with the same A number have the same value of S_Λ. Nevertheless, the experimental values in Table I show that the values of S_Λ in mirror nuclei, such as ${}^4_\Lambda$H and ${}^4_\Lambda$He, are not the same. This is known as Charge Symmetry Breaking (CSB) and it is tied to the fact that the interaction of the Λ-baryon with protons could be slightly different to that with neutrons. The objective of this section will be to implement this phenomenon in our formula.

In the first place, the CSB new term should be proportional to $N - Z$, knowing that for nuclei with the same number of protons and neutrons the CSB contribution is zero. In the second place, we will normalize the contribution by the number of interacting Λn and Λp pairs in the hypernucleus, namely $N + Z = A - 1$. With this, the new formula for the separation energy of the Λ including the charge symmetry breaking term becomes:

$$\begin{aligned} S_\Lambda(A, Z) &= S_\Lambda(A) + S_\Lambda^{CSB}(A, Z) \\ &= S_\Lambda(A) + \alpha_{CSB} \cdot \frac{(N - Z)}{(A - 1)} \\ &= S_\Lambda(A) + \alpha_{CSB} \cdot \left(1 - \frac{2Z}{A - 1}\right). \end{aligned} \tag{9}$$

At this point we can obtain the value of $\alpha_{CSB}^{(A=4)}$, an estimation of the CSB coefficient that reproduces the difference between the experimental values of the $A = 4$ Λ-hypernuclei:

$$S_\Lambda^{CSB}({}^4_\Lambda\text{He}) - S_\Lambda^{CSB}({}^4_\Lambda\text{H}) \approx 0.35 \text{ MeV}.$$

Taking $Z = 2$ and $A = 4$ for ${}^4_\Lambda$He and $Z = 1$ and $A = 4$ for ${}^4_\Lambda$H in Eq. (9), we obtain

$$\alpha_{CSB}^{(A=4)} = -\frac{3}{2} \left[S_\Lambda^{CSB}({}^4_\Lambda\text{He}) - S_\Lambda^{CSB}({}^4_\Lambda\text{H}) \right] = -0.525.$$

III. RESULTS

In this third section, we present the results of the study. We have fitted the experimental values listed in Ref. [4], which amount to $n = 35$ values covering hypernuclei from $A = 4$ to $A = 208$. This has been done by employing the theoretical models, explained in the previous section, using Python as our computational tool.

A. Second and third-order approximations

In the first computational part of the study, we have assumed the dependence of Eq. (6) between the separation energy and the atomic mass of the hypernuclei. This is a polynomial equation in terms of $x \equiv A^{-2/3}$, yielding

$$S_\Lambda = a_0 + a_1 x + a_2 x^2 + a_3 x^3.$$

The coefficients are obtained using the function `polyfit`, that employs the least-squares approximation to fit the given polynomial function on the experimental values, returning the coefficients that minimize the squared residual SR:

$$SR = \sum_{j=0}^{k} |S_\Lambda(x_j) - y_j|^2,$$

with $k = 34$ being the number of the experimental values used, x_j the A value of the hypernucleus, y_j the experimental value of the separation energy and $S_\Lambda(x_j)$ the value of the separation energy obtained with the coefficients calculated with the program.

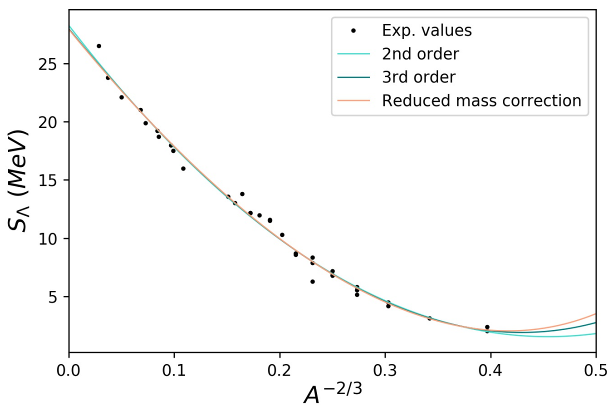

FIG. 1: Λ separation energy as a function of $A^{-2/3}$. The figure shows the experimental values of S_Λ and three polynomial fits as: the second and third order approximations and the reduced mass correction.

The coefficients from the second and third order approximations in the expansion over x are presented in the second and third columns of Table II respectively, as well as their estimated standard deviation σ, defined as

$$\sigma = \sqrt{\frac{SR}{n-1}}.$$

We find that the standard deviation from the second-order is larger than the third-order one. The third-order model is a novelty comparing to the results of Ref. [4] and it provides a better fit, in particular, it makes a significant difference for the smallest A values, an effect shown in Fig. 2. This is the reason why we have taken the third-order approximation as a conceptual basis for the subsequent corrections.

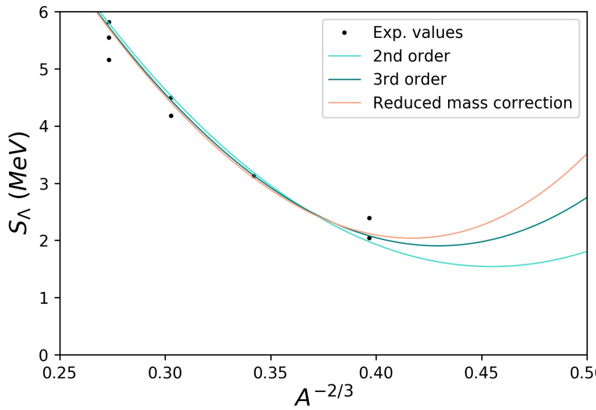

FIG. 2: This figure represents a zoom of the Fig. 1 in the low mass region to visualize better the differences between the various approaches.

B. Reduced mass correction

In the second computational part of the study, the base expression is that of Eq. (8). Therefore, we write

$$S_\Lambda = a_0 + a_1 A^{-2/3} + a_2\, A^{-4/3} + a_{5/3}\, A^{-5/3} + a_3\, A^{-6/3} + a_{7/3}\, A^{-7/3} + a_{9/3}\, A^{-9/3}. \quad (10)$$

Since this equation is not polynomial in terms of $x \equiv A^{-2/3}$, we can no longer use the `polyfit` function and instead we employ the Ridge regression method [9] to obtain the coefficients. This probabilistic method has a control parameter that blocks the coefficients from growing without physical sense, preventing the function from being over-fitted.

The coefficients obtained are shown in the third column of Table II, together with the standard deviation which is $\sigma = 0.557$. The corresponding fit is represented by the orange line in Fig. 1 and Fig. 2. As expected, we observe that the reduced mass correction affects mainly the results for low mass hypernuclei.

C. CSB correction

We will base the computational implementation of the CSB correction on Eq. (9) taking S_Λ from the third-order approximation of Eq. (6) because the reduced mass correction terms provide only a minor improvement. With the second computational method, we have obtained new coefficients that are shown in the fourth column of Table II. In this case S_Λ depends on both A and Z, which makes it impossible to do a graphic representation in our 2D plots. The CSB coefficient obtained with the computational method is

$$\alpha^c_{CSB} = 0.432,$$

shown in Table II, different in sign from the value $\alpha^{(A=4)}_{CSB} = -0.525$ extracted from the experimental Λ separation energies of $A = 4$ hypernuclei.

To understand this difference, we present in Figs. 3 and 4 the energies of mirror hypernuclei with $A = 4$ and $A = 7$, respectively. The dotted line in each figure displays the result of the CSB-S_Λ fit but setting the CSB term to zero. The dark green lines with the dot represent the experimental value of S_Λ for each hypernucleus. The light blue lines on the left represent the values of S_Λ obtained when the CSB term with the coefficient $\alpha^{(A=4)}_{CSB}$ is added to the dotted line, while the orange lines are similarly obtained but using α^c_{CSB}.

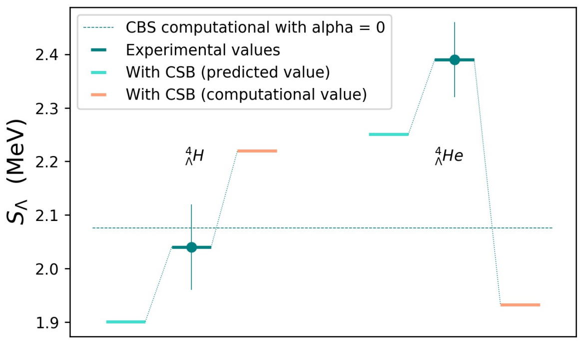

FIG. 3: Separation Λ energies for hypernuclei with $A = 4$. The horizontal dotted line corresponds to the S_Λ CSB value but setting the CSB term to zero. The points represent the experimental values of S_Λ with error. The light blue lines corresponds to the value obtained adding the predicted CSB term to the dotted line, and the orange line to the value obtained adding the computational CSB term.

Let us focus on Fig. 3 for $A = 4$. The first thing we notice is that the increase of Z implies an increase in the experimental Λ separation energy. Ideally, the dotted line should be located right in the middle of the two experimental values but this is not the case because the coefficients of the global fit try to accommodate the data of all hypernuclei. In any case, we see that the CSB contribution with $\alpha^{(A=4)}_{CSB}$ lowers the value of $S_\Lambda(^4_\Lambda\text{H})$ and rises the value of $S_\Lambda(^4_\Lambda\text{He})$ with respect to the dotted line, so that the light blue lines follow the experimental trend, as expected. Surprisingly, the global fit provides a positive value for α^c_{CSB} and, therefore, the shift of the orange lines goes in the opposite direction exhibiting an inverted behavior compared with the experimental values.

The reason for that can be understood by inspecting Fig. 4 for $A = 7$ hypernuclei. We focus on the cases of $^7_\Lambda$He and $^7_\Lambda$Be, since $^7_\Lambda$Li has $N = Z$ and is not affected by CSB effects. We see that the experimental value of S_Λ decreases when increasing the Z from $^7_\Lambda$He to $^7_\Lambda$Be, contrary to what is observed for the $A = 4$ hypernuclei. This is precisely the type of behavior that the computational method tries to model.

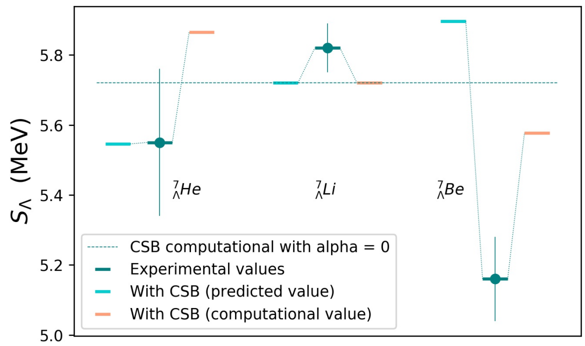

FIG. 4: The same as Fig. 3 but for hypernuclei with $A = 7$.

Coefficients' order	Approximation			
	2nd o.	3rd o.	μ_Λ	CSB
0	28.29	28.00	27.95	27.89
-2/3	-117.7	-111.0	-110.8	-108.9
-4/3	129.4	92.59	102.0	81.80
-6/3		56.91	1.251	72.06
-5/3			-4.577	
-7/3			36.09	
-9/3			66.61	
α^c_{CSB}				0.432
σ	0.577	0.564	0.557	0.557

TABLE II: Values of the coefficients for the different fits. The standard deviation is included in the last row.

IV. CONCLUSIONS

After studying the A-dependence of the separation energy of the Λ-hyperon in Λ-hypernuclei by fitting the parameters of a semi-empirical mass formula, we can conclude that the expansion based on the hypernuclear finite-size gives a proper focus to this study. Also, the two different corrections to the S_Λ presented, the reduced mass correction and the CSB correction, are non-negligible, especially in the case of finite hypernuclei.

The value of the lowest-order coefficient $a_0 \approx 28$ MeV, corresponding to the theoretical value of the Λ well potential depth, is very stable in all the approaches as we can see in Table II. This value would represent the separation energy of a Λ in a hypothetical infinite-size nuclear matter system, widely studied in many works in the literature.

The second-order approximation model, previously studied in other works [4] reproduces precisely the behavior of the experimental values. Nevertheless, we have also considered the third-order term and we have seen that it provides a better estimation of S_Λ, reducing the standard deviation compared to the second-order approach. This new term makes the model more accurate for the lightest hypernuclei as we can see in Fig. 2, where the third-order fit passes in between the two experimental values corresponding to $S_\Lambda({}^4_\Lambda\mathrm{He})$ and $S_\Lambda({}^4_\Lambda\mathrm{H})$.

In the reduced mass correction the standard deviation is even smaller. This means that this correction is valid and provides a better fit to the experimental values.

We can conclude that, in comparison to other works in the literature [4–6], the corrections implemented in the present study have provided a semi-empirical mass formula for the Λ separation energy in hypernuclei which adjusts better to the available experimental data.

In relation to the CSB correction, our study has revealed that attributing the energy differences between mirror nuclei to CSB effects is premature because there is not a coherence between the observations in $A = 4$ and $A = 7$ hypernuclei. This is presently a hot subject of study in the hypernuclear physics field. New experiments are being planned [7] to reduce the experimental errors and bring new light into this issue.

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